Sorbonne Université and South China University of Technology

École doctorale de sciences mathématiques de Paris centre

Doctoral thesis

(jointly supervised)

Discipline : Mathematics

presented by

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Some results on beta-expansions and generalized Thue-Morse sequences

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Acknowledgements

First of all I would like to deeply thank my supervisors Jean-Paul Allouche, Bing Li and Michel Pocchiola. Since I met Bing Li at South China University of Technology, he has been giving me guidance. I became his student, and under his recommendation I got a funding to study at Université Paris-Est Marne-la-Vallée for my first year in France. In 2018, Jean-Paul Allouche and Bing Li helped me successfully apply for the Oversea Study Program of Guangzhou Elite Project, which gave me the precious opportunity to study at Sorbonne Université as a joint PhD student. During my two years at Sorbonne Université and until now, Jean-Paul Allouche and Michel Pocchiola guide me and help me a lot.

I greatly acknowledge Michel Rigo and Thomas Stoll for their valuable reports on my thesis, and Hélène Frankowska, Anna Frid, Yining Hu, Thomas Stoll, Wen Wu and Reem Yassawi for accepting to be on the jury of my defence.

Besides, I owe great thanks to Lingmin Liao for his guidance during my first year in France, and to Tuomas Sahlsten for our joint work (Section 1.2 in this thesis) together with Bing Li. I would also like to thank the teachers and students I met in China and France these years for imparting knowledge, communication and help.

I am extremely grateful to the Oversea Study Program of Guangzhou Elite Project and the Guangzhou Elite Scholarship Council. Without them, I would not have the opportunity for preparing this jointly supervised thesis.

Finally, a special thanks goes to my family for encouraging me to do what I want.

Abstract

This thesis consists of three chapters including ten sections, which focus on beta-expansions, related digit frequencies, generalized Thue-Morse sequences and their relations.

Chapter 1 is devoted to greedy beta-expansions and related digit frequencies. In Section 1.1, we study the distributions and numbers of full and non-full words in greedy beta-expansions. In Sections 1.2 and 1.3, by studying Bernoulli-type measures and variational formulae respectively, we obtain some exact formulae for the Hausdorff dimension of some digit frequency sets in greedy beta-expansions.

Chapter 2 is devoted to general beta-expansions (not only the greedy ones) and related digit frequencies. In Section 2.1, we systematically study expansions in multiple bases, which are natural generalizations of usual expansions in one base. From Section 2.2 we return to expansions in one base and consider digit frequencies. In Section 2.2, we give three small results on the digit frequencies of general beta-expansions. In Section 2.3, we study Bernoulli-type measures in a framework similar to Section 1.2, and as an application we obtain the Hausdorff dimension of some frequency subsets of the set of univoque sequences.

Chapter 3 is devoted to some generalizations of the famous Thue-Morse sequence, including their relations to beta-expansions and digit frequencies. In Section 3.1, we show that a class of generalized shifted Thue-Morse sequences is strongly related to a bifurcation phenomenon on the digit frequencies of unique beta-expansions. In Section 3.2, we study expansions of generalized Thue-Morse numbers, which are defined by further generalizations of the generalized shifted Thue-Morse sequences given in Section 3.1. Finally we consider another class of generalizations of the Thue-Morse sequence in Sections 3.3 and 3.4, and respectively we study related infinite products and generalized Koch curves.

Keywords

beta-expansions, digit frequencies, generalized Thue-Morse sequences.

Résumé

Cette thèse se compose de trois chapitres comprenant dix sections, qui se concentrent sur les bêta-expansions, les fréquences de chiffres associées, les suites de Thue-Morse généralisées et leurs relations.

Le chapitre 1 est consacré aux bêta-expansions gloutonnes et aux fréquences de chiffres associées. Dans la section 1.1, nous étudions les distributions et les nombres de mots pleins et non-pleins dans les bêta-expansions gloutonnes. Dans les sections 1.2 et 1.3, en étudiant respectivement les mesures de Bernoulli-type et les formules variationnelles, nous obtenons des formules exactes pour la dimension de Hausdorff de certains ensembles de fréquences de chiffres en bêta-expansions gloutonnes.

Le chapitre 2 est consacré aux bêta-expansions générales (pas seulement les plus gloutonnes) et aux fréquences de chiffres associées. Dans la section 2.1, nous étudions systématiquement les expansions dans plusieurs bases, qui sont des généralisations naturelles d'expansions habituelles dans une base. À partir de la section 2.2 nous revenons aux expansions dans une base et considérons les fréquences de chiffres. Dans la section 2.2, nous donnons trois petits résultats sur les fréquences de chiffres des bêta-expansions générales. Dans la section 2.3, nous étudions les mesures de Bernoulli-type dans un cadre similaire à la section 1.2, et comme application nous obtenons la dimension de Hausdorff de certains sous-ensembles de fréquences de l'ensemble des séquences univoques.

Le chapitre 3 est consacré à certaines généralisations de la célèbre suite de Thue-Morse, y compris leurs relations avec les bêta-expansions et les fréquences de chiffres. Dans la section 3.1, nous montrons qu'une classe de suites de Thue-Morse décalées généralisées est fortement liée à un phénomène de bifurcation sur les fréquences de chiffres des bêta-expansions uniques. Dans la section 3.2, nous étudions les expansions des nombres de Thue-Morse généralisés, qui sont définis par d'autres généralisations des suites de Thue-Morse décalées généralisées données dans la section 3.1. Enfin, nous considérons une autre classe de généralisations de la suite de Thue-Morse dans les sections 3.3 et 3.4, et nous étudions respectivement les produits infinis associés et les courbes de Koch généralisées.

Mots-clés

bêta-expansions, fréquences de chiffres, suites de Thue-Morse généralisées.

摘要

本论文由三章共十节组成, 着重研究beta-展式, 相关的数字频率, 广义Thue-Morse序列和它们之间的联系.

第一章着重于贪婪的beta-展式和相关的数字频率. 在1.1节中, 我们研究贪婪beta-展式中满词和不满词的分布和数量. 在1.2和1.3节中, 分别通过对Bernoulli型测度和变分公式进行研究, 我们得到了贪婪beta-展式中一些数字频率集的Hausdorff维数的精确公式.

第二章着重于一般的beta-展式(不仅是贪婪的)和相关的数字频率. 在2.1节中, 我们系统地研究多重基底下的展式, 这是常用的一个基底下的展式的自然推广. 从2.2节开始, 我们回到一个基底下的展式并考虑数字频率. 在2.2节中, 我们给出关于一般beta-展式的数字频率的三个小结果. 在2.3节中, 我们在与1.2节相似的框架下研究Bernoulli型测度, 并且作为应用, 我们得到了univoque序列所组成集合的一些频率子集的Hausdorff维数.

第三章着重于著名的Thue-Morse序列的一些推广, 包括它们与beta-展式和数字频率的关系. 在3.1节中, 我们展示了一类广义推移Thue-Morse序列和唯一beta-展式中数字频率的一个分叉现象有很强的联系. 在3.2节中, 我们研究广义Thue-Morse数的展式, 这是由3.1节中的广义推移Thue-Morse序列的进一步推广所定义的. 最后, 我们在3.3和3.4节中考虑另一类广义Thue-Morse序列, 并且分别研究与之相关的无穷乘积和广义Koch曲线.

关键词

beta-展式, 数字频率, 广义Thue-Morse序列.

Contents

Introduction

To represent real numbers, the most common way is to use expansions in integer bases. For example, expansions in base 10 are used in our daily lives and expansions in base 2 are used in computer systems. As a natural generalization, expansions in non-integer bases were introduced by Rényi [102] in 1957, and then attracted a lot of attention until now. See for examples [5, 9, 29, 35, 66, 69, 91, 99, 104, 105].

Let $\mathbb{N} := \{1, 2, 3, \cdots\}$ be the set of positive integers and $\mathbb{R}$ be the set of real numbers. Given $m \in \mathbb{N}$, a base $\beta \in (1, m+1]$ and $x \in \mathbb{R}$, in general, a sequence $w = (w_n)_{n\geq 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ is called a *β-expansion* of x if

$$x = \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}.$$

It is known that x has a β-expansion if and only if $x \in [0, \frac{m}{\beta-1}]$ (see for examples [23, 24, 25, 102]).

An interesting phenomenon is that an x may have many β-expansions. For examples, [61, Theorem 3] shows that if $\beta \in (1, \frac{1+\sqrt{5}}{2})$, every $x \in (0, \frac{1}{\beta-1})$ has a continuum of different β-expansions, and [107, Theorem 1] shows that if $\beta \in (1, 2)$, Lebesgue almost every $x \in [0, \frac{1}{\beta-1}]$ has a continuum of different β-expansions. For more on the cardinality of β-expansions, see for examples [26, 62, 70]. We study the most common beta-expansions, which are called greedy beta-expansions in Chapter 1 and then return to general beta-expansions from Chapter 2.

Chapter 1 consists of three sections which are devoted to greedy beta-expansions and related digit frequencies.

In Section 1.1 we completely characterize the structures of admissible words and then study the distributions and numbers of full and non-full words (cylinders). Concretely, on the one hand, the precise lengths of all the maximal runs of full and non-full words among admissible words with same order are obtained, which generalizes the result on the distribution of full cylinders given by Bugeaud and Wang [37] in 2014, and on the other hand, delighted by the result on the total number of admissible words given by Rényi [102] in 1957, for any base $\beta > 1$, we prove that the number of full words with length n is comparable to β^n, and this conclusion is also true for the non-full words if β is not an

integer.

Section 1.2 is a joint work with Mr. Bing Li and Mr. Tuomas Sahlsten at the end of my master at Université Paris-Est Marne-la-Vallée (UPEM) under the guidance of Mr. Lingmin Liao. Most of the content has already appeared in my master thesis at UPEM. I still present it in this thesis for completeness and also for the convenience of the reader. We study Bernoulli-type measures related to greedy beta-expansions, study their invariance as dynamical properties and find out the unique equivalent ergodic probability measure with respect to the β-transformation when the greedy β-expansion of 1 is finite. Then we study the modified lower local dimension of measures related to β-expansions. As an application, we prove that the Hausdorff dimension [64] of three kinds of frequency sets are equal and obtain the exact formula when the greedy β-expansion of 1 is 10^m10^∞ for any non-negative integer m. This generalizes the relative well known result for β equal to the golden ratio $(\sqrt{5}+1)/2$.

In Section 1.3, we first give a proof of the useful folklore: for any $\beta > 1$, the Hausdorff dimension of an arbitrary set in the shift space S_β is equal to the Hausdorff dimension of its natural projection in $[0, 1]$. It has been used in some former papers without explicit proof (see for example [114, Section 5]). We will prove it by applying a covering property given by Bugeaud and Wang [37] on the distribution of full cylinders. Then we clarify that for calculating the Hausdorff dimension of frequency sets using variational formulae, one only needs to focus on the Markov measures of explicit order when the greedy β-expansion of 1 is finite. Concretely, it suffices to optimize a function with finitely many variables under some restrictions. As an application, we obtain an exact formula for the Hausdorff dimension of frequency sets for an important class of β's, which are called pseudo-golden ratios (also called multinacci numbers).

From Chapter 2, which consists of three sections, we return to general beta-expansions, not only the greedy ones, and we also study related digit frequencies.

Usually we expand real numbers in one given base. In Section 2.1, we begin to systematically study expansions in multiple given bases in a reasonable way, which is a generalization in the sense that if all the bases are taken to be the same, we return to the classical expansions in one base. In particular, we focus on greedy, quasi-greedy, lazy, quasi-lazy and unique expansions in multiple bases, and give lexicographic characterizations for greedy, lazy and unique expansions. These recover some relative well known results on expansions in one base including Parry's criterion [99]. Note that Neunhäuserer began the study of expansions in two bases in his recent paper [98], where he focused on the cardinality of the expansions.

In Section 2.2, we return to expansions in one base and study their digit frequencies. Consider the alphabet $\{0, 1, \cdots, m\}$ and $\beta \in (1, m+1) \setminus \mathbb{N}$. First we show that Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has a β-expansion of a given frequency if and only if Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has infinitely many β-expansions of the same given frequency.

Then delighted by [25, Theorem 4.1] and [24, Theorem 2.1], which are given by Baker and Kong, on the one hand we prove that Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has infinitely many balanced β-expansions, where an infinite sequence on the finite alphabet $\{0, 1, \cdots, m\}$ is called balanced if the frequency of the digit k is equal to the frequency of the digit $m-k$ for all $k \in \{0, 1, \cdots, m\}$, and on the other hand we consider variable frequency and prove that for every pseudo-golden ratio $\beta \in (1, 2)$, there exists a constant $c = c(\beta) > 0$ such that for any $p \in [\frac{1}{2} - c, \frac{1}{2} + c]$, Lebesgue almost every $x \in [0, \frac{1}{\beta-1}]$ has infinitely many β-expansions on $\{0, 1\}$ with frequency of 0's equal to p.

In Section 2.3, for integer $m \geq 3$, we study the dynamical system $(\Lambda^{(m)}, \sigma_m)$ where $\Lambda^{(m)} := \left\{w \in \{0,1\}^{\mathbb{N}} : w \text{ does not contain } 0^m \text{ or } 1^m\right\}$ and σ_m is the shift map on $\{0,1\}^{\mathbb{N}}$ restricted to $\Lambda^{(m)}$, study the Bernoulli-type measures on $\Lambda^{(m)}$ and find out the unique equivalent σ_m-invariant ergodic probability measure in a framework similar to Section 1.2. As an application, we obtain the Hausdorff dimension of the set of univoque sequences, the Hausdorff dimension of the set of sequences in which the lengths of consecutive 0's and consecutive 1's are bounded, and the Hausdorff dimension of their frequency subsets. Here we call $\Gamma := \left\{w \in \{0,1\}^{\mathbb{N}} : \overline{w} \prec \sigma^k w \prec w \text{ for all } k \geq 1\right\}$ the set of *univoque sequences* since Erdös, Joó and Komornik [61] proved in 1990 that a sequence $\alpha = (\alpha_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ is the unique expansion of 1 in some base $\beta \in (1, 2)$ if and only if $\alpha \in \Gamma$.

Chapter 3 consists of four sections, which are devoted to some generalizations of the famous Thue-Morse sequence, including their relations to beta-expansions, related infinite products and generalized Koch curves.

Let $(t_n)_{n\geq 0}$ be the well known classical *Thue-Morse sequence*

$$0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 0110\ 1001\ \cdots.$$

Since the work of Thue [115, 116] and Morse [97], this sequence has been widely studied [4, 12, 15, 51, 56, 71, 96]. There are several equivalent definitions of this sequence. One is to define the shifted Thue-Morse sequence $(t_n)_{n\geq 1}$ as follows:

$$t_1 := 1, \quad t_2 := \overline{t_1}^+, \quad t_3t_4 := \overline{t_1t_2}^+, \quad t_5t_6t_7t_8 := \overline{t_1t_2t_3t_4}^+, \quad \cdots$$

where $\overline{0} := 1$, $\overline{1} := 0$ and $w^+ := w_1 \cdots w_{n-1}(w_n + 1)$ for any finite word $w = w_1 \cdots w_n$. The unique $\mathfrak{q} \in (1, 2)$ such that $\sum_{n=1}^{\infty} \frac{t_n}{\mathfrak{q}^n} = 1$ is the well known Komornik-Loreti constant.

In Section 3.1, according to the above definition, we define generalized shifted Thue-Mores sequences on alphabets with more than two digits, and we show that corresponding generalized Komornik-Loreti constants are critical values of β's, above which the digit frequencies in unique β-expansions are much more flexible and opposite below them.

In Section 3.2, we generalize the concepts of generalized shifted Thue-Morse sequences and generalized Komornik-Loreti constants in Section 3.1 a bit more, and then we introduce generalized Thue-Morse numbers of the form $\pi_\beta(\theta) := \sum_{n=1}^{\infty} \frac{\theta_n}{\beta^n}$ where $\theta = (\theta_n)_{n\geq 1}$ is a

generalized shifted Thue-Morse sequence and $\beta \in (1, \infty)$. This is a natural generalization of the classical Thue-Morse number $\sum_{n=1}^{\infty} \frac{t_n}{2^n}$. We study when θ will be the unique, greedy, lazy, quasi-greedy and quasi-lazy β-expansions of $\pi_\beta(\theta)$. In particular, we deduce that the classical shifted Thue-Morse sequence $(t_n)_{n\geq 1}$ is the unique β-expansion of $\sum_{n=1}^{\infty} \frac{t_n}{\beta^n}$ if and only if it is the greedy expansion, if and only if it is the lazy expansion, if and only if it is the quasi-greedy expansion, if and only if it is the quasi-lazy expansion, and if and only if β is no less than the classical Komornik-Loreti constant.

One of the other equivalent definitions of the classical Thue-Morse sequence $(t_n)_{n\geq 0}$ is that it is the unique fixed point of the morphism

$$0 \mapsto 01$$
$$1 \mapsto 10$$

beginning with $t_0 := 0$. A natural generalization is: given $m \in \mathbb{N}$ and $\theta_1, \cdots, \theta_m \in \{0, 1\}$, we define the generalized Thue-Morse sequence $(\theta_n)_{n\geq 0}$ to be the unique fixed point of the morphism

$$0 \mapsto 0\theta_1 \cdots \theta_m$$
$$1 \mapsto 1\overline{\theta_1} \cdots \overline{\theta_m}$$

beginning with $\theta_0 := 0$, where $\overline{0} := 1$ and $\overline{1} := 0$.

In Section 3.3, for ad hoc rational functions R, we evaluate infinite products of the forms $\prod (R(n))^{(-1)^{\theta_n}}$ and $\prod (R(n))^{\theta_n}$. This generalizes relevant results given by Allouche, Riasat and Shallit [13] in 2019 on infinite products related to the classical Thue-Morse sequence $(t_n)_{n\geq 0}$ of the forms $\prod (R(n))^{(-1)^{t_n}}$ and $\prod (R(n))^{t_n}$.

Since the 1982-1983 work of Coquet and Dekking, it is known that the classical Thue-Morse sequence is strongly related to the famous Koch curve. As a natural generalization, in Section 3.4, we use the above mentioned generalized Thue-Morse sequences to define generalized Koch curves, and we prove that generalized Koch curves are the attractors of corresponding iterated function systems. For special cases, the open set condition holds, and then we obtain the Hausdorff, packing and box dimension of corresponding generalized Koch curves. This recovers the result on the classical Koch curve.

Chapter 1

Greedy beta-expansions and related digit frequencies

In this chapter we focus on greedy beta-expansions. For simplification, we use the term "beta/β-expansion" instead of "greedy beta/β-expansion" throughout this chapter.

In Section 1.1, we study distributions and numbers of full and non-full words in beta-expansions. Then in Section 1.2 we study Bernoulli-type measures related to beta-expansions and apply them to obtain the Hausdorff dimension of some frequency sets. Finally we use variational formulae to study the Hausdorff dimension of frequency sets for more β's in Section 1.3 to end this chapter.

1.1 Distributions and numbers of full and non-full words

Let $\beta > 1$ be a real number. Denoted by Σ_β^n the set of all admissible words with length $n \in \mathbb{N}$. The projection to $[0,1)$ of any word in Σ_β^n is a cylinder of order n (also say a fundamental interval), which is a left-closed and right-open interval in $[0,1)$. The lengths of cylinders are irregular for $\beta \notin \mathbb{N}$, meanwhile, they are all regular for $\beta \in \mathbb{N}$, namely, the length of any cylinder of order n equals β^{-n}.

A cylinder with order n is said to be full if it is mapped by the n-th iteration of β-transformation T_β^n onto $[0,1)$ (see Definition 1.1.6 below, [44] or [120]) or equivalently its length is maximal, that is, equal to β^{-n} (see Proposition 1.1.8 below, [37] or [66]). An admissible word is said to be full if the corresponding cylinder is full. Full words and cylinders have very good properties. For example, Walters [120] proved that for any given $N > 0$, $[0,1)$ is covered by the full cylinders of order at least N. Fan and Wang [66] obtained some good properties of full cylinders (see Propositions 1.1.8 and 1.1.9 below). Bugeaud and Wang [37] studied the distribution of full cylinders, showed that for any integer $n \geq 1$, among every $(n+1)$ consecutive cylinders of order n, there exists at least one full cylinder, and used it to prove a modified mass distribution principle to estimate

the Hausdorff dimension of sets defined in terms of β-expansions. Zheng, Wu and Li proved that the extremely irregular set is residual with the help of the full cylinders (for details see [129]).

In this section, we are interested in the distributions and numbers of full and non-full words in Σ_β^n, i.e., the distributions and numbers of full and non-full cylinders of order n in $[0,1)$. More precisely, we consider the lexicographically ordered sequence of all order n admissible words, count the numbers of successive full words and successive non-full words, and estimate the total numbers of full words and non-full words separately. Or, in what amounts to the same thing, we look at all the fundamental intervals of order n, arranged in increasing order along the unit interval, ask about numbers of successive intervals where T_β^n is onto and where it is not onto, and estimate the total number of each kind of these intervals.

Firstly Theorem 1.1.14 gives a unique and clear form of any admissible word, and Corollaries 1.1.15 and 1.1.16 provide some convenient ways to check whether an admissible word is full or not. Secondly in Definition 1.1.19 we introduce the concept of maximal run, which is a new way to study the distributions of full and non-full words and cylinders, and then Theorem 1.1.22 describes all the precise lengths of the maximal runs of full words, which indicates that such lengths rely on the nonzero terms in the β-expansion of 1. Consequently, the maximal and minimal lengths of the maximal runs of full words are given in Corollaries 1.1.27 and 1.1.28 respectively. Thirdly by introducing a function τ_β in Definition 1.1.30, a similar concept of numeration system and greedy algorithm, we obtain a convenient way to count the consecutive non-full words in Lemma 1.1.34, which can easily give the maximal length of the runs of non-full words in Corollary 1.1.36 and generalize the result of Bugeaud and Wang mentioned above (see Remark 1.1.39). Finally, all the precise lengths of the maximal runs of non-full words are stated in Theorem 1.1.40, which depends on the positions of nonzero terms in the β-expansion of 1. Furthermore, the minimal lengths of the maximal runs of non-full words are obtained in Corollary 1.1.41. Moveover, the numbers of all full words and all non-full words are separately estimated in Theorem 1.1.43.

This section is organized as follows. In Subsection 1.1.1, we introduce some basic notation and preliminary work needed. In Subsection 1.1.2, we study the structures of admissible words, including full words and non-full words. In Subsections 1.1.3 and 1.1.4, we obtain all the precise lengths of the maximal runs of full words and non-full words respectively. Finally Subsection 1.1.5 is devoted to the numbers of full and non-full words.

1.1.1 Notation and preliminaries

For any $x \in \mathbb{R}$, we use $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the greatest integer no larger than x and the smallest integer no less than x respectively throughout this thesis.

Let $\beta > 1$. Define the *alphabet* $\mathcal{A}_\beta := \{0, 1, \cdots, \lceil \beta \rceil - 1\}$ and let $\mathcal{A}_\beta^{\mathbb{N}}$ be the set of

infinite sequences on $\mathcal{A}_\beta$. Define the *β-transformation* $T_\beta : [0,1) \to [0,1)$ by

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor \quad \text{for } x \in [0,1). \tag{1.1}$$

Given $x \in [0,1)$, for all $n \in \mathbb{N}$, let

$$\varepsilon_n(x,\beta) := \lfloor \beta T_\beta^{n-1}(x) \rfloor \in \mathcal{A}_\beta.$$

Then

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x,\beta)}{\beta^n}.$$

The sequence $\varepsilon(x,\beta) := \varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots\varepsilon_n(x,\beta)\cdots$ is called the *greedy β-expansion* of x. For simplification, we call it *β-expansion* throughout this chapter. The system $([0,1), T_\beta)$ is a *β-dynamical system.*

Define

$$T_\beta(1) := \beta - \lfloor \beta \rfloor \quad \text{and} \quad \varepsilon_n(1,\beta) := \lfloor \beta T_\beta^{n-1}(1) \rfloor \quad \text{for all } n \in \mathbb{N}.$$

Then the number 1 can also be expanded into a series, denoted by

$$1 = \sum_{n=1}^{\infty} \frac{\varepsilon_n(1,\beta)}{\beta^n}.$$

The sequence $\varepsilon(1,\beta) := \varepsilon_1(1,\beta)\varepsilon_2(1,\beta)\cdots\varepsilon_n(1,\beta)\cdots$ is called the (*greedy*) *β-expansion* of 1. For simplicity, we use $\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots$ to denote the digits of $\varepsilon(1,\beta)$ throughout this section.

If there are infinitely many n with $\varepsilon_n \neq 0$, we say that $\varepsilon(1,\beta)$ is infinite. Otherwise, there exists $m \in \mathbb{N}$ such that $\varepsilon_m \neq 0$ with $\varepsilon_j = 0$ for all $j > m$, $\varepsilon(1,\beta)$ is said to be finite, and we say that $\varepsilon(1,\beta)$ is finite with length m.

Let $\varepsilon^*(1,\beta) := \varepsilon_1^*(1,\beta)\varepsilon_2^*(1,\beta)\cdots\varepsilon_n^*(1,\beta)\cdots$ be the *quasi-greedy β-expansion* of 1 defined by

$$\varepsilon^*(1,\beta) := \begin{cases} \varepsilon(1,\beta) & \text{if } \varepsilon(1,\beta) \text{ is infinite;} \\ (\varepsilon_1\cdots\varepsilon_{m-1}(\varepsilon_m - 1))^\infty & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m. \end{cases}$$

Here for a finite word $w = w_1w_2\cdots w_n$, we use w^∞ to denote the periodic sequence

$$w_1w_2\cdots w_n\ w_1w_2\cdots w_n\ w_1w_2\cdots w_n\ \cdots.$$

Throughout this section, we use $\varepsilon_1^*\varepsilon_2^*\cdots\varepsilon_n^*\cdots$ to denote the digits of $\varepsilon^*(1,\beta)$ no matter whether $\varepsilon(1,\beta)$ is finite or not. Moreover, for any finite word or infinite sequence w, we always use w_n to denote its nth term.

Let $\prec$ and $\preceq$ be the *lexicographic order* in $\mathcal{A}_\beta^{\mathbb{N}}$. More precisely, $w \prec w'$ means that there exists $k \in \mathbb{N}$ such that $w_i = w'_i$ for all $1 \le i < k$ and $w_k < w'_k$. Besides, $w \preceq w'$ means that $w \prec w'$ or $w = w'$. Similarly, the definitions of $\prec$ and $\preceq$ are extended to the finite words of the same length by identifying a word w with the sequence $w0^\infty$.

For any $w \in \mathcal{A}_\beta^{\mathbb{N}}$, we use $w|_k$ to denote the prefix of w with length k, i.e., $w_1 w_2 \cdots w_k$ where $k \in \mathbb{N}$. For any $w \in \mathcal{A}_\beta^n$, we use $|w| := n$ to denote the *length* of w and $w|_k$ to denote the prefix of w with length k where $1 \le k \le |w|$.

Let $\sigma : \mathcal{A}_\beta^{\mathbb{N}} \to \mathcal{A}_\beta^{\mathbb{N}}$ be the *shift map* defined by

$$\sigma(w_1 w_2 \cdots) := w_2 w_3 \cdots \quad \text{for } w \in \mathcal{A}_\beta^{\mathbb{N}} \tag{1.2}$$

and $\pi_\beta : \mathcal{A}_\beta^{\mathbb{N}} \to \mathbb{R}$ be the *natural projection map* defined by

$$\pi_\beta(w) := \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots \quad \text{for } w \in \mathcal{A}_\beta^{\mathbb{N}}. \tag{1.3}$$

Definition 1.1.1 (Admissibility)**.**

(1) *A word $w \in \mathcal{A}_\beta^n$ for some $n \in \mathbb{N}$ is called admissible, if there exists $x \in [0,1)$ such that $\varepsilon_i(x,\beta) = w_i$ for all $i \in \{1, \cdots, n\}$. We define*

$$\Sigma_\beta^n := \{w \in \mathcal{A}_\beta^n : w \text{ is admissible}\} \quad \text{and} \quad \Sigma_\beta^* := \bigcup_{n=1}^{\infty} \Sigma_\beta^n.$$

(2) *A sequence $w \in \mathcal{A}_\beta^{\mathbb{N}}$ is called admissible, if there exists $x \in [0,1)$ such that $\varepsilon_i(x,\beta) = w_i$ for all $i \in \mathbb{N}$. We define*

$$\Sigma_\beta := \{w \in \mathcal{A}_\beta^{\mathbb{N}} : w \text{ is admissible}\}.$$

Obviously, if $w \in \Sigma_\beta$, then $w|_n \in \Sigma_\beta^n$ and $w_{n+1} w_{n+2} \cdots \in \Sigma_\beta$ for any $n \in \mathbb{N}$. We prove the following basic property for self-contained.

Lemma 1.1.2. *For any $n \in \mathbb{N}$, $\varepsilon^*(1,\beta)|_n \in \Sigma_\beta^n$ and is maximal in Σ_β^n with lexicographic order.*

Proof. (1) Prove that for all $k \in \mathbb{N}$ we have $\frac{\varepsilon^*_{k+1}}{\beta} + \frac{\varepsilon^*_{k+2}}{\beta^2} + \cdots \le 1$.
① If $\varepsilon(1,\beta)$ is infinite, then $\frac{\varepsilon^*_{k+1}}{\beta} + \frac{\varepsilon^*_{k+2}}{\beta^2} + \cdots = \frac{\varepsilon_{k+1}}{\beta} + \frac{\varepsilon_{k+2}}{\beta^2} + \cdots = T_\beta^k 1 < 1$.

② If $\varepsilon(1,\beta)$ is finite with length $m \in \mathbb{N}$, let $p \geq 0$ such that $pm \leq k \leq (p+1)m-1$. Then

$$\begin{aligned}
&\frac{\varepsilon^*_{k+1}}{\beta} + \frac{\varepsilon^*_{k+2}}{\beta^2} + \cdots \\
=&\frac{\varepsilon^*_{k-pm+1}}{\beta} + \cdots + \frac{\varepsilon^*_m}{\beta^{(p+1)m-k}} + \frac{1}{\beta^{(p+1)m-k}}\Big(\frac{\varepsilon^*_1}{\beta} + \cdots + \frac{\varepsilon^*_m}{\beta^m} + \frac{\varepsilon^*_1}{\beta^{m+1}} + \cdots + \frac{\varepsilon^*_m}{\beta^{2m}} + \cdots\Big) \\
=&\frac{\varepsilon_{k-pm+1}}{\beta} + \cdots + \frac{\varepsilon_m}{\beta^{(p+1)m-k}} = T_\beta^{k-pm}1 \leq 1.
\end{aligned}$$

(2) Prove that for all $n \in \mathbb{N}$ we have $\varepsilon^*(1,\beta)|_n \in \Sigma^n_\beta$.
Let $x := \frac{\varepsilon^*_1}{\beta} + \cdots + \frac{\varepsilon^*_n}{\beta^n} \in [0,1)$. It suffices to prove $\varepsilon_i(x,\beta) = \varepsilon^*_i$ for all $i \in \{1,\cdots,n\}$. First we have

$$\varepsilon_1(x,\beta) = \lfloor \beta x \rfloor = \lfloor \varepsilon^*_1 + \frac{\varepsilon^*_2}{\beta} + \cdots + \frac{\varepsilon^*_n}{\beta^{n-1}} \rfloor = \varepsilon^*_1,$$

where the last equality follows from

$$\frac{\varepsilon^*_2}{\beta} + \cdots + \frac{\varepsilon^*_n}{\beta^{n-1}} < \frac{\varepsilon^*_2}{\beta} + \frac{\varepsilon^*_3}{\beta^2} + \cdots \overset{\text{by (1)}}{\leq} 1. \tag{1.4}$$

Then we have

$$\varepsilon_2(x,\beta) = \lfloor \beta T_\beta x \rfloor \overset{\text{use (1.4)}}{=\!=\!=} \lfloor \varepsilon^*_2 + \frac{\varepsilon^*_3}{\beta} + \cdots + \frac{\varepsilon^*_n}{\beta^{n-2}} \rfloor = \varepsilon^*_2,$$

where the last equality follows from

$$\frac{\varepsilon^*_3}{\beta} + \cdots + \frac{\varepsilon^*_n}{\beta^{n-2}} < \frac{\varepsilon^*_3}{\beta} + \frac{\varepsilon^*_4}{\beta^2} + \cdots \overset{\text{by (1)}}{\leq} 1.$$

$\cdots$ Repeating the above process we get $\varepsilon_i(x,\beta) = \varepsilon^*_i$ for all $i \in \{1,\cdots,n\}$.
(3) Prove that for all $n \in \mathbb{N}$, $\varepsilon^*(1,\beta)|_n$ is maximal in Σ^n_β.
(By contradiction) Assume that there exists $w_1 \cdots w_n \in \Sigma^n_\beta$ such that $\varepsilon^*_1 \cdots \varepsilon^*_n \prec w_1 \cdots w_n$. Then there exists $k \in \{1,\cdots,n\}$ such that $\varepsilon^*_1 \cdots \varepsilon^*_{k-1} = w_1 \cdots w_{k-1}$ and $\varepsilon^*_k + 1 \leq w_k$. By $w_1 \cdots w_n \in \Sigma^n_\beta$, there exists $x \in [0,1)$ such that $\varepsilon(x,\beta)|_n = w_1 \cdots w_n$. Then

$$x \geq \frac{w_1}{\beta} + \cdots + \frac{w_k}{\beta^k} \geq \frac{\varepsilon^*_1}{\beta} + \cdots + \frac{\varepsilon^*_{k-1}}{\beta^{k-1}} + \frac{\varepsilon^*_k + 1}{\beta^k} \overset{\text{by (1)}}{\geq} \frac{\varepsilon^*_1}{\beta} + \frac{\varepsilon^*_2}{\beta^2} + \cdots = 1,$$

which contradicts $x \in [0,1)$. □

The following criterion for admissible sequence is due to Parry.

Lemma 1.1.3 ([99]). *Let $\beta > 1$ and $w \in \mathcal{A}^{\mathbb{N}}_\beta$. Then w is admissible (that is, $w \in \Sigma_\beta$) if and only if*

$$\sigma^k(w) \prec \varepsilon^*(1,\beta) \quad \textit{for all } k \geq 0.$$

The next lemma can be found in [86, Theorem 2.1].

Lemma 1.1.4. *Let w be a sequence of non-negative integers. Then w is the β-expansion of 1 for some $\beta > 1$ if and only if $\sigma^k w \prec w$ for all $k \geq 1$. Moreover, such β satisfies $w_1 \leq \beta < w_1 + 1$.*

Definition 1.1.5 (Cylinder)**.** *Let $n \in \mathbb{N}$ and $w \in \Sigma_\beta^n$. We call*

$$[w] := \left\{v \in \Sigma_\beta : v_1 = w_1, \cdots, v_n = w_n\right\}$$

the cylinder of order n in Σ_β generated by w and

$$I(w) := \pi_\beta([w]) = \left\{x \in [0,1) : \varepsilon_1(x,\beta) = w_1, \cdots, \varepsilon_n(x,\beta) = w_n\right\}$$

the cylinder of order n in $[0,1)$ generated by w. For any $x \in [0,1)$, the cylinder of order n containing x is denoted by

$$I_n(x) := I(\varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots\varepsilon_n(x,\beta)).$$

Definition 1.1.6 (Full and non-full words and cylinders)**.** *Let $w \in \Sigma_\beta^n$ for some $n \in \mathbb{N}$. If $T_\beta^n I(w) = [0,1)$, we call the word w and the cylinders $[w], I(w)$ full. Otherwise, we call them non-full.*

Lemma 1.1.7 ([91], [66], [37])**.** *Suppose the word $w_1 \cdots w_n$ is admissible and $w_n \neq 0$. Then $w_1 \cdots w_{n-1} w_n'$ is full for any $w_n' < w_n$.*

1.1.2 The structures of admissible words, full words and non-full words

The following proposition is a criterion of full words. The equivalence of (1), (2) and (4) can be found in [66]. We give some proofs for self-contained and more characterizations (3), (5), (6) are given here.

Proposition 1.1.8. *Let $w \in \Sigma_\beta^n$ for some $n \in \mathbb{N}$. Then the following are equivalent.*

(1) *w is full, i.e., $T_\beta^n I(w) = [0,1)$;*

(2) *$|I(w)| = \beta^{-n}$;*

(3) *The sequence ww' is admissible for any $w' \in \Sigma_\beta$;*

(4) *The word ww' is admissible for any $w' \in \Sigma_\beta^*$;*

(5) *The word $w\varepsilon_1^* \cdots \varepsilon_k^*$ is admissible for any $k \geq 1$;*

(6) *$\sigma^n[w] = \Sigma_\beta$.*

Proof. (1) $\Rightarrow$ (2) Since w is full, $T_\beta^n I(w) = [0,1)$. Noting that

$$x = \frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n} + \frac{T_\beta^n x}{\beta^n} \quad \text{for any } x \in I(w),$$

we can get

$$I(w) = [\frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n}, \frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n} + \frac{1}{\beta^n}).$$

Therefore $|I(w)| = \beta^{-n}$.
(2) $\Rightarrow$ (3) Let $x, x' \in [0,1)$ such that $\varepsilon(x,\beta) = w0^\infty$ and $\varepsilon(x',\beta) = w'$. Then

$$x = \frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n} \quad \text{and} \quad x' = \frac{w_1'}{\beta} + \frac{w_2'}{\beta^2} + \cdots .$$

Let

$$y = x + \frac{x'}{\beta^n} = \frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n} + \frac{w_1'}{\beta^{n+1}} + \frac{w_2'}{\beta^{n+2}} \cdots .$$

We need to prove $ww' \in \Sigma_\beta$. It suffices to prove $y \in [0,1)$ and $\varepsilon(y,\beta) = ww'$. In fact, since $I(w)$ is a left-closed and right-open interval with $\frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n}$ as its left endpoint and $|I(w)| = \beta^{-n}$, we get

$$I(w) = [\frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n}, \frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n} + \frac{1}{\beta^n}) = [x, x + \frac{1}{\beta^n}).$$

So $y \in I(w) \subset [0,1)$ and $\varepsilon_1(y,\beta) = w_1, \cdots, \varepsilon_n(y,\beta) = w_n$. That is

$$y = \frac{w_1}{\beta} + \cdots + \frac{w_n}{\beta^n} + \frac{T_\beta^n y}{\beta^n} = x + \frac{T_\beta^n y}{\beta^n},$$

which implies $T_\beta^n y = x'$. Then for any $k \geq 1$,

$$\varepsilon_{n+k}(y,\beta) = \lfloor \beta T_\beta^{n+k-1} y \rfloor = \lfloor \beta T_\beta^{k-1} x' \rfloor = \varepsilon_k(x',\beta) = w_k'.$$

Thus $\varepsilon(y,\beta) = ww'$. Therefore $ww' \in \Sigma_\beta$.
(3) $\Rightarrow$ (4) is obvious.
(4) $\Rightarrow$ (5) follows from $\varepsilon_1^* \cdots \varepsilon_k^* \in \Sigma_\beta^*$ for any $k \geq 1$.
(5) $\Rightarrow$ (1) We need to prove $T_\beta^n I(w) = [0,1)$. It suffices to show $T_\beta^n I(w) \supset [0,1)$ since the reverse inclusion is obvious. Indeed, let $x \in [0,1)$ and $u = w_1 \cdots w_n \varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots$.
At first, we prove $u \in \Sigma_\beta$. By Lemma 1.1.3, it suffices to prove $\sigma^k(u) \prec \varepsilon^*(1,\beta)$ for any $k \geq 0$ below.
① If $k \geq n$, we have

$$\sigma^k(u) = \varepsilon_{k-n+1}(x,\beta)\varepsilon_{k-n+2}(x,\beta)\cdots = \sigma^{k-n}(\varepsilon(x,\beta)) \overset{\text{by Lemma 1.1.3}}{\prec} \varepsilon^*(1,\beta).$$

② If $0 \le k \le n-1$, we have

$$\sigma^k(u) = w_{k+1} \cdots w_n \varepsilon_1(x,\beta)\varepsilon_2(x,\beta)\cdots .$$

Since $\varepsilon(x,\beta) \prec \varepsilon^*(1,\beta)$, there exists $m \in \mathbb{N}$ such that $\varepsilon_1(x,\beta) = \varepsilon_1^*, \cdots, \varepsilon_{m-1}(x,\beta) = \varepsilon_{m-1}^*$ and $\varepsilon_m(x,\beta) < \varepsilon_m^*$. Combining $w\varepsilon_1^* \cdots \varepsilon_m^* \in \Sigma_\beta^*$ and Lemma 1.1.3, we get

$$\sigma^k(u) \prec w_{k+1} \cdots w_n \varepsilon_1^* \cdots \varepsilon_m^* 0^\infty = \sigma^k(w\varepsilon_1^* \cdots \varepsilon_m^* 0^\infty) \prec \varepsilon^*(1,\beta).$$

Therefore $u \in \Sigma_\beta$.
Let $y \in [0,1)$ such that $\varepsilon(y,\beta) = u$. Then $y \in I(w)$. Since

$$\varepsilon_k(T_\beta^n y, \beta) = \lfloor \beta T_\beta^{n+k-1} y \rfloor = \varepsilon_{n+k}(y,\beta) = \varepsilon_k(x,\beta) \quad \text{for any } k \in \mathbb{N},$$

we get $x = T_\beta^n y \in T_\beta^n I(w)$.
(1) $\Leftrightarrow$ (6) follows from the facts that the function $\varepsilon(\cdot,\beta) : [0,1) \to \Sigma_\beta$ is bijective and the commutativity $\varepsilon(T_\beta x, \beta) = \sigma(\varepsilon(x,\beta))$. □

Proposition 1.1.9. *Let $w, w' \in \Sigma_\beta^*$ be full and $|w| = n \in \mathbb{N}$. Then*

(1) *the word ww' is full;*

(2) *the word $\sigma^k(w) := w_{k+1} \cdots w_n$ is full for any k with $1 \le k < n$;*

(3) *the digit $w_n < \lfloor \beta \rfloor$ if $\beta \notin \mathbb{N}$. In particular, $w_n = 0$ if $1 < \beta < 2$.*

Proof. (1) A proof has been given in [37]. We give another proof here to be self-contained. Since w' is full, by Proposition 1.1.8 (5) we get $w'\varepsilon_1^* \cdots \varepsilon_m^* \in \Sigma_\beta^*$ for any $m \ge 1$. Then $ww'\varepsilon_1^* \cdots \varepsilon_m^* \in \Sigma_\beta^*$ by the fullness of w and Proposition 1.1.8 (4), which implies that ww' is full by Proposition 1.1.8 (5).

(2) Since w is full , by Proposition 1.1.8 (5) we get $w_1 \cdots w_n \varepsilon_1^* \cdots \varepsilon_m^* \in \Sigma_\beta^*$, and also $w_{k+1} \cdots w_n \varepsilon_1^* \cdots \varepsilon_m^* \in \Sigma_\beta^*$ for any $m \ge 1$. Therefore $w_{k+1} \cdots w_n$ is full by Proposition 1.1.8 (5).

(3) Since w is full, by (2) we know that $\sigma^{n-1} w = w_n$ is full. Then $|I(w_n)| = 1/\beta$ by Proposition 1.1.8 (2). Suppose $w_n = \lfloor \beta \rfloor$, then $I(w_n) = I(\lfloor \beta \rfloor) = [\lfloor \beta \rfloor / \beta, 1)$ and $|I(w_n)| = 1 - \lfloor \beta \rfloor / \beta < 1/\beta$ which is a contradiction. Therefore $w_n \ne \lfloor \beta \rfloor$. So $w_n < \lfloor \beta \rfloor$ noting that $w_n \le \lfloor \beta \rfloor$.

□

Proposition 1.1.10.

(1) *Any truncation of $\varepsilon(1,\beta)$ is not full (if it is admissible). That is, $\varepsilon(1,\beta)|_k$ is not full for any $k \in \mathbb{N}$ (if it is admissible).*

(2) *Let $k \in \mathbb{N}$. Then $\varepsilon^*(1,\beta)|_k$ is full if and only if $\varepsilon(1,\beta)$ is finite with length m which exactly divides k, i.e., $m|k$.*

Proof. (1) We show the conclusion by the cases that $\varepsilon(1,\beta)$ is finite or infinite.
Cases 1. $\varepsilon(1,\beta)$ is finite with length m.
① If $k \geq m$, then $\varepsilon(1,\beta)|_k = \varepsilon_1 \cdots \varepsilon_m 0^{k-m}$ is not admissible.
② If $1 \leq k \leq m-1$, combining $\varepsilon_{k+1} \cdots \varepsilon_m 0^\infty = \varepsilon(T_\beta^k 1, \beta) \in \Sigma_\beta$, $\varepsilon_1 \cdots \varepsilon_k \varepsilon_{k+1} \cdots \varepsilon_m 0^\infty = \varepsilon(1,\beta) \notin \Sigma_\beta$ and Proposition 1.1.8 (1) (3), we know that $\varepsilon(1,\beta)|_k = \varepsilon_1 \cdots \varepsilon_k$ is not full.
Cases 2. $\varepsilon(1,\beta)$ is infinite. It follows from the similar proof with Case 1 ②.

(2) $\boxed{\Leftarrow}$ Let $p \in \mathbb{N}$ with $k = pm$. For any $n \geq 1$, we know that $\varepsilon_1^* \cdots \varepsilon_{pm}^* \varepsilon_1^* \cdots \varepsilon_n^* = \varepsilon^*(1,\beta)|_{k+n}$ is admissible by Lemma 1.1.2. Therefore $\varepsilon^*(1,\beta)|_k = \varepsilon_1^* \cdots \varepsilon_{pm}^*$ is full by Proposition 1.1.8 (1) (5).
$\boxed{\Rightarrow}$ (By contradiction) Suppose that the conclusion is not true, that is, either $\varepsilon(1,\beta)$ is infinite or finite with length m, but m does not divide k exactly.
① If $\varepsilon(1,\beta)$ is infinite, then $\varepsilon^*(1,\beta)|_k = \varepsilon(1,\beta)|_k$ is not full by (1), which contradicts our condition.
② If $\varepsilon(1,\beta)$ is finite with length m, but $m \nmid k$, then there exists $p \geq 0$ such that $pm < k < pm + m$. Since $\varepsilon^*(1,\beta)|_k$ is full, combining

$$\varepsilon_{k-pm+1} \cdots \varepsilon_m 0^\infty = \varepsilon(T_\beta^{k-pm} 1, \beta) \in \Sigma_\beta,$$

and Proposition 1.1.8 (1) (3), we get

$$\varepsilon_1^* \cdots \varepsilon_k^* \varepsilon_{k-pm+1} \cdots \varepsilon_{m-1} \varepsilon_m 0^\infty \in \Sigma_\beta, \quad \text{i.e.,} \quad \varepsilon_1^* \cdots \varepsilon_{pm}^* \varepsilon_1 \cdots \varepsilon_{m-1} \varepsilon_m 0^\infty \in \Sigma_\beta,$$

which is false since $\pi_\beta(\varepsilon_1^* \cdots \varepsilon_{pm}^* \varepsilon_1 \cdots \varepsilon_{m-1} \varepsilon_m 0^\infty) = 1$. □

The following lemma is a convenient way to show that an admissible word is not full.

Lemma 1.1.11. *Any admissible word ends with a prefix of $\varepsilon(1,\beta)$ is not full. That is, if there exists $s \in \{1, \cdots, n\}$ such that $w = w_1 \cdots w_{n-s} \varepsilon_1 \cdots \varepsilon_s \in \Sigma_\beta^n$, then w is not full.*

Proof. It follows from Proposition 1.1.9 (2) and Proposition 1.1.10 (1). □

Notation 1.1.12. *Denote the first position where w and $\varepsilon(1,\beta)$ are different by*

$$\mathfrak{m}(w) := \min\{k \geq 1 : w_k < \varepsilon_k\} \quad \textit{for } w \in \Sigma_\beta$$

and

$$\mathfrak{m}(w) := \mathfrak{m}(w0^\infty) \quad \textit{for } w \in \Sigma_\beta^*.$$

Remark 1.1.13.

(1) *Let $\varepsilon(1,\beta)$ be finite with the length m. Then $\mathfrak{m}(w) \leq m$ for any w in Σ_β or Σ_β^*.*

(2) *Let* $w \in \Sigma_\beta^n$ *and* $\mathfrak{m}(w) \geq n$. *Then* $w = \varepsilon_1 \cdots \varepsilon_{n-1} w_n$ *with* $w_n \leq \varepsilon_n$.

Proof. (1) follows from $w \prec \varepsilon(1, \beta)$.

(2) follows from $w_1 = \varepsilon_1, \cdots, w_{n-1} = \varepsilon_{n-1}$ and $w \in \Sigma_\beta^n$. □

We give the complete characterizations of the structures of admissible words, full words and non-full words by the following theorem and two corollaries.

Theorem 1.1.14 (The structure of admissible words)**.** *Let* $w \in \Sigma_\beta^n$ *for some* $n \in \mathbb{N}$. *Then* $w = w_1 w_2 \cdots w_n$ *can be uniquely decomposed to the form*

$$\varepsilon_1 \cdots \varepsilon_{k_1-1} w_{n_1} \ \varepsilon_1 \cdots \varepsilon_{k_2-1} w_{n_2} \ \cdots \ \varepsilon_1 \cdots \varepsilon_{k_p-1} w_{n_p} \ \varepsilon_1 \cdots \varepsilon_{l-1} w_n, \tag{1.5}$$

where $p \geq 0$, $k_1, \cdots, k_p, l \in \mathbb{N}$, $n = k_1 + ... + k_p + l$, $n_j = k_1 + \cdots + k_j$, $w_{n_j} < \varepsilon_{k_j}$ *for all* $1 \leq j \leq p$, $w_n \leq \varepsilon_l$ *and the words* $\varepsilon_1 \cdots \varepsilon_{k_1-1} w_{n_1}, \cdots, \varepsilon_1 \cdots \varepsilon_{k_p-1} w_{n_p}$ *are all full.*

Moreover, if $\varepsilon(1, \beta)$ *is finite with length* m, *then* $k_1, \cdots, k_p, l \leq m$. *For the case* $l = m$, *we must have* $w_n < \varepsilon_m$.

Corollary 1.1.15 (The structural criterion of full words)**.** *Let* $w \in \Sigma_\beta^n$ *for some* $n \in \mathbb{N}$ *and* $w_* := \varepsilon_1 \cdots \varepsilon_{l-1} w_n$ *be the suffix of* w *as in Theorem 1.1.14. Then*

$$w \text{ is full} \Longleftrightarrow w_* \text{ is full} \Longleftrightarrow w_n < \varepsilon_l.$$

Corollary 1.1.16. *Let* $w \in \Sigma_\beta^n$ *for some* $n \in \mathbb{N}$. *Then* w *is not full if and only if it ends with a prefix of* $\varepsilon(1, \beta)$. *That is, when* $\varepsilon(1, \beta)$ *is infinite (finite with length* m*), there exists* $1 \leq s \leq n$ *(* $1 \leq s \leq \min\{m-1, n\}$ *respectively) such that* $w = w_1 \cdots w_{n-s} \varepsilon_1 \cdots \varepsilon_s$.

Proof. $\boxed{\Rightarrow}$ follows from Theorem 1.1.14 and Corollary 1.1.15.
$\boxed{\Leftarrow}$ follows from Lemma 1.1.11. □

Proof of Theorem 1.1.14. Firstly, we show the decomposition by the cases that $\varepsilon(1, \beta)$ is infinite or finite.
Case 1. $\varepsilon(1, \beta)$ is infinite.
Compare w and $\varepsilon(1, \beta)$. If $\mathfrak{m}(w) \geq n$, then w has the form (1.5) with $w = \varepsilon_1 \cdots \varepsilon_{n-1} w_n$ by Remark 1.1.13 (2). If $\mathfrak{m}(w) < n$, let $n_1 = k_1 = \mathfrak{m}(w) \geq 1$. Then $w|_{n_1} = \varepsilon_1 \cdots \varepsilon_{k_1-1} w_{n_1}$ with $w_{n_1} < \varepsilon_{k_1}$. Continue to compare the tail of w and $\varepsilon(1, \beta)$. If $\mathfrak{m}(w_{n_1+1} \cdots w_n) \geq n - n_1$, then $w_{n_1+1} \cdots w_n = \varepsilon_1 \cdots \varepsilon_{n-n_1-1} w_n$ with $w_n \leq \varepsilon_{n-n_1}$ by Remark 1.1.13 (2) and w has the form (1.5) with $w = \varepsilon_1 \cdots \varepsilon_{k_1-1} w_{n_1} \varepsilon_1 \cdots \varepsilon_{n-n_1-1} w_n$. If $\mathfrak{m}(w_{n_1+1} \cdots w_n) < n - n_1$, let $k_2 = \mathfrak{m}(w_{n_1+1} \cdots w_n) \geq 1$ and $n_2 = n_1 + k_2$. Then $w|_{n_2} = \varepsilon_1 \cdots \varepsilon_{k_1-1} w_{n_1} \varepsilon_1 \cdots \varepsilon_{k_2-1} w_{n_2}$ with $w_{n_2} < \varepsilon_{k_2}$. Continue to compare the tail of w and $\varepsilon(1, \beta)$ for finite times. Then we can get that w must have the form (1.5).
Case 2. $\varepsilon(1, \beta)$ is finite with length m.
By Remark 1.1.13(1), we get $\mathfrak{m}(w)$,$\mathfrak{m}(w_{n_1+1} \cdots w_n), \cdots, \mathfrak{m}(w_{n_j+1} \cdots w_n), \cdots, \mathfrak{m}(w_{n_p+1} \cdots w_n) \leq$

m in Case 1. That is, $k_1, k_2, \cdots, k_p, l \leq m$ in (1.5). For the case $l = m$, combining $w_{n_p+1} = \varepsilon_1, \cdots, w_{n-1} = \varepsilon_{m-1}$ and $w_{n_p+1} \cdots w_n \prec \varepsilon_1 \cdots \varepsilon_m$, we get $w_n < \varepsilon_m$.
Secondly, $\varepsilon_1 \cdots \varepsilon_{k_1-1} w_{n_1}, \cdots, \varepsilon_1 \cdots \varepsilon_{k_p-1} w_{n_p}$ are obviously full by Lemma 1.1.7. □

Proof of Corollary 1.1.15. By Proposition 1.1.9 (1) (2), we know that w is full $\Longleftrightarrow$ w_* is full. So it suffices to prove that w_* is full $\Longleftrightarrow$ $w_n < \varepsilon_{|w_*|}$.
$\boxed{\Rightarrow}$ By $w_* \in \Sigma_\beta^*$, we get $w_n \leq \varepsilon_l$. Suppose $w_n = \varepsilon_l$, then $w_* = \varepsilon_1 \cdots \varepsilon_l$ is not full by Proposition 1.1.10 (1), which contradicts our condition. Therefore $w_n < \varepsilon_l$.
$\boxed{\Leftarrow}$ Let $w_n < \varepsilon_l$. We show that w_* is full by the cases that $\varepsilon(1, \beta)$ is infinite or finite.
Case 1. When $\varepsilon(1, \beta)$ is infinite. we know that w_* is full by $\varepsilon_1 \cdots \varepsilon_{l-1} \varepsilon_l \in \Sigma_\beta^*, w_n < \varepsilon_l$ and Lemma 1.1.7.
Case 2. When $\varepsilon(1, \beta)$ is finite with length m, we know $l \leq m$ by Theorem 1.1.14.
If $l < m$, we get $\varepsilon_1 \cdots \varepsilon_{l-1} \varepsilon_l \in \Sigma_\beta^*$. Then w_* is full by $w_n < \varepsilon_l$ and Lemma 1.1.7.
If $l = m$, we know that $\varepsilon_1 \cdots \varepsilon_{l-1}(\varepsilon_l - 1) = \varepsilon_1 \cdots \varepsilon_{m-1}(\varepsilon_m - 1) = \varepsilon_1^* \cdots \varepsilon_m^*$ is full by Proposition 1.1.10 (2). Then w_* is full by $w_n \leq \varepsilon_l - 1$ and Lemma 1.1.7. □

From Theorem 1.1.14, Corollaries 1.1.15 and 1.1.16 above, we can understand the structures of admissible words, full words and non-full words clearly, and judge whether an admissible word is full or not conveniently. They will be used for many times in the following sections.

1.1.3 The lengths of the runs of full words

Definition 1.1.17. *Let $\beta > 1$. Define $\{n_i(\beta)\}$ to be those positions of $\varepsilon(1, \beta)$ that are nonzero. That is,*

$$n_1(\beta) := \min\{k \geq 1 : \varepsilon_k \neq 0\} \text{ and } n_{i+1}(\beta) := \min\{k > n_i : \varepsilon_k \neq 0\}$$

if there exists $k > n_i$ such that $\varepsilon_k \neq 0$ for $i \geq 1$. We call $\{n_i(\beta)\}$ the nonzero sequence of β, also denote it by $\{n_i\}$ if there is no confusion.

Remark 1.1.18. *Let $\beta > 1$, $\{n_i\}$ be the nonzero sequence of β. Then the following are obviously true.*

(1) $n_1 = 1$;

(2) *$\varepsilon(1, \beta)$ is finite if and only if $\{n_i\}$ is finite;*

(3) $\varepsilon(1, \beta) = \varepsilon_{n_1} 0 \cdots 0 \varepsilon_{n_2} 0 \cdots 0 \varepsilon_{n_3} 0 \cdots$.

Definition 1.1.19 (Run and maximal run)**.**

(1) *Denote by $[w^{(1)}, \cdots, w^{(l)}]$ the l consecutive words from small to large in Σ_β^n with lexicographic order, which is called a run of words and l is the length of the run of words. If $w^{(1)}, \cdots, w^{(l)}$ are all full, we call $[w^{(1)}, \cdots, w^{(l)}]$ a run of full words.*

(2) *A run of full words* $[w^{(1)}, \cdots, w^{(l)}]$ *is said to be maximal, if it can not be elongated, i.e., " the previous word of* $w^{(1)}$ *in* Σ_β^n *is not full or* $w^{(1)} = 0^n$ *" and " the next word of* $w^{(l)}$ *is not full or* $w^{(l)} = \varepsilon^*(1,\beta)|_n$ *".*

In a similar way, we can define a run of non-full words and a maximal run of non-full words.

Definition 1.1.20. *We use* $\mathcal{F}_\beta^n$ *to denote the set of all the maximal runs of full words in* Σ_β^n *and* F_β^n *to denote the length set of* $\mathcal{F}_\beta^n$*, i.e.,*

$$F_\beta^n := \{l \in \mathbb{N} : \text{ there exists } [w^{(1)}, \cdots, w^{(l)}] \in \mathcal{F}_\beta^n\}.$$

Similarly, we use $\mathcal{N}_\beta^n$ *to denote the set of all the maximal runs of non-full words and* N_β^n *to denote the length set of* $\mathcal{N}_\beta^n$*.*
In $\mathcal{F}_\beta^n \cup \mathcal{N}_\beta^n$*, we use* S_{max}^n *to denote the maximal run with* $\varepsilon^*(1,\beta)|_n$ *as its last element.*

Remark 1.1.21. *For any* $w \in \Sigma_\beta^n$ *with* $w \neq 0^n$ *and* $w_n = 0$*, the previous word of* w *in the lexicographic order in* Σ_β^n *is* $w_1 \cdots w_{k-1}(w_k - 1)\varepsilon_1^* \cdots \varepsilon_{n-k}^*$ *where* $k = \max\{1 \le i \le n-1 : w_i \neq 0\}$*.*

Notice that we will use the basic fact above for many times in the proofs of the following results in this section.

Theorem 1.1.22 (The lengths of the maximal runs of full words)**.** *Let* $\beta > 1$ *with* $\beta \notin \mathbb{N}$*,* $\{n_i\}$ *be the nonzero sequence of* β*. Then*

$$F_\beta^n = \begin{cases} \{\varepsilon_{n_i} : n_i \le n\} & \text{if } \varepsilon(1,\beta) \text{ is infinite or finite with length } m \ge n; \\ \{\varepsilon_{n_i}\} \cup \{\varepsilon_1 + \varepsilon_m\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m < n \text{ and } m|n; \\ \{\varepsilon_{n_i} : n_i \neq m\} \cup \{\varepsilon_1 + \varepsilon_m\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m < n \text{ and } m \nmid n. \end{cases}$$

Proof. It follows from Definition 1.1.19, Lemma 1.1.24, Lemma 1.1.25 and the fact that $n_i \le m$ for any i when $\varepsilon(1,\beta)$ is finite with length m. □

Remark 1.1.23. *By Theorem 1.1.22, when* $1 < \beta < 2$*, we have*

$$F_\beta^n = \begin{cases} \{1\} & \text{if } \varepsilon(1,\beta) \text{ is infinite or finite with length } m \ge n; \\ \{1,2\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m < n. \end{cases}$$

Lemma 1.1.24. *Let* $\beta > 1$ *with* $\beta \notin \mathbb{N}$*,* $\{n_i\}$ *be the nonzero sequence of* β*. Then the length set of* $\mathcal{F}_\beta^n \backslash \{S_{max}^n\}$*, i.e.,* $\{l \in \mathbb{N} : \text{ there exists } [w^{(1)}, \cdots, w^{(l)}] \in \mathcal{F}_\beta^n \backslash \{S_{max}^n\}\}$ *is*

$$\begin{cases} \{\varepsilon_{n_i} : n_i \le n\} & \text{if } \varepsilon(1,\beta) \text{ is infinite or finite with length } m > n; \\ \{\varepsilon_{n_i} : n_i \neq m\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m = n; \\ \{\varepsilon_{n_i} : n_i \neq m\} \cup \{\varepsilon_1 + \varepsilon_m\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m < n. \end{cases}$$

Proof. Let $[w^{(l)}, w^{(l-1)}, \cdots, w^{(2)}, w^{(1)}] \in \mathcal{F}_\beta^n \backslash \{S_{max}^n\}$ and w which is not full be the next word of $w^{(1)}$. By Corollary 1.1.16, there exist $1 \le s \le n$, $0 \le a \le n-1$ with $a+s=n$ ($s \le m-1$, when $\varepsilon(1,\beta)$ is finite with length m), such that $w = w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_s$.

(1) If $s = 1$, that is, $w = w_1 \cdots w_{n-1}\varepsilon_1$, then $w^{(1)} = w_1 \cdots w_{n-1}(\varepsilon_1 - 1)$, $w^{(2)} = w_1 \cdots w_{n-1}(\varepsilon_1 - 2)$, $\cdots$, $w^{(\varepsilon_1)} = w_1 \cdots w_{n-1}0$ are full by Lemma 1.1.7.
① If $n = 1$ or $w_1 \cdots w_{n-1} = 0^{n-1}$, it is obvious that $l = \varepsilon_1$.
② If $n \ge 2$ and $w_1 \cdots w_{n-1} \ne 0^{n-1}$, there exists $1 \le k \le n-1$ such that $w_k \ne 0$ and $w_{k+1} = \cdots = w_{n-1} = 0$. Then the previous word of $w^{(\varepsilon_1)}$ is

$$w^{(\varepsilon_1+1)} = w_1 \cdots w_{k-1}(w_k - 1)\varepsilon_1^* \cdots \varepsilon_{n-k}^*.$$

i) If $\varepsilon(1,\beta)$ is infinite or finite with length $m \ge n$, then $w^{(\varepsilon_1+1)} = w_1 \cdots w_{k-1}(w_k - 1)\varepsilon_1 \cdots \varepsilon_{n-k}$ is not full by Lemma 1.1.11. Therefore $l = \varepsilon_1$.

ii) If $\varepsilon(1,\beta)$ is finite with length $m < n$, we divide this case into two parts according to $m \nmid n-k$ or $m | n-k$.
ⓐ If $m \nmid n-k$, then $\varepsilon_1^* \cdots \varepsilon_{n-k}^*$ is not full by Proposition 1.1.10 (2) and $w^{(\varepsilon_1+1)}$ is also not full by Proposition 1.1.9 (2). Therefore $l = \varepsilon_1$.
ⓑ If $m | n-k$, then $\varepsilon_1^* \cdots \varepsilon_{n-k}^*$ is full by Proposition 1.1.10 (2) and $w^{(\varepsilon_1+1)}$ is also full by Lemma 1.1.7 and Proposition 1.1.9 (1). Let $w_1' \cdots w_{n-m}' := w_1 \cdots w_{k-1}(w_k - 1)\varepsilon_1^* \cdots \varepsilon_{n-k-m}^*$. Then

$$w^{(\varepsilon_1+1)} = w_1' \cdots w_{n-m}' \varepsilon_1 \cdots \varepsilon_{m-1}(\varepsilon_m - 1).$$

The consecutive previous words

$$\begin{aligned} w^{(\varepsilon_1+2)} &= w_1' \cdots w_{n-m}' \varepsilon_1 \cdots \varepsilon_{m-1}(\varepsilon_m - 2) \\ w^{(\varepsilon_1+3)} &= w_1' \cdots w_{n-m}' \varepsilon_1 \cdots \varepsilon_{m-1}(\varepsilon_m - 3) \\ &\cdots \\ w^{(\varepsilon_1+\varepsilon_m)} &= w_1' \cdots w_{n-m}' \varepsilon_1 \cdots \varepsilon_{m-1} 0 \end{aligned}$$

are all full by Lemma 1.1.7. Since $\varepsilon_1 \ne 0$ and $m > 1$, there exists $1 \le t \le m-1$ such that $\varepsilon_t \ne 0$ and $\varepsilon_{t+1} = \cdots = \varepsilon_{m-1} = 0$. Then, as the previous word of $w^{(\varepsilon_1+\varepsilon_m)}$,

$$w^{(\varepsilon_1+\varepsilon_m+1)} = w_1' \cdots w_{n-m}' \varepsilon_1 \cdots \varepsilon_{t-1}(\varepsilon_t - 1)\varepsilon_1 \cdots \varepsilon_{m-t}$$

is not full by Lemma 1.1.11. Therefore $l = \varepsilon_1 + \varepsilon_m$.

(2) If $2 \le s \le n$, we divide this case into two parts according to $\varepsilon_s = 0$ or not.
① If $\varepsilon_s = 0$, there exists $1 \le t \le s-1$ such that $\varepsilon_t \ne 0$ and $\varepsilon_{t+1} = \cdots = \varepsilon_s = 0$ by $\varepsilon_1 \ne 0$. Then $w = w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_t 0^{s-t}$, and $w^{(1)} = w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_{t-1}(\varepsilon_t - 1)\varepsilon_1 \cdots \varepsilon_{s-t}$ is not full by Lemma 1.1.11, which contradicts our assumption.

② If $\varepsilon_s \neq 0$, then

$$\begin{aligned} w^{(1)} &= w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_{s-1}(\varepsilon_s - 1) \\ w^{(2)} &= w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_{s-1}(\varepsilon_s - 2) \\ &\cdots \\ w^{(\varepsilon_s)} &= w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_{s-1} 0 \end{aligned}$$

are full by Lemma 1.1.7. By nearly the same way of ①, we can prove that the previous word of $w^{(\varepsilon_s)}$ is not full. Therefore $l = \varepsilon_s$.

i) If $\varepsilon(1, \beta)$ is infinite or finite with length $m > n$, combining $2 \leq s \leq n$ and $\varepsilon_s \neq 0$, we know that the set of all values of $l = \varepsilon_s$ is $\{\varepsilon_{n_i} : 2 \leq n_i \leq n\}$.

ii) If $\varepsilon(1, \beta)$ finite with length $m \leq n$, combining $2 \leq s \leq m - 1$ and $\varepsilon_s \neq 0$, we know that the set of all values of $l = \varepsilon_s$ is $\{\varepsilon_{n_i} : 2 \leq n_i < m\}$.

By the discussion above, we can see that in every case, every value of l can be achieved. Combining $n_i \leq m$ for any i when $\varepsilon(1, \beta)$ is finite with length m, $\varepsilon_{n_1} = \varepsilon_1$ and all the cases discussed above, we get the conclusion of this lemma. □

Lemma 1.1.25. *Let $\beta > 1$ with $\beta \notin \mathbb{N}$. If $\varepsilon(1, \beta)$ is finite with length m and $m|n$, then $S^n_{max} \in \mathcal{F}^n_\beta$ and the length of S^n_{max} is ε_m. Otherwise, $S^n_{max} \in \mathcal{N}^n_\beta$.*

Proof. Let $w^{(1)} = \varepsilon^*_1 \cdots \varepsilon^*_n$.
If $\varepsilon(1, \beta)$ is finite with length m and $m|n$, then $w^{(1)}$ is full by Proposition 1.1.10 (2). We get $S^n_{max} \in \mathcal{F}^n_\beta$. Let $p = n/m - 1 \geq 0$. As the consecutive previous words of $w^{(1)}$, $w^{(2)} = (\varepsilon_1 \cdots \varepsilon_{m-1}(\varepsilon_m - 1))^p \varepsilon_1 \cdots \varepsilon_{m-1}(\varepsilon_m - 2)$, $\cdots$, $w^{(\varepsilon_m)} = (\varepsilon_1 \cdots \varepsilon_{m-1}(\varepsilon_m - 1))^p \varepsilon_1 \cdots \varepsilon_{m-1} 0$ are full by Lemma 1.1.7. By nearly the same way in the proof of Lemma 1.1.24 (2) ①, we know that the previous word of $w^{(\varepsilon_m)}$ is not full. Therefore the number of S^n_{max} is ε_m.
Otherwise, $w^{(1)}$ is not full by Proposition 1.1.10 (2). We get $S^n_{max} \in \mathcal{N}^n_\beta$. □

Remark 1.1.26. *All the locations of all the lengths in Theorem 1.1.22 can be found in the proof of Lemma 1.1.24 and Lemma 1.1.25.*

Corollary 1.1.27 (The maximal length of the runs of full words)**.** *Let $\beta > 1$ with $\beta \notin \mathbb{N}$. Then*

$$\max F^n_\beta = \begin{cases} \lfloor \beta \rfloor + \varepsilon_m & \text{if } \varepsilon(1, \beta) \text{ is finite with length } m < n; \\ \lfloor \beta \rfloor & \text{if } \varepsilon(1, \beta) \text{ is infinite or finite with length } m \geq n. \end{cases}$$

Proof. It follows from $\varepsilon_{n_i} \leq \varepsilon_{n_1} = \varepsilon_1 = \lfloor \beta \rfloor$ for any i and Theorem 1.1.22. □

Corollary 1.1.28 (The minimal length of the maximal runs of full words)**.** *Let $\beta > 1$ with $\beta \notin \mathbb{N}$, $\{n_i\}$ be the nonzero sequence of β. Then*

$$\min F_\beta^n = \begin{cases} \min\limits_{n_i<m} \varepsilon_{n_i} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m < n \text{ and } m \nmid n; \\ \min\limits_{n_i\le n} \varepsilon_{n_i} & \text{otherwise.} \end{cases}$$

Proof. It follows from $n_i \le m$ for any i when $\varepsilon(1,\beta)$ is finite with length m and Theorem 1.1.22. □

Remark 1.1.29. *It follows from Theorem 1.1.22 that the lengths of maximal runs of full words rely on the nonzero terms in $\varepsilon(1,\beta)$, i.e., $\{\varepsilon_{n_i}\}$.*

1.1.4 The lengths of the runs of non-full words

Let $\{n_i\}$ be the nonzero sequence of β. We will use a similar concept of numeration system and greedy algorithm in the sense of [16, Section 3.1] to define the function τ_β below. For any $s \in \mathbb{N}$, we can write $s = \sum_{i\ge1} a_i n_i$ greedily and uniquely where $a_i \in \mathbb{N} \cup \{0\}$ for any i and then define $\tau_\beta(s) = \sum_{i\ge1} a_i$. Equivalently, we have the following.

Definition 1.1.30 (The function τ_β)**.** *Let $\beta > 1$, $\{n_i\}$ be the nonzero sequence of β and $s \in \mathbb{N}$. Define $\tau_\beta(s)$ to be the number needed to add up to s greedily by $\{n_i\}$ with repetition. We define it precisely below.*
Let $n_{i_1} = \max\{n_i : n_i \le s\}$. (Notice $n_1 = 1$.)
If $n_{i_1} = s$, define $\tau_\beta(s) := 1$.
If $n_{i_1} < s$, let $t_1 = s - n_{i_1}$ and $n_{i_2} = \max\{n_i : n_i \le t_1\}$.
If $n_{i_2} = t_1$, define $\tau_\beta(s) := 2$.
If $n_{i_2} < t_1$, let $t_2 = t_1 - n_{i_2}$ and $n_{i_3} = \max\{n_i : n_i \le t_2\}$.
...
Generally for $j \in \mathbb{N}$. If $n_{i_j} = t_{j-1} (t_0 := s)$, define $\tau_\beta(s) := j$.
If $n_{i_j} < t_{j-1}$, let $t_j = t_{j-1} - n_{i_j}$ and $n_{i_{j+1}} = \max\{n_i : n_i \le t_j\}$.
...
Noting that $n_1 = 1$, it is obvious that there exist $n_{i_1} \ge n_{i_2} \ge \cdots \ge n_{i_d}$ all in $\{n_i\}$ such that $s = n_{i_1} + n_{i_2} + \cdots + n_{i_d}$, i.e., $n_{i_d} = t_{d-1}$. Define $\tau_\beta(s) := d$.

In the following we give an example to show how to calculate τ_β.

Example 1.1.31. *Let $\beta > 1$ such that $\varepsilon(1,\beta) = 302000010^\infty$ (such β exists by Lemma 1.1.4). Then the nonzero sequence of β is $\{1,3,8\}$. The way to add up to 7 greedily with repetition is $7 = 3 + 3 + 1$. Therefore $\tau_\beta(7) = 3$.*

Proposition 1.1.32 (Properties of τ_β)**.** *Let $\beta > 1$, $\{n_i\}$ be the nonzero sequence of β and $n \in \mathbb{N}$. Then*

(1) *$\tau_\beta(n_i) = 1$ for any i;*

(2) $\tau_\beta(s) = s$ *for any* $1 \le s \le n_2 - 1$, *and* $\tau_\beta(s) \le s$ *for any* $s \in \mathbb{N}$;

(3) $\{1, 2, \cdots, k\} \subset \{\tau_\beta(s) : 1 \le s \le n\}$ *for any* $k \in \{\tau_\beta(s) : 1 \le s \le n\}$;

(4) $\{\tau_\beta(s) : 1 \le s \le n\} = \{1, 2, \cdots, \max\limits_{1 \le s \le n} \tau_\beta(s)\}$.

Proof. (1) and (2) follow from Definition 1.1.30 and $n_1 = 1$.

(3) Let $k \in \{\tau_\beta(s) : 1 \le s \le n\}$. If $k = 1$, the conclusion is obviously true. If $k \ge 2$, let $2 \le t_0 \le n$ such that $k = \tau_\beta(t_0)$, $n_{i_1} = \max\{n_i : n_i \le t_0\}$ and $t_1 = t_0 - n_{i_1}$. Then $1 \le t_1 < t_0 \le n$ and it is obvious that $k - 1 = \tau_\beta(t_1) \in \{\tau_\beta(s) : 1 \le s \le n\}$ by Definition 1.1.30. By the same way, we can get $k-2, k-3, \cdots, 1 \in \{\tau_\beta(s) : 1 \le s \le n\}$. Therefore $\{1, 2, \cdots, k\} \subset \{\tau_\beta(s) : 1 \le s \le n\}$.

(4) The inclusion $\{\tau_\beta(s) : 1 \le s \le n\} \subset \{1, 2, \cdots, \max\limits_{1 \le s \le n} \tau_\beta(s)\}$ is obvious and the reverse inclusion follows from $\max\limits_{1 \le s \le n} \tau_\beta(s) \in \{\tau_\beta(s) : 1 \le s \le n\}$ and (3).

□

For $n \in \mathbb{N}$, we use $r_n(\beta)$ to denote the maximal length of the strings of 0's in $\varepsilon_1^* \cdots \varepsilon_n^*$ as in [68], [73] and [117], i.e.,

$$r_n(\beta) = \max\{k \ge 1 : \varepsilon_{i+1}^* = \cdots = \varepsilon_{i+k}^* = 0 \text{ for some } 0 \le i \le n-k\}$$

with the convention that $\max \emptyset = 0$.

The following relation between $\tau_\beta(s)$ and $r_s(\beta)$ will be used in the proof of Corollary 1.1.38.

Proposition 1.1.33. *Let* $\beta > 1$. *If* $\varepsilon(1, \beta)$ *is infinite, then* $\tau_\beta(s) \le r_s(\beta) + 1$ *for any* $s \ge 1$. *If* $\varepsilon(1, \beta)$ *is finite with length* m, *then* $\tau_\beta(s) \le r_s(\beta) + 1$ *is true for any* $1 \le s \le m$.

Proof. Let $\{n_i\}$ be the nonzero sequence of β and $n_{i_1} = \max\{n_i : n_i \le s\}$. No matter $\varepsilon(1, \beta)$ is infinite with $s \ge 1$ or finite with length $m \ge s \ge 1$, we have

$$\tau_\beta(s) - 1 = \tau_\beta(s - n_{i_1}) \le s - n_{i_1} \le r_s(\beta)$$

since $s - n_{i_1} = 0$ or $\varepsilon_{n_{i_1}+1}^* \varepsilon_{n_{i_1}+2}^* \cdots \varepsilon_s^* = \varepsilon_{n_{i_1}+1} \varepsilon_{n_{i_1}+2} \cdots \varepsilon_s = 0^{s - n_{i_1}}$. □

Lemma 1.1.34. *Let* $n \in \mathbb{N}$, $\beta > 1$ *with* $\beta \notin \mathbb{N}$ *and* $w \in \Sigma_\beta^n$ *end with a prefix of* $\varepsilon(1, \beta)$, *i.e.,* $w = w_1 \cdots w_{n-s} \varepsilon_1 \cdots \varepsilon_s$ *where* $1 \le s \le n$. *Then the previous consecutive* $\tau_\beta(s)$ *words starting from* w *in* Σ_β^n *are not full, but the previous* $(\tau_\beta(s) + 1)$*-th word is full.*

Remark 1.1.35. *Notice that* $w = w_1 \cdots w_{n-s} \varepsilon_1 \cdots \varepsilon_s$ *does not imply that* $w_1 \cdots w_{n-s}$ *is full. For example, when* $\beta > 1$ *with* $\varepsilon(1, \beta) = 1010010^\infty$, *let* $w = 001010 = w_1 \cdots w_4 \varepsilon_1 \varepsilon_2$. *But* $w_1 \cdots w_4 = 0010$ *is not full by Lemma 1.1.11.*

Proof of Lemma 1.1.34. Let $\{n_i\}$ be the nonzero sequence of β and

$$w^{(1)} := w_1^{(1)} \cdots w_{a_1}^{(1)} \varepsilon_1 \cdots \varepsilon_s := w_1 \cdots w_{n-s} \varepsilon_1 \cdots \varepsilon_s = w,$$

where $a_1 = n - s$. It is not full by Lemma 1.1.11.

...

Generally for any $j \geq 1$, suppose $w^{(j)}, w^{(j-1)}, \cdots, w^{(2)}, w^{(1)}$ to be j consecutive non-full words in Σ_β^n where $w^{(j)} = w_1^{(j)} \cdots w_{a_j}^{(j)} \varepsilon_1 \cdots \varepsilon_{t_{j-1}}$, $t_{j-1} > 0$ ($t_0 := s$). Let $w^{(j+1)} \in \Sigma_\beta^n$ be the previous word of $w^{(j)}$ and $n_{i_j} := \max\{n_i : n_i \leq t_{j-1}\}$.
If $n_{i_j} = t_{j-1}$, then $\varepsilon_{t_{j-1}} > 0$ and $w^{(j+1)} = w_1^{(j)} \cdots w_{a_j}^{(j)} \varepsilon_1 \cdots \varepsilon_{t_{j-1}-1} (\varepsilon_{t_{j-1}} - 1)$ is full by Lemma 1.1.7. We get the conclusion of this lemma since $\tau_\beta(s) = j$ at this time.
If $n_{i_j} < t_{j-1}$, let $t_j = t_{j-1} - n_{i_j}$. Then $w^{(j)} = w_1^{(j)} \cdots w_{a_j}^{(j)} \varepsilon_1 \cdots \varepsilon_{n_{i_j}} 0^{t_j}$ and the previous word is

$$w^{(j+1)} = w_1^{(j)} \cdots w_{a_j}^{(j)} \varepsilon_1 \cdots \varepsilon_{n_{i_j}-1} (\varepsilon_{n_{i_j}} - 1) \varepsilon_1 \cdots \varepsilon_{t_j} =: w_1^{(j+1)} \cdots w_{a_{j+1}}^{(j+1)} \varepsilon_1 \cdots \varepsilon_{t_j},$$

where $a_{j+1} = a_j + n_{i_j}$. By Lemma 1.1.11, $w^{(j+1)}$ is also not full. At this time, $w^{(j+1)}$, $w^{(j)}$, $\cdots$, $w^{(2)}$, $w^{(1)}$ are $j+1$ consecutive non-full words in Σ_β^n.

...

Noting that $n_1 = 1$, it is obvious that there exist $d \in \mathbb{N}$ such that $w^{(d)}, \cdots, w^{(1)}$ are not full, and $s = n_{i_1} + n_{i_2} + \cdots + n_{i_d}$, i.e., $n_{i_d} = t_{d-1}$. Then $\varepsilon_{t_{d-1}} > 0$ and $w^{(d+1)} = w_1^{(d)} \cdots w_{a_d}^{(d)} \varepsilon_1 \cdots \varepsilon_{t_{d-1}-1} (\varepsilon_{t_{d-1}} - 1)$ is full by Lemma 1.1.7. We get the conclusion since $\tau_\beta(s) = d$. □

Corollary 1.1.36 (The maximal length of the runs of non-full words)**.** *Let $\beta > 1$ with $\beta \notin \mathbb{N}$. Then*

$$\max N_\beta^n = \begin{cases} \max\{\tau_\beta(s) : 1 \leq s \leq n\} & \text{if } \varepsilon(1,\beta) \text{ is infinite;} \\ \max\{\tau_\beta(s) : 1 \leq s \leq min\{m-1, n\}\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m. \end{cases}$$

Proof. Let $l \in N_\beta^n$ and $[w^{(l)}, w^{(l-1)}, \cdots, w^{(2)}, w^{(1)}] \in \mathcal{N}_\beta^n$. Then, by Corollary 1.1.16, there exists

$$\begin{cases} 1 \leq s_0 \leq n & \text{if } \varepsilon(1,\beta) \text{ is infinite} \\ 1 \leq s_0 \leq \min\{m-1, n\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m \end{cases}$$

such that $w^{(1)} = w_1^{(1)} \cdots w_{n-s_0}^{(1)} \varepsilon_1 \cdots \varepsilon_{s_0}$ and we have $l = \tau_\beta(s_0)$ by Lemma 1.1.34. Therefore

$$\max N_\beta^n \leq \begin{cases} \max\{\tau_\beta(s) : 1 \leq s \leq n\} & \text{if } \varepsilon(1,\beta) \text{ is infinite} \\ \max\{\tau_\beta(s) : 1 \leq s \leq min\{m-1, n\}\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m \end{cases}$$

by the randomicity of the selection of l. On the other hand, the equality follows from the

fact that $0^{n-t_0}\varepsilon_1\cdots\varepsilon_{t_0}\in\Sigma_\beta^n$ included, the previous consecutive $\tau_\beta(t_0)$ words are not full by Lemma 1.1.34 where

$$\tau_\beta(t_0)=\begin{cases}\max\{\tau_\beta(s):1\le s\le n\} & \text{if } \varepsilon(1,\beta) \text{ is infinite;}\\ \max\{\tau_\beta(s):1\le s\le min\{m-1,n\}\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } m.\end{cases}$$

□

In the following we give an example to show how to calculate the maximal length of the runs of non-full words in Σ_β^n.

Example 1.1.37. *Let $n=8$ and $\varepsilon(1,\beta)=\varepsilon_{n_1}0\varepsilon_{n_2}000\varepsilon_{n_3}0\cdots0\varepsilon_{n_4}0\cdots0\varepsilon_{n_5}0\cdots$, where $n_1=1,n_2=3,n_3=7,n_4>8,\varepsilon_{n_i}\neq0$ for any i. Then, by Corollary 1.1.36, the maximal length of the runs of non-full words in Σ_β^8 is $\max\{\tau_\beta(s):1\le s\le 8\}$. Since*

$1=1 \quad\Rightarrow\tau_\beta(1)=1;\quad 2=1+1 \quad\Rightarrow\tau_\beta(2)=2;\quad 3=3 \quad\Rightarrow\tau_\beta(3)=1;$
$4=3+1 \quad\Rightarrow\tau_\beta(4)=2;\quad 5=3+1+1 \quad\Rightarrow\tau_\beta(5)=3;\quad 6=3+3 \quad\Rightarrow\tau_\beta(6)=2;$
$7=7 \quad\Rightarrow\tau_\beta(7)=1;\quad 8=7+1 \quad\Rightarrow\tau_\beta(8)=2,$

we get that $\max\{\tau_\beta(s):1\le s\le 8\}=3$ is the maximal length.

Corollary 1.1.38. *Let $\beta>1$. We have $\max N_\beta^n\le r_n(\beta)+1$ for any $n\in\mathbb{N}$. Moreover, if $\varepsilon(1,\beta)$ is finite with length m, then $\max N_\beta^n\le r_{m-1}(\beta)+1$ for any $n\in\mathbb{N}$.*

Proof. If $\varepsilon(1,\beta)$ is infinite, then

$$\max N_\beta^n=\max\{\tau_\beta(s):1\le s\le n\}\le\max\{r_s(\beta)+1:1\le s\le n\}=r_n(\beta)+1.$$

If $\varepsilon(1,\beta)$ is finite with length m, then

$$\max N_\beta^n=\max\{\tau_\beta(s):1\le s\le \min\{m-1,n\}\}\le\max\{r_s(\beta)+1:1\le s\le \min\{m-1,n\}\}.$$

and we have $\max N_\beta^n\le r_n(\beta)+1$ and $\max N_\beta^n\le r_{m-1}(\beta)+1$. □

Remark 1.1.39. *Combining Corollary 1.1.36 and $\tau_\beta(n)\le n$ (or Corollary 1.1.38 and $r_n(\beta)+1\le n$), we have $\max N_\beta^n\le n$ for any $n\in\mathbb{N}$ which contains the result about the distribution of full cylinders given by Bugeaud and Wang [37, Theorem 1.2]. Moreover, if $\varepsilon(1,\beta)$ is finite with length m, then $\max N_\beta^n\le m-1$ for any $n\in\mathbb{N}$. If $\beta\in A_0$ which is a class of β given by Li and Wu [91], then $\max N_\beta^n$ has the upper bound $\max\limits_{s\ge1} r_s(\beta)+1$ which does not rely on n.*

Theorem 1.1.40 (The lengths of the maximal runs of non-full words)**.** *Let $\beta>1$ with*

$\beta \notin \mathbb{N}$ and $\{n_i\}$ be the nonzero sequence of β. Then N_β^n is given by the following table.

Condition				*Conclusion*	*Case*
β	$\varepsilon(1,\beta)$			$N_\beta^n =$	
$\beta > 2$	*infinite*			D_1	*(1)*
	finite with length m			D_2	*(2)*
$1 < \beta < 2$	*infinite*	$n < n_2$		$\{n\}$	*(3)*
		$n \geq n_2$		D_5	*(4)*
	finite with length m	$n_2 = m$	$n < m$	$\{n\}$	*(5)*
			$n = m$	$\{m-1\}$	*(6)*
			$n > m$	D_4	*(7)*
		$n_2 < m$	$n < n_2$	$\{n\}$	*(8)*
			$n_2 \leq n < m$	D_5	*(9)*
			$n \geq m$	D_3	*(10)*

Here $D_1 = \{1, 2, \cdots, \max\{\tau_\beta(s) : 1 \leq s \leq n\}\}$;
$D_2 = \{1, 2, \cdots, \max\{\tau_\beta(s) : 1 \leq s \leq \min\{m-1, n\}\}\}$;
$D_3 = \{1, 2, \cdots, \max\{\tau_\beta(s) : 1 \leq s \leq m-1\}\}$;
$D_4 = \{1, 2, \cdots, \min\{n-m, m-1\}\} \cup \{m-1\}$;
$D_5 = \{1, 2, \cdots, \min\{n_2 - 1, n - n_2 + 1\}\} \cup \{\tau_\beta(s) : n_2 - 1 \leq s \leq n\}$.

Corollary 1.1.41 (The minimal length of the maximal runs of non-full words)**.** *Let $\beta > 1$ with $\beta \notin \mathbb{N}$ and $\{n_i\}$ be the nonzero sequence of β. Then*

$$\min N_\beta^n = \begin{cases} m-1 & \text{if } 1 < \beta < 2 \text{ and } \varepsilon(1,\beta) \text{ is finite with length } m = n_2 = n; \\ n & \text{if } 1 < \beta < 2 \text{ and } n < n_2; \\ 1 & \text{otherwise.} \end{cases}$$

Proof. It follows from Theorem 1.1.40. □

Proof of Theorem 1.1.40. We prove the conclusions for the cases (1)-(10) from simple ones to complicate as below.

Cases (3), (5) and (8) can be proved together. When $1 < \beta < 2$ and $n < n_2$, no matter $\varepsilon(1,\beta)$ is finite or not, noting that $\lfloor \beta \rfloor = 1$ and $\varepsilon(1,\beta)|_{n_2} = 10^{n_2-2}1$, we get $\varepsilon_1 \cdots \varepsilon_n = 10^{n-1}$. Then all the elements in Σ_β^n from small to large are 0^n, $0^{n-1}1$, $0^{n-2}10$, $\cdots$, 10^{n-1}, where 0^n is full and the others are all not full by Lemma 1.1.11. Therefore $N_\beta^n = \{n\}$.

Case (6). When $1 < \beta < 2$, $\varepsilon(1,\beta)$ is finite with length m and $n = n_2 = m$, noting that $\lfloor \beta \rfloor = 1$ and $\varepsilon(1,\beta) = 10^{m-2}10^\infty$, all the elements in Σ_β^n from small to large are 0^m, $0^{m-1}1$, $0^{m-2}10$, $\cdots$, 010^{m-2}, 10^{m-1}, where 0^m is full, 10^{m-1} is also full by Proposition 1.1.10 (2) and the others are all not full by Lemma 1.1.11. Therefore $N_\beta^n = \{m-1\}$.

Case (1). When $\beta > 2$ and $\varepsilon(1,\beta)$ is infinite, it suffices to prove $N_\beta^n \supset D_1$ since the reverse inclusion follows immediately from Corollary 1.1.36. By Proposition 1.1.32 (4), it suffices to show $N_\beta^n \supset \{\tau_\beta(s) : 1 \le s \le n\}$. In fact:

① For any $s \in \{1, \cdots, n-1\}$, let $u = 0^{n-s-1}10^s$. It is full by $\varepsilon_1 = \lfloor \beta \rfloor \ge 2$ and Corollary 1.1.16. The previous word $u^{(1)} = 0^{n-s}\varepsilon_1 \cdots \varepsilon_s$ is not full by Lemma 1.1.11. So $\tau_\beta(s) \in N_\beta^n$ by Lemma 1.1.34.

② For $s = n$, combining the fact that $\varepsilon_1 \cdots \varepsilon_s$ is maximal in Σ_β^n and Lemma 1.1.34, we get $\tau_\beta(s) \in N_\beta^n$.

Therefore $N_\beta^n = D_1$.

Case (2) can be proved by similar way as Case (1).

Case (10). When $1 < \beta < 2$, $\varepsilon(1,\beta)$ is finite with length m and $n_2 < m \le n$, we have $\varepsilon(1,\beta) = 10^{n_2-2}1\varepsilon_{n_2+1}\cdots\varepsilon_m 0^\infty$. It suffices to prove $N_\beta^n \supset D_3$ since the reverse inclusion follows immediately from Corollary 1.1.36. By Proposition 1.1.32 (4), it suffices to show $N_\beta^n \supset \{\tau_\beta(s) : 1 \le s \le m-1\}$. In fact:

① For any $n_2 - 1 \le s \le m-1$, let $u = 0^{n-s-1}10^s$. It is full by $s \ge n_2 - 1$ and Corollary 1.1.16. The previous word $u^{(1)} = 0^{n-s}\varepsilon_1^* \cdots \varepsilon_s^* = 0^{n-s}\varepsilon_1 \cdots \varepsilon_s$ is not full by Lemma 1.1.11. So $\tau_\beta(s) \in N_\beta^n$ by Lemma 1.1.34.

② For any $1 \le s \le n_2 - 2$, we get $n_2 - 1 \le n_3 - n_2$ by Lemma 1.1.4. So $1 \le s \le n_2 - 2 \le n_3 - n_2 - 1 \le m - n_2 - 1 \le n - n_2 - 1$ and then $n - n_2 - s \ge 1$. Let

$$u = 0^{n-n_2-s}10^{n_2+s-1}.$$

It is full by $n_2+s-1 \ge n_2-1$ and Corollary 1.1.16. Noting that $n_2 \le n_2+s-1 < n_3$, the previous word of u is

$$\begin{aligned} u^{(1)} &= 0^{n-n_2-s+1}\varepsilon_1^* \cdots \varepsilon_{n_2+s-1}^* \\ &= 0^{n-n_2-s+1}\varepsilon_1 \cdots \varepsilon_{n_2+s-1} \\ &= 0^{n-n_2-s+1}10^{n_2-2}10^{s-1} \\ &= 0^{n-n_2-s+1}10^{n_2-2}\varepsilon_1 \cdots \varepsilon_s \end{aligned}$$

which is not full by Lemma 1.1.11. So $\tau_\beta(s) \in N_\beta^n$ by Lemma 1.1.34.

Therefore $N_\beta^n = D_3$.

Case (7). When $1 < \beta < 2$, $\varepsilon(1,\beta)$ is finite with length m and $n > n_2 = m$, we have $\varepsilon(1,\beta) = 10^{m-2}10^\infty$.

On the one hand, we prove $N_\beta^n \subset D_4$. Let $l \in N_\beta^n$ and $[w^{(l)}, w^{(l-1)}, \cdots, w^{(2)}, w^{(1)}] \in \mathcal{N}_\beta^n$. By Corollary 1.1.16, there exist $1 \le s \le m-1$, $2 \le n-m+1 \le a \le n-1$ such that $a + s = n$ and $w^{(1)} = w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_s$. Then $l = \tau_\beta(s) = s$ by Lemma 1.1.34 and $s \le n_2 - 1$. Moreover, $w^{(1)} = w_1 \cdots w_a 10^{s-1}$.

① If $w_1 \cdots w_a = 0^a$, then the next word of $w^{(1)}$ is $w := 0^{a-1}10^s$ which is full by $[w^{(l)}, w^{(l-1)}, \cdots, w^{(2)}, w^{(1)}] \in \mathcal{N}_\beta^n$. Combining $s \le m-1$ and Corollary 1.1.16, we get $s = m-1$. Hence $l = m-1 \in D_4$.

② If $w_1 \cdots w_a \ne 0^a$, we get $a \ge m$ by $w_{k+1} \cdots w_a 10^\infty \prec \varepsilon(1,\beta) = 10^{m-2}10^\infty$ for any $k \ge 0$. Hence $s \le n-m$ and $l = s \in D_4$.

On the other hand, we prove $N_\beta^n \supset D_4$.

① For $m-1$, let $u = 0^{n-m}10^{m-1}$ which is full by Corollary 1.1.16. The consecutive previous words are $u^{(1)} = 0^{n-m+1}10^{m-2}, \cdots, u^{(m-1)} = 0^{n-1}1, u^{(m)} = 0^n$ where $u^{(1)}, \cdots, u^{(m-1)}$ are not full by Lemma 1.1.11, and $u^{(m)}$ is full. Therefore $m-1 \in N_\beta^n$.

② For any $1 \le s \le \min\{n-m, m-1\}$, let

$$u^{(1)} = 0^{n-m-s}\varepsilon_1^* \cdots \varepsilon_{m+s}^* = 0^{n-m-s}10^{m-1}10^{s-1} = 0^{n-m-s}10^{m-1}\varepsilon_1 \cdots \varepsilon_s.$$

i) If $s = n-m$, then $u^{(1)} = \varepsilon_1^* \cdots \varepsilon_{m+s}^*$ is maximal in Σ_β^n.
ii) If $s < n-m$, i.e.,$n-m-s-1 \ge 0$, then the next word of $u^{(1)}$ is $0^{n-m-s-1}10^{m+s}$ which is full by Corollary 1.1.16.
Hence we must have $s = \tau_\beta(s) \in N_\beta^n$ by $s \le n_2 - 1$ and Lemma 1.1.34.

Therefore $N_\beta^n = D_4$.

Cases (4) and (9) can be proved together. When $1 < \beta < 2$, $\varepsilon(1,\beta)$ is infinite with $n \ge n_2$ or $\varepsilon(1,\beta)$ is finite with length m and $n_2 \le n < m$, we have $\varepsilon(1,\beta) = 10^{n_2-2}1\varepsilon_{n_2+1}\varepsilon_{n_2+2}\cdots$. By Proposition 1.1.32 (2), we get

$$D_5 = \{\tau_\beta(s) : 1 \le s \le \min\{n_2 - 1, n - n_2 + 1\} \text{ or } n_2 - 1 \le s \le n\}.$$

On the one hand, we prove $N_\beta^n \subset D_5$. Let $l \in N_\beta^n$ and $[w^{(l)}, w^{(l-1)}, \cdots, w^{(2)}, w^{(1)}] \in \mathcal{N}_\beta^n$. By Corollary 1.1.16, there exist $1 \le s \le n$, $0 \le a \le n-1$ such that $a + s = n$ and $w^{(1)} = w_1 \cdots w_a \varepsilon_1 \cdots \varepsilon_s$. Then $l = \tau_\beta(s)$ by Lemma 1.1.34.

① If $a = 0$, then $s = n$ and $l = \tau_\beta(n) \in D_5$.

② If $a \ge 1$, we divide it into two cases.
i) If $w_1 \cdots w_a = 0^a$, then the next word of $w^{(1)}$ is $0^{a-1}10^s$ which is full by $[w^{(l)}, w^{(l-1)}, \cdots, w^{(2)}, w^{(1)}] \in \mathcal{N}_\beta^n$. Combining $\varepsilon(1,\beta) = 10^{n_2-2}1\varepsilon_{n_2+1}\varepsilon_{n_2+2}\cdots$ and Corollary 1.1.16, we get $s \ge n_2 - 1$. Hence $l = \tau_\beta(s) \in D_5$.
ii) If $w_1 \cdots w_a \ne 0^a$, by

$$w_{k+1} \cdots w_a 10^\infty \prec \varepsilon(1,\beta) = 10^{n_2-2}1\varepsilon_{n_2+1}\varepsilon_{n_2+2}\cdots \quad \text{for any } k \ge 0,$$

we get $a \geq n_2 - 1$ Hence $s \leq n - n_2 + 1$.
ⓐ If $s \geq n_2 - 1$, then $l = \tau_\beta(s) \in \{\tau_\beta(s) : n_2 - 1 \leq s \leq n\} \subset D_5$.
ⓑ If $s \leq n_2 - 1$, then $l = \tau_\beta(s) \in \{\tau_\beta(s) : 1 \leq s \leq \min\{n_2 - 1, n - n_2 + 1\}\} \subset D_5$.

On the other hand, we prove $N_\beta^n \supset D_5$.

① For any $n_2 - 1 \leq s \leq n$, let $u^{(1)} = 0^{n-s}\varepsilon_1^* \cdots \varepsilon_s^*$. No matter whether $\varepsilon(1, \beta)$ is infinite or finite with length $m > n$ (which implies $s < m$), we get $u^{(1)} = 0^{n-s}\varepsilon_1 \cdots \varepsilon_s$ which is not full by Lemma 1.1.11.
i) If $s = n$, then $u^{(1)} = \varepsilon_1^* \cdots \varepsilon_n^*$ is maximal in Σ_β^n.
ii) If $n_2 - 1 \leq s \leq n - 1$, then the next word of $u^{(1)}$ is $0^{n-s-1}10^s$ which is full by $s \geq n_2 - 1$ and Corollary 1.1.16.
Hence we must have $\tau_\beta(s) \in N_\beta^n$ by Lemma 1.1.34.

② For any $1 \leq s \leq \min\{n_2 - 1, n - n_2 + 1\}$, let

$$u^{(1)} = 0^{n-n_2-s+1}\varepsilon_1^* \cdots \varepsilon_{n_2+s-1}^*.$$

No matter $\varepsilon(1, \beta)$ is infinite or finite with length $m > n$ (which implies $n_2 + s - 1 \leq n < m$), we get

$$u^{(1)} = 0^{n-n_2-s+1}\varepsilon_1 \cdots \varepsilon_{n_2+s-1}.$$

Since Lemma 1.1.4 implies $n_2 - 1 \leq n_3 - n_2$, we get $1 \leq s \leq n_2 - 1 \leq n_3 - n_2$ and then $n_2 \leq n_2 + s - 1 < n_3$. Hence

$$u^{(1)} = 0^{n-n_2-s+1}10^{n_2-2}10^{s-1}$$

$$= 0^{n-n_2-s+1}10^{n_2-2}\varepsilon_1 \cdots \varepsilon_s$$

which is not full by Lemma 1.1.11.
i) If $s = n - n_2 + 1$, then $u^{(1)} = \varepsilon_1^* \cdots \varepsilon_n^*$ is maximal in Σ_β^n.
ii) If $s < n - n_2 + 1$, i.e., $n - n_2 - s \geq 0$, then the next word of $u^{(1)}$ is $0^{n-n_2-s}10^{n_2+s-1}$ which is full by Corollary 1.1.16.
Hence we must have $\tau_\beta(s) \in N_\beta^n$ by Lemma 1.1.34.

Therefore $N_\beta^n = D_5$. □

Remark 1.1.42. *It follows from Theorem 1.1.40 that the lengths of the maximal runs of non-full words rely on the positions of nonzero terms in $\varepsilon(1, \beta)$, i.e., $\{n_i\}$.*

1.1.5 Numbers of full and non-full words

In 1957, Rényi [102] estimated the number of all the admissible words with the same length (see Lemma 1.1.44 below). By applying the results in Subsection 1.1.2, we estimate the numbers of full words and non-full words separately in this subsection.

We say that two sequences $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ in $(0,+\infty)$ are *comparable*, and denote it by $x_n \backsimeq y_n$, if there exist $c_1, c_2 > 0$ such that $c_1 x_n \leq y_n \leq c_2 x_n$ for all $n \in \mathbb{N}$. It is not difficult to see that $\backsimeq$ is an equivalent relation.

Denote the sets of admissible words, full words and non-full words with length n by Σ_β^n, $\Sigma_{\beta,F}^n$ and $\Sigma_{\beta,N}^n$ respectively. The result from Rényi means that $\#\Sigma_\beta^n$ (where $\#$ denotes the cardinality) is comparable to β^n which is an exponential growth. As the main result of this subsection, the following theorem claims that $\#\Sigma_{\beta,F}^n$ is also comparable to β^n, and if $\beta \notin \mathbb{N}$, $\#\Sigma_{\beta,N}^n$ is also comparable to β^n.

Theorem 1.1.43. *Let $\beta > 1$. Then*

$$\#\Sigma_{\beta,F}^n \backsimeq \#\Sigma_\beta^n \backsimeq \beta^n.$$

Moreover, if $\beta \notin \mathbb{N}$, then

$$\#\Sigma_{\beta,N}^n \backsimeq \#\Sigma_{\beta,F}^n \backsimeq \#\Sigma_\beta^n \backsimeq \beta^n.$$

This is a combination of the following lemmas.

Lemma 1.1.44 ([102])**.** *For all $\beta > 1$ and $n \in \mathbb{N}$,*

$$\beta^n \leq \#\Sigma_\beta^n < \frac{\beta^{n+1}}{\beta - 1}.$$

Lemma 1.1.45. *For all $\beta > 1$ and $n \in \mathbb{N}$,*

$$\#\Sigma_{\beta,F}^n \leq \beta^n.$$

Proof. It follows immediately from $\#\Sigma_{\beta,F}^n \cdot \frac{1}{\beta^n} \leq 1$, where $\frac{1}{\beta^n}$ is the length of any full cylinder of order n (see Proposition 1.1.8). □

Lemma 1.1.46. *Let $\beta > 1$ and $n \in \mathbb{N}$.*

(1) *If $\beta \in \mathbb{N}$, then*

$$\#\Sigma_{\beta,F}^n = \beta^n.$$

(2) *If $\beta > 2$, then*

$$\#\Sigma_{\beta,F}^n > \frac{\beta - 2}{\beta - 1} \cdot \beta^n.$$

(3) *If $1 < \beta < 2$, then*

$$\#\Sigma_{\beta,F}^n > \Big(\prod_{i=1}^{\infty}(1 - \frac{1}{\beta^i})\Big) \cdot \beta^n,$$

where $\prod_{i=1}^{\infty}(1 - \frac{1}{\beta^i}) > 0$

Proof. For all $n \in \mathbb{N}$, let $a_n := \#\Sigma^n_{\beta,F}$ and $b_n := \#\Sigma^n_{\beta,N}$. Then $a_n + b_n = \#\Sigma^n_\beta$. Statement (1) is obvious. We prove (2) and (3) as follows.
(2) Suppose $\beta > 2$.

① For $n = 1$, we have $\#\Sigma^1_{\beta,F} = \lfloor\beta\rfloor > \beta - 1 > \frac{\beta-2}{\beta-1} \cdot \beta$.

② For $n \geq 2$, by Lemma 1.1.44, we get

$$a_n + b_n \geq \beta^n, \quad a_{n-1} + b_{n-1} < \frac{\beta^n}{\beta - 1} \quad \text{and then} \quad a_n + b_n - a_{n-1} - b_{n-1} > \beta^n - \frac{\beta^n}{\beta - 1}.$$

Since every cylinder has at most one non-full sub-cylinder, we have $\#\Sigma^n_{\beta,N} \leq \#\Sigma^{n-1}_\beta$, i.e., $b_n \leq a_{n-1} + b_{n-1}$. Therefore $a_n > \beta^n - \frac{\beta^n}{\beta-1} = \frac{\beta-2}{\beta-1} \cdot \beta^n$.

(3) Suppose $1 < \beta < 2$. For all $n \in \mathbb{N}$, let

$$c_n := (1 - \frac{1}{\beta})(1 - \frac{1}{\beta^2}) \cdots (1 - \frac{1}{\beta^n}) > \prod_{i=1}^{\infty}(1 - \frac{1}{\beta^i}).$$

It suffices to prove

$$a_n > c_n \beta^n. \tag{1.6}$$

(By induction) When $n = 1$, $1 > (1 - \frac{1}{\beta})\beta$ implies that (1.6) is true. Assume that $n \geq 2$ and (1.6) is true for $1, 2, \cdots, n-1$, i.e.,

$$a_1 > c_1\beta, \quad a_2 > c_2\beta^2, \quad \cdots, \quad a_{n-1} > c_{n-1}\beta^{n-1}. \tag{1.7}$$

Let $\{n_i\}$ denote the nonzero sequence of β. By $1 < \beta < 2$ we know $\varepsilon_{n_1} = \varepsilon_{n_2} = \varepsilon_{n_3} = \cdots = 1$. For the fixed $n \geq 2$, there exists a maximal $k \in \mathbb{N}$ such that $n_k \leq n$. By Proposition 1.1.9 (1), Theorem 1.1.14 and Corollary 1.1.15, we get a classification of the full words

$$\Sigma^n_{\beta,F} = \Sigma^n_{\beta,F,1} \cup \Sigma^n_{\beta,F,2} \cup \cdots \cup \Sigma^n_{\beta,F,k}$$

where

$$\begin{aligned}
\Sigma^n_{\beta,F,1} &:= \Big\{0w_2 \cdots w_n : w_2 \cdots w_n \text{ is a full word with length } n-1\Big\},\\
\Sigma^n_{\beta,F,2} &:= \Big\{\varepsilon_1 \cdots \varepsilon_{n_2-1} 0 w_{n_2+1} \cdots w_n : w_{n_2+1} \cdots w_n \text{ is a full word with length } n - n_2\Big\},\\
&\cdots,\\
\Sigma^n_{\beta,F,k} &:= \Big\{\varepsilon_1 \cdots \varepsilon_{n_k-1} 0 w_{n_k+1} \cdots w_n : w_{n_k+1} \cdots w_n \text{ is a full (or empty) word with length } n - n_k\Big\}
\end{aligned}$$

are all disjoint. Therefore

$$a_n \quad = \quad a_{n-1} + a_{n-n_2} + \cdots + a_{n-n_k} \qquad \text{(if } n = n_k\text{, define } a_{n-n_k} := 1)$$

$$\begin{aligned}
&\overset{(*)}{\geq} \quad c_{n-1}\cdot\beta^{n-1}+c_{n-n_2}\cdot\beta^{n-n_2}+\cdots+c_{n-n_k}\cdot\beta^{n-n_k} \quad (\text{if } n=n_k, \text{ define } c_{n-n_k}:=1)\\
&\geq \quad c_{n-1}\cdot(\beta^{n-1}+\beta^{n-n_2}+\cdots+\beta^{n-n_k})\\
&\overset{(**)}{>} \quad c_n\cdot\beta^n.
\end{aligned}$$

where $(*)$ follows from (1.7) and $(**)$ is equivalent to

$$1-(\frac{1}{\beta}+\frac{1}{\beta^{n_2}}+\cdots+\frac{1}{\beta^{n_k}})<\frac{1}{\beta^n}. \tag{1.8}$$

Thus it suffices to prove (1.8) in the following.

① If $\varepsilon(1,\beta)=\varepsilon_1\varepsilon_2\cdots\varepsilon_{n_k}0^\infty$, then $1=\frac{1}{\beta}+\frac{1}{\beta^{n_2}}+\cdots+\frac{1}{\beta^{n_k}}$, which implies (1.8).

② If there exists $m>k$ such that $\varepsilon(1,\beta)=\varepsilon_1\varepsilon_2\cdots\varepsilon_{n_k}\cdots\varepsilon_{n_m}0^\infty$, then

$$1-(\frac{1}{\beta}+\frac{1}{\beta^{n_2}}+\cdots+\frac{1}{\beta^{n_k}})=\frac{1}{\beta^{n_{k+1}}}+\cdots+\frac{1}{\beta^{n_m}}<\frac{1}{\beta^{n_{k+1}-1}},$$

where the last inequality follows from the fact that the β-expansion of 1 is greedy. Since k is the maximal integer such that $n_k\leq n$, we have $n\leq n_{k+1}-1$ and then $\frac{1}{\beta^{n_{k+1}-1}}\leq\frac{1}{\beta^n}$. Thus we get (1.8).

③ If $\varepsilon(1,\beta)$ is infinite, in a way similar to ②, we can get (1.8).

□

Lemma 1.1.47. *Let $\beta>1$ with $\beta\notin\mathbb{N}$. Then*

$$\#\Sigma^n_{\beta,N}\backsimeq\beta^n.$$

Proof. (1) We have $\#\Sigma^n_{\beta,N}\leq\#\Sigma^n_\beta<\frac{\beta}{\beta-1}\cdot\beta^n$, where the last inequality follows from Lemma 1.1.44

(2) Prove that there exists $c_\beta>0$ such that for all $n\in\mathbb{N}$, we have $\#\Sigma^n_{\beta,N}\geq c_\beta\cdot\beta^n$.

① When $n=1$, by $\beta\notin\mathbb{N}$ we get $\#\Sigma^1_{\beta,N}=1\geq\frac{1}{\beta}\cdot\beta$.

② When $n\geq 2$, it follows from $\beta\notin\mathbb{N}$ that every full cylinder of order $n-1$ has a non-full sub-cylinder of order n. Thus $\#\Sigma^n_{\beta,N}\geq\#\Sigma^{n-1}_{\beta,F}$. Since Lemma 1.1.46 says that there exists $c=c(\beta)>0$ such that $\#\Sigma^{n-1}_{\beta,F}\geq c\cdot\beta^{n-1}$, we get $\#\Sigma^n_{\beta,N}\geq\frac{c}{\beta}\cdot\beta^n$.

□

1.2 Bernoulli-type measures and frequency sets

This section is a joint work with Mr. Bing Li and Mr. Tuomas Sahlsten at the end of my master at Université Paris-Est Marne-la-Vallée (UPEM) under the guidance of Mr.

Lingmin Liao. Most of the content in this section has already appeared in my master thesis at UPEM. For completeness and for the convenience of the reader, I still present it here.

Let $\beta > 1$. Recall that $\mathcal{A}_\beta^{\mathbb{N}}$ is the set of infinite sequences on $\mathcal{A}_\beta = \{0, 1, \cdots, \lceil \beta \rceil - 1\}$ and Σ_β is the set of admissible sequences. Define the *usual metric* d_β on $\mathcal{A}_\beta^{\mathbb{N}}$ (also on Σ_β) by

$$d_\beta(w, v) := \beta^{-\inf\{k \geq 0:\ w_{k+1} \neq v_{k+1}\}} \quad \text{for } w = (w_i)_{i \geq 1}, v = (v_i)_{i \geq 1} \in \mathcal{A}_\beta^{\mathbb{N}}. \tag{1.9}$$

Let $\mathcal{B}(\Sigma_\beta)$ be the Borel sigma-algebra on the metric space (Σ_β, d_β). Recall that we use $[w]$ to denote the cylinder in Σ_β generated by the admissible word w. Given $\beta \in (1, 2]$, for $p \in (0, 1)$, we define the $(p, 1-p)$ *Bernoulli-type measure* μ_p on the measurable space $(\Sigma_\beta, \mathcal{B}(\Sigma_\beta))$ as follows:

I. Let
$$\mu_p(\emptyset) := 0, \quad \mu_p(\Sigma_\beta) := 1, \quad \mu_p[0] := p, \quad \text{and} \quad \mu_p[1] := 1 - p.$$

II. Suppose μ_p has been defined for all cylinders of order $k \in \mathbb{N}$. For any admissible word w with length k, if $w1$ is admissible, we define
$$\mu_p[w0] := p\mu_p[w] \quad \text{and} \quad \mu_p[w1] := (1-p)\mu_p[w];$$
if $w1$ is not admissible, then naturally
$$\mu_p[w0] = \mu_p[w].$$

III. By Carathéodory's extension theorem, we extend the definition of μ_p from the family of cylinders to $\mathcal{B}(\Sigma_\beta)$ by
$$\mu_p(A) := \inf\Big\{\sum_n \mu_p[w^{(n)}] : w^{(n)} \in \Sigma_\beta^*, A \subset \bigcup_n [w^{(n)}]\Big\}$$
for any $A \in \mathcal{B}(\Sigma_\beta)$.

The corresponding image measure

$$\nu_p := \pi_\beta \mu_p := \mu_p \circ \pi_\beta^{-1}$$

is called the $(p, 1-p)$ *Bernoulli-type measure* on $([0, 1), \mathcal{B}[0, 1))$, where $\mathcal{B}[0, 1)$ is the Borel sigma-algebra on $[0, 1)$ and $\pi_\beta : \Sigma_\beta \to [0, 1)$ is the natural projection map defined by (1.3) restricted to Σ_β (so $\pi_\beta^{-1} A \subset \Sigma_\beta$ for any $A \subset [0, 1)$). Moreover, we use $\sigma_\beta : \Sigma_\beta \to \Sigma_\beta$ to denote the shift map σ defined by (1.2) restricted to Σ_β (so $\sigma_\beta^{-1} A \subset \Sigma_\beta$ for any $A \subset \Sigma_\beta$), and recall that T_β is the β-transformation on $[0, 1)$ defined by (1.1). It is straightforward to see that μ_p may not be σ_β-invariant and ν_p may not be T_β-invariant. For example, if

$\beta = \frac{1+\sqrt{5}}{2}$ is the golden ratio, then 11 is not an admissible word. We have

$$\mu_p[1] = 1 - p, \quad \text{but} \quad \mu_p(\sigma_\beta^{-1}[1]) = \mu_p[01] = p(1-p).$$

Correspondingly,

$$\nu_p[\frac{1}{\beta}, 1) = 1 - p, \quad \text{but} \quad \nu_p(T_\beta^{-1}[\frac{1}{\beta}, 1)) = p(1-p).$$

Hence we consider the following concepts.

Definition 1.2.1 (Quasi-invariance)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space and T be a measurable transformation on it. Then*

(1) *μ is quasi-invariant with respect to the transformation T if μ and its image measure $T\mu$ are mutually absolutely continuous (i.e. equivalent), that is,*

$$\mu \ll T\mu \ll \mu \quad (i.e.\ T\mu \sim \mu);$$

(2) *μ is strongly quasi-invariant with respect to the transformation T if there exists a constant $C > 0$ such that*

$$C^{-1}\mu(A) \leq T^k\mu(A) \leq C\mu(A)$$

for any $k \in \mathbb{N}$ and $A \in \mathcal{F}$. We also say μ is C-strongly quasi-invariant if we know such a C.

Definition 1.2.2 (Quasi-Bernoulli)**.** *A measure μ on $(\Sigma_\beta, \mathcal{B}(\Sigma_\beta))$ is called quasi-Bernoulli if there exists a constant $C > 0$ such that*

$$C^{-1}\mu[w]\mu[w'] \leq \mu[ww'] \leq C\mu[w]\mu[w']$$

for every pair $w, w' \in \Sigma_\beta^*$ satisfying $ww' \in \Sigma_\beta^*$.

As the first main result of this section, the following theorem focuses on the invariance of Bernoulli-type measures as dynamical properties. Recall from Section 1.1 that we use $\varepsilon(x, \beta)$ to denote the β-expansion of x.

Theorem 1.2.3. *Let $\beta \in (1, 2]$ and $p \in (0, 1)$. Then*

(1) *μ_p is quasi-invariant with respect to σ_β;*

(2) *$\varepsilon(1, \beta)$ is finite if and only if μ_p is quasi-Bernoulli;*

(3) *$\varepsilon(1, \beta)$ is finite if and only if μ_p is strongly quasi-invariant with respect to σ_β.*

By $\pi_\beta \circ \sigma_\beta = T_\beta \circ \pi_\beta$, we get the following.

Corollary 1.2.4. *Let $\beta \in (1,2]$ and $p \in (0,1)$. Then*

(1) *ν_p is quasi-invariant with respect to T_β;*

(2) *$\varepsilon(1,\beta)$ is finite if and only if ν_p is strongly quasi-invariant with respect to T_β.*

As the second main result of this section, we have the following.

Theorem 1.2.5. *Let $\beta \in (1,2]$ and $p \in (0,1)$. If $\varepsilon(1,\beta)$ is finite, then there exists a unique T_β-ergodic probability measure m_p on $([0,1),\mathcal{B}[0,1))$ equivalent to ν_p, where m_p is given by*

$$m_p(B) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T_\beta^k \nu_p(B) \quad \text{for } B \in \mathcal{B}[0,1).$$

In the following, we consider digit frequencies. Given $\beta > 1$, for any $a \in [0,1]$, define the frequency set, lower frequency set and upper frequency set by

$$F_{\beta,a} := \Big\{x \in [0,1) : \lim_{n\to\infty} \frac{\#\{1 \le k \le n : \varepsilon_k(x,\beta) = 0\}}{n} = a\Big\},$$

$$\underline{F}_{\beta,a} := \Big\{x \in [0,1) : \varliminf_{n\to\infty} \frac{\#\{1 \le k \le n : \varepsilon_k(x,\beta) = 0\}}{n} = a\Big\}$$

and

$$\overline{F}_{\beta,a} := \Big\{x \in [0,1) : \varlimsup_{n\to\infty} \frac{\#\{1 \le k \le n : \varepsilon_k(x,\beta) = 0\}}{n} = a\Big\}$$

respectively. As an application of the above Theorem 1.2.5 and Theorem 1.2.33 in Subsection 1.2.4, we have the following as the third main result of this section, where $\dim_H$ denotes the Hausdorff dimension.

Theorem 1.2.6. *Let $\beta \in (1,2)$ such that $\varepsilon(1,\beta) = 10^m 10^\infty$ for some integer $m \ge 0$.*
(1) *If $0 \le a < \frac{m+1}{m+2}$, then $F_{\beta,a} = \underline{F}_{\beta,a} = \overline{F}_{\beta,a} = \emptyset$.*
(2) *If $\frac{m+1}{m+2} \le a \le 1$, then $\dim_H F_{\beta,a} = \dim_H \underline{F}_{\beta,a} = \dim_H \overline{F}_{\beta,a}$*

$$= \frac{(ma-m+a)\log(ma-m+a) - (ma-m+2a-1)\log(ma-m+2a-1) - (1-a)\log(1-a)}{\log\beta}.$$

In particular, $\dim_H F_{\beta,\frac{m+1}{m+2}} = \dim_H \underline{F}_{\beta,\frac{m+1}{m+2}} = \dim_H \overline{F}_{\beta,\frac{m+1}{m+2}} = \dim_H F_{\beta,1} = \dim_H \underline{F}_{\beta,1} = \dim_H \overline{F}_{\beta,1} = 0$.

Remark 1.2.7. *Taking $m = 0$ in Theorem 1.2.6, we get the well known result*

$$\dim_H F_{\beta,a} = \frac{a\log a - (2a-1)\log(2a-1) - (1-a)\log(1-a)}{\log\beta}$$

where $\beta = \frac{\sqrt{5}+1}{2}$ is the golden ratio and $\frac{1}{2} \le a \le 1$. See for examples [67, 92]. Note that when $0 \le a < \frac{1}{2}$, $F_{\beta,a} = \emptyset$.

This section is organized as follows. In Subsection 1.2.1, on the basis of Section 1.1, we give more necessary notation and preliminaries on beta-expansions and measure theory. In Subsection 1.2.2, we study some digit occurence parameters and their properties which are useful for studying Bernoulli-type measures. In Subsection 1.2.3, we study Bernoulli-type measures and prove Theorems 1.2.3 and 1.2.5. In Subsection 1.2.4, according to the structure of cylinders, we define and study the modified lower local dimension of finite Borel measures, where the main result Theorem 1.2.33 implies the modified mass distribution principle given by Bugeaud and Wang [37]. It is a useful tool to estimate the upper and lower bounds for the Hausdorff dimension of some sets defined in terms of beta-expansions. In Subsection 1.2.5, we apply the Bernoulli-type measures and the modified lower local dimension to prove the digit frequency result Theorem 1.2.6.

1.2.1 Notation and preliminaries

Let $\beta > 1$. For simplification, we still use $\varepsilon_1\varepsilon_2\cdots\varepsilon_n\cdots$ and $\varepsilon_1^*\varepsilon_2^*\cdots\varepsilon_n^*\cdots$ to denote the digits of $\varepsilon(1,\beta)$ and $\varepsilon^*(1,\beta)$ respectively as in Section 1.1.

For $n \in \mathbb{N}$, let $l_n(\beta)$ denote the number of 0's following ε_n^*, i.e.,

$$l_n(\beta) := \sup\{k \geq 1 : \varepsilon_{n+j}^* = 0 \text{ for all } 1 \leq j \leq k\}$$

where by convention $\sup\emptyset := 0$. The set of $\beta > 1$ such that the length of the strings of 0's in $\varepsilon^*(1,\beta)$ is bounded is denoted by

$$A_0 := \{\beta > 1 : (l_n(\beta))_{n\geq 1} \text{ is bounded}\}.$$

Proposition 1.2.8 ([91])**.** *Let $\beta > 1$. Then $\beta \in A_0$ if and only if there exists a constant $c > 0$ such that for all $x \in [0,1)$ and $n \in \mathbb{N}$,*

$$c \cdot \frac{1}{\beta^n} \leq |I_n(x)| \leq \frac{1}{\beta^n}.$$

The following covering property is deduced from the length and distribution of full cylinders.

Proposition 1.2.9. *([37, Proposition 4.1]) Let $\beta > 1$. For any $x \in [0,1)$ and $n \in \mathbb{N}$, the interval $[x - \frac{1}{\beta^n}, x + \frac{1}{\beta^n}]$ intersected with $[0,1)$ can be covered by at most $4(n+1)$ cylinders of order n.*

By the structure of cylinders, the following lemma follows from a similar proof of Lemma 1 (i) in [120].

Lemma 1.2.10. *Any cylinder (in Σ_β or $[0,1)$) can be written as a countable disjoint union of full cylinders.*

Definition 1.2.11. *Let $\mathcal{C}$ be a family of certain subsets of a set X.*

(1) *$\mathcal{C}$ is called a monotone class on X if*

① $\{A_n\}_{n\geq 1} \subset \mathcal{C}$ *and* $A_1 \subset A_2 \subset \cdots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$;

② $\{A_n\}_{n\geq 1} \subset \mathcal{C}$ *and* $A_1 \supset A_2 \supset \cdots \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{C}$.

(2) *$\mathcal{C}$ is called a semi-algebra on X if*

① $\emptyset \in \mathcal{C}$;

② $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$;

③ $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}_{\Sigma f}$

where $A^c := X \setminus A$ *and* $\mathcal{C}_{\Sigma f} := \left\{ \bigcup_{i=1}^{n} C_i : C_1, \cdots, C_n \in \mathcal{C} \text{ are disjoint, } n \in \mathbb{N} \right\}$.
(The subscript $_{\Sigma f}$ means finite disjoint union.)

(3) *$\mathcal{C}$ is called an algebra on X if*

① $\emptyset, X \in \mathcal{C}$;

② $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$;

③ $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$.

(4) *$\mathcal{C}$ is called a sigma-algebra on X if*

① $\emptyset, X \in \mathcal{C}$;

② $A \in \mathcal{C} \Rightarrow A^c \in \mathcal{C}$;

③ $A_1, A_2, A_3, \cdots \in \mathcal{C} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{C}$.

In order to extend some properties from a small family to a larger one in some proofs in Subsection 1.2.3, we recall the following well known theorem as basic knowledge of measure theory. For more details, see for examples [39] and [57].

Theorem 1.2.12 (Monotone class theorem)**.** *Let $\mathcal{A}$ be an algebra and $M(\mathcal{A})$ be the smallest monotone class containing $\mathcal{A}$. Then $M(\mathcal{A})$ is precisely the sigma-algebra generated by $\mathcal{A}$, i.e., $sig(\mathcal{A}) = M(\mathcal{A})$.*

The following useful approximation lemma follows from Theorems 0.1 and 0.7 in [121].

Lemma 1.2.13. *Let $(X, \mathcal{B}, \mu)$ be a probability space, $\mathcal{C}$ be a semi-algebra which generates the sigma-algebra $\mathcal{B}$ and $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. Then*

(1) $\mathcal{A} = \mathcal{C}_{\Sigma f} := \{\bigcup_{i=1}^{n} C_i : C_1, \cdots, C_n \in \mathcal{C} \text{ are disjoint, } n \in \mathbb{N}\}$;

(2) *for each $\varepsilon > 0$ and each $B \in \mathcal{B}$, there is some $A \in \mathcal{A}$ with $\mu(A \triangle B) < \varepsilon$.*

We recall some well known concepts and theorems (see for examples [81, 121, 127]) needed to be used.

Theorem 1.2.14 (Carathéodory's measure extension theorem). *Let $\mathcal{C}$ be a semi-algebra on X and $\mu : \mathcal{C} \to [0, +\infty]$ such that for all sets $A \in \mathcal{C}$ for which there exists a countable decomposition $A = \cup_{i=1}^{\infty} A_i$ in disjoint sets $A_i \in \mathcal{C}$ for $i \in \mathbb{N}$, we have $\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$. Then μ can be extended to become a measure μ' on $sig(\mathcal{C})$ (the smallest sigma-algebra containing $\mathcal{C}$). That is, there exists a measure $\mu' : sig(\mathcal{C}) \to [0, +\infty]$ such that its restriction to $\mathcal{C}$ is equal to μ (i.e., $\mu'|_{\mathcal{C}} = \mu$). Moreover, if $X \in \mathcal{C}$ and $\mu(X) < +\infty$, then the extension μ' is unique.*

Theorem 1.2.15 (Dominated convergence theorem). *Let $(X, \mathcal{F}, \mu)$ be a probability space and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued measurable functions on X satisfying*

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for } \mu\text{-almost every } x \in X.$$

If there exists a real-valued integrable function g on X such that for all $n \in \mathbb{N}$, $|f_n(x)| \leq g(x)$ for μ-almost every $x \in X$, then f is integrable and

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

Theorem 1.2.16 (Vitali-Hahn-Saks Theorem). *Let $(X, \mathcal{F}, \mu)$ be a probability space and $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures such that $\lambda_n \ll \mu$ for all $n \in \mathbb{N}$. If the finite $\lim_{n\to\infty} \lambda_n(B) = \lambda(B)$ exists for every $B \in \mathcal{F}$, then λ is countable additive on $\mathcal{F}$.*

Definition 1.2.17 (Invariance and ergodicity). *Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving dynamical system, that is, $(X, \mathcal{F}, \mu)$ is a probability space and μ is T-invariant, i.e., $T\mu = \mu$. We say that the probability measure μ is ergodic with respect to T if for every $A \in \mathcal{F}$ satisfying $T^{-1}A = A$ (such a set is called T-invariant), we have $\mu(A) = 0$ or 1. We also say that $(X, \mathcal{F}, \mu, T)$ is ergodic.*

Theorem 1.2.18 (Birkhoff's ergodic theorem). *Let $(X, \mathcal{F}, \mu, T)$ be a measure-preserving dynamical system where the probability measure μ is ergodic with respect to T. Then for any real-valued integrable function $f : X \to \mathbb{R}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int f d\mu$$

for μ almost every $x \in X$.

1.2.2 Digit occurence parameters

Definition 1.2.19 (Digit occurence parameters)**.** *Let $\beta \in (1,2]$. Define*

$$\mathcal{N}_0(w) := \{k \geq 0 : w_{k+1} = 0 \text{ and } w_1 w_2 \dots w_k 1 \text{ is admissible}\} \quad \text{for any } w \in \Sigma_\beta,$$

$$\mathcal{N}_0(w) := \{0 \leq k < |w| : w_{k+1} = 0 \text{ and } w_1 w_2 \dots w_k 1 \text{ is admissible}\} \quad \text{for any } w \in \Sigma_\beta^*,$$

$$\mathcal{N}_1(w) := \{k \geq 1 : w_k = 1\} \quad \text{for any } w \in \Sigma_\beta,$$

$$\mathcal{N}_1(w) := \{1 \leq k \leq |w| : w_k = 1\} \quad \text{for any } w \in \Sigma_\beta^*$$

and let

$$N_0(w) := \#\mathcal{N}_0(w), \quad N_1(w) := \#\mathcal{N}_1(w) \quad \text{for any } w \in \Sigma_\beta^* \cup \Sigma_\beta,$$

$$N_0(x,n) := N_0(\varepsilon(x,\beta)|_n), \quad N_1(x,n) := N_1(\varepsilon(x,\beta)|_n) \quad \text{for any } x \in [0,1)$$

where $\#\mathcal{N}$ denotes the cardinality of the set $\mathcal{N}$.

Noting that $N_1(w)$ is just the number of the digit 1 appearing in w, it is immediate from the definition that if $w, w' \in \Sigma_\beta^*$ such that $ww' \in \Sigma_\beta^*$, then

$$N_1(ww') = N_1(w) + N_1(w').$$

Notation 1.2.20. *Let $\beta > 1$. Denote the first position where w and $\varepsilon^*(1,\beta)$ are different by*

$$\mathfrak{m}(w) := \min\{k \geq 1 : w_k < \varepsilon_k^*\} \quad \text{for } w \in \Sigma_\beta$$

$$\text{and} \quad \mathfrak{m}(w) := \mathfrak{m}(w0^\infty) \quad \text{for } w \in \Sigma_\beta^*.$$

For any $w \in \Sigma_\beta$, combing the facts $w \prec \varepsilon^*(1,\beta)$, $\varepsilon^*(1,\beta)|_n \in \Sigma_\beta^*$ for all $n \in \mathbb{N}$ and Lemma 1.1.7, we know that there exists $k \in \mathbb{N}$ such that $w|_k$ is full. Therefore we can define

$$\tau(w) := \min\{k \geq 1 : w|_k \text{ is full}\} \quad \text{for any } w \in \Sigma_\beta,$$

$$\text{and} \quad \tau(w) := \tau(w0^\infty) \quad \text{for any } w \in \Sigma_\beta^*.$$

For any $w \in \Sigma_\beta^*$, regarding $w|_0$ as the empty word which is full, we define

$$\tau'(w) := \max\{0 \leq k \leq |w| : w|_k \text{ is full}\}.$$

Lemma 1.2.21. *Let $\beta > 1$. For any $w \in \Sigma_\beta \cup \Sigma_\beta^*$, we have*

$$\tau(w) = \begin{cases} \mathfrak{m}(w) & \text{if } \varepsilon(1,\beta) \text{ is infinite;} \\ \min\{\mathfrak{m}(w), M\} & \text{if } \varepsilon(1,\beta) \text{ is finite with length } M. \end{cases}$$

Proof. For any $w \in \Sigma_\beta \cup \Sigma_\beta^*$. Let $k = \mathfrak{m}(w)$. Then $w|_k = \varepsilon_1^* \cdots \varepsilon_{k-1}^* w_k$ and $w_k < \varepsilon_k^*$. (When $w \in \Sigma_\beta^*$ and $k > |w|$, we regard $w|_k = w_1 \cdots w_k$ as $w_1 \cdots w_{|w|} 0^{k-|w|}$). By $\varepsilon_1^* \cdots \varepsilon_{k-1}^* \varepsilon_k^* \in \Sigma_\beta^*$ and Lemma 1.1.7, $w|_k$ is full.

(1) When $\varepsilon(1,\beta)$ is infinite, for any $i \in \{1, \cdots, k-1\}$, we have $w|_i = \varepsilon^*(1,\beta)|_i = \varepsilon(1,\beta)|_i$ which is not full by Proposition 1.1.10. Therefore $\tau(w) = k = \mathfrak{m}(w)$.

(2) when $\varepsilon(1,\beta) = \varepsilon_1 \cdots \varepsilon_M 0^\infty$ with $\varepsilon_M \neq 0$:
① If $k \leq M$, then for any $i \in \{1, \cdots, k-1\}$, we have $w|_i = \varepsilon^*(1,\beta)|_i$ which is not full by Proposition 1.1.10. Therefore $\tau(w) = k = \mathfrak{m}(w)$.
② If $k > M$, then $w|_M = \varepsilon^*(1,\beta)|_M$ is full by Proposition 1.1.10. For any $i \in \{1, \cdots, M-1\}$, we have $w|_i = \varepsilon^*(1,\beta)|_i$ which is not full by Proposition 1.1.10. Therefore $\tau(w) = M$. □

Lemma 1.2.22. *Let $\beta > 1$ and $w \in \Sigma_\beta$. Then*

(1) *there exists a strictly increasing sequence $(n_j)_{j\geq 1}$ such that $w|_{n_j}$ is full for any $j \in \mathbb{N}$;*

(2) *$N_0(w) = +\infty$ if $1 < \beta \leq 2$.*

Proof.

(1) Let $k_1 := \mathfrak{m}(w)$, $n_1 := k_1$, $k_j := \mathfrak{m}(\sigma^{n_{j-1}} w)$ and $n_j := n_{j-1} + k_j$ for any $j \geq 2$. Then n_j is strictly increasing. By $\varepsilon_1^* \cdots \varepsilon_{k_1-1}^* \varepsilon_{k_1}^* \in \Sigma_\beta^*$, $w_{n_1} < \varepsilon_{k_1}^*$ and Lemma 1.1.7, we know that $w_1 \cdots w_{n_1-1} w_{n_1} = \varepsilon_1^* \cdots \varepsilon_{k_1-1}^* w_{n_1}$ is full. Similarly for any $j \geq 2$, by $\varepsilon_1^* \cdots \varepsilon_{k_j-1}^* \varepsilon_{k_j}^* \in \Sigma_\beta^*$, $w_{n_j} < \varepsilon_{k_j}^*$ and Lemma 1.1.7, we know that $w_{n_{j-1}+1} \cdots w_{n_j-1} w_{n_j} = \varepsilon_1^* \cdots \varepsilon_{k_j-1}^* w_{n_j}$ is full. Therefore, by Proposition 1.1.9 (1), $w|_{n_j}$ is full for any $j \in \mathbb{N}$.

(2) Noting that $1 < \beta \leq 2$, by $w_{n_j} < \varepsilon_{k_j}^*$, we get $w_{n_j} = 0, \varepsilon_{k_j}^* = 1$ for any $j \in \mathbb{N}$. Thus

$$w_1 \cdots w_{n_j-1} 1 = \varepsilon_1^* \cdots \varepsilon_{k_1-1}^* w_{n_1} \cdots\cdots \varepsilon_1^* \cdots \varepsilon_{k_{j-1}-1}^* w_{n_{j-1}} \varepsilon_1^* \cdots \varepsilon_{k_j-1}^* \varepsilon_{k_j}^* \in \Sigma_\beta^*$$

for any $j \in \mathbb{N}$ by Proposition 1.1.9 (1) and Proposition 1.1.8 (5). Therefore $N_0(w) = +\infty$. □

Lemma 1.2.23. *Let $\beta \in (1,2]$ and $w, w' \in \Sigma_\beta^*$ with $ww' \in \Sigma_\beta^*$. Then*

(1) *$N_0(w) \leq N_0(ww') \leq N_0(w) + N_0(w')$;*

(2) *when w is full, we have $N_0(ww') = N_0(w) + N_0(w')$;*

(3) *when $\varepsilon(1,\beta) = \varepsilon_1 \cdots \varepsilon_M 0^\infty$ with $\varepsilon_M \neq 0$, we have $N_0(ww') \geq N_0(w) + N_0(w') - M$.*

Proof. Let $a = |w|$ and $b = |w'|$. Then $ww' = w_1 \cdots w_a w_1' \cdots w_b'$.
(1) The first inequality $N_0(w) \leq N_0(ww')$ follows from $\mathcal{N}_0(w) \subset \mathcal{N}_0(ww')$. In the following we prove the second inequality $N_0(ww') \leq N_0(w) + N_0(w')$.

① We prove $\mathcal{N}_0(ww') \subset \mathcal{N}_0(w) \cup (\mathcal{N}_0(w') + a)$ first. Let $k \in \mathcal{N}_0(ww')$.
If $0 \le k < a$, then $w_{k+1} = 0$ and $w_1 \cdots w_k 1 \in \Sigma_\beta^*$. We get $k \in \mathcal{N}_0(w)$.
If $a \le k < a + b$, then $w'_{k-a+1} = 0$ and $w_1 \cdots w_a w'_1 \cdots w'_{k-a} 1 \in \Sigma_\beta^*$. It follows from $w'_1 \cdots w'_{k-a} 1 \in \Sigma_\beta^*$ that $k - a \in \mathcal{N}_0(w')$ and $k \in \mathcal{N}_0(w') + a$.

② Combining $\mathcal{N}_0(w) \cap (\mathcal{N}_0(w') + a) = \emptyset$, $\#(\mathcal{N}_0(w') + a) = \#\mathcal{N}_0(w')$ and i), we get $N_0(ww') \le N_0(w) + N_0(w')$.

(2) We need to prove $N_0(ww') \ge N_0(w) + N_0(w')$. By $\#\mathcal{N}_0(w') = \#(\mathcal{N}_0(w') + a)$, it suffices to prove $\mathcal{N}_0(ww') \supset \mathcal{N}_0(w) \cup (\mathcal{N}_0(w') + a)$. For each $k \in \mathcal{N}_0(w)$, obviously $k \in \mathcal{N}_0(ww')$. On the other hand, if $k \in (\mathcal{N}_0(w') + a)$, then $k - a \in \mathcal{N}_0(w')$, $w'_{k-a+1} = 0$ and $w'_1 \cdots w'_{k-a} 1 \in \Sigma_\beta^*$. Since w is full, by Proposition 1.1.8, we get $ww'_1 \cdots w'_{k-a} 1 \in \Sigma_\beta^*$ and then $k \in \mathcal{N}_0(ww')$.

(3) ① First we divide ww' into three segments.

i) Let $k_0 := \tau'(w)$, then $0 \le k_0 \le a$. If $k_0 = a$, w is full. Then the conclusion follows from (2) immediately. Therefore we assumes $0 \le k_0 < a$ in the following proof. Let $u^{(1)} := w_1 \cdots w_{k_0}$ be full and $|u^{(1)}| = k_0$. (When $k_0 = 0$, we regard $u^{(1)}$ as the empty word and $N_0(u^{(1)}) := 0$.)

ii) Consider $w_{k_0+1} \cdots w_a w'_1 \cdots w'_b \in \Sigma_\beta^*$ (the admissibility follows from $ww' \in \Sigma_\beta^*$).
Let $k_1 := \tau(w_{k_0+1} \cdots w_a w'_1 \cdots w'_b) \ge 1$. By the definition of $k_0 = \tau'(w)$ and Proposition 1.1.9, we get $k_1 > a - k_0$. In the following, we assume $k_1 \le a - k_0 + b$ first. The case $k_1 > a - k_0 + b$ will be considered at the end of the proof. Let $u^{(2)} := w_{k_0+1} \cdots w_a w'_1 \cdots w'_{k_0+k_1-a}$, then $|u^{(2)}| = k_1$.

iii) Let $u^{(3)} := w'_{k_0+k_1-a+1} \cdots w'_b$. (When $k_0 + k_1 - a = b$, we regard $u^{(3)}$ as the empty word and $N_0(u^{(3)}) := 0$.)

Up to now, we write $ww' = u^{(1)} u^{(2)} u^{(3)}$ as:

$$\underbrace{w_1 \cdots w_{k_0}}_{|u^{(1)}|=k_0} \underbrace{w_{k_0+1} \cdots w_a w'_1 \cdots w'_{k_0+k_1-a}}_{|u^{(2)}|=k_1} \underbrace{w'_{k_0+k_1-a+1} \cdots w'_b}_{|u^{(3)}|}.$$

② Estimate $N_0(ww'), N_0(w)$ and $N_0(w')$.

i) $N_0(ww') = N_0(u^{(1)} u^{(2)} u^{(3)}) \underset{\text{by (2)}}{\overset{u^{(1)} \text{ full}}{=\!=}} N_0(u^{(1)}) + N_0(u^{(2)} u^{(3)}) \underset{\text{by (2)}}{\overset{u^{(2)} \text{ full}}{=\!=}} N_0(u^{(1)}) + N_0(u^{(2)}) + N_0(u^{(3)})$.

ii) $N_0(w) \underset{\text{by (2)}}{\overset{u^{(1)} \text{ full}}{=\!=}} N_0(u^{(1)}) + N_0(w_{k_0+1} \cdots w_a) \overset{\text{by (1)}}{\le} N_0(u^{(1)}) + N_0(u^{(2)})$.

iii) $N_0(w') \overset{\text{by (1)}}{\le} N_0(w'_1 \cdots w'_{k_0+k_1-a}) + N_0(u^{(3)}) \le M + N_0(u^{(3)})$ where the last inequality follows from

$$N_0(w'_1 \cdots w'_{k_0+k_1-a}) \le k_0 + k_1 - a \le k_1 = \tau(w_{k_0+1} \cdots w_a w'_1 \cdots w'_b) \overset{\text{by Lemma 1.2.21}}{\le} M.$$

Combining i), ii) and iii), we get $N_0(ww') \geq N_0(w) + N_0(w') - M$.

To end the proof, it suffices to consider the case $k_1 > a - k_0 + b$ below. We define $u^{(1)}$ as before and define $u^{(2)} := w_{k_0+1} \cdots w_a w'_1 \cdots w'_b$ which is not full. Then $|u^{(2)}| = a - k_0 + b$. We do not define $u^{(3)}$.

① Prove $N_0(u^{(2)}) = 0$.

By contradiction, we suppose $N_0(u^{(2)}) \neq 0$, then there exists $k \in \mathcal{N}_0(u^{(2)})$, $0 \leq k < a - k_0 + b$ such that $u^{(2)}_{k+1} = 0$ and $u^{(2)}_1 \cdots u^{(2)}_k 1 \in \Sigma^*_\beta$. By Lemma 1.1.7, $u^{(2)}_1 \cdots u^{(2)}_{k+1}$ is full which contradicts $\tau(u^{(2)}) = k_1 > a - k_0 + b$.

② Estimate $N_0(ww'), N_0(w)$ and $N_0(w')$.

i) $N_0(ww') = N_0(u^{(1)}u^{(2)}) \xlongequal[\text{by (2)}]{u^{(1)} \text{ full}} N_0(u^{(1)}) + N_0(u^{(2)}) \xlongequal{\text{by ①}} N_0(u^{(1)})$.

ii) $N_0(w) \xlongequal[\text{by (2)}]{u^{(1)} \text{ full}} N_0(u^{(1)}) + N_0(w_{k_0+1} \cdots w_a) = N_0(u^{(1)})$ where the last equality follows from $N_0(w_{k_0+1} \cdots w_a) \leq N_0(u^{(2)}) = 0$.

iii) $N_0(w') \leq b < a - k_0 + b < k_1 = \tau(u^{(2)}) \overset{\text{by Lemma 1.2.21}}{\leq} M$.

Combining i), ii) and iii), we get $N_0(ww') \geq N_0(w) + N_0(w') - M$. □

1.2.3 Bernoulli-type measures μ_p and ν_p

Let $\beta \in (1, 2]$. Recall the definitions of the Bernoulli-type measures μ_p and ν_p from the beginning of this section.

Remark 1.2.24. (1) *We have*

$$\begin{aligned}
\nu_p(I(w)) = \mu_p[w] &= p^{N_0(w)}(1-p)^{N_1(w)} \quad \textit{for any } w \in \Sigma^*_\beta; \\
\nu_p(I(w|_n)) = \mu_p[w|_n] &= p^{N_0(w|_n)}(1-p)^{N_1(w|_n)} \quad \textit{for any } w \in \Sigma_\beta \textit{ and } n \in \mathbb{N}; \\
\nu_p(I_n(x)) = \mu_p[\varepsilon(x,\beta)|_n] &= p^{N_0(x,n)}(1-p)^{N_1(x,n)} \quad \textit{for any } x \in [0,1) \textit{ and } n \in \mathbb{N}.
\end{aligned}$$

(2) *For any $w \in \Sigma_\beta$, as $n \to +\infty$, by Lemma 1.2.22 (2) we get $N_0(w|_n) \to +\infty$ and then $\mu_p[w|_n] \to 0$.*

Proposition 1.2.25. *The measures μ_p, $\sigma^k_\beta \mu_p$, ν_p and $T^k_\beta \nu_p$ have no atoms. That is, $\mu_p(\{w\}) = \sigma^k_\beta \mu_p(\{w\}) = \nu_p(\{x\}) = T^k_\beta \nu_p(\{x\}) = 0$ for any single point $w \in \Sigma_\beta$, $x \in [0,1)$ and $k \in \mathbb{N}$.*

Proof. It follows immediately from $\mu_p[w|_n] \to 0$, $\#\sigma^{-k}_\beta\{w\} \leq 2^k$, $\#\pi^{-1}_\beta\{x\} = 1$ and $\#T^{-k}_\beta\{x\} \leq 2^k$ for any $w \in \Sigma_\beta$ and $x \in [0,1)$. □

Combing Remark 1.2.24 (1), Lemma 1.2.23 and the fact that $N_1(ww') = N_1(w) + N_1(w')$ for any $w, w' \in \Sigma^*_\beta$ satisfying $ww' \in \Sigma^*_\beta$, we have the following.

Lemma 1.2.26. *Let $\beta \in (1,2]$, $p \in (0,1)$ and $w, w' \in \Sigma_\beta^*$ with $ww' \in \Sigma_\beta^*$.*

(1) *We have*
$$\mu_p[w] \geq \mu_p[ww'] \geq \mu_p[w]\mu_p[w'].$$

(2) *When w is full, we have*
$$\mu_p[ww'] = \mu_p[w]\mu_p[w'].$$

(3) *When $\varepsilon(1,\beta) = \varepsilon_1 \cdots \varepsilon_M 0^\infty$ with $\varepsilon_M \neq 0$, we have*
$$\mu_p[ww'] \leq p^{-M}\mu_p[w]\mu_p[w'],$$
and then μ_p is quasi-Bernoulli.

Now we can begin to prove our first main result.

Proof of Theorem 1.2.3. (1) ① First we prove $\mu_p \ll \sigma_\beta \mu_p$.
Let $A \in \mathcal{B}(\Sigma_\beta)$ with $\sigma_\beta\mu_p(A) = 0$. It suffices to prove $\mu_p(A) = 0$. For any $\varepsilon > 0$, by
$$\mu_p(\sigma_\beta^{-1}A) = \inf\Big\{\sum_n \mu_p[w^{(n)}] : w^{(n)} \in \Sigma_\beta^*, \sigma_\beta^{-1}A \subset \bigcup_n [w^{(n)}]\Big\} = 0,$$
there exists $\{w^{(n)}\} \subset \Sigma_\beta^*$ such that
$$\sigma_\beta^{-1}A \subset \bigcup_n [w^{(n)}] \text{ and } \sum_n \mu_p[w^{(n)}] < \varepsilon.$$
Since ε can be small enough such that $\mu_p[0] = p$ and $\mu_p[1] = 1-p > \varepsilon$, we can assume $a_n := |w^{(n)}| \geq 2$ for any n without loss of generality. By the fact that $\sigma_\beta : \Sigma_\beta \to \Sigma_\beta$ is surjective, we get
$$A = \sigma_\beta(\sigma_\beta^{-1}A) \subset \sigma_\beta(\bigcup_n [w^{(n)}]) \subset \bigcup_n \sigma_\beta[w^{(n)}] = \bigcup_n \sigma_\beta[w_1^{(n)}w_2^{(n)}\cdots w_{a_n}^{(n)}] \subset \bigcup_n [w_2^{(n)}\cdots w_{a_n}^{(n)}].$$
Therefore
$$\begin{aligned}
\mu_p(A) &\leq \sum_n \mu_p[w_2^{(n)}\cdots w_{a_n}^{(n)}] \\
&\leq \frac{1}{\min\{p, 1-p\}} \sum_n \mu_p[w_1^{(n)}]\mu_p[w_2^{(n)}\cdots w_{a_n}^{(n)}] \\
&\leq \frac{1}{\min\{p, 1-p\}} \sum_n \mu_p[w^{(n)}] \\
&< \frac{\varepsilon}{\min\{p, 1-p\}}
\end{aligned}$$
for any $\varepsilon > 0$. This implies $\mu_p(A) = 0$.

② Now we prove $\sigma_\beta\mu_p \ll \mu_p$.
Let $B \in \mathcal{B}(\Sigma_\beta)$ with $\mu_p(B) = 0$. It suffices to prove $\sigma_\beta\mu_p(B) = 0$. For any integer $m \geq 2$, we define $B_m := B \setminus [\varepsilon_2^* \cdots \varepsilon_m^*]$.

i) Prove that $\sigma_\beta\mu_p(B_m)$ increase to $\sigma_\beta\mu_p(B)$.
ⓐ If $\varepsilon(1,\beta)$ is finite, then $\varepsilon_2^*\varepsilon_3^*\varepsilon_4^*\cdots \notin \Sigma_\beta$, $[\varepsilon_2^* \cdots \varepsilon_m^*]$ decrease to $\emptyset$, B_m increase to B and $\sigma_\beta\mu_p(B_m)$ increase to $\sigma_\beta\mu_p(B)$.
ⓑ If $\varepsilon(1,\beta)$ is infinite, then $\varepsilon_2^*\varepsilon_3^*\varepsilon_4^*\cdots = \varepsilon_2\varepsilon_3\varepsilon_4\cdots = \varepsilon(T_\beta 1,\beta) \in \Sigma_\beta$, $[\varepsilon_2^* \cdots \varepsilon_m^*]$ decrease to $\{\varepsilon_2^*\varepsilon_3^*\varepsilon_4^*\cdots\}$ (a single point set), B_m increase to $(B \setminus \{\varepsilon_2^*\varepsilon_3^*\varepsilon_4^*\cdots\})$ and $\sigma_\beta\mu_p(B_m)$ increase to $\sigma_\beta\mu_p(B\setminus\{\varepsilon_2^*\varepsilon_3^*\varepsilon_4^*\cdots\})$. Since $\sigma_\beta\mu_p$ has no atom (by Proposition 1.2.25), we get $\sigma_\beta\mu_p(B_m)$ increase to $\sigma_\beta\mu_p(B)$.

ii) In order to get $\sigma_\beta\mu_p(B) = 0$, by i) it suffices to prove that for any integer $m \geq 2$, $\sigma_\beta\mu_p(B_m) = 0$.
Fix an integer $m \geq 2$. By $\mu_p(B_m) \leq \mu_p(B) = 0$, we get

$$\inf\Big\{\sum_n \mu_p[w^{(n)}] : w^{(n)} \in \Sigma_\beta^*, B_m \subset \bigcup_n [w^{(n)}]\Big\} = 0.$$

For any $\varepsilon > 0$, there exists $\{w^{(n)}\}_{n\in N'} \subset \Sigma_\beta^*$ with

$$B_m \subset \bigcup_{n\in N'} [w^{(n)}] \quad \text{such that} \quad \sum_{n\in N'} \mu_p[w^{(n)}] < \varepsilon$$

where N' is an index set with cardinality at most countable. Since ε can be small enough such that

$$\delta_m := \min\{\mu_p[w] : w \in \Sigma_\beta^*, |w| \leq m-1\} > \varepsilon,$$

we can assume $a_n := |w^{(n)}| \geq m$ for all $n \in N'$. Let

$$N := \{n \in N' : w^{(n)}|_{m-1} \neq \varepsilon_2^* \cdots \varepsilon_m^*\} \subset N'.$$

By the facts that for any $n \in N$, $[w^{(n)}] \cap [\varepsilon_2^* \cdots \varepsilon_m^*] = \emptyset$ and for any $n \in N' \setminus N$, $[w^{(n)}] \subset [\varepsilon_2^* \cdots \varepsilon_m^*]$, we get

$$\begin{aligned} B_m &= B_m \setminus [\varepsilon_2^* \cdots \varepsilon_m^*] \subset \bigcup_{n\in N'} \big([w^{(n)}] \setminus [\varepsilon_2^* \cdots \varepsilon_m^*]\big) \\ &= \Big(\bigcup_{n\in N} \big([w^{(n)}] \setminus [\varepsilon_2^* \cdots \varepsilon_m^*]\big)\Big) \bigcup \Big(\bigcup_{n\in N'\setminus N} \big([w^{(n)}] \setminus [\varepsilon_2^* \cdots \varepsilon_m^*]\big)\Big) = \bigcup_{n\in N} [w^{(n)}] \end{aligned}$$

and then $\sigma_\beta^{-1}B_m \subset \bigcup_{n\in N} \sigma_\beta^{-1}[w^{(n)}]$. Let

$$N_0 := \{n \in N : 1w^{(n)} \notin \Sigma_\beta^*\} \text{ and } N_1 := \{n \in N : 1w^{(n)} \in \Sigma_\beta^*\}.$$

Then for any $n \in N_0$, $\sigma_\beta^{-1}[w^{(n)}] = [0w^{(n)}]$ and for any $n \in N_1$, $\sigma_\beta^{-1}[w^{(n)}] = [0w^{(n)}] \cup [1w^{(n)}]$. Thus

$$\sigma_\beta^{-1} B_m \subset \Big(\bigcup_{n\in N} [0w^{(n)}] \Big) \bigcup \Big(\bigcup_{n\in N_1} [1w^{(n)}] \Big)$$

and

$$\mu_p(\sigma_\beta^{-1} B_m) \le \sum_{n\in N} \mu_p[0w^{(n)}] + \sum_{n\in N_1} \mu_p[1w^{(n)}] =: R_1 + R_2$$

where by Lemma 1.2.26 (2),

$$R_1 := \sum_{n\in N} p\mu_p[w^{(n)}] \le p \sum_{n\in N'} \mu_p[w^{(n)}] < p\varepsilon.$$

Now we estimate the upper bounded of R_2.
For each $n \in N_1 \subset N$, by $1w_1^{(n)} \cdots w_{m-1}^{(n)} \ne \varepsilon_1^* \varepsilon_2^* \cdots \varepsilon_m^*$, there exists $1 \le k_n \le m-1$ such that $1 = \varepsilon_1^*, w_1^{(n)} = \varepsilon_2^*, \cdots w_{k_n-1}^{(n)} = \varepsilon_{k_n}^*$ and $w_{k_n}^{(n)} < \varepsilon_{k_n+1}^*$. Since $\varepsilon_1^* \cdots \varepsilon_{k_n}^* \varepsilon_{k_n+1}^* \in \Sigma_\beta^*$, by Lemma 1.1.7 and Proposition 1.1.9 (2), we know that both $1w_1^{(n)} \cdots w_{k_n}^{(n)}$ and $w_1^{(n)} \cdots w_{k_n}^{(n)}$ are full. It follows from Lemma 1.2.26 (2) that

$$\mu_p[1w^{(n)}] = \mu_p[1w_1^{(n)} \cdots w_{k_n}^{(n)}] \mu_p[w_{k_n+1}^{(n)} \cdots w_{a_n}^{(n)}]$$

and

$$\mu_p[w^{(n)}] = \mu_p[w_1^{(n)} \cdots w_{k_n}^{(n)}] \mu_p[w_{k_n+1}^{(n)} \cdots w_{a_n}^{(n)}].$$

Let

$$C_m := \max \Big\{ \frac{\mu_p[1w]}{\mu_p[w]} : w \in \Sigma_\beta^* \text{ with } 1w \in \Sigma_\beta^* \text{ and } 1 \le |w| \le m-1 \Big\} < \infty.$$

By $k_n \le m-1$, we get $\mu_p[1w^{(n)}] \le C_m \mu_p[w^{(n)}]$ for any $n \in N_1$. This implies

$$R_2 := \sum_{n\in N_1} \mu_p[1w^{(n)}] \le C_m \sum_{n\in N_1} \mu_p[w^{(n)}] \le C_m \sum_{n\in N'} \mu_p[w^{(n)}] < C_m \varepsilon.$$

Therefore $\mu_p(\sigma_\beta^{-1} B_m) < (p + C_m)\varepsilon$ for any $0 < \varepsilon < \delta_m$. We conclude that $\sigma_\beta \mu_p(B_m) = 0$.

(2) $\boxed{\Rightarrow}$ follows from Lemma 1.2.26.
$\boxed{\Leftarrow}$ (By contradiction) Assume that $\varepsilon(1, \beta) = \varepsilon_1 \varepsilon_2 \varepsilon_3 \cdots$ is infinite. By $\varepsilon_2 \varepsilon_3 \cdots = \varepsilon(T_\beta 1, \beta) \in \Sigma_\beta$ and Lemma 1.2.22 (2), we get $N_0(\varepsilon_2 \varepsilon_3 \cdots) = +\infty$. Then for any $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $N_0(\varepsilon_2 \varepsilon_3 \cdots \varepsilon_n) \ge N$. Let $w := \varepsilon_1 = 1$ and $w' := \varepsilon_2 \varepsilon_3 \cdots \varepsilon_n$. Then $ww' = \varepsilon_1 \cdots \varepsilon_n$ and obviously

$$N_0(ww') = 0 = 0 + N - N \le N_0(w) + N_0(w') - N.$$

By Remark 1.2.24 (1) and $N_1(ww') = N_1(w) + N_1(w')$, we get

$$\mu_p[ww'] \geq p^{-N}\mu_p[w]\mu_p[w'].$$

Since for any $N \in \mathbb{N}$, there exists w, w' which satisfy the above inequality and p^{-N} can be arbitrarily large, we know that μ_p is not quasi-Bernoulli.

(3) $\boxed{\Leftarrow}$ (By contradiction) Assume that $\varepsilon(1,\beta) = \varepsilon_1\varepsilon_2\varepsilon_3\cdots$ is infinite. By $\varepsilon_2\varepsilon_3\cdots = \varepsilon(T_\beta 1, \beta) \in \Sigma_\beta$ and Lemma 1.2.22 (2), we get $N_0(\varepsilon_2\varepsilon_3\cdots) = +\infty$. Then for any $N \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $N_0(\varepsilon_2\varepsilon_3\cdots\varepsilon_n) \geq N$. Let $w := \varepsilon_2\cdots\varepsilon_n$. Then

$$\sigma_\beta\mu_p[w] = \mu_p[0w] + \mu_p[1w] \geq \mu_p[\varepsilon_1\varepsilon_2\cdots\varepsilon_n] = p^{N_0(\varepsilon_1\cdots\varepsilon_n)}(1-p)^{N_1(\varepsilon_1\cdots\varepsilon_n)} = (1-p)^{N_1(\varepsilon_1\cdots\varepsilon_n)}$$

and

$$\mu_p[w] = p^{N_0(w)}(1-p)^{N_1(w)} \leq p^N(1-p)^{N_1(\varepsilon_1\cdots\varepsilon_n)-1}.$$

Thus

$$\sigma_\beta\mu_p[w] \geq (1-p)p^{-N}\mu_p[w].$$

Since for any $N \in \mathbb{N}$, there exists w which satisfies the above inequality and $(1-p)p^{-N}$ can be arbitrarily large, we know that μ_p is not strongly quasi-invariant.
$\boxed{\Rightarrow}$ Let $\varepsilon(1,\beta) = \varepsilon_1\cdots\varepsilon_M 0^\infty$ with $\varepsilon_M \neq 0$ and $c = p^{-M} > 0$.
① Prove $c^{-1}\mu_p[w] \leq \sigma_\beta^k\mu_p[w] \leq c\mu_p[w]$ for all $k \in \mathbb{N}$ and $w \in \Sigma_\beta^*$.
Notice that

$$\sigma_\beta^{-k}[w] = \bigcup_{u_1\cdots u_k w \in \Sigma_\beta^*} [u_1\cdots u_k w]$$

is a disjoint union.

i) Estimate the upper bound of $\sigma_\beta^k\mu_p[w]$:

$$\begin{aligned}
\mu_p\sigma_\beta^{-k}[w] &= \sum_{u_1\cdots u_k w \in \Sigma_\beta^*} \mu_p[u_1\cdots u_k w] \\
&\overset{ⓐ}{\leq} \sum_{u_1\cdots u_k w \in \Sigma_\beta^*} p^{-M}\mu_p[u_1\cdots u_k]\mu_p[w] \\
&\leq p^{-M} \sum_{u_1\cdots u_k \in \Sigma_\beta^*} \mu_p[u_1\cdots u_k]\mu_p[w] \\
&= p^{-M}\mu_p[w].
\end{aligned}$$

where ⓐ follows from Lemma 1.2.26.

ii) Estimate the lower bound of $\sigma_\beta^k \mu_p[w]$:

$$\mu_p \sigma_\beta^{-k}[w] = \sum_{u_1\cdots u_k w \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_k w] \geq \sum_{u_1\cdots u_{k-M} 0^M w \in \Sigma_\beta^*} \mu_p[u_1 \cdots_{k-M} 0^M w].$$

(Without loss of generality, we assume $k \geq M$. Otherwise, we consider $0^k w$ instead of $u_1 \cdots u_{k-M} 0^M w$). By Proposition 1.1.16, $u_1 \cdots u_{k-m} 0^M$ is full for any $u_1 \cdots u_{k-m} \in \Sigma_\beta^*$. Then by Proposition 1.1.8 (4), we get

$$u_1 \cdots u_{k-M} 0^M w \in \Sigma_\beta^* \Longleftrightarrow u_1 \cdots u_{k-M} \in \Sigma_\beta^*.$$

Therefore

$$\begin{aligned}
\mu_p \sigma_\beta^{-k}[w] &\geq \sum_{u_1\cdots u_{k-M} \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_{k-M} 0^M w] \\
&\overset{ⓑ}{=} \sum_{u_1\cdots u_{k-M} \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_{k-M} 0^M] \mu_p[w] \\
&\overset{ⓒ}{\geq} \sum_{u_1\cdots u_{k-M} \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_{k-M}] p^M \mu_p[w] \\
&= p^M \mu_p[w]
\end{aligned}$$

where ⓑ and ⓒ follow from Lemma 1.2.26 (2) and (1) respectively.

② Prove $c^{-1}\mu_p(B) \leq \sigma_\beta^k \mu_p(B) \leq c\mu_p(B)$ for all $k \in \mathbb{N}$ and $B \in \mathcal{B}(\Sigma_\beta)$.
Let $\mathcal{C} := \{[w] : w \in \Sigma_\beta^*\} \cup \{\emptyset\}$, $\mathcal{C}_{\Sigma f} := \{\bigcup_{i=1}^n C_i : C_1, \cdots, C_n \in \mathcal{C}$ are disjoint, $n \in \mathbb{N}\}$ and

$$\mathcal{G} := \{B \in \mathcal{B}(\Sigma_\beta) : c^{-1}\mu_p(B) \leq \sigma_\beta^k \mu_p(B) \leq c\mu_p(B) \text{ for all } k \in \mathbb{N}\}.$$

Then $\mathcal{C}$ is a semi-algebra, $\mathcal{C}_{\Sigma f}$ is the algebra generated by $\mathcal{C}$ (by Lemma 1.2.13 (1)) and $\mathcal{G}$ is a monotone class. Since in ① we have already showed $\mathcal{C} \subset \mathcal{G}$, it is obvious that $\mathcal{C}_{\Sigma f} \subset \mathcal{G} \subset \mathcal{B}(\Sigma_\beta)$. By monotone class theorem (Theorem 1.2.12), we get $\mathcal{G} = \mathcal{B}(\Sigma_\beta)$. □

To prove our second main result Theorem 1.2.5, we need the following lemmas.

Lemma 1.2.27 ([58]). *Let $(X, \mathcal{B}, \mu)$ be a probability space and T be a measurable transformation on X. If there exists a constant M such that for any $E \in \mathcal{B}$ and any $n \geq 1$,*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(T^{-k}E) \leq M\mu(E),$$

then for any real integrable function f on X, the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x)$$

exists for μ-almost every $x \in X$.

Lemma 1.2.28. *Let $\beta \in (1,2]$ and $p \in (0,1)$.*

(1) *If $B \in \mathcal{B}(\Sigma_\beta)$ with $\sigma_\beta^{-1}B = B$, then $\mu_p(B) = 0$ or 1.*

(2) *If $B \in \mathcal{B}[0,1)$ with $T_\beta^{-1}B = B$, then $\nu_p(B) = 0$ or 1.*

Proof.

(1) Let $\mathcal{F} := \{w \in \Sigma_\beta^* : w \text{ is full}\}$.

① Let $w \in \mathcal{F}$ with $|w| = n$. We prove $\mu_p([w] \cap \sigma_\beta^{-n}A) = \mu_p[w]\mu_p(A)$ for any $A \in \mathcal{B}(\Sigma_\beta)$ as follows.
Since w is full and $[ww'] = [w] \cap \sigma_\beta^{-n}[w']$ for any $w' \in \Sigma_\beta^*$, we get

$$\mu_p([w] \cap \sigma_\beta^{-n}[w']) = \mu_p[ww'] \underset{\text{Lemma 1.2.26 (2)}}{\overset{\text{by}}{=\!=\!=}} \mu_p[w]\mu_p[w'].$$

Let $\mathcal{C} := \{[w'] : w' \in \Sigma_\beta^*\} \cup \{\emptyset\}$ and $\mathcal{G} := \{A \in \mathcal{B}(\Sigma_\beta) : \mu_p([w] \cap \sigma_\beta^{-n}A) = \mu_p[w]\mu_p(A)\}$. In the same way as the end of the *Proof of Theorem 1.2.3*, we get $\mathcal{G} = \mathcal{B}(\Sigma_\beta)$.

② We use B^c to denote the complement of B in Σ_β. For any $\delta > 0$, by Lemma 1.2.13 and Lemma 1.2.10, there exists a countable disjoint union of full cylinders $E_\delta = \bigcup_i [w^{(i)}]$ with $\{w^{(i)}\} \subset \mathcal{F}$ such that $\mu_p(B^c \triangle E_\delta) < \delta$.

③ Let $B \in \mathcal{B}(\Sigma_\beta)$ with $\sigma_\beta^{-1}B = B$. Then $B = \sigma_\beta^{-n}B$ and by ① we get

$$\mu_p(B \cap [w]) = \mu_p(\sigma_\beta^{-n}B \cap [w]) = \mu_p(B)\mu_p[w]$$

for any $w \in \mathcal{F}$ where $n = |w|$. Thus

$$\mu_p(B \cap E_\delta) = \mu_p(B \cap \bigcup_i [w^{(i)}]) = \sum_i \mu_p(B \cap [w^{(i)}]) = \sum_i \mu_p(B)\mu_p[w^{(i)}] = \mu_p(B)\mu_p(E_\delta).$$

Let $a = \mu_p((B \cup E_\delta)^c)$, $b = \mu_p(B \cap E_\delta)$, $c = \mu_p(B \setminus E_\delta)$ and $d = \mu_p(E_\delta \setminus B)$. Then

$$b = (b+c)(b+d), \quad a + b < \delta \text{ (by ②)} \quad \text{and} \quad a + b + c + d = 1.$$

By

$$(b+c)(a+d-\delta) \le (b+c)(b+d) = b < \delta,$$

we get

$$(b+c)(a+d) < (1+b+c)\delta \leq 2\delta$$

which implies $\mu_p(B)\mu_p(B^c) \leq 2\delta$ for any $\delta > 0$. Therefore $\mu_p(B) = 0$ or $\mu_p(B^c) = 0$.

(2) follows from (1). In fact, let $B \in \mathcal{B}[0,1)$ with $T_\beta^{-1}B = B$. By $\sigma_\beta^{-1}\pi_\beta^{-1}B = \pi_\beta^{-1}T_\beta^{-1}B = \pi_\beta^{-1}B \in \mathcal{B}(\Sigma_\beta)$ and (1), we get $\mu_p(\pi_\beta^{-1}B) = 0$ or 1, i.e., $\nu_p(B) = 0$ or 1. □

Proof of Theorem 1.2.5. (1) For any $n \in \mathbb{N}$ and $B \in \mathcal{B}[0,1)$, define

$$m_p^n(B) := \frac{1}{n}\sum_{k=0}^{n-1} \nu_p(T_\beta^{-k}B).$$

Then m_p^n is a probability measure on $([0,1), \mathcal{B}[0,1))$. By Corollary 1.2.4, there exists $c > 0$ such that

$$c^{-1}\nu_p(B) \leq m_p^n(B) \leq c\nu_p(B) \quad \text{for any } B \in \mathcal{B}[0,1) \text{ and } n \in \mathbb{N}. \tag{1.10}$$

(2) For any $B \in \mathcal{B}[0,1)$, prove that $\lim_{n\to\infty} m_p^n(B)$ exists. In fact,

$$\begin{aligned}
\lim_{n\to\infty} m_p^n(B) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \int \mathbb{1}_{T_\beta^{-k}B} d\nu_p \\
&= \lim_{n\to\infty} \int \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_B(T_\beta^k x) d\nu_p(x) \\
&= \int \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_B(T_\beta^k x) d\nu_p(x),
\end{aligned}$$

noting that the last equality follows from the dominated convergence theorem where the ν_p-a.e. existence of $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_B(T_\beta^k x)$ follows from Lemma 1.2.27 and (1.10).

(3) For any $B \in \mathcal{B}[0,1)$, define $m_p(B) := \lim_{n\to\infty} m_p^n(B)$. Then by Theorem 1.2.16, m_p is a probability measure on $([0,1), \mathcal{B}[0,1))$.

(4) $m_p \sim \nu_p$ on $\mathcal{B}[0,1)$ follows from (1.10) and the definition of m_p.

(5) Prove that m_p is T_β-invariant.
For any $B \in \mathcal{B}[0,1)$ and $n \in \mathbb{N}$, we have

$$m_p^n(T_\beta^{-1}B) = \frac{1}{n}\sum_{k=1}^{n} \nu_p(T_\beta^{-k}B) = \frac{n+1}{n} m_p^{n+1}(B) - \frac{\nu_p(B)}{n}.$$

As $n \to \infty$, we get $m_p(T_\beta^{-1}B) = m_p(B)$.

(6) Prove that $([0,1), \mathcal{B}[0,1), m_p, T_\beta)$ is ergodic.
Let $B \in \mathcal{B}[0,1)$ such that $T_\beta^{-1}B = B$. Then by Lemma 1.2.28 (2), we get $\nu_p(B) = 0$ or

$\nu_p(B^c) = 0$ which implies $m_p(B) = 0$ or $m_p(B^c) = 0$ since $m_p \sim \nu_p$. Noting that m_p is T_β-invariant, we know that m_p is ergodic with respect to T_β.

(7) Prove that such m_p is unique on $\mathcal{B}[0,1)$.
Let m'_p be a T_β-ergodic probability measure on $([0,1), \mathcal{B}[0,1))$ equivalent to ν_p. Then for any $B \in \mathcal{B}[0,1)$, by Birkhoff's ergodic theorem, we get

$$m_p(B) = \int \mathbb{1}_B dm_p = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(T_\beta^k x) \quad \text{for } m_p\text{-a.e. } x \in [0,1)$$

and

$$m'_p(B) = \int \mathbb{1}_B dm'_p = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(T_\beta^k x) \quad \text{for } m'_p\text{-a.e. } x \in [0,1).$$

Since $m_p \sim \nu_p \sim m'_p$, there exists $x \in [0,1)$ such that $m_p(B) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(T_\beta^k x) = m'_p(B)$. Thus $m'_p = m_p$. □

1.2.4 Modified lower local dimension related to beta-expansions

Let ν be a finite Borel measure on $\mathbb{R}^n$. The *lower local dimension* of ν at $x \in \mathbb{R}^n$ is defined by

$$\underline{\dim}_{loc}\nu(x) := \underline{\lim}_{r\to 0} \frac{\log \nu(B(x,r))}{\log r},$$

where $B(x,r)$ is the closed ball centered at x with radius r. Theoretically, we can use the lower local dimension to estimate the upper and lower bounds of the Hausdorff dimension by the following proposition.

Proposition 1.2.29. *([63, Proposition 2.3]) Let $s \geq 0$, $E \subset \mathbb{R}^n$ be a Borel set and ν be a finite Borel measure on $\mathbb{R}^n$.*

(1) *If $\underline{\dim}_{loc}\nu(x) \leq s$ for all $x \in E$ then $\dim_H E \leq s$.*

(2) *If $\underline{\dim}_{loc}\nu(x) \geq s$ for all $x \in E$ and $\nu(E) > 0$ then $\dim_H E \geq s$.*

But in the definition of the lower local dimension, the Bernoulli-type measure of a ball $\nu_p(B(x,r))$ is difficult to estimate. Therefore, we use the measure of a cylinder $\nu(I_n(x))$ instead of $\nu(B(x,r))$ to define the *modified lower local dimension related to β-expansions* of a measure at a point.

Definition 1.2.30. *Let $\beta > 1$ and ν be a finite Borel measure on $[0,1)$. The modified lower local dimension of ν at $x \in [0,1)$ is defined by*

$$\underline{\dim}^{\beta}_{loc}\nu(x) := \underline{\lim}_{n\to\infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|}$$

where $I_n(x)$ is the cylinder of order n containing x.

Combining Proposition 1.2.29 (1) and the following proposition, we can estimate the upper bound of the Hausdorff dimension by the modified lower local dimension.

Proposition 1.2.31. *Let $\beta > 1$ and ν be a finite Borel measure on $[0,1)$. Then for any $x \in [0,1)$, we have*

$$\underline{\dim}^{\beta}_{loc}(\nu, x) \geq \underline{\dim}_{loc}(\nu, x).$$

Proof. For any $x \in [0,1)$ and $n \in \mathbb{N}$. Let $r_n := |I_n(x)|$, then $I_n(x) \subset B(x, r_n)$, $\nu(I_n(x)) \leq \nu(B(x, r_n))$ and $-\log \nu(I_n(x)) \geq -\log \nu(B(x, r_n))$. We get

$$\frac{-\log \nu(I_n(x))}{-\log |I_n(x)|} \geq \frac{-\log \nu(B(x, r_n))}{-\log r_n}.$$

Therefore

$$\varliminf_{n\to\infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|} \geq \varliminf_{n\to\infty} \frac{\log \nu(B(x, r_n))}{\log r_n} \geq \underline{\dim}_{loc}\nu(x).$$

□

Remark 1.2.32. *The reverse inequality in Proposition 1.2.31, i.e., $\underline{\dim}^{\beta}_{loc}(\nu, x) \leq \underline{\dim}_{loc}(\nu, x)$ is not always true. For example, let β be the golden ratio $(\sqrt{5}+1)/2$, $x = \beta^{-1}$ and $\nu = \nu_p$ be the $(p, 1-p)$ Bernoulli-type measure with $0 < p < 1/2$. For any $n \in \mathbb{N}$, let $r_n = |I_n(x)|$ and J_n be the left consecutive cylinder of $I_n(x)$ with the same order n. When $n \geq 2$, we have $r_n = \beta^{-n} \geq |J_n|$ and $B(x, r_n) \supset J_n$. Then $\nu_p(B(x, r_n)) \geq \nu_p(J_n) \geq p(1-p)^{n-1}$ and $\nu_p(I_n(x)) = (1-p)p^{n-2}$ which implies*

$$\underline{\dim}^{\beta}_{loc}\nu_p(x) = \varliminf_{n\to\infty} \frac{\log(1-p)p^{n-2}}{\log \beta^{-n}} = \frac{-\log p}{\log \beta}$$

and

$$\underline{\dim}_{loc}\nu_p(x) \leq \varliminf_{n\to\infty} \frac{\log \nu_p(B(x, r_n))}{\log r_n} \leq \varliminf_{n\to\infty} \frac{\log p(1-p)^{n-1}}{\log \beta^{-n}} = \frac{-\log(1-p)}{\log \beta}.$$

When $0 < p < 1/2$, we have $\underline{\dim}^{\beta}_{loc}(\nu_p, x) > \underline{\dim}_{loc}(\nu_p, x)$.

Although the reverse inequality in Proposition 1.2.31 is not always true, we are going to establish the following theorem for estimating both of the upper and lower bounds of the Hausdorff dimension by the modified lower local dimension.

Theorem 1.2.33. *Let $\beta > 1$, $s \geq 0$, $E \subset [0,1)$ be a Borel set and ν be a finite Borel measure on $[0,1)$.*

(1) *If $\underline{\dim}^{\beta}_{loc}\nu(x) \leq s$ for all $x \in E$, then $\dim_H E \leq s$.*

(2) *If $\underline{\dim}^{\beta}_{loc}\nu(x) \geq s$ for all $x \in E$ and $\nu(E) > 0$, then $\dim_H E \geq s$.*

Proof. (1) follows from Proposition 1.2.29 (1) and Proposition 1.2.31.
(2) follows from the following Lemma 1.2.35. In fact, if $s = 0$, $\dim_H E \geq s$ is obvious. If $s > 0$, let $0 < t < s$. For any $x \in E$, by $\underline{\lim}_{n\to\infty} \frac{\log \nu(I_n(x))}{\log |I_n(x)|} \geq s > t$, there exists $N \in \mathbb{N}$ such that any $n > N$ implies $\frac{\log \nu(I_n(x))}{\log |I_n(x)|} > t$ and $\nu(I_n(x)) < |I_n(x)|^t$. So $\overline{\lim}_{n\to\infty} \frac{\nu(I_n(x))}{|I_n(x)|^t} \leq 1 < 2$. For any $0 < \varepsilon < t$, by Lemma 1.2.35, we get $\mathcal{H}^{t-\varepsilon}(E) > 0$ (where $\mathcal{H}^s(E)$ denotes the s-dimensional Hausdorff measure of the set E.) and then $\dim_H E \geq t - \varepsilon$. So $\dim_H E \geq t$ for any $t < s$. Therefore $\dim_H E \geq s$. □

Remark 1.2.34. *The statement (2) in Theorem 1.2.33 obviously implies the Proposition 1.3 in [37] which is called the modified mass distribution principle.*

Recall that we use $\mathcal{H}^s(E)$ to denotes the s-dimensional Hausdorff measure of the set E.

Lemma 1.2.35. *Let $\beta > 1, s > 0, c > 0$, $E \subset [0, 1)$ be a Borel set and ν be a finite Borel measure on $[0, 1)$. If $\overline{\lim}_{n\to\infty} \frac{\nu(I_n(x))}{|I_n(x)|^s} < c$ for all $x \in E$, then for any $\varepsilon \in (0, s)$, we have $\mathcal{H}^{s-\varepsilon}(E) \geq c^{-1}\nu(E)$.*

This lemma is a combination of the next two. First we introduce the following concept. Let $\beta > 1$, $s \geq 0$ and $E \subset [0, 1)$. For any $\delta > 0$, we define

$$\mathcal{H}^{s,\beta}_\delta(E) := \inf\Big\{ \sum_k |J_k|^s : |J_k| \leq \delta, E \subset \bigcup_k J_k, \{J_k\} \text{ are countable cylinders}\Big\}.$$

It is increasing as $\delta \searrow 0$. We call $\mathcal{H}^{s,\beta}(E) := \lim_{\delta\to 0} \mathcal{H}^{s,\beta}_\delta(E)$ the s-dimensional Hausdorff measure of E related to the cylinder net of β.

Lemma 1.2.36. *Let $\beta > 1$, $s > 0$ and $E \subset [0, 1)$. Then for any $\varepsilon \in (0, s)$ we have $\mathcal{H}^{s,\beta}(E) \leq \mathcal{H}^{s-\varepsilon}(E)$.*

Proof. Fix $0 < \varepsilon < s$.

(1) Choose $\delta_0 > 0$ small enough as below.
Since $\beta^{(n+1)\varepsilon} \to \infty$ much faster than $8\beta^s n \to \infty$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $8\beta^s n \leq \beta^{(n+1)\varepsilon}$. By $\frac{-\log \delta}{\log \beta} - 1 \to \infty$ as $\delta \to 0^+$, there exists $\delta_0 > 0$ small enough such that $\frac{-\log \delta_0}{\log \beta} - 1 > n_0$. Then for any $n > \frac{-\log \delta_0}{\log \beta} - 1$, we will have $8\beta^s n \leq \beta^{(n+1)\varepsilon}$.

(2) Fix $\delta \in (0, \delta_0)$. Let $\{U_i\}$ be a δ-cover of E, i.e., $0 < |U_i| \leq \delta$ and $E \subset \cup_i U_i$. Then for each U_i, there exists $n_i \in \mathbb{N}$ such that $\beta^{-n_i-1} < |U_i| \leq \beta^{-n_i}$. By Proposition 1.2.9, U_i can be covered by at most $8n_i$ cylinders $I_{i,1}, I_{i,2}, \cdots, I_{i,8n_i}$ of order n_i. Noting that

$$|I_{i,j}| \leq \beta^{-n_i} < \beta|U_i| \leq \beta\delta \text{ and } E \subset \bigcup_i \bigcup_{j=1}^{8n_i} I_{i,j},$$

we get

$$\mathcal{H}^{s,\beta}_{\beta\delta}(E) \le \sum_i \sum_{j=1}^{8n_i} |I_{i,j}|^s \le \sum_i \frac{8n_i}{\beta^{n_i s}} \overset{(\star)}{\le} \sum_i \frac{1}{\beta^{(n_i+1)(s-\varepsilon)}} < \sum_i |U_i|^{s-\varepsilon}. \tag{1.11}$$

where $(\star)$ is because $\frac{1}{\beta^{n_i+1}} < |U_i| < \delta_0$ implies $n_i > \frac{-\log \delta_0}{\log \beta} - 1$ and then $8n_i\beta^s \le \beta^{(n_i+1)\varepsilon}$ by (1). Taking inf on the right of (1.11), we get $\mathcal{H}^{s,\beta}_{\beta\delta}(E) \le \mathcal{H}^{s-\varepsilon}_{\delta}(E)$. It follows from letting $\delta \to 0$ that $\mathcal{H}^{s,\beta}(E) \le \mathcal{H}^{s-\varepsilon}(E)$. □

Lemma 1.2.37. *Let $\beta > 1, s \ge 0, c > 0$, $E \subset [0,1)$ be a Borel set and ν be a finite Borel measure on $[0,1)$. If $\overline{\lim}_{n\to\infty} \frac{\nu(I_n(x))}{|I_n(x)|^s} < c$ for all $x \in E$, then $\mathcal{H}^{s,\beta}(E) \ge c^{-1}\nu(E)$.*

Proof. For any $\delta > 0$, let $E_\delta := \{x \in E : |I_n(x)| < \delta \text{ implies } \nu(I_n(x)) < c|I_n(x)|^s\}$.
(1) Prove that when $\delta \searrow 0$, $E_\delta \nearrow E$ as below.
① If $0 < \delta_2 < \delta_1$, then obviously $E_{\delta_2} \supset E_{\delta_1}$.
② It suffices to prove $E = \bigcup_{\delta>0} E_\delta$.
$\boxed{\supset}$ follows from $E \supset E_\delta$ for all $\delta > 0$.
$\boxed{\subset}$ Let $x \in E$. By $\overline{\lim}_{n\to\infty} \frac{\nu(I_n(x))}{|I_n(x)|^s} < c$, there exists $N_x \in \mathbb{N}$ such that any $n > N_x$ will have $\nu(I_n(x)) < c|I_n(x)|^s$. Let $\delta_x = |I_{N_x}(x)|$, then $|I_n(x)| < \delta_x$ will imply $n > N_x$ and $\nu(I_n(x)) < c|I_n(x)|^s$. Therefore $x \in E_{\delta_x} \subset \bigcup_{\delta>0} E_\delta$.
(2) Fix $\delta > 0$. Let $\{J_k\}_{k\in K}$ be countable cylinders such that $|J_k| < \delta$ and $\bigcup_{k\in K} J_k \supset E \supset E_\delta$. Let $K' = \{k \in K : J_k \cap E_\delta \ne \emptyset\}$. For any $k \in K'$, there exists $x_k \in J_k \cup E_\delta$. By the definition of E_δ, we get $\nu(J_k) < c|J_k|^s$. So

$$\nu(E_\delta) \le \nu(\bigcup_{k\in K'} J_k) \le \sum_{k\in K'} \nu(J_k) < \sum_{k\in K'} c|J_k|^s \le c\sum_{k\in K} |J_k|^s.$$

Taking inf on the right, we get $\nu(E_\delta) \le c\mathcal{H}^{s,\beta}_{\delta}(E) \le c\mathcal{H}^{s,\beta}(E)$. Let $\delta \to 0$ on the left, by $E_\delta \nearrow E$, we conclude that $\nu(E) \le c\mathcal{H}^{s,\beta}(E)$. □

1.2.5 Hausdorff dimension of some frequency sets

We apply the Bernoulli-type measures and the modified lower local dimension related to β-expansions to give some results on the Hausdorff dimension of frequency sets and prove Theorem 1.2.6 in this subsection. First we prove the following.

Theorem 1.2.38 (Upper bound of the Hausdorff dimension of frequency sets)**.** *Let $\beta \in (1,2]$ and $a \in [0,1]$. Then*

$$\dim_H F_{\beta,a},\ \dim_H \underline{F}_{\beta,a},\ \dim_H \overline{F}_{\beta,a} \ \le\ \frac{-a\log a - (1-a)\log(1-a)}{\log\beta}.$$

In particular, $\dim_H F_{\beta,0} = \dim_H \underline{F}_{\beta,0} = \dim_H \overline{F}_{\beta,0} = \dim_H F_{\beta,1} = \dim_H \underline{F}_{\beta,1} = \dim_H \overline{F}_{\beta,1} = 0$.

Proof. We consider $0 < a < 1$ first.
For any $x \in [0,1)$ and $n \in \mathbb{N}$, it follows from $\nu_a(I_n(x)) = a^{N_0(x,n)}(1-a)^{N_1(x,n)}$ that

$$-\log \nu_a(I_n(x)) = N_0(x,n)(-\log a) + N_1(x,n)(-\log(1-a))$$

$$\leq (n - N_1(x,n))(-\log a) + N_1(x,n)(-\log(1-a)).$$

By $|I_n(x)| \leq \beta^{-n}$, we get

$$\frac{-\log \nu_a(I_n(x))}{-\log |I_n(x)|} \leq \frac{(1-\frac{N_1(x,n)}{n})(-\log a) + \frac{N_1(x,n)}{n}(-\log(1-a))}{\log \beta}. \tag{1.12}$$

(1) For any $x \in \underline{F}_{\beta,a}$, it follows from $\underline{\lim}_{n\to\infty}(1-\frac{N_1(x,n)}{n}) = a$ and $\overline{\lim}_{n\to\infty}\frac{N_1(x,n)}{n} = 1-a$ that

$$\begin{aligned}\underline{\lim}_{n\to\infty}\frac{\log \nu_a(I_n(x))}{\log |I_n(x)|} &\leq \frac{\underline{\lim}_{n\to\infty}(1-\frac{N_1(x,n)}{n})(-\log a) + \overline{\lim}_{n\to\infty}\frac{N_1(x,n)}{n}(-\log(1-a))}{\log \beta} \\ &= \frac{-a\log a - (1-a)\log(1-a)}{\log \beta}.\end{aligned}$$

By Theorem 1.2.33 (1), we get

$$\dim_H \underline{F}_{\beta,a} \leq \frac{-a\log a - (1-a)\log(1-a)}{\log \beta}.$$

(2) For any $x \in \overline{F}_{\beta,a}$, it follows from $\overline{\lim}_{n\to\infty}(1-\frac{N_1(x,n)}{n}) = a$ and $\underline{\lim}_{n\to\infty}\frac{N_1(x,n)}{n} = 1-a$ that

$$\begin{aligned}\underline{\lim}_{n\to\infty}\frac{\log \nu_a(I_n(x))}{\log |I_n(x)|} &\leq \frac{\overline{\lim}_{n\to\infty}(1-\frac{N_1(x,n)}{n})(-\log a) + \underline{\lim}_{n\to\infty}\frac{N_1(x,n)}{n}(-\log(1-a))}{\log \beta} \\ &= \frac{-a\log a - (1-a)\log(1-a)}{\log \beta}.\end{aligned}$$

By Theorem 1.2.33 (1), we get

$$\dim_H \overline{F}_{\beta,a} \leq \frac{-a\log a - (1-a)\log(1-a)}{\log \beta}.$$

Therefore, it follows from $F_{\beta,a} = \underline{F}_{\beta,a} \cap \overline{F}_{\beta,a}$ that

$$\dim_H F_{\beta,a},\ \dim_H \underline{F}_{\beta,a},\ \dim_H \overline{F}_{\beta,a} \ \leq\ \frac{-a\log a - (1-a)\log(1-a)}{\log \beta}.$$

Before proving $\dim_H F_{\beta,0} = \dim_H \underline{F}_{\beta,0} = \dim_H \overline{F}_{\beta,0} = \dim_H F_{\beta,1} = \dim_H \underline{F}_{\beta,1} = \dim_H \overline{F}_{\beta,1} = 0$, we establish the following.

Lemma 1.2.39. *Let $\beta \in (1,2]$ and $a \in (0,1)$.*

(1) *Let*
$$\underline{F}_{\beta,\leq a} := \Big\{x \in [0,1) : \varliminf_{n\to\infty} \frac{\#\{1 \leq k \leq n : \varepsilon_k(x,\beta) = 0\}}{n} \leq a\Big\}.$$
Then
$$\dim_H \underline{F}_{\beta,\leq a} \leq \frac{-a\log a - \log(1-a)}{\log\beta}.$$

(2) *Let*
$$\overline{F}_{\beta,\geq a} := \Big\{x \in [0,1) : \varlimsup_{n\to\infty} \frac{\#\{1 \leq k \leq n : \varepsilon_k(x,\beta) = 0\}}{n} \geq a\Big\}.$$
Then
$$\dim_H \overline{F}_{\beta,\geq a} \leq \frac{-\log a - (1-a)\log(1-a)}{\log\beta}.$$

Proof.

(1) For any $x \in \underline{F}_{\beta,\leq a}$, it follows from (1.12), $\varliminf_{n\to\infty}(1 - \frac{N_1(x,n)}{n}) \leq a$ and $\frac{N_1(x,n)}{n} \leq 1$ ($\forall n \in \mathbb{N}$) that
$$\varliminf_{n\to\infty} \frac{\log \nu_a(I_n(x))}{\log|I_n(x)|} \leq \frac{-a\log a - \log(1-a)}{\log\beta}.$$
By Theorem 1.2.33 (1), we get
$$\dim_H \underline{F}_{\beta,\leq a} \leq \frac{-a\log a - \log(1-a)}{\log\beta}.$$

(2) For any $x \in \overline{F}_{\beta,\geq a}$, it follows from (1.12), $\varliminf_{n\to\infty} \frac{N_1(x,n)}{n} \leq 1-a$ and $1 - \frac{N_1(x,n)}{n} \leq 1$ ($\forall n \in \mathbb{N}$) that
$$\varliminf_{n\to\infty} \frac{\log \nu_a(I_n(x))}{\log|I_n(x)|} \leq \frac{-\log a - (1-a)\log(1-a)}{\log\beta}.$$
By Theorem 1.2.33 (1), we get
$$\dim_H \overline{F}_{\beta,\geq a} \leq \frac{-\log a - (1-a)\log(1-a)}{\log\beta}.$$

□

Now we prove $\dim_H F_{\beta,0} = \dim_H \underline{F}_{\beta,0} = \dim_H \overline{F}_{\beta,0} = \dim_H F_{\beta,1} = \dim_H \underline{F}_{\beta,1} = \dim_H \overline{F}_{\beta,1} = 0$.
(1) For any $0 < a < 1$, $F_{\beta,0} = \overline{F}_{\beta,0} \subset \underline{F}_{\beta,0} \subset \underline{F}_{\beta,\leq a}$ implies $\dim_H F_{\beta,0} = \dim_H \overline{F}_{\beta,0} \leq \dim_H \underline{F}_{\beta,0} \leq \dim_H \underline{F}_{\beta,\leq a}$. Let $a \to 0$, by Lemma 1.2.39 (1), we get $\dim_H F_{\beta,0} = \dim_H \overline{F}_{\beta,0} = \dim_H \underline{F}_{\beta,0} = 0$.
(2) For any $0 < a < 1$, $F_{\beta,1} = \underline{F}_{\beta,1} \subset \overline{F}_{\beta,1} \subset \overline{F}_{\beta,\geq a}$ implies $\dim_H F_{\beta,1} = \dim_H \underline{F}_{\beta,1} \leq$

$\dim_H \overline{F}_{\beta,1} \le \dim_H \overline{F}_{\beta,\ge a}$. Let $a \to 0$, by Lemma 1.2.39 (2), we get $\dim_H F_{\beta,1} = \dim_H \underline{F}_{\beta,1} = \dim_H \overline{F}_{\beta,1} = 0$. □

Before proving Theorem 1.2.6, we state the following two lemmas, which will be proved at the end of this subsection.

Lemma 1.2.40. *Let $\beta \in (1,2)$ such that $\varepsilon(1,\beta) = 10^m 10^\infty$ for some integer $m \ge 0$. Then for any $x \in [0,1)$ and integer $n \ge m+2$, we have*

$$n \le N_0(x,n) + (m+2)N_1(x,n) \le n+m+1.$$

Lemma 1.2.41. *Let $\beta \in (1,2)$ such that $\varepsilon(1,\beta) = 10^m 10^\infty$ for some integer $m \ge 0$. Then for any $p \in (0,1)$, we have*

$$m_p[0, \frac{1}{\beta}) = \frac{m(1-p)+1}{(m+1)(1-p)+1},$$

where m_p is given by Theorem 1.2.5.

Proof of Theorem 1.2.6. (1) For any $x \in [0,1)$, by Lemma 1.1.3, each digit 1 in $\varepsilon(x,\beta)$ must be followed by at least $m+1$ consecutive 0's. Thus

$$\overline{\lim_{n\to\infty}} \frac{N_1(x,n)}{n} \le \frac{1}{m+2} \quad \text{and then} \quad \underline{\lim_{n\to\infty}} \frac{\#\{1 \le k \le n : \varepsilon_k(x,\beta) = 0\}}{n} \ge \frac{m+1}{m+2}$$

for any $x \in [0,1)$. If $0 \le a < \frac{m+1}{m+2}$, we get $F_{\beta,a} = \underline{F}_{\beta,a} = \overline{F}_{\beta,a} = \emptyset$.
(2) ① First, we consider $\frac{m+1}{m+2} < a < 1$.
For any $x \in [1,0)$ and $n \in \mathbb{N}$, by Proposition 1.2.8, we get

$$\frac{1}{n\log\beta - \log c} \le \frac{1}{-\log|I_n(x)|} \le \frac{1}{n\log\beta}.$$

Let $p := \frac{ma-m+2a-1}{ma-m+a}$. By $\frac{m+1}{m+2} < a < 1$ we get $0 < p < 1$. Let ν_p be the $(p, 1-p)$ Bernoulli-type measure on $[0,1)$. It follows from

$$-\log \nu_p(I_n(x)) = N_0(x,n)(-\log p) + N_1(x,n)(-\log(1-p))$$

that

$$\frac{\frac{N_0(x,n)}{n}(-\log p) + \frac{N_1(x,n)}{n}(-\log(1-p))}{\log\beta - \frac{\log c}{n}} \le \frac{\log \nu_p(I_n(x))}{\log|I_n(x)|} \le \frac{\frac{N_0(x,n)}{n}(-\log p) + \frac{N_1(x,n)}{n}(-\log(1-p))}{\log\beta}. \tag{1.13}$$

Taking $\underline{\lim}_{n\to\infty}$, we get

$$\underline{\dim}^{\beta}_{loc}\nu_p(x) = \underline{\lim_{n\to\infty}} \frac{\frac{N_0(x,n)}{n}(-\log p) + \frac{N_1(x,n)}{n}(-\log(1-p))}{\log\beta}.$$

i) Prove $\dim_H \underline{F}_{\beta,a} \leq \frac{(1-(m+2)(1-a))(-\log p)+(1-a)(-\log(1-p))}{\log\beta}$.
For any $x \in \underline{F}_{\beta,a}$, we have $\overline{\lim}_{n\to\infty} \frac{N_1(x,n)}{n} = 1-a$ and then by Lemma 1.2.40, $\underline{\lim}_{n\to\infty} \frac{N_0(x,n)}{n} = 1-(m+2)(1-a)$. Thus

$$\begin{aligned}\underline{\dim}^{\beta}_{loc}\nu_p(x) &\leq \frac{\underline{\lim}_{n\to\infty} \frac{N_0(x,n)}{n}(-\log p) + \overline{\lim}_{n\to\infty} \frac{N_1(x,n)}{n}(-\log(1-p))}{\log\beta} \\ &= \frac{(1-(m+2)(1-a))(-\log p) + (1-a)(-\log(1-p))}{\log\beta}.\end{aligned}$$

Then we apply Theorem 1.2.33 (1).

ii) Prove $\dim_H \overline{F}_{\beta,a} \leq \frac{(1-(m+2)(1-a))(-\log p)+(1-a)(-\log(1-p))}{\log\beta}$.
For any $x \in \overline{F}_{\beta,a}$, we have $\underline{\lim}_{n\to\infty} \frac{N_1(x,n)}{n} = 1-a$ and then by Lemma 1.2.40, $\overline{\lim}_{n\to\infty} \frac{N_0(x,n)}{n} = 1-(m+2)(1-a)$. Thus

$$\begin{aligned}\underline{\dim}^{\beta}_{loc}\nu_p(x) &\leq \frac{\overline{\lim}_{n\to\infty} \frac{N_0(x,n)}{n}(-\log p) + \underline{\lim}_{n\to\infty} \frac{N_1(x,n)}{n}(-\log(1-p))}{\log\beta} \\ &= \frac{(1-(m+2)(1-a))(-\log p) + (1-a)(-\log(1-p))}{\log\beta}.\end{aligned}$$

Then we apply Theorem 1.2.33 (1).

iii) Prove $\dim_H F_{\beta,a} \geq \frac{(1-(m+2)(1-a))(-\log p)+(1-a)(-\log(1-p))}{\log\beta}$.
For any $x \in F_{\beta,a}$, we have $\lim_{n\to\infty} \frac{N_1(x,n)}{n} = 1-a$ and then by Lemma 1.2.40, $\lim_{n\to\infty} \frac{N_0(x,n)}{n} = 1-(m+2)(1-a)$. Thus

$$\underline{\dim}^{\beta}_{loc}\nu_p(x) = \frac{(1-(m+2)(1-a))(-\log p) + (1-a)(-\log(1-p))}{\log\beta}.$$

By Theorem 1.2.33 (2), it suffices to prove $\nu_p(F_{\beta,a}) = 1 > 0$. Noting that

$$\varepsilon_k(x,\beta) = 0 \Leftrightarrow \lfloor \beta T_\beta^{k-1}x \rfloor = 0 \Leftrightarrow 0 \leq T_\beta^{k-1}x \leq \frac{1}{\beta} \Leftrightarrow \mathbb{1}_{[0,\frac{1}{\beta})}(T_\beta^{k-1}x) = 1,$$

we get

$$\frac{1}{n}\#\{1 \leq k \leq n : \varepsilon_k(x,\beta) = 0\} = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{[0,\frac{1}{\beta})}(T_\beta^{k-1}x).$$

Since $([0,1), \mathcal{B}[0,1), m_p, T_\beta)$ is ergodic and the indicator function $\mathbb{1}_{[0,\frac{1}{\beta})}$ is m_p-integrable, it follows from Birkhoff's ergodic theorem that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{[0,\frac{1}{\beta})}(T_\beta^{k-1}x) &= \int \mathbb{1}_{[0,\frac{1}{\beta})} dm_p = m_p[0,\frac{1}{\beta}) \\ &\xlongequal[\text{Lemma 1.2.41}]{\text{by}} \frac{m(1-p)+1}{(m+1)(1-p)+1} \xlongequal[\text{definition of } p]{\text{by the}} a\end{aligned}$$

for m_p-*a.e.* $x \in [0,1)$. Therefore $m_p(F_{\beta,a}) = 1$. By $m_p \sim \nu_p$, we get $\nu_p(F_{\beta,a}) = 1 > 0$.

Combining i), ii), iii) and $F_{\beta,a} = \underline{F}_{\beta,a} \cap \overline{F}_{\beta,a}$, we get

$$\dim_H F_{\beta,a} = \dim_H \underline{F}_{\beta,a} = \dim_H \overline{F}_{\beta,a} = \frac{(1-(m+2)(1-a))(-\log p) + (1-a)(-\log(1-p))}{\log \beta}.$$

We draw the conclusion by $p = \frac{ma-m+2a-1}{ma-m+a}$.
② For $a = 1$, by Theorem 1.2.38 we get $\dim_H F_{\beta,1} = \dim_H \underline{F}_{\beta,1} = \dim_H \overline{F}_{\beta,1} = 0$.
③ Prove $\dim_H F_{\beta,\frac{m+1}{m+2}} = \dim_H \underline{F}_{\beta,\frac{m+1}{m+2}} = \dim_H \overline{F}_{\beta,\frac{m+1}{m+2}} = 0$.
By $\underline{\lim}_{n\to\infty} \frac{\#\{1\le k\le n:\varepsilon_k(x,\beta)=0\}}{n} \ge \frac{m+1}{m+2}$ for any $x \in [0,1)$ in (1), we get $F_{\beta,\frac{m+1}{m+2}} = \overline{F}_{\beta,\frac{m+1}{m+2}}$.
Since $F_{\beta,\frac{m+1}{m+2}} \subset \underline{F}_{\beta,\frac{m+1}{m+2}}$, it suffices to prove $\dim \underline{F}_{\beta,\frac{m+1}{m+2}} = 0$.
For $\frac{m+1}{m+2} < a < 1$, let $p := \frac{ma-m+2a-1}{ma-m+a}$. Then $0 < p < 1$. For any $x \in \underline{F}_{\beta,\le a}$ (see Lemma 1.2.39 (1) for definition), we have $\overline{\lim}_{n\to\infty} \frac{N_1(x,n)}{n} \ge 1-a$ and then by Lemma 1.2.40, $\underline{\lim}_{n\to\infty} \frac{N_0(x,n)}{n} \le 1-(m+2)(1-a)$. It follows from $\frac{N_1(x,n)}{n} \le 1$ ($\forall n \in \mathbb{N}$) and (1.13) that

$$\underline{\lim}_{n\to\infty} \frac{\log \nu_p(I_n(x))}{\log |I_n(x)|} \le -\frac{(1-(m+2)(1-a))\log p + \log(1-p)}{\log \beta}$$

for any $x \in \underline{F}_{\beta,\le a}$. By Theorem 1.2.33 (1) and the definition of p, we get $\dim_H \underline{F}_{\beta,\le a} \le$

$$-\frac{(ma-m+2a-1)\log(ma-m+2a-1) - (ma-m+2a-1)\log(ma-m+a) + \log(1-p)}{\log \beta}.$$

For any $\frac{m+1}{m+2} < a < 1$, $\underline{F}_{\beta,\frac{m+1}{m+2}} \subset \underline{F}_{\beta,\le a}$ implies $\dim_H \underline{F}_{\beta,\frac{m+1}{m+2}} \le \dim_H \underline{F}_{\beta,\le a}$. Let $a \to \frac{m+1}{m+2}$, then $p \to 0$ and we get $\dim_H \underline{F}_{\beta,\frac{m+1}{m+2}} = 0$. □

Proof of Lemma 1.2.40. Let $w \in \Sigma_\beta^n$. It suffices to prove

$$n \overset{(1)}{\le} N_0(w) + (m+2)N_1(w) \overset{(2)}{\le} n+m+1.$$

(1) Let

$$\mathcal{N}_{10}(w) := \{2 \le k \le n : w_{k-1}w_k = 10\}, \quad \mathcal{N}_{100}(w) := \{3 \le k \le n : w_{k-2}w_{k-1}w_k = 100\},$$

$$\cdots, \quad \mathcal{N}_{10^{m+1}}(w) := \{m+2 \le k \le n : w_{k-m-1}\cdots w_k = 10^{m+1}\}$$

and let

$$N_{10}(w) := \#\mathcal{N}_{10}(w), \quad N_{100}(w) := \#\mathcal{N}_{100}(w), \quad \cdots, \quad N_{10^{m+1}}(w) := \#\mathcal{N}_{10^{m+1}}(w).$$

Noting that by Proposition 1.1.16, $u0^{m+1}$ is full for any $u \in \Sigma_\beta^*$ and then $u0^{m+1}1$ is

admissible, we get

$$\{1 \le k \le n : w_k = 0\} = (\mathcal{N}_0(w) + 1) \cup \mathcal{N}_{10}(w) \cup \mathcal{N}_{100}(w) \cup \cdots \cup \mathcal{N}_{10^{m+1}}$$

which is a disjoint union. Thus

$$\#\{1 \le k \le n : w_k = 0\} = N_0(w) + N_{10}(w) + N_{100}(w) + \cdots + N_{10^{m+1}}(w)$$

and then

$$n = N_0(w) + N_{10}(w) + N_{100}(w) + \cdots + N_{10^{m+1}}(w) + N_1(w).$$

By $N_{10}(w), N_{100}(w), \cdots, N_{10^{m+1}}(w) \le N_1(w)$, we get $n \le N_0(w) + (m+2)N_1(w)$.
(2) If $N_1(w) = 0$, the conclusion is obvious. If $N_1(w) \ge 1$, except for the last digit 1 in w, by Lemma 1.1.3, the other 1's must be followed by at least $m+1$ consecutive 0's, and non of these 0's can be replaced by 1 to get an admissible word. Therefore

$$N_1(w) + (m+1)(N_1(w) - 1) + N_0(w) \le n, \quad i.e., \quad N_0(w) + (m+2)N_1(w) \le n + m + 1.$$

□

Proof of Lemma 1.2.41. Notice that $m_p[0, \frac{1}{\beta}) = 1 - m_p[\frac{1}{\beta}, 1)$ where

$$m_p[\frac{1}{\beta}, 1) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \nu_p T_\beta^{-k}[\frac{1}{\beta}, 1) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu_p \sigma_\beta^{-k}[1]$$

by Theorem 1.2.5. For any integer $k \ge 0$, let

$$a_k := \mu_p \sigma_\beta^{-k}[1] = \sum_{u_1\cdots u_k 1 \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_k 1]$$

and

$$b_k := \mu_p \sigma_\beta^{-k}[0^{m+1}] = \sum_{u_1\cdots u_k 0^{m+1} \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_k 0^{m+1}].$$

By Theorem 1.2.5, the limits

$$a := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k \quad \text{and} \quad b := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} b_k$$

exist.

(1) Prove $a = (1-p)b$. In fact,

$$b_{k+1} = \sum_{u_1\cdots u_k 00^{m+1} \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_k 00^{m+1}] + \sum_{u_1\cdots u_k 10^{m+1} \in \Sigma_\beta^*} \mu_p[u_1 \cdots u_k 10^{m+1}]$$

$$= \sum_{u_1\cdots u_k 0^{m+1}\in\Sigma_\beta^*} \mu_p[u_1\cdots u_k 0^{m+1}0] + \sum_{u_1\cdots u_k 1\in\Sigma_\beta^*} \mu_p[u_1\cdots u_k 10^{m+1}].$$

On the one hand, by Proposition 1.1.16, $u_1\cdots u_k 0^{m+1}$ is full and then $u_1\cdots u_k 0^{m+1}1 \in \Sigma_\beta^*$. On the other hand, by Lemma 1.1.3, for any $0 \le s \le m$, $u_1\cdots u_k 10^s 10^{m-s} \notin \Sigma_\beta^*$ and then $[u_1\cdots u_k 10^{m+1}] = [u_1\cdots u_k 1]$. Thus, it follows from the definition of μ_p that

$$b_{k+1} = p \sum_{u_1\cdots u_k 0^{m+1}\in\Sigma_\beta^*} \mu_p[u_1\cdots u_k 0^{m+1}] + \sum_{u_1\cdots u_k 1\in\Sigma_\beta^*} \mu_p[u_1\cdots u_k 1] = pb_k + a_k.$$

Let $n \to \infty$ in

$$\frac{1}{n}\sum_{k=0}^{n-1} b_{k+1} = p\cdot\frac{1}{n}\sum_{k=0}^{n-1} b_k + \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$

We get $b = pb + a$.

(2) Prove $b + (m+1)a = 1$. It follows from

$$\begin{aligned}
& \Big(\bigcup_{u_1\cdots u_k 0^{m+1}\in\Sigma_\beta^*} [u_1\cdots u_k 0^{m+1}]\Big) \cup \Big(\bigcup_{u_1\cdots u_k 1\in\Sigma_\beta^*} [u_1\cdots u_k 1]\Big) \\
\cup\ & \Big(\bigcup_{u_1\cdots u_{k+1} 1\in\Sigma_\beta^*} [u_1\cdots u_{k+1} 1]\Big) \cup\cdots\cup \Big(\bigcup_{u_1\cdots u_{k+m} 1\in\Sigma_\beta^*} [u_1\cdots u_{k+m} 1]\Big) \\
=\ & \Big(\bigcup_{u_1\cdots u_k 0^{m+1}\in\Sigma_\beta^*} [u_1\cdots u_k 0^{m+1}]\Big) \cup \Big(\bigcup_{u_1\cdots u_k 10^{m}\in\Sigma_\beta^*} [u_1\cdots u_k 10^{m}]\Big) \\
\cup\ & \Big(\bigcup_{u_1\cdots u_{k+1} 10^{m-1}\in\Sigma_\beta^*} [u_1\cdots u_{k+1} 10^{m-1}]\Big) \cup\cdots\cup \Big(\bigcup_{u_1\cdots u_{k+m} 1\in\Sigma_\beta^*} [u_1\cdots u_{k+m} 1]\Big) \\
=\ & \Sigma_\beta
\end{aligned}$$

that $b_k + a_k + a_{k+1} + \cdots + a_{k+m} = 1$. Let $n \to \infty$ in

$$\frac{1}{n}\sum_{k=0}^{n-1} b_k + \frac{1}{n}\sum_{k=0}^{n-1} a_k + \frac{1}{n}\sum_{k=0}^{n-1} a_{k+1} + \cdots + \frac{1}{n}\sum_{k=0}^{n-1} a_{k+m} = 1.$$

We get $b + a + a + \cdots + a = 1$.

(3) It follows from (1) and (2) that $a = \frac{1-p}{(m+1)(1-p)+1}$. Therefore

$$m_p[0, \frac{1}{\beta}) = 1 - a = \frac{m(1-p)+1}{(m+1)(1-p)+1}.$$

□

1.3 Hausdorff dimension of frequency sets

Recall that Σ_β is the set of admissible sequences and let S_β be its closure in the metric space $(\mathcal{A}_\beta^{\mathbb{N}}, d_\beta)$, where $\mathcal{A}_\beta$ is the alphabet $\{0, 1, \cdots, \lceil\beta\rceil - 1\}$ and d_β is the usual metric on $\mathcal{A}_\beta^{\mathbb{N}}$ (also on S_β) defined by (1.9). In this section, we use $\pi_\beta : S_\beta \to [0, 1]$ to denote the natural projection map defined by (1.3) restricted to S_β (so $\pi_\beta^{-1} A \subset S_\beta$ for any $A \subset [0, 1]$).

As the first main result in this section, the following theorem is a folklore result used in some former papers without explicit proof (for example [114, Section 5]).

Theorem 1.3.1. *Let $\beta > 1$. The Hausdorff dimension of any set Z in (S_β, d_β) is equal to the Hausdorff dimension of its natural projection in $[0, 1]$, i.e.,*

$$\dim_H(Z, d_\beta) = \dim_H \pi_\beta(Z).$$

It is worth to note that $\dim_H(Z, d_\beta) \geq \dim_H \pi_\beta(Z)$ follows immediately from the fact that π_β is Lipschitz continuous. But even if omitting countable many points to make π_β invertible, the inverse is not Lipschitz continuous. This makes the proof of the inverse inequality much more intricate. We will prove it by using a covering property (see Proposition 1.2.9) given by a recent result on the distribution of full cylinders.

In the following, we consider the digit frequencies of the expansions. This is a classical research topic began by Borel in 1909. His well known normal number theorem [31] implies that, for Lebesgue almost every $x \in [0, 1]$, the digit frequency of 0's in its binary expansion is equal to $\frac{1}{2}$. Given $\beta > 1$, for any $a \in [0, 1]$, recall from Section 1.2 that those x's in $[0, 1)$ with digit frequencies of 0's equal to a in their β-expansions constitute the frequency set

$$F_{\beta,a} := \Big\{x \in [0, 1) : \lim_{n\to\infty} \frac{\#\{1 \leq k \leq n : \varepsilon_k(x, \beta) = 0\}}{n} = a\Big\},$$

where $\varepsilon_k(x, \beta)$ is the kth digit in the β-expansion of x and $\#$ denotes the cardinality. For $\beta = 2$, Borel's normal number theorem means that $F_{2,\frac{1}{2}}$ is of full Lebesgue measure, and implies that $F_{2,a}$ is of zero Lebesgue measure for all $a \neq \frac{1}{2}$. This leaves a natural question: How large is $F_{2,a}$ in the sense of dimension? Forty years later, another well known result given by Eggleston [59] showed that

$$\dim_H F_{2,a} = \frac{-a \log a - (1 - a) \log(1 - a)}{\log 2} \quad \text{for all } a \in [0, 1].$$

For the case that β is not an integer, the above question, about giving concrete formulae for the Hausdorff dimension of frequency sets, is almost entirely open. Although the Hausdorff dimension of frequency sets can be given by some variational formulae (see for examples [65, 111, 113]), they are abstract and concrete formulae are very scarce. Except for Theorem 1.2.6 in this thesis, the previously known concrete formula is only the one in Remark 1.2.7.

As the second main result in this section, the next theorem takes a step from abstraction to concreteness. It means that for calculating the Hausdorff dimension of frequency sets, we only need to focus on the entropy (see [121] for definition) with respect to Markov measures of explicit order (see Definition 1.3.11) when $\beta \in (1,2)$ and the β-expansion of 1 is finite. More concretely, it suffices to optimize a function with finitely many variables under some restrictions.

For $\beta > 1$, recall that Σ_β^n is the set of admissible words with length $n \in \mathbb{N}$ and $\Sigma_\beta^* := \cup_{n=1}^\infty \Sigma_\beta^n$. For any $w \in \Sigma_\beta^*$, we use

$$[w] := \{v \in S_\beta : v \text{ begins with } w\}$$

to denote the cylinder in S_β (not Σ_β as in Sections 1.1 and 1.2) generated by w throughout this section.

Recall that σ is the shift map on $\mathcal{A}_\beta^{\mathbb{N}}$ defined by (1.2), and we also use it to denote its restriction on S_β for simplification throughout this section (so $\sigma^{-1}A \subset S_\beta$ for any $A \subset S_\beta$). Let $\mathcal{M}_\sigma(S_\beta)$ be the set of σ-invariant Borel probability measures on S_β and $h_\mu(\sigma)$ be the *measure-theoretic entropy* of σ with respect to the measure μ.

In the following, we regard $0\log 0$, $0\log\frac{0}{0}$, $\max\emptyset$ and $\sup\emptyset$ as 0.

Theorem 1.3.2. *Let $\beta > 1$ such that $\varepsilon(1,\beta) = \varepsilon_1(1,\beta)\cdots\varepsilon_m(1,\beta)0^\infty$ for some integer $m \geq 2$ with $\varepsilon_m(1,\beta) \neq 0$ and let $a \in [0,1]$. Then*

$$\dim_H F_{\beta,a} = \frac{1}{\log\beta}\cdot\max\Big\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(S_\beta), \mu[0] = a, \mu \text{ is an } (m-1)\text{-Markov measure}\Big\}.$$

More concretely,

$$\dim_H F_{\beta,a} = \frac{1}{\log\beta}\cdot\max\Big\{\mathfrak{h}_\mu(\beta,m) : \mu \text{ is a } (\beta,m,a)\text{-coordinated set function}\Big\},$$

where for a set function μ defined from $\{[w] : w \in \cup_{n=1}^m \Sigma_\beta^n\}$ to $[0,1]$,

$$\mathfrak{h}_\mu(\beta,m) := -\sum_{w_1\cdots w_m \in \Sigma_\beta^m} \mu[w_1\cdots w_m]\log\frac{\mu[w_1\cdots w_m]}{\mu[w_1\cdots w_{m-1}]},$$

and μ is called (β,m,a)-coordinated if

$$\mu[0] = a, \quad \sum_{v\in\mathcal{A}_\beta}\mu[v] = 1, \quad \sum_{\substack{v\in\mathcal{A}_\beta\\ wv\in\Sigma_\beta^*}}\mu[wv] = \mu[w] \quad \text{and} \quad \sum_{\substack{u\in\mathcal{A}_\beta\\ uw\in\Sigma_\beta^*}}\mu[uw] = \mu[w]$$

for all $w \in \cup_{n=1}^{m-1}\Sigma_\beta^n$.

Note that for any $(m-1)$-Markov measure $\mu \in \mathcal{M}_\sigma(S_\beta)$, $h_\mu(\sigma)$ is exactly equal to $\mathfrak{h}_\mu(\beta,m)$ (see Proposition 1.3.12).

As applications of the above theorem, we can obtain exact formulae for the Hausdorff dimension of frequency sets for the β's in Theorem 1.2.6 and for another important class of β's in the following theorem, which are called *pseudo-golden ratios*

Theorem 1.3.3. *Let $\beta \in (1,2)$ such that $\varepsilon(1,\beta) = 1^m 0^\infty$ for some integer $m \geq 3$.*
(1) *If $0 \leq a < \frac{1}{m}$, then $F_{\beta,a} = \emptyset$ and $\dim_H F_{\beta,a} = 0$.*
(2) *If $\frac{1}{m} \leq a \leq 1$, then*

$$\dim_H F_{\beta,a} = \frac{1}{\log\beta} \cdot \max_{x_1,\cdots,x_{m-2}} f_a(x_1,\cdots,x_{m-2})$$

where $f_a(x_1,\cdots,x_{m-2})$

$$\begin{aligned}
= a\log a &- (a-x_1)\log(a-x_1) \\
&- (x_1-x_2)\log(x_1-x_2) \\
&\cdots \\
&- (x_{m-3}-x_{m-2})\log(x_{m-3}-x_{m-2}) \\
&- (1-a-x_1-\cdots-x_{m-2})\log(1-a-x_1-\cdots-x_{m-2}) \\
&- (x_1+\cdots+x_{m-3}+2x_{m-2}+a-1)\log(x_1+\cdots+x_{m-3}+2x_{m-2}+a-1)
\end{aligned}$$

and the maximum is taken over $x_1,\cdots,x_{m-2}$ such that all terms in the $\log$*'s are non-negative. That is, $a \geq x_1 \geq x_2 \geq \cdots \geq x_{m-2} \geq 0$ and $x_1+\cdots+x_{m-3}+x_{m-2} \leq 1-a \leq x_1+\cdots+x_{m-3}+2x_{m-2}$.*

In particular, $\dim_H F_{\beta,\frac{1}{m}} = \dim_H F_{\beta,1} = 0$.

Remark 1.3.4. *For the case $m = 3$, i.e., $\varepsilon(1,\beta) = 1110^\infty$, given any $a \in [\frac{1}{3},1]$, by calculating the derivative of $f_a(x_1)$, it is straightforward to get*

$$\begin{aligned}
dim_H F_{\beta,a} = \frac{1}{\log\beta}\Big(a\log a &- \frac{10a-3-\sqrt{-8a^2+12a-3}}{6}\log\frac{10a-3-\sqrt{-8a^2+12a-3}}{6} \\
&- \frac{-2a+3-\sqrt{-8a^2+12a-3}}{6}\log\frac{-2a+3-\sqrt{-8a^2+12a-3}}{6} \\
&- \frac{-a+\sqrt{-8a^2+12a-3}}{3}\log\frac{-a+\sqrt{-8a^2+12a-3}}{3}\Big).
\end{aligned}$$

In particular, $\dim_H F_{\beta,\frac{1}{3}} = \dim_H F_{\beta,1} = 0$.

Base on Sections 1.1 and 1.2, we give additional and necessary preliminaries in Subsection 1.3.1, and then prove Theorems 1.3.1, 1.3.2 and 1.3.3 in Subsections 1.3.2, 1.3.3 and 1.3.4 respectively.

1.3.1 Notation and preliminaries

In Lemma 1.1.3, we introduce Parry's criterion for Σ_β. Here we also need the criterion for S_β.

Lemma 1.3.5 ([99]). *Let $\beta > 1$ and w be a sequence in $\mathcal{A}_\beta^{\mathbb{N}}$. Then*

$$w \in \Sigma_\beta \iff \sigma^k(w) \prec \varepsilon^*(1,\beta) \quad \text{for all } k \geq 0$$

and

$$w \in S_\beta \iff \sigma^k(w) \preceq \varepsilon^*(1,\beta) \quad \text{for all } k \geq 0$$

where $\prec$ and $\preceq$ denote the lexicographic order in $\mathcal{A}_\beta^{\mathbb{N}}$.

We prove the following useful proposition.

Proposition 1.3.6. *Let $\beta > 1$ such that $\varepsilon(1,\beta) = \varepsilon_1(1,\beta)\cdots\varepsilon_m(1,\beta)0^m$ for some integer $m \geq 2$ with $\varepsilon_m(1,\beta) \neq 0$ and $w_1\cdots w_n \in \mathcal{A}_\beta^n$ for some integer $n \geq m$, then*

$$w_1\cdots w_n \in \Sigma_\beta^* \quad \text{if and only if} \quad w_1\cdots w_m, w_2\cdots w_{m+1}, \cdots, w_{n-m+1}\cdots w_n \in \Sigma_\beta^*.$$

Proof. $\boxed{\Rightarrow}$ Obvious.
$\boxed{\Leftarrow}$ For simplification we use $\varepsilon_1, \cdots, \varepsilon_m$ instead of $\varepsilon_1(1,\beta), \cdots, \varepsilon_m(1,\beta)$ in the following. Suppose

$$w_1\cdots w_m, w_2\cdots w_{m+1}, \cdots, w_{n-m+1}\cdots w_n \in \Sigma_\beta^*.$$

By Lemma 1.3.5 we get

$$w_1\cdots w_m, w_2\cdots w_{m+1}, \cdots, w_{n-m+1}\cdots w_n \preceq \varepsilon_1\cdots\varepsilon_{m-1}(\varepsilon_m - 1).$$

In order to get $w_1\cdots w_n \in \Sigma_\beta^*$, by Lemma 1.3.5, it suffices to check

$$\sigma^k(w_1\cdots w_n 0^\infty) \prec (\varepsilon_1\cdots\varepsilon_{m-1}(\varepsilon_m - 1))^\infty \quad \text{for all } k \geq 0.$$

If $k \geq n$, this is obvious. We consider $k \leq n-1$ in the following. Let $l \geq 0$ be the greatest integer such that $k + lm \leq n-1$. Then

$$\begin{aligned}
\sigma^k(w_1\cdots w_n 0^\infty) &= (w_{k+1}\cdots w_{k+m})(w_{k+m+1}\cdots w_{k+2m}) \\
&\quad \cdots (w_{k+(l-1)m+1}\cdots w_{k+lm})(w_{k+lm+1}\cdots w_n 0^{k+(l+1)m-n})0^\infty \\
&\preceq (\varepsilon_1\cdots\varepsilon_{m-1}(\varepsilon_m - 1))^l (w_{k+lm+1}\cdots w_n 0^{k+(l+1)m-n})0^\infty \\
&\prec (\varepsilon_1\cdots\varepsilon_{m-1}(\varepsilon_m - 1))^\infty,
\end{aligned}$$

where the last inequality follows from

$$w_{k+lm+1}\cdots w_n 0^{k+(l+1)m-n} \preceq \varepsilon_1\cdots\varepsilon_{m-1}(\varepsilon_m - 1), \tag{1.14}$$

which can be proved as follows. In fact, by $w_{n-m+1}\cdots w_n \in \Sigma_\beta^*$ and Lemma 1.3.5, we get

$$\sigma^{k+(l+1)m-n}(w_{n-m+1}\cdots w_n 0^\infty) \prec (\varepsilon_1\cdots\varepsilon_{m-1}(\varepsilon_m - 1))^\infty.$$

This implies (1.14). □

Definition 1.3.7 (Hausdorff measure and dimension in metric space)**.** *Let (X, d) be a metric space. For any $U \subset X$, denote the diameter of U by $|U| := \sup_{x,y\in U} d(x, y)$. For any $A \subset X, s \geq 0$ and $\delta > 0$, let*

$$\mathcal{H}^s_\delta(A, d) := \inf\Big\{\sum_{i=1}^{\infty} |U_i|^s : A \subset \bigcup_{i=1}^{\infty} U_i \text{ and } |U_i| \leq \delta \text{ for all } i \in \mathbb{N}\Big\}.$$

We define the s-dimensional Hausdorff measure of A in (X, d) by

$$\mathcal{H}^s(A, d) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A, d)$$

and the Hausdorff dimension of A in (X, d) by

$$\dim_H(A, d) := \sup\{s \geq 0 : \mathcal{H}^s(A, d) = \infty\}.$$

In the space of real numbers $\mathbb{R}$ (equipped with the usual metric), we use $\mathcal{H}^s(A)$ and $\dim_H A$ to denote the s-dimensional Hausdorff measure and the Hausdorff dimension of A respectively for simplification.

Definition 1.3.8 (Lipschitz continuous)**.** *Let (X, d) and (X', d') be two metric spaces. A map $f : X \to X'$ is called Lipschitz continuous if there exists a constant $c > 0$ such that*

$$d'(f(x), f(y)) \leq c \cdot d(x, y) \quad \text{for all } x, y \in X.$$

The following basic proposition can be deduced directly from the definitions.

Proposition 1.3.9. *If the map $f : (X, d) \to (X', d')$ between two metric spaces is Lipschitz continuous, then for any $A \subset X$, we have*

$$\dim_H(f(A), d') \leq \dim_H(A, d).$$

Recall that $\mathcal{M}_\sigma(S_\beta)$ is the set of σ-invariant Borel probability measures on S_β. The following is a consequence of Carathéodory's measure extension theorem and the fact that for verifying the σ-invariance of measures on S_β, one only needs to check it for the cylinders.

Proposition 1.3.10. *Let $\beta > 1$. Any set function μ from $\{[w] : w \in \Sigma^*_\beta\}$ to $[0, 1]$ satisfying*

$$\sum_{v\in\mathcal{A}_\beta} \mu[v] = 1, \quad \sum_{\substack{v\in\mathcal{A}_\beta \\ wv\in\Sigma^*_\beta}} \mu[wv] = \mu[w] \quad \text{and} \quad \sum_{\substack{u\in\mathcal{A}_\beta \\ uw\in\Sigma^*_\beta}} \mu[uw] = \mu[w]$$

for all $w \in \Sigma^*_\beta$ can be uniquely extended to be a measure in $\mathcal{M}_\sigma(S_\beta)$.

The following concept is well known (see for examples [65, Section 2] and [80, Section 6.2]).

Definition 1.3.11 (k-Markov measure)**.** *Let $\beta > 1$, $k \in \mathbb{N}$ and $\mu \in \mathcal{M}_\sigma(S_\beta)$. We call μ a k-Markov measure if*

$$\mu[w_1 \cdots w_n] = \mu[w_1 \cdots w_{n-1}] \cdot \frac{\mu[w_{n-k} \cdots w_n]}{\mu[w_{n-k} \cdots w_{n-1}]}$$

for all $w_1 \cdots w_n \in \Sigma_\beta^n$ with $n > k$.

Recall that $h_\mu(\sigma)$ is the measure-theoretic entropy of σ with respect to the measure μ. Using $\mathcal{P} := \{[v] : v \in \mathcal{A}_\beta\}$ as a partition generator of the Borel sigma-algebra on S_β, the proof of the following proposition is straightforward.

Proposition 1.3.12. *Let $\beta > 1$, $k \in \mathbb{N}$ and $\mu \in \mathcal{M}_\sigma(S_\beta)$ be a k-Markov measure, then*

$$h_\mu(\sigma) = -\sum_{w_1 \cdots w_{k+1} \in \Sigma_\beta^{k+1}} \mu[w_1 \cdots w_{k+1}] \log \frac{\mu[w_1 \cdots w_{k+1}]}{\mu[w_1 \cdots w_k]}.$$

1.3.2 Proof of Theorem 1.3.1

The main we need to prove is the following technical lemma.

Lemma 1.3.13. *Let $\beta > 1$, $s > 0$ and $Z \subset S_\beta$. Then for any $\varepsilon \in (0, s)$, we have*

$$\mathcal{H}^s(Z, d_\beta) \le \mathcal{H}^{s-\varepsilon}(\pi_\beta(Z)).$$

Proof. Fix $\varepsilon \in (0, s)$. Let $Z_0 := Z \cap \Sigma_\beta$. Since $S_\beta \setminus \Sigma_\beta$ is countable, we only need to prove $\mathcal{H}^s(Z_0, d_\beta) \le \mathcal{H}^{s-\varepsilon}(\pi_\beta(Z_0))$.
(1) Choose $\delta_0 \in (0, \frac{1}{\beta})$ small enough as follows. Since $\beta^{(n+1)\varepsilon} \to \infty$ much faster than $8\beta^s n \to \infty$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $8\beta^s n \le \beta^{(n+1)\varepsilon}$. By $\frac{-\log \delta}{\log \beta} - 1 \to \infty$ as $\delta \to 0^+$, there exists $\delta_0 \in (0, \frac{1}{\beta})$ small enough such that $\frac{-\log \delta_0}{\log \beta} - 1 > n_0$. Then for any $n > \frac{-\log \delta_0}{\log \beta} - 1$, we will have $8\beta^s n \le \beta^{(n+1)\varepsilon}$.
(2) For any $\delta \in (0, \delta_0)$, let $\{U_i\}$ be a δ-cover of $\pi_\beta(Z_0)$, i.e., $0 < |U_i| \le \delta$ and $\pi_\beta(Z_0) \subset \cup_i U_i$. Then for each U_i, there exists $n_i \in \mathbb{N}$ such that $\frac{1}{\beta^{n_i+1}} < |U_i| \le \frac{1}{\beta^{n_i}}$. By Proposition 1.2.9, U_i can be covered by at most $8n_i$ cylinders $I_{i,1}, I_{i,2}, \cdots, I_{i,8n_i}$ of order n_i. It follows from

$$|\Sigma_\beta \cap \pi_\beta^{-1} I_{i,j}| = \frac{1}{\beta^{n_i}} < \beta |U_i| \le \beta\delta \quad \text{and} \quad Z_0 \subset \Sigma_\beta \cap \bigcup_i \pi_\beta^{-1} U_i \subset \bigcup_i \bigcup_{j=1}^{8n_i} (\Sigma_\beta \cap \pi_\beta^{-1} I_{i,j})$$

that

$$\mathcal{H}^s_{\beta\delta}(Z_0, d_\beta) \le \sum_i \sum_{j=1}^{8n_i} |\Sigma_\beta \cap \pi_\beta^{-1} I_{i,j}|^s = \sum_i \frac{8n_i}{\beta^{n_i s}} \overset{(\star)}{\le} \sum_i \frac{1}{\beta^{(n_i+1)(s-\varepsilon)}} < \sum_i |U_i|^{s-\varepsilon}, \quad (1.15)$$

where $(\star)$ is because $\frac{1}{\beta^{n_i+1}} < |U_i| < \delta_0$ implies $n_i > \frac{-\log \delta_0}{\log \beta} - 1$, and then by (1) we have $8n_i\beta^s \leq \beta^{(n_i+1)\varepsilon}$. Taking inf on the right of (1.15), we get $\mathcal{H}^s_{\beta\delta}(Z_0, d_\beta) \leq \mathcal{H}^{s-\varepsilon}_\delta(\pi_\beta(Z_0))$. It follows from letting $\delta \to 0$ that $\mathcal{H}^s(Z_0, d_\beta) \leq \mathcal{H}^{s-\varepsilon}(\pi_\beta(Z_0))$. □

Proof of Theorem 1.3.1. The inequality $\dim_H(Z, d_\beta) \geq \dim_H \pi_\beta(Z)$ follows from Proposition 1.3.9 and the fact that π_β is Lipschitz continuous. The inverse inequality follows from Lemma 1.3.13. In fact, for any $t < \dim_H(Z, d_\beta)$, there exists s such that $t < s < \dim_H(Z, d_\beta)$. By $\mathcal{H}^s(Z, d_\beta) = \infty$ and Lemma 1.3.13, we get $\mathcal{H}^t(\pi_\beta(Z)) = \infty$. Thus $t \leq \dim_H \pi_\beta(Z)$. It means that $\dim_H(Z, d_\beta) \leq \dim_H \pi_\beta(Z)$. □

1.3.3 Proof of Theorem 1.3.2

We will deduce Theorem 1.3.2 from the following proposition, which is essentially from [101].

Proposition 1.3.14. *Let $\beta > 1$ and $a \in [0, 1]$. Then*

$$\dim_H F_{\beta,a} = \frac{1}{\log \beta} \cdot \sup\Big\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(S_\beta), \mu[0] = a\Big\}.$$

For the convenience of the readers, we recall some definitions and show how Proposition 1.3.14 comes from [101].

Definition 1.3.15. *Let $\beta > 1$.*
(1) *For any $w \in S_\beta$ and $n \in \mathbb{N}$, the empirical measure is defined by*

$$\mathcal{E}_n(w) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i w}$$

where δ_w is the Dirac probability measure concentrated on w.
(2) *Let $\mathcal{A}$ be an arbitrary non-empty parameter set and let*

$$\mathcal{F} := \Big\{(f_\alpha, c_\alpha, d_\alpha) : \alpha \in \mathcal{A}\Big\}$$

where $f_\alpha : S_\beta \to \mathbb{R}$ is continuous and $c_\alpha, d_\alpha \in \mathbb{R}$ with $c_\alpha \leq d_\alpha$ for all $\alpha \in \mathcal{A}$. Define

$$S_{\beta,\mathcal{F}} := \Big\{w \in S_\beta : \forall \alpha \in \mathcal{A}, c_\alpha \leq \varliminf_{n\to\infty} \int f_\alpha \, d\mathcal{E}_n(w) \leq \varlimsup_{n\to\infty} \int f_\alpha \, d\mathcal{E}_n(w) \leq d_\alpha\Big\}$$

and

$$\mathcal{M}_{\beta,\mathcal{F}} := \Big\{\mu \in \mathcal{M}_\sigma(S_\beta) : \forall \alpha \in \mathcal{A}, c_\alpha \leq \int f_\alpha \, d\mu \leq d_\alpha\Big\}.$$

Combining Theorems 5.2 and 5.3 in [101], we get the following.

Lemma 1.3.16. *Let $\beta > 1$. If $\mathcal{M}_{\beta,\mathcal{F}}$ is a non-empty closed connected set, then*

$$h_{top}(S_{\beta,\mathcal{F}}, \sigma) = \sup\Big\{h_\mu(\sigma) : \mu \in \mathcal{M}_{\beta,\mathcal{F}}\Big\}$$

where $h_{top}(S_{\beta,\mathcal{F}}, \sigma)$ is the topological entropy of $S_{\beta,\mathcal{F}}$ in the dynamical system $(S_\beta, d_\beta, \sigma)$. (See [32] for the definition of the topological entropy for non-compact sets.)

For $\beta > 1$ and $a \in [0, 1]$, let

$$S_{\beta,a} := \Big\{w \in S_\beta : \lim_{n\to\infty} \frac{\#\{1 \le k \le n : w_k = 0\}}{n} = a\Big\}.$$

In Definition 1.3.15 (2), let $\mathcal{F}$ be the singleton $\{(\mathbb{1}_{[0]}, a, a)\}$, where the characteristic function $\mathbb{1}_{[0]} : S_\beta \to \mathbb{R}$ is continuous. (Here we note that another characteristic function $\mathbb{1}_{[0,\frac{1}{\beta}]} : [0, 1] \to \mathbb{R}$ is not continuous, which means that some other similar variational formulae corresponding to dynamical systems on [0,1] can not be applied directly in our case.) We get the following lemma as a special case of the above one.

Lemma 1.3.17.

$$h_{top}(S_{\beta,a}, \sigma) = \sup\Big\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(S_\beta), \mu[0] = a\Big\}.$$

Hence, Proposition 1.3.14 follows from

$$\begin{aligned}
\dim_H F_{\beta,a} &\underset{\text{is countable}}{\overset{\pi_\beta(S_{\beta,a})\setminus F_{\beta,a}}{=\!=\!=}} \dim_H \pi_\beta(S_{\beta,a}) \\
&\underset{\text{Theorem 1.3.1}}{\overset{\text{by}}{=\!=\!=}} \dim_H(S_{\beta,a}, d_\beta) \\
&\underset{\text{Lemma 1.3.18}}{\overset{\text{by}}{=\!=\!=}} \frac{1}{\log\beta} \cdot h_{top}(S_{\beta,a}, \sigma),
\end{aligned}$$

where $\pi_\beta(S_{\beta,a}) \setminus F_{\beta,a}$ is countable since we can check $\pi_\beta(S_{\beta,a}) \setminus F_{\beta,a} \subset \pi_\beta(S_\beta \setminus \Sigma_\beta)$ and Lemma 1.3.5 implies that $S_\beta \setminus \Sigma_\beta$ is countable.

Lemma 1.3.18. *([114, Lemma 5.3]) Let $\beta > 1$. For any $Z \subset S_\beta$, we have*

$$\dim_H(Z, d_\beta) = \frac{1}{\log\beta} \cdot h_{top}(Z, \sigma).$$

We give the following proofs to end this subsection.

Proof of Lemma 1.3.17. In Definition 1.3.15 (2), let $\mathcal{F}$ be the singleton $\{(\mathbb{1}_{[0]}, a, a)\}$. Then

$$S_{\beta,\mathcal{F}} = \Big\{w \in S_\beta : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_{[0]}(\sigma^i w) = a\Big\} = S_{\beta,a}$$

and

$$\mathcal{M}_{\beta,\mathcal{F}} = \Big\{\mu \in \mathcal{M}_\sigma(S_\beta) : \mu[0] = a\Big\} \overset{\text{denote}}{\underset{\text{by}}{=\!=\!=}} : \mathcal{M}_{\beta,a}.$$

(1) If $\mathcal{M}_{\beta,a} = \emptyset$, we can prove $S_{\beta,a} = \emptyset$ (and then the conclusion follows).
(By contradiction) If $S_{\beta,a} \neq \emptyset$, there exists $w \in S_{\beta,a}$. For any $n \in \mathbb{N}$, let

$$\mu_n := \mathcal{E}_n(w) \in \mathcal{M}(S_\beta) := \{\text{Borel probability measures on } S_\beta\}.$$

Since $\mathcal{M}(S_\beta)$ is compact, there exists subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}} \subset \{\mu_n\}_{n\in\mathbb{N}}$ and $\mu \in \mathcal{M}(S_\beta)$ such that $\mu_{n_k} \overset{w^*}{\to} \mu$ (i.e. μ_{n_k} converge to μ under the weak* topology). By $\mu_{n_k} \circ \sigma^{-1} \overset{w^*}{\to} \mu \circ \sigma^{-1}$ and $\mu_{n_k} \circ \sigma^{-1} - \mu_{n_k} \overset{w^*}{\to} 0$, we get $\mu \circ \sigma^{-1} = \mu$ and then $\mu \in \mathcal{M}_\sigma(S_\beta)$. It follows from

$$\mu[0] = \int \mathbb{1}_{[0]}\, d\mu = \lim_{k\to\infty} \int \mathbb{1}_{[0]}\, d\mu_{n_k} = \lim_{k\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \mathbb{1}_{[0]}(\sigma^i w) \overset{w\in S_{\beta,a}}{=\!=\!=} a$$

that $\mu \in \mathcal{M}_{\beta,a}$, which contradicts $\mathcal{M}_{\beta,a} = \emptyset$.

(2) If $\mathcal{M}_{\beta,a} \neq \emptyset$, by Lemma 1.3.16, it suffices to prove that $\mathcal{M}_{\beta,a}$ is a closed connected set in $\mathcal{M}_\sigma(S_\beta)$.

① Prove that $\mathcal{M}_{\beta,a}$ is closed.
Let $\{\mu_n, n \in \mathbb{N}\} \subset \mathcal{M}_{\beta,a}$ and $\mu \in \mathcal{M}_\sigma(S_\beta)$ such that $\mu_n \overset{w^*}{\to} \mu$. It follows from

$$\mu[0] = \int \mathbb{1}_{[0]}\, d\mu = \lim_{n\to\infty} \int \mathbb{1}_{[0]}\, d\mu_n = \lim_{n\to\infty} \mu_n[0] = a$$

that $\mu \in \mathcal{M}_{\beta,a}$.

② Prove that $\mathcal{M}_{\beta,a}$ is connected.
It suffices to prove that $\mathcal{M}_{\beta,a}$ is path connected. In fact, for any $\mu_0, \mu_1 \in \mathcal{M}_{\beta,a}$, we define the path $f : [0,1] \to \mathcal{M}_{\beta,a}$ by $f(s) := \mu_s := (1-s)\mu_0 + s\mu_1$ for $s \in [0,1]$. Then $f(0) = \mu_0$, $f(1) = \mu_1$ and $f([0,1]) \subset \mathcal{M}_{\beta,a}$. It remains to show that f is continuous. Let $\{s, s_n, n \geq 1\} \subset [0,1]$ such that $s_n \to s$. We only need to prove $f(s_n) \to f(s)$, i.e., $\mu_{s_n} \overset{w^*}{\to} \mu_s$. Let $\phi : S_\beta \to \mathbb{R}$ be a continuous function. It suffices to check $\int \phi\, d\mu_{s_n} \to \int \phi\, d\mu_s$, i.e.,

$$(1-s_n)\int \phi\, d\mu_0 + s_n \int \phi\, d\mu_1 \to (1-s)\int \phi\, d\mu_0 + s\int \phi\, d\mu_1.$$

This follows immediately from $s_n \to s$.

□

Proof of Theorem 1.3.2. By Proposition 1.3.14 it suffices to consider the following (1), (2) and (3).

(1) We have

$$\begin{aligned}
&\sup\Big\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(S_\beta), \mu[0] = a, \mu \text{ is an } (m-1)\text{-Markov measure}\Big\}\\
\le &\sup\Big\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(S_\beta), \mu[0] = a\Big\}\\
\le &\sup\Big\{\mathfrak{h}_\mu(\beta, m) : \mu \text{ is a } (\beta, m, a)\text{-coordinated set function}\Big\}.
\end{aligned}$$

Since the first inequality is obvious, we only prove the second one as follows. Let $\mu \in \mathcal{M}_\sigma(S_\beta)$ such that $\mu[0] = a$. Restricted to $\{[w] : w \in \cup_{n=1}^m \Sigma_\beta^n\}$, μ is obviously a (β, m, a)-coordinated set function. It suffices to prove $h_\mu(\sigma) \le \mathfrak{h}_\mu(\beta, m)$. Using $\mathcal{P} := \{[v] : v \in \mathcal{A}_\beta\}$ as a partition generator of the Borel sigma-algebra on (S_β, d_β), by simple calculation, we get that the conditional entropy of $\mathcal{P}$ given $\bigvee_{k=1}^{m-1} \sigma^{-k}\mathcal{P}$ with respect to μ, denoted by $H_\mu\Big(\mathcal{P} \mid \bigvee_{k=1}^{m-1} \sigma^{-k}\mathcal{P}\Big)$, is equal to $\mathfrak{h}_\mu(\beta, m)$. Since $H_\mu\Big(\mathcal{P} \mid \bigvee_{k=1}^{n-1} \sigma^{-k}\mathcal{P}\Big)$ decreases as n increases and [121, Theorem 4.14] says that it converges to $h_\mu(\sigma)$, we get $h_\mu(\sigma) \le \mathfrak{h}_\mu(\beta, m)$. In the following we attached the calculation mentioned above.

$$\begin{aligned}
&H_\mu\Big(\mathcal{P} \mid \bigvee_{k=1}^{m-1} \sigma^{-k}\mathcal{P}\Big) = H_\mu\Big(\mathcal{P} \mid \sigma^{-1}(\bigvee_{k=0}^{m-2} \sigma^{-k}\mathcal{P})\Big)\\
&= -\sum_{P\in\mathcal{P},\ Q\in\bigvee_{k=0}^{m-2}\sigma^{-k}\mathcal{P}} \mu(P \cap \sigma^{-1}Q) \log \frac{\mu(P \cap \sigma^{-1}Q)}{\mu(\sigma^{-1}Q)}\\
&= -\sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \frac{\mu[w_1 \cdots w_m]}{\mu(\sigma^{-1}[w_2 \cdots w_m])}\\
&= \sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \mu[w_2 \cdots w_m] - \sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \mu[w_1 \cdots w_m]\\
&= \sum_{w_2\cdots w_m\in\Sigma_\beta^*} \mu[w_2 \cdots w_m] \log \mu[w_2 \cdots w_m] - \sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \mu[w_1 \cdots w_m]\\
&= \sum_{w_1\cdots w_{m-1}\in\Sigma_\beta^*} \mu[w_1 \cdots w_{m-1}] \log \mu[w_1 \cdots w_{m-1}] - \sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \mu[w_1 \cdots w_m]\\
&= \sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \mu[w_1 \cdots w_{m-1}] - \sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \mu[w_1 \cdots w_m]\\
&= -\sum_{w_1\cdots w_m\in\Sigma_\beta^*} \mu[w_1 \cdots w_m] \log \frac{\mu[w_1 \cdots w_m]}{\mu[w_1 \cdots w_{m-1}]} = \mathfrak{h}_\mu(\beta, m).
\end{aligned}$$

(2) Prove

$$\begin{aligned}
&\Big\{h_\mu(\sigma) : \mu \in \mathcal{M}_\sigma(S_\beta), \mu[0] = a, \mu \text{ is an } (m-1)\text{-Markov measure}\Big\}\\
&= \Big\{\mathfrak{h}_\mu(\beta, m) : \mu \text{ is a } (\beta, m, a)\text{-coordinated set function}\Big\}.
\end{aligned}$$

$\boxed{\subset}$ follows from the facts that every $(m-1)$-Markov measure $\mu \in M_\sigma(S_\beta)$ with $\mu[0] = a$

restricted to $\{[w] : w \in \cup_{n=1}^{m} \Sigma_\beta^n\}$ is a (β, m, a)-coordinated set function and Proposition 1.3.12 implies $h_\mu(\sigma) = \mathfrak{h}_\mu(\beta, m)$.

$\boxed{\supset}$ Let μ be a (β, m, a)-coordinated set function. By the entropy formula Proposition 1.3.12, it suffices to show that μ can be extended to be an $(m-1)$-Markov measure in $\mathcal{M}_\sigma(S_\beta)$. Note that μ is already defined on all the cylinders of order $\leq m$. Suppose that for some $n \geq m$, μ is already defined on $\{[w_1 \cdots w_n] : w_1 \cdots w_n \in \Sigma_\beta^n\}$. Then for all $w_1 \cdots w_{n+1} \in \Sigma_\beta^{n+1}$ we define

$$\mu[w_1 \cdots w_{n+1}] := \mu[w_1 \cdots w_n] \cdot \frac{\mu[w_{n-m+2} \cdots w_{n+1}]}{\mu[w_{n-m+2} \cdots w_n]}$$

where the right hand side is regarded as 0 if one of $\mu[w_1 \cdots w_n]$, $\mu[w_{n-m+2} \cdots w_n]$ and $\mu[w_{n-m+2} \cdots w_{n+1}]$ is 0. By Proposition 1.3.10 it suffices to check

$$① \sum_{\substack{v \in \mathcal{A}_\beta \\ wv \in \Sigma_\beta^*}} \mu[wv] = \mu[w] \quad \text{and} \quad ② \sum_{\substack{u \in \mathcal{A}_\beta \\ uw \in \Sigma_\beta^*}} \mu[uw] = \mu[w]$$

for all $w \in \Sigma_\beta^n$ with $n \geq m$. (Note that for $n \leq m-1$, ① and ② are already guaranteed by the condition that μ is (β, m, a)-coordinated.)

① Let $n \geq m$ and $w_1 \cdots w_n \in \Sigma_\beta^n$. Then

$$\sum_{\substack{v \in \mathcal{A}_\beta \\ w_1 \cdots w_n v \in \Sigma_\beta^*}} \mu[w_1 \cdots w_n v] = \sum_{\substack{v \in \mathcal{A}_\beta \\ w_1 \cdots w_n v \in \Sigma_\beta^*}} \mu[w_1 \cdots w_n] \cdot \frac{\mu[w_{n-m+2} \cdots w_n v]}{\mu[w_{n-m+2} \cdots w_n]} \overset{(\star)}{=} \mu[w_1 \cdots w_n],$$

where $(\star)$ can be proved as follows.

i) If $\mu[w_1 \cdots w_n] = 0$, then $(\star)$ is obvious.

ii) If $\mu[w_{n-m+2} \cdots w_n] = 0$, since the fact that μ is (β, m, a)-coordinated implies $\mu[w_{n-m+1} \cdots w_n] \leq \mu[w_{n-m+2} \cdots w_n]$, we get $\mu[w_{n-m+1} \cdots w_n] = 0$. Then

$$\mu[w_1 \cdots w_n] = \mu[w_1 \cdots w_{n-1}] \cdot \frac{\mu[w_{n-m+1} \cdots w_n]}{\mu[w_{n-m+1} \cdots w_{n-1}]} = 0$$

and $(\star)$ follows.

iii) If $\mu[w_1 \cdots w_n] \neq 0$ and $\mu[w_{n-m+2} \cdots w_n] \neq 0$, then $(\star)$ follows from

$$\sum_{\substack{v \in \mathcal{A}_\beta \\ w_1 \cdots w_n v \in \Sigma_\beta^*}} \mu[w_{n-m+2} \cdots w_n v] \overset{(\star\star)}{=} \sum_{\substack{v \in \mathcal{A}_\beta \\ w_{n-m+2} \cdots w_n v \in \Sigma_\beta^*}} \mu[w_{n-m+2} \cdots w_n v] = \mu[w_{n-m+2} \cdots w_n],$$

where the last equality follows from the fact that μ is (β, m, a)-coordinated, and $(\star\star)$ follows from the fact that $w_1 \cdots w_n \in \Sigma_\beta^*$ and Proposition 1.3.6 imply the equivalence

of $w_1 \cdots w_n v \in \Sigma_\beta^*$ and $w_{n-m+2} \cdots w_n v \in \Sigma_\beta^*$.

② Prove $\sum_{\substack{u \in \mathcal{A}_\beta \\ uw_1 \cdots w_n \in \Sigma_\beta^*}} \mu[uw_1 \cdots w_n] = \mu[w_1 \cdots w_n]$ for all $w_1 \cdots w_n \in \Sigma_\beta^n$ and $n \geq m$ by induction. Since μ is (β, m, a)-coordinated, the conclusion is true for $n = m - 1$. Now suppose that the conclusion is already true for some $n \geq m - 1$. We consider $n + 1$ in the following. Let $w_1 \cdots w_{n+1} \in \Sigma_\beta^{n+1}$. Then

$$\begin{aligned}
\sum_{\substack{u \in \mathcal{A}_\beta \\ uw_1 \cdots w_{n+1} \in \Sigma_\beta^*}} \mu[uw_1 \cdots w_{n+1}] &\overset{(\star)}{=} \sum_{\substack{u \in \mathcal{A}_\beta \\ uw_1 \cdots w_n \in \Sigma_\beta^*}} \mu[uw_1 \cdots w_{n+1}] \\
&= \sum_{\substack{u \in \mathcal{A}_\beta \\ uw_1 \cdots w_n \in \Sigma_\beta^*}} \mu[uw_1 \cdots w_n] \cdot \frac{\mu[w_{n-m+2} \cdots w_{n+1}]}{\mu[w_{n-m+2} \cdots w_n]} \\
&\overset{(\star\star)}{=} \mu[w_1 \cdots w_n] \cdot \frac{\mu[w_{n-m+2} \cdots w_{n+1}]}{\mu[w_{n-m+2} \cdots w_n]} = \mu[w_1 \cdots w_{n+1}],
\end{aligned}$$

where $(\star)$ follows from the fact that $w_1 \cdots w_{n+1} \in \Sigma_\beta^*$ and Proposition 1.3.6 imply the equivalence of $uw_1 \cdots w_{n+1} \in \Sigma_\beta^*$ and $uw_1 \cdots w_n \in \Sigma_\beta^*$, and $(\star\star)$ follows from inductive hypothesis.

(3) By the definition of (β, m, a)-coordinated set functions and $\mathfrak{h}_\mu(\beta, m)$, it is straightforward to see that the supremum of

$$\Big\{ \mathfrak{h}_\mu(\beta, m) : \mu \text{ is a } (\beta, m, a)\text{-coordinated set function} \Big\}$$

can be achieved as a maximum. □

1.3.4 Proof of Theorem 1.3.3

We need the following lemma which follows immediately from the convexity of the function $x \log x$.

Lemma 1.3.19. *Let $\phi : [0, \infty) \to \mathbb{R}$ be defined by*

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0; \\ -x \log x & \text{if } x > 0. \end{cases}$$

Then for all $x, y \in [0, \infty)$ and $a, b \geq 0$ with $a + b = 1$,

$$a\phi(x) + b\phi(y) \leq \phi(ax + by).$$

The equality holds if and only if $x = y$, $a = 0$ or $b = 0$.

Proof of Theorem 1.3.3.

(1) By $\varepsilon^*(1,\beta) = (1^{m-1}0)^\infty$ and Lemma 1.3.5, we know that for any $x \in [0,1)$, every m consecutive digits in $\varepsilon(x,\beta)$ must contain at least one 0. This implies

$$\#\{1 \le k \le n : \varepsilon_k(x,\beta) = 0\} \ge \lfloor \frac{n}{m} \rfloor$$

for all $n \in \mathbb{N}$, and then

$$\varliminf_{n\to\infty} \frac{\#\{1 \le k \le n : \varepsilon_k(x,\beta) = 0\}}{n} \ge \frac{1}{m}$$

for any $x \in [0,1)$. If $0 \le a < \frac{1}{m}$, we get $F_{\beta,a} = \emptyset$.

(2) When $\frac{1}{m} \le a \le 1$, f_a is a continuous function on its domain of definition

$$\begin{aligned} D_{m,a} &:= \Big\{(x_1, x_2, \cdots, x_{m-2}) \in \mathbb{R}^{m-2} : \text{all terms in the } \log\text{'s in } f_a \text{ are non-negative}\Big\} \\ &= \Big\{(x_1, x_2, \cdots, x_{m-2}) \in \mathbb{R}^{m-2} : a \ge x_1 \ge x_2 \ge \cdots \ge x_{m-2} \ge 0 \quad \text{and} \\ &\qquad\qquad x_1 + \cdots + x_{m-3} + x_{m-2} \le 1 - a \le x_1 + \cdots + x_{m-3} + 2x_{m-2}\Big\}, \end{aligned}$$

which is closed and non-empty since

$$\begin{cases} (a, \frac{1-2a}{m-2}, \cdots, \frac{1-2a}{m-2}) \in D_{m,a} & \text{if } \frac{1}{m} \le a < \frac{1}{2}; \\ (1-a, 0, \cdots, 0) \in D_{m,a} & \text{if } a \ge \frac{1}{2}. \end{cases}$$

Therefore $\max_{(x_1,\cdots,x_{m-2})\in D_{m,a}} f_a(x_1, \cdots, x_{m-2})$ exists.

In order to get our conclusion, by Theorem 1.3.2, it suffices to prove

$$\max\Big\{\mathfrak{h}_\mu(\beta,m) : \mu \text{ is a } (\beta,m,a)\text{-coordinated set function}\Big\} = \max_{(x_1,\cdots,x_{m-2})\in D_{m,a}} f_a(x_1, \cdots, x_{m-2}) \tag{1.16}$$

in the following ① and ②, which are enlightened by drawing figures of the cylinders in $[0,1)$ and understanding their relations.

① Prove the inequality "$\le$" in (1.16).

Let μ be a (β,m,a)-coordinated set function. By Lemma 1.3.5 we get $\Sigma_\beta^m = \{0,1\}^m \backslash \{1^m\}$, $\mu[1^{m-1}0] = \mu[1^{m-1}]$ and then

$$\begin{aligned} \mathfrak{h}_\mu(\beta,m) &= -\sum_{\substack{i_1,\cdots,i_m\in\{0,1\} \\ i_2\cdots i_{m-1}\neq 1^{m-2}}} \mu[i_1\cdots i_m] \log \frac{\mu[i_1\cdots i_m]}{\mu[i_1\cdots i_{m-1}]} \\ &\quad -\mu[01^{m-2}0] \log \frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}] \log \frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}. \end{aligned}$$

For $i_2\cdots i_{m-1}\neq 1^{m-2}$ and $i_m\in\{0,1\}$, we can prove

$$-\mu[0i_2\cdots i_m]\log\frac{\mu[0i_2\cdots i_m]}{\mu[0i_2\cdots i_{m-1}]}-\mu[1i_2\cdots i_m]\log\frac{\mu[1i_2\cdots i_m]}{\mu[1i_2\cdots i_{m-1}]}\leq -\mu[i_2\cdots i_m]\log\frac{\mu[i_2\cdots i_m]}{\mu[i_2\cdots i_{m-1}]}. \tag{1.17}$$

In fact, if $\mu[0i_2\cdots i_{m-1}]=0$, then $\mu[0i_2\cdots i_m]=0$. We get $\mu[1i_2\cdots i_{m-1}]=\mu[i_2\cdots i_{m-1}]-\mu[0i_2\cdots i_{m-1}]=\mu[i_2\cdots i_{m-1}]$ and $\mu[1i_2\cdots i_m]=\mu[i_2\cdots i_m]-\mu[0i_2\cdots i_m]=\mu[i_2\cdots i_m]$, which imply (1.17). If $\mu[1i_2\cdots i_{m-1}]=0$, in the same way we can get (1.17). If $\mu[0i_2\cdots i_{m-1}]\neq 0$ and $\mu[1i_2\cdots i_{m-1}]\neq 0$, then $\mu[i_2\cdots i_{m-1}]\neq 0$ and (1.17) follows from

$$\begin{aligned}
&-\mu[0i_2\cdots i_m]\log\frac{\mu[0i_2\cdots i_m]}{\mu[0i_2\cdots i_{m-1}]}-\mu[1i_2\cdots i_m]\log\frac{\mu[1i_2\cdots i_m]}{\mu[1i_2\cdots i_{m-1}]}\\
=\;&\mu[i_2\cdots i_{m-1}]\Big(\frac{\mu[0i_2\cdots i_{m-1}]}{\mu[i_2\cdots i_{m-1}]}\Big(-\frac{\mu[0i_2\cdots i_m]}{\mu[0i_2\cdots i_{m-1}]}\log\frac{\mu[0i_2\cdots i_m]}{\mu[0i_2\cdots i_{m-1}]}\Big)\\
&\qquad+\frac{\mu[1i_2\cdots i_{m-1}]}{\mu[i_2\cdots i_{m-1}]}\Big(-\frac{\mu[1i_2\cdots i_m]}{\mu[1i_2\cdots i_{m-1}]}\log\frac{\mu[1i_2\cdots i_m]}{\mu[1i_2\cdots i_{m-1}]}\Big)\Big)\\
\leq\;&-\mu[i_2\cdots i_m]\log\frac{\mu[i_2\cdots i_m]}{\mu[i_2\cdots i_{m-1}]},
\end{aligned}$$

where the last inequality follows from Lemma 1.3.19. Thus

$$\begin{aligned}
\mathfrak{h}_\mu(\beta,m)\;\leq\;&-\sum_{\substack{i_2,\cdots,i_m\in\{0,1\}\\ i_2\cdots i_{m-1}\neq 1^{m-2}}}\mu[i_2\cdots i_m]\log\frac{\mu[i_2\cdots i_m]}{\mu[i_2\cdots i_{m-1}]}\\
&-\mu[01^{m-2}0]\log\frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]}-\mu[01^{m-1}]\log\frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}\\
=\;&-\sum_{\substack{i_1,\cdots,i_{m-1}\in\{0,1\}\\ i_1\cdots i_{m-2}\neq 1^{m-2}}}\mu[i_1\cdots i_{m-1}]\log\frac{\mu[i_1\cdots i_{m-1}]}{\mu[i_1\cdots i_{m-2}]}\\
&-\mu[01^{m-2}0]\log\frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]}-\mu[01^{m-1}]\log\frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}\\
=\;&-\sum_{\substack{i_1,\cdots,i_{m-1}\in\{0,1\}\\ i_2\cdots i_{m-2}\neq 1^{m-3}}}\mu[i_1\cdots i_{m-1}]\log\frac{\mu[i_1\cdots i_{m-1}]}{\mu[i_1\cdots i_{m-2}]}\\
&-\mu[01^{m-3}0]\log\frac{\mu[01^{m-3}0]}{\mu[01^{m-3}]}-\mu[01^{m-2}]\log\frac{\mu[01^{m-2}]}{\mu[01^{m-3}]}\\
&-\mu[01^{m-2}0]\log\frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]}-\mu[01^{m-1}]\log\frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}.
\end{aligned}$$

For $i_2\cdots i_{m-2}\neq 1^{m-3}$ and $i_{m-1}\in\{0,1\}$, in the same way as proving (1.17), we get

$$-\mu[0i_2\cdots i_{m-1}]\log\frac{\mu[0i_2\cdots i_{m-1}]}{\mu[0i_2\cdots i_{m-2}]}-\mu[1i_2\cdots i_{m-1}]\log\frac{\mu[1i_2\cdots i_{m-1}]}{\mu[1i_2\cdots i_{m-2}]}\leq -\mu[i_2\cdots i_{m-1}]\log\frac{\mu[i_2\cdots i_{m-1}]}{\mu[i_2\cdots i_{m-2}]}.$$

Thus

$$
\begin{aligned}
\mathfrak{h}_\mu(\beta, m) &\leq -\sum_{\substack{i_2,\cdots,i_{m-1}\in\{0,1\}\\ i_2\cdots i_{m-2}\neq 1^{m-3}}} \mu[i_2\cdots i_{m-1}]\log\frac{\mu[i_2\cdots i_{m-1}]}{\mu[i_2\cdots i_{m-2}]} \\
&\quad -\mu[01^{m-3}0]\log\frac{\mu[01^{m-3}0]}{\mu[01^{m-3}]} - \mu[01^{m-2}]\log\frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
&\quad -\mu[01^{m-2}0]\log\frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}]\log\frac{\mu[01^{m-1}]}{\mu[01^{m-2}]} \\
&= -\sum_{\substack{i_1,\cdots,i_{m-2}\in\{0,1\}\\ i_1\cdots i_{m-3}\neq 1^{m-3}}} \mu[i_1\cdots i_{m-2}]\log\frac{\mu[i_1\cdots i_{m-2}]}{\mu[i_1\cdots i_{m-3}]} \\
&\quad -\mu[01^{m-3}0]\log\frac{\mu[01^{m-3}0]}{\mu[01^{m-3}]} - \mu[01^{m-2}]\log\frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
&\quad -\mu[01^{m-2}0]\log\frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}]\log\frac{\mu[01^{m-1}]}{\mu[01^{m-2}]} \\
&= -\sum_{\substack{i_1,\cdots,i_{m-2}\in\{0,1\}\\ i_2\cdots i_{m-3}\neq 1^{m-4}}} \mu[i_1\cdots i_{m-2}]\log\frac{\mu[i_1\cdots i_{m-2}]}{\mu[i_1\cdots i_{m-3}]} \\
&\quad -\mu[01^{m-4}0]\log\frac{\mu[01^{m-4}0]}{\mu[01^{m-4}]} - \mu[01^{m-3}]\log\frac{\mu[01^{m-3}]}{\mu[01^{m-4}]} \\
&\quad -\mu[01^{m-3}0]\log\frac{\mu[01^{m-3}0]}{\mu[01^{m-3}]} - \mu[01^{m-2}]\log\frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
&\quad -\mu[01^{m-2}0]\log\frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}]\log\frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}.
\end{aligned}
$$

...

Repeat the above process a finite number of times. Finally we get

$$
\begin{aligned}
\mathfrak{h}_\mu(\beta, m) &\leq -\mu[00]\log\frac{\mu[00]}{\mu[0]} - \mu[01]\log\frac{\mu[01]}{\mu[0]} \\
&\quad -\mu[010]\log\frac{\mu[010]}{\mu[01]} - \mu[011]\log\frac{\mu[011]}{\mu[01]} \\
&\quad \cdots \\
&\quad -\mu[01^{m-3}0]\log\frac{\mu[01^{m-3}0]}{\mu[01^{m-3}]} - \mu[01^{m-2}]\log\frac{\mu[01^{m-2}]}{\mu[01^{m-3}]} \\
&\quad -\mu[01^{m-2}0]\log\frac{\mu[01^{m-2}0]}{\mu[01^{m-2}]} - \mu[01^{m-1}]\log\frac{\mu[01^{m-1}]}{\mu[01^{m-2}]}.
\end{aligned}
$$

Since μ is (β, m, a)-coordinated, we have

$$\begin{cases} \mu[0] = a, & \mu[1] = 1-a, \\ \mu[00] + \mu[01] = \mu[0], & \mu[01] + \mu[11] = \mu[1], \\ \mu[010] + \mu[011] = \mu[01], & \mu[011] + \mu[111] = \mu[11], \\ \cdots, & \cdots, \\ \mu[01^{m-3}0] + \mu[01^{m-2}] = \mu[01^{m-3}], & \mu[01^{m-2}] + \mu[1^{m-1}] = \mu[1^{m-2}], \\ \mu[01^{m-2}0] + \mu[01^{m-1}] = \mu[01^{m-2}], & \mu[01^{m-1}] = \mu[1^{m-1}]. \end{cases}$$

Let $y_1 := \mu[01], y_2 := \mu[011], \cdots, y_{m-2} := \mu[01^{m-2}]$. Then we have

$$\begin{cases} \mu[0] = a, \mu[00] = a - y_1, \mu[010] = y_1 - y_2, \mu[0110] = y_2 - y_3, \cdots, \mu[01^{m-3}0] = y_{m-3} - y_{m-2}, \\ \mu[1] = 1-a, \mu[11] = 1 - a - y_1, \cdots, \mu[1^{m-1}] = 1 - a - y_1 - y_2 - \cdots - y_{m-2}, \\ \mu[01^{m-1}] = 1 - a - y_1 - y_2 - \cdots - y_{m-2}, \mu[01^{m-2}0] = y_1 + y_2 + \cdots + y_{m-3} + 2y_{m-2} + a - 1. \end{cases}$$

By a simple calculation, we get

$$\mathfrak{h}_\mu(\beta, m) \le f_a(y_1, \cdots, y_{m-2}).$$

It follows from $\mu[00], \mu[010], \cdots, \mu[01^{m-3}0], \mu[01^{m-2}0], \mu[01^{m-1}] \ge 0$ that $(y_1, \cdots, y_{m-2}) \in D_{m,a}$. Therefore

$$\mathfrak{h}_\mu(\beta, m) \le \max_{(x_1, \cdots, x_{m-2}) \in D_{m,a}} f_a(x_1, \cdots, x_{m-2}).$$

② Prove that the inequality "$\le$" in (1.16) can achieve "$=$" by some (β, m, a)-coordinated set function.

Let $(y_1, \cdots, y_{m-2}) \in D_{m,a}$ such that

$$f_a(y_1, \cdots, y_{m-2}) = \max_{(x_1, \cdots, x_{m-2}) \in D_{m,a}} f_a(x_1, \cdots, x_{m-2}).$$

Define

$$\begin{array}{rcl} \mu[0] := a, & & \mu[1] := 1-a, \\ \mu[00] := a - y_1, & \mu[01] = \mu[10] := y_1, & \mu[11] := 1 - a - y_1, \\ \mu[010] := y_1 - y_2, & \mu[011] = \mu[110] := y_2, & \mu[111] := 1 - a - y_1 - y_2, \\ \cdots, & \cdots, & \cdots, \\ \mu[01^{m-3}0] := y_{m-3} - y_{m-2}, & \mu[01^{m-2}] = \mu[1^{m-2}0] := y_{m-2}, & \mu[1^{m-1}] := 1 - a - y_1 - \cdots - y_{m-2}, \end{array}$$

$$\mu[01^{m-2}0] := y_1 + \cdots + y_{m-3} + 2y_{m-2} + a - 1, \mu[01^{m-1}] = \mu[1^{m-1}0] := 1 - a - y_1 - \cdots - y_{m-2}$$

and

$$\mu[uwv] := \frac{\mu[uw] \cdot \mu[wv]}{\mu[w]} \quad \text{for } u, v \in \{0,1\} \quad \text{and} \quad w \in \bigcup_{k=1}^{m-2} \Big(\{0,1\}^k \setminus \{1^k\} \Big) \tag{1.18}$$

where $\mu[uwv]$ is defined to be 0 if one of $\mu[w]$, $\mu[uw]$ and $\mu[wv]$ is 0. Then μ is a (β, m, a)-coordinated set function. By (1.18) and Lemma 1.3.19, it is straightforward to check that in the proof of ①, all the "$\leq$" in the upper bound estimation of $\mathfrak{h}_\mu(\beta, m)$ can take "$=$" and then

$$\mathfrak{h}_\mu(\beta, m) = f_a(y_1, \cdots, y_{m-2}) = \max_{(x_1, \cdots, x_{m-2}) \in D_{m,a}} f_a(x_1, \cdots, x_{m-2}).$$

□

Chapter 2

General beta-expansions and related digit frequencies

In this chapter, we return to general beta-expansions, not only the greedy ones. First we systematically study expansions of real numbers in multiple bases in Section 2.1. Then we return to expansions in one base and study their digit frequencies in Section 2.2. Finally we study frequency sets of univoque sequences in Section 2.3 to end this chapter.

2.1 Expansions in multiple bases

Until Neunhäuserer [98] began the study of expansions in two bases recently in 2019, all expansions studied were in one base. In this section, we begin the study of expansions in multiple bases. Note that a lot of content (including Theorem 2.1.3, Proposition 2.1.11 and Proposition 2.1.15) in this section has been generalized to expansions in multiple bases over general alphabets by Zou, Komornik and Lu recently in [130]

Recall the concept of expansion in one base first. Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $x \in \mathbb{R}$. A sequence $w = (w_i)_{i\geq 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ is called a *β-expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{w_i}{\beta^i}.$$

The following question is natural to be thought of: Given $m \in \mathbb{N}$, $\beta_0, \beta_1, \cdots, \beta_m > 1$, $x \in \mathbb{R}$ and $w = (w_i)_{i\geq 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$, in which case should we say that w is a $(\beta_0, \beta_1, \cdots, \beta_m)$-expansion of x, such that when $\beta_0, \beta_1, \cdots, \beta_m$ are taken to be the same β, we have $x = \sum_{i=1}^{\infty} \frac{w_i}{\beta^i}$? Proposition 2.1.1 may answer this question.

Let us give some notation first. For all $m \in \mathbb{N}$ and $\beta_0, \beta_1, \cdots, \beta_m > 1$, define

$$a_k := \frac{k}{\beta_k} \quad \text{and} \quad b_k := \frac{k}{\beta_k} + \frac{m}{\beta_k(\beta_m - 1)} \quad \text{for all } k \in \{0, \cdots, m\}.$$

Note that $a_0 = 0$ and $b_m = \frac{m}{\beta_m - 1}$. For all $m \in \mathbb{N}$, let

$$D_m := \Big\{(\beta_0, \cdots, \beta_m) : \beta_0, \cdots, \beta_m > 1 \text{ and } a_k < a_{k+1} \le b_k < b_{k+1} \text{ for all } k,\ 0 \le k \le m-1\Big\}.$$

It is worth to note that D_m is large enough to ensure that $(\overbrace{\beta, \cdots, \beta}^{m+1}) \in D_m$ for all $\beta \in (1, m+1]$ and $m \in \mathbb{N}$, and $(\beta_0, \beta_1) \in D_1$ for all $\beta_0, \beta_1 \in (1, 2]$.

Proposition 2.1.1. *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ and $x \in \mathbb{R}$. Then $x \in [0, \frac{m}{\beta_m - 1}]$ if and only if there exists a sequence $w \in \{0, \cdots, m\}^{\mathbb{N}}$ such that*

$$x = \sum_{i=1}^{\infty} \frac{w_i}{\beta_{w_1}\beta_{w_2}\cdots\beta_{w_i}}.$$

Thus we give the following.

Definition 2.1.2 (Expansions in multiple bases)**.** *Let $m \in \mathbb{N}$, $\beta_0, \cdots, \beta_m > 1$ and $x \in \mathbb{R}$. We say that the sequence $w \in \{0, \cdots, m\}^{\mathbb{N}}$ is a $(\beta_0, \cdots, \beta_m)$-expansion of x if*

$$x = \sum_{i=1}^{\infty} \frac{w_i}{\beta_{w_1}\beta_{w_2}\cdots\beta_{w_i}}.$$

On the one hand, it is straightforward to see that when $\beta_0, \cdots, \beta_m$ are taken to be the same β, $(\beta_0, \cdots, \beta_m)$-expansions are just β-expansions. On the other hand, we will see in Subsection 2.1.1 that many properties of β-expansions can be generalized to $(\beta_0, \cdots, \beta_m)$-expansions. This further confirms that our definition of expansions in multiple bases is reasonable.

Let σ be the *shift map* defined by $\sigma(w_1 w_2 \cdots) := w_2 w_3 \cdots$ for any sequence $(w_i)_{i \ge 1}$. Given $\beta_0, \cdots, \beta_m > 1$, for every integer $k \in \{0, \cdots, m\}$, we define the map T_k by

$$T_k(x) := \beta_k x - k \quad \text{for } x \in \mathbb{R}.$$

The main results in this section are the following theorem and corollaries, in which g^* and l^* denote the quasi-greedy and quasi-lazy expansion maps respectively (see Definition 2.1.7 (2) and (4)), and $\prec, \preceq, \succ, \succeq$ denote the lexicographic order. These results focus on determining greedy, lazy and unique expansions in multiple bases (see Definition 2.1.7 (1) and (3)), and generalize some classical results on expansions in one base in some former well known papers.

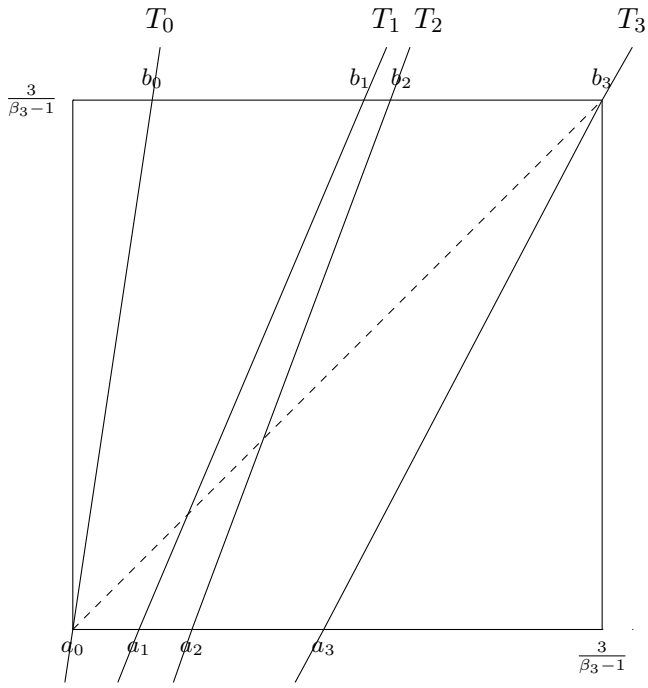


Figure 2.1: The graph of T_0, T_1, T_2 and T_3 for some $(\beta_0, \beta_1, \beta_2, \beta_3) \in D_3$.

Theorem 2.1.3. *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$, $x \in [0, \frac{m}{\beta_m - 1}]$, $w \in \{0, \cdots, m\}^{\mathbb{N}}$ be a $(\beta_0, \cdots, \beta_m)$-expansion of x and*

$$\xi_+ := \max_{0 \le k \le m-1} T_k(a_{k+1}), \quad \xi_- := \min_{0 \le k \le m-1} T_k(a_{k+1}),$$

$$\eta_+ := \max_{1 \le k \le m} T_k(b_{k-1}), \quad \eta_- := \min_{1 \le k \le m} T_k(b_{k-1}).$$

(1) ① *If w is a greedy expansion, then $\sigma^n w \prec g^*(\xi_+)$ whenever $w_n < m$.*

② *If $\sigma^n w \prec g^*(\xi_-)$ whenever $w_n < m$, then w is a greedy expansion.*

(2) ① *If w is a lazy expansion, then $\sigma^n w \succ l^*(\eta_-)$ whenever $w_n > 0$.*

② *If $\sigma^n w \succ l^*(\eta_+)$ whenever $w_n > 0$, then w is a lazy expansion.*

(3) ① *If w is a unique expansion, then*

$$\sigma^n w \prec g^*(\xi_+) \text{ whenever } w_n < m \quad \text{and} \quad \sigma^n w \succ l^*(\eta_-) \text{ whenever } w_n > 0.$$

② *If*

$$\sigma^n w \prec g^*(\xi_-) \text{ whenever } w_n < m \quad \text{and} \quad \sigma^n w \succ l^*(\eta_+) \text{ whenever } w_n > 0,$$

then w is a unique expansion.

For the case that there are at most two different bases, we get the following criteria directly from Theorem 2.1.3.

Corollary 2.1.4. *Let $\beta_0, \beta_1 \in (1,2]$, $x \in [0, \frac{1}{\beta_1 - 1}]$ and $w \in \{0,1\}^{\mathbb{N}}$ be a (β_0, β_1)-expansion of x. Then*

(1) *w is a greedy expansion if and only if $\sigma^n w \prec g^*(\frac{\beta_0}{\beta_1})$ whenever $w_n = 0$;*

(2) *w is a lazy expansion if and only if $\sigma^n w \succ l^*(\frac{\beta_1}{\beta_0(\beta_1 - 1)} - 1)$ whenever $w_n = 1$;*

(3) *w is a unique expansion if and only if*

$$\sigma^n w \prec g^*(\frac{\beta_0}{\beta_1}) \text{ whenever } w_n = 0 \quad \text{and} \quad \sigma^n w \succ l^*(\frac{\beta_1}{\beta_0(\beta_1 - 1)} - 1) \text{ whenever } w_n = 1.$$

The following corollary provide some ways to determine whether an expansion is greedy, lazy or unique by the quasi-greedy expansion of 1 and the quasi-lazy expansion of $\frac{m}{\beta_m - 1} - 1$.

Corollary 2.1.5. *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$, $x \in [0, \frac{m}{\beta_m - 1}]$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$ be a $(\beta_0, \cdots, \beta_m)$-expansion of x.*

(1) ① *Suppose $\beta_0 \le \beta_1 \le \cdots \le \beta_m$. If w is a greedy expansion, then $\sigma^n w \prec g^*(1)$ whenever $w_n < m$.*

② *Suppose $\beta_0 \ge \beta_1 \ge \cdots \ge \beta_m$. If $\sigma^n w \prec g^*(1)$ whenever $w_n < m$, then w is a greedy expansion.*

(2) ① *Suppose $\beta_0 \le \beta_1 \le \cdots \le \beta_m$. If w is a lazy expansion, then $\sigma^n w \succ l^*(\frac{m}{\beta_m - 1} - 1)$ whenever $w_n > 0$.*

② *Suppose $\beta_0 \ge \beta_1 \ge \cdots \ge \beta_m$. If $\sigma^n w \succ l^*(\frac{m}{\beta_m - 1} - 1)$ whenever $w_n > 0$, then w is a lazy expansion.*

(3) ① *Suppose $\beta_0 \le \beta_1 \le \cdots \le \beta_m$. If w is a unique expansion, then*

$$\sigma^n w \prec g^*(1) \quad \text{whenever } w_n < m \quad \text{and} \quad \sigma^n w \succ l^*(\frac{m}{\beta_m - 1} - 1) \quad \text{whenever } w_n > 0.$$

② *Suppose $\beta_0 \ge \beta_1 \ge \cdots \ge \beta_m$. If*

$$\sigma^n w \prec g^*(1) \quad \text{whenever } w_n < m \quad \text{and} \quad \sigma^n w \succ l^*(\frac{m}{\beta_m - 1} - 1) \quad \text{whenever } w_n > 0,$$

then w is a unique expansion.

Take $\beta_0, \cdots, \beta_m$ to be the same β. By Corollary 2.1.5, Proposition 2.1.18, Lemma 2.1.19 and Proposition 2.1.14, we get the following corollary, in which $\overline{k} := m - k$ for all $k \in \{0, \cdots, m\}$ and $\overline{w} := (\overline{w_i})_{i \ge 1}$ for all $w = (w_i)_{i \ge 1} \in \{0, \cdots, m\}^{\mathbb{N}}$.

Corollary 2.1.6. *Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$, $x \in [0, \frac{m}{\beta-1}]$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$ be a β-expansion of x. Then:*

(1) ① *w is a greedy expansion if and only if $\sigma^n w \prec g^*(1)$ whenever $w_n < m$;*

② *w is a lazy expansion if and only if $\sigma^n w \succ \overline{g^*(1)}$ whenever $w_n > 0$;*

③ *w is a unique expansion if and only if*

$$\sigma^n w \prec g^*(1) \text{ whenever } w_n < m \quad \text{and} \quad \sigma^n w \succ \overline{g^*(1)} \text{ whenever } w_n > 0.$$

(2) ① *$0 \le x < 1$ and w is a greedy expansion if and only if $\sigma^n w \prec g^*(1)$ for all $n \ge 0$;*

② *$\frac{m}{\beta-1} - 1 < x \le \frac{m}{\beta-1}$ and w is a lazy expansion if and only if $\sigma^n w \succ \overline{g^*(1)}$ for all $n \ge 0$;*

③ *$\frac{m}{\beta-1} - 1 < x < 1$ and w is a unique expansion if and only if*

$$\overline{g^*(1)} \prec \sigma^n w \prec g^*(1) \quad \text{for all } n \ge 0.$$

This corollary recovers some classical results. See for examples [53, Theorem 1.1], [70, Lemma 4] and [99, Theorem 3]. See also [11, Theorem 2.1] and [108, Lemma 2.11]).

Many papers on β-expansions are restricted to bases belonging to $(m, m+1]$ or expansion sequences belonging to $\{0, 1, \cdots, \lceil\beta\rceil - 1\}^{\mathbb{N}}$ (see for examples [52, 53, 82]), where $\lceil\beta\rceil$ denotes the smallest integer no less than β. Even if all $\beta_0, \cdots, \beta_m$ are taken to be the same β throughout this section, we are working under a more general framework: bases belonging to $(1, m+1]$ and expansion sequences belonging to $\{0, 1, \cdots, m\}^{\mathbb{N}}$ (for examples Corollary 2.1.6 and Proposition 2.1.18. See also [23, 55, 76]).

This section is organized as follows. In Subsection 2.1.1, we give some notation and study some basic properties of greedy, quasi-greedy, lazy and quasi-lazy expansions in multiple-bases. Subsection 2.1.2 is devoted to the proof our main results. In the last subsection, we present some further questions.

2.1.1 Greedy, quasi-greedy, lazy and quasi-lazy expansions

Let $m \in \mathbb{N}$ and $\beta_0, \cdots, \beta_m > 1$. We define the *projection* $\pi_{\beta_0,\cdots,\beta_m}$ by

$$\pi_{\beta_0,\cdots,\beta_m}(w_1 \cdots w_n) := \sum_{i=1}^{n} \frac{w_i}{\beta_{w_1}\beta_{w_2}\cdots\beta_{w_i}}$$

for $w_1 \cdots w_n \in \{0, \cdots, m\}^n$ and $n \in \mathbb{N}$, and

$$\pi_{\beta_0,\cdots,\beta_m}(w) = \pi_{\beta_0,\cdots,\beta_m}(w_1 w_2 \cdots) := \lim_{n\to\infty} \pi_{\beta_0,\cdots,\beta_m}(w_1 \cdots w_n) = \sum_{i=1}^{\infty} \frac{w_i}{\beta_{w_1}\beta_{w_2}\cdots\beta_{w_i}}$$

for $w = (w_i)_{i\geq 1} \in \{0, \cdots, m\}^{\mathbb{N}}$. When $\beta_0, \cdots, \beta_m$ are understood from the context, we usually use π instead of $\pi_{\beta_0,\cdots,\beta_m}$ for simplification.

Definition 2.1.7 (Transformations and expansions)**.** *Let $m \in \mathbb{N}$ and $(\beta_0, \cdots, \beta_m) \in D_m$.*

(1) *The greedy $(\beta_0, \cdots, \beta_m)$-transformation $G_{\beta_0,\cdots,\beta_m} : [0, \frac{m}{\beta_m - 1}] \to [0, \frac{m}{\beta_m - 1}]$ is defined by*

$$x \mapsto G_{\beta_0,\cdots,\beta_m} x := \begin{cases} T_k x & \text{if } x \in [a_k, a_{k+1}) \text{ for some } k \in \{0, \cdots, m-1\}; \\ T_m x & \text{if } x \in [a_m, b_m]. \end{cases}$$

For all $x \in [0, \frac{m}{\beta_m - 1}]$ and $n \in \mathbb{N}$, let

$$g_n(x; \beta_0, \cdots, \beta_m) := \begin{cases} k & \text{if } G^{n-1}_{\beta_0,\cdots,\beta_m} x \in [a_k, a_{k+1}) \text{ for some } k \in \{0, \cdots, m-1\}; \\ m & \text{if } G^{n-1}_{\beta_0,\cdots,\beta_m} x \in [a_m, b_m]. \end{cases}$$

We call the sequence $g(x; \beta_0, \cdots, \beta_m) := (g_n(x; \beta_0, \cdots, \beta_m))_{n\geq 1}$ the greedy $(\beta_0, \cdots, \beta_m)$-expansion of x.

(2) *The quasi-greedy $(\beta_0, \cdots, \beta_m)$-transformation $G^*_{\beta_0,\cdots,\beta_m} : [0, \frac{m}{\beta_m - 1}] \to [0, \frac{m}{\beta_m - 1}]$ is defined by*

$$x \mapsto G^*_{\beta_0,\cdots,\beta_m} x := \begin{cases} T_0 x & \text{if } x \in [0, a_1]; \\ T_k x & \text{if } x \in (a_k, a_{k+1}] \text{ for some } k \in \{1, \cdots, m-1\}; \\ T_m x & \text{if } x \in (a_m, b_m]. \end{cases}$$

For all $x \in [0, \frac{m}{\beta_m - 1}]$ and $n \in \mathbb{N}$, let

$$g^*_n(x; \beta_0, \cdots, \beta_m) := \begin{cases} 0 & \text{if } (G^*_{\beta_0,\cdots,\beta_m})^{n-1} x \in [0, a_1]; \\ k & \text{if } (G^*_{\beta_0,\cdots,\beta_m})^{n-1} x \in (a_k, a_{k+1}] \text{ for some } k \in \{1, \cdots, m-1\}; \\ m & \text{if } (G^*_{\beta_0,\cdots,\beta_m})^{n-1} x \in (a_m, b_m]. \end{cases}$$

We call the sequence $g^*(x; \beta_0, \cdots, \beta_m) := (g^*_n(x; \beta_0, \cdots, \beta_m))_{n\geq 1}$ the quasi-greedy $(\beta_0, \cdots, \beta_m)$-expansion of x.

(3) *The lazy $(\beta_0, \cdots, \beta_m)$-transformation $L_{\beta_0,\cdots,\beta_m} : [0, \frac{m}{\beta_m - 1}] \to [0, \frac{m}{\beta_m - 1}]$ is defined by*

$$x \mapsto L_{\beta_0,\cdots,\beta_m} x := \begin{cases} T_0 x & \text{if } x \in [0, b_0]; \\ T_k x & \text{if } x \in (b_{k-1}, b_k] \text{ for some } k \in \{1, \cdots, m\}. \end{cases}$$

For all $x \in [0, \frac{m}{\beta_m - 1}]$ and $n \in \mathbb{N}$, let

$$l_n(x; \beta_0, \cdots, \beta_m) := \begin{cases} 0 & \text{if } L^{n-1}_{\beta_0,\cdots,\beta_m} x \in [0, b_0]; \\ k & \text{if } L^{n-1}_{\beta_0,\cdots,\beta_m} x \in (b_{k-1}, b_k] \text{ for some } k \in \{1, \cdots, m\}. \end{cases}$$

We call the sequence $l(x;\beta_0,\cdots,\beta_m) := (l_n(x;\beta_0,\cdots,\beta_m))_{n\geq 1}$ *the lazy* $(\beta_0,\cdots,\beta_m)$*-expansion of* x.

(4) *The quasi-lazy* $(\beta_0,\cdots,\beta_m)$*-transformation* $L^*_{\beta_0,\cdots,\beta_m} : [0,\frac{m}{\beta_m-1}] \to [0,\frac{m}{\beta_m-1}]$ *is defined by*

$$x \mapsto L^*_{\beta_0,\cdots,\beta_m}x := \begin{cases} T_0x & \text{if } x\in[0,b_0); \\ T_kx & \text{if } x\in[b_{k-1},b_k) \text{ for some } k\in\{1,\cdots,m-1\}; \\ T_mx & \text{if } x\in[b_{m-1},b_m]. \end{cases}$$

For all $x\in[0,\frac{m}{\beta_m-1}]$ *and* $n\in\mathbb{N}$, *let*

$$l^*_n(x;\beta_0,\cdots,\beta_m) := \begin{cases} 0 & \text{if } (L^*_{\beta_0,\cdots,\beta_m})^{n-1}x\in[0,b_0); \\ k & \text{if } (L^*_{\beta_0,\cdots,\beta_m})^{n-1}x\in[b_{k-1},b_k) \text{ for some } k\in\{1,\cdots,m-1\}; \\ m & \text{if } (L^*_{\beta_0,\cdots,\beta_m})^{n-1}x\in[b_{m-1},b_m]. \end{cases}$$

We call the sequence $l^*(x;\beta_0,\cdots,\beta_m) := (l^*_n(x;\beta_0,\cdots,\beta_m))_{n\geq 1}$ *the quasi-lazy* $(\beta_0,\cdots,\beta_m)$*-expansion of* x.

Generally, let $\mathcal{I}_{\beta_0,\cdots,\beta_m}$ *be the set of tuples* $(I_0,\cdots,I_m)$ *which satisfy*

$$I_0 \in \Big\{[0,c_1],[0,c_1)\Big\},$$

$$I_k \in \Big\{[c_k,c_{k+1}],[c_k,c_{k+1}),(c_k,c_{k+1}],(c_k,c_{k+1})\Big\}$$

for all $k\in\{1,\cdots,m-1\}$, *and*

$$I_m \in \Big\{[c_m,\frac{m}{\beta_m-1}],(c_m,\frac{m}{\beta_m-1}]\Big\},$$

where

$$c_k\in[a_k,b_{k-1}] \quad \text{for all } k\in\{1,\cdots,m\}$$

such that $c_1<c_2<\cdots<c_m$, $I_0\cup I_1\cup\cdots\cup I_m=[0,\frac{m}{\beta_m-1}]$ *and* $I_0,I_1,\cdots,I_m$ *are all disjoint. For any* $(I_0,\cdots,I_m)\in\mathcal{I}_{\beta_0,\cdots,\beta_m}$, *we define the* $(I_0,\cdots,I_m)$*-*$(\beta_0,\cdots,\beta_m)$*-transformation* $T^{I_0,\cdots,I_m}_{\beta_0,\cdots,\beta_m} : [0,\frac{m}{\beta_m-1}] \to [0,\frac{m}{\beta_m-1}]$ *by*

$$T^{I_0,\cdots,I_m}_{\beta_0,\cdots,\beta_m}(x) := T_k(x) \quad \text{for } x\in I_k \text{ where } k\in\{0,\cdots,m\}.$$

For all $x\in[0,\frac{m}{\beta_m-1}]$ *and* $n\in\mathbb{N}$, *let*

$$t_n(x;\beta_0,\cdots,\beta_m;I_0,\cdots,I_m) := k \quad \text{if } (T^{I_0,\cdots,I_m}_{\beta_0,\cdots,\beta_m})^{n-1}x\in I_k \text{ where } k\in\{0,\cdots,m\}.$$

We call the sequence $t(x;\beta_0,\cdots,\beta_m;I_0,\cdots,I_m) := (t_n(x;\beta_0,\cdots,\beta_m;I_0,\cdots,I_m))_{n\geq 1}$ *the* $(I_0,\cdots,I_m)$*-*$(\beta_0,\cdots,\beta_m)$*-expansion of* x.

It is straightforward to see that greedy, quasi-greedy, lazy and quasi-lazy $(\beta_0, \cdots, \beta_m)$-transformations/expansions are special cases of some $(I_0, \cdots, I_m)$-$(\beta_0, \cdots, \beta_m)$-transformations/expansions. For simplification, on the one hand, if $\beta_0, \cdots, \beta_m$ are understood from the context, we use $G, G^*, L, L^*, g(x), g^*(x), l(x)$ and $l^*(x)$ instead of $G_{\beta_0,\cdots,\beta_m}$, $G^*_{\beta_0,\cdots,\beta_m}$, $L_{\beta_0,\cdots,\beta_m}$, $L^*_{\beta_0,\cdots,\beta_m}$, $g(x;\beta_0,\cdots,\beta_0)$, $g^*(x;\beta_0,\cdots,\beta_0)$, $l(x;\beta_0,\cdots,\beta_0)$ and $l^*(x;\beta_0,\cdots,\beta_0)$ respectively, and if x is also understood, we use g_n, g^*_n, l_n and l^*_n instead of $g_n(x;\beta_0,\cdots,\beta_m)$, $g^*_n(x;\beta_0,\cdots,\beta_m)$, $l_n(x;\beta_0,\cdots,\beta_m)$ and $l^*_n(x;\beta_0,\cdots,\beta_m)$ respectively for all $n \in \mathbb{N}$; on the other hand, if $\beta_0, \cdots, \beta_m$ and $I_0, \cdots, I_m$ are understood, we use T and $t(x)$ instead of $T^{I_0,\cdots,I_m}_{\beta_0,\cdots,\beta_m}$ and $t(x;\beta_0,\cdots,\beta_m;I_0,\cdots,I_m)$ respectively, and if x is also understood, we use t_n instead of $t_n(x;\beta_0,\cdots,\beta_m;I_0,\cdots,I_m)$ for all $n \in \mathbb{N}$.

For the case of a single base, greedy β-transformations and expansions were studied in Chapter 1 and also in many papers [29, 33, 37, 66, 69, 104, 105]), lazy β-transformations and expansions can be found in [42, 43, 54, 61, 77], and quasi-greedy β-expansions were introduced in [86, 90, 100].

In Proposition 2.1.9, we will see that the above definition really give $(\beta_0, \cdots, \beta_m)$-expansions coincide with Definition 2.1.2. First we prove the following useful lemma.

Lemma 2.1.8. *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ and $x \in [0, \frac{m}{\beta_m - 1}]$. If $(I_0, \cdots, I_m) \in \mathcal{I}_{\beta_0,\cdots,\beta_m}$, then for all $n \in \mathbb{N}$, we have*

$$x = \pi(t_1 \cdots t_n) + \frac{T^n x}{\beta_{t_1} \cdots \beta_{t_n}}.$$

In particular, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} x &= \pi(g_1 \cdots g_n) + \frac{G^n x}{\beta_{g_1} \cdots \beta_{g_n}} = \pi(g^*_1 \cdots g^*_n) + \frac{(G^*)^n x}{\beta_{g^*_1} \cdots \beta_{g^*_n}} \\ &= \pi(l_1 \cdots l_n) + \frac{L^n x}{\beta_{l_1} \cdots \beta_{l_n}} = \pi(l^*_1 \cdots l^*_n) + \frac{(L^*)^n x}{\beta_{l^*_1} \cdots \beta_{l^*_n}}. \end{aligned}$$

Proof. (By induction) Let $k \in \{0, \cdots, m\}$ such that $x \in I_k$. Then $t_1 = k$, $Tx = T_k x = \beta_k x - k$ and we have

$$\pi(t_1) + \frac{Tx}{\beta_{t_1}} = \frac{t_1 + Tx}{\beta_{t_1}} = \frac{\beta_k x}{\beta_k} = x.$$

Suppose that the conclusion is true for some $n \in \mathbb{N}$, we prove that it is also true for $n+1$ as follows. In fact, we have

$$\begin{aligned} \pi(t_1 \cdots t_{n+1}) + \frac{T^{n+1} x}{\beta_{t_1} \cdots \beta_{t_{n+1}}} &= \pi(t_1 \cdots t_n) + \frac{t_{n+1} + T^{n+1} x}{\beta_{t_1} \cdots \beta_{t_{n+1}}} \\ &\overset{(\star)}{=} \pi(t_1 \cdots t_n) + \frac{\beta_{t_{n+1}} T^n x}{\beta_{t_1} \cdots \beta_{t_{n+1}}} \\ &= x, \end{aligned}$$

where the last equality follows from the inductive hypothesis and $(\star)$ can be proved as follows. Let $k \in \{0, \cdots, m\}$ such that $T^n x \in I_k$. Then $t_{n+1} = k$ and

$$t_{n+1} + T^{n+1}x = t_{n+1} + T_k(T^n x) = k + (\beta_k T^n x - k) = \beta_{t_{n+1}} T^n x.$$

□

Proposition 2.1.9. *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ and $x \in [0, \frac{m}{\beta_m - 1}]$. If $(I_0, \cdots, I_m) \in \mathcal{I}_{\beta_0, \cdots, \beta_m}$, then the $(I_0, \cdots, I_m)$-$(\beta_0, \cdots, \beta_m)$-expansion of x is a $(\beta_0, \cdots, \beta_m)$-expansion of x, i.e.,*

$$x = \pi(t(x)),$$

and for all $n \in \mathbb{N}$ we have

$$T^n x = \pi(t_{n+1} t_{n+2} \cdots).$$

In particular, greedy, quasi-greedy, lazy and quasi-lazy $(\beta_0, \cdots, \beta_m)$-expansions of x are all $(\beta_0, \cdots, \beta_m)$-expansions of x, i.e.,

$$x = \pi(g(x)) = \pi(g^*(x)) = \pi(l(x)) = \pi(l^*(x)),$$

and for all $n \in \mathbb{N}$ we have

$$G^n x = \pi(g_{n+1} g_{n+2} \cdots), \quad (G^*)^n x = \pi(g^*_{n+1} g^*_{n+2} \cdots),$$

$$L^n x = \pi(l_{n+1} l_{n+2} \cdots), \quad (L^*)^n x = \pi(l^*_{n+1} l^*_{n+2} \cdots).$$

Proof. By Lemma 2.1.8 and

$$\frac{T^n x}{\beta_{t_1} \cdots \beta_{t_n}} \leq \frac{\frac{m}{\beta_m - 1}}{(\min\{\beta_0, \cdots, \beta_m\})^n} \to 0$$

as $n \to \infty$, we get $x = \lim_{n \to \infty} \pi(t_1 \cdots t_n) = \pi(t(x))$. That is,

$$x = \pi(t_1 \cdots t_n) + \frac{\pi(t_{n+1} t_{n+2} \cdots)}{\beta_{t_1} \cdots \beta_{t_n}}.$$

It follows from Lemma 2.1.8 that $T^n x = \pi(t_{n+1} t_{n+2} \cdots)$. □

Greedy, quasi-greedy, lazy and quasi-lazy expansions are not necessarily identical. A real number may have many different expansions even in one given base as mentioned at the beginning of Chapter 1.

Proof of Proposition 2.1.1. $\boxed{\Rightarrow}$ follows from Proposition 2.1.9.
$\boxed{\Leftarrow}$ Let $w \in \{0, \cdots, m\}^{\mathbb{N}}$ and $x = \pi(w)$. It suffices to prove $x \leq \frac{m}{\beta_m - 1}$ in the following. (By contradiction) We assume $x > \frac{m}{\beta_m - 1}$.

(1) Prove that for all $v \in \{0, \cdots, m\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we have

$$T_{v_n} \circ \cdots \circ T_{v_1} x > \cdots > T_{v_2} \circ T_{v_1} x > T_{v_1} x > x.$$

Let $k \in \{0, \cdots, m-1\}$, by $(\beta_0, \cdots, \beta_m) \in D_m$, we get

$$\frac{k}{\beta_k} + \frac{m}{\beta_k(\beta_m - 1)} = b_k < b_{k+1} < \cdots < b_m = \frac{m}{\beta_m - 1},$$

which implies $\frac{k}{\beta_k - 1} < \frac{m}{\beta_m - 1}$. Thus for all $y > \frac{m}{\beta_m - 1}$ and $k \in \{0, \cdots, m\}$, we have $y > \frac{k}{\beta_k - 1}$, i.e., $T_k y > y$. Then we perform the maps $T_{v_1}, \cdots, T_{v_n}$ to x one by one to get the conclusion.

(2) Let $s \in \{0, \cdots, m\}$ such that $T_s x = \min_{0 \le k \le m} T_k x$. For all $n \in \mathbb{N}$, we prove

$$T_{w_{n+1}} \circ \cdots \circ T_{w_1} x - T_{w_n} \circ \cdots \circ T_{w_1} x > T_s x - x.$$

In fact, it suffices to prove

$$T_{w_{n+1}} \circ \cdots \circ T_{w_1} x - T_{w_n} \circ \cdots \circ T_{w_1} x > T_{w_{n+1}} x - x.$$

This follows from

$$\begin{aligned} T_{w_{n+1}} \circ T_{w_n} \circ \cdots \circ T_{w_1} x - T_{w_{n+1}} x &= (\beta_{w_{n+1}} T_{w_n} \circ \cdots \circ T_{w_1} x - w_{n+1}) - (\beta_{w_{n+1}} x - w_{n+1}) \\ &= \beta_{w_{n+1}} (T_{w_n} \circ \cdots \circ T_{w_1} x - x) \\ &> T_{w_n} \circ \cdots \circ T_{w_1} x - x \end{aligned}$$

where the last inequality follows from $\beta_{w_{n+1}} > 1$ and $T_{w_n} \circ \cdots \circ T_{w_1} x - x > 0$ (by (1)).

(3) Deduce a contradiction.
On the one hand, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} T_{w_n} \circ \cdots \circ T_{w_1} x =& (T_{w_n} \circ \cdots \circ T_{w_1} x - T_{w_{n-1}} \circ \cdots \circ T_{w_1} x) \\ &+ (T_{w_{n-1}} \circ \cdots \circ T_{w_1} x - T_{w_{n-2}} \circ \cdots \circ T_{w_1} x) \\ &+ \cdots \\ &+ (T_{w_2} \circ T_{w_1} x - T_{w_1} x) \\ &+ (T_{w_1} x - x) + x \\ \overset{\text{by (2)}}{\ge}& n(T_s x - x) + x, \end{aligned}$$

where $T_s x - x > 0$ by (1). This implies $T_{w_n} \circ \cdots \circ T_{w_1} x \to \infty$ as $n \to \infty$.

On the other hand, by

$$x = \sum_{i=1}^{\infty} \frac{w_i}{\beta_{w_1} \cdots \beta_{w_i}},$$

we get

$$T_{w_1} x = \sum_{i=2}^{\infty} \frac{w_i}{\beta_{w_2} \cdots \beta_{w_i}},$$

$$T_{w_2} \circ T_{w_1} x = \sum_{i=3}^{\infty} \frac{w_i}{\beta_{w_3} \cdots \beta_{w_i}},$$

$$\cdots,$$

and then for all $n \in \mathbb{N}$,

$$\begin{aligned} T_{w_n} \circ \cdots \circ T_{w_1} x &= \sum_{i=n+1}^{\infty} \frac{w_i}{\beta_{w_{n+1}} \cdots \beta_{w_i}} \\ &\leq \sum_{i=n+1}^{\infty} \frac{m}{(\min\{\beta_0, \cdots, \beta_m\})^{i-n}} \\ &= \frac{m}{\min\{\beta_0, \cdots, \beta_m\} - 1} < \infty, \end{aligned}$$

which contradicts $T_{w_n} \circ \cdots \circ T_{w_1} x \to \infty$ as $n \to \infty$.

□

We should keep the following lemma in mind.

Lemma 2.1.10. *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$. Then $w = m^\infty$ if and only if $\pi(w) = \frac{m}{\beta_m - 1}$.*

Proof. $\boxed{\Rightarrow}$ is obvious.
$\boxed{\Leftarrow}$ (By contradiction) Suppose $w \neq m^\infty$ and

$$\sum_{i=1}^{\infty} \frac{w_i}{\beta_{w_1} \cdots \beta_{w_i}} = \frac{m}{\beta_m - 1}. \tag{2.1}$$

Then there exists $k \in \mathbb{N}$ such that $w_1 \cdots w_{k-1} = m^{k-1}$ and $w_k < m$. By applying T_m^{k-1} to (2.1), we get

$$\frac{w_k}{\beta_{w_k}} + \sum_{i=k+1}^{\infty} \frac{w_i}{\beta_{w_k} \cdots \beta_{w_i}} = \frac{m}{\beta_m - 1}.$$

It follows from applying T_{w_k} to the above equality that

$$\sum_{i=1}^{\infty} \frac{w_{k+i}}{\beta_{w_{k+1}} \cdots \beta_{w_{k+i}}} = \frac{m\beta_{w_k}}{\beta_m - 1} - w_k. \tag{2.2}$$

On the one hand, by Proposition 2.1.1 we know

$$\sum_{i=1}^{\infty} \frac{w_{k+i}}{\beta_{w_{k+1}} \cdots \beta_{w_{k+i}}} \leq \frac{m}{\beta_m - 1}. \tag{2.3}$$

On the other hand, by $(\beta_0, \cdots, \beta_m) \in D_m$ and $w_k < m$, we get

$$\frac{w_k}{\beta_{w_k}} + \frac{m}{\beta_{w_k}(\beta_m - 1)} = b_{w_k} < b_{w_k+1} < \cdots < b_m = \frac{m}{\beta_m - 1},$$

which implies

$$\frac{m\beta_{w_k}}{\beta_m - 1} - w_k > \frac{m}{\beta_m - 1}.$$

This contradicts (2.2) and (2.3). □

The following useful criteria generalize [61, Lemma 1].

Proposition 2.1.11 (Basic criteria of greedy, quasi-greedy, lazy and quasi-lazy expansions)**.** *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$, $x \in [0, \frac{m}{\beta_m - 1}]$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$ be a $(\beta_0, \cdots, \beta_m)$-expansion of x.*

(1) *w is the greedy expansion if and only if*

$$\pi(w_n w_{n+1} \cdots) < a_{w_n+1} \quad \textit{whenever } w_n < m.$$

(2) *When $x \neq 0$, w is the quasi-greedy expansion if and only if it does not end with 0^∞ and*

$$\pi(w_n w_{n+1} \cdots) \leq a_{w_n+1} \quad \textit{whenever } w_n < m.$$

(3) *w is the lazy expansion if and only if*

$$\pi(w_n w_{n+1} \cdots) > b_{w_n-1} \quad \textit{whenever } w_n > 0.$$

(4) *When $x \neq \frac{m}{\beta_m - 1}$, w is the quasi-lazy expansion if and only if it does not end with m^∞ and*

$$\pi(w_n w_{n+1} \cdots) \geq b_{w_n-1} \quad \textit{whenever } w_n > 0.$$

Proof. (1) $\boxed{\Rightarrow}$ Suppose that w is the greedy $(\beta_0, \cdots, \beta_m)$-expansion of x, i.e., $(w_i)_{i\geq 1} = (g_i)_{i\geq 1}$, and suppose $w_n < m$. By $g_n = w_n$ and the definition of g_n, we get $G^{n-1}x < a_{w_n+1}$. It follows from Proposition 2.1.9 that $\pi(g_n g_{n+1} \cdots) < a_{w_n+1}$. Thus $\pi(w_n w_{n+1} \cdots) < a_{w_n+1}$.

$\boxed{\Leftarrow}$ We prove $(w_i)_{i\geq 1} = (g_i)_{i\geq 1}$ by induction. Recall that

$$g_1 := \begin{cases} k & \text{if } x \in [a_k, a_{k+1}) \text{ for some } k \in \{0, \cdots, m-1\} \\ m & \text{if } x \in [a_m, b_m] \end{cases}$$

and $(w_i)_{i\geq 1}$ is a $(\beta_0, \cdots, \beta_m)$-expansion of x, which implies $x \geq a_{w_1}$.

i) If $w_1 = m$, then $x \geq a_m$, which implies $g_1 = m = w_1$.

ii) If $w_1 < m$, by condition $\pi(w_1 w_2 \cdots) < a_{w_1+1}$ we get $x < a_{w_1+1}$. It follows from $x \geq a_{w_1}$ that $g_1 = w_1$.

Suppose $w_1 \cdots w_{n-1} = g_1 \cdots g_{n-1}$ for some $n \geq 2$. We need to prove $w_n = g_n$ in the following. Recall

$$g_n := \begin{cases} k & \text{if } G^{n-1}x \in [a_k, a_{k+1}) \text{ for some } k \in \{0, \cdots, m-1\}; \\ m & \text{if } G^{n-1}x \in [a_m, b_m]. \end{cases}$$

Since the fact that $(w_i)_{i\geq 1}$ is a $(\beta_0, \cdots, \beta_m)$-expansion of x implies

$$x = \pi(w_1 \cdots w_{n-1}) + \frac{\pi(w_n w_{n+1} \cdots)}{\beta_{w_1} \cdots \beta_{w_{n-1}}},$$

by Lemma 2.1.8 we know $G^{n-1}x = \pi(w_n w_{n+1} \cdots)$. This implies $G^{n-1}x \geq a_{w_n}$.

i) If $w_n = m$, then $G^{n-1}x \geq a_m$, which implies $g_n = m = w_n$.

ii) If $w_n < m$, by condition $\pi(w_n w_{n+1} \cdots) < a_{w_n+1}$ we get $G^{n-1}x < a_{w_n+1}$. It follows from $G^{n-1}x \geq a_{w_n}$ that $g_n = w_n$.

(2) $\boxed{\Rightarrow}$ Suppose that w is the quasi-greedy $(\beta_0, \cdots, \beta_m)$-expansion of x, i.e., $(w_i)_{i\geq 1} = (g_i^*)_{i\geq 1}$.

i) Prove that w does not end with 0^∞.
(By contradiction) Assume that there exists $n \in \mathbb{N}$ such that $w_{n+1}w_{n+2}\cdots = 0^\infty$. By Proposition 2.1.9, we get $(G^*)^n x = \pi(0^\infty) = 0$. It follows from the definition of G^* that $(G^*)^{n-1}x = 0$, $(G^*)^{n-2}x = 0$, $\cdots$, $G^*x = 0$ and $x = 0$, which contradicts $x \neq 0$.

ii) Suppose $w_n < m$. Similarly to (1) $\boxed{\Rightarrow}$, we get $\pi(w_n w_{n+1} \cdots) \leq a_{w_n+1}$.

$\boxed{\Leftarrow}$ follows in a way similar to (1) $\boxed{\Leftarrow}$.
(3) and (4) follow in a way similar to (1) and (2) noting Lemma 2.1.10. □

Proposition 2.1.12 (Lexicographic order on greedy, quasi-greedy, lazy and quasi-lazy expansions)**.** *Let* $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ *and* $x \in [0, \frac{m}{\beta_m - 1}]$.

(1) *Among all the* $(\beta_0, \cdots, \beta_m)$*-expansions of* x*, the greedy expansion and the lazy expansion are maximal and minimal respectively in lexicographic order.*

(2) *Among all the* $(\beta_0, \cdots, \beta_m)$*-expansions of* x *which do not end with* 0^∞*, the quasi-greedy expansion is maximal in lexicographic order.*

(3) *Among all the* $(\beta_0, \cdots, \beta_m)$*-expansions of* x *which do not end with* m^∞*, the quasi-lazy expansion is minimal in lexicographic order.*

Proof. (1) Let $v \in \{0, \cdots, m\}^{\mathbb{N}}$ be a $(\beta_0, \cdots, \beta_m)$-expansion of x.

① Prove $v \preceq g(x)$.
(By contradiction) Assume $v \succ g(x)$. Then there exists $n \in \mathbb{N}$ such that $v_1 \cdots v_{n-1} = g_1 \cdots g_{n-1}$ and $v_n > g_n$. Since Proposition 2.1.11 (1) implies $\pi(g_n g_{n+1} \cdots) < a_{g_n+1}$ and $(\beta_0, \cdots, \beta_m) \in D_m$ implies $a_{g_n+1} \leq a_{g_n+2} \leq \cdots \leq a_{v_n} = \frac{v_n}{\beta_{v_n}}$, we get $\pi(g_n g_{n+1} \cdots) < \frac{v_n}{\beta_{v_n}}$ and then

$$\begin{aligned} x = \pi(g(x)) &= \pi(g_1 \cdots g_{n-1}) + \frac{\pi(g_n g_{n+1} \cdots)}{\beta_{g_1} \cdots \beta_{g_{n-1}}} \\ &< \pi(v_1 \cdots v_{n-1}) + \frac{v_n}{\beta_{v_1} \cdots \beta_{v_{n-1}} \beta_{v_n}} \\ &= \pi(v_1 \cdots v_n) \\ &\leq \pi(v). \end{aligned}$$

This contradicts $x = \pi(v)$.

② We can prove $v \succeq l(x)$ in a way similar to ① noting that Proposition 2.1.1 implies $\frac{m}{\beta_m - 1} \geq \pi(v_{n+1} v_{n+2} \cdots)$.

(2) and (3) follow in the same way as (1), noting that v does not end with 0^∞ implies $\pi(v_1 \cdots v_n) < \pi(v)$, and v does not end with m^∞ implies $\frac{m}{\beta_m - 1} > \pi(v_{n+1} v_{n+2} \cdots)$ by Proposition 2.1.1 and Lemma 2.1.8 for all $n \in \mathbb{N}$. □

The following definition on admissibility is a natural generalization of Definition 1.1.1 (2) (see also [91, Definition 2.1]).

Definition 2.1.13 (Admissibility)**.** *Let* $m \in \mathbb{N}$ *and* $(\beta_0, \cdots, \beta_m) \in D_m$*. For fixed* $(I_0, \cdots, I_m) \in \mathcal{I}_{\beta_0, \cdots, \beta_m}$*, a sequence* $w \in \{0, \cdots, m\}^{\mathbb{N}}$ *is called* $(I_0, \cdots, I_m)$*-admissible if there exists* $x \in [0, \frac{m}{\beta_m - 1}]$ *such that* $w = t(x)$*. We let* $\mathcal{T} = \mathcal{T}(\beta_0, \cdots, \beta_m; I_0, \cdots, I_m)$ *denote the set of* $(I_0, \cdots, I_m)$*-admissible sequences. In particular, a sequence* $w \in \{0, \cdots, m\}^{\mathbb{N}}$ *is called greedy, quasi-greedy, lazy and quasi-lazy (admissible) if there exists* $x \in [0, \frac{m}{\beta_m - 1}]$ *such that* $w = g(x)$*,* $g^*(x)$*,* $l(x)$ *and* $l^*(x)$ *respectively. The sets of greedy, quasi-greedy, lazy and quasi-lazy sequences are denoted respectively by* $\mathcal{G} = \mathcal{G}(\beta_0, \cdots, \beta_m)$*,* $\mathcal{G}^* = \mathcal{G}^*(\beta_0, \cdots, \beta_m)$*,* $\mathcal{L} = \mathcal{L}(\beta_0, \cdots, \beta_m)$ *and* $\mathcal{L}^* = \mathcal{L}^*(\beta_0, \cdots, \beta_m)$*.*

Proposition 2.1.14 (Commutativity)**.** *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ and $(I_0, \cdots, I_m) \in \mathcal{I}_{\beta_0,\cdots,\beta_m}$. Then*

(1) $\pi \circ \sigma(w) = T \circ \pi(w)$ *for all* $w \in \mathcal{T}$ *and* $t \circ T(x) = \sigma \circ t(x)$ *for all* $x \in [0, \frac{m}{\beta_m - 1}]$*;*

(2) $\sigma(\mathcal{T}) = \mathcal{T}$ *and* $T([0, \frac{m}{\beta_m-1}]) = [0, \frac{m}{\beta_m-1}]$*;*

(3) $t \circ \pi(w) = w$ *for all* $w \in \mathcal{T}$ *and* $\pi \circ t(x) = x$ *for all* $x \in [0, \frac{m}{\beta_m-1}]$*;*

(4) $\pi|_{\mathcal{T}} : \mathcal{T} \to [0, \frac{m}{\beta_m-1}]$ *and* $t : [0, \frac{m}{\beta_m-1}] \to \mathcal{T}$ *are both increasing bijections.*

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\sigma"] \arrow[d, shift right, "\pi"'] & \mathcal{T} \arrow[d, shift right, "\pi"'] \\
{[0, \frac{m}{\beta_m-1}]} \arrow[r, "T"] \arrow[u, shift right, "t"'] & {[0, \frac{m}{\beta_m-1}]} \arrow[u, shift right, "t"']
\end{tikzcd}$$

In particular, the above properties hold for the greedy, quasi-greedy, lazy and quasi-lazy cases.

Proof. (1) ① Let $w \in \mathcal{T}$. We need to prove $\pi \circ \sigma(w) = T \circ \pi(w)$. In fact, there exists $x \in [0, \frac{m}{\beta_m-1}]$ such that $w = t(x)$, and then $\pi(w) = x$ by Proposition 2.1.9. On the one hand,

$$\pi \circ \sigma(w) = \pi(w_2 w_3 \cdots) = \sum_{i=2}^{\infty} \frac{w_i}{\beta_{w_2} \cdots \beta_{w_i}}.$$

On the other hand,

$$T \circ \pi(w) = Tx \overset{(\star)}{=} T_{w_1} x = \beta_{w_1} x - w_1 = \beta_{w_1} \sum_{i=1}^{\infty} \frac{w_i}{\beta_{w_1} \cdots \beta_{w_i}} - w_1 = \sum_{i=2}^{\infty} \frac{w_i}{\beta_{w_2} \cdots \beta_{w_i}},$$

where $(\star)$ follows from the fact that $t_1(x) = w_1$ implies $x \in I_{w_1}$.
② Let $x \in [0, \frac{m}{\beta_m-1}]$. We need to prove $t \circ T(x) = \sigma \circ t(x)$. In fact, it follows immediately from the definition of t that $t_n(Tx) = t_1(T^{n-1}(Tx)) = t_1(T^n x) = t_{n+1}(x)$ for all $n \in \mathbb{N}$.
(2) $T([0, \frac{m}{\beta_m-1}]) = [0, \frac{m}{\beta_m-1}]$ follows from the definition of T. We prove $\sigma(\mathcal{T}) = \mathcal{T}$ as follows.
$\boxed{\subset}$ Let $w \in \mathcal{T}$. Then there exists $x \in [0, \frac{m}{\beta_m-1}]$ such that $w = t(x)$. Thus $\sigma w = \sigma \circ t(x) \overset{\text{by (1)}}{=\!=} t \circ T(x) \in \mathcal{T}$.
$\boxed{\supset}$ Let $w \in \mathcal{T}$. Then there exists $y \in [0, \frac{m}{\beta_m-1}]$ such that $w = t(y)$ and there exists $x \in [0, \frac{m}{\beta_m-1}]$ such that $y = Tx$. It follows from $w = t(y) = t(Tx) \overset{\text{by (1)}}{=\!=} \sigma(t(x))$ and $t(x) \in \mathcal{T}$ that $w \in \sigma(\mathcal{T})$.
(3) ① For any $w \in \mathcal{T}$, there exists $x \in [0, \frac{m}{\beta_m-1}]$ such that $w = t(x)$ and $\pi(w) = x$, which implies $t \circ \pi(w) = t(x) = w$.
② For any $x \in [0, \frac{m}{\beta_m-1}]$, $\pi(t(x)) = x$ follows from Proposition 2.1.9.
(4) By (3), it suffices to prove that $\pi|_{\mathcal{T}}$ is increasing.

Let $w, v \in \mathcal{T}$ such that $w \prec v$. Then there exists $n \geq 0$ such that $w_1 \cdots w_n = v_1 \cdots v_n$ and $w_{n+1} < v_{n+1}$. Let $x, y \in [0, \frac{m}{\beta_m - 1}]$ such that $w = t(x)$ and $v = t(y)$. We need to prove $x < y$. In fact, by Lemma 2.1.8 we get

$$x = \pi(w_1 \cdots w_n) + \frac{T^n x}{\beta_{w_1} \cdots \beta_{w_n}} \quad \text{and} \quad y = \pi(v_1 \cdots v_n) + \frac{T^n y}{\beta_{v_1} \cdots \beta_{v_n}}. \tag{2.4}$$

Since $t_{n+1}(x) = w_{n+1}$ and $t_{n+1}(y) = v_{n+1}$ imply $T^n x \in I_{w_{n+1}}$ and $T^n y \in I_{v_{n+1}}$, by $w_{n+1} < v_{n+1}$ we get $T^n x < T^n y$. It follows from (2.4) and $w_1 \cdots w_n = v_1 \cdots v_n$ that $x < y$. □

The following is a generalization of [22, Proposition 3.4].

Proposition 2.1.15 (Relations between greedy/lazy and quasi-greedy/quasi-lazy expansions)**.** *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ and $x \in [0, \frac{m}{\beta_m - 1}]$.*
(1) *Suppose $x \neq 0$.*

① *$g(x)$ does not end with 0^∞ if and only if $g^*(x) = g(x)$.*

② *If $g(x)$ ends with 0^∞, then*

$$\begin{aligned} g^*(x) &= g_1(x) \cdots g_{n-1}(x) g^*(a_{g_n(x)}) \\ &= g_1(x) \cdots g_{n-1}(x)(g_n(x) - 1) g^*(T_{g_n(x)-1}(a_{g_n(x)})) \end{aligned}$$

where n is the greatest integer such that $g_n(x) > 0$.

(2) *Suppose $x \neq \frac{m}{\beta_m - 1}$.*

① *$l(x)$ does not end with m^∞ if and only if $l^*(x) = l(x)$.*

② *If $l(x)$ ends with m^∞, then*

$$\begin{aligned} l^*(x) &= l_1(x) \cdots l_{n-1}(x) l^*(b_{l_n(x)}) \\ &= l_1(x) \cdots l_{n-1}(x)(l_n(x) + 1) l^*(T_{l_n(x)+1}(b_{l_n(x)})) \end{aligned}$$

where n is the greatest integer such that $l_n(x) < m$.

Proof. (1) ① $\boxed{\Leftarrow}$ follows from Proposition 2.1.11 (2).
$\boxed{\Rightarrow}$ (By contradiction) Assume $(g_i)_{i \geq 1} \neq (g_i^*)_{i \geq 1}$. Then there exists $n \in \mathbb{N}$ such that $g_1 \cdots g_{n-1} = g_1^* \cdots g_{n-1}^*$ and $g_n \neq g_n^*$. Recall the definitions of g, g^*, G and G^*. By $x \neq 0$ and $g_1 = g_1^*$, we get $x \notin \{a_0, \cdots, a_m\}$, and then $Gx = G^* x \neq 0$. By $g_2 = g_2^*$, we get $Gx = G^* x \notin \{a_0, \cdots, a_m\}$, and then $G^2 x = (G^*)^2 x \neq 0. \cdots$ By repeating the above process, we get $G^{n-1} x = (G^*)^{n-1} x \neq 0$. It follows from

$$G^{n-1} x \in \begin{cases} [a_{g_n}, a_{g_n+1}) & \text{if } 0 \leq g_n \leq m-1, \\ [a_m, \frac{m}{\beta_m - 1}] & \text{if } g_n = m, \end{cases}$$

and $g_n \neq g_n^*$ that $G^{n-1}x = a_{g_n}$ This implies $G^n x = 0$, and then for all $i \geq n$, $G^i x = 0$. Thus $g_{n+1}g_{n+2}\cdots = 0^\infty$, which contradicts that $(g_i)_{i\geq 1}$ does not end with 0^∞.
② Suppose that $g(x)$ ends with 0^∞ and n is the greatest integer such that $g_n > 0$. We need to consider the following i), ii) and iii).

i) Prove $g_1^* \cdots g_{n-1}^* = g_1 \cdots g_{n-1}$.
(By contradiction) Assume $g_1^* \cdots g_{n-1}^* \neq g_1 \cdots g_{n-1}$. Then there exists $k \in \{1, \cdots, n-1\}$ such that $g_1^* \cdots g_{k-1}^* = g_1 \cdots g_{k-1}$ but $g_k^* \neq g_k$. By Lemma 2.1.8 we get $(G^*)^{k-1}x = G^{k-1}x$. Since $g_k^* \neq g_k$, there must exist $j \in \{1, \cdots, m\}$ such that $G^{k-1}x = a_j$. This implies $G^k x = 0$, and then for all $i \geq k$ we have $G^i x = 0$. Thus $g_{k+1}g_{k+2}\cdots = 0^\infty$, which contradicts $g_n > 0$.

ii) Prove $g_n^* g_{n+1}^* \cdots = g^*(a_{g_n})$. In fact, we have

$$\sigma^{n-1}(g^*(x)) \overset{(\star)}{=} g^*((G^*)^{n-1}x) \overset{(\star\star)}{=} g^*(a_{g_n}),$$

where $(\star)$ follows from Proposition 2.1.14 (1), and $(\star\star)$ follows from $(G^*)^{n-1}x = a_{g_n}$, which is a consequence of i), Lemma 2.1.8 and

$$x = \pi(g_1 \cdots g_n) = \pi(g_1 \cdots g_{n-1}) + \frac{a_{g_n}}{\beta_{g_1} \cdots \beta_{g_{n-1}}}.$$

iii) Prove $g^*(a_{g_n}) = (g_n - 1)g^*(T_{g_n-1}(a_{g_n}))$.
In fact, on the one hand, $g_1^*(a_{g_n}) = g_n - 1$ follows directly from the definition of g_1^*. On the other hand, we have

$$\sigma(g^*(a_{g_n})) \overset{(\star)}{=} g^*(G^*(a_{g_n})) \overset{(\star\star)}{=} g^*(T_{g_n-1}(a_{g_n})),$$

where $(\star)$ follows from Proposition 2.1.14 (1), and $(\star\star)$ follows from $g_n > 0$ and the definition of G^*.

(2) follows in a way similar to (1). □

In the proof of our main results, we need the following.

Proposition 2.1.16 (Interactive increase)**.** *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$ and $x, y \in [0, \frac{m}{\beta_m - 1}]$.*

(1) *Let $(I_0, \cdots, I_m), (I_0', \cdots, I_m') \in \mathcal{I}_{\beta_0, \cdots, \beta_m}$ such that for all $k \in \{0, \cdots, m\}$, the intervals I_k and I_k' are at most different at the end points (i.e., they have the same closure), $t(x)$ be the $(I_0, \cdots, I_m)$-$(\beta_0, \cdots, \beta_m)$-expansion of x and $t'(y)$ be the $(I_0', \cdots, I_m')$-$(\beta_0, \cdots, \beta_m)$-expansion of y. If $x < y$, then $t(x) \prec t'(y)$.*

(2) *In particular, if $x < y$, we have $g(x) \prec g^*(y)$ and $l^*(x) \prec l(y)$.*

Proof. We only need to prove (1). Suppose $0 \le x < y \le \frac{m}{\beta_m - 1}$. Since $t(x) = t'(y)$ will imply $x = \pi(t(x)) = \pi(t'(y)) = y$ which contradicts $x < y$, we must have $t(x) \ne t'(y)$. Thus there exists $n \ge 0$ such that $t_1(x) \cdots t_n(x) = t'_1(y) \cdots t'_n(y)$ and $t_{n+1}(x) \ne t'_{n+1}(y)$. It suffices to prove $t_{n+1}(x) < t'_{n+1}(y)$ by contradiction.

In fact, by $x < y$ and Lemma 2.1.8, we get $T^n x < (T')^n y$, where T is the $(I_0, \cdots, I_m)$-$(\beta_0, \cdots, \beta_m)$-transformation and T' is the $(I'_0, \cdots, I'_m)$-$(\beta_0, \cdots, \beta_m)$-transformation. If $t_{n+1}(x) > t'_{n+1}(y)$, by $T^n x \in I_{t_{n+1}(x)}$ and $(T')^n y \in I'_{t'_{n+1}(y)}$ we get

$$T^n x \ge \inf I_{t_{n+1}(x)} \ge \sup I'_{t'_{n+1}(y)} \ge (T')^n y,$$

which contradicts $T^n x < (T')^n y$. □

Given $x \in [0, \frac{m}{\beta_m - 1}]$, let

$$\Sigma_{\beta_0, \cdots, \beta_m}(x) := \Big\{ (w_i)_{i \ge 1} \in \{0, \cdots, m\}^{\mathbb{N}} : (w_i)_{i \ge 1} \text{ is a } (\beta_0, \cdots, \beta_m)\text{-expansion of } x \Big\}$$

and

$$\Omega_{\beta_0, \cdots, \beta_m}(x) := \Big\{ (S_i)_{i \ge 1} \in \{T_0, \cdots, T_m\}^{\mathbb{N}} : (S_n \circ \cdots \circ S_1)(x) \in \Big[0, \frac{m}{\beta_m - 1}\Big] \text{ for all } n \in \mathbb{N} \Big\}.$$

As a generalization of [24, Lemma 3.1] and [25, Lemma 2.1] (see also [23]), the following is a dynamical interpretation of $(\beta_0, \cdots, \beta_m)$-expansions.

Proposition 2.1.17 (Dynamical interpretation)**.** *Let $m \in \mathbb{N}$ and $(\beta_0, \cdots, \beta_m) \in D_m$. For all $x \in [0, \frac{m}{\beta_m - 1}]$, the map which sends $(w_i)_{i \ge 1}$ to $(T_{w_i})_{i \ge 1}$ is a bijection from $\Sigma_{\beta_0, \cdots, \beta_m}(x)$ to $\Omega_{\beta_0, \cdots, \beta_m}(x)$.*

Proof. (1) Prove that the mentioned map is well-defined.
Let $(w_i)_{i \ge 1} \in \{0, \cdots, m\}^{\mathbb{N}}$ be a $(\beta_0, \cdots, \beta_m)$-expansion of x and $n \in \mathbb{N}$. It suffices to prove $T_{w_n} \circ \cdots \circ T_{w_1} x \in [0, \frac{m}{\beta_m - 1}]$. In fact, by a simple calculation as in (3) in the proof of Proposition 2.1.1, we get

$$T_{w_n} \circ \cdots \circ T_{w_1} x = \sum_{i=n+1}^{\infty} \frac{w_i}{\beta_{w_{n+1}} \cdots \beta_{w_i}}.$$

Thus

$$T_{w_n} \circ \cdots \circ T_{w_1} x = \sum_{i=1}^{\infty} \frac{w_{n+i}}{\beta_{w_{n+1}} \cdots \beta_{w_{n+i}}} = \pi(w_{n+1} w_{n+2} \cdots) \in [0, \frac{m}{\beta_m - 1}]$$

by Proposition 2.1.1.
(2) The mentioned map is obviously injective. We prove that it is surjective as follows.

Let $(w_i)_{i\geq 1} \in \{0, \cdots, m\}^{\mathbb{N}}$ such that $T_{w_n} \circ \cdots \circ T_{w_1} x \in [0, \frac{m}{\beta_m - 1}]$ for all $n \in \mathbb{N}$. By

$$0 \leq T_{w_n} \circ \cdots \circ T_{w_1} x \leq \frac{m}{\beta_m - 1},$$

we get

$$\frac{w_n}{\beta_{w_n}} \leq T_{w_{n-1}} \circ \cdots \circ T_{w_1} x \leq \frac{w_n}{\beta_{w_n}} + \frac{m}{\beta_{w_n}(\beta_m - 1)},$$

$$\frac{w_{n-1}}{\beta_{w_{n-1}}} + \frac{w_n}{\beta_{w_{n-1}}\beta_{w_n}} \leq T_{w_{n-2}} \circ \cdots \circ T_{w_1} x \leq \frac{w_{n-1}}{\beta_{w_{n-1}}} + \frac{w_n}{\beta_{w_{n-1}}\beta_{w_n}} + \frac{m}{\beta_{w_{n-1}}\beta_{w_n}(\beta_m - 1)},$$

$$\cdots,$$

$$\frac{w_1}{\beta_{w_1}} + \frac{w_2}{\beta_{w_1}\beta_{w_2}} + \cdots + \frac{w_n}{\beta_{w_1}\cdots\beta_{w_n}} \leq x \leq \frac{w_1}{\beta_{w_1}} + \frac{w_2}{\beta_{w_1}\beta_{w_2}} + \cdots + \frac{w_n}{\beta_{w_1}\cdots\beta_{w_n}} + \frac{m}{\beta_{w_1}\cdots\beta_{w_n}(\beta_m - 1)},$$

which implies

$$\pi(w_1 \cdots w_n) \leq x \leq \pi(w_1 \cdots w_n) + \frac{m}{(\beta_m - 1)(\min\{\beta_0, \cdots, \beta_m\})^n}$$

for all $n \in \mathbb{N}$. Let $n \to \infty$, we get $x = \pi(w_1 w_2 \cdots)$. Thus $(w_i)_{i\geq 1} \in \Sigma_{\beta_0, \cdots, \beta_m}(x)$. □

The following proposition on expansions in one base, which will be used in the proof of Corollary 2.1.6, implies that w is lazy if and only if $\overline{w}$ is greedy (recall Definition 2.1.13) for all $w = (w_i)_{i\geq 1} \in \{0, \cdots, m\}^{\mathbb{N}}$, where $\overline{w} := (\overline{w_i})_{i\geq 1}$ and $\overline{k} := m - k$ for all $k \in \{0, \cdots, m\}$. By Proposition 2.1.12 (1), we recover [45, Theorem 2.1] and [78, Lemma 1].

Proposition 2.1.18 (Reflection principle in one base)**.** *Let $m \in \mathbb{N}$ and $\beta \in (1, m+1]$. For all $x \in [0, \frac{m}{\beta-1}]$, we have*

$$l\Big(\frac{m}{\beta - 1} - x\Big) = \overline{g(x)} \quad \textit{and} \quad l^*\Big(\frac{m}{\beta - 1} - x\Big) = \overline{g^*(x)}.$$

Proof. (1) Prove $l(\frac{m}{\beta-1} - x) = \overline{g(x)}$. Let $w = g(x)$. By Proposition 2.1.11 (1) we get

$$\pi(w_n w_{n+1} \cdots) < a_{w_n+1} \quad \text{whenever } w_n < m.$$

It follows from $\pi(w_n w_{n+1} \cdots) + \pi(\overline{w}_n \overline{w}_{n+1} \cdots) = \frac{m}{\beta-1}$ and $a_{w_n+1} + b_{w_n-1} = \frac{m}{\beta-1}$ that

$$\pi(\overline{w}_n \overline{w}_{n+1} \cdots) > b_{w_n-1} \quad \text{whenever } \overline{w}_n > 0. \tag{2.5}$$

Since $w = g(x)$ implies $\pi(\overline{w}) = \frac{m}{\beta-1} - x$, by Proposition 2.1.11 (3) and (2.5) we get $\overline{w} = l(\frac{m}{\beta-1} - x)$.
(2) $l^*(\frac{m}{\beta-1} - x) = \overline{g^*(x)}$ follows in a way similar to (1) by applying Proposition 2.1.11 (2) and (4). □

2.1.2 Proofs of the main results

First we give the following lemma, which is essentially stronger than Theorem 2.1.3 (1) ①, (2) ① and (3) ①.

Lemma 2.1.19. *Let $m \in \mathbb{N}$, $(\beta_0, \cdots, \beta_m) \in D_m$, $x \in [0, \frac{m}{\beta_m - 1}]$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$ be a $(\beta_0, \cdots, \beta_m)$-expansion of x.*

(1) *If w is the greedy expansion and $w \neq m^\infty$, then*

$$\sigma^n w \prec g^*(\xi_+) \quad \textit{for all } n \geq p,$$

where $p := \min\{i \geq 0 : G^i x < \xi_+\}$ exists, and $\xi_+ := \max_{0 \leq k \leq m-1} T_k(a_{k+1})$.

(2) *If w is the lazy expansion and $w \neq 0^\infty$, then*

$$\sigma^n w \succ l^*(\eta_-) \quad \textit{for all } n \geq q,$$

where $q := \min\{i \geq 0 : L^i x > \eta_-\}$ exists, and $\eta_- := \min_{1 \leq k \leq m} T_k(b_{k-1})$.

Proof. (1) By $(\beta_0, \cdots, \beta_m) \in D_m$, we get

$$a_k < a_{k+1} \leq b_k$$

for all $k \in \{0, \cdots, m-1\}$. This implies $0 < \xi_+ \leq \frac{m}{\beta_m - 1}$.

① Prove that there exists $i \geq 0$ such that $G^i x < \xi_+$.
(By contradiction) Assume $G^i x \geq \xi_+$ for all $i \geq 0$. Let r be the greatest integer in $\{0, \cdots, m\}$ such that $a_r \leq \xi_+$ and

$$c = c(x) := \begin{cases} x - \beta_m x + m & \text{if } r = m; \\ \min\{x - \beta_m x + m, a_{r+1} - \xi_+\} & \text{if } r \leq m-1. \end{cases}$$

It follows from $w \neq m^\infty$ (which implies $x < \frac{m}{\beta_m - 1}$ by Lemma 2.1.10) and the definition of r that $c > 0$.

i) Prove that for all $y \in [\xi_+, x]$, we have $y - Gy \geq c$.
In fact, if $y \geq a_m$, then $y - Gy = y - \beta_m y + m \geq x - \beta_m x + m \geq c$. We only need to consider $\xi_+ \leq y < a_m$ in the following. By $\xi_+ < a_m$, we know $r \leq m-1$ and

$$[\xi_+, a_m) \subset [a_r, a_{r+1}) \cup [a_{r+1}, a_{r+2}) \cup \cdots \cup [a_{m-1}, a_m).$$

There exists $k \in \{r, r+1, \cdots, m-1\}$ such that $y \in [a_k, a_{k+1})$. Thus

$$\begin{aligned} y - Gy &= y - (\beta_k y - k) = (1 - \beta_k) y + k > (1 - \beta_k) a_{k+1} + k \\ &= a_{k+1} - T_k(a_{k+1}) \geq a_{r+1} - \xi_+ \geq c. \end{aligned}$$

ii) Deduce a contradiction.
Recall that we have assumed $G^i x \geq \xi_+$ for all $i \geq 0$. First by $x \geq \xi_+$ and i), we get $x - Gx \geq c$. Then by $\xi_+ \leq Gx \leq x$ and i) again, we get $Gx - G^2 x \geq c$. $\cdots$ For all $n \geq 1$, we can get $G^{n-1}x - G^n x \geq c$. It follows from the summation of the above inequalities that $x - G^n x \geq nc$, where $nc \to +\infty$ as $n \to +\infty$. This contradicts $G^i x \geq \xi_+$ for all $i \geq 0$.

② For all $n \geq p$, $\sigma^n w \prec g^*(\xi_+)$ follows from

$$\sigma^n w = \sigma^n(g(x)) \overset{(\star)}{=} g(G^n x) \overset{(\star\star)}{\prec} g^*(\xi_+),$$

where $(\star)$ follows from Proposition 2.1.14 (1), and $(\star\star)$ follows from Proposition 2.1.16 and $G^n x < \xi_+$, which can be proved as follows. First we have $G^p x < \xi_+$ by the definition of p. It suffices to prove that for all $y \in [0, \xi_+)$, we have $Gy < \xi_+$. In fact, let $y \in [0, \xi_+) \subset [0, \frac{m}{\beta_m - 1})$. If $y \geq a_m$, then

$$Gy = T_m y = \beta_m y - m < y < \xi_+.$$

If $y < a_m$, then there exists $k \in \{0, \cdots, m-1\}$ such that $y \in [a_k, a_{k+1})$ and we have

$$Gy = T_k y < T_k(a_{k+1}) \leq \xi_+.$$

(2) follows in a way similar to (1) by using $a_k \leq b_{k-1} < b_k$ instead of $a_k < a_{k+1} \leq b_k$ for all $k \in \{1, \cdots, m\}$. □

Proof of Theorem 2.1.3. (1) ① Suppose that w is the greedy $(\beta_0, \cdots, \beta_m)$-expansion of x and $w_n < m$. Then $G^{n-1}x \in [a_{w_n}, a_{w_n+1})$ and

$$G^n x = G(G^{n-1}x) = T_{w_n}(G^{n-1}x) < T_{w_n}(a_{w_n+1}) \leq \xi_+.$$

It follows from Lemma 2.1.19 (1) that $\sigma^n w \prec g^*(\xi_+)$.
② Suppose $w_n < m$. By Proposition 2.1.11 (1), we only need to prove $\pi(w_n w_{n+1} \cdots) < a_{w_n+1}$, which is equivalent to $\pi(w_{n+1} w_{n+2} \cdots) < T_{w_n}(a_{w_n+1})$.

For simplification, we use g_i^* to denote $g_i^*(\xi_-)$ for all $i \in \mathbb{N}$ in the following.

First by condition $\sigma^n w \prec g^*(\xi_-)$, we get $w_{n+1} w_{n+2} \cdots \prec g_1^* g_2^* \cdots$. Then there exist $s_1 \in \mathbb{N}$ and $n_1 = n + s_1$ such that

$$w_{n+1} \cdots w_{n_1 - 1} = g_1^* \cdots g_{s_1 - 1}^* \quad \text{and} \quad w_{n_1} < g_{s_1}^*.$$

By condition $\sigma^{n_1} w \prec g^*(\xi_-)$, we get $w_{n_1+1} w_{n_1+2} \cdots \prec g_1^* g_2^* \cdots$. Then there exist $s_2 \in \mathbb{N}$

and $n_2 = n_1 + s_2$ such that

$$w_{n_1+1} \cdots w_{n_2-1} = g_1^* \cdots g_{s_2-1}^* \quad \text{and} \quad w_{n_2} < g_{s_2}^*.$$

For general $j \geq 2$, if there already exist $s_j \in \mathbb{N}$ and $n_j = n_{j-1} + s_j$ such that

$$w_{n_{j-1}+1} \cdots w_{n_j-1} = g_1^* \cdots g_{s_j-1}^* \quad \text{and} \quad w_{n_j} < g_{s_j}^*,$$

by condition $\sigma^{n_j} w \prec g^*(\xi_-)$ we get $w_{n_j+1} w_{n_j+2} \cdots \prec g_1^* g_2^* \cdots$. Then there exist $s_{j+1} \in \mathbb{N}$ and $n_{j+1} = n_j + s_{j+1}$ such that

$$w_{n_j+1} \cdots w_{n_{j+1}-1} = g_1^* \cdots g_{s_{j+1}-1}^* \quad \text{and} \quad w_{n_{j+1}} < g_{s_{j+1}}^*.$$

For all $i \geq 1$, s_i and n_i are well defined by the above process. Since

$$\pi(w_{n+1} w_{n+2} \cdots) = \sum_{i=0}^{\infty} \frac{\pi(w_{n_i+1} \cdots w_{n_{i+1}})}{\beta_{w_{n+1}} \beta_{w_{n+2}} \cdots \beta_{w_{n_i}}}$$

and

$$T_{w_n}(a_{w_n+1}) = \sum_{i=0}^{\infty} \Big(\frac{T_{w_{n_i}}(a_{w_{n_i}+1})}{\beta_{w_{n+1}} \beta_{w_{n+2}} \cdots \beta_{w_{n_i}}} - \frac{T_{w_{n_{i+1}}}(a_{w_{n_{i+1}}+1})}{\beta_{w_{n+1}} \beta_{w_{n+2}} \cdots \beta_{w_{n_{i+1}}}} \Big)$$

where $n_0 := n$ and $\beta_{w_{n+1}} \beta_{w_{n+2}} \cdots \beta_{w_{n_0}} := 1$, we only need to prove

$$\pi(w_{n_i+1} \cdots w_{n_{i+1}}) < T_{w_{n_i}}(a_{w_{n_i}+1}) - \frac{T_{w_{n_{i+1}}}(a_{w_{n_{i+1}}+1})}{\beta_{w_{n_i+1}} \beta_{w_{n_i+2}} \cdots \beta_{w_{n_{i+1}}}},$$

$$\text{i.e.,} \quad \pi(w_{n_i+1} \cdots w_{n_{i+1}-1}) + \frac{a_{w_{n_{i+1}}+1}}{\beta_{w_{n_i+1}} \beta_{w_{n_i+2}} \cdots \beta_{w_{n_{i+1}-1}}} < T_{w_{n_i}}(a_{w_{n_i}+1}) \quad \text{for all } i \geq 0.$$

In fact, for all $i \geq 0$, by $w_{n_i+1} \cdots w_{n_{i+1}-1} = g_1^* \cdots g_{s_{i+1}-1}^*$ and $w_{n_{i+1}} + 1 \leq g_{s_{i+1}}^*$ (which implies $a_{w_{n_{i+1}}+1} \leq a_{g_{s_{i+1}}^*}$), we get

$$\begin{aligned} \pi(w_{n_i+1} \cdots w_{n_{i+1}-1}) + \frac{a_{w_{n_{i+1}}+1}}{\beta_{w_{n_i+1}} \beta_{w_{n_i+2}} \cdots \beta_{w_{n_{i+1}-1}}} &\leq \pi(g_1^* \cdots g_{s_{i+1}-1}^*) + \frac{a_{g_{s_{i+1}}^*}}{\beta_{g_1^*} \beta_{g_2^*} \cdots \beta_{g_{s_{i+1}-1}^*}} \\ &= \pi(g_1^* \cdots g_{s_{i+1}}^*) \\ &\overset{(\star)}{<} \pi(g^*(\xi_-)) = \xi_- \leq T_{w_{n_i}}(a_{w_{n_i}+1}), \end{aligned}$$

where $(\star)$ follows from the fact that $g^*(\xi_-)$ does not end with 0^∞ (recalling Proposition 2.1.11 (2)).

(2) follows in a way similar to (1).

(3) follows immediately from (1), (2) and Proposition 2.1.12 (1). □

Corollary 2.1.4 follows directly from Theorem 2.1.3.

Corollary 2.1.5 follows from Theorem 2.1.3, the facts that $\beta_0 \le \beta_1 \le \cdots \le \beta_m$ implies $\xi_+ \le 1$ and $\eta_- \ge \frac{m}{\beta_m - 1} - 1$, $\beta_0 \ge \beta_1 \ge \cdots \ge \beta_m$ implies $\xi_- \ge 1$ and $\eta_+ \le \frac{m}{\beta_m - 1} - 1$, and the increase of g^* and l^* (by Proposition 2.1.14 (4)).

Proof of Corollary 2.1.6. Since (1) follows immediately from Corollary 2.1.5 and Proposition 2.1.18, in the following we only prove (2).
① $\boxed{\Rightarrow}$ follows from Lemma 2.1.19 (1), in which $\xi_+ = 1$ and $p = 0$.
$\boxed{\Leftarrow}$ First by (1) ①, we know that w is the greedy expansion $g(x)$. Then it follows from $g(x) = w < g^*(1) \le g(1)$ and the strictly increase of g (by Proposition 2.1.14 (4)) that $x < 1$.
② $\boxed{\Rightarrow}$ follows from Proposition 2.1.18 and Lemma 2.1.19 (2), in which $\eta_- = \frac{m}{\beta - 1} - 1$ and $q = 0$.
$\boxed{\Leftarrow}$ First by (1) ②, we know that w is the lazy expansion $l(x)$. Then it follows from $l(x) = w > \overline{g^*(1)} = l^*(\frac{m}{\beta-1} - 1) \ge l(\frac{m}{\beta-1} - 1)$ and the strictly increase of l (by Proposition 2.1.14 (4)) that $x > \frac{m}{\beta-1} - 1$.
③ follows from ①, ② and Proposition 2.1.12 (1). □

2.1.3 Further questions

On the one hand, although necessary and sufficient conditions for sequences to be greedy, lazy and unique expansions in two bases and one base are obtained in Corollaries 2.1.4 and 2.1.6 respectively, for general cases, i.e., in more than two bases, Theorem 2.1.3 and Corollary 2.1.5 can only give necessary conditions and sufficient conditions separately. We look forward to getting necessary and sufficient conditions for general cases. (This was answered by Zou, Komornik and Lu very recently in [130, Theorem 1.2].)

2.2 Digit frequencies of beta-expansions

From this section, we return to expansions in one base and consider digit frequencies. Let $m \in \mathbb{N}$ and $\beta \in (1, m+1]$. Given $x \in \mathbb{R}$, recall that a sequence $w = (w_i)_{i\ge1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ is called a *β-expansion* of x if

$$x = \sum_{i=1}^{\infty} \frac{w_i}{\beta^i}.$$

It is known that x has a β-expansion if and only if $x \in [0, \frac{m}{\beta-1}]$.

For any sequence $w = (w_i)_{i\ge1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$, we define the *upper-frequency*, *lower-frequency* and *frequency* of the digit k by

$$\overline{\mathrm{Freq}}_k(w) := \overline{\lim_{n\to\infty}} \frac{\#\{i : 1 \le i \le n, w_i = k\}}{n},$$

$$\underline{\mathrm{Freq}}_k(w) := \underline{\lim_{n\to\infty}} \frac{\#\{i : 1 \le i \le n, w_i = k\}}{n}$$

and

$$\mathrm{Freq}_k(w) := \lim_{n\to\infty} \frac{\#\{i : 1 \le i \le n, w_i = k\}}{n}$$

(assuming the limit exists) respectively, where # denotes the cardinality. If $\overline{p} = (\overline{p}_0, \cdots, \overline{p}_m)$, $\underline{p} = (\underline{p}_0, \cdots, \underline{p}_m) \in [0,1]^{m+1}$ satisfy

$$\overline{\mathrm{Freq}}_k(w) = \overline{p}_k \quad \text{and} \quad \underline{\mathrm{Freq}}_k(w) = \underline{p}_k \quad \text{for all } k \in \{0, 1, \cdots, m\},$$

we say that w is of frequency $(\overline{p}, \underline{p})$. The following theorem is the first main result in this section.

Theorem 2.2.1. *For all $m \in \mathbb{N}$, $\beta \in (1, m+1) \setminus \mathbb{N}$ and $\overline{p}, \underline{p} \in [0,1]^{m+1}$, Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has a β-expansion of frequency $(\overline{p}, \underline{p})$ if and only if Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has infinitely many β-expansions of frequency $(\overline{p}, \underline{p})$.*

As the second main result, the next theorem focuses on a special kind of frequency. Given $m \in \mathbb{N}$, a sequence w on $\{0, 1, \cdots, m\}$ is called *balanced* if $\mathrm{Freq}_k(w) = \mathrm{Freq}_{m-k}(w)$ for all $k \in \{0, 1, \cdots, m\}$.

Theorem 2.2.2. *For all $m \in \mathbb{N}$ and $\beta \in (1, m+1) \setminus \mathbb{N}$, Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has infinitely many balanced β-expansions.*

In the following, we consider variable frequency. Recently, Baker proved in [24] that for any $\beta \in (1, \frac{1+\sqrt{5}}{2})$, there exists $c = c(\beta) > 0$ such that for any $p \in [\frac{1}{2} - c, \frac{1}{2} + c]$ and $x \in (0, \frac{1}{\beta-1})$, there exists a β-expansion of x on $\{0,1\}$ with frequency of zeros equal to p. This result is sharp, since for any $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$, there exists an $x \in (0, \frac{1}{\beta-1})$ such that for any β-expansion of x on $\{0,1\}$ its frequency of zeros exists and is equal to either 0 or $\frac{1}{2}$ (see the statements between Theorems 1.1 and 1.2 in [25]). It is natural to ask for which $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$, the result can be true for almost every $x \in (0, \frac{1}{\beta-1})$. We give a class of such β in Theorem 2.2.3 as the third main result in this section. They are the *pseudo-golden ratios*, i.e., the $\beta \in (1,2)$ such that $\beta^m - \beta^{m-1} - \cdots - \beta - 1 = 0$ for some integer $m \ge 2$. Note that the smallest pseudo-golden ratio is the golden ratio $\frac{1+\sqrt{5}}{2}$.

Theorem 2.2.3. *Let $\beta \in (1,2)$ such that $\beta^m - \beta^{m-1} - \cdots - \beta - 1 = 0$ for some integer $m \ge 2$ and let $c = \frac{(m-1)(2-\beta)}{2(m\beta+\beta-2m)}$ (> 0). Then for any $p \in [\frac{1}{2} - c, \frac{1}{2} + c]$, Lebesgue almost every $x \in [0, \frac{1}{\beta-1}]$ has infinitely many β-expansions on $\{0,1\}$ with frequency of zeros equal to p.*

We give some notation and preliminaries in Subsection 2.2.1, prove the main results in Subsection 2.2.2 and end this section with further questions in the last subsection.

2.2.1 Notation and preliminaries

Let $m \in \mathbb{N}$ and $\beta \in (1, m+1]$. For all $k \in \{0, \cdots, m\}$, we define the maps $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(x) := \beta x - k \quad \text{for } x \in \mathbb{R}$$

. Given $x \in [0, \frac{m}{\beta-1}]$, let

$$\Sigma_{\beta,m}(x) := \Big\{(w_i)_{i\geq 1} \in \{0, \cdots, m\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{w_i}{\beta^i} = x\Big\}$$

and

$$\Omega_{\beta,m}(x) := \Big\{(a_i)_{i\geq 1} \in \{T_0, \cdots, T_m\}^{\mathbb{N}} : (a_n \circ \cdots \circ a_1)(x) \in [0, \frac{m}{\beta-1}] \text{ for all } n \in \mathbb{N}\Big\}.$$

The following lemma given by Baker is a dynamical interpretation of β-expansions.

Lemma 2.2.4 ([23, 24])**.** *For any $x \in [0, \frac{m}{\beta-1}]$, we have $\#\Sigma_{\beta,m}(x) = \#\Omega_{\beta,m}(x)$. Moreover, the map which sends $(w_i)_{i\geq 1}$ to $(T_{w_i})_{i\geq 1}$ is a bijection from $\Sigma_{\beta,m}(x)$ to $\Omega_{\beta,m}(x)$.*

2.2.2 Proof of the main results

Proof of Theorem 2.2.1. Let $m \in \mathbb{N}$, $\beta \in (1, m+1) \setminus \mathbb{N}$ and $\overline{p}, \underline{p} \in [0,1]^{m+1}$. The "if" part is obvious. We only need to prove the "only if" part. Let $\mathcal{L}$ denote the Lebesgue measure. Suppose that $\mathcal{L}$-a.e. (almost every) $x \in [0, \frac{m}{\beta-1}]$ has a β-expansion of frequency $(\overline{p}, \underline{p})$. Let

$$\mathcal{U}_{\beta,m} := \Big\{x \in [0, \frac{m}{\beta-1}] : x \text{ has a unique } \beta\text{-expansion}\Big\}$$

and

$$\mathcal{N}_{\beta,m}^{\overline{p},\underline{p}} := \Big\{x \in [0, \frac{m}{\beta-1}] : x \text{ has no } \beta\text{-expansions of frequency } (\overline{p}, \underline{p})\Big\}.$$

On the one hand, it is well known that $\mathcal{L}(\mathcal{U}_{\beta,m}) = 0$ (see for example [87]). On the other hand, by condition we know $\mathcal{L}(\mathcal{N}_{\beta,m}^{\overline{p},\underline{p}}) = 0$. Let

$$\Psi := \Big(\mathcal{U}_{\beta,m} \cup \mathcal{N}_{\beta,m}^{\overline{p},\underline{p}}\Big) \cup \bigcup_{n=1}^{\infty} \bigcup_{w_1,\cdots,w_n \in \{0,\cdots,m\}} T_{w_n}^{-1} \circ \cdots \circ T_{w_1}^{-1}\Big(\mathcal{U}_{\beta,m} \cup \mathcal{N}_{\beta,m}^{\overline{p},\underline{p}}\Big).$$

Then $\mathcal{L}(\Psi) = 0$. Let $x \in [0, \frac{m}{\beta-1}] \setminus \Psi$. It suffices to prove that x has infinitely many different β-expansions of frequency $(\overline{p}, \underline{p})$.

Let $(w_i)_{i\geq 1}$ be a β-expansions of x. Since $x \notin \Psi$ implies $x \notin \mathcal{U}_{\beta,m}$, x has another β-expansion $(v_i^{(1)})_{i\geq 1}$. There exists $n_1 \in \mathbb{N}$ such that $v_1^{(1)} \cdots v_{n_1-1}^{(1)} = w_1 \cdots w_{n_1-1}$ and

$v_{n_1}^{(1)} \neq w_{n_1}$. By

$$T_{v_{n_1}^{(1)}} \circ T_{w_{n_1}-1} \circ \cdots \circ T_{w_1} x = T_{v_{n_1}^{(1)}} \circ \cdots \circ T_{v_1^{(1)}} x = \sum_{i=1}^{\infty} \frac{v_{n_1+i}^{(1)}}{\beta^i},$$

we know that $(v_{n_1+i}^{(1)})_{i\geq 1}$ is a β-expansion of $T_{v_{n_1}^{(1)}} \circ T_{w_{n_1}-1} \circ \cdots \circ T_{w_1} x$. Since $x \notin \Psi$ implies $T_{v_{n_1}^{(1)}} \circ T_{w_{n_1}-1} \circ \cdots \circ T_{w_1} x \notin \mathcal{N}_{\beta,m}^{\overline{p},\underline{p}}$, $T_{v_{n_1}^{(1)}} \circ T_{w_{n_1}-1} \circ \cdots \circ T_{w_1} x$ has a β-expansion $(w_{n_1+i}^{(1)})_{i\geq 1}$ of frequency $(\overline{p},\underline{p})$. Let $w_1^{(1)} \cdots w_{n_1-1}^{(1)} w_{n_1}^{(1)} := w_1 \cdots w_{n_1-1} v_{n_1}^{(1)}$. Then $(w_i^{(1)})_{i\geq 1}$ is a β-expansion of x of frequency $(\overline{p},\underline{p})$ with $w_{n_1}^{(1)} \neq w_{n_1}$, which implies that $(w_i)_{i\geq 1}$ and $(w_i^{(1)})_{i\geq 1}$ are different.

Note that $(w_{n_1+i})_{i\geq 1}$ is a β-expansion of $T_{w_{n_1}} \circ \cdots \circ T_{w_1} x$. Since $x \notin \Psi$ implies $T_{w_{n_1}} \circ \cdots \circ T_{w_1} x \notin \mathcal{U}_{\beta,m}$, $T_{w_{n_1}} \circ \cdots \circ T_{w_1} x$ has another β-expansion $(v_{n_1+i}^{(2)})_{i\geq 1}$. There exists $n_2 > n_1$ such that $v_{n_1+1}^{(2)} \cdots v_{n_2-1}^{(2)} = w_{n_1+1} \cdots w_{n_2-1}$ and $v_{n_2}^{(2)} \neq w_{n_2}$. By

$$T_{v_{n_2}^{(2)}} \circ T_{w_{n_2}-1} \circ \cdots \circ T_{w_1} x = T_{v_{n_2}^{(2)}} \circ \cdots \circ T_{v_{n_1+1}^{(2)}} \circ (T_{w_{n_1}} \circ \cdots \circ T_{w_1} x) = \sum_{i=1}^{\infty} \frac{v_{n_2+i}^{(2)}}{\beta^i},$$

we know that $(v_{n_2+i}^{(2)})_{i\geq 1}$ is a β-expansion of $T_{v_{n_2}^{(2)}} \circ T_{w_{n_2}-1} \circ \cdots \circ T_{w_1} x$. Since $x \notin \Psi$ implies $T_{v_{n_2}^{(2)}} \circ T_{w_{n_2}-1} \circ \cdots \circ T_{w_1} x \notin \mathcal{N}_{\beta,m}^{\overline{p},\underline{p}}$, $T_{v_{n_2}^{(2)}} \circ T_{w_{n_2}-1} \circ \cdots \circ T_{w_1} x$ has a β-expansion $(w_{n_2+i}^{(2)})_{i\geq 1}$ of frequency $(\overline{p},\underline{p})$. Let $w_1^{(2)} \cdots w_{n_2-1}^{(2)} w_{n_2}^{(2)} := w_1 \cdots w_{n_2-1} v_{n_2}^{(2)}$. Then $(w_i^{(2)})_{i\geq 1}$ is a β-expansion of x of frequency $(\overline{p},\underline{p})$ with $w_{n_1}^{(2)} = w_{n_1}$ and $w_{n_2}^{(2)} \neq w_{n_2}$, which implies that $(w_i)_{i\geq 1}$, $(w_i^{(1)})_{i\geq 1}$ and $(w_i^{(2)})_{i\geq 1}$ are all different.

$\cdots$ Generally, suppose that for some $j \in \mathbb{N}$ we have already constructed $(w_i^{(1)})_{i\geq 1}$, $(w_i^{(2)})_{i\geq 1}$, $\cdots$, $(w_i^{(j)})_{i\geq 1}$, which are all β-expansions of x of frequency $(\overline{p},\underline{p})$ such that

$$\begin{cases} w_{n_1}^{(1)} \neq w_{n_1}, \\ w_{n_1}^{(2)} = w_{n_1}, w_{n_2}^{(2)} \neq w_{n_2}, \\ w_{n_1}^{(3)} = w_{n_1}, w_{n_2}^{(3)} = w_{n_2}, w_{n_3}^{(3)} \neq w_{n_3}, \\ \cdots \\ w_{n_1}^{(j)} = w_{n_1}, w_{n_2}^{(j)} = w_{n_2}, \cdots, w_{n_{j-1}}^{(j)} = w_{n_{j-1}}, w_{n_j}^{(j)} \neq w_{n_j}. \end{cases}$$

Note that $(w_{n_j+i})_{i\geq 1}$ is a β-expansion of $T_{w_{n_j}} \circ \cdots \circ T_{w_1} x$. Since $x \notin \Psi$ implies $T_{w_{n_j}} \circ \cdots \circ T_{w_1} x \notin \mathcal{U}_{\beta,m}$, $T_{w_{n_j}} \circ \cdots \circ T_{w_1} x$ has another β-expansion $(v_{n_j+i}^{(j+1)})_{i\geq 1}$. There exists $n_{j+1} > n_j$ such that $v_{n_j+1}^{(j+1)} \cdots v_{n_{j+1}-1}^{(j+1)} = w_{n_j+1} \cdots w_{n_{j+1}-1}$ and $v_{n_{j+1}}^{(j+1)} \neq w_{n_{j+1}}$. By

$$T_{v_{n_{j+1}}^{(j+1)}} \circ T_{w_{n_{j+1}}-1} \circ \cdots \circ T_{w_1} x = T_{v_{n_{j+1}}^{(j+1)}} \circ \cdots \circ T_{v_{n_j+1}^{(j+1)}} \circ (T_{w_{n_j}} \circ \cdots \circ T_{w_1} x) = \sum_{i=1}^{\infty} \frac{v_{n_{j+1}+i}^{(j+1)}}{\beta^i},$$

we know that $(v^{(j+1)}_{n_{j+1}+i})_{i\geq 1}$ is a β-expansion of $T_{v^{(j+1)}_{n_{j+1}}} \circ T_{w_{n_{j+1}}-1} \circ \cdots \circ T_{w_1}x$. Since $x \notin \Psi$ implies $T_{v^{(j+1)}_{n_{j+1}}} \circ T_{w_{n_{j+1}}-1} \circ \cdots \circ T_{w_1}x \notin \mathcal{N}^{\overline{p},\underline{p}}_{\beta,m}$, $T_{v^{(j+1)}_{n_{j+1}}} \circ T_{w_{n_{j+1}}-1} \circ \cdots \circ T_{w_1}x$ has a β-expansion $(w^{(j+1)}_{n_{j+1}+i})_{i\geq 1}$ of frequency $(\overline{p},\underline{p})$. Let $w^{(j+1)}_1 \cdots w^{(j+1)}_{n_{j+1}-1} w^{(j+1)}_{n_{j+1}} := w_1 \cdots w_{n_{j+1}-1} v^{(j+1)}_{n_{j+1}}$. Then $(w^{(j+1)}_i)_{i\geq 1}$ is a β-expansion of x of frequency $(\overline{p},\underline{p})$ with $w^{(j+1)}_{n_1} = w_{n_1}, \cdots, w^{(j+1)}_{n_j} = w_{n_j}$ and $w^{(j+1)}_{n_{j+1}} \neq w_{n_{j+1}}$, which implies that $(w_i)_{i\geq 1}$, $(w^{(1)}_i)_{i\geq 1}$, $\cdots$, $(w^{(j+1)}_i)_{i\geq 1}$ are all different.

$\cdots$ It follows from repeating the above process that x has infinitely many different β-expansions of frequency $(\overline{p},\underline{p})$. $\square$

Proof of Theorem 2.2.2. Let $m \in \mathbb{N}$ and $\beta \in (1, m+1) \setminus \mathbb{N}$. By Theorem 2.2.1, it suffices to prove that $\mathcal{L}$-a.e. $x \in [0, \frac{m}{\beta-1}]$ has a balanced β-expansion. Let

$$z_- := \frac{m}{2(\beta-1)} - \frac{1}{2} \quad \text{and} \quad z_+ := \frac{m}{2(\beta-1)} + \frac{1}{2}.$$

For all $k \in \{1, \cdots, m\}$, define

$$z_k := \frac{m}{2\beta(\beta-1)} + \frac{2k-1}{2\beta}.$$

Then $T_1(z_1) = T_2(z_2) = \cdots = T_m(z_m) = z_-$ and $T_0(z_1) = T_1(z_2) = \cdots = T_{m-1}(z_m) = z_+$.

First we prove that $\mathcal{L}$-a.e. $x \in [z_-, z_+]$ has a balanced β-expansion. If $\beta \in (1,2)$ and m is odd, let $a_- := \frac{m-1}{2(\beta-1)}$ and $a_+ := \frac{m+1}{2(\beta-1)}$. Then $T_{\frac{m-1}{2}}(a_-) = a_-$ and $T_{\frac{m+1}{2}}(a_+) = a_+$. Considering $T_{\frac{m-1}{2}}$ restricted on $[a_-, a_+ - \frac{1}{\beta}]$ and $T_{\frac{m+1}{2}}$ restricted on $[a_- + \frac{1}{\beta}, a_+]$, by [25, Theorem 4.1] and Lemma 2.2.4, we know that $\mathcal{L}$-a.e. $x \in [a_-, a_+]$ ($\supset [z_-, z_+]$) has a β-expansion w on $\{\frac{m-1}{2}, \frac{m+1}{2}\}$ satisfying $\mathrm{Freq}_{\frac{m-1}{2}}(w) = \mathrm{Freq}_{\frac{m+1}{2}}(w) = \frac{1}{2}$. Thus we only need to consider that $\beta > 2$ or m is even in the following.

Define $T : [0, \frac{m}{\beta-1}] \to [0, \frac{m}{\beta-1}]$ by

$$T(x) := \begin{cases} T_0(x) = \beta x & \text{for } x \in [0, z_1), \\ T_k(x) = \beta x - k & \text{for } x \in [z_k, z_{k+1}) \text{ and } k \in \{1, 2, \cdots, m-1\}, \\ T_m(x) = \beta x - m & \text{for } x \in [z_m, \frac{m}{\beta-1}]. \end{cases}$$

We consider the restriction $T|_{[z_-,z_+)} : [z_-, z_+) \to [z_-, z_+)$. By Theorem 5.2 in [124], there exists a $T|_{[z_-,z_+)}$-invariant ergodic Borel probability measure μ on $[z_-, z_+)$ equivalent to $\mathcal{L}$. Let r be the smallest in $\{1, 2, \cdots, m\}$ such that $z_- < z_r$. Then $m+1-r$ is the largest in $\{1, 2, \cdots, m\}$ such that $z_{m+1-r} < z_+$. Let

$$z'_{r-1} := z_-, \ z'_{m-r+2} := z_+ \text{ and } z'_k := z_k \text{ for all } k \in \{r, r+1, \cdots, m-r+1\}.$$

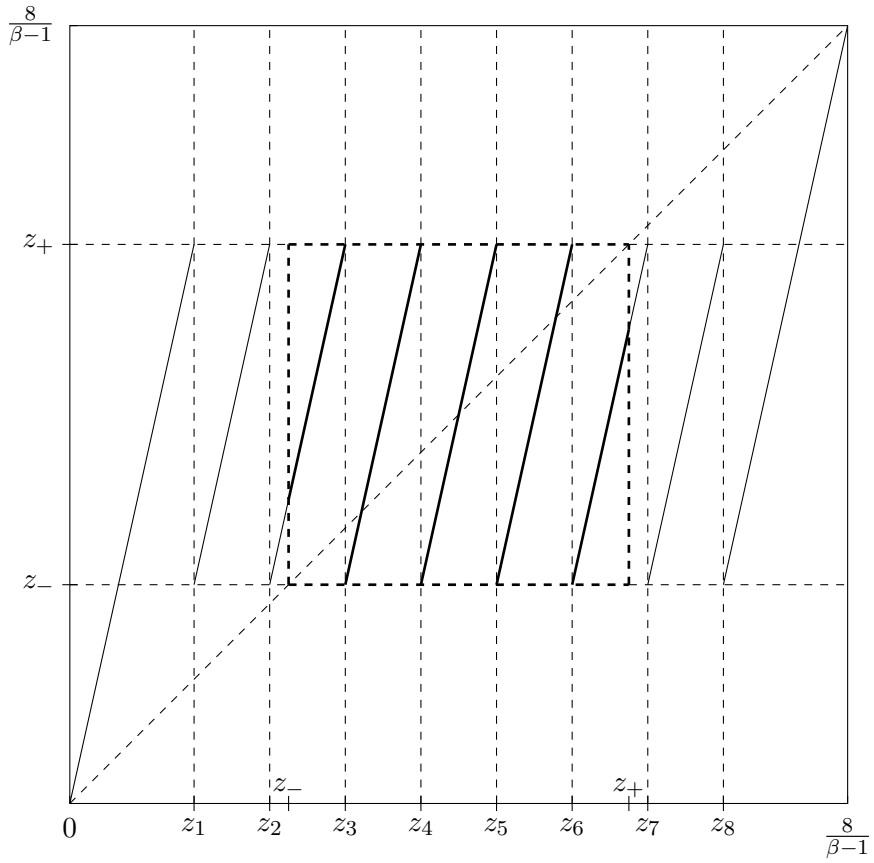


Figure 2.2: The graph of T and $T|_{[z_-,z_+)}$ for $m = 8$ and some $\beta \in (4, 5)$.

Then $z'_{r-1} < z'_r < \cdots < z'_{m-r+1} < z'_{m-r+2}$. For any $x \in [z_-, z_+)$ which is not a preimage of a discontinuity point of $T|_{[z_-,z_+)}$, by symmetry, we know that for any $k \in \{r-1, r, \cdots, m-r+1\}$ and $i \in \{0, 1, 2, \cdots\}$,

$$T^i(x) \in (z'_k, z'_{k+1}) \Leftrightarrow T^i\Big(\frac{m}{\beta-1} - x\Big) \in (z'_{m-k}, z'_{m-k+1}).$$

For all $k \in \{r-1, r, \cdots, m-r+1\}$, it follows from Birkhoff's ergodic theorem that for $\mathcal{L}$-a.e. $x \in [z_-, z_+)$,

$$\mu((z'_k, z'_{k+1})) = \int_{z_-}^{z_+} \mathbb{1}_{(z'_k, z'_{k+1})} d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(z'_k, z'_{k+1})}\Big(T^i(x)\Big) \tag{2.6}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(z'_{m-k}, z'_{m-k+1})}\Big(T^i\big(\frac{m}{\beta-1} - x\big)\Big), \tag{2.7}$$

and for $\mathcal{L}$-a.e. $y \in [z_-, z_+)$,

$$\mu((z'_{m-k}, z'_{m-k+1})) = \int_{z_-}^{z_+} \mathbb{1}_{(z'_{m-k}, z'_{m-k+1})} d\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(z'_{m-k}, z'_{m-k+1})}\Big(T^i(y)\Big),$$

which implies that for $\mathcal{L}$-a.e. $(\frac{m}{\beta-1} - x) \in (z_-, z_+)$,

$$\mu((z'_{m-k}, z'_{m-k+1})) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{(z'_{m-k}, z'_{m-k+1})}\Big(T^i\big(\frac{m}{\beta-1} - x\big)\Big).$$

So this is also true for $\mathcal{L}$-a.e $x \in (z_-, z_+)$. Recalling (2.7), we get

$$\mu((z'_k, z'_{k+1})) = \mu((z'_{m-k}, z'_{m-k+1})) \quad \text{for } k \in \{r-1, r, \cdots, m-r+1\}. \tag{2.8}$$

For every $x \in [z_-, z_+)$ and $i \in \mathbb{N}$, there exists $k_i \in \{r-1, r, \cdots, m-r, m-r+1\}$ such that $T^{i-1}x \in [z'_{k_i}, z'_{k_i+1})$, then we define $\varepsilon_i(x) := k_i$ and denote $\varepsilon(x) := (\varepsilon_i(x))_{i\geq 1} \in \{r-1, r, \cdots, m-r+1\}^{\mathbb{N}}$. For all $k \in \{r-1, r, \cdots, m-r+1\}$, $i \in \{0, 1, 2, \cdots\}$ and $x \in [z_-, z_+)$, we have

$$\mathbb{1}_{[z'_k, z'_{k+1})}(T^i x) = 1 \Leftrightarrow T^i x \in [z'_k, z'_{k+1}) \Leftrightarrow \varepsilon_{i+1}(x) = k.$$

By (2.6), we know that for all $k \in \{r-1, r, \cdots, m-r+1\}$ and $\mathcal{L}$-a.e. $x \in [z_-, z_+)$,

$$\mathrm{Freq}_k(\varepsilon(x)) = \lim_{n\to\infty} \frac{\#\{i : 1 \leq i \leq n, \varepsilon_i(x) = k\}}{n} = \mu((z'_k, z'_{k+1})).$$

It follows from (2.8) that for all $k \in \{r-1, r, \cdots, m-r+1\}$ and $\mathcal{L}$-a.e. $x \in [z_-, z_+)$,

$$\mathrm{Freq}_k(\varepsilon(x)) = \mathrm{Freq}_{m-k}(\varepsilon(x)). \tag{2.9}$$

(1) For any $x \in [z_-, z_+)$, we prove that $\varepsilon(x)$ is a β-expansion of x, i.e., $\sum_{i=1}^{\infty} \frac{\varepsilon_i(x)}{\beta^i} = x$. In fact, by Lemma 2.2.4, it suffices to show $T_{\varepsilon_n(x)} \circ \cdots \circ T_{\varepsilon_1(x)}(x) \in [0, \frac{m}{\beta-1}]$ for all $n \in \mathbb{N}$. We only need to prove $T_{\varepsilon_n(x)} \circ \cdots \circ T_{\varepsilon_1(x)}(x) = T^n(x)$ by induction as follows. For $x \in [z_-, z_+)$, let $k_1 \in \{r-1, r, \cdots, m-r, m-r+1\}$ such that $x \in [z'_{k_1}, z'_{k_1+1})$. Then $\varepsilon_1(x) = k_1$ and

$$T_{\varepsilon_1(x)}(x) = T_{k_1}(x) = T(x).$$

Assume that for some $n \in \mathbb{N}$ we have $T_{\varepsilon_n(x)} \circ \cdots \circ T_{\varepsilon_1(x)}(x) = T^n(x)$. Let $k_{n+1} \in \{r-1, r, \cdots, m-r, m-r+1\}$ such that $T^n(x) \in [z'_{k_{n+1}}, z'_{k_{n+1}+1})$. Then $\varepsilon_{n+1}(x) = k_{n+1}$ and

$$T_{\varepsilon_{n+1}(x)} \circ T_{\varepsilon_n(x)} \circ \cdots \circ T_{\varepsilon_1(x)}(x) = T_{k_{n+1}} \circ T^n(x) = T^{n+1}(x).$$

Combining (1) and (2.9), we know that $\mathcal{L}$-a.e. $x \in [z_-, z_+]$ has a balanced β-expansion.

Let

$$N := \big\{x \in [0, \frac{m}{\beta-1}] : x \text{ has no balanced } \beta\text{-expansions}\big\}.$$

We have already proved $\mathcal{L}(N \cap [z_-, z_+]) = 0$. To end the proof, we need to show $\mathcal{L}(N) = 0$. In fact, it suffices to prove $\mathcal{L}(N \cap (0, z_-)) = \mathcal{L}(N \cap (z_+, \frac{m}{\beta-1})) = 0$.

i) Prove $\mathcal{L}(N \cap (0, z_-)) = 0$.
By $\mathcal{L}(N \cap [z_-, z_+]) = 0$, we know that for any $n \in \mathbb{N}$ and $v_1, \cdots, v_n \in \{0, \cdots, r-1\}$, $\mathcal{L}(T_{v_1}^{-1} \circ \cdots \circ T_{v_n}^{-1}(N \cap [z_-, z_+])) = 0$. It suffices to prove

$$N \cap (0, z_-) \subset \bigcup_{n=1}^{\infty} \bigcup_{v_1, \cdots, v_n \in \{0, \cdots, r-1\}} T_{v_1}^{-1} \circ \cdots \circ T_{v_n}^{-1}(N \cap [z_-, z_+]).$$

(By contradiction) Let $x \in N \cap (0, z_-)$ and assume that x is not contained in the right hand side. By $x \in (0, z_-)$, one can verify that there exist $v_1, \cdots, v_k \in \{0, \cdots, r-1\}$ such that $T_{v_k} \circ \cdots \circ T_{v_1}(x) \in [z_-, z_+]$. (In fact, it suffices to use $T|_{[0,z_-)}$.) Since $x \notin T_{v_1}^{-1} \circ \cdots \circ T_{v_k}^{-1}(N \cap [z_-, z_+])$, we must have $T_{v_k} \circ \cdots \circ T_{v_1}(x) \notin N$. This means that there exists a balanced sequence $(w_i)_{i \geq 1}$ on $\{0, \cdots, m\}$ such that $T_{v_k} \circ \cdots \circ T_{v_1}(x) = \sum_{i=1}^{\infty} \frac{w_i}{\beta^i}$, and then

$$x = \frac{v_1}{\beta} + \frac{v_2}{\beta^2} + \cdots + \frac{v_k}{\beta^k} + \sum_{i=1}^{\infty} \frac{w_i}{\beta^{k+i}} =: \sum_{i=1}^{\infty} \frac{v_i}{\beta^i}$$

where $v_{k+i} := w_i$ for $i \geq 1$. It follows that $(v_i)_{i \geq 1}$ is a balanced β-expansion of x, which contradicts $x \in N$.

ii) The fact $\mathcal{L}(N \cap (z_+, \frac{m}{\beta-1})) = 0$ follows in a way similar to i) by applying T_m, T_{m-1}, $\cdots$, T_{m-r+1} instead of T_0, T_1, $\cdots$, T_{r-1}.

□

Proof of Theorem 2.2.3. Let $\beta \in (1, 2)$ such that $\beta^m - \beta^{m-1} - \cdots - \beta - 1 = 0$ for some integer $m \geq 2$ and let $c = \frac{(m-1)(2-\beta)}{2(m\beta+\beta-2m)}$. We have $c > 0$ since $m - 1 > 0$, $2 - \beta > 0$ and $m\beta + \beta - 2m > 0$, which is a consequence of

$$m + 1 < 2m < 2(\beta^{m-1} + \cdots + \beta + 1) = 2\beta^m = \frac{2}{2-\beta},$$

where the equalities follows from

$$\beta^m = \beta^{m-1} + \cdots + \beta + 1 = \frac{\beta^m - 1}{\beta - 1}.$$

For any $x \in [0, \frac{1}{\beta-1} - 1]$, define

$$f(x) := \frac{(\beta-1)(1-(m-1)x)}{m\beta+\beta-2m}.$$

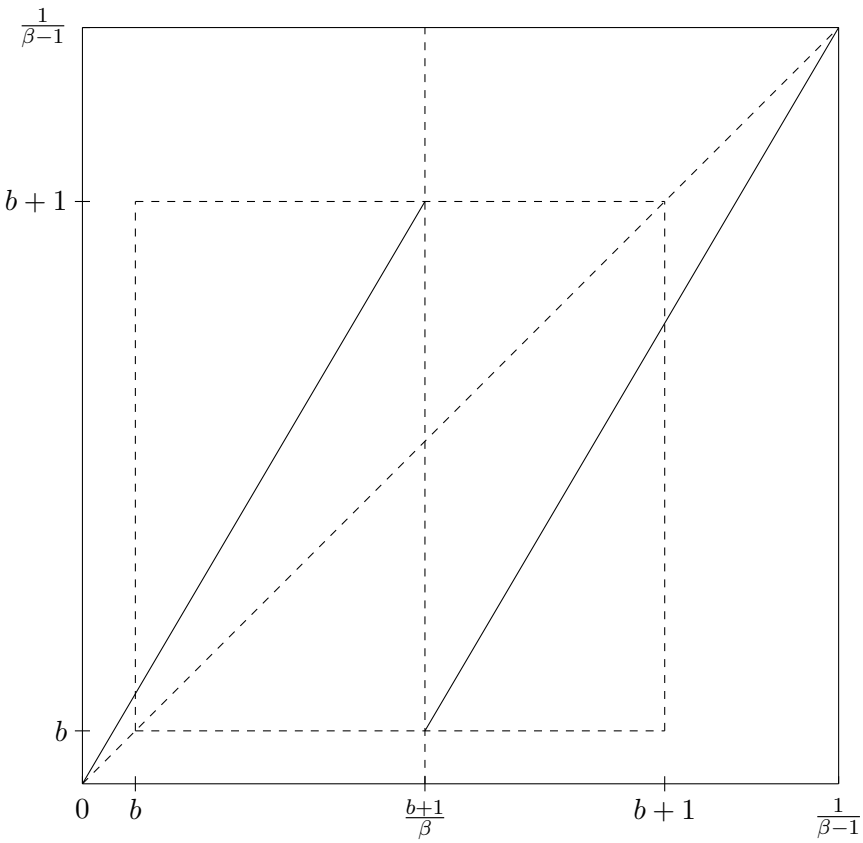


Figure 2.3: The graph of T.

Then

$$f(0) = \frac{\beta-1}{m\beta+\beta-2m} = \frac{1}{2}+c \quad \text{and} \quad f(\frac{1}{\beta-1}-1) = \frac{m\beta+1-2m}{m\beta+\beta-2m} = \frac{1}{2}-c,$$

i.e., $[f(\frac{1}{\beta-1}-1), f(0)] = [\frac{1}{2}-c, \frac{1}{2}+c]$. Since f is continuous, for any $p \in [\frac{1}{2}-c, \frac{1}{2}+c]$, there exists $b \in [0, \frac{1}{\beta-1}-1]$ such that $f(b) = p$. We only consider $b \in [0, \frac{1}{\beta-1}-1)$ in the following, since the proof for the case $b \in (0, \frac{1}{\beta-1}-1]$ is similar. Define $T : [0, \frac{1}{\beta-1}] \to [0, \frac{1}{\beta-1}]$ by

$$T(x) := \begin{cases} T_0(x) = \beta x & \text{for } x \in [0, \frac{b+1}{\beta}), \\ T_1(x) = \beta x - 1 & \text{for } x \in [\frac{b+1}{\beta}, \frac{1}{\beta-1}]. \end{cases}$$

Noting that $T_0(\frac{b+1}{\beta}) = b+1$ and $T_1(\frac{b+1}{\beta}) = b$, by Section 3 in [89], there exists a

T-invariant ergodic measure $\mu \ll \mathcal{L}$ on $[0, \frac{1}{\beta-1}]$ such that for $\mathcal{L}$-a.e. $x \in [0, \frac{1}{\beta-1}]$,

$$\frac{d\mu}{d\mathcal{L}}(x) = \sum_{n=0}^{\infty} \frac{\mathbb{1}_{[0,T^n(b+1)]}(x)}{\beta^n} - \sum_{n=0}^{\infty} \frac{\mathbb{1}_{[0,T^n(b)]}(x)}{\beta^n} \tag{2.10}$$

and $\nu := \frac{1}{\mu([0,\frac{1}{\beta-1}])} \cdot \mu$ is a T-invariant ergodic probability measure on $[0, \frac{1}{\beta-1}]$.

(1) For $1 \le n \le m-1$, prove $T^n(b) = \beta^n b < \frac{b+1}{\beta} \le \beta^n b + \beta^n - \beta^{n-1} - \cdots - \beta - 1 = T^n(b+1)$. Note that $\beta^m = \beta^{m-1} + \cdots + \beta + 1 = \frac{\beta^m - 1}{\beta - 1}$.

① By $b < \frac{1}{\beta-1} - 1 = \frac{1}{\beta^m - 1} \le \frac{1}{\beta^{n+1}-1}$, we get $\beta^n b < \frac{b+1}{\beta}$.

② By $\frac{1}{\beta} + \cdots + \frac{1}{\beta^{n+1}} \le \frac{1}{\beta} + \cdots + \frac{1}{\beta^m} = 1$, we get $\beta^n + \cdots + \beta + 1 \le \beta^{n+1}$ and then $\beta^n + \cdots + \beta + 1 + b \le \beta^{n+1} + \beta^{n+1} b$ which implies $\frac{b+1}{\beta} \le \beta^n b + \beta^n - \beta^{n-1} - \cdots - \beta - 1$.

(2) For $n \ge m$, prove $T^n(b) = T^n(b+1)$.
It suffices to prove $T^m(b) = T^m(b+1)$. In fact, this follows from (1) and $\beta^m b = \beta^m b + \beta^m - \beta^{m-1} - \cdots - \beta - 1$.

Combining (2.10) and (2), we know that for $\mathcal{L}$-a.e. $x \in [0, \frac{1}{\beta-1}]$,

$$\frac{d\mu}{d\mathcal{L}}(x) = \sum_{n=0}^{m-1} \frac{\mathbb{1}_{[0,T^n(b+1)]}(x) - \mathbb{1}_{[0,T^n(b)]}(x)}{\beta^n}. \tag{2.11}$$

Thus

$$\begin{aligned}
\mu[0, \frac{b+1}{\beta}) &= \int_0^{\frac{b+1}{\beta}} \frac{d\mu}{d\mathcal{L}}(x)dx \\
&= \sum_{n=0}^{m-1} \frac{\min\{T^n(b+1), \frac{b+1}{\beta}\} - \min\{T^n(b), \frac{b+1}{\beta}\}}{\beta^n} \\
&\overset{\text{by (1)}}{=\!=} \sum_{n=0}^{m-1} \frac{\frac{b+1}{\beta} - \beta^n b}{\beta^n} \\
&= 1 - (m-1)b
\end{aligned}$$

where the last equality follows from $\frac{1}{\beta} + \cdots + \frac{1}{\beta^m} = 1$. By

$$\begin{aligned}
\mu([0, \frac{1}{\beta-1}]) &= \int_0^{\frac{1}{\beta-1}} \frac{d\mu}{d\mathcal{L}}(x)dx \\
&= \sum_{n=0}^{m-1} \frac{T^n(b+1) - T^n(b)}{\beta^n} \\
&\overset{\text{by (1)}}{=\!=} 1 + \sum_{n=1}^{m-1} \frac{\beta^n - \beta^{n-1} - \cdots - \beta - 1}{\beta^n}
\end{aligned}$$

$$\begin{aligned} &= 1+\sum_{n=1}^{m-1}(1-\frac{1}{\beta}-\cdots-\frac{1}{\beta^n}) \\ &= m-\frac{m-1}{\beta}-\frac{m-2}{\beta^2}-\cdots-\frac{1}{\beta^{m-1}}, \end{aligned}$$

we get

$$\frac{1}{\beta}\cdot\mu([0,\frac{1}{\beta-1}])=\frac{m}{\beta}-\frac{m-1}{\beta^2}-\frac{m-2}{\beta^3}-\cdots-\frac{1}{\beta^m}.$$

It follows from the subtraction of the above two equalities that $\mu([0,\frac{1}{\beta-1}])=\frac{m\beta+\beta-2m}{\beta-1}$. Therefore $\nu=\frac{\beta-1}{m\beta+\beta-2m}\cdot\mu$ and

$$\nu[0,\frac{b+1}{\beta})=\frac{(\beta-1)(1-(m-1)b)}{m\beta+\beta-2m}=f(b)=p.$$

Since $T:[0,\frac{1}{\beta-1}]\to[0,\frac{1}{\beta-1}]$ is ergodic with respect to ν, it follows from Birkhoff's Ergodic Theorem that for ν-a.e. $x\in[0,\frac{1}{\beta-1}]$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_{[0,\frac{b+1}{\beta})}T^k(x)=\int_0^{\frac{1}{\beta-1}}\mathbb{1}_{[0,\frac{b+1}{\beta})}d\nu=\nu[0,\frac{b+1}{\beta})=p,$$

which implies that for ν-a.e. $x\in[b,b+1]$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_{[0,\frac{b+1}{\beta})}T^k(x)=p.$$

By (2.11) and (1), we know that for $\mathcal{L}$-a.e. $x\in[b,b+1]$, $\frac{d\mu}{d\mathcal{L}}(x)\geq 1$. This implies $\mathcal{L}\ll\mu(\sim\nu)$ on $[b,b+1]$, and then for $\mathcal{L}$-a.e. $x\in[b,b+1]$, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_{[0,\frac{b+1}{\beta})}T^k(x)=p.$$

For every $x\in[0,\frac{1}{\beta-1}]$, define a sequence $\varepsilon(x)=(\varepsilon_i(x))_{i\geq1}\in\{0,1\}^{\mathbb{N}}$ by

$$\varepsilon_i(x):=\begin{cases}0 & \text{if } T^{i-1}x\in[0,\frac{b+1}{\beta}) \\ 1 & \text{if } T^{i-1}x\in[\frac{b+1}{\beta},\frac{1}{\beta-1}]\end{cases}\qquad\text{for all } i\geq1.$$

Then by

$$\mathbb{1}_{[0,\frac{b+1}{\beta})}(T^kx)=1\Leftrightarrow T^kx\in[0,\frac{b+1}{\beta})\Leftrightarrow\varepsilon_{k+1}(x)=0,$$

we know that for $\mathcal{L}$-a.e. $x\in[b,b+1]$,

$$\lim_{n\to\infty}\frac{\#\{1\leq i\leq n:\varepsilon_i(x)=0\}}{n}=p,\quad\text{i.e.,}\quad \mathrm{Freq}_0(\varepsilon(x))=p. \tag{2.12}$$

By the same way as in the proof of Theorem 2.2.2, we know that for every $x \in [0, \frac{1}{\beta-1}]$, the $\varepsilon(x)$ defined above is a β-expansion of x, and $\mathcal{L}$-a.e. $x \in [0, \frac{1}{\beta-1}]$ has a β-expansion with frequency of zeros equal to p. Then we finish the proof by applying Theorem 2.2.1. $\square$

2.2.3 Further questions

First we wonder whether Theorem 2.2.1 can be generalized.

Question 2.2.5. *Let $m \in \mathbb{N}$, $\beta \in (1, m+1) \setminus \mathbb{N}$ and $\overline{p}, \underline{p} \in [0,1]^{m+1}$. Is it true that Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has a β-expansion of frequency $(\overline{p}, \underline{p})$ if and only if Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has a continuum of β-expansions of frequency $(\overline{p}, \underline{p})$?*

If a positive answer is given to this question, by Theorem 2.2.2, there is also a positive answer to the following question.

Question 2.2.6. *Let $m \in \mathbb{N}$ and $\beta \in (1, m+1) \setminus \mathbb{N}$. Is it true that Lebesgue almost every $x \in [0, \frac{m}{\beta-1}]$ has a continuum of balanced β-expansions?*

Even if a negative answer is given to Question 2.2.5, there may be a positive answer to Question 2.2.6 when $m \geq 2$. An intuitive reason is that, when $\#\{0, 1, \cdots, m\} \geq 3$, balanced β-expansions are much more flexible than simply normal β-expansions (see [25, Theorem 4.1]).

The last question we want to ask is on the variability of the frequency related to Theorem 2.2.3. Let $\beta > 1$. If there exists $c = c(\beta) > 0$ such that for any $p_0, p_1, \cdots, p_{\lceil\beta\rceil-1} \in [\frac{1}{\lceil\beta\rceil} - c, \frac{1}{\lceil\beta\rceil} + c]$ with $p_0 + p_1 + \cdots + p_{\lceil\beta\rceil-1} = 1$, every $x \in (0, \frac{\lceil\beta\rceil-1}{\beta-1})$ has a β-expansion $w = (w_i)_{i\geq 1}$ with

$$\mathrm{Freq}_0(w) = p_0,\ \mathrm{Freq}_1(w) = p_1, \cdots,\ \mathrm{Freq}_{\lceil\beta\rceil-1}(w) = p_{\lceil\beta\rceil-1},$$

we say that β is a *variational frequency* base. Similarly, if there exists $c = c(\beta) > 0$ such that for any $p_0, p_1, \cdots, p_{\lceil\beta\rceil-1} \in [\frac{1}{\lceil\beta\rceil} - c, \frac{1}{\lceil\beta\rceil} + c]$ with $p_0 + p_1 + \cdots + p_{\lceil\beta\rceil-1} = 1$, Lebesgue almost every $x \in [0, \frac{\lceil\beta\rceil-1}{\beta-1}]$ has a β-expansion $w = (w_i)_{i\geq 1}$ with

$$\mathrm{Freq}_0(w) = p_0,\ \mathrm{Freq}_1(w) = p_1, \cdots,\ \mathrm{Freq}_{\lceil\beta\rceil-1}(w) = p_{\lceil\beta\rceil-1},$$

we say that β is an *almost variational frequency* base.

Obviously, all variational frequency bases are almost variational frequency bases. Baker's results (see the statements between Theorems 2.2.2 and 2.2.3) say that all numbers in $(1, \frac{1+\sqrt{5}}{2})$ are variational frequency bases and all numbers in $[\frac{1+\sqrt{5}}{2}, 2)$ are not variational frequency bases. Fortunately, Theorem 2.2.3 says that pseudo-golden ratios (which are all in $[\frac{1+\sqrt{5}}{2}, 2)$) are almost variational frequency bases. We wonder whether all numbers in $[\frac{1+\sqrt{5}}{2}, 2)$ are almost variational frequency bases.

For all integers $\beta > 1$, we know that Lebesgue almost every $x \in [0,1]$ has a unique β-expansion $w = (w_i)_{i\geq 1}$, and this expansion satisfies

$$\mathrm{Freq}_0(w) = \mathrm{Freq}_1(w) = \cdots = \mathrm{Freq}_{\beta-1}(w) = \frac{1}{\beta}$$

by Borel's normal number theorem. Therefore all integers are not almost variational frequency bases. It is natural to ask the following question.

Question 2.2.7. *Is it true that all non-integers greater than* 1 *are almost variational frequency bases?*

2.3 Bernoulli-type measures and frequency sets of univoque sequences

Let $\{0,1\}^* := \bigcup_{n=1}^{\infty}\{0,1\}^n$ be the set of finite words and recall that $\{0,1\}^{\mathbb{N}}$ is the set of infinite sequences on $\{0,1\}$. For any integer $m \geq 3$, define

$$\Lambda^{(m)} := \Big\{w \in \{0,1\}^{\mathbb{N}} : w \text{ does not contain } 0^m \text{ or } 1^m\Big\},$$

$$\Lambda^{(m),*} := \Big\{w \in \{0,1\}^{*} : w \text{ does not contain } 0^m \text{ or } 1^m\Big\}$$

and

$$\Lambda^{(m),n} := \Big\{w \in \{0,1\}^{n} : w \text{ does not contain } 0^m \text{ or } 1^m\Big\}$$

where $n \in \mathbb{N}$. Given $w \in \Lambda^{(m),n}$, we call

$$[w] := \Big\{v \in \Lambda^{(m)} : v_1 = w_1, \cdots, v_n = w_n\Big\}$$

the *cylinder* of order n in $\Lambda^{(m)}$ generated by w.

Let $\mathcal{B}(\Lambda^{(m)})$ be the Borel sigma-algebra on $\Lambda^{(m)}$ (equipped with the usual metric d_2) and $p \in (0,1)$. We define the $(p, 1-p)$ *Bernoulli-type measure* μ_p on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ as follows:

I. Let
$$\mu_p(\emptyset) := 0, \quad \mu_p(\Lambda^{(m)}) := 1, \quad \mu_p[0] := p, \quad \text{and} \quad \mu_p[1] := 1-p.$$

II. Suppose that μ_p has been defined for all cylinders of order $n \in \mathbb{N}$. For any $w \in \Lambda^{(m),n}$, if $w0, w1 \in \Lambda^{(m),n+1}$, we define

$$\mu_p[w0] := p\mu_p[w] \quad \text{and} \quad \mu_p[w1] := (1-p)\mu_p[w];$$

if $w0 \in \Lambda^{(m),n+1}$ but $w1 \notin \Lambda^{(m),n+1}$, then $[w0] = [w]$ and naturally we have

$$\mu_p[w0] = \mu_p[w];$$

if $w1 \in \Lambda^{(m),n+1}$ but $w0 \notin \Lambda^{(m),n+1}$, then $[w1] = [w]$ and naturally we have

$$\mu_p[w1] = \mu_p[w].$$

III. By Carathéodory's measure extension theorem, we uniquely extend μ_p from its definition on the family of cylinders to become a measure on $\mathcal{B}(\Lambda^{(m)})$.

Let $\sigma_m : \Lambda^{(m)} \to \Lambda^{(m)}$ be the *shift map* defined by

$$\sigma_m(w_1 w_2 w_3 \cdots) := w_2 w_3 w_4 \cdots \quad \text{for } (w_n)_{n\geq 1} \in \Lambda^{(m)}.$$

The first main result in this section is the following.

Theorem 2.3.1. *Let $m \geq 3$ be an integer and $p \in (0,1)$. Then there exists a unique σ_m-invariant ergodic probability measure λ_p on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ equivalent to μ_p, where λ_p is given by*

$$\lambda_p(B) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sigma_m^k \mu_p(B) \quad \textit{for } B \in \mathcal{B}(\Lambda^{(m)}).$$

As an application of this theorem, we consider frequency sets of univoque sequences in the following. Define

$$\Gamma := \left\{ w \in \{0,1\}^{\mathbb{N}} : \overline{w} \prec \sigma^k w \prec w \text{ for all } k \geq 1 \right\}$$

where σ is the shift map on $\{0,1\}^{\mathbb{N}}$, $\overline{0} := 1$, $\overline{1} := 0$ and $\overline{w} := \overline{w}_1 \overline{w}_2 \cdots$ for all $w = w_1 w_2 \cdots \in \{0,1\}^{\mathbb{N}}$.

The set Γ is strongly related to two well known research topics, iterations of unimodal functions and unique expansions of real numbers (see [9] for more details).

On the one hand, in 1985, Cosnard [41] proved that a sequence $\alpha = (\alpha_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ is the kneading sequence of 1 for some unimodal function if and only if $\tau(\alpha) \in \Gamma'$, where $\tau : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is a bijection defined by $\tau(w) := (\sum_{i=1}^n w_i \pmod 2)_{n\geq 1}$ and

$$\Gamma' := \left\{ w \in \{0,1\}^{\mathbb{N}} : \overline{w} \preceq \sigma^k w \preceq w \text{ for all } k \geq 0 \right\}$$

is similar to Γ in the sense that $\Gamma' \setminus \{\text{periodic sequences}\} = \Gamma$. The structure of $\Gamma' \setminus \{(10)^\infty\}$ was studied in detail by Allouche [3] (see also [7]). The generalizations of Γ and Γ' (to more than two digits) were studied in [3, 10].

On the other hand, in 1990, Erdös, Joó and Komornik [61] proved that a sequence $\alpha = (\alpha_n)_{n\geq 1} \in \{0,1\}^{\mathbb{N}}$ is the unique expansion of 1 in some base $\beta \in (1,2)$ if and only if

$\alpha \in \Gamma$. Thus we call Γ the set of *univoque sequences* in this section. Note that the term "univoque sequence" is different in some papers [45, 52, 53].

Recall from Section 2.2 that for any sequence $w = (w_n)_{n\geq 1}$, we use $\mathrm{Freq}_k(w)$, $\underline{\mathrm{Freq}}_k(w)$ and $\overline{\mathrm{Freq}}_k(w)$ to denote respectively the frequency, lower frequency and upper frequency of the digit k in w.

Given $a \in [0,1]$, define the *frequency subsets* of Γ by

$$\Gamma_a := \Big\{w \in \Gamma : \mathrm{Freq}_0(w) = a\Big\},$$

$$\underline{\Gamma}_a := \Big\{w \in \Gamma : \underline{\mathrm{Freq}}_0(w) = a\Big\},$$

$$\overline{\Gamma}_a := \Big\{w \in \Gamma : \overline{\mathrm{Freq}}_0(w) = a\Big\},$$

and define the *frequency subsets* of

$$\Lambda := \Big\{w \in \{0,1\}^{\mathbb{N}} : \text{the lengths of consecutive 0's and consecutive 1's in } w \text{ are bounded}\Big\}$$

by

$$\Lambda_a := \Big\{w \in \Lambda : \mathrm{Freq}_0(w) = a\Big\},$$

$$\underline{\Lambda}_a := \Big\{w \in \Lambda : \underline{\mathrm{Freq}}_0(w) = a\Big\},$$

$$\overline{\Lambda}_a := \Big\{w \in \Lambda : \overline{\mathrm{Freq}}_0(w) = a\Big\}.$$

It is straightforward to check $\Gamma \subset \Lambda$. Let

$$\mathcal{U} := \Big\{\beta \in (1,2) : 1 \text{ has a unique } \beta\text{-expansion on } \{0,1\}\Big\}$$

be the set of *univoque bases*. It is proved in [46, 83] that $\mathcal{U}$ is of full Hausdorff dimension. That is,

$$\dim_H \mathcal{U} = 1.$$

For more research on $\mathcal{U}$, we refer the reader to [55, 86, 88].

On frequency sets, recall the well known result given by Eggleston [59], which says that for any $a \in [0,1]$, the *classical Eggleston-Besicovitch set* has Hausdorff dimension

$$\dim_H \Big\{x \in [0,1) : \mathrm{Freq}_0(\varepsilon(x)) = a\Big\} = \frac{-a\log a - (1-a)\log(1-a)}{\log 2}, \tag{2.13}$$

where $\varepsilon(x) := \varepsilon_1(x)\varepsilon_2(x)\cdots\varepsilon_n(x)\cdots$ is the *greedy binary expansion* of x, and $0\log 0 := 0$.

Motivated by the above mentioned results, correspondingly, we study the set of univoque sequences Γ, the larger set Λ, and their frequency subsets Γ_a, $\underline{\Gamma}_a$, $\overline{\Gamma}_a$, Λ_a, $\underline{\Lambda}_a$, $\overline{\Lambda}_a$. By applying Theorem 2.3.1, we give the next theorem as the second main result in this section.

Let $\dim_H(\cdot, d_2)$ denote the Hausdorff dimension in $\{0,1\}^{\mathbb{N}}$ equipped with the usual metric d_2.

Theorem 2.3.2. (1) *We have* $\dim_H(\Gamma, d_2) = \dim_H(\Lambda, d_2) = 1$.
(2) *For all* $a \in [0,1]$ *we have*

$$\begin{aligned}&\dim_H(\Gamma_a, d_2) = \dim_H(\underline{\Gamma}_a, d_2) = \dim_H(\overline{\Gamma}_a, d_2)\\ =&\dim_H(\Lambda_a, d_2) = \dim_H(\underline{\Lambda}_a, d_2) = \dim_H(\overline{\Lambda}_a, d_2) = \frac{-a\log a - (1-a)\log(1-a)}{\log 2},\end{aligned}$$

where $0\log 0 := 0$.

It is known that by defining Bernoulli measures, and then calculating the lower local dimension of the measures and using Billingsley Lemma [63, Proposition 2.3], the Hausdorff dimension of classical Eggleston-Besicovitch sets mentioned above can be obtained. But this is based on the fact that only expansions in integer bases are considered in classical Eggleston-Besicovitch sets, there are no forbidden words in the symbolic space and the Bernoulli measures are invariant and ergodic with respect to the shift map. Ergodicity garuantees that classical Eggleston-Besicovitch sets have positive Bernoulli measures, which is a condition needed for applying Billingsley Lemma to get the lower bound of the Hausdorff dimension. If there are forbidden words, such as expansions in non-integer bases in Section 1.2, the corresponding Bernoulli-type measures are not ergodic (actually not invariant). This makes some difficulties to be overcome. In Section 1.2, after defining Bernoulli-type measures, we found out the equivalent invariant ergodic measures, studied the relation between the equivalent measures and the original measures and obtained the Hausdorff dimension of Eggleston-Besicovitch (frequency) sets for a class of non-integer bases (see Theorem 1.2.6) by applying an avatar of the Billingsley Lemma. This section follows a similar framework and construction, but most of the details we need to confirm are different.

For any $a \in [0,1]$ we define the *global frequency sets* in $\{0,1\}^{\mathbb{N}}$ by

$$G_a := \Big\{w \in \{0,1\}^{\mathbb{N}} : \mathrm{Freq}_0(w) = a\Big\},$$

$$\underline{G}_a := \Big\{w \in \{0,1\}^{\mathbb{N}} : \underline{\mathrm{Freq}}_0(w) = a\Big\},$$

$$\overline{G}_a := \Big\{w \in \{0,1\}^{\mathbb{N}} : \overline{\mathrm{Freq}}_0(w) = a\Big\},$$

and for any integer $m \geq 3$ we let

$$\Lambda_a^{(m)} := \Lambda^{(m)} \cap G_a.$$

Here we give an outline for the proof of Theorem 2.3.2 (2) to explain how the concepts in this section interact. Following the simple argument at the beginning of the *Proof of*

Theorem 2.3.2 in Subsection 2.3.4, we know that it suffices to consider the lower bound of $\dim_H(\Gamma_a, d_2)$. Since (2.16) says that $\dim_H(\Gamma_a, d_2) \geq \dim_H(\Lambda_a^{(m)}, d_2)$ for any integer $m \geq 3$, we only need to find a good lower bound for $\dim_H(\Lambda_a^{(m)}, d_2)$. Hence we apply the Bernoulli-type measure μ_p to the Billingsley Lemma in metric space (Proposition 2.3.5), and the unique equivalent σ_m-invariant ergodic measure λ_p in Theorem 2.3.1 (with a suitable p) can guarantee that $\Lambda_a^{(m)}$ has positive measure, which is needed by the Billingsley Lemma. Then we obtain a good lower bound of $\dim_H(\Lambda_a^{(m)}, d_2)$ in Lemma 2.3.16.

This section is organized as follows. In Subsection 2.3.1, we recall some basic notation and preliminaries. In Subsection 2.3.2, we study related digit occurrence parameters and their properties which will be used later. In Subsection 2.3.3, we study Bernoulli-type measures and prove Theorem 2.3.1. Finally we prove Theorem 2.3.2 in Subsection 2.3.4.

2.3.1 Notation and preliminaries

For a finite word $w \in \{0,1\}^*$, we use $|w|$, $|w|_0$ and $|w|_1$ to denote its length, the number of 0's in w and the number of 1's in w respectively. Recall that $w|_k := w_1 w_2 \cdots w_k$ denotes the prefix of w with length k for $w \in \{0,1\}^{\mathbb{N}}$ or $w \in \{0,1\}^n$ where $n \geq k$.

First we recall the following concept.

Definition 2.3.3. *Let μ be a finite Borel measure on a metric space (X, d). The lower local dimension of μ at $x \in X$ is defined by*

$$\underline{\dim}_{loc}\mu(x) := \underline{\lim}_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ is the closed ball centered at x with radius r.

In $\mathbb{R}^n$, recall that we can use the lower local dimension to estimate the upper and lower bounds of the Hausdorff dimension by the following proposition, which is called Billingsley Lemma.

Proposition 2.3.4 ([63] Proposition 2.3)**.** *Let $E \subset \mathbb{R}^n$ be a Borel set, μ be a finite Borel measure on $\mathbb{R}^n$ and $s \geq 0$.*

(1) *If $\underline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$, then $\dim_H E \leq s$.*

(2) *If $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $\dim_H E \geq s$.*

We need to use the following version which is a generalization to metric spaces. For the sake of completeness we give a self-contained proof.

Proposition 2.3.5. *Let (X, d) be a metric space, $E \subset X$ be a Borel set, μ be a finite Borel measure on X and $s \geq 0$. If $\mu(E) > 0$ and $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$, then $\dim_H(E, d) \geq s$.*

The main we need to prove is the following.

Lemma 2.3.6. *Let (X, d) be a metric space, $E \subset X$ be a Borel set, μ be a finite Borel measure on X, $s \geq 0$ and $c > 0$. If $\overline{\lim}_{r\to 0} \frac{\mu(B(x,r))}{r^s} < c$ for all $x \in E$, then $\mathcal{H}^s(E, d) \geq \frac{\mu(E)}{c}$.*

Proof. For any $\delta > 0$, let

$$E_\delta := \{x \in E : \mu(B(x, r)) \leq cr^s \text{ for all } r \in (0, \delta)\}.$$

(1) Prove that E_δ is a Borel set. We define

$$F_q := \{x \in E : \mu(B(x, q)) \leq cq^s\} \quad \text{for } q \in \mathbb{Q}.$$

It suffices to prove the following ① and ②.

① Prove $E_\delta = \bigcap_{q\in\mathbb{Q}\cap(0,\delta)} F_q$.
$\boxed{\subset}$ follows from $E_\delta \subset F_q$ for all $q \in \mathbb{Q} \cap (0, \delta)$.
$\boxed{\supset}$ Let $x \in \bigcap_{q\in\mathbb{Q}\cap(0,\delta)} F_q$. For any $r \in (0, \delta)$, there exist $q_1, q_2, \cdots, q_n, \cdots \in \mathbb{Q}\cap(0, \delta)$ decreasing to r. By $x \in \bigcap_{n=1}^{\infty} F_{q_n}$ we get $\mu(B(x, q_n)) \leq cq_n^s$ for all $n \in \mathbb{N}$. Thus

$$\mu(B(x, r)) = \mu(\bigcap_{n=1}^{\infty} B(x, q_n)) = \lim_{n\to\infty} \mu(B(x, q_n)) \leq \lim_{n\to\infty} cq_n^s = cr^s.$$

This implies $x \in E_\delta$.

② Prove that F_q is a Borel set.
Define $f(x) := \mu(B(x, q))$ for $x \in X$. Then $F_q = E \cap f^{-1}(-\infty, cq^s]$. We only need to prove that f is a Borel function. For any $a \in \mathbb{R}$, it suffices to prove that $f^{-1}(-\infty, a)$ is an open set. If $f^{-1}(-\infty, a) = \emptyset$, it is obviously open. We only need to consider $f^{-1}(-\infty, a) \neq \emptyset$ in the following.

Let $x_0 \in f^{-1}(-\infty, a)$. Then $\mu(B(x_0, q)) < a$. Since $\mu(B(x_0, q + \delta))$ decreases to $\mu(B(x_0, q))$ as δ decreases to 0, there exists $\delta_0 > 0$ such that $\mu(B(x_0, q + \delta_0)) < a$. It suffices to prove that the open ball $B^o(x_0, \delta_0) := \{x \in X : d(x, x_0) < \delta_0\} \subset f^{-1}(-\infty, a)$.

In fact, for any $x \in B^o(x_0, \delta_0)$, by $B(x, q) \subset B(x_0, q + \delta_0)$ we get $\mu(B(x, q)) \leq \mu(B(x_0, q + \delta_0)) < a$, which implies $x \in f^{-1}(-\infty, a)$.

(2) Prove that E_δ increases to E as δ decreases to 0.

① If $0 < \delta_2 < \delta_1$, then obviously $E_{\delta_1} \subset E_{\delta_2}$.

② Prove $E = \cup_{\delta>0} E_\delta$.
$\boxed{\supset}$ follows from $E \supset E_\delta$ for all $\delta > 0$.

$\boxed{\subset}$ Let $x \in E$. By $\overline{\lim}_{r\to 0} \frac{\mu(B(x,r))}{r^s} < c$, there exists $\delta_x > 0$ such that for all $r \in (0, \delta_x)$, $\mu(B(x,r)) \le cr^s$. Thus $x \in E_{\delta_x} \subset \cup_{\delta>0} E_\delta$.

(3) Prove $\mathcal{H}^s(E,d) \ge \frac{\mu(E)}{c}$.
Fix $\delta > 0$. Let $\{U_k\}_{k\in K}$ be a countable δ-cover of E, i.e.,

$$|U_k| \le \delta \text{ for all } k \in K \quad \text{and} \quad \bigcup_{k\in K} U_k \supset E \ (\supset E_\delta).$$

Let $K' := \{k \in K : U_k \cap E_\delta \ne \emptyset\}$. Then $\bigcup_{k\in K'} U_k \supset E_\delta$. For any $k \in K'$, let $x_k \in U_k \cap E_\delta$ and $B_k := B(x_k, |U_k|) \supset U_k$. Then $\bigcup_{k\in K'} B_k \supset E_\delta$. It follows that

$$\sum_{k\in K} |U_k|^s \ge \frac{1}{c} \sum_{k\in K'} c|U_k|^s \overset{(\star)}{\ge} \frac{1}{c} \sum_{k\in K'} \mu(B(x_k, |U_k|)) \ge \frac{1}{c} \cdot \mu(\bigcup_{k\in K'} B_k) \ge \frac{\mu(E_\delta)}{c}$$

where $(\star)$ follows from $x_k \in E_\delta$. By the randomness of the choice of the δ-cover $\{U_k\}_{k\in K}$, we get $\mathcal{H}^s_\delta(E,d) \ge \frac{\mu(E_\delta)}{c}$ and then $\mathcal{H}^s(E,d) \ge \frac{\mu(E_\delta)}{c}$. Let $\delta \to 0$, by (1) and (2) we get $\mathcal{H}^s(E,d) \ge \frac{\mu(E)}{c}$. □

Proof of Proposition 2.3.5. If $s = 0$, the conclusion is obvious. If $s > 0$, let $t \in (0,s)$. For any $x \in E$, by $\underline{\lim}_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} > t$, there exists $\delta_x \in (0,1)$ such that for any $r \in (0,\delta_x)$, $\frac{\log \mu(B(x,r))}{\log r} > t$ and then $\mu(B(x,r)) < r^t$. Thus $\overline{\lim}_{r\to 0} \frac{\mu(B(x,r))}{r^t} \le 1 < 2$ for all $x \in E$. By Lemma 2.3.6, we get $\mathcal{H}^t(E,d) \ge \frac{\mu(E)}{2} > 0$. Thus $\dim_H(E,d) \ge t$ for all $t \in (0,s)$, which implies $\dim_H(E,d) \ge s$. □

2.3.2 Digit occurrence parameters

The digit occurrence parameters and their properties studied in this subsection will be used in Subsections 2.3.3 and 2.3.4.

Definition 2.3.7 (Digit occurrence parameters)**.** *Let $m \ge 3$ be an integer. For any $w \in \Lambda^{(m),*}$, define*

$$\mathcal{N}_0^{(m)}(w) := \Big\{k : 1 \le k \le |w|, w_k = 0 \text{ and } w_1 \dots w_{k-1}1 \in \Lambda^{(m),*}\Big\},$$

$$\mathcal{N}_1^{(m)}(w) := \Big\{k : 1 \le k \le |w|, w_k = 1 \text{ and } w_1 \dots w_{k-1}0 \in \Lambda^{(m),*}\Big\},$$

and let

$$N_0^{(m)}(w) := \#\mathcal{N}_0^{(m)}(w) \quad \text{and} \quad N_1^{(m)}(w) := \#\mathcal{N}_1^{(m)}(w).$$

Proposition 2.3.8. *Let $m \ge 3$ be an integer and $w, v \in \Lambda^{(m),*}$ such that $wv \in \Lambda^{(m),*}$. Then*

(1) $N_0^{(m)}(w) + N_0^{(m)}(v) - 1 \le N_0^{(m)}(wv) \le N_0^{(m)}(w) + N_0^{(m)}(v)$;

(2) $N_1^{(m)}(w) + N_1^{(m)}(v) - 1 \le N_1^{(m)}(wv) \le N_1^{(m)}(w) + N_1^{(m)}(v)$.

Proof. Let $a = |w|$ and $b = |v|$.

(1) ① Prove $N_0^{(m)}(wv) \le N_0^{(m)}(w) + N_0^{(m)}(v)$.
It suffices to prove $\mathcal{N}_0^{(m)}(wv) \subset \mathcal{N}_0^{(m)}(w) \cup (\mathcal{N}_0^{(m)}(v) + a)$, where $\mathcal{N}_0^{(m)}(v) + a := \{j + a : j \in \mathcal{N}_0^{(m)}(v)\}$. Let $k \in \mathcal{N}_0^{(m)}(wv)$.

i) If $1 \le k \le a$, then $w_k = 0$, $w_1 \cdots w_{k-1}1 \in \Lambda^{(m),*}$ and we get $k \in \mathcal{N}_0^{(m)}(w)$.

ii) If $a + 1 \le k \le a + b$, then $v_{k-a} = 0$ and $w_1 \cdots w_a v_1 \cdots v_{k-a-1}1 \in \Lambda^{(m),*}$. It follows from $v_1 \cdots v_{k-a-1}1 \in \Lambda^{(m),*}$ that $k - a \in \mathcal{N}_0^{(m)}(v)$ and $k \in \mathcal{N}_0^{(m)}(v) + a$.

② Prove $N_0^{(m)}(w) + N_0^{(m)}(v) \le N_0^{(m)}(wv) + 1$.
When $v = 1^b$, we get $N_0^{(m)}(v) = 0$ and then the conclusion follows immediately from $N_0^{(m)}(w) \le N_0^{(m)}(wv)$. Thus it suffices to consider $v \ne 1^b$ in the following. Let $s \in \{1, \cdots, b\}$ be the smallest such that $v_1 = \cdots = v_{s-1} = 1$ and $v_s = 0$. In order to get the conclusion, it suffices to show $\mathcal{N}_0^{(m)}(w) \cup (a + \mathcal{N}_0^{(m)}(v)) \subset \mathcal{N}_0^{(m)}(wv) \cup \{a + s\}$. Since $\mathcal{N}_0^{(m)}(w) \subset \mathcal{N}_0^{(m)}(wv)$, we only need to prove $(a + \mathcal{N}_0^{(m)}(v)) \subset \mathcal{N}_0^{(m)}(wv) \cup \{a + s\}$. Let $k \in \mathcal{N}_0^{(m)}(v) \setminus \{s\}$. It suffices to check $a + k \in \mathcal{N}_0^{(m)}(wv)$. By $v_k = 0$, we only need to prove $w_1 \cdots w_a v_1 \cdots v_{k-1}1 \in \Lambda^{(m),*}$. (By contradiction) Assume $w_1 \cdots w_a v_1 \cdots v_{k-1}1 \notin \Lambda^{(m),*}$. Then $w_1 \cdots w_a v_1 \cdots v_{k-1}1$ contains 0^m or 1^m.

i) If $w_1 \cdots w_a v_1 \cdots v_{k-1}1$ contains 0^m, then $w_1 \cdots w_a v_1 \cdots v_{k-1}$ contains 0^m. This contradicts $wv \in \Lambda^{(m),*}$.

ii) If $w_1 \cdots w_a v_1 \cdots v_{k-1}1$ contains 1^m, by $k \ge s + 1$, we know that

$$w_1 \cdots w_a v_1 \cdots v_{s-1} 0 v_{s+1} \cdots v_{k-1} 1$$

contains 1^m. Thus $w_1 \cdots w_a v_1 \cdots v_{s-1}$ contains 1^m or $v_{s+1} \cdots v_{k-1}1$ contains 1^m. But $w_1 \cdots w_a v_1 \cdots v_{s-1}$ contains 1^m will contradict $wv \in \Lambda^{(m),*}$, and $v_{s+1} \cdots v_{k-1}1$ contains 1^m will imply $v_1 \cdots v_{k-1}1$ contains 1^m which contradicts $k \in \mathcal{N}_0^{(m)}(v)$.

(2) follows in the same way as (1). □

Proposition 2.3.9. *Let $m \ge 3$ be an integer and $w \in \Lambda^{(m),*}$. Then*

(1) $m \cdot |w|_0 \le (m-1)N_0^{(m)}(w) + |w|$;

(2) $m \cdot |w|_1 \le (m-1)N_1^{(m)}(w) + |w|$.

Proof. (1) Let $n = |w|$. If $n \le m - 1$, the conclusion follows immediately from $N_0^{(m)}(w) = |w|_0$. In the following, we assume $n \ge m$. Recall

$$\mathcal{N}_0^{(m)}(w) = \left\{k : 1 \le k \le n, w_k = 0, w_1 \cdots w_{k-1}1 \in \Lambda^{(m),*}\right\} \quad \text{and} \quad N_0^{(m)}(w) = \#\mathcal{N}_0^{(m)}(w).$$

We define

$$\mathcal{N}^{(m)}_{1^{m-1}0}(w) := \left\{k : m \le k \le n, w_{k-m+1}\cdots w_{k-1}w_k = 1^{m-1}0\right\} \quad \text{and} \quad N^{(m)}_{1^{m-1}0} := \#\mathcal{N}^{(m)}_{1^{m-1}0}(w).$$

① Prove $\{k : 1 \le k \le n, w_k = 0\} = \mathcal{N}^{(m)}_0(w) \cup \mathcal{N}^{(m)}_{1^{m-1}0}(w)$.
$\boxed{\supset}$ Obvious.
$\boxed{\subset}$ Let $k \in \{1, \cdots, n\}$ such that $w_k = 0$. If $k \notin \mathcal{N}^{(m)}_0(w)$, then $k \ge m$ and $w_1 \cdots w_{k-1}1 \notin \Lambda^{(m),*}$. By $w_1 \cdots w_{k-1} \in \Lambda^{(m),*}$, we get $w_{k-m+1}\cdots w_{k-1} = 1^{m-1}$. This implies $k \in \mathcal{N}^{(m)}_{1^{m-1}0}(w)$.

② Prove $\mathcal{N}^{(m)}_0(w) \cap \mathcal{N}^{(m)}_{1^{m-1}0}(w) = \emptyset$.
(By contradiction) Assume that there exists $k \in \mathcal{N}^{(m)}_0(w) \cap \mathcal{N}^{(m)}_{1^{m-1}0}(w)$. Then $k \ge m$, $w_{k-m+1}\cdots w_{k-1} = 1^{m-1}$ and $w_1 \cdots w_{k-1}1 \in \Lambda^{(m),*}$. These imply $w_1 \cdots w_{k-m}1^m \in \Lambda^{(m),*}$, which contradicts the definition of $\Lambda^{(m),*}$.

Combining ① and ②, we get $|w|_0 = N^{(m)}_0(w) + N^{(m)}_{1^{m-1}0}(w)$. It follows from $(m-1)N^{(m)}_{1^{m-1}0}(w) \le |w|_1 = |w| - |w|_0$ that $(m-1)(|w|_0 - N^{(m)}_0(w)) \le |w| - |w|_0$, i.e., $m \cdot |w|_0 \le (m-1)N^{(m)}_0(w) + |w|$.
(2) follows in the same way as (1). □

2.3.3 Proof of Theorem 2.3.1

Let $p \in (0,1)$. Recall the definition of the Bernoulli-type measure μ_p from the introduction.

Remark 2.3.10. *We have*

$$\mu_p[w] = p^{N^{(m)}_0(w)}(1-p)^{N^{(m)}_1(w)} \quad \textit{for all } w \in \Lambda^{(m),*}.$$

Note that μ_p is not σ_m-invariant. In fact, for all $p \in (0,1)$, we have

$$\mu_p[0^{m-2}1] = p^{m-2}(1-p),$$

but

$$\mu_p(\sigma_m^{-1}[0^{m-2}1]) = \mu_p[0^{m-1}1] + \mu_p[10^{m-2}1] = p^{m-1} + p^{m-2}(1-p)^2 \ne p^{m-2}(1-p).$$

Combing Remark 2.3.10 and Proposition 2.3.8, we have the following.

Lemma 2.3.11. *Let $m \ge 3$ be an integer, $p \in (0,1)$ and $w, v \in \Lambda^{(m),*}$ such that $wv \in \Lambda^{(m),*}$. Then*

$$\mu_p[w]\mu_p[v] \le \mu_p[wv] \le p^{-1}(1-p)^{-1}\mu_p[w]\mu_p[v].$$

The proof of Theorem 2.3.1 is based on the following lemmas.

Lemma 2.3.12. *Let $m \geq 3$ be an integer and $p \in (0,1)$. Then there exists a constant $c > 1$ such that*

$$c^{-1}\mu_p(B) \leq \sigma_m^k \mu_p(B) \leq c\mu_p(B)$$

for all $k \in \mathbb{N}$ and $B \in \mathcal{B}(\Lambda^{(m)})$.

Proof. Let $c = p^{-2}(1-p)^{-2} > 1$.
(1) Prove $c^{-1}\mu_p[w] \leq \sigma_m^k \mu_p[w] \leq c\mu_p[w]$ for any $k \in \mathbb{N}$ and $w \in \Lambda^{(m),*}$.
Fix $w \in \Lambda^{(m),*}$ and $k \in \mathbb{N}$. Note that

$$\sigma_m^{-k}[w] = \bigcup_{u_1 \cdots u_k w \in \Lambda^{(m),*}} [u_1 \cdots u_k w]$$

is a disjoint union.
① Estimate the upper bound of $\sigma_m^k \mu_p[w]$:

$$\begin{aligned}
\mu_p \sigma_m^{-k}[w] &= \sum_{u_1 \cdots u_k w \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_k w] \\
&\overset{(\star)}{\leq} \sum_{u_1 \cdots u_k w \in \Lambda^{(m),*}} p^{-1}(1-p)^{-1} \mu_p[u_1 \cdots u_k]\mu_p[w] \\
&\leq p^{-1}(1-p)^{-1} \sum_{u_1 \cdots u_k \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_k]\mu_p[w] \\
&= p^{-1}(1-p)^{-1}\mu_p[w] \\
&\leq c\mu_p[w]
\end{aligned}$$

where $(\star)$ follows from Lemma 2.3.11.
② Estimate the lower bound of $\sigma_m^k \mu_p[w]$:

i) Prove $\mu_p \sigma_m^{-k}[0] \geq p^2(1-p)$ and $\mu_p \sigma_m^{-k}[1] \geq p(1-p)^2$. In fact, when $k = 1$, the conclusion is obvious. When $k \geq 2$, we have

$$\begin{aligned}
\mu_p \sigma_m^{-k}[0] &= \sum_{u_1 \cdots u_k 0 \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_k 0] \\
&\geq \sum_{u_1 \cdots u_{k-1} \overline{u}_{k-1} 0 \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1} \overline{u}_{k-1} 0] \\
&\overset{(\star)}{=} \sum_{u_1 \cdots u_{k-1} \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1} \overline{u}_{k-1} 0] \\
&\overset{(\star\star)}{\geq} \mu_p[0] \sum_{u_1 \cdots u_{k-1} \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1}]\mu_p[\overline{u}_{k-1}] \\
&\geq p \sum_{u_1 \cdots u_{k-1} \in \Lambda^{(m),*}} \mu_p[u_1 \cdots u_{k-1}] \cdot p(1-p) \\
&= p^2(1-p),
\end{aligned}$$

where $(\star)$ follows from

$$u_1 \cdots u_{k-1}\overline{u}_{k-1}0 \in \Lambda^{(m),*} \Leftrightarrow u_1 \cdots u_{k-1} \in \Lambda^{(m),*}$$

and $(\star\star)$ follows from Lemma 2.3.11. In the same way, we can get $\mu_p\sigma_m^{-k}[1] \geq p(1-p)^2$.

ii) Prove $\mu_p\sigma_m^{-k}[w] \geq c^{-1}\mu_p[w]$. In fact, when $w_1 = 0$, we have

$$\begin{aligned}
\mu_p\sigma_m^{-k}[w] &= \sum_{u_1\cdots u_k w\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_k w] \\
&\geq \sum_{u_1\cdots u_{k-1}1w\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}1w] \\
&\overset{(\star)}{=} \sum_{u_1\cdots u_{k-1}1\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}1w] \\
&\overset{(\star\star)}{\geq} \sum_{u_1\cdots u_{k-1}1\in\Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}1]\mu_p[w] \\
&= \mu_p\sigma_m^{-(k-1)}[1]\mu_p[w] \\
&\overset{(\star\star\star)}{\geq} p(1-p)^2\mu_p[w].
\end{aligned}$$

where $(\star)$ follows from $w_1 = 0$ and $w \in \Lambda^{(m),*}$, $(\star\star)$ follows from Lemma 2.3.11 and $(\star\star\star)$ follows from i). When $w_1 = 1$, in the same way, we can get $\mu_p\sigma_m^{-k}[w] \geq p^2(1-p)\mu_p[w]$.

(2) Prove $c^{-1}\mu_p(B) \leq \sigma_m^k\mu_p(B) \leq c\mu_p(B)$ for all $k \in \mathbb{N}$ and $B \in \mathcal{B}(\Lambda^{(m)})$. Let

$$\mathcal{C} := \Big\{[w] : w \in \Lambda^{(m),*}\Big\} \cup \Big\{\emptyset\Big\},$$

$$\mathcal{C}_{\Sigma f} := \Big\{\bigcup_{i=1}^{n} C_i : C_1, \cdots, C_n \in \mathcal{C} \text{ are disjoint, } n \in \mathbb{N}\Big\}$$

and

$$\mathcal{G} := \Big\{B \in \mathcal{B}(\Lambda^{(m)}) : c^{-1}\mu_p(B) \leq \sigma_m^k\mu_p(B) \leq c\mu_p(B) \text{ for all } k \in \mathbb{N}\Big\}.$$

Then $\mathcal{C}$ is a semi-algebra on $\Lambda^{(m)}$, $\mathcal{C}_{\Sigma f}$ is the algebra generated by $\mathcal{C}$ (by Lemma 1.2.13 (1)) and $\mathcal{G}$ is a monotone class. Since in (1) we have already proved $\mathcal{C} \subset \mathcal{G}$, it follows that $\mathcal{C}_{\Sigma f} \subset \mathcal{G} \subset \mathcal{B}(\Lambda^{(m)})$. Noting that $\mathcal{B}(\Lambda^{(m)})$ is the smallest sigma-algebra containing $\mathcal{C}_{\Sigma f}$, it follows from the Monotone Class Theorem (Theorem 1.2.12) that $\mathcal{G} = \mathcal{B}(\Lambda^{(m)})$. □

Lemma 2.3.13. *Let $m \geq 3$ be an integer and $p \in (0,1)$. For any $B \in \mathcal{B}(\Lambda^{(m)})$ satisfying $\sigma_m^{-1}B = B$, we have $\mu_p(B) = 0$ or 1.*

Proof. Let $\alpha = p^2(1-p)^2 > 0$.

(1) Let $w \in \Lambda^{(m),*}$ and $n = |w|$. For any $A \in \mathcal{B}(\Lambda^{(m)})$, we prove $\alpha\mu_p[w]\mu_p(A) \le \mu_p([w] \cap \sigma_m^{-(n+2)}A)$.

① For any $v \in \Lambda^{(m),*}$, prove $\alpha\mu_p[w]\mu_p[v] \le \mu_p([w] \cap \sigma_m^{-(n+2)}[v])$.
In fact, it follows from $w\overline{w}_n\overline{v}_1 v \in \Lambda^{(m),*}$ and $[w] \cap \sigma_m^{-(n+2)}[v] \supset [w\overline{w}_n\overline{v}_1 v]$ that

$$\mu_p([w]\cap\sigma_m^{-(n+2)}[v]) \ge \mu_p[w\overline{w}_n\overline{v}_1 v] \overset{(\star)}{\ge} \mu_p[w]\mu_p[\overline{w}_n]\mu_p[\overline{v}_1]\mu_p[v] \ge (p(1-p))^2\mu_p[w]\mu_p[v]$$

where $(\star)$ follows from Lemma 2.3.11.

② Let

$$\mathcal{C} := \Big\{[v] : v \in \Lambda^{(m),*}\Big\} \cup \Big\{\emptyset\Big\}$$

and

$$\mathcal{G}_w := \Big\{A \in \mathcal{B}(\Lambda^{(m)}) : \alpha\mu_p[w]\mu_p(A) \le \mu_p([w] \cap \sigma_m^{-(n+2)}A)\Big\}.$$

Then $\mathcal{G}_w$ is a monotone class. Since in ① we have already proved $\mathcal{C} \subset \mathcal{G}_w$, in the same way as the end of the proof of Lemma 2.3.12, we get $\mathcal{G}_w = \mathcal{B}(\Lambda^{(m)})$.

(2) We use B^c to denote the complement of B in $\Lambda^{(m)}$. For any $\varepsilon > 0$, by Lemma 1.2.13, there exist finitely many disjoint cylinders $\big\{[w^{(i)}]\big\} \subset \mathcal{C}$ such that $\mu_p(B^c\Delta E_\varepsilon) < \varepsilon$ where $E_\varepsilon = \bigcup_i [w^{(i)}]$.

(3) Let $B \in \mathcal{B}(\Lambda^{(m)})$ with $\sigma_m^{-1}B = B$. For any $w \in \Lambda^{(m),*}$, by $B = \sigma_m^{-(|w|+2)}B$ and (1) we get

$$\alpha\mu_p(B)\mu_p[w] \le \mu(\sigma_m^{-(|w|+2)}B \cap [w]) = \mu_p(B \cap [w]).$$

Thus

$$\alpha\mu_p(B)\mu_p(E_\varepsilon) = \sum_i \alpha\mu_p(B)\mu_p[w^{(i)}] \le \sum_i \mu_p(B\cap[w^{(i)}]) = \mu_p(B\cap\bigcup_i[w^{(i)}]) = \mu_p(B\cap E_\varepsilon).$$

Let $a = \mu_p((B \cup E_\varepsilon)^c)$, $b = \mu_p(B \cap E_\varepsilon)$, $c = \mu_p(B \setminus E_\varepsilon)$ and $d = \mu_p(E_\varepsilon \setminus B)$. Then we already have

$$\alpha(b+c)(b+d) \le b, \quad a+b < \varepsilon \text{ (by } \mu_p(B^c\Delta E_\varepsilon) < \varepsilon) \quad \text{and} \quad a+b+c+d = 1.$$

It follows from

$$\alpha(b+c)(a+d-\varepsilon) \le \alpha(b+c)(b+d) \le b < \varepsilon$$

that

$$(b+c)(a+d) < (\frac{1}{\alpha} + b + c)\varepsilon \le (\frac{1}{\alpha} + 1)\varepsilon.$$

This implies $\mu_p(B)\mu_p(B^c) \le (\frac{1}{\alpha}+1)\varepsilon$ for any $\varepsilon > 0$. Therefore $\mu_p(B)(1-\mu_p(B)) = 0$ and then $\mu_p(B) = 0$ or 1. □

Proof of Theorem 2.3.1. (1) For any $n \in \mathbb{N}$ and $B \in \mathcal{B}(\Lambda^{(m)})$, define

$$\lambda_p^n(B) := \frac{1}{n}\sum_{k=0}^{n-1} \mu_p(\sigma_m^{-k}B).$$

Then λ_p^n is a probability measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$. By Lemma 2.3.12, there exists $c > 0$ such that

$$c^{-1}\mu_p(B) \le \lambda_p^n(B) \le c\mu_p(B) \quad \text{for any } B \in \mathcal{B}(\Lambda^{(m)}) \text{ and } n \in \mathbb{N}. \tag{2.14}$$

(2) For any $B \in \mathcal{B}(\Lambda^{(m)})$, prove that $\lim_{n\to\infty} \lambda_p^n(B)$ exists.
Let $\mathbb{1}_B : \Lambda^{(m)} \to \{0,1\}$ be defined by

$$\mathbb{1}_B(w) := \begin{cases} 1 & \text{if } w \in B \\ 0 & \text{if } w \notin B \end{cases}$$

for any $w \in \Lambda^{(m)}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \lambda_p^n(B) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \int \mathbb{1}_{\sigma_m^{-k}B}\, d\mu_p \\
&= \lim_{n\to\infty} \int \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_B(\sigma_m^k w)\, d\mu_p(w) \\
&= \int \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_B(\sigma_m^k w)\, d\mu_p(w)
\end{aligned}$$

where the last equality is an application of the dominated convergence theorem, in which the μ_p-a.e. (almost every) existence of $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_B(\sigma_m^k w)$ follows from Lemma 1.2.27, Lemma 2.3.12 and (2.14).

(3) For any $B \in \mathcal{B}(\Lambda^{(m)})$, define

$$\lambda_p(B) := \lim_{n\to\infty} \lambda_p^n(B).$$

By the well known Vitali-Hahn-Saks Theorem, λ_p is a probability measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$.

(4) The fact $\lambda_p \sim \mu_p$ on $\mathcal{B}(\Lambda^{(m)})$ follows from (2.14) and the definition of λ_p.

(5) Prove that λ_p is σ_m-invariant.
In fact, for any $B \in \mathcal{B}(\Lambda^{(m)})$ and $n \in \mathbb{N}$, we have

$$\lambda_p^n(\sigma_m^{-1}B) = \frac{1}{n}\sum_{k=1}^{n} \mu_p(\sigma_m^{-k}B) = \frac{1}{n}\sum_{k=0}^{n} \mu_p(\sigma_m^{-k}B) - \frac{\mu_p(B)}{n} = \frac{n+1}{n}\lambda_p^{n+1}(B) - \frac{\mu_p(B)}{n}.$$

Let $n \to \infty$, we get $\lambda_p(\sigma_m^{-1}B) = \lambda_p(B)$.

(6) Prove that $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}), \lambda_p, \sigma_m)$ is ergodic.
In fact, for any $B \in \mathcal{B}(\Lambda^{(m)})$ satisfying $\sigma_m^{-1}B = B$, by Lemma 2.3.13 we get $\mu_p(B) = 0$ or 1, which implies $\lambda_p(B) = 0$ or 1 since $\lambda_p \sim \mu_p$.

(7) Prove that such λ_p is unique on $\mathcal{B}(\Lambda^{(m)})$.
Let λ_p' be a σ_m-invariant ergodic probability measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ equivalent to μ_p. Then for any $B \in \mathcal{B}(\Lambda^{(m)})$, by the Birkhoff Ergodic Theorem, we get

$$\lambda_p'(B) = \int \mathbb{1}_B \, d\lambda_p' = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma_m^k w) \quad \text{for } \lambda_p'\text{-a.e. } w \in \Lambda^{(m)}$$

and

$$\lambda_p(B) = \int \mathbb{1}_B \, d\lambda_p = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma_m^k w) \quad \text{for } \lambda_p\text{-a.e. } w \in \Lambda^{(m)}.$$

Since $\lambda_p' \sim \mu_p \sim \lambda_p$, there exists $w \in \Lambda^{(m)}$ such that $\lambda_p'(B) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_B(\sigma_m^k w) = \lambda_p(B)$. It means that λ_p' and λ_p are the same on $\mathcal{B}(\Lambda^{(m)})$. □

2.3.4 Proof of Theorem 2.3.2

For any $a \in [0,1]$, recall the definition of the global frequency sets $G_a, \underline{G}_a$ and $\overline{G}_a$ from the introduction. The following lemma follows immediately from (2.13), Theorem 1.2.38 and the invariance of Hausdorff dimension under the projection π_2.

Lemma 2.3.14. *For any $a \in [0,1]$, we have*

$$\dim_H(G_a, d_2) = \dim_H(\underline{G}_a, d_2) = \dim_H(\overline{G}_a, d_2) = \frac{-a\log a - (1-a)\log(1-a)}{\log 2}.$$

To prove Theorem 2.3.2, we also need the next two lemmas, which will be proved later.

Lemma 2.3.15. *Let $m \geq 3$ be an integer, $p \in (0,1)$ and λ_p be the measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$ defined in Theorem 2.3.1. Then*

$$\lambda_p[0] = \frac{p - p^m}{1 - p^m - (1-p)^m}.$$

For any integer $m \geq 3$ and $a \in [0,1]$, recall $\Lambda_a^{(m)} = \Lambda^{(m)} \cap G_a$.

Lemma 2.3.16. *Let $a \in (0,1)$ and $m \geq 3$ be an integer large enough such that $\frac{1}{m} < a < 1 - \frac{1}{m}$. Define $f_m : (0,1) \to \mathbb{R}$ by*

$$f_m(x) := \frac{x - x^m}{1 - x^m - (1-x)^m} \quad \textit{for } x \in (0,1).$$

Then there exists $p_m \in (0,1)$ *such that* $f_m(p_m) = a$ *and*

$$\dim_H(\Lambda_a^{(m)}, d_2) \geq \frac{-(ma-1)\log p_m - (m-ma-1)\log(1-p_m)}{(m-1)\log 2}.$$

Moreover, $p_m \to a$ *as* $m \to \infty$.

Proof of Theorem 2.3.2. First we prove (2). Let $a \in [0,1]$. Since it is straightforward to check $\Gamma \subset \Lambda$, we have

$$\Gamma_a \subset \Lambda_a \subset G_a, \quad \Gamma_a \subset \underline{\Gamma}_a \subset \underline{\Lambda}_a \subset \underline{G}_a \quad \text{and} \quad \Gamma_a \subset \overline{\Gamma}_a \subset \overline{\Lambda}_a \subset \overline{G}_a.$$

By Lemma 2.3.14, we only need to prove

$$\dim_H(\Gamma_a, d_2) \geq \frac{-a\log a - (1-a)\log(1-a)}{\log 2}. \tag{2.15}$$

If $a = 0$ or 1, this follows immediately from $0\log 0 := 0$ and $1\log 1 = 0$. So we only need to consider $0 < a < 1$ in the following. For any integer $m \geq 3$, we define

$$\Theta_a^{(m)} := \Big\{ w \in G_a : w_1 \cdots w_{2m} = 1^{2m}, w_{km+1} \cdots w_{km+m} \notin \{0^m, 1^m\} \text{ for all } k \geq 2 \Big\}$$

and

$$\Xi_a^{(m)} := \Big\{ w \in G_a : w_{km+1} \cdots w_{km+m} \notin \{0^m, 1^m\} \text{ for all } k \geq 0 \Big\}.$$

Then

$$\dim_H(\Gamma_a, d_2) \overset{(\star)}{\geq} \dim_H(\Theta_a^{(m)}, d_2) \overset{(\star\star)}{\geq} \dim_H(\Xi_a^{(m)}, d_2) \overset{(\star\star\star)}{\geq} \dim_H(\Lambda_a^{(m)}, d_2) \tag{2.16}$$

where $(\star)$ follows from $\Gamma_a \supset \Theta_a^{(m)}$, $(\star\star\star)$ follows from $\Xi_a^{(m)} \supset \Lambda_a^{(m)}$, and $(\star\star)$ follows from $\sigma^{2m}(\Theta_a^{(m)}) = \Xi_a^{(m)}$ and the fact that σ^{2m} is Lipschitz continuous (since $d_2(\sigma^{2m}(w), \sigma^{2m}(v)) \leq 2^{2m} d_2(w,v)$ for all $w, v \in \{0,1\}^{\mathbb{N}}$). By (2.16) and Lemma 2.3.16, for m large enough, there exists $p_m \in (0,1)$ such that $p_m \to a$ (as $m \to \infty$) and

$$\dim_H(\Gamma_a, d_2) \geq \frac{-(ma-1)\log p_m - (m-ma-1)\log(1-p_m)}{(m-1)\log 2}.$$

Let $m \to \infty$, we get (2.15).

Finally we deduce (1) from (2). In fact, since (2) implies $\dim_H(\Gamma_{\frac{1}{2}}, d_2) = 1$, it follows from $\Gamma_{\frac{1}{2}} \subset \Gamma \subset \Lambda \subset \{0,1\}^{\mathbb{N}}$ that $\dim_H(\Gamma, d_2) = \dim_H(\Lambda, d_2) = 1$. □

Finally we prove Lemmas 2.3.15 and 2.3.16 to end this section.

Proof of Lemma 2.3.16. Since f_m is continuous on $(0,1)$, $\lim_{x\to 0^+} f_m(x) = \frac{1}{m}$, $\lim_{x\to 1^-} f_m(x) = 1 - \frac{1}{m}$ and $\frac{1}{m} < a < 1 - \frac{1}{m}$, there exists $p_m \in (0,1)$ such that $f_m(p_m) = a$.

(1) Prove $p_m \to a$ as $m \to \infty$. Notice that

$$|p_m - a| = |p_m - f_m(p_m)| = \Big|\frac{p_m^m(1-p_m) - p_m(1-p_m)^m}{1 - p_m^m - (1-p_m)^m}\Big|.$$

Let

$$g_m(x) := \frac{x^m(1-x) - x(1-x)^m}{1 - x^m - (1-x)^m} \quad \text{for } x \in (0,1).$$

Then

$$|p_m - a| = |g_m(p_m)| \le \sup_{x\in(0,1)} |g_m(x)|.$$

In order to prove $p_m \to a$, it suffices to check $|g_m(x)| \le \frac{1}{m}$ for all $x \in (0,1)$. That is,

$$m \cdot |x^m(1-x) - x(1-x)^m| \le 1 - x^m - (1-x)^m \quad \text{for all } x \in (0,1).$$

① When $x \in (0, \frac{1}{2}]$, we get $x^m(1-x) - x(1-x)^m \le 0$. It suffices to prove $(m - mx - 1)x^m + 1 - (mx+1)(1-x)^m \ge 0$. Since $m - mx - 1 > 0$, we only need to prove $h_m(x) := (mx+1)(1-x)^m \le 1$ for all $x \in [0, \frac{1}{2}]$. This follows from $h_m(0) = 1$ and $h_m'(x) = -m(m+1)x(1-x)^{m-1} \le 0$ for all $x \in [0, \frac{1}{2}]$.

② When $x \in (\frac{1}{2}, 1)$, we get $x^m(1-x) - x(1-x)^m \ge 0$. It suffices to prove $(mx - 1)(1-x)^m + 1 - (1 + m - mx)x^m \ge 0$. Since $mx - 1 > 0$, we only need to prove $h_m(x) := (1 + m - mx)x^m \le 1$ for all $x \in [\frac{1}{2}, 1]$. This follows from $h_m(1) = 1$ and $h_m'(x) = m(m+1)(1-x)x^{m-1} \ge 0$ for all $x \in [\frac{1}{2}, 1]$.

(2) We apply Proposition 2.3.5 to get the lower bound of $\dim_H(\Lambda_a^{(m)}, d_2)$. Let μ_{p_m} be the $(p_m, 1-p_m)$ Bernoulli-type measure on $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}))$.
① The fact that $\Lambda_a^{(m)} = \Lambda^{(m)} \cap G_a$ is a Borel set in $(\Lambda^{(m)}, d_2)$ follows from the fact that G_a is a Borel set in $(\{0,1\}^{\mathbb{N}}, d_2)$.
② Prove $\mu_{p_m}(\Lambda_a^{(m)}) = 1$.
Let λ_{p_m} be the measure defined in Theorem 2.3.1 such that $(\Lambda^{(m)}, \mathcal{B}(\Lambda^{(m)}), \lambda_{p_m}, \sigma_m)$ is ergodic. It follows from Birkhoff's ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{[0]}(\sigma_m^k w) = \int \mathbb{1}_{[0]} d\lambda_{p_m} = \lambda_{p_m}[0] \underset{\text{Lemma 2.3.15}}{\overset{\text{by}}{=\!=\!=}} \frac{p_m - p_m^m}{1 - p_m^m - (1-p_m)^m} = f_m(p_m) = a$$

for λ_{p_m}-almost every $w \in \Lambda^{(m)}$. By $\frac{|w_1\cdots w_n|_0}{n} = \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{[0]}(\sigma_m^k w)$, we get

$$\lim_{n\to\infty} \frac{|w_1 \cdots w_n|_0}{n} = a \quad \text{for } \lambda_{p_m}\text{-almost every } w \in \Lambda^{(m)},$$

which implies $\lambda_{p_m}(\Lambda_a^{(m)}) = 1$. It follows from $\lambda_{p_m} \sim \mu_{p_m}$ that $\mu_{p_m}(\Lambda_a^{(m)}) = 1$.

③ For all $w \in \Lambda_a^{(m)}$, we have

$$\begin{aligned}
&\varliminf_{r\to\infty} \frac{\log \mu_{p_m}(B(w,r))}{\log r} \\
&\overset{(\star)}{\geq} \varliminf_{n\to\infty} \frac{\log \mu_{p_m}[w_1\cdots w_n]}{\log 2^{-n}} \\
&= \varliminf_{n\to\infty} \frac{-\log p_m^{N_0^{(m)}(w_1\cdots w_n)}(1-p_m)^{N_1^{(m)}(w_1\cdots w_n)}}{n\log 2} \\
&\geq \frac{\varliminf_{n\to\infty} \frac{N_0^{(m)}(w_1\cdots w_n)}{n}(-\log p_m) + \varliminf_{n\to\infty} \frac{N_1^{(m)}(w_1\cdots w_n)}{n}(-\log(1-p_m))}{\log 2} \\
&\overset{(\star\star)}{\geq} \frac{\varliminf_{n\to\infty} \left(\frac{m\cdot|w_1\cdots w_n|_0}{(m-1)n} - \frac{1}{m-1}\right)(-\log p_m) + \varliminf_{n\to\infty} \left(\frac{m\cdot|w_1\cdots w_n|_1}{(m-1)n} - \frac{1}{m-1}\right)(-\log(1-p_m))}{\log 2} \\
&\overset{(\star\star\star)}{=} \frac{-(ma-1)\log p_m - (m-ma-1)\log(1-p_m)}{(m-1)\log 2}
\end{aligned}$$

where $(\star\star\star)$ follows from $w \in \Lambda_a^{(m)}$, $(\star\star)$ follows from Proposition 2.3.9 and $(\star)$ can be proved as follows. For any $r \in (0,1)$, there exists $n = n(r) \in \mathbb{N}$ such that $\frac{1}{2^n} \leq r < \frac{1}{2^{n-1}}$. Then by $B(w,r) = [w_1\cdots w_n]$ and $\log \mu_{p_m}[w_1\cdots w_n] < 0$, we get $\frac{\log \mu_{p_m}(B(w,r))}{\log r} \geq \frac{\log \mu_{p_m}[w_1\cdots w_n]}{\log 2^{-n}}$. (In fact, $(\star)$ can take "=".)

Thus the lower bound of $\dim_H(\Lambda_a^{(m)}, d_2)$ follows from ①, ②, ③ and Proposition 2.3.5. □

Proof of Lemma 2.3.15. By the definition of λ_p, we know

$$\lambda_p[0] = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu_p\sigma^{-k}[0].$$

For any integer $k \geq 0$, let

$$a_k := \mu_p\sigma^{-k}[0] = \sum_{u_1\cdots u_k 0 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 0], \quad b_k := \mu_p\sigma^{-k}[1] = \sum_{u_1\cdots u_k 1 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 1],$$

$$c_k := \mu_p\sigma^{-k}[01] = \sum_{u_1\cdots u_k 01 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 01], \quad d_k := \mu_p\sigma^{-k}[10] = \sum_{u_1\cdots u_k 10 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 10].$$

By Theorem 2.3.1, the following limits exist:

$$a := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} a_k = \lambda_p[0], \quad b := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} b_k = \lambda_p[1],$$

$$c := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} c_k = \lambda_p[01], \quad d := \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} d_k = \lambda_p[10].$$

(1) We have $a+b=1$ since $\lambda_p[0]+\lambda_p[1]=\lambda_p(\Lambda^{(m)})$.
(2) We have $c=d$ since $\lambda_p[00]+\lambda_p[01]=\lambda_p[0]=\lambda_p\sigma^{-1}[0]=\lambda_p[00]+\lambda_p[10]$.
(3) Prove $(1-p)a+p^{m-1}d=c$ and $pb+(1-p)^{m-1}c=d$.
① For $k\geq m$, we have

$$a_k=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+p^{m-2}d_{k-m+1},$$

since

$$\begin{aligned}
&\sum_{u_1\cdots u_k0\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_k0]\\
&=\sum_{u_1\cdots u_{k-1}10\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-1}10]+\sum_{u_1\cdots u_{k-1}00\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-1}00]\\
&=d_{k-1}+\sum_{u_1\cdots u_{k-2}100\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-2}100]+\sum_{u_1\cdots u_{k-2}000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-2}000]\\
&\overset{(\star)}{=}d_{k-1}+\sum_{u_1\cdots u_{k-2}10\in\Lambda^{(m),*}}p\mu_p[u_1\cdots u_{k-2}10]+\sum_{u_1\cdots u_{k-2}000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-2}000]\\
&=d_{k-1}+pd_{k-2}+\sum_{u_1\cdots u_{k-3}1000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-3}1000]+\sum_{u_1\cdots u_{k-3}0000\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-3}0000]\\
&\overset{(\star\star)}{=}d_{k-1}+pd_{k-2}+\sum_{u_1\cdots u_{k-3}10\in\Lambda^{(m),*}}p^2\mu_p[u_1\cdots u_{k-3}10]+\sum_{u_1\cdots u_{k-3}0^4\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-3}0^4]\\
&=\cdots\\
&=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+2}0^{m-1}\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-m+2}0^{m-1}]\\
&=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+1}10^{m-1}\in\Lambda^{(m),*}}\mu_p[u_1\cdots u_{k-m+1}10^{m-1}]\\
&\overset{(\star\star\star)}{=}d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+\sum_{u_1\cdots u_{k-m+1}10\in\Lambda^{(m),*}}p^{m-2}\mu_p[u_1\cdots u_{k-m+1}10]\\
&=d_{k-1}+pd_{k-2}+\cdots+p^{m-3}d_{k-m+2}+p^{m-2}d_{k-m+1},
\end{aligned}$$

where $(\star)$, $(\star\star)$ and $(\star\star\star)$ follow from

$$\begin{aligned}
&u_1\cdots u_{k-2}100\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-2}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-2}101\in\Lambda^{(m),*},
\end{aligned}$$

$$\begin{aligned}
&u_1\cdots u_{k-3}1000\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-3}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-3}101,u_1\cdots u_{k-3}1001\in\Lambda^{(m),*}
\end{aligned}$$

and

$$\begin{aligned}
&u_1\cdots u_{k-m+1}10^{m-1}\in\Lambda^{(m),*}\Leftrightarrow u_1\cdots u_{k-m+1}10\in\Lambda^{(m),*}\\
&\Rightarrow u_1\cdots u_{k-m+1}101,u_1\cdots u_{k-m+1}1001,\cdots,u_1\cdots u_{k-m+1}10^{m-2}1\in\Lambda^{(m),*}
\end{aligned}$$

respectively, recalling the definition of μ_p.

② For $k \geq m$, we have

$$c_k = (1-p)d_{k-1} + (1-p)pd_{k-2} + \cdots + (1-p)p^{m-3}d_{k-m+2} + p^{m-2}d_{k-m+1},$$

since

$$\begin{aligned}
&\sum_{u_1\cdots u_k 01 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_k 01] \\
&= \sum_{u_1\cdots u_{k-1}101 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}101] + \sum_{u_1\cdots u_{k-1}001 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}001] \\
&\overset{(\star)}{=} \sum_{u_1\cdots u_{k-1}10 \in \Lambda^{(m),*}} (1-p)\mu_p[u_1\cdots u_{k-1}10] + \sum_{u_1\cdots u_{k-1}001 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-1}001] \\
&= (1-p)d_{k-1} + \sum_{u_1\cdots u_{k-2}1001 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-2}1001] + \sum_{u_1\cdots u_{k-2}0001 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-2}0001] \\
&\overset{(\star\star)}{=} (1-p)d_{k-1} + \sum_{u_1\cdots u_{k-2}10 \in \Lambda^{(m),*}} p(1-p)\mu_p[u_1\cdots u_{k-2}10] + \sum_{u_1\cdots u_{k-2}0^3 1 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-2}0^3 1] \\
&= (1-p)d_{k-1} + p(1-p)d_{k-2} + \sum_{u_1\cdots u_{k-3}10^3 1 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-3}10^3 1] + \sum_{u_1\cdots u_{k-3}0^4 1 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-3}0^4 1] \\
&= \cdots \\
&= (1-p)d_{k-1} + (1-p)pd_{k-2} + \cdots + (1-p)p^{m-3}d_{k-m+2} + \sum_{u_1\cdots u_{k-m+2}0^{m-1}1 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-m+2}0^{m-1}1] \\
&= (1-p)d_{k-1} + (1-p)pd_{k-2} + \cdots + (1-p)p^{m-3}d_{k-m+2} + \sum_{u_1\cdots u_{k-m+1}10^{m-1}1 \in \Lambda^{(m),*}} \mu_p[u_1\cdots u_{k-m+1}10^{m-1}1] \\
&\overset{(\star\star\star)}{=} (1-p)d_{k-1} + (1-p)pd_{k-2} + \cdots + (1-p)p^{m-3}d_{k-m+2} + \sum_{u_1\cdots u_{k-m+1}10 \in \Lambda^{(m),*}} p^{m-2}\mu_p[u_1\cdots u_{k-m+1}10] \\
&= (1-p)d_{k-1} + (1-p)pd_{k-2} + \cdots + (1-p)p^{m-3}d_{k-m+2} + p^{m-2}d_{k-m+1},
\end{aligned}$$

where $(\star)$, $(\star\star)$ and $(\star\star\star)$ follow from

$$\begin{aligned}
&u_1\cdots u_{k-1}101 \in \Lambda^{(m),*} \Leftrightarrow u_1\cdots u_{k-1}10 \in \Lambda^{(m),*} \\
&\Rightarrow u_1\cdots u_{k-1}100 \in \Lambda^{(m),*},
\end{aligned}$$

$$\begin{aligned}
&u_1\cdots u_{k-2}1001 \in \Lambda^{(m),*} \Leftrightarrow u_1\cdots u_{k-2}10 \in \Lambda^{(m),*} \\
&\Rightarrow u_1\cdots u_{k-2}101, u_1\cdots u_{k-2}1000 \in \Lambda^{(m),*}
\end{aligned}$$

and

$$\begin{aligned}
&u_1\cdots u_{k-m+1}10^{m-1}1 \in \Lambda^{(m),*} \Leftrightarrow u_1\cdots u_{k-m+1}10 \in \Lambda^{(m),*} \\
&\Rightarrow u_1\cdots u_{k-m+1}101, \cdots, u_1\cdots u_{k-m+1}10^{m-2}1 \in \Lambda^{(m),*} \\
&\text{but } u_1\cdots u_{k-m+1}10^{m-1}0 \notin \Lambda^{(m),*}
\end{aligned}$$

respectively, recalling the definition of μ_p.

Combining ① and ② we get $(1-p)(a_k - p^{m-2}d_{k-m+1}) = c_k - p^{m-2}d_{k-m+1}$,

$$\text{i.e.,} \quad (1-p)a_k + p^{m-1}d_{k-m+1} = c_k \quad \text{for any } k \geq m.$$

That is,

$$(1-p)a_{k+m} + p^{m-1}d_{k+1} = c_{k+m} \quad \text{for any } k \geq 0,$$

which implies

$$(1-p)\frac{1}{n}\sum_{k=0}^{n-1} a_{k+m} + p^{m-1}\frac{1}{n}\sum_{k=0}^{n-1} d_{k+1} = \frac{1}{n}\sum_{k=0}^{n-1} c_{k+m}.$$

Let $n \to \infty$, we get $(1-p)a+p^{m-1}d = c$. It follows in the same way that $pb+(1-p)^{m-1}c = d$.

Combining (1), (2) and (3) we get $a = \frac{p-p^m}{1-p-(1-p)^m}$. $\square$

Chapter 3

Generalized Thue-Morse sequences

In this chapter, we study some generalizations of the well known Thue-Morse sequence, including their relations to beta-expansions in Sections 3.1 and 3.2, related infinite products in Section 3.3 and generalized Koch curves in Section 3.4.

3.1 Bifurcations of digit frequencies in unique expansions

Let $(t_n)_{n\geq 0}$ be the famous *Thue-Morse sequence*

$$0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 0110\ 1001\ \cdots.$$

It is well known that there are several equivalent definitions of this sequence [15]. One of them is

$$t_0 := 0, \quad t_1 := \overline{t_0}, \quad t_2t_3 := \overline{t_0t_1}, \quad t_4t_5t_6t_7 := \overline{t_0t_1t_2t_3}, \quad \cdots$$

where $\overline{0} := 1$ and $\overline{1} := 0$. Hence it is straightforward to see that the *shifted Thue-Morse sequence* $(t_n)_{n\geq 1}$,

$$1101\ 0011\ 0010\ 1101\ 0010\ 1100\ 1101\ 0011\ \cdots, \tag{3.1}$$

can be defined by

$$t_1 := 1, \quad t_2 := \overline{t_1}^{+}, \quad t_3t_4 := \overline{t_1t_2}^{+}, \quad t_5t_6t_7t_8 := \overline{t_1t_2t_3t_4}^{+}, \quad \cdots$$

where $w^+ := w_1\cdots w_{n-1}(w_n+1)$ for any finite word $w = w_1\cdots w_n$.

First we generalize the shifted Thue-Morse sequence according to the above definition. For any $m \in \mathbb{N}$ and $k \in \{1, \cdots, m\}$, we define a sequence of finite words $\{\theta^{(n)}_{m;k}\}_{n\geq 0}$ by induction as follows:

$$\theta^{(0)}_{m;k} := k \quad \text{and} \quad \theta^{(n+1)}_{m;k} := \theta^{(n)}_{m;k}\overline{\theta^{(n)}_{m;k}}^{+} \text{ for all } n \geq 0, \tag{3.2}$$

where $\overline{w} := \overline{w_1} \cdots \overline{w_i}$ for any word $w = w_1 \cdots w_i$ and $\overline{j} := m - j$ for any $j \in \{0, 1, \cdots, m\}$. When m and k are understood from the context, we use $\theta^{(n)}$ instead of $\theta^{(n)}_{m;k}$ for simplification. We call the infinite sequence

$$\theta = (\theta_i)_{i\geq 1} := \lim_{n\to\infty} \theta^{(n)} = k(\overline{k}+1)\overline{k}k\ \overline{k}(k-1)k(\overline{k}+1)\ \overline{k}(k-1)k\overline{k}\ k(\overline{k}+1)\overline{k}k\ \cdots$$

the $(m;k)$-*shifted-Thue-Morse sequence*, and call the unique $\mathfrak{q} = \mathfrak{q}_{m;k} \in (1, m+1)$ such that

$$\sum_{i=1}^{\infty} \frac{\theta_i}{\mathfrak{q}^i} = 1$$

the $(m;k)$-*Komornik-Loreti constant.*

Note that the $(1;1)$-shifted-Thue-Morse sequence is exactly the classical shifted Thue-Morse sequence $(t_n)_{n\geq 1}$ and the $(1;1)$-Komornik-Loreti constant is exactly the classical Komornik-Loreti constant [8, 84].

In the following, we will study the relation between the above generalized Komornik-Loreti constants and digit frequencies in unique expansions.

Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $x \in \mathbb{R}$. Recall that a sequence $w = (w_i)_{i\geq 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ is called a β-*expansion* of x if

$$x = \pi_\beta(w) := \sum_{i=1}^{\infty} \frac{w_i}{\beta^i}.$$

An x may have many different β-expansions, or it may have a unique β-expansion. We focus on unique expansions, which got a lot of attention in the last three decades [2, 5, 45, 53, 60, 83, 86]. For $m \in \mathbb{N}$ and $\beta \in (1, m+1]$, let

$$\Gamma_{m,\beta} := \left\{ w \in \{0, 1, \cdots, m\}^{\mathbb{N}} : w \text{ is the unique } \beta\text{-expansion of } \pi_\beta(w) \right\} \setminus \left\{ 0^\infty, m^\infty \right\}$$

be the set of unique β-expansions except 0^∞ and m^∞.

For any $m \in \mathbb{N}$, let

$$G_m := \begin{cases} p+1 & \text{if } m = 2p \text{ for some integer } p \geq 1 \\ \frac{p+1+\sqrt{p^2+6p+5}}{2} & \text{if } m = 2p+1 \text{ for some integer } p \geq 0 \end{cases}$$

be the *generalized golden ratio.* Baker [23] showed that:

(1) for all $\beta \in (1, G_m)$, we have $\Gamma_{m,\beta} = \emptyset$;

(2) for all $\beta \in (G_m, m+1]$, we have $\Gamma_{m,\beta} \neq \emptyset$.

We study digit frequencies of the sequences in $\Gamma_{m,\beta}$. Baker's result make us only need to consider $\beta \in (G_m, m+1]$. Recall from Section 2.2 that for any infinite sequence w,

the frequency, lower-frequency and upper-frequency of the digit k in w are denoted by $\mathrm{Freq}_k(w)$, $\underline{\mathrm{Freq}}_k(w)$ and $\overline{\mathrm{Freq}}_k(w)$ respectively.

Let β_1 be the unique zero in $(1,2)$ of the polynomial x^3-x^2-2x+1. It is straightforward to check that β_1 is strictly larger than the golden ratio G_1. In [75, Lemma 2.3] Jordan, Shmerkin and Solomyak showed that:

(1) if $\beta \in (G_1, \beta_1]$, then for all $w \in \Gamma_{1,\beta}$,

$$\mathrm{Freq}_1(w) \text{ and } \mathrm{Freq}_0(w) \text{ exist and are equal to } \frac{1}{2};$$

(2) if $\beta \in (\beta_1, 2)$, then

$$\dim_H \Big\{ w \in \Gamma_{1,\beta} : \mathrm{Freq}_1(w) \text{ and } \mathrm{Freq}_0(w) \text{ do not exist} \Big\} > 0,$$

and there exists $c = c(\beta) > 0$ such that for all $r \in (-c, c)$,

$$\dim_H \Big\{ w \in \Gamma_{1,\beta} : \mathrm{Freq}_1(w) - \mathrm{Freq}_0(w) = r \Big\} > 0,$$

where $\dim_H$ denotes the Hausdorff dimension in $\{0,1\}^{\mathbb{N}}$ equipped with the usual metric d_2.

This is a bifurcation phenomenon of digit frequencies in unique expansions on the alphabet $\{0,1\}$. We are going to show similar bifurcation phenomenons on larger alphabets. Interestingly, in our first main result, the bifurcations are exactly the generalized Komornik-Loreti constants, which are defined by the generalized shifted Thue-Morse sequences.

Theorem 3.1.1. *Let $m \geq 2$ be an integer, $k \in \{\lceil \frac{m}{2} \rceil + 1, \cdots, m\}$ and $\beta \in (G_m, m+1]$.*

(1) *If $\beta \in (G_m, \mathfrak{q}_{m;k}]$, then for all $w \in \Gamma_{m,\beta}$,*

$$\mathrm{Freq}_k(w) \textit{ and } \mathrm{Freq}_{\overline{k}}(w) \textit{ exist and are equal.}$$

(2) *If $\beta \in (\mathfrak{q}_{m;k}, m+1]$, then*

$$\dim_H \Big\{ w \in \Gamma_{m,\beta} : \mathrm{Freq}_k(w) \textit{ and } \mathrm{Freq}_{\overline{k}}(w) \textit{ do not exist} \Big\} > 0,$$

where $\dim_H$ denotes the Hausdorff dimension in $\{0,1,\cdots,m\}^{\mathbb{N}}$ equipped with the usual metric d_{m+1}.

For integer $m \geq 2$ and $k \in \{\lceil \frac{m}{2} \rceil + 1, \cdots, m\}$, let

$$\beta_{m;k} := \frac{k+1+\sqrt{k^2-6k+4m+5}}{2}$$

be the unique zero in $(1, m+1)$ of the polynomial $x^2 - (k+1)x + 2k - m - 1$. One can verify $\beta_{m;k} > \mathfrak{q}_{m;k} > G_m$ for all $k \in \{\lceil \frac{m}{2} \rceil + 1, \cdots, m\}$. The following is our second main result.

Theorem 3.1.2. *Let $m \geq 2$ be an integer, $k \in \{\lceil \frac{m}{2} \rceil + 1, \cdots, m\}$ and $\beta \in (G_m, m+1]$.*

(1) *If $\beta \in (G_m, \beta_{m;k}]$, then for all $w \in \Gamma_{m,\beta}$, we have*

$$\overline{\mathrm{Freq}}_k(w) = \overline{\mathrm{Freq}}_{\overline{k}}(w) \quad \textit{and} \quad \underline{\mathrm{Freq}}_k(w) = \underline{\mathrm{Freq}}_{\overline{k}}(w).$$

(2) *If $\beta \in (\beta_{m;k}, m+1]$, then there exists $c = c(\beta) > 0$ such that for all $r \in (-c, c)$, we have*

$$\dim_H \left\{ w \in \Gamma_{m,\beta} : \mathrm{Freq}_k(w) - \mathrm{Freq}_{\overline{k}}(w) = r \right\} > 0,$$

where $\dim_H$ denotes the Hausdorff dimension in $\{0, 1, \cdots, m\}^{\mathbb{N}}$ equipped with the usual metric d_{m+1}.

Remark 3.1.3. *The domains $\beta \in (G_m, \mathfrak{q}_{m;k}]$ in Theorem 3.1.1 (1) and $\beta \in (G_m, \beta_{m;k}]$ in Theorem 3.1.2 (1) can be extended to $\beta \in (1, \mathfrak{q}_{m;k}]$ and $\beta \in (1, \beta_{m;k}]$ respectively. In fact, on the one hand, the condition $\beta > G_m$ has not been used in the proof of Theorem 3.1.1 or 3.1.2, and is just used to guarantee $\Gamma_{m,\beta} \neq \emptyset$; on the other hand, even if $\beta \leq G_m$ makes $\Gamma_{m,\beta} = \emptyset$, the statements of Theorem 3.1.1 (1) and Theorem 3.1.2 (1) still hold.*

We will give some notation and preliminaries in Subsection 3.1.1, and then prove Theorems 3.1.1 and 3.1.2 in the last subsection.

3.1.1 Notation and preliminaries

Given a finite word w, recall that we use $|w|$ and $|w|_k$ to denote its length and the number of the digit k in w respectively. If $w = w_1 \cdots w_{n-1} w_n$, we define $w^* := w_1 \cdots w_{n-1}$, $w^+ := w_1 \cdots w_{n-1}(w_n + 1)$ and $w^- := w_1 \cdots w_{n-1}(w_n - 1)$. For $m \in \mathbb{N}$ and $k \in \{0, 1, \cdots, m\}$, the *bar* operation is defined by $\overline{k} := m - k$, extended to all infinite sequences $w = w_1 w_2 \cdots \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ by $\overline{w} := \overline{w}_1 \overline{w}_2 \cdots$ and extended to all finite words $w = w_1 \cdots w_n \in \{0, 1, \cdots, m\}^n$ by $\overline{w} := \overline{w}_1 \cdots \overline{w}_n$ for all $n \in \mathbb{N}$.

Let $m \in \mathbb{N}$. On $\{0, 1, \cdots, m\}^{\mathbb{N}}$, recall that the *usual metric* d_{m+1} is defined by

$$d_{m+1}(w, v) := (m+1)^{-\inf\{n \geq 0 : w_{n+1} \neq v_{n+1}\}} \quad \text{for } w, v \in \{0, 1, \cdots, m\}^{\mathbb{N}},$$

and the *shift map* σ is defined by

$$\sigma(w) := w_2 w_3 w_4 \cdots \quad \text{for } w = w_1 w_2 w_3 \cdots \in \{0, 1, \cdots, m\}^{\mathbb{N}}.$$

For $\beta \in (1, m+1]$, we use $g^*(1, \beta) = (g_n^*(1, \beta))_{n \geq 1} \in \{0, 1, \cdots, m\}^{\mathbb{N}}$ to denote the

quasi-greedy β-expansion of 1 (the largest expansion in lexicographic order among all the β-expansions of 1 which do not end with 0^∞).

Between two infinite sequences or two finite words with the same length, we use $<$, $\leq$, $>$ and $\geq$ to denote the lexicographic order. The following lexicographic criteria for unique expansions can be found in [55, Theorem 2.5], [85, Theorem 3.1], [76, Lemma 2.2] and Corollary 2.1.6 in this thesis.

Lemma 3.1.4. *Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $\varepsilon \in \{0, \cdots, m\}^{\mathbb{N}}$ be a β-expansion of 1. Then ε is the unique expansion if and only if*

$$\sigma^n \varepsilon < \varepsilon \text{ whenever } \varepsilon_n < m \quad \text{and} \quad \sigma^n \varepsilon > \overline{\varepsilon} \text{ whenever } \varepsilon_n > 0.$$

Lemma 3.1.5. *Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$, $x \in [0, \frac{m}{\beta-1}]$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$ be a β-expansion of x. Then w is the unique expansion if and only if*

$$\sigma^n w < g^*(1, \beta) \text{ whenever } w_n < m \quad \text{and} \quad \sigma^n w > \overline{g^*(1, \beta)} \text{ whenever } w_n > 0.$$

The next lemma follows from [55, Proposition 2.3] (see also [22, Theorem 2.2]).

Lemma 3.1.6. *Let $m \in \mathbb{N}$ and $\beta_1, \beta_2 \in (1, m+1]$. If $\beta_1 < \beta_2$, then $g^*(1, \beta_1) < g^*(1, \beta_2)$.*

The following lemma on Cesàro limit can be proved straightforwardly.

Lemma 3.1.7. *Let $a_1, a_2, \cdots \geq 0$. If $a_n \to \infty$ then $\frac{a_1 + \cdots + a_n}{n} \to \infty$ as $n \to \infty$.*

Proof. Fix any $M > 0$. By $a_n \to \infty$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n > 2M$. Then for all $n > 2N$, we have

$$\frac{a_1 + \cdots + a_n}{n} \geq \frac{a_{N+1} + \cdots + a_n}{n} > \frac{(n-N) \cdot 2M}{n} = 2M - \frac{2NM}{n} > M.$$

□

The following concept and basic property are well known [64].

Definition 3.1.8 (Hölder continuity)**.** *Let (X, d), (X', d') be two metric spaces and $\alpha > 0$. A map $f : X \to X'$ is called α-Hölder continuous if there exists a constant $c > 0$ such that*

$$d'(f(x), f(y)) \leq c \cdot (d(x, y))^\alpha \quad \text{for all } x, y \in X.$$

Proposition 3.1.9. *Let (X, d), (X', d') be two metric spaces, $\alpha > 0$ and $f : X \to X'$ be an α-Hölder continuous map. Then for any $E \subset X$, we have*

$$\dim_H(E, d) \geq \alpha \cdot \dim_H(f(E), d').$$

Besides, we recall two useful basic results (see for examples Lemma 2.3.14 and [24, 59]).

Proposition 3.1.10. *For all $a \in (0,1)$, we have*

$$\dim_H \left(\left\{w \in \{0,1\}^{\mathbb{N}} : \mathrm{Freq}_0(w) = a\right\}, d_2\right) > 0.$$

Proposition 3.1.11.

$$\dim_H \left(\left\{w \in \{0,1\}^{\mathbb{N}} : \mathrm{Freq}_0(w) \textit{ and } \mathrm{Freq}_1(w) \textit{ do not exist}\right\}, d_2\right) = 1.$$

3.1.2 Proofs of the main results

Throughout this subsection, $m \geq 2$ and $k \in \{\lceil \frac{m}{2} \rceil + 1, \cdots, m\}$ are given integers. Recall from the introduction that $\theta^{(0)}, \theta^{(1)}, \cdots, \theta^{(n)}, \cdots$ are defined by (3.2), and $\theta = (\theta_i)_{i\geq 1} = \lim_{n\to\infty} \theta^{(n)}$ is the $(m;k)$-shifted-Thue-Morse sequence. Before proving Theorem 3.1.1, we give some necessary technical lemmas first.

Lemma 3.1.12. *For all integers $n \geq 0$, we have the following.*

(1) $|\theta^{(n)}| = 2^n$.

(2) $|\theta^{(n)-}|_k = |\theta^{(n)-}|_{\overline{k}} = \begin{cases} (2^n-1)/3 & \textit{if } n \textit{ is even,} \\ (2^n+1)/3 & \textit{if } n \textit{ is odd.} \end{cases}$

(3) $|\theta^{(n)}|_k = \begin{cases} (2^n+2)/3 & \textit{if } n \textit{ is even,} \\ (2^n+1)/3 & \textit{if } n \textit{ is odd,} \end{cases}$ *and* $|\theta^{(n)}|_{\overline{k}} = \begin{cases} (2^n-1)/3 & \textit{if } n \textit{ is even,} \\ (2^n-2)/3 & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. (1) follows from the definition of $\theta^{(n)}$.
(2) ① Prove $|\theta^{(n)-}|_k = |\theta^{(n)-}|_{\overline{k}}$.
For $n = 0$, by $\theta^{(0)-} = k-1$ and $\overline{k} < k-1 < k$, we get $|\theta^{(0)-}|_k = |\theta^{(0)-}|_{\overline{k}} = 0$. For $n \geq 1$, it follows from $\theta^{(n)-} = \theta^{(n-1)}\overline{\theta^{(n-1)}}$ that $|\theta^{(n)-}|_k = |\theta^{(n)-}|_{\overline{k}}$.
② Let $|\theta^{(n)-}|_{k,\overline{k}} := |\theta^{(n)-}|_k + |\theta^{(n)-}|_{\overline{k}}$. By ① it remains to prove

$$|\theta^{(n)-}|_{k,\overline{k}} = \begin{cases} 2(2^n-1)/3 & \text{if } n \text{ is even,} \\ 2(2^n+1)/3 & \text{if } n \text{ is odd.} \end{cases}$$

In fact we can prove that

$$\begin{cases} |\theta^{(n)-}|_{k,\overline{k}} = 2(2^n-1)/3 \text{ and } \theta^{(n)-} \text{ ends with } k-1 & \text{if } n \text{ is even,} \\ |\theta^{(n)-}|_{k,\overline{k}} = 2(2^n+1)/3 \text{ and } \theta^{(n)-} \text{ ends with } \overline{k} & \text{if } n \text{ is odd,} \end{cases} \tag{3.3}$$

by induction. For $n = 0$, (3.3) is true since $\theta^{(0)-} = k-1$ and $\overline{k} < k-1 < k$. Suppose that (3.3) is true for some $n \geq 0$.

i) If n is even, then $|\theta^{(n)-}|_{k,\overline{k}} = 2(2^n-1)/3$ and $\theta^{(n)-}$ ends with $k-1$, which implies that $\theta^{(n)}$ ends with k. By $\theta^{(n+1)-} = \theta^{(n)}\overline{\theta^{(n)}}$, we know that $|\theta^{(n+1)-}|_{k,\overline{k}} = 2(|\theta^{(n)-}|_{k,\overline{k}} +$

1) $= 2(2^{n+1}+1)/3$ and $\theta^{(n+1)-}$ ends with $\overline{k}$, where $n+1$ is odd. Thus (3.3) is true for $n+1$.

ii) If n is odd, then $|\theta^{(n)-}|_{k,\overline{k}} = 2(2^n+1)/3$ and $\theta^{(n)-}$ ends with $\overline{k}$, which implies that $\theta^{(n)}$ ends with $\overline{k}+1$. By $\theta^{(n+1)-} = \theta^{(n)}\overline{\theta^{(n)}}$, we know that $|\theta^{(n+1)-}|_{k,\overline{k}} = 2(|\theta^{(n)-}|_{k,\overline{k}}-1) = 2(2^{n+1}-1)/3$ and $\theta^{(n+1)-}$ ends with $k-1$, where $n+1$ is even. Thus (3.3) is true for $n+1$.

(3) follows from (2) and (3.3). □

By [10, Theorem 1], [3, Part 3, pp. 74, Lemma 3] and (3.3), we get the following.

Lemma 3.1.13. (1) *For all $n \geq 1$, we have $\overline{\theta} < \sigma^n\theta < \theta$.*
(2) *Let $j \in \mathbb{N}$ and u, v be finite words on $\{0, \cdots, m\}$ such that $\theta^{(j)*} = uv$, where u is non-empty and v may be empty. Then*

$$\begin{cases} \overline{uv}\overline{k} < \overline{v}\overline{k}u < \overline{v}(\overline{k}+1)\overline{u} < uvk & \text{if } j \text{ is even,} \\ \overline{uv}(k-1) < \overline{v}(k-1)u < \overline{v}k\overline{u} < uv(\overline{k}+1) & \text{if } j \text{ is odd.} \end{cases}$$

Lemma 3.1.14. *Let n, s be integers such that $0 \leq n < s$. Then $\theta^{(s)*}$ begins with $\theta^{(n)}\overline{\theta^{(n)*}}$.*

Proof. It follows immediately from $\theta^{(s)*} = \theta^{(s-1)}\overline{\theta^{(s-1)*}}$, $\theta^{(s-1)*} = \theta^{(s-2)}\overline{\theta^{(s-2)*}}$, $\cdots$, $\theta^{(n+2)*} = \theta^{(n+1)}\overline{\theta^{(n+1)*}}$ and $\theta^{(n+1)*} = \theta^{(n)}\overline{\theta^{(n)*}}$. □

Lemma 3.1.15. *For any $n \in \mathbb{N}$, there exist integers $l_1 > l_2 > \cdots > l_t \geq 0$ such that $n = 2^{l_1} + 2^{l_2} + \cdots + 2^{l_t}$ and*

$$\theta_1 \cdots \theta_n = \begin{cases} \theta^{(l_1)}\overline{\theta^{(l_2)}} \cdots \theta^{(l_{t-1})}\overline{\theta^{(l_t)}} & \text{if } t \text{ is even,} \\ \theta^{(l_1)}\overline{\theta^{(l_2)}} \cdots \theta^{(l_{t-2})}\overline{\theta^{(l_{t-1})}}\theta^{(l_t)} & \text{if } t \text{ is odd.} \end{cases}$$

Proof. Let $n \in \mathbb{N}$. Then there exists $l_1 \in \{0, 1, 2, \cdots\}$ such that $2^{l_1} \leq n \leq 2^{l_1+1}-1$. By the definition of θ, we know that $\theta_1 \cdots \theta_{2^{l_1}} = \theta^{(l_1)}$ and

$$\theta_{2^{l_1}+1}\theta_{2^{l_1}+2}\cdots \text{ begins with } \overline{\theta^{(l_1)*}}. \tag{3.4}$$

If $n = 2^{l_1}$, then $\theta_1 \cdots \theta_n = \theta^{(l_1)}$ and the conclusion follows.
If $n > 2^{l_1}$, by $n - 2^{l_1} \leq 2^{l_1} - 1$, there exists $l_2 \in \{0, \cdots, l_1 - 1\}$ such that $2^{l_2} \leq n - 2^{l_1} \leq 2^{l_2+1} - 1$. By Lemma 3.1.14 we know that $\overline{\theta^{(l_1)*}}$ begins with $\overline{\theta^{(l_2)}}\theta^{(l_2)*}$. It follows from (3.4) that $\theta_{2^{l_1}+1} \cdots \theta_{2^{l_1}+2^{l_2}} = \overline{\theta^{(l_2)}}$ and

$$\theta_{2^{l_1}+2^{l_2}+1}\theta_{2^{l_1}+2^{l_2}+2}\cdots \text{ begins with } \theta^{(l_2)*}.$$

...

For general $j \geq 2$, suppose that there already exist integers $l_1 > l_2 > \cdots > l_j \geq 0$ such

that $2^{l_j} \le n - 2^{l_1} - \cdots - 2^{l_{j-1}} \le 2^{l_j+1} - 1$, $\theta_1 \cdots \theta_{2^{l_1}+\cdots+2^{l_j}} = \theta^{(l_1)}\overline{\theta^{(l_2)}} \cdots \theta^{(l_{j-1})}\overline{\theta^{(l_j)}}$ (we only consider that j is even since the case that j is odd is similar), and

$$\theta_{2^{l_1}+\cdots+2^{l_j}+1}\theta_{2^{l_1}+\cdots+2^{l_j}+2}\cdots \text{ begins with } \theta^{(l_j)*}. \tag{3.5}$$

If $n = 2^{l_1} + \cdots + 2^{l_j}$, then $\theta_1 \cdots \theta_n = \theta^{(l_1)}\overline{\theta^{(l_2)}} \cdots \theta^{(l_{j-1})}\overline{\theta^{(l_j)}}$ and the conclusion follows. If $n > 2^{l_1} + \cdots + 2^{l_j}$, by $n - 2^{l_1} - \cdots - 2^{l_j} \le 2^{l_j} - 1$, there exists $l_{j+1} \in \{0, \cdots, l_j - 1\}$ such that $2^{l_{j+1}} \le n - 2^{l_1} - \cdots - 2^{l_j} \le 2^{l_{j+1}+1} - 1$. By Lemma 3.1.14 we know that $\theta^{(l_j)*}$ begins with $\theta^{(l_{j+1})}\overline{\theta^{(l_{j+1})*}}$. It follows from (3.5) that $\theta_{2^{l_1}+\cdots+2^{l_j}+1} \cdots \theta_{2^{l_1}+\cdots+2^{l_{j+1}}} = \theta^{(l_{j+1})}$ and

$$\theta_{2^{l_1}+\cdots+2^{l_{j+1}}+1}\theta_{2^{l_1}+\cdots+2^{l_{j+1}}+2}\cdots \text{ begins with } \overline{\theta^{(l_{j+1})*}}.$$

...

The above process must stop in a finite number of times since n is finite. Therefore the conclusion follows. □

To show Theorem 3.1.1 (1), the main we need to prove is the following.

Lemma 3.1.16. *Let $m \ge 2$ be an integer, $k \in \{\lceil \frac{m}{2} \rceil + 1, \cdots, m\}$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$ such that*

$$\sigma^n w < \theta \quad \textit{whenever } w_n < m \tag{3.6}$$

and

$$\sigma^n w > \overline{\theta} \quad \textit{whenever } w_n > 0. \tag{3.7}$$

(1) ① *For all $s \in \mathbb{N}$ such that $w_s < m$ and $w_{s+1} = k$, there exist integers $j_0 \ge 1$ and $j_1, j_2, \cdots \ge 0$ such that $w_{s+1}w_{s+2}\cdots = \theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}\cdots$.*
② *For all $s \in \mathbb{N}$ such that $w_s > 0$ and $w_{s+1} = \overline{k}$, there exist integers $j_0 \ge 1$ and $j_1, j_2, \cdots \ge 0$ such that $w_{s+1}w_{s+2}\cdots = \overline{\theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}}\cdots$.*

(2) *For all integers $s \ge 0$, $j_0 \ge 1$ and $j_1, j_2, \cdots \ge 0$ such that*

$$w_{s+1}w_{s+2}\cdots = \theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}\cdots \quad (\textit{or } \overline{\theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}}\cdots),$$

we have the following.

① *$j_{n+1} \ge j_n - 1$ for all $n \ge 0$.*
② *If $j_{n+1} = j_n - 1$ for some $n \ge 0$, then $j_{n+2} \ge j_n$.*
③ *If $j_{n+1} = j_n - 1$ and $j_{n+2} = j_n$ for some $n \ge 0$, then $j_{n+3} \ge j_n + 1$.*
④ *If $\{j_n\}_{n\ge 0}$ is bounded, then w ends with $(\theta^{(M)-})^\infty$ (or $(\overline{\theta^{(M)-}})^\infty$) where $M = \max_{n\ge 0} j_n$.*

⑤ *If* $\{j_n\}_{n\geq 0}$ *is not bounded, then* $j_n \to \infty$ *as* $n \to \infty$.

(3) *If* $w \notin \{0^\infty, m^\infty\}$, *then* $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w)$.

Proof. (1) Since the proofs of ① and ② are similar, we only prove ① as follows.
i) For all $s \in \mathbb{N}$ such that $w_s < m$ and $w_{s+1} = k$, prove that there exists $j \in \mathbb{N}$ such that $w_{s+1} \cdots w_{s+2^j} = \theta^{(j)-}$.

Note that $\theta_1 \cdots \theta_{2^i} = \theta^{(i)}$ for all $i \geq 0$ and $\theta^{(0)} = k$. On the one hand it follows from

$$\theta^{(0)} w_{s+2} = w_{s+1} w_{s+2} \overset{\text{by (3.6)}}{\leq} \theta_1 \theta_2 = \theta^{(1)} = \theta^{(0)} \overline{\theta^{(0)}}^+$$

that

$$w_{s+2} \leq \overline{\theta^{(0)}}^+.$$

On the other hand it follows from (3.7) that

$$w_{s+2} \geq \overline{\theta_1} = \overline{\theta^{(0)}}.$$

Thus $w_{s+2} = \overline{\theta^{(0)}}$ or $\overline{\theta^{(0)}}^+$. If $w_{s+2} = \overline{\theta^{(0)}}$, then

$$w_{s+1} w_{s+2} = \theta^{(0)} \overline{\theta^{(0)}} = \theta^{(1)-}$$

will complete the proof. If $w_{s+2} = \overline{\theta^{(0)}}^+$, then

$$w_{s+1} w_{s+2} = \theta^{(0)} \overline{\theta^{(0)}}^+ = \theta^{(1)}.$$

On the one hand it follows from

$$\theta^{(1)} w_{s+3} w_{s+4} = w_{s+1} \cdots w_{s+4} \overset{\text{by (3.6)}}{\leq} \theta_1 \cdots \theta_4 = \theta^{(2)} = \theta^{(1)} \overline{\theta^{(1)}}^+$$

that

$$w_{s+3} w_{s+4} \leq \overline{\theta^{(1)}}^+.$$

On the other hand it follows from (3.7) that

$$w_{s+3} w_{s+4} \geq \overline{\theta_1 \theta_2} = \overline{\theta^{(1)}}.$$

Thus $w_{s+3} w_{s+4} = \overline{\theta^{(1)}}$ or $\overline{\theta^{(1)}}^+$. If $w_{s+3} w_{s+4} = \overline{\theta^{(1)}}$, then

$$w_{s+1} \cdots w_{s+4} = \theta^{(1)} \overline{\theta^{(1)}} = \theta^{(2)-}$$

will complete the proof. If $w_{s+3} w_{s+4} = \overline{\theta^{(1)}}^+$, then

$$w_{s+1} \cdots w_{s+4} = \theta^{(1)} \overline{\theta^{(1)}}^+ = \theta^{(2)}.$$

...

For general $i \in \mathbb{N}$, if we have already had $w_{s+1} \cdots w_{s+2^i} = \theta^{(i)}$, on the one hand it follows from

$$\theta^{(i)} w_{s+2^i+1} \cdots w_{s+2^{i+1}} = w_{s+1} \cdots w_{s+2^{i+1}} \overset{\text{by (3.6)}}{\leq} \theta_1 \cdots \theta_{2^{i+1}} = \theta^{(i+1)} = \theta^{(i)} \overline{\theta^{(i)}}^{+}$$

that

$$w_{s+2^i+1} \cdots w_{s+2^{i+1}} \leq \overline{\theta^{(i)}}^{+};$$

on the other hand it follows from (3.7) that

$$w_{s+2^i+1} \cdots w_{s+2^{i+1}} \geq \overline{\theta_1 \cdots \theta_{2^i}} = \overline{\theta^{(i)}}.$$

Thus $w_{s+2^i+1} \cdots w_{s+2^{i+1}} = \overline{\theta^{(i)}}$ or $\overline{\theta^{(i)}}^{+}$. If $w_{s+2^i+1} \cdots w_{s+2^{i+1}} = \overline{\theta^{(i)}}$, then

$$w_{s+1} \cdots w_{s+2^{i+1}} = \theta^{(i)} \overline{\theta^{(i)}} = \theta^{(i+1)-}$$

will complete the proof. If $w_{s+2^i+1} \cdots w_{s+2^{i+1}} = \overline{\theta^{(i)}}^{+}$, then

$$w_{s+1} \cdots w_{s+2^{i+1}} = \theta^{(i)} \overline{\theta^{(i)}}^{+} = \theta^{(i+1)}.$$

...

The above process must end in a finite number of times (otherwise we get $w_{s+1} w_{s+2} \cdots = \lim_{i \to \infty} \theta^{(i)} = \theta$, which contradicts (3.6)). Thus there must exist $j \in \mathbb{N}$ such that $w_{s+1} \cdots w_{s+2^j} = \theta^{(j)-}$.

ii) Let $s \in \mathbb{N}$ such that $w_s < m$ and $w_{s+1} = k$. Prove that there exist integers $j_0 \geq 1$ and $j_1, j_2, \cdots \geq 0$ such that $w_{s+1} w_{s+2} \cdots = \theta^{(j_0)-} \theta^{(j_1)-} \theta^{(j_2)-} \cdots$.

In fact, by the definition of $\theta^{(i)}$ and induction, it is straightforward to check that for all $i \in \mathbb{N}$, we have

$$\theta^{(i)-} \text{ ends with } k\overline{k} \quad \text{if } i \text{ is odd}$$

and

$$\theta^{(i)-} \text{ ends with } (\overline{k}+1)\overline{k}(k-1) \quad \text{if } i \text{ is even.}$$

Recall from i) that there exists $j_0 \in \mathbb{N}$ such that $w_{s+1} \cdots w_{s+2^{j_0}} = \theta^{(j_0)-}$.

ⓐ If j_0 is even, then $\theta^{(j_0)-}$ ends with $(\overline{k}+1)\overline{k}(k-1)$ and

$$w_{s+2^{j_0}-2} w_{s+2^{j_0}-1} w_{s+2^{j_0}} \cdots = (\overline{k}+1)\overline{k}(k-1) w_{s+2^{j_0}+1} w_{s+2^{j_0}+2} \cdots .$$

On the one hand by (3.6) we get

$$w_{s+2^{j_0}+1} w_{s+2^{j_0}+2} \cdots < \theta = k(\overline{k}+1) \cdots ,$$

which implies $w_{s+2^{j_0}+1} \le k$. On the other hand by (3.7) we get

$$\overline{k}(k-1)w_{s+2^{j_0}+1}w_{s+2^{j_0}+2}\cdots > \overline{\theta} = \overline{k}(k-1)k\overline{k}\cdots,$$

which implies $w_{s+2^{j_0}+1} \ge k$. Thus $w_{s+2^{j_0}+1} = k$. Since $w_{s+2^{j_0}} = k-1 < m$, by applying i), there exists $j_1 \in \mathbb{N}$ such that $w_{s+2^{j_0}+1}\cdots w_{s+2^{j_0}+2^{j_1}} = \theta^{(j_1)-}$ and then

$$w_{s+1}\cdots w_{s+2^{j_0}+2^{j_1}} = \theta^{(j_0)-}\theta^{(j_1)-}.$$

ⓑ If j_0 is odd, then $\theta^{(j_0)-}$ ends with $k\overline{k}$ and

$$w_{s+2^{j_0}-1}w_{s+2^{j_0}}\cdots = k\overline{k}w_{s+2^{j_0}+1}w_{s+2^{j_0}+2}\cdots.$$

By (3.6) we get

$$w_{s+2^{j_0}+1}w_{s+2^{j_0}+2}\cdots < \theta = k(\overline{k}+1)\cdots,$$

which implies $w_{s+2^{j_0}+1} \le k$.
I. If $w_{s+2^{j_0}+1} = k$, by i) there exists $j_1 \in \mathbb{N}$ such that $w_{s+2^{j_0}+1}\cdots w_{s+2^{j_0}+2^{j_1}} = \theta^{(j_1)-}$ and then

$$w_{s+1}\cdots w_{s+2^{j_0}+2^{j_1}} = \theta^{(j_0)-}\theta^{(j_1)-}.$$

II. If $w_{s+2^{j_0}+1} \le k-1$, it follows from

$$\overline{k}w_{s+2^{j_0}+1}w_{s+2^{j_0}+2}\cdots \overset{\text{by (3.7)}}{>} \overline{\theta} = \overline{k}(k-1)k\cdots$$

that $w_{s+2^{j_0}+1} = k-1$ and $w_{s+2^{j_0}+2} \ge k$. Since (3.6) implies $w_{s+2^{j_0}+2} \le \theta_1 = k$, we get $w_{s+2^{j_0}+2} = k$, and then $w_{s+2^{j_0}+1}w_{s+2^{j_0}+2} = (k-1)k$. By i) there exists $j_2 \in \mathbb{N}$ such that

$$w_{s+2^{j_0}+2}\cdots w_{s+2^{j_0}+2^{j_2}+1} = \theta^{(j_2)-}.$$

Let $j_1 = 0$. Then $w_{s+2^{j_0}+1} = k-1 = \theta^{(j_1)-}$ and

$$w_{s+1}\cdots w_{s+2^{j_0}+2^{j_1}+2^{j_2}} = \theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}.$$

By applying i) and repeating the above process again and again, we know that there exist $j_1, j_2, \cdots \ge 0$ such that $w_{s+1}w_{s+2}\cdots = \theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}\cdots$.

(2) Let $s \ge 0$, $j_0 \ge 1$ and $j_1, j_2, \cdots \ge 0$ such that $w_{s+1}w_{s+2}\cdots = \theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}\cdots$ (the case $w_{s+1}w_{s+2}\cdots = \overline{\theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}}\cdots$ is similar). For all $n \ge 0$ such that $j_n \ge 1$, by

$\theta^{(j_n)-} = \theta^{(j_n-1)}\overline{\theta^{(j_n-1)}}$ we get

$$w_{s+1}w_{s+2}\cdots = \theta^{(j_0)-}\cdots\theta^{(j_{n-2})-}\theta^{(j_{n-1})-}\theta^{(j_n-1)}\overline{\theta^{(j_n-1)}}\theta^{(j_{n+1})-}\theta^{(j_{n+2})-}\cdots.$$

It follows from (3.7) that

$$\overline{\theta^{(j_n-1)}}\theta^{(j_{n+1})-}\theta^{(j_{n+2})-}\cdots > \overline{\theta} \quad \text{whenever } j_n \geq 1. \tag{3.8}$$

① Let $n \geq 0$ be an integer. If $j_n \leq 1$, then $j_{n+1} \geq j_n - 1$ is obvious. We only need to consider $j_n \geq 2$ and prove $j_{n+1} \geq j_n - 1$ in the following.
(By contradiction) Assume $j_{n+1} < j_n - 1$. Since θ begins with $\theta^{(j_n-1)}\overline{\theta^{(j_n-1)}}^{+}$, we know that $\overline{\theta}$ begins with $\overline{\theta^{(j_n-1)}}\theta^{(j_n-1)-}$, where $\theta^{(j_n-1)-}$ begins with $\theta^{(j_{n+1})}$ by Lemma 3.1.14. Thus $\overline{\theta}$ begins with $\overline{\theta^{(j_n-1)}}\theta^{(j_{n+1})}$. This contradicts (3.8).

② Suppose $j_{n+1} = j_n - 1$ for some $n \geq 0$. We need to prove $j_{n+2} \geq j_n$ in the following.
(By contradiction) Assume $j_{n+2} < j_n$. Since θ begins with

$$\theta^{(j_n)}\overline{\theta^{(j_n)}}^{+} = \theta^{(j_n-1)}\overline{\theta^{(j_n-1)}}^{+}\overline{\theta^{(j_n)}}^{+},$$

we know that $\overline{\theta}$ begins with

$$\overline{\theta^{(j_n-1)}}\theta^{(j_n-1)-}\theta^{(j_n)-} = \overline{\theta^{(j_n-1)}}\theta^{(j_{n+1})-}\theta^{(j_n)-},$$

where $\theta^{(j_n)-}$ begins with $\theta^{(j_{n+2})}$ by Lemma 3.1.14. Thus $\overline{\theta}$ begins with $\overline{\theta^{(j_n-1)}}\theta^{(j_{n+1})-}\theta^{(j_{n+2})}$. This contradicts (3.8).

③ Suppose $j_{n+1} = j_n - 1$ and $j_{n+2} = j_n$ for some $n \geq 0$. We need to prove $j_{n+3} \geq j_n + 1$ in the following.
(By contradiction) Assume $j_{n+3} \leq j_n$. Since θ begins with

$$\theta^{(j_n+1)}\overline{\theta^{(j_n+1)}}^{+} = \theta^{(j_n)}\overline{\theta^{(j_n)}}^{+}\overline{\theta^{(j_n)}}\theta^{(j_n)} = \theta^{(j_n-1)}\overline{\theta^{(j_n-1)}}^{+}\overline{\theta^{(j_n)}}^{+}\overline{\theta^{(j_n)}}\theta^{(j_n)},$$

we know that $\overline{\theta}$ begins with

$$\overline{\theta^{(j_n-1)}}\theta^{(j_n-1)-}\theta^{(j_n)-}\theta^{(j_n)} = \overline{\theta^{(j_n-1)}}\theta^{(j_{n+1})-}\theta^{(j_{n+2})-}\theta^{(j_n)},$$

where $\theta^{(j_n)} = \theta^{(j_{n+3})}$ if $j_{n+3} = j_n$ and $\theta^{(j_n)}$ begins with $\theta^{(j_{n+3})}$ if $j_{n+3} < j_n$ by Lemma 3.1.14. Thus $\overline{\theta}$ begins with $\overline{\theta^{(j_n-1)}}\theta^{(j_{n+1})-}\theta^{(j_{n+2})-}\theta^{(j_{n+3})}$. This contradicts (3.8).

④ If $\{j_n\}_{n\geq 0}$ is bounded, let $M = \max_{n\geq 0} j_n$. Then there exists $p \geq 0$ such that $j_p = M$. By ① we get $j_{p+1} \geq M - 1$. Thus $j_{p+1} = M - 1$ or M. If $j_{p+1} = M - 1$ $(= j_p - 1)$, it follows from ② that $j_{p+2} \geq j_p$ $(= M)$, which implies $j_{p+2} = j_p$. Then by ③ we get $j_{p+3} \geq j_p + 1$ $(= M + 1)$. This contradicts the definition of M. Thus $j_{p+1} \neq M - 1$

and we must have $j_{p+1} = M$. In the same way we can get $j_{p+2} = M, j_{p+3} = M, \cdots$.

⑤ Fix any $M > 0$. Since $\{j_n\}_{n\geq 1}$ is not bounded, there exists $N \in \mathbb{N}$ such that $j_N \geq M + 2$. It suffices to prove $j_n > M$ for all $n \geq N$. Let $p \geq N$ be the smallest integer such that $j_p = \min_{n\geq N} j_n$. We only need to prove $j_p > M$. It suffices to prove $j_p \geq j_N - 1$.
(By contradiction) Assume $j_p \leq j_N - 2$. Then $p \neq N$. By $p \geq N$ we get $p \geq N + 1$, i.e., $p - 1 \geq N$. It follows from the definition of p that $j_{p-1} \geq j_p + 1$. By ① we get $j_{p-1} = j_p + 1$. This implies $j_{p-1} \leq j_N - 1$, and then by $p - 1 \geq N$ we must have $p - 1 \geq N + 1$, i.e., $p - 2 \geq N$. It follows from the definition of p that $j_{p-2} \geq j_p + 1$ $(= j_{p-1})$. Since ① implies $j_{p-2} \leq j_{p-1} + 1$, we get $j_{p-2} = j_{p-1}$ or $j_{p-1} + 1$. If $j_{p-2} = j_{p-1} + 1$, by ② we get $j_p \geq j_{p-2}$, which contradicts $j_{p-2} = j_{p-1} + 1 = j_p + 2$. Thus we must have $j_{p-2} = j_{p-1} = j_p + 1$.

For general $i \geq 2$, if we have already had $p - i \geq N$ and $j_{p-i} = j_{p-i+1} = j_p + 1$, by $j_p \leq j_N - 2$ we get $j_{p-i} \leq j_N - 1$, and then by $p - i \geq N$ we must have $p - i \geq N + 1$, i.e., $p - i - 1 \geq N$. It follows from the definition of p that $j_{p-i-1} \geq j_p + 1$ $(= j_{p-i})$. Since ① implies $j_{p-i-1} \leq j_{p-i} + 1$, we get $j_{p-i-1} = j_{p-i}$ or $j_{p-i} + 1$. If $j_{p-i-1} = j_{p-i} + 1$, by ② we get $j_{p-i+1} \geq j_{p-i-1}$, which contradicts $j_{p-i-1} = j_{p-i} + 1 = j_{p-i+1} + 1$. Thus we must have $j_{p-i-1} = j_{p-i} = j_p + 1$. This implies $j_{p-i-1} \leq j_N - 1$, and then by $p - i - 1 \geq N$ we must have $p - i - 1 \geq N + 1$, i.e., $p - i - 2 \geq N$.

By induction we get $p - i \geq N$ for all $i \in \mathbb{N}$. This is impossible.

(3) ① If $k = m$, it suffices to prove $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w)$.

i) If $w_1 = 0$, by $w \neq 0^\infty$, there exists $s \in \mathbb{N}$ such that $w_1 \cdots w_s = 0^s$ and $w_{s+1} > 0$.
ⓐ When $w_{s+1} = m$ $(= k)$, we have $w = 0^s m w_{s+2} w_{s+3} \cdots$. By (1) ① there exist $j_0 \geq 1$ and $j_1, j_2, \cdots \geq 0$ such that $w = 0^s \theta^{(j_0)-} \theta^{(j_1)-} \theta^{(j_2)-} \cdots$.

I. If $\{j_n\}_{n\geq 0}$ is bounded, let $M = \max_{n\geq 0} j_n$. By (2) ④ we know that w ends with $(\theta^{(M)-})^\infty$, which implies that both $\mathrm{Freq}_0(w)$ and $\mathrm{Freq}_m(w)$ exist. Since Lemma 3.1.12 (2) implies $|\theta^{(M)-}|_0 = |\theta^{(M)-}|_m$, we get $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w)$.

II. If $\{j_n\}_{n\geq 0}$ is not bounded, we can prove $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w) = \frac{1}{3}$. Since the proofs of $\mathrm{Freq}_0(w) = \frac{1}{3}$ and $\mathrm{Freq}_m(w) = \frac{1}{3}$ are similar, we only prove $\mathrm{Freq}_0(w) = \frac{1}{3}$ as follows. Let $v := \theta^{(j_1)-} \theta^{(j_2)-} \theta^{(j_3)-} \cdots$. It suffices to prove $\mathrm{Freq}_0(v) = \frac{1}{3}$, i.e., $\lim_{n\to\infty} \frac{|v_1 \cdots v_n|_0}{n} = \frac{1}{3}$.
Let $\varepsilon > 0$. Since $\frac{t}{2^{t-1}} \to 0$ as $t \to \infty$, there exists $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$ we have

$$\frac{t}{2^{t-1}} < \varepsilon. \tag{3.9}$$

By the fact that $2^{j_1}+\cdots+2^{j_p}\to\infty$ as $p\to\infty$, there exists $p_0\in\mathbb{N}$ such that for all $p\geq p_0$ we have

$$\frac{t_0}{2^{j_1}+\cdots+2^{j_p}}<\varepsilon. \tag{3.10}$$

Since $\{j_p\}_{p\geq 0}$ is not bounded, by (2) ⑤ we get $j_p\to\infty$ as $p\to\infty$, which implies $2^{j_p}\to\infty$, and then $\frac{2^{j_1}+\cdots+2^{j_p}}{p}\to\infty$ by Lemma 3.1.7. Thus there exists $p_1\geq p_0$ such that for all $p\geq p_1$ we have

$$\frac{p}{2^{j_1}+\cdots+2^{j_p}}<\frac{\varepsilon}{2}. \tag{3.11}$$

Let $N_{p_1}:=|\theta^{(j_1)}\theta^{(j_2)}\cdots\theta^{(j_{p_1})}|=2^{j_1}+2^{j_2}+\cdots+2^{j_{p_1}}$. Then for any $n>N_{p_1}$, we only need to check $|\frac{|v_1\cdots v_n|_0}{n}-\frac{1}{3}|<\varepsilon$.

In fact, for any $n>N_{p_1}$, there exists $p\geq p_1$ such that $|\theta^{(j_1)}\cdots\theta^{(j_p)}|\leq n<|\theta^{(j_1)}\cdots\theta^{(j_p)}\theta^{(j_{p+1})}|$. Let $r:=n-|\theta^{(j_1)}\cdots\theta^{(j_p)}|<|\theta^{(j_{p+1})}|$. Since the proof for the case $r=0$ is similar and more straightforward, we only consider $r\geq 1$ in the following. By $\theta^{(j_{p+1})*}=\theta_1\cdots\theta_{2^{j_{p+1}}-1}$ and Lemma 3.1.15, there exist integers $l_1>l_2>\cdots>l_t\geq 0$ such that $r=2^{l_1}+2^{l_2}+\cdots+2^{l_t}$ and

$$v_1\cdots v_n=\begin{cases}\theta^{(j_1)-}\cdots\theta^{(j_p)-}\theta^{(l_1)}\overline{\theta^{(l_2)}}\cdots\theta^{(l_{t-1})}\overline{\theta^{(l_t)}} & \text{if } t \text{ is even,}\\ \theta^{(j_1)-}\cdots\theta^{(j_p)-}\theta^{(l_1)}\overline{\theta^{(l_2)}}\cdots\theta^{(l_{t-2})}\overline{\theta^{(l_{t-1})}}\theta^{(l_t)} & \text{if } t \text{ is odd.}\end{cases}$$

It follows from Lemma 3.1.12 that

$$|v_1\cdots v_n|_0\leq\frac{2^{j_1}+2}{3}+\cdots+\frac{2^{j_p}+2}{3}+\frac{2^{l_1}+2}{3}+\cdots+\frac{2^{l_t}+2}{3}.$$

By $n=2^{j_1}+\cdots+2^{j_p}+2^{l_1}+\cdots+2^{l_t}$ we get

$$\frac{|v_1\cdots v_n|_0}{n}\leq\frac{1}{3}+\frac{2(p+t)}{3n}\leq\frac{1}{3}+\frac{2}{3}\Big(\frac{p}{2^{j_1}+\cdots+2^{j_p}}+\frac{t}{n}\Big)\overset{\text{by (3.11)}}{<}\frac{1}{3}+\frac{\varepsilon}{3}+\frac{2t}{3n}. \tag{3.12}$$

If $t\leq t_0$, then

$$\frac{t}{n}\leq\frac{t}{2^{j_1}+\cdots+2^{j_p}}\leq\frac{t_0}{2^{j_1}+\cdots+2^{j_p}}\overset{\text{by (3.10)}}{<}\varepsilon.$$

It follows from (3.12) that $\frac{|v_1\cdots v_n|_0}{n}<\frac{1}{3}+\varepsilon$.
If $t\geq t_0+1$, then

$$\frac{t}{n}\leq\frac{t}{2^{l_1}}\overset{(\star)}{\leq}\frac{t}{2^{t-1}}\overset{\text{by (3.9)}}{<}\varepsilon,$$

where $(\star)$ follows from $l_1\geq t-1$ (recall $l_1>l_2>\cdots>l_t\geq 0$). By (3.12) we get $\frac{|v_1\cdots v_n|_0}{n}<\frac{1}{3}+\varepsilon$.

It follows in the same way that $\frac{|v_1\cdots v_n|_0}{n}>\frac{1}{3}-\varepsilon$. Thus $|\frac{|v_1\cdots v_n|_0}{n}-\frac{1}{3}|<\varepsilon$ for all

$n > N_{p_1}$.

ⓑ When $1 \le w_{s+1} \le m-1$, since $w_{s+2}, w_{s+3}, w_{s+4}, \cdots \notin \{0, m\}$ will imply $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w) = 0$ directly, we only need to consider that there exists $t \ge s+1$ such that $w_{t+1} \in \{0, m\}$. Assume that such t is the smallest one. Then $w_{s+1}, w_{s+2}, \cdots, w_t \notin \{0, m\}$.

I. If $w_{t+1} = 0$, then

$$w = 0^s w_{s+1} \cdots w_t 0 w_{t+2} w_{t+3} \cdots .$$

By (1) ②, there exist $j_0 \ge 1$ and $j_1, j_2, \cdots \ge 0$ such that

$$w = 0^s w_{s+1} \cdots w_t \overline{\theta^{(j_0)} - \theta^{(j_1)} - \theta^{(j_2)} -} \cdots .$$

In the same way as ⓐ, we get $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w)$.

II. If $w_{t+1} = m$, then

$$w = 0^s w_{s+1} \cdots w_t m w_{t+2} w_{t+3} \cdots .$$

By (1) ①, there exist $j_0 \ge 1$ and $j_1, j_2, \cdots \ge 0$ such that

$$w = 0^s w_{s+1} \cdots w_t \theta^{(j_0)} - \theta^{(j_1)} - \theta^{(j_2)} - \cdots .$$

In the same way as ⓐ, we get $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w)$.

ii) If $w_1 = m$, then $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w)$ follows in the same way as i).

iii) If $1 \le w_1 \le m-1$, since $w_2, w_3, w_4, \cdots \notin \{0, m\}$ will obviously imply $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w) = 0$, we only need to consider that there exists a smallest $s \ge 1$ such that $w_{s+1} \in \{0, m\}$ but $w_1, w_2, \cdots, w_s \notin \{0, m\}$. By (1) ① and ②, there exist $j_0 \ge 1$ and $j_1, j_2, \cdots \ge 0$ such that

$$w = \begin{cases} w_1 \cdots w_s \theta^{(j_0)} - \theta^{(j_1)} - \theta^{(j_2)} - \cdots & \text{if } w_{s+1} = m; \\ w_1 \cdots w_s \overline{\theta^{(j_0)} - \theta^{(j_1)} - \theta^{(j_2)} -} \cdots & \text{if } w_{s+1} = 0. \end{cases}$$

It follows in the same way as i) ⓐ that $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w)$.

② If $\lceil \frac{m}{2} \rceil + 1 \le k \le m-1$, we need to prove $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w)$. Since $w_1, w_2, w_3, \cdots \notin \{\overline{k}, \overline{k}+1, \cdots, k-1, k\}$ will imply $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w) = 0$ directly, we only need to consider that there exists $t \in \mathbb{N}$ such that $0 < \overline{k} \le w_t \le k < m$. By (3.6) and (3.7), we get $0 < \overline{k} \le w_{t+1} \le k < m$. By (3.6) and (3.7) again, we get $0 < \overline{k} \le w_{t+2} \le k < m$. $\cdots$ By induction we get $0 < \overline{k} \le w_n \le k < m$ for all $n \ge t$. Since $w_{t+1}, w_{t+2}, w_{t+3}, \cdots \notin \{\overline{k}, k\}$ will obviously imply $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w) = 0$, it suffices to consider that there exists $s \ge t$ such that $w_{s+1} \in \{\overline{k}, k\}$. By $0 < w_s < m$, (1) ① and ②, there exist $j_0 \ge 1$ and

$j_1, j_2, \cdots \geq 0$ such that

$$w = \begin{cases} w_1 \cdots w_s \theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-} \cdots & \text{if } w_{s+1} = k; \\ w_1 \cdots w_s \overline{\theta^{(j_0)-}\theta^{(j_1)-}\theta^{(j_2)-}} \cdots & \text{if } w_{s+1} = \overline{k}. \end{cases}$$

It follows in the same way as ① i) ⓐ that $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w)$. □

Proof of Theorem 3.1.1. By Lemma 3.1.13 (1) and Lemma 3.1.4 we know that θ is the unique $\mathfrak{q}_{m;k}$-expansion of 1.
(1) Let $\beta \in (G_m, \mathfrak{q}_{m;k}]$ and $w \in \Gamma_{m,\beta}$. By Lemma 3.1.5 we get

$$\sigma^n w < g^*(1,\beta) \text{ whenever } w_n < m \quad \text{and} \quad \sigma^n w > \overline{g^*(1,\beta)} \text{ whenever } w_n > 0.$$

It follows from $\beta \leq \mathfrak{q}_{m;k}$ and Lemma 3.1.6 that

$$\sigma^n w < g^*(1,\mathfrak{q}_{m;k}) \text{ whenever } w_n < m \quad \text{and} \quad \sigma^n w > \overline{g^*(1,\mathfrak{q}_{m;k})} \text{ whenever } w_n > 0.$$

Since θ is the unique $\mathfrak{q}_{m;k}$-expansion of 1, we have

$$\sigma^n w < \theta \text{ whenever } w_n < m \quad \text{and} \quad \sigma^n w > \overline{\theta} \text{ whenever } w_n > 0.$$

It follows from Lemma 3.1.16 (3) that $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w)$.
(2) Let $\beta \in (\mathfrak{q}_{m;k}, m+1]$. Since θ is the unique $\mathfrak{q}_{m;k}$-expansion of 1, by Lemma 3.1.6 we get $g^*(1,\beta) > \theta$ and then $g_1^*(1,\beta) \geq k$.
① If $g_1^*(1,\beta) \geq k+1$, by Lemma 3.1.5 we get $\{k, \overline{k}\}^{\mathbb{N}} \subset \Gamma_{m,\beta}$. Define

$$\Lambda_{k,\overline{k}}^{\nexists} := \Big\{ w \in \{k, \overline{k}\}^{\mathbb{N}} : \mathrm{Freq}_k(w) \text{ and } \mathrm{Freq}_{\overline{k}}(w) \text{ do not exist} \Big\}.$$

Then

$$\Lambda_{k,\overline{k}}^{\nexists} \subset \Big\{ w \in \Gamma_{m,\beta} : \mathrm{Freq}_k(w) \text{ and } \mathrm{Freq}_{\overline{k}}(w) \text{ do not exist} \Big\}.$$

It suffice to prove $\dim_H(\Lambda_{k,\overline{k}}^{\nexists}, d_{m+1}) > 0$. In fact, this follows from

$$\dim_H(\Lambda_{k,\overline{k}}^{\nexists}, d_{m+1}) \overset{(\star)}{\geq} \frac{\log 2}{\log(m+1)} \cdot \dim_H(\Lambda_{k,\overline{k}}^{\nexists}, d_2) \overset{(\star\star)}{>} 0,$$

where $(\star)$ follows from Proposition 3.1.9 and the fact that the identity map from $(\Lambda_{k,\overline{k}}^{\nexists}, d_{m+1})$ to $(\Lambda_{k,\overline{k}}^{\nexists}, d_2)$ is $\frac{\log 2}{\log(m+1)}$-Hölder continuous, and $(\star\star)$ follows from Proposition 3.1.11.
② If $g_1^*(1,\beta) = k$, by $g^*(1,\beta) > \theta$, there exists $n \geq 2$ such that

$$g_1^*(1,\beta) \cdots g_{n-1}^*(1,\beta) = \theta_1 \cdots \theta_{n-1} \quad \text{and} \quad g_n^*(1,\beta) > \theta_n.$$

Let $j \geq 1$ be an integer large enough such that $2^{j+1} \geq n$. Then

$$g^*(1,\beta) \geq g_1^*(1,\beta)\cdots g_n^*(1,\beta)0^\infty > \theta_1\cdots\theta_n m^\infty \geq \theta_1\cdots\theta_{2^{j+1}}m^\infty = \theta^{(j+1)}m^\infty. \quad (3.13)$$

Define

$$\begin{aligned}\Xi_j &:= \left\{\theta^{(j)-}\theta^{(j)-}, \theta^{(j)}\overline{\theta^{(j)}}\right\}^{\mathbb{N}} \\ &= \left\{w \in \{0,\cdots,m\}^{\mathbb{N}} : w_{n\cdot 2^{j+1}+1}\cdots w_{(n+1)\cdot 2^{j+1}} = \theta^{(j)-}\theta^{(j)-} \text{ or } \theta^{(j)}\overline{\theta^{(j)}} \text{ for all } n \geq 0\right\}.\end{aligned}$$

i) Prove $\Xi_j \subset \Gamma_{m,\beta}$.
Let $w \in \Xi_j$. By Lemma 3.1.5 and (3.13), it suffices to prove that for all $n \in \mathbb{N}$, we have $\overline{\theta^{(j+1)}}0^\infty < \sigma^n w < \theta^{(j+1)}m^\infty$, i.e.,

$$\overline{\theta^{(j)}}\theta^{(j)-}0^\infty < \sigma^n w < \theta^{(j)}\overline{\theta^{(j)}}^+ m^\infty. \quad (3.14)$$

If n is a multiple of $|\theta^{(j)}|$, by the definition of Ξ_j, $\sigma^n w$ must begin with $\theta^{(j)-}$, $\theta^{(j)}\overline{\theta^{(j)}}$, $\overline{\theta^{(j)}}\theta^{(j)}$ or $\overline{\theta^{(j)}}\theta^{(j)-}\theta^{(j)-}$. This implies (3.14). If n is not a multiple of $|\theta^{(j)}|$, then there exist finite words u and v, where u is non-empty and v may be empty, such that $\theta^{(j)*} = uv$ and $\sigma^n w$ begins with

$$\begin{cases} v(k-1)u, vk\overline{u} \text{ or } \overline{v}\overline{k}u & \text{if } j \text{ is even (implies that } \theta^{(j)-} \text{ ends with } k-1 \text{ by (3.3))}; \\ v\overline{k}u, v(\overline{k}+1)\overline{u} \text{ or } \overline{v}(k-1)u & \text{if } j \text{ is odd (implies that } \theta^{(j)-} \text{ ends with } \overline{k} \text{ by (3.3))}. \end{cases}$$

It follows from Lemma 3.1.13 (2) that (3.14) is true.

For any $v \in \{0,1\}^{\mathbb{N}}$, we define $\Psi(v) := \psi(v_1)\psi(v_2)\cdots$ where $\psi(0) := \theta^{(j)-}\theta^{(j)-}$ and $\psi(1) := \theta^{(j)}\overline{\theta^{(j)}}$. Let

$$\Xi_j^\sharp := \left\{w \in \Xi_j : \mathrm{Freq}_k(w) \text{ and } \mathrm{Freq}_{\overline{k}}(w) \text{ do not exist}\right\}.$$

By i) we get

$$\Xi_j^\sharp \subset \left\{w \in \Gamma_{m,\beta} : \mathrm{Freq}_k(w) \text{ and } \mathrm{Freq}_{\overline{k}}(w) \text{ do not exist}\right\}.$$

It suffices to prove $\dim_H(\Xi_j^\sharp, d_{m+1}) > 0$. Let

$$\Lambda_{0,1}^\sharp := \left\{v \in \{0,1\}^{\mathbb{N}} : \mathrm{Freq}_0(v) \text{ and } \mathrm{Freq}_1(v) \text{ do not exist}\right\}.$$

Then we have

$$\dim_H(\Xi_j^\sharp, d_{m+1}) \overset{(\star)}{\geq} \dim_H(\Psi(\Lambda_{0,1}^\sharp), d_{m+1}) \overset{(\star\star)}{\geq} \frac{\log 2}{2^{j+2}\log(m+1)} \cdot \dim_H(\Lambda_{0,1}^\sharp, d_2) \overset{(\star\star\star)}{>} 0,$$

where $(\star\star\star)$ follows from Proposition 3.1.11, $(\star\star)$ follows from Proposition 3.1.9 and the facts that $\Psi : \{0,1\}^{\mathbb{N}} \to \Xi_j$ is bijective and $\Psi^{-1} : (\Xi_j, d_{m+1}) \to (\{0,1\}^{\mathbb{N}}, d_2)$ is $\frac{\log 2}{2^{j+2}\log(m+1)}$-Hölder continuous, and $(\star)$ follows from $\Psi(\Lambda_{0,1}^{\nexists}) \subset \Xi_j^{\nexists}$, which can be proved as follows.

Let $v \in \Lambda_{0,1}^{\nexists}$ and $w := \Psi(v)$. It suffices to prove $w \in \Xi_j^{\nexists}$. Since the proofs of $\overline{\mathrm{Freq}}_k(w) \neq \underline{\mathrm{Freq}}_k(w)$ and $\overline{\mathrm{Freq}}_{\overline{k}}(w) \neq \underline{\mathrm{Freq}}_{\overline{k}}(w)$ are similar. We only prove $\overline{\mathrm{Freq}}_k(w) \neq \underline{\mathrm{Freq}}_k(w)$ in the following.

ⓐ If j is odd, by Lemma 3.1.12 we get

$$|\theta^{(j)-}|_k = \frac{2^j+1}{3}, \quad |\theta^{(j)}|_k = \frac{2^j+1}{3} \quad \text{and} \quad |\overline{\theta^{(j)}}|_k = |\theta^{(j)}|_{\overline{k}} = \frac{2^j-2}{3}.$$

Then

$$|\psi(0)|_k = 2|\theta^{(j)-}|_k = \frac{2^{j+1}+2}{3} \quad \text{and} \quad |\psi(1)|_k = |\theta^{(j)}|_k + |\overline{\theta^{(j)}}|_k = \frac{2^{j+1}-1}{3}.$$

Note that Lemma 3.1.17 implies

$$\overline{\mathrm{Freq}}_k(w) = \frac{|\psi(0)|_k}{2^{j+1}} \cdot \overline{\mathrm{Freq}}_0(v) + \frac{|\psi(1)|_k}{2^{j+1}} \cdot (1 - \overline{\mathrm{Freq}}_0(v))$$

and

$$\underline{\mathrm{Freq}}_k(w) = \frac{|\psi(0)|_k}{2^{j+1}} \cdot \underline{\mathrm{Freq}}_0(v) + \frac{|\psi(1)|_k}{2^{j+1}} \cdot (1 - \underline{\mathrm{Freq}}_0(v)).$$

By $v \in \Lambda_{0,1}^{\nexists}$ we get $\overline{\mathrm{Freq}}_0(v) \neq \underline{\mathrm{Freq}}_0(v)$. It follows from $|\psi(0)|_k \neq |\psi(1)|_k$ that $\overline{\mathrm{Freq}}_k(w) \neq \underline{\mathrm{Freq}}_k(w)$.

ⓑ If j is even, by Lemma 3.1.12 we get

$$|\theta^{(j)-}|_k = \frac{2^j-1}{3}, \quad |\theta^{(j)}|_k = \frac{2^j+2}{3} \quad \text{and} \quad |\overline{\theta^{(j)}}|_k = |\theta^{(j)}|_{\overline{k}} = \frac{2^j-1}{3}.$$

Then

$$|\psi(0)|_k = 2|\theta^{(j)-}|_k = \frac{2^{j+1}-2}{3} \quad \text{and} \quad |\psi(1)|_k = |\theta^{(j)}|_k + |\overline{\theta^{(j)}}|_k = \frac{2^{j+1}+1}{3}.$$

Note that Lemma 3.1.17 implies

$$\overline{\mathrm{Freq}}_k(w) = \frac{|\psi(1)|_k}{2^{j+1}} \cdot \overline{\mathrm{Freq}}_1(v) + \frac{|\psi(0)|_k}{2^{j+1}} \cdot (1 - \overline{\mathrm{Freq}}_1(v))$$

and

$$\underline{\mathrm{Freq}}_k(w) = \frac{|\psi(1)|_k}{2^{j+1}} \cdot \underline{\mathrm{Freq}}_1(v) + \frac{|\psi(0)|_k}{2^{j+1}} \cdot (1 - \underline{\mathrm{Freq}}_1(v)).$$

By $v \in \Lambda_{0,1}^{\nexists}$ we get $\overline{\mathrm{Freq}}_1(v) \neq \underline{\mathrm{Freq}}_1(v)$. It follows from $|\psi(1)|_k \neq |\psi(0)|_k$ that $\overline{\mathrm{Freq}}_k(w) \neq \underline{\mathrm{Freq}}_k(w)$.

□

Proof of Theorem 3.1.2. Let $m \geq 2$ be an integer and $k \in \{\lceil \frac{m}{2} \rceil + 1, \cdots, m\}$. Since $\beta_{m;k} \in (1, m+1)$ is a zero of the polynomial $x^2 - (k+1)x + 2k - m - 1$, we get

$$\frac{k}{\beta_{m;k}} + \frac{\overline{k}+1}{\beta_{m;k}^2} + \frac{\overline{k}+1}{\beta_{m;k}^3} + \frac{\overline{k}+1}{\beta_{m;k}^4} + \cdots = 1.$$

It follows from Lemma 3.1.4 that $k(\overline{k}+1)^\infty$ is the unique $\beta_{m;k}$-expansion of 1.
(1) Let $\beta \in (G_m, \beta_{m;k}]$ and $w \in \Gamma_{m,\beta}$. In the same way as the proof of Theorem 3.1.1 (1), we get

$$\sigma^n w < k(\overline{k}+1)^\infty \quad \text{whenever } w_n < m \tag{3.15}$$

and

$$\sigma^n w > \overline{k}(k-1)^\infty \quad \text{whenever } w_n > 0. \tag{3.16}$$

① If $k = m$, then we have

$$\sigma^n w < m1^\infty \quad \text{whenever } w_n < m \tag{3.17}$$

and

$$\sigma^n w > 0(m-1)^\infty \quad \text{whenever } w_n > 0. \tag{3.18}$$

It suffices to prove $\overline{\mathrm{Freq}}_0(w) = \overline{\mathrm{Freq}}_m(w)$ and $\underline{\mathrm{Freq}}_0(w) = \underline{\mathrm{Freq}}_m(w)$.

i) If $w_1 = 0$, by $w \neq 0^\infty$, there exists $s \geq 2$ such that $w_1 \cdots w_{s-1} = 0^{s-1}$ and $w_s > 0$.
ⓐ When $w_s = m$ we have

$$w = 0^{s-1} m w_{s+1} w_{s+2} \cdots .$$

By (3.17) we get $m w_{s+1} w_{s+2} \cdots < m1^\infty$, which implies that there exists $i_1 \geq s+1$ such that $w_{s+1} \cdots w_{i_1-1} = 1^{i_1-s-1}$, $w_{i_1} = 0$ and

$$w = 0^{s-1} m 1^{i_1-s-1} 0 w_{i_1+1} w_{i_1+2} \cdots .$$

It follows from (3.18) that $0 w_{i_1+1} w_{i_1+2} \cdots > 0(m-1)^\infty$. Thus there exists $i_2 \geq i_1+1$ such that $w_{i_1+1} \cdots w_{i_2-1} = (m-1)^{i_2-i_1-1}$, $w_{i_2} = m$ and

$$w = 0^{s-1} m 1^{i_1-s-1} 0 (m-1)^{i_2-i_1-1} m w_{i_2+1} w_{i_2+2} \cdots .$$

$\cdots$ Repeating the above process, we get

$$w = 0^{s-1}m1^{j_1}0(m-1)^{j_2}m1^{j_3}0(m-1)^{j_4}\cdots$$

for some integers $j_1, j_2, j_3, j_4, \cdots \geq 0$. Therefore

$$\begin{aligned}
\left|\overline{\mathrm{Freq}}_0(w) - \overline{\mathrm{Freq}}_m(w)\right| &= \Big|\lim_{p\to\infty}\Big(\sup_{n\geq p}\frac{|w_1\cdots w_n|_0}{n}\Big) - \lim_{p\to\infty}\Big(\sup_{n\geq p}\frac{|w_1\cdots w_n|_m}{n}\Big)\Big| \\
&= \lim_{p\to\infty}\Big|\sup_{n\geq p}\frac{|w_1\cdots w_n|_0}{n} - \sup_{n\geq p}\frac{|w_1\cdots w_n|_m}{n}\Big| \\
&\leq \lim_{p\to\infty}\sup_{n\geq p}\frac{\big||w_1\cdots w_n|_0 - |w_1\cdots w_n|_m\big|}{n} \\
&\leq \lim_{p\to\infty}\frac{s-1}{p} = 0,
\end{aligned}$$

and $\left|\underline{\mathrm{Freq}}_0(w) - \underline{\mathrm{Freq}}_m(w)\right| = 0$ follows in a similar way noting that

$$\Big|\inf_{n\geq p}\frac{|w_1\cdots w_n|_0}{n} - \inf_{n\geq p}\frac{|w_1\cdots w_n|_m}{n}\Big| \leq \sup_{n\geq p}\frac{\big||w_1\cdots w_n|_0 - |w_1\cdots w_n|_m\big|}{n} \quad \text{for all } p\in\mathbb{N}.$$

ⓑ When $1 \leq w_s \leq m-1$, since $w_{s+1}, w_{s+2}, w_{s+3}, \cdots \notin \{0, m\}$ will imply $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w) = 0$ directly, we only need to consider that there exists $i_1 \geq s+1$ such that $w_{i_1} \in \{0, m\}$. Assume that such i_1 is the smallest one. Then $w_s, w_{s+1}, \cdots, w_{i_1-1} \notin \{0, m\}$.

I. If $w_{i_1} = 0$, then

$$w = 0^{s-1}w_s\cdots w_{i_1-1}0w_{i_1+1}w_{i_1+2}\cdots.$$

By (3.18) we get $0w_{i_1+1}w_{i_1+2}\cdots > 0(m-1)^\infty$, which implies that there exists $i_2 \geq i_1+1$ such that $w_{i_1+1}\cdots w_{i_2-1} = (m-1)^{i_2-i_1-1}$, $w_{i_2} = m$ and

$$w = 0^{s-1}w_s\cdots w_{i_1-1}0(m-1)^{i_2-i_1-1}mw_{i_2+1}w_{i_2+2}\cdots.$$

It follows from (3.17) that $mw_{i_2+1}w_{i_2+2}\cdots < m1^\infty$. Thus there exists $i_3 \geq i_2+1$ such that $w_{i_2+1}\cdots w_{i_3-1} = 1^{i_3-i_2-1}$, $w_{i_3} = 0$ and

$$w = 0^{s-1}w_s\cdots w_{i_1-1}0(m-1)^{i_2-i_1-1}m1^{i_3-i_2-1}0w_{i_3+1}w_{i_3+2}\cdots.$$

$\cdots$ Repeating the above process, we get

$$w = 0^{s-1}w_s\cdots w_{i_1-1}0(m-1)^{j_1}m1^{j_2}0(m-1)^{j_3}m1^{j_4}\cdots$$

for some integers $j_1, j_2, j_3, j_4, \cdots \geq 0$. In the same way as ⓐ, the conclusion

follows.

II. If $w_{i_1} = m$, in the same way as I, we get

$$w = 0^{s-1} w_s \cdots w_{i_1-1} m 1^{j_1} 0 (m-1)^{j_2} m 1^{j_3} 0 (m-1)^{j_4} \cdots$$

for some integers $j_1, j_2, j_3, j_4, \cdots \geq 0$, and then the conclusion follows.

ii) If $w_1 = m$, the conclusion follows in the same way as i).

iii) If $1 \leq w_1 \leq m-1$, since $w_2, w_3, w_4, \cdots \notin \{0, m\}$ will obviously imply $\mathrm{Freq}_0(w) = \mathrm{Freq}_m(w) = 0$, we only need to consider that there exists a smallest $i_1 \geq 2$ such that $w_{i_1} \in \{0, m\}$ but $w_1, w_2, \cdots, w_{i_1-1} \notin \{0, m\}$. In the same way as i) ⓑ I and II, we get

$$w = \begin{cases} w_1 \cdots w_{i_1-1} 0 (m-1)^{j_1} m 1^{j_2} 0 (m-1)^{j_3} m 1^{j_4} \cdots & \text{if } w_{i_1} = 0, \\ w_1 \cdots w_{i_1-1} m 1^{j_1} 0 (m-1)^{j_2} m 1^{j_3} 0 (m-1)^{j_4} \cdots & \text{if } w_{i_1} = m, \end{cases}$$

for some integers $j_1, j_2, j_3, j_4, \cdots \geq 0$, and then the conclusion follows.

② If $\lceil \frac{m}{2} \rceil + 1 \leq k \leq m-1$, we need to prove $\overline{\mathrm{Freq}}_k(w) = \overline{\mathrm{Freq}}_{\overline{k}}(w)$ and $\underline{\mathrm{Freq}}_k(w) = \underline{\mathrm{Freq}}_{\overline{k}}(w)$. Since $w_1, w_2, w_3, \cdots \notin \{\overline{k}, \overline{k}+1, \cdots, k-1, k\}$ will imply $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w) = 0$ directly, we only need to consider that there exists $t \in \mathbb{N}$ such that $0 < \overline{k} \leq w_t \leq k < m$. By (3.15) and (3.16), we get $0 < \overline{k} \leq w_{t+1} \leq k < m$. By (3.15) and (3.16) again, we get $0 < \overline{k} \leq w_{t+2} \leq k < m$. $\cdots$ By induction we get

$$0 < \overline{k} \leq w_n \leq k < m \quad \text{for all } n \geq t. \tag{3.19}$$

Since $w_{t+1}, w_{t+2}, w_{t+3}, \cdots \notin \{\overline{k}, k\}$ will obviously imply $\mathrm{Freq}_k(w) = \mathrm{Freq}_{\overline{k}}(w) = 0$, it suffices to consider that there exists $s \geq t+1$ such that $w_s \in \{\overline{k}, k\}$.

i) If $w_s = k$, by $w_{s-1} < m$ and (3.15) we get $k w_{s+1} w_{s+2} \cdots < k(\overline{k}+1)^\infty$. It follows from (3.19) that there exists $i_1 \geq s+1$ such that $w_{s+1} \cdots w_{i_1-1} = (\overline{k}+1)^{i_1-s-1}$, $w_{i_1} = \overline{k}$ and

$$w = w_1 \cdots w_{s-1} k (\overline{k}+1)^{i_1-s-1} \overline{k} w_{i_1+1} w_{i_1+2} \cdots .$$

By (3.16) we get $\overline{k} w_{i_1+1} w_{i_1+2} \cdots > \overline{k}(k-1)^\infty$. It follows again from (3.19) that there exists $i_2 \geq i_1+1$ such that $w_{i_1+1} \cdots w_{i_2-1} = (k-1)^{i_2-i_1-1}$, $w_{i_2} = k$ and

$$w = w_1 \cdots w_{s-1} k (\overline{k}+1)^{i_1-s-1} \overline{k} (k-1)^{i_2-i_1-1} k w_{i_2+1} w_{i_2+2} \cdots .$$

$\cdots$ Repeating the above process, we get

$$w = w_1 \cdots w_{s-1} k (\overline{k}+1)^{j_1} \overline{k} (k-1)^{j_2} k (\overline{k}+1)^{j_3} \overline{k} (k-1)^{j_4} \cdots$$

for some integers $j_1, j_2, j_3, j_4, \cdots \geq 0$. In the same way as ① i) ⓐ, the conclusion

follows.

ii) If $w_s = \overline{k}$, in the same way as above, we get

$$w = w_1 \cdots w_{s-1}\overline{k}(k-1)^{j_1}k(\overline{k}+1)^{j_2}\overline{k}(k-1)^{j_3}k(\overline{k}+1)^{j_4}\cdots$$

for some integers $j_1, j_2, j_3, j_4, \cdots \geq 0$ and then the conclusion follows.

(2) Let $\beta \in (\beta_{m;k}, m+1]$. Since $k(\overline{k}+1)^\infty$ is the unique $\beta_{m;k}$-expansion of 1, by Lemma 3.1.6 we get $g^*(1,\beta) > k(\overline{k}+1)^\infty$ and then $g_1^*(1,\beta) \geq k$.
① If $g_1^*(1,\beta) \geq k+1$, by Lemma 3.1.5 we get $\{k,\overline{k}\}^{\mathbb{N}} \subset \Gamma_{m,\beta}$. Let $c = 1$. For any $r \in (-c, c)$, we define

$$\Lambda_{k,\overline{k}}^r := \Big\{w \in \{k,\overline{k}\}^{\mathbb{N}} : \mathrm{Freq}_k(w) - \mathrm{Freq}_{\overline{k}}(w) = r\Big\}.$$

Then

$$\Lambda_{k,\overline{k}}^r \subset \Big\{w \in \Gamma_{m,\beta} : \mathrm{Freq}_k(w) - \mathrm{Freq}_{\overline{k}}(w) = r\Big\}.$$

It suffices to prove $\dim_H(\Lambda_{k,\overline{k}}^r, d_{m+1}) > 0$. In fact, this follows from

$$\dim_H(\Lambda_{k,\overline{k}}^r, d_{m+1}) \overset{(\star)}{\geq} \frac{\log 2}{\log(m+1)} \cdot \dim_H(\Lambda_{k,\overline{k}}^r, d_2) \overset{(\star\star)}{>} 0,$$

where $(\star)$ follows from Proposition 3.1.9 and the fact that the identity map from $(\Lambda_{k,\overline{k}}^r, d_{m+1})$ to $(\Lambda_{k,\overline{k}}^r, d_2)$ is $\frac{\log 2}{\log(m+1)}$-Hölder continuous, and $(\star\star)$ follows from combining

$$\Lambda_{k,\overline{k}}^r = \Big\{w \in \{k,\overline{k}\}^{\mathbb{N}} : \mathrm{Freq}_k(w) = \frac{1+r}{2}\Big\},$$

Proposition 3.1.10 and $0 < \frac{1+r}{2} < 1$.
② If $g_1^*(1,\beta) = k$, by $g^*(1,\beta) > k(\overline{k}+1)^\infty$, there exists $s \in \mathbb{N}$ such that

$$g_1^*(1,\beta)g_2^*(1,\beta)\cdots g_s^*(1,\beta) = k(\overline{k}+1)^{s-1} \quad \text{and} \quad g_{s+1}^*(1,\beta) > \overline{k}+1.$$

Let

$$\begin{aligned}\Xi_{k,\overline{k}} &:= \Big\{k\overline{k}k(\overline{k}+1)^s, \overline{k}k\overline{k}(k-1)^s\Big\}^{\mathbb{N}} \\ &= \Big\{w \in \{0,\cdots,m\}^{\mathbb{N}} : w_{n(s+3)+1}\cdots w_{(n+1)(s+3)} = k\overline{k}k(\overline{k}+1)^s \text{ or } \overline{k}k\overline{k}(k-1)^s \text{ for all } n \geq 0\Big\}.\end{aligned}$$

Then by Lemma 3.1.5 we get

$$\Xi_{k,\overline{k}} \subset \Gamma_{m,\beta}.$$

For any $v \in \{k,\overline{k}\}^{\mathbb{N}}$, define $\Psi(v) := \psi(v_1)\psi(v_2)\cdots$ where $\psi(k) := k\overline{k}k(\overline{k}+1)^s$ and $\psi(\overline{k}) :=$

$\overline{k}k\overline{k}(k-1)^s$. Let $c = \frac{1}{s+3}$. For any $r \in (-c, c)$, we define

$$\Xi^r_{k,\overline{k}} := \Big\{ w \in \Xi_{k,\overline{k}} : \mathrm{Freq}_k(w) - \mathrm{Freq}_{\overline{k}}(w) = r \Big\}.$$

Then

$$\Xi^r_{k,\overline{k}} \subset \Big\{ w \in \Gamma_{m,\beta} : \mathrm{Freq}_k(w) - \mathrm{Freq}_{\overline{k}}(w) = r \Big\}.$$

It suffices to prove $\dim_H \big(\Xi^r_{k,\overline{k}}, d_{m+1}\big) > 0$. Let

$$\Lambda^{(s+3)r}_{k,\overline{k}} := \Big\{ v \in \{k, \overline{k}\}^{\mathbb{N}} : \mathrm{Freq}_k(v) - \mathrm{Freq}_{\overline{k}}(v) = (s+3)r \Big\}.$$

Since $\Psi : \{k, \overline{k}\}^{\mathbb{N}} \to \Xi_{k,\overline{k}}$ is bijective and $\Psi^{-1} : \big(\Xi_{k,\overline{k}}, d_{m+1}\big) \to \big(\{k, \overline{k}\}^{\mathbb{N}}, d_2\big)$ is $\frac{\log 2}{(s+3)\log(m+1)}$-Hölder continuous, by Proposition 3.1.9 we get

$$\begin{aligned}\dim_H(\Xi^r_{k,\overline{k}}, d_{m+1}) &\geq \frac{\log 2}{(s+3)\log(m+1)} \cdot \dim_H(\Psi^{-1}(\Xi^r_{k,\overline{k}}), d_2) \\ &\geq \frac{\log 2}{(s+3)\log(m+1)} \cdot \dim_H(\Lambda^{(s+3)r}_{k,\overline{k}}, d_2)\end{aligned}$$

where the last inequality follows from $\Psi^{-1}(\Xi^r_{k,\overline{k}}) \supset \Lambda^{(s+3)r}_{k,\overline{k}}$, which can be directly proved by Lemma 3.1.17. It follows from

$$\Lambda^{(s+3)r}_{k,\overline{k}} = \Big\{ v \in \{k, \overline{k}\}^{\mathbb{N}} : \mathrm{Freq}_k(v) = \frac{1 + (s+3)r}{2} \Big\},$$

Proposition 3.1.10 and $0 < \frac{1+(s+3)r}{2} < 1$ that $\dim_H(\Lambda^{(s+3)r}_{k,\overline{k}}, d_2) > 0$. Thus $\dim_H(\Xi^r_{k,\overline{k}}, d_{m+1}) > 0$. □

To end this section, we prove the following lemma, which has already been used in the proofs of Theorem 3.1.1 (2) and Theorem 3.1.2 (2).

Lemma 3.1.17. *Let a, b be two digits, $s \in \mathbb{N}$ and $v \in \{a, b\}^{\mathbb{N}}$. Define $\Psi(v) := \psi(v_1)\psi(v_2)\cdots$ where $\psi(a)$ and $\psi(b)$ are two finite words satisfying $|\psi(a)| = |\psi(b)| = s$. If $|\psi(a)|_\xi \geq |\psi(b)|_\xi$ for some digit ξ, then*

$$\overline{\mathrm{Freq}}_\xi(\Psi(v)) = \frac{|\psi(a)|_\xi}{s} \cdot \overline{\mathrm{Freq}}_a(v) + \frac{|\psi(b)|_\xi}{s} \cdot (1 - \overline{\mathrm{Freq}}_a(v)) \tag{3.20}$$

and

$$\underline{\mathrm{Freq}}_\xi(\Psi(v)) = \frac{|\psi(a)|_\xi}{s} \cdot \underline{\mathrm{Freq}}_a(v) + \frac{|\psi(b)|_\xi}{s} \cdot (1 - \underline{\mathrm{Freq}}_a(v)). \tag{3.21}$$

Proof. Let $v \in \{a, b\}^{\mathbb{N}}$ and $w := \Psi(v)$. Since the proofs of (3.20) and (3.21) are similar,

we only prove (3.20) in the following. Let $p := \overline{\mathrm{Freq}}_a(v)$. It suffices to prove

$$\lim_{j\to\infty}\sup_{i\geq j}\frac{|w_1\cdots w_i|_\xi}{i} = \frac{|\psi(a)|_\xi}{s}\cdot p + \frac{|\psi(b)|_\xi}{s}\cdot(1-p).$$

Fix any $\varepsilon > 0$. By

$$\lim_{n\to\infty}\sup_{t\geq n}\frac{|v_1\cdots v_t|_a}{t} = p,$$

there exists integer $N > \frac{1}{\varepsilon}$ such that for all $n \geq N$ we have

$$\sup_{t\geq n}\frac{|v_1\cdots v_t|_a}{t} < p+\varepsilon \tag{3.22}$$

and

$$\sup_{t\geq n}\frac{|v_1\cdots v_t|_a}{t} > p-\varepsilon. \tag{3.23}$$

Let $j > Ns$ ($> \frac{s}{\varepsilon}$) be an integer. It suffices to prove

$$\Big|\sup_{i\geq j}\frac{|w_1\cdots w_i|_\xi}{i} - \Big(\frac{|\psi(a)|_\xi}{s}\cdot p + \frac{|\psi(b)|_\xi}{s}\cdot(1-p)\Big)\Big| < 2\varepsilon.$$

Recall that for any $x\in\mathbb{R}$, $\lceil x\rceil$ and $\lfloor x\rfloor$ denote the smallest integer no less than x and the greatest integer no larger than x respectively. On the one hand we have

$$\begin{aligned}
\sup_{i\geq j}\frac{|w_1\cdots w_i|_\xi}{i} &\leq \sup_{i\geq j}\frac{|w_1\cdots w_{\lfloor\frac{i}{s}\rfloor\cdot s}|_\xi + s}{i}\\
&\leq \sup_{i\geq j}\frac{|\Psi(v_1\cdots v_{\lfloor\frac{i}{s}\rfloor})|_\xi}{i} + \frac{s}{j}\\
&< \sup_{i\geq j}\frac{|v_1\cdots v_{\lfloor\frac{i}{s}\rfloor}|_a\cdot|\psi(a)|_\xi + |v_1\cdots v_{\lfloor\frac{i}{s}\rfloor}|_b\cdot|\psi(b)|_\xi}{i} + \varepsilon\\
&= \sup_{i\geq j}\frac{|v_1\cdots v_{\lfloor\frac{i}{s}\rfloor}|_a\cdot|\psi(a)|_\xi + (\lfloor\frac{i}{s}\rfloor - |v_1\cdots v_{\lfloor\frac{i}{s}\rfloor}|_a)\cdot|\psi(b)|_\xi}{i} + \varepsilon\\
&\leq \sup_{i\geq j}\frac{|v_1\cdots v_{\lfloor\frac{i}{s}\rfloor}|_a\cdot(|\psi(a)|_\xi - |\psi(b)|_\xi) + \lfloor\frac{i}{s}\rfloor\cdot|\psi(b)|_\xi}{\lfloor\frac{i}{s}\rfloor\cdot s} + \varepsilon\\
&= \frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}\cdot\sup_{i\geq j}\frac{|v_1\cdots v_{\lfloor\frac{i}{s}\rfloor}|_a}{\lfloor\frac{i}{s}\rfloor} + \frac{|\psi(b)|_\xi}{s} + \varepsilon\\
&\overset{(\star)}{\leq} \frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}\cdot\sup_{t\geq\lfloor\frac{j}{s}\rfloor}\frac{|v_1\cdots v_t|_a}{t} + \frac{|\psi(b)|_\xi}{s} + \varepsilon\\
&\overset{\text{by (3.22)}}{<} \frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}\cdot(p+\varepsilon) + \frac{|\psi(b)|_\xi}{s} + \varepsilon
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|\psi(a)|_\xi}{s}\cdot p + \frac{|\psi(b)|_\xi}{s}\cdot(1-p) + \Big(\frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}+1\Big)\varepsilon \\
&\le \frac{|\psi(a)|_\xi}{s}\cdot p + \frac{|\psi(b)|_\xi}{s}\cdot(1-p) + 2\varepsilon,
\end{aligned}
$$

where $(\star)$ follows from the fact that $i \ge j$ implies $\lfloor \frac{i}{s} \rfloor \ge \lfloor \frac{j}{s} \rfloor$. On the other hand we have

$$
\begin{aligned}
\sup_{i\ge j}\frac{|w_1\cdots w_i|_\xi}{i} &\ge \sup_{i\ge j}\frac{|w_1\cdots w_{\lceil \frac{i}{s}\rceil\cdot s}|_\xi - s}{i} \\
&\ge \sup_{i\ge j}\frac{|\Psi(v_1\cdots v_{\lceil \frac{i}{s}\rceil})|_\xi}{i} - \frac{s}{j} \\
&> \sup_{i\ge j}\frac{|v_1\cdots v_{\lceil \frac{i}{s}\rceil}|_a\cdot|\psi(a)|_\xi + |v_1\cdots v_{\lceil \frac{i}{s}\rceil}|_b\cdot|\psi(b)|_\xi}{i} - \varepsilon \\
&= \sup_{i\ge j}\frac{|v_1\cdots v_{\lceil \frac{i}{s}\rceil}|_a\cdot|\psi(a)|_\xi + (\lceil \frac{i}{s}\rceil - |v_1\cdots v_{\lceil \frac{i}{s}\rceil}|_a)\cdot|\psi(b)|_\xi}{i} - \varepsilon \\
&\ge \sup_{i\ge j}\frac{|v_1\cdots v_{\lceil \frac{i}{s}\rceil}|_a\cdot(|\psi(a)|_\xi - |\psi(b)|_\xi) + \lceil \frac{i}{s}\rceil\cdot|\psi(b)|_\xi}{\lceil \frac{i}{s}\rceil\cdot s} - \varepsilon \\
&= \frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}\cdot\sup_{i\ge j}\frac{|v_1\cdots v_{\lceil \frac{i}{s}\rceil}|_a}{\lceil \frac{i}{s}\rceil} + \frac{|\psi(b)|_\xi}{s} - \varepsilon \\
&\overset{(\star\star)}{\ge} \frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}\cdot\sup_{t\ge\lceil \frac{j}{s}\rceil}\frac{|v_1\cdots v_t|_a}{t} + \frac{|\psi(b)|_\xi}{s} - \varepsilon \\
&\overset{\text{by (3.23)}}{>} \frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}\cdot(p-\varepsilon) + \frac{|\psi(b)|_\xi}{s} - \varepsilon \\
&= \frac{|\psi(a)|_\xi}{s}\cdot p + \frac{|\psi(b)|_\xi}{s}\cdot(1-p) - \Big(\frac{|\psi(a)|_\xi - |\psi(b)|_\xi}{s}+1\Big)\varepsilon \\
&\ge \frac{|\psi(a)|_\xi}{s}\cdot p + \frac{|\psi(b)|_\xi}{s}\cdot(1-p) - 2\varepsilon,
\end{aligned}
$$

where $(\star\star)$ follows from the fact that $t \ge \lceil \frac{j}{s}\rceil$ implies $ts \ge j$. □

3.2 Expansions of generalized Thue-Morse numbers

Base on the generalized shifted Thue-Morse sequences defined in the last section, we generalize this concept a bit more first. For any $m, q \in \mathbb{N}$ and $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \ne 0$, we define a sequence of finite words $\{\theta^{(n)}_{m;\theta_1,\cdots,\theta_q}\}_{n\ge 0}$ by induction as follows:

$$
\theta^{(0)}_{m;\theta_1,\cdots,\theta_q} := \theta_1\cdots\theta_q \quad\text{and}\quad \theta^{(n+1)}_{m;\theta_1,\cdots,\theta_q} := \theta^{(n)}_{m;\theta_1,\cdots,\theta_q}\overline{\theta^{(n)}_{m;\theta_1,\cdots,\theta_q}}^{+} \text{ for all } n \ge 0,
$$

where $w^+ := w_1\cdots w_{i-1}(w_i+1)$ and $\overline{w} := \overline{w_1}\cdots\overline{w_i}$ for any finite word $w = w_1\cdots w_i$ and $\overline{k} := m-k$ for any $k \in \{0, 1, \cdots, m\}$. When $m, \theta_1, \cdots, \theta_q$ are understood from the context,

we use $\theta^{(n)}$ instead of $\theta^{(n)}_{m;\theta_1,\cdots,\theta_q}$ for simplification. We call the infinite sequence

$$\theta = (\theta_i)_{i\geq 1} := \lim_{n\to\infty} \theta^{(n)} = \theta_1\cdots\theta_q\ \overline{\theta_1\cdots\theta_q}^{+}\ \overline{\theta_1\cdots\theta_q}\ \theta_1\cdots\theta_q\ \cdots$$

the $(m;\theta_1,\cdots,\theta_q)$-*shifted-Thue-Morse sequence*, and call the unique $\beta_\theta \in (1, m+1)$ such that

$$\sum_{i=1}^{\infty} \frac{\theta_i}{\beta_\theta^i} = 1$$

the $(m;\theta_1,\cdots,\theta_q)$-*Komornik-Loreti constant*. An equivalent definition of the $(m;\theta_1,\cdots,\theta_q)$-shifted-Thue-Morse sequence $\theta = (\theta_i)_{i\geq 1}$ is that for all integers $l \geq 0$,

$$\theta_{2^l q+j} := \overline{\theta}_j \quad \text{for } j \in \{1,\cdots,2^l q-1\},$$

$$\theta_{2^{2l}q} := \theta_q \quad \text{and} \quad \theta_{2^{2l+1}q} := \overline{\theta}_q^{+}.$$

It is worth to note that these generalized shifted Thue-Morse sequences were previously studied in [3, 53, 87, 88] in different terms. The classical shifted Thue-Morse sequence $(t_n)_{n\geq 1}$ given in (3.1) is not only the $(1;1)$ but also the $(1;1,1)$-shifted-Thue-Morse sequence in our terms.

For any $(m;\theta_1,\cdots,\theta_q)$-shifted-Thue-Morse sequence $\theta = (\theta_i)_{i\geq 1}$ and $\beta \in (1,\infty)$, we call

$$\pi_\beta(\theta) := \sum_{i=1}^{\infty} \frac{\theta_i}{\beta^i}$$

the β-$(m;\theta_1,\cdots,\theta_q)$-*Thue-Morse number*. The classical Thue-Morse number $\sum_{n=1}^{\infty} \frac{t_n}{2^n}$ is exactly the 2-$(1;1)$-Thue-Morse number (also the 2-$(1;1,1)$-Thue-Morse number), and more generally the Thue-Morse(-Mahler) number $\sum_{n=1}^{\infty} \frac{t_n}{b^n}$ for integer $b \geq 2$ is exactly the b-$(1;1)$-Thue-Morse number (also the b-$(1;1,1)$-Thue-Morse number) in our terms. These numbers are transcendental [16, 47, 95] and received a lot of attention recently [1, 20, 21, 34, 36].

Recall that for $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $x \in \mathbb{R}$, a sequence $w = (w_n)_{n\geq 1} \in \{0,1,\cdots,m\}^{\mathbb{N}}$ is called a β-*expansion* of x if

$$x = \sum_{n=1}^{\infty} \frac{w_n}{\beta^n}.$$

An $(m;\theta_1,\cdots,\theta_q)$-shifted-Thue-Morse sequence θ is naturally a β-expansion of the β-$(m;\theta_1,\cdots,\theta_q)$-Thue-Morse number $\pi_\beta(\theta)$. Our goal in this section is to study when will this expansion be unique, greedy, lazy, quasi-greedy and quasi-lazy.

Recall that σ is the shift map on $\{0,\cdots,m\}^{\mathbb{N}}$, and $<, \leq, >, \geq$ denote the lexicographic order between infinite sequences and also between finite words with the same length. First we have the following purely combinatorial proposition, which generalizes [87, Theorem

4.4] since one can directly verify that a block $s_1 \cdots s_q \in \{0, \cdots, m\}^q$ with $s_q \neq m$ is called admissible in [87] if and only if $s_1 \cdots s_q^+$ satisfies (1) in the following proposition.

Proposition 3.2.1. *Let $m \geq 1$, $q \geq 2$ be integers, $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$ and θ be the $(m; \theta_1, \cdots, \theta_q)$-shifted-Thue-Morse sequence. The following are all equivalent.*

(1) *For all $n \in \{1, \cdots, q-1\}$ we have $\overline{\theta_1 \cdots \theta_{q-n}} < \theta_{n+1} \cdots \theta_q \leq \theta_1 \cdots \theta_{q-n}$.*

(2) *For all $n \geq 1$ we have $\overline{\theta} < \sigma^n \theta < \theta$.*

(3) *For all $n \geq 1$ we have $\sigma^n \theta < \theta$.*

(4) *For all $n \geq 1$ we have $\sigma^n \theta > \overline{\theta}$.*

(5) *Whenever $\theta_n < m$ we have $\sigma^n \theta < \theta$.*

(6) *Whenever $\theta_n > 0$ we have $\sigma^n \theta > \overline{\theta}$.*

The following is our main result.

Theorem 3.2.2. *Let $m \geq 1, q \geq 2$ be integers, $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$, θ be the $(m; \theta_1, \cdots, \theta_q)$-shifted-Thue-Morse sequence and β_θ be the $(m; \theta_1, \cdots, \theta_q)$-Komornik-Loreti constant.*
(1) *Let $\beta \in (1, m+1]$. If θ is the greedy, lazy, quasi-greedy, quasi-lazy or unique β-expansion of $\pi_\beta(\theta)$, then $\beta \geq \beta_\theta$.*
(2) *The following are all equivalent.*

① *For all $n \in \{1, \cdots, q-1\}$ we have $\overline{\theta_1 \cdots \theta_{q-n}} < \theta_{n+1} \cdots \theta_q \leq \theta_1 \cdots \theta_{q-n}$.*

② *θ is the unique β_θ-expansion of 1.*

③ *θ is the greedy β_θ-expansion of 1.*

④ *θ is the lazy β_θ-expansion of 1.*

⑤ *θ is the quasi-greedy β_θ-expansion of 1.*

⑥ *θ is the quasi-lazy β_θ-expansion of 1.*

⑦ *$\{\beta \in (1, m+1] : \theta$ is the unique β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.*

⑧ *$\{\beta \in (1, m+1] : \theta$ is the greedy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.*

⑨ *$\{\beta \in (1, m+1] : \theta$ is the lazy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.*

⑩ *$\{\beta \in (1, m+1] : \theta$ is the quasi-greedy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.*

⑪ *$\{\beta \in (1, m+1] : \theta$ is the quasi-lazy β-expansion of $\pi_\beta(\theta)\} = [\beta_\theta, m+1]$.*

Recall that $(t_n)_{n\geq 0}$ is the classical Thue-Morse sequence, $\pi_\beta((t_n)_{n\geq 1}) = \sum_{n=1}^{\infty} \frac{t_n}{\beta^n}$ is the β-$(1;1,1)$-Thue-Morse number and the classical Komornik-Loreti constant is the $(1;1,1)$-Komornik-Loreti constant. Since the $(1;1,1)$-shifted-Thue-Morse sequence (i.e., the classical shifted Thue-Morse sequence $(t_n)_{n\geq 1}$) satisfies Theorem 3.2.2 (2) ①, by ⑦, ⑧, ⑨, ⑩ and ⑪, we get the following.

Corollary 3.2.3. *Let $\beta \in (1,2]$ and consider the alphabet $\{0,1\}$. The following are all equivalent.*

(1) *$(t_n)_{n\geq 1}$ is the unique β-expansion of $\pi_\beta((t_n)_{n\geq 1})$.*

(2) *$(t_n)_{n\geq 1}$ is the greedy β-expansion of $\pi_\beta((t_n)_{n\geq 1})$.*

(3) *$(t_n)_{n\geq 1}$ is the lazy β-expansion of $\pi_\beta((t_n)_{n\geq 1})$.*

(4) *$(t_n)_{n\geq 1}$ is the quasi-greedy β-expansion of $\pi_\beta((t_n)_{n\geq 1})$.*

(5) *$(t_n)_{n\geq 1}$ is the quasi-lazy β-expansion of $\pi_\beta((t_n)_{n\geq 1})$.*

(6) *β is no less than the classical Komornik-Loreti constant.*

We recall some notation and preliminaries in Subsection 3.2.1, and then prove Proposition 3.2.1 and Theorem 3.2.2 in the last subsection.

3.2.1 Notation and preliminaries

Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $x \in [0, \frac{m}{\beta-1}]$. Recall the definitions of greedy, lazy, quasi-greedy and quasi-lazy β-expansions of x by taking $\beta_0, \cdots, \beta_m$ to be the same β in Definition 2.1.7 in Section 2.1. Note that Proposition 2.1.12 gives equivalent definitions: among all $w \in \{0, \cdots, m\}^{\mathbb{N}}$ satisfying $\pi_\beta(w) = x$, the lexicographically largest and smallest ones are called the *greedy* and *lazy* β-expansions of x respectively; among all $w \in \{0, \cdots, m\}^{\mathbb{N}}$ not end with 0^∞ and satisfying $\pi_\beta(w) = x$, the lexicographically largest one is called the *quasi-greedy* β-expansion of x; among all $w \in \{0, \cdots, m\}^{\mathbb{N}}$ not end with m^∞ and satisfying $\pi_\beta(w) = x$, the lexicographically smallest one is called the *quasi-lazy* β-expansion of x.

Recall that given $m \in \mathbb{N}$, for any digit $k \in \{0, \cdots, m\}$, $\overline{k}$ denotes $m-k$. The following criterion for greedy, quasi-greedy, lazy and quasi-lazy expansions, which is a direct consequence of Proposition 2.1.11 (see also [61, Lemma 1]), plays an important role in the proof of Theorem 3.2.2.

Proposition 3.2.4. *Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $w = (w_i)_{i\geq 1} \in \{0, \cdots, m\}^{\mathbb{N}}$.*

(1) *w is the greedy β-expansion of $\pi_\beta(w)$ if and only if*

$$\sum_{i=1}^{\infty} \frac{w_{n+i}}{\beta^i} < 1 \quad \textit{whenever } w_n < m.$$

(2) *When $w \neq 0^\infty$, it is the quasi-greedy β-expansion of $\pi_\beta(w)$ if and only if it does not end with 0^∞ and*

$$\sum_{i=1}^{\infty} \frac{w_{n+i}}{\beta^i} \leq 1 \quad \textit{whenever } w_n < m.$$

(3) *w is the lazy β-expansion of $\pi_\beta(w)$ if and only if*

$$\sum_{i=1}^{\infty} \frac{\overline{w}_{n+i}}{\beta^i} < 1 \quad \textit{whenever } w_n > 0.$$

(4) *When $w \neq m^\infty$, it is the quasi-lazy β-expansion of $\pi_\beta(w)$ if and only if it does not end with m^∞ and*

$$\sum_{i=1}^{\infty} \frac{\overline{w}_{n+i}}{\beta^i} \leq 1 \quad \textit{whenever } w_n > 0.$$

Almost immediately we get the following.

Proposition 3.2.5. *Let $m \in \mathbb{N}$, $\beta_0 \in (1, m+1]$ and $w \in \{0, \cdots, m\}^{\mathbb{N}}$. Then*

$$w \textit{ is the greedy/quasi-greedy/lazy/quasi-lazy/unique } \beta_0\textit{-expansion of } \pi_{\beta_0}(w)$$

if and only if for all $\beta \in [\beta_0, m+1]$,

$$w \textit{ is the greedy/quasi-greedy/lazy/quasi-lazy/unique } \beta\textit{-expansion of } \pi_\beta(w).$$

Proof. $\boxed{\Leftarrow}$ Obvious.
$\boxed{\Rightarrow}$ We only prove the greedy case since the others are similar. Suppose that w is the greedy β_0-expansion of $\pi_{\beta_0}(w)$. Let $\beta \in [\beta_0, m+1]$. Suppose $w_n < m$ for some $n \in \mathbb{N}$. By Proposition 3.2.4 (1), it suffices to check $\sum_{i=1}^{\infty} \frac{w_{n+i}}{\beta^i} < 1$. In fact, since w is the greedy β_0-expansion of $\pi_{\beta_0}(w)$, by Proposition 3.2.4 (1) we get $\sum_{i=1}^{\infty} \frac{w_{n+i}}{\beta_0^i} < 1$. It follows from $\beta \geq \beta_0$ that $\sum_{i=1}^{\infty} \frac{w_{n+i}}{\beta^i} < 1$. □

For the sake of completeness we prove the following basic combinatorial fact, in which (1) and (3) are mentioned in [55, Proposition 2.2 and Theorem 2.5] and [61, Remark 1].

Proposition 3.2.6. *Let $m \in \mathbb{N}$ and $w = (w_i)_{i \geq 1} \in \{0, \cdots, m\}^{\mathbb{N}}$.*

(1) *We have*

$$\sigma^n w < w \quad \textit{for all } n \geq 1$$

if and only if

$$w \neq m^\infty \quad \textit{and} \quad \sigma^n w < w \quad \textit{whenever } w_n < m.$$

(2) *We have*

$$w = 0^k w_{k+1} m^\infty \quad \textit{for some } k \geq 1 \quad \textit{or} \quad \sigma^n w > \overline{w} \quad \textit{for all } n \geq 1$$

if and only if

$$w \neq 0^\infty \quad \textit{and} \quad \sigma^n w > \overline{w} \quad \textit{whenever } w_n > 0.$$

(3) *We have*

$$\overline{w} < \sigma^n w < w \quad \textit{for all } n \geq 1$$

if and only if

$$w \neq m^\infty, \quad \sigma^n w < w \quad \textit{whenever } w_n < m \quad \textit{and} \quad \sigma^n w > \overline{w} \quad \textit{whenever } w_n > 0.$$

Proof. (1) $\boxed{\Rightarrow}$ is obvious.
$\boxed{\Leftarrow}$ It suffices to consider $w_n = m$ for some $n \geq 1$ and prove $\sigma^n w < w$.

① If $w_1 \cdots w_n = m^n$, we need to prove $w_{n+1} w_{n+2} \cdots < m^n w_{n+1} w_{n+2} \cdots$. This follows immediately from $w \neq m^\infty$.

② If $w_1 \cdots w_n \neq m^n$, recalling $w_n = m$, there exists a largest $k \in \{1, \cdots, n-1\}$ such that $w_k \neq m$ but $w_{k+1} = \cdots = w_n = m$. Thus

$$\sigma^n w = w_{n+1} w_{n+2} \cdots \leq m^{n-k} w_{n+1} w_{n+2} \cdots = w_{k+1} w_{k+2} \cdots < w,$$

where the last inequality follows from the condition $\sigma^k w < w$ when $w_k < m$.

(2) $\boxed{\Rightarrow}$ We have the following two cases.

① If $w = 0^k w_{k+1} m^\infty$ for some $k \geq 1$, then $w \neq 0^\infty$, and for all $n \geq 1$ with $w_n > 0$ we have $\sigma^n w = m^\infty > \overline{w}$.

② If $\sigma^n w > \overline{w}$ for all $n \geq 1$, we obviously have $w \neq 0^\infty$ and $\sigma^n w > \overline{w}$ whenever $w_n > 0$.

$\boxed{\Leftarrow}$ Suppose $w \neq 0^\infty$ and

$$\sigma^n w > \overline{w} \quad \text{whenever } w_n > 0 \tag{3.24}$$

① If $w_1 = 0$, by $w \neq 0^\infty$, there exists $k \in \mathbb{N}$ such that $w_1 \cdots w_k = 0^k$ and $w_{k+1} > 0$. By (3.24) we get $\sigma^{k+1} w > \overline{w}$, which implies $w_{k+2} \geq \overline{w}_1 = m$. By (3.24) again we get $\sigma^{k+2} w > \overline{w}$, which implies $w_{k+3} \geq \overline{w}_1 = m$. $\cdots$ Finally we get $w_{k+2} w_{k+3} \cdots = m^\infty$ and $w = 0^k w_{k+1} m^\infty$.

② If $w_1 \neq 0$, it suffices to consider $w_n = 0$ for some $n \geq 1$ and prove $\sigma^n w > \overline{w}$. By $w_1 \neq 0$ and $w_n = 0$, there exists a largest $k \in \{1, \cdots, n-1\}$ such that $w_k \neq 0$ but

$w_{k+1} = \cdots = w_n = 0$. It follows that

$$\sigma^n w = w_{n+1}w_{n+2}\cdots \geq 0^{n-k}w_{n+1}w_{n+2}\cdots = w_{k+1}w_{k+2}\cdots \overset{\text{by (3.24)}}{>} \overline{w}.$$

(3) $\boxed{\Rightarrow}$ is obvious.
$\boxed{\Leftarrow}$ follows from (1) and (2), noting that $\sigma^n w < w$ whenever $w_n < m$ implies $w \neq 0^\infty$ and $w \neq 0^k w_{k+1} m^\infty$ for any $k \geq 1$. □

Besides, we need the following lemmas.

Lemma 3.2.7. *([85, Theorem 2.1]) Let $m \in \mathbb{N}$, $\beta \in (1, m+1]$ and $\varepsilon \in \{0, \cdots, m\}^{\mathbb{N}}$ be a β-expansion of 1. Then ε is the greedy expansion if and only if*

$$\sigma^n \varepsilon < \varepsilon \quad \textit{whenever } \varepsilon_n < m.$$

The next lemma follows from [3, Page 72, Theorem a)] with different notation.

Lemma 3.2.8. *For $s \in \mathbb{N}$ and alphabet $\mathcal{A} = \{a_0, a_1, \cdots, a_s\}$ where digits $a_0 < a_1 < \cdots < a_s$, let*

$$\Gamma(s, \mathcal{A}) = \{w \in \mathcal{A}^{\mathbb{N}} : w_1 = a_s \textit{ and for all } n \geq 0, \overline{w} \leq \sigma^n w \leq w\}.$$

Then for any integer $q \geq 2$ and q-mirror sequence $u = (u_n)_{n\geq 1}$ on the alphabet $\mathcal{A}$ with $u_1 = a_s$ and $u_q = a_i$ $(i \neq 0)$, we have

$$u \in \Gamma(s, \mathcal{A}) \quad \textit{if and only if} \quad (u_1 \cdots u_{q-1} a_{i-1})^\infty \in \Gamma(s, \mathcal{A}) \textit{ with smallest period } q.$$

3.2.2 Proofs of Proposition 3.2.1 and Theorem 3.2.2

Let $m \geq 1$, $q \geq 2$ be integers, $\theta_1, \cdots, \theta_q \in \{0, \cdots, m\}$ with $\theta_q \neq 0$ and θ be the $(m; \theta_1, \cdots, \theta_q)$-shifted-Thue-Morse sequence. By [3, Page 70, 3)] we know that θ is not eventually periodic. In particular θ does not end with 0^∞ or m^∞.

Proof of Proposition 3.2.1. (3)⇔(5) follows from Proposition 3.2.6 (1) noting that $\theta \neq m^\infty$.
(4)⇔(6) follows from Proposition 3.2.6 (2) noting that $\theta \neq 0^\infty$ and θ does not end with m^∞.
(2)⇔"(3) and (4)" is obvious.

In the following we only need to prove (3)⇔(4) and (1)⇔(2).
(3)⇒(4) Let $n \geq 1$. We need to prove $\overline{\sigma^n \theta} < \theta$ in the following.

① If $n \leq q-1$, since $\overline{\sigma^n \theta}$ begins with

$$\overline{\theta_{n+1} \cdots \theta_q} < \overline{\theta_{n+1} \cdots \theta_q}^+ = \theta_{q+n+1} \cdots \theta_{2q},$$

we get

$$\overline{\sigma^n\theta} < \sigma^{q+n}\theta \overset{\text{by (3)}}{<} \theta.$$

② If $n \geq q$, then there exists integer $k \geq 0$ and $j \in \{0, 1, \cdots, 2^k q - 1\}$ such that $n = 2^k q + j$. Since $\overline{\sigma^n\theta} = \overline{\sigma^{2^k q+j}\theta}$ begins with

$$\overline{\theta_{2^k q+j+1} \cdots \theta_{2^k q+2^k q}} = \theta_{j+1} \cdots \theta^-_{2^k q} < \theta_{j+1} \cdots \theta_{2^k q},$$

we get

$$\overline{\sigma^n\theta} < \sigma^j\theta \overset{\text{by (3)}}{\leq} \theta.$$

(4)⇒(3) Suppose

$$\overline{\sigma^i\theta} < \theta \quad \text{for all } i \geq 1. \tag{3.25}$$

Let $n \geq 1$. We need to prove $\sigma^n\theta < \theta$ in the following. For all $k \geq 0$ large enough such that $n + 1 \leq 2^k q - 1$, $\sigma^n\theta$ begins with

$$\theta_{n+1} \cdots \theta_{2^k q-1} = \overline{\theta_{2^k q+n+1} \cdots \theta_{2^k q+2^k q-1}}.$$

Since (3.25) implies

$$\overline{\theta_{2^k q+n+1} \cdots \theta_{2^k q+2^k q-1}} \leq \theta_1 \cdots \theta_{2^k q-n-1},$$

we get

$$\theta_{n+1} \cdots \theta_{2^k q-1} \leq \theta_1 \cdots \theta_{2^k q-n-1} \quad \text{for all } k \text{ large enough.}$$

Thus $\theta_{n+1}\theta_{n+2} \cdots \leq \theta_1\theta_2 \cdots$, i.e., $\sigma^n\theta \leq \theta$. Since θ is not periodic, we get $\sigma^n\theta \neq \theta$ and then $\sigma^n\theta < \theta$.

(2)⇒(1) For all $n \in \{1, \cdots, q-1\}$, we get $\theta_{n+1} \cdots \theta_q \leq \theta_1 \cdots \theta_{q-n}$ immediately from $\sigma^n\theta < \theta$, and thus we only need to prove $\overline{\theta}_1 \cdots \overline{\theta}_{q-n} < \theta_{n+1} \cdots \theta_q$ in the following. Noting that (2) implies $\overline{\theta}_1 \leq \theta_n \leq \theta_1$ for all $n \in \mathbb{N}$, we only need to consider the alphabet $\{\overline{\theta}_1, \overline{\theta}_1 + 1, \cdots, \theta_1 - 1, \theta_1\}$. Since $\theta_q \neq \overline{\theta}_1$ (otherwise $\theta_{2q} = \overline{\theta}^+_q = \theta^+_1 > \theta_1$), by applying Lemma 3.2.8 we get

$$\overline{(\theta_1 \cdots \theta_{q-1}\theta^-_q)^\infty} \leq \sigma^n((\theta_1 \cdots \theta_{q-1}\theta^-_q)^\infty) \leq (\theta_1 \cdots \theta_{q-1}\theta^-_q)^\infty$$

for all $n \geq 0$. This implies that for all $n \in \{1, \cdots, q-1\}$ we have

$$\overline{\theta_1 \cdots \theta_{q-1}\theta^-_q} \leq \theta_{n+1} \cdots \theta_{q-1}\theta^-_q\theta_1 \cdots \theta_n$$

and then

$$\overline{\theta_1 \cdots \theta_{q-n}} \leq \theta_{n+1} \cdots \theta_{q-1}\theta^-_q,$$

which is equivalent to

$$\overline{\theta_1 \cdots \theta_{q-n}} < \theta_{n+1} \cdots \theta_q.$$

(1)⇒(2) Noting that (1) and the definition of θ imply $\overline{\theta}_1 \le \theta_n \le \theta_1$ for all $n \in \mathbb{N}$, we only need to consider the alphabet $\{\overline{\theta}_1, \overline{\theta}_1 + 1, \cdots, \theta_1 - 1, \theta_1\}$. Since θ is not periodic, we have $\sigma^n\theta \ne \theta$ and $\sigma^n\theta \ne \overline{\theta}$ (otherwise $\sigma^{2n}\theta = \theta$) for all $n \ge 1$. In order to prove (2), by applying Lemma 3.2.8, noting that (1) implies $\theta_q \ne \overline{\theta}_1$, we only need to prove the following ① and ②.

① Prove $\overline{(\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty} \le \sigma^n((\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty) \le (\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty$ for all $n \ge 0$. It suffices to prove that for all $n \in \{1, \cdots, q-1\}$ we have

$$\overline{\theta_1 \cdots \theta_{q-1}\theta_q^-} \le \theta_{n+1} \cdots \theta_{q-1}\theta_q^-\theta_1 \cdots \theta_n \le \theta_1 \cdots \theta_{q-1}\theta_q^-,$$

where the second inequality follows from the fact that (1) implies $\theta_{n+1} \cdots \theta_{q-1}\theta_q^- < \theta_1 \cdots \theta_{q-n}$. We only need to prove the first inequality in the following. Let $n \in \{1, \cdots, q-1\}$. Replacing n by $q-n$ in the first inequality in (1), we get

$$\overline{\theta_1 \cdots \theta_n} < \theta_{q-n+1} \cdots \theta_q$$

and then

$$\overline{\theta_1 \cdots \theta_n} \le \theta_{q-n+1} \cdots \theta_{q-1}\theta_q^-,$$

which is equivalent to

$$\overline{\theta_{q-n+1} \cdots \theta_{q-1}\theta_q^-} \le \theta_1 \cdots \theta_n.$$

Since the first inequality in (1) also implies

$$\overline{\theta_1 \cdots \theta_{q-n}} \le \theta_{n+1} \cdots \theta_{q-1}\theta_q^-,$$

we get

$$\overline{\theta_1 \cdots \theta_{q-1}\theta_q^-} = \overline{\theta_1 \cdots \theta_{q-n}}\ \overline{\theta_{q-n+1} \cdots \theta_{q-1}\theta_q^-} \le \theta_{n+1} \cdots \theta_{q-1}\theta_q^-\ \theta_1 \cdots \theta_n.$$

② Prove that the smallest period of $(\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty$ is q. In fact, for all $n \in \{1, \cdots, q-1\}$ we have

$$\sigma^n((\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty) = (\theta_{n+1} \cdots \theta_{q-1}\theta_q^-\theta_1 \cdots \theta_n)^\infty < \theta_{n+1} \cdots \theta_q 0^\infty \overset{\text{by (1)}}{<} (\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty$$

which implies

$$\sigma^n((\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty) \ne (\theta_1 \cdots \theta_{q-1}\theta_q^-)^\infty.$$

□

Proof of Theorem 3.2.2. (1) Let $\beta \in (1, m+1]$.

① Suppose that θ is the greedy β-expansion of $\pi_\beta(\theta)$. For all $j \in \mathbb{N}$, by the definition of θ we get

$$\theta_{3\cdot 2^j q} = \theta_{2^{j+1}q+2^j q} = \overline{\theta}_{2^j q} = \overline{\theta}_q \text{ or } \theta_q^- < m.$$

It follows from Proposition 3.2.4 (1) that

$$\sum_{i=1}^{\infty} \frac{\theta_{3\cdot 2^j q+i}}{\beta^i} < 1,$$

which implies

$$\frac{\theta_{3\cdot 2^j q+1}}{\beta} + \frac{\theta_{3\cdot 2^j q+2}}{\beta^2} + \cdots + \frac{\theta_{3\cdot 2^j q+2^j q-1}}{\beta^{2^j q-1}} < 1$$

for all $j \in \mathbb{N}$. By

$$\begin{aligned}
\theta_{3\cdot 2^j q+1}\theta_{3\cdot 2^j q+2}\cdots\theta_{3\cdot 2^j q+2^j q-1} &= \theta_{2^{j+1}q+2^j q+1}\theta_{2^{j+1}q+2^j q+2}\cdots\theta_{2^{j+1}q+2^j q+2^j q-1} \\
&= \overline{\theta_{2^j q+1}\theta_{2^j q+2}\cdots\theta_{2^j q+2^j q-1}} \\
&= \theta_1\theta_2\cdots\theta_{2^j q-1}
\end{aligned}$$

we get

$$\frac{\theta_1}{\beta} + \frac{\theta_2}{\beta^2} + \cdots + \frac{\theta_{2^j q-1}}{\beta^{2^j q-1}} < 1$$

for all $j \in \mathbb{N}$. Thus $\sum_{i=1}^{\infty} \frac{\theta_i}{\beta^i} \le 1$. It follows from $\sum_{i=1}^{\infty} \frac{\theta_i}{\beta_\theta^i} = 1$ that $\beta \ge \beta_\theta$.

② Suppose that θ is the lazy β-expansion of $\pi_\beta(\theta)$. For all $j \in \mathbb{N}$, by the definition of θ we get

$$\theta_{2^j q} = \theta_q \text{ or } \overline{\theta}_q^+ > 0.$$

It follows from Proposition 3.2.4 (3) that

$$\sum_{i=1}^{\infty} \frac{\overline{\theta}_{2^j q+i}}{\beta^i} < 1,$$

which implies

$$\frac{\overline{\theta}_{2^j q+1}}{\beta} + \frac{\overline{\theta}_{2^j q+2}}{\beta^2} + \cdots + \frac{\overline{\theta}_{2^j q+2^j q-1}}{\beta^{2^j q-1}} < 1$$

for all $j \in \mathbb{N}$. By $\overline{\theta_{2^j q+1}\theta_{2^j q+2}\cdots\theta_{2^j q+2^j q-1}} = \theta_1\theta_2\cdots\theta_{2^j q-1}$ we get

$$\frac{\theta_1}{\beta} + \frac{\theta_2}{\beta^2} + \cdots + \frac{\theta_{2^j q-1}}{\beta^{2^j q-1}} < 1$$

for all $j \in \mathbb{N}$. Thus $\sum_{i=1}^{\infty} \frac{\theta_i}{\beta^i} \le 1$. It follows from $\sum_{i=1}^{\infty} \frac{\theta_i}{\beta_\theta^i} = 1$ that $\beta \ge \beta_\theta$.

③ The "quasi-greedy" case follows in a way similar to ① by applying Proposition 3.2.4 (2) instead of (1).

④ The "quasi-lazy" case follows in a way similar to ② by applying Proposition 3.2.4 (4) instead of (3).

⑤ The "unique" case follows immediately from ①, ②, ③ or ④.

(2) Since ②⇒⑦, ③⇒⑧, ④⇒⑨, ⑤⇒⑩, ⑥⇒⑪ follow from (1) and Proposition 3.2.5, and their reverses follow immediately from $\pi_{\beta_\theta}(\theta) = 1$, we only need to check the equivalence of ①, ②, ③, ④, ⑤ and ⑥. In fact we can show ①⇒②⇒④⇒⑥⇒⑤⇒③⇒① as follows.

① ⇒ ② follows from Lemma 3.1.4 and Proposition 3.2.1.

② ⇒ ④ is obvious.

④ ⇒ ⑥ follows from the fact that θ does not end with m^∞.

⑥ ⇒ ⑤ Suppose that θ is the quasi-lazy β_θ-expansion of 1. We need to prove that θ is the quasi-greedy β_θ-expansion of 1. Suppose $\theta_n < m$ for some $n \in \mathbb{N}$. By Proposition 3.2.4 (2), it suffices to prove

$$\sum_{i=1}^{\infty} \frac{\theta_{n+i}}{\beta_\theta^i} \leq 1.$$

In fact, let $s \in \mathbb{N}$ such that $n < 2^s q$. Then for all $j > s$ we have $\theta_{2^j q+n} = \overline{\theta}_n > 0$. It follows from Proposition 3.2.4 (4) that

$$\sum_{i=1}^{\infty} \frac{\overline{\theta}_{2^j q+n+i}}{\beta_\theta^i} \leq 1,$$

which implies

$$\frac{\overline{\theta}_{2^j q+n+1}}{\beta_\theta} + \frac{\overline{\theta}_{2^j q+n+2}}{\beta_\theta^2} + \cdots + \frac{\overline{\theta}_{2^j q+2^j q-1}}{\beta_\theta^{2^j q-n-1}} < 1$$

for all $j > s$. By $\overline{\theta_{2^j q+n+1}\theta_{2^j q+n+2}\cdots\theta_{2^j q+2^j q-1}} = \theta_{n+1}\theta_{n+2}\cdots\theta_{2^j q-1}$ we get

$$\frac{\theta_{n+1}}{\beta_\theta} + \frac{\theta_{n+2}}{\beta_\theta^2} + \cdots + \frac{\theta_{2^j q-1}}{\beta_\theta^{2^j q-n-1}} < 1$$

for all $j > s$. Thus $\sum_{i=1}^{\infty} \frac{\theta_{n+i}}{\beta_\theta^i} \leq 1$.

⑤ ⇒ ③ Suppose that θ is the quasi-greedy β_θ-expansion of 1. If the greedy β_θ-expansion of 1 ends with 0^∞, then the quasi-greedy β_θ-expansion of 1 must be periodic (see for examples Proposition 2.1.15 (1) ② and [22, Proposition 3.4 (b)]). This contradicts

that θ is not periodic. Thus the greedy β_θ-expansion of 1 must not end with 0^∞. It follows that the greedy and quasi-greedy β_θ-expansions of 1 are the same.

③ $\Rightarrow$ ① follows immediately from Lemma 3.2.7 and the equivalence of (1) and (5) in Proposition 3.2.1.

□

3.3 Infinite products related to generalized Thue-Morse sequences

In this and the next sections, we consider another class of generalizations of the famous Thue-Morse sequence. We study infinite products related to these generalized Thue-Morse sequences in this section.

At the beginning of Section 3.1, we introduce one of the equivalent definitions of the classical Thue-Morse sequence

$$0110\ 1001\ 1001\ 0110\ 1001\ 0110\ 0110\ 1001\ \cdots.$$

Here we consider another one: the Thue-Morse sequence $(t_n)_{n\geq 0}$ is the unique fixed point of the morphism

$$0 \mapsto 01$$
$$1 \mapsto 10$$

beginning with $t_0 := 0$. A natural generalization is the following: given any integer $q \geq 2$ and $\theta_1, \cdots, \theta_{q-1} \in \{0, 1\}$, we call the unique fixed point of the morphism

$$0 \mapsto 0\theta_1 \cdots \theta_{q-1}$$
$$1 \mapsto 1\overline{\theta_1} \cdots \overline{\theta_{q-1}}$$

beginning with $\theta_0 := 0$ the $(0, \theta_1, \cdots, \theta_{q-1})$-*Thue-Morse sequence*, where $\overline{0} := 1$ and $\overline{1} := 0$. Note that the classical Thue-Morse sequence $(t_n)_{n\geq 0}$ is exactly the $(0, 1)$-Thue-Morse sequence in our terms.

Generalized Thue-Morse sequences defined above are essentially contained in the concept of generalized Morse sequences in [79]. In fact, given any integer $q \geq 2$ and $\theta_1, \cdots, \theta_{q-1} \in \{0, 1\}$, by Proposition 3.3.15 (1) and inductive, one can check that the $(0, \theta_1, \cdots, \theta_{q-1})$-Thue-Morse sequence $\theta = (\theta_n)_{n\geq 0}$ is exactly

$$(0, \theta_1, \cdots, \theta_{q-1}) \times (0, \theta_1, \cdots, \theta_{q-1}) \times (0, \theta_1, \cdots, \theta_{q-1}) \times \cdots$$

where we use the notation of products of blocks mentioned in [79]. It follows from [79, Lemma 1] that θ is periodic if and only if $\theta = 0^\infty$ or $(01)^\infty$. Therefore, if θ is not the

trivial 0^∞ or $(01)^\infty$, it is a generalized Morse sequence in the sense of [79].

Recently, for rational functions R, Allouche, Riasat and Shallit [13] studied infinite products related to the classical Thue-morse sequence $(t_n)_{n\geq 0}$ of the forms

$$\prod_{n=1}^{\infty}\Big(R(n)\Big)^{(-1)^{t_n}} \quad \text{and} \quad \prod_{n=1}^{\infty}\Big(R(n)\Big)^{t_n},$$

obtained a class of equalities involving variables in [13, Theorem 2.2 and Corollary 2.3], and obtained many concrete equalities in [13, Corollary 2.4 and Theorem 4.2]. In this section, we generalize these results by studying infinite products related to the generalized Thue-Morse sequence $(\theta_n)_{n\geq 0}$ of the forms

$$\prod_{n=1}^{\infty}\Big(R(n)\Big)^{(-1)^{\theta_n}} \quad \text{and} \quad \prod_{n=1}^{\infty}\Big(R(n)\Big)^{\theta_n}.$$

Let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{C}$ be the sets of positive integers $1, 2, 3, \cdots$, non-negative integers $0, 1, 2, \cdots$ and complex numbers respectively. Moreover, for simplification we define $\delta_n := (-1)^{\theta_n} \in \{+1, -1\}$ for all $n \in \mathbb{N}_0$ throughout this section.

First we have the following convergence theorem, which is a generalization of [13, Lemmas 2.1 and 4.1] (see also [103, Lemma 1]) and guarantees the convergence of all the infinite products given in the results in this section.

Theorem 3.3.1. *Let $q \geq 2$ be an integer, $\theta_0 = 0$, $(\theta_1, \cdots, \theta_{q-1}) \in \{0,1\}^{q-1} \setminus \{0^{q-1}\}$, $(\theta_n)_{n\geq 0}$ be the $(0, \theta_1, \cdots, \theta_{q-1})$-Thue-Morse sequence and $R \in \mathbb{C}(X)$ be a rational function such that the values $R(n)$ are defined and non-zero for all $n \in \mathbb{N}$. Then:*

(1) *the infinite product $\prod_{n=1}^{\infty}(R(n))^{\delta_n}$ converges if and only if the numerator and the denominator of R have the same degree and the same leading coefficient;*

(2) *the infinite product $\prod_{n=1}^{\infty}(R(n))^{\theta_n}$ converges if and only if the numerator and the denominator of R have the same degree, the same leading coefficient and the same sum of roots (in $\mathbb{C}$).*

Although Theorem 3.3.1 is a natural generalization of [13, Lemmas 2.1 and 4.1], the proof is more intricate and relies on Proposition 3.3.15 as we will see.

In the following Subsections 3.3.1 and 3.3.2, we introduce our results concerning products of the forms $\prod(R(n))^{\delta_n}$ and $\prod(R(n))^{\theta_n}$ respectively. Then we give some preliminaries in Subsection 3.3.3 and prove all the results in Subsection 3.3.4.

3.3.1 Products of the form $\prod(R(n))^{\delta_n}$

In order to study the infinite product $\prod_{n=1}^{\infty}(R(n))^{\delta_n}$, by Theorem 3.3.1 (1), it suffices to study products of the form

$$f(a,b) := \prod_{n=1}^{\infty}\Big(\frac{n+a}{n+b}\Big)^{\delta_n},$$

where $a, b \in \mathbb{C}\setminus\{-1,-2,-3,\cdots\}$. For the $(0,1)$-Thue-Morse sequence $(t_n)_{n\geq 0}$, the special form $f(\frac{x}{2}, \frac{x+1}{2}) = \prod_{n=1}^{\infty}(\frac{2n+x}{2n+x+1})^{(-1)^{t_n}}$ is used to define new functions and is further studied in [13, Theorem 2.2] and [103, Definition 1] (see also [38, Remark 6.5]). For infinite products involving the first 2^m terms of $(t_n)_{n\geq 0}$, see the equalities (23) and (24) in [38, Section 6].

As the first main result in this section, the following theorem generalizes [13, Theorem 2.2 and Corollary 2.3 (i)] (see also [103, Lemma 2] and the equalities (6) and (7) in [103, Section 4]).

Theorem 3.3.2. *Let $q \geq 2$ be an integer, $\theta_0 = 0$, $(\theta_1, \cdots, \theta_{q-1}) \in \{0,1\}^{q-1}\setminus\{0^{q-1}\}$ and $(\theta_n)_{n\geq 0}$ be the $(0,\theta_1,\cdots,\theta_{q-1})$-Thue-Morse sequence. Then for all $a, b \in \mathbb{C}\setminus\{-1,-2,-3,\cdots\}$, we have*

$$f(a,b) = \Big(\frac{a+1}{b+1}\Big)^{\delta_1}\cdots\Big(\frac{a+q-1}{b+q-1}\Big)^{\delta_{q-1}} f(\frac{a}{q},\frac{b}{q})\Big(f(\frac{a+1}{q},\frac{b+1}{q})\Big)^{\delta_1}\cdots\Big(f(\frac{a+q-1}{q},\frac{b+q-1}{q})\Big)^{\delta_{q-1}}$$

which is equivalent to

$$\prod_{n=1}^{\infty}\Big(\frac{n+a}{n+b}\cdot\frac{qn+b}{qn+a}(\frac{qn+b+1}{qn+a+1})^{\delta_1}\cdots(\frac{qn+b+q-1}{qn+a+q-1})^{\delta_{q-1}}\Big)^{\delta_n} = \Big(\frac{a+1}{b+1}\Big)^{\delta_1}\cdots\Big(\frac{a+q-1}{b+q-1}\Big)^{\delta_{q-1}}.$$

This theorem implies many neat equalities.

Corollary 3.3.3. *Let $q \geq 2$ be an integer, $\theta_0 = 0$, $(\theta_1, \cdots, \theta_{q-1}) \in \{0,1\}^{q-1}\setminus\{0^{q-1}\}$ and $(\theta_n)_{n\geq 0}$ be the $(0,\theta_1,\cdots,\theta_{q-1})$-Thue-Morse sequence.*

(1) *For all $a, b \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}$, we have*

$$\prod_{n=0}^{\infty}\Big(\frac{n+a}{n+b}\cdot\frac{qn+b}{qn+a}(\frac{qn+b+1}{qn+a+1})^{\delta_1}\cdots(\frac{qn+b+q-1}{qn+a+q-1})^{\delta_{q-1}}\Big)^{\delta_n} = 1.$$

(2) *For all $a \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}$, we have*

$$\prod_{n=0}^{\infty}\Big(\frac{n+a}{n+a+1}\cdot\frac{qn+a+1}{qn+a}(\frac{qn+a+2}{qn+a+1})^{\delta_1}(\frac{qn+a+3}{qn+a+2})^{\delta_2}\cdots(\frac{qn+a+q}{qn+a+q-1})^{\delta_{q-1}}\Big)^{\delta_n} = 1$$

and

$$\prod_{n=0}^{\infty}\Big(\frac{qn+qa}{qn+a}(\frac{qn+1}{qn+a+1})^{\delta_1}(\frac{qn+2}{qn+a+2})^{\delta_2}\cdots(\frac{qn+q-1}{qn+a+q-1})^{\delta_{q-1}}\Big)^{\delta_n} = q.$$

(3) *We have*

$$\prod_{n=0}^{\infty}\Big(\frac{qn+q}{qn+1}\big(\frac{qn+1}{qn+2}\big)^{\delta_1}\big(\frac{qn+2}{qn+3}\big)^{\delta_2}\cdots\big(\frac{qn+q-1}{qn+q}\big)^{\delta_{q-1}}\Big)^{\delta_n}=q.$$

Remark 3.3.4. *It may seem that the conditions on the domains of a and b in Corollary 3.3.3 are more restrictive than Theorem 3.3.2. In fact they are equivalent, since Corollary 3.3.3 (1) is the case that $a\neq 0$ and $b\neq 0$ in Theorem 3.3.2, the second equality in (2) of Corollary 3.3.3 is the case that $a\neq 0$ and $b=0$ (the same as $a=0$ and $b\neq 0$) in Theorem 3.3.2, and obviously the case that $a=b=0$ in Theorem 3.3.2 is trivial.*

Let $q\geq 2$ be an integer. For $k=1,2,\cdots,q-1$, define $N_{k,q}(n)$ to be the number of occurrences of the digit k in the base q expansion of the non-negative integer n, and let

$$s_q(n):=\sum_{k=1}^{q-1}kN_{k,q}(n)$$

be the sum of digits. It is obtained in [18, Example 11 and Corollary 5] (see also [109, 110]) respectively that

$$\prod_{n=0}^{\infty}\Big(\frac{qn+k}{qn+k+1}\Big)^{(-1)^{N_{k,q}(n)}}=\frac{1}{\sqrt{q}} \tag{3.26}$$

for $k=1,2,\cdots,q-1$, and

$$\prod_{n=0}^{\infty}\prod_{\substack{0<k<q\\ k\text{ odd}}}\Big(\frac{qn+k}{qn+k+1}\Big)^{(-1)^{s_q(n)}}=\frac{1}{\sqrt{q}}. \tag{3.27}$$

For more infinite products related to $(s_q(n))_{n\geq 0}$, see for example [93, Propositions 6 and 7]. Equalities (3.26) and (3.27) are two ways to represent $\frac{1}{\sqrt{q}}$ in the form of infinite products and generalize the well known Woods-Robbins product [125, 126]

$$\prod_{n=0}^{\infty}\Big(\frac{2n+1}{2n+2}\Big)^{(-1)^{t_n}}=\frac{1}{\sqrt{2}} \tag{3.28}$$

where $(t_n)_{n\geq 0}$ is the $(0,1)$-Thue-Morse sequence. We give one more such way in the first equality in the following corollary.

Corollary 3.3.5. *Let $q\geq 2$ be an integer, $k\in\{1,2,\cdots,q-1\}$, $\theta_0=\theta_1=\cdots=\theta_{k-1}=0$, $\theta_k=\theta_{k+1}=\cdots=\theta_{q-1}=1$ and $(\theta_n)_{n\geq 0}$ be the $(0,\theta_1,\cdots,\theta_{q-1})$-Thue-Morse sequence. Then*

$$\prod_{n=0}^{\infty}\Big(\frac{qn+k}{qn+q}\Big)^{\delta_n}=\frac{1}{\sqrt{q}}$$

and

$$\prod_{n=0}^{\infty}\Big(\frac{(n+a)(qn+a+k)^2}{(n+a+1)(qn+a)(qn+a+q)}\Big)^{\delta_n}=1$$

for all $a \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$.

For more generalizations of the Woods-Robbins product (3.28), we refer the reader to [6, 14, 106].

Note that for any integer $q \geq 2$, the $(\overbrace{0, \cdots, 0}^{q})$-Thue-Morse sequence is the trivial 0^∞. For $q = 2$, the only nontrivial case, related to the $(0, 1)$-Thue-Morse sequence, is already studied in [103] and [13, Section 2]. In the following three examples, we study nontrivial cases for $q = 3$ in detail, related to the $(0, 0, 1)$, $(0, 1, 1)$ and $(0, 1, 0)$-Thue-Morse sequences.

Example 3.3.6. *Let* $(\theta_n)_{n\geq 0}$ *be the* $(0, 0, 1)$*-Thue-Morse sequence.*
(1) *For all* $a, b \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ *we have*

$$\prod_{n=0}^{\infty} \Big(\frac{(n+a)(3n+b)(3n+b+1)(3n+a+2)}{(n+b)(3n+a)(3n+a+1)(3n+b+2)} \Big)^{\delta_n} = 1.$$

(2) *For all* $a \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ *we have*

$$① \quad \prod_{n=0}^{\infty} \Big(\frac{(n+a)(3n+a+2)^2}{(n+a+1)(3n+a)(3n+a+3)} \Big)^{\delta_n} = 1,$$

$$② \quad \prod_{n=0}^{\infty} \Big(\frac{(3n+1)(3n+3a)(3n+a+2)}{(3n+2)(3n+a)(3n+a+1)} \Big)^{\delta_n} = 3,$$

$$③ \quad \prod_{n=0}^{\infty} \Big(\frac{(3n+1)(3n+3a)(3n+a+2)}{(3n+3)(3n+a)(3n+a+1)} \Big)^{\delta_n} = \sqrt{3},$$

$$④ \quad \prod_{n=0}^{\infty} \Big(\frac{(6n+1)(3n+3a)(3n+a+2)}{(6n+5)(3n+a)(3n+a+1)} \Big)^{\delta_n} = 1.$$

(3) *The following concrete equalities hold.*

$$① \quad \prod_{n=0}^{\infty} \Big(\frac{3n+2}{3n+3} \Big)^{\delta_n} = \frac{1}{\sqrt{3}}, \qquad ② \quad \prod_{n=0}^{\infty} \Big(\frac{(6n-3)(6n+3)}{(6n-1)(6n+5)} \Big)^{\delta_n} = 1,$$

$$③ \quad \prod_{n=0}^{\infty} \Big(\frac{(3n+1)(6n+5)}{(3n+2)(6n+1)} \Big)^{\delta_n} = 3, \qquad ④ \quad \prod_{n=0}^{\infty} \Big(\frac{(3n+1)(6n+5)}{(3n+3)(6n+1)} \Big)^{\delta_n} = \sqrt{3},$$

$$⑤ \quad \prod_{n=0}^{\infty} \Big(\frac{(6n+7)^2}{(6n+3)(6n+15)} \Big)^{\delta_n} = 1, \qquad ⑥ \quad \prod_{n=0}^{\infty} \Big(\frac{(9n+3)(9n+8)}{(9n+2)(9n+5)} \Big)^{\delta_n} = 3,$$

$$⑦ \quad \prod_{n=0}^{\infty} \Big(\frac{(18n+3)(18n+17)}{(18n+5)(18n+11)} \Big)^{\delta_n} = 1, \qquad ⑧ \quad \prod_{n=0}^{\infty} \Big(\frac{(2n+3)(3n+1)(6n+7)}{(2n+1)(3n+2)(6n+5)} \Big)^{\delta_n} = 3,$$

$$⑨ \quad \prod_{n=0}^{\infty} \Big(\frac{(n+1)(3n+3)^2}{(n+2)(3n+1)(3n+4)} \Big)^{\delta_n} = 1, \qquad ⑩ \quad \prod_{n=0}^{\infty} \Big(\frac{(n+1)(3n+2)(3n+3)}{(n+2)(3n+1)(3n+4)} \Big)^{\delta_n} = \frac{1}{\sqrt{3}},$$

$$⑪ \quad \prod_{n=0}^{\infty} \Big(\frac{(n+1)(3n+2)^2}{(n+2)(3n+1)(3n+4)} \Big)^{\delta_n} = \frac{1}{3}, \qquad ⑫ \quad \prod_{n=0}^{\infty} \Big(\frac{(3n+2)^3}{(3n+1)(3n+4)(3n+6)} \Big)^{\delta_n} = \frac{1}{3\sqrt{3}},$$

⑬ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+2)(3n+4)^2}{(n+3)(3n+2)(3n+5)}\Big)^{\delta_n}=1,$ ⑭ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+2)(3n+4)^2}{(n+3)(3n+3)(3n+5)}\Big)^{\delta_n}=\frac{1}{\sqrt{3}},$

⑮ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+2)(9n+4)(9n+7)}{(n+1)(9n+6)(9n+10)}\Big)^{\delta_n}=1,$ ⑯ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(3n+1)(6n+3)(6n-3)}{(3n+2)(6n+1)(6n-1)}\Big)^{\delta_n}=3.$

Example 3.3.7. *Let $(\theta_n)_{n\geq 0}$ be the $(0,1,1)$-Thue-Morse sequence.*
(1) *For all $a,b\in\mathbb{C}\setminus\{0,-1,-2,\cdots\}$ we have*

$$\prod_{n=0}^{\infty}\Big(\frac{(n+a)(3n+b)(3n+a+1)(3n+a+2)}{(n+b)(3n+a)(3n+b+1)(3n+b+2)}\Big)^{\delta_n}=1.$$

(2) *For all $a\in\mathbb{C}\setminus\{0,-1,-2,\cdots\}$ we have*

① $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+a)(3n+a+1)^2}{(n+a+1)(3n+a)(3n+a+3)}\Big)^{\delta_n}=1,$

② $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(3n+a+1)(3n+a+2)(3n+3a)}{(3n+1)(3n+2)(3n+a)}\Big)^{\delta_n}=3,$

③ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(3n+a+1)(3n+a+2)(3n+3a)}{(3n+2)(3n+3)(3n+a)}\Big)^{\delta_n}=\sqrt{3}.$

(3) *The following concrete equalities hold.*

① $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{3n+1}{3n+3}\Big)^{\delta_n}=\frac{1}{\sqrt{3}},$ ② $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(3n+4)(3n+6)}{(3n+2)^2}\Big)^{\delta_n}=\sqrt{3},$

③ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(6n+5)^2}{(6n+3)(6n+15)}\Big)^{\delta_n}=1,$ ④ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+4)(9n+7)}{(9n+1)(9n+6)}\Big)^{\delta_n}=3,$

⑤ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+5)(9n+8)}{(9n+2)(9n+3)}\Big)^{\delta_n}=3,$ ⑥ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+5)(9n+8)}{(9n+2)(9n+9)}\Big)^{\delta_n}=\sqrt{3},$

⑦ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(24n+7)(24n+13)}{(24n+5)(24n+23)}\Big)^{\delta_n}=1,$ ⑧ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+2)(3n+3)^2}{(n+3)(3n+2)(3n+5)}\Big)^{\delta_n}=1,$

⑨ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+2)(3n+1)(3n+3)}{(n+3)(3n+2)(3n+5)}\Big)^{\delta_n}=\frac{1}{\sqrt{3}},$ ⑩ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+2)(3n+1)^2}{(n+3)(3n+2)(3n+5)}\Big)^{\delta_n}=\frac{1}{3},$

⑪ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+3)(3n+4)(3n+5)}{(n+1)(3n+1)(3n+2)}\Big)^{\delta_n}=3,$ ⑫ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+3)(3n+4)(3n+5)}{(n+1)(3n+2)(3n+3)}\Big)^{\delta_n}=\sqrt{3},$

⑬ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(3n+4)(3n+5)(3n+9)}{(3n+1)^2(3n+2)}\Big)^{\delta_n}=3\sqrt{3},$ ⑭ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(2n-1)(6n+1)^2}{(2n+1)(6n-1)(6n+5)}\Big)^{\delta_n}=1,$

⑮ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(2n+2)(6n+1)(6n+4)}{(2n+1)(6n+3)(6n+5)}\Big)^{\delta_n}=\frac{1}{\sqrt{3}},$ ⑯ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(2n+3)(6n+5)(6n+7)}{(2n+1)(6n+2)(6n+4)}\Big)^{\delta_n}=3.$

Note that the $(0,1,0)$-Thue-Morse sequence is exactly $01010101\cdots$, which implies $\delta_n:=(-1)^{\theta_n}=(-1)^n$ for all $n\in\mathbb{N}_0$. The next example is deduced from Corollary 3.3.3, and can

also be deduced from Theorem 3.3.13 and Proposition 3.3.14, which are classical results on the Gamma function.

Example 3.3.8. (1) *For all odd $q \geq 3$, we have*

$$\prod_{n=0}^{\infty}\Big(\frac{(qn+1)(qn+3)\cdots(qn+q-2)}{(qn+2)(qn+4)\cdots(qn+q-1)}\Big)^{(-1)^n}=\frac{1}{\sqrt{q}}.$$

(2) *For all odd $q \geq 3$ and all $a \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}$ we have*

$$\prod_{n=0}^{\infty}\Big(\frac{(qn+a)(qn+a+2)(qn+a+4)\cdots(qn+a+q-1)}{(qn+qa)(qn+a+1)(qn+a+3)\cdots(qn+a+q-2)}\Big)^{(-1)^n}=\frac{1}{\sqrt{q}}.$$

(3) *The following concrete equalities hold.*

① $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{3n+1}{3n+2}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

② $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+1)(3n+5)}{(n+3)(3n+4)}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

③ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(3n+2)(3n+4)}{(3n+3)(3n+6)}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

④ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(3n+1)(3n+5)}{(3n+6)(3n+9)}\Big)^{(-1)^n}=\frac{1}{3\sqrt{3}},$

⑤ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+2)(9n+8)}{(9n+5)(9n+6)}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

⑥ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+2)(9n+8)}{(9n+3)(9n+5)}\Big)^{(-1)^n}=1,$

⑦ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+1)(9n+7)}{(9n+3)(9n+4)}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

⑧ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+1)(9n+7)}{(9n+4)(9n+6)}\Big)^{(-1)^n}=\frac{1}{3},$

⑨ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+2)(9n+11)}{(9n+3)(9n+15)}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

⑩ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+1)(9n+10)}{(9n+6)(9n+12)}\Big)^{(-1)^n}=\frac{1}{3\sqrt{3}},$

⑪ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n+5)(9n+11)}{(9n+8)(9n+15)}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

⑫ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(9n-1)(9n+8)}{(9n-3)(9n+3)}\Big)^{(-1)^n}=\frac{1}{\sqrt{3}},$

⑬ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(5n+1)(5n+3)}{(5n+2)(5n+4)}\Big)^{(-1)^n}=\frac{1}{\sqrt{5}},$

⑭ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(10n+1)(10n+9)}{(10n+3)(10n+7)}\Big)^{(-1)^n}=\frac{1}{\sqrt{5}},$

⑮ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+1)(5n+3)(5n+7)}{(n+3)(5n+4)(5n+6)}\Big)^{(-1)^n}=\frac{1}{\sqrt{5}},$

⑯ $\displaystyle\prod_{n=0}^{\infty}\Big(\frac{(n+1)(5n+2)(5n+7)}{(n+3)(5n+1)(5n+6)}\Big)^{(-1)^n}=1.$

In [74] Hu studied infinite sums of the form

$$\sum_{n\geq 0}\Big((-1)^{a_{w,B}(n)}\sum_{(l,c_l)\in L_{w,B}}c_l f(l(n))\Big)$$

where $a_{w,B}(n)$ denote the number of occurrences of the word w in the base B expansion of the non-negative integer n, f is any function that verifies certain convergence conditions, and $L_{w,B}$ is a computable finite set of pairs (l,c_l) where l is a polynomial with integer coefficients of degree 1 and c_l is an integer. If f is taken to be an appropriate composition of a logarithmic function and a rational function, after exponentiating, some infinite products of the form $\prod_n (R(n))^{(-1)^{a_{w,B}(n)}}$ can be obtained, where R is a rational function depending

on the sequence $(a_{w,B}(n))_{n\geq 0}$. For instance the above Example 3.3.8 (3) ① is also obtained in [74, Section 5] (see also [18, Section 4.4]).

3.3.2 Products of the form $\prod(R(n))^{\theta_n}$

In order to study the infinite product $\prod_{n=1}^{\infty}(R(n))^{\theta_n}$, by Theorem 3.3.1 (2), it suffices to study products of the form

$$\mathfrak{f}(a_1,\cdots,a_d;b_1,\cdots,b_d) := \prod_{n=1}^{\infty}\Big(\frac{(n+a_1)\cdots(n+a_d)}{(n+b_1)\cdots(n+b_d)}\Big)^{\theta_n}$$

where $d\in\mathbb{N}$ and $a_1,\cdots,a_d,b_1,\cdots,b_d\in\mathbb{C}\setminus\{-1,-2,-3,\cdots\}$ satisfy $a_1+\cdots+a_d=b_1+\cdots+b_d$. As the second main result in this section, the following theorem (which implies Corollary 3.3.11) generalizes [13, Theorem 4.2].

Theorem 3.3.9. *Let $q\geq 2$ be an integer, $\theta_0=0$, $(\theta_1,\cdots,\theta_{q-1})\in\{0,1\}^{q-1}\setminus\{0^{q-1}\}$ and $(\theta_n)_{n\geq 0}$ be the $(0,\theta_1,\cdots,\theta_{q-1})$-Thue-Morse sequence. Then for all $d\in\mathbb{N}$ and $a_1,\cdots,a_d$, $b_1,\cdots,b_d\in\mathbb{C}\setminus\{-1,-2,-3,\cdots\}$ satisfying $a_1+\cdots+a_d=b_1+\cdots+b_d$, we have*

$$\mathfrak{f}(a_1,\cdots,a_d;b_1,\cdots,b_d)=\prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{\Gamma(\frac{b_i+k}{q})}{\Gamma(\frac{a_i+k}{q})}\Big)^{\theta_k}\cdot\prod_{k=0}^{q-1}\Big(\mathfrak{f}(\frac{a_1+k}{q},\cdots,\frac{a_d+k}{q};\frac{b_1+k}{q},\cdots,\frac{b_d+k}{q})\Big)^{(-1)^{\theta_k}}$$

which is equivalent to

$$\prod_{n=1}^{\infty}\Big(\prod_{i=1}^{d}\Big(\frac{n+a_i}{n+b_i}\cdot\prod_{k=0}^{q-1}(\frac{qn+b_i+k}{qn+a_i+k})^{(-1)^{\theta_k}}\Big)\Big)^{\theta_n}=\prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{\Gamma(\frac{b_i+k}{q})}{\Gamma(\frac{a_i+k}{q})}\Big)^{\theta_k},$$

where Γ denotes the Gamma function.

This theorem implies a large number of equalities for products of the form $\prod(R(n))^{\theta_n}$ as we will see in the following corollaries, which can also be viewed as special examples.

Corollary 3.3.10. *Let $q\geq 2$ be an integer, $\theta_0=0$, $(\theta_1,\cdots,\theta_{q-1})\in\{0,1\}^{q-1}\setminus\{0^{q-1}\}$ and $(\theta_n)_{n\geq 0}$ be the $(0,\theta_1,\cdots,\theta_{q-1})$-Thue-Morse sequence.*
(1) *For all $a,b,c\in\mathbb{C}$ such that $a,b,a+c,b+c\notin\{-1,-2,-3,\cdots\}$ we have*

$$\prod_{n=1}^{\infty}\Big(\frac{(n+a)(n+b+c)}{(n+b)(n+a+c)}\cdot\prod_{k=0}^{q-1}\Big(\frac{(qn+b+k)(qn+a+c+k)}{(qn+a+k)(qn+b+c+k)}\Big)^{(-1)^{\theta_k}}\Big)^{\theta_n}=\prod_{k=1}^{q-1}\Big(\frac{\Gamma(\frac{b+k}{q})\Gamma(\frac{a+c+k}{q})}{\Gamma(\frac{a+k}{q})\Gamma(\frac{b+c+k}{q})}\Big)^{\theta_k}.$$

(2) *For all $d\in\mathbb{N}$ and $a_1,\cdots,a_d\in\mathbb{C}\setminus\{-1,-2,-3,\cdots\}$ such that $a_1+\cdots+a_d=0$ we have*

$$\prod_{n=1}^{\infty}\Big(\prod_{i=1}^{d}\Big(\frac{qn+qa_i}{qn+a_i}\cdot\prod_{k=1}^{q-1}\Big(\frac{qn+k}{qn+a_i+k}\Big)^{(-1)^{\theta_k}}\Big)\Big)^{\theta_n}=\prod_{k=1}^{q-1}\Big(\frac{(\Gamma(\frac{k}{q}))^d}{\Gamma(\frac{a_1+k}{q})\cdots\Gamma(\frac{a_d+k}{q})}\Big)^{\theta_k}.$$

(3) *For all* $a \in \mathbb{C} \setminus \mathbb{Z}$ *we have*

$$\prod_{n=1}^{\infty}\Big(\frac{(qn+qa)(qn-qa)}{(qn+a)(qn-a)}\cdot\prod_{k=1}^{q-1}\Big(\frac{(qn+k)^2}{(qn+a+k)(qn-a+k)}\Big)^{(-1)^{\theta_k}}\Big)^{\theta_n}=\prod_{k=1}^{q-1}\Big(\frac{(\Gamma(\frac{k}{q}))^2}{\Gamma(\frac{k+a}{q})\Gamma(\frac{k-a}{q})}\Big)^{\theta_k}.$$

In particular for the well known $(0,1)$-Thue-Morse sequence, we have the following corollary, in which (5) ②, ③ and ④ recover [13, Theorem 4.2].

Corollary 3.3.11. *Let* $(t_n)_{n\geq 0}$ *be the* $(0,1)$*-Thue-Morse sequence.*
(1) *For all* $d \in \mathbb{N}$ *and* $a_1, \cdots, a_d, b_1, \cdots, b_d \in \mathbb{C}\setminus\{-1,-2,-3,\cdots\}$ *such that* $a_1+\cdots+a_d = b_1+\cdots+b_d$ *we have*

$$\prod_{n=1}^{\infty}\Big(\prod_{i=1}^{d}\frac{(n+a_i)(2n+b_i)(2n+a_i+1)}{(n+b_i)(2n+a_i)(2n+b_i+1)}\Big)^{t_n}=\prod_{i=1}^{d}\frac{\Gamma(\frac{b_i+1}{2})}{\Gamma(\frac{a_i+1}{2})}.$$

(2) *For all* $a,b,c \in \mathbb{C}$ *such that* $a,b,a+c,b+c \notin \{-1,-2,-3,\cdots\}$ *we have*

$$\prod_{n=1}^{\infty}\Big(\frac{(n+a)(n+b+c)(2n+b)(2n+a+1)(2n+a+c)(2n+b+c+1)}{(n+b)(n+a+c)(2n+a)(2n+b+1)(2n+b+c)(2n+a+c+1)}\Big)^{t_n}=\frac{\Gamma(\frac{b+1}{2})\Gamma(\frac{a+c+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+c+1}{2})}.$$

(3) ① *For all* $a,b \in \mathbb{C}$ *such that* $a,b,a+b \notin \{-1,-2,-3,\cdots\}$ *we have*

$$\prod_{n=1}^{\infty}\Big(\frac{2(n+a)(n+b)(2n+a+1)(2n+b+1)(2n+a+b)}{(2n+1)(n+a+b)(2n+a)(2n+b)(2n+a+b+1)}\Big)^{t_n}=\frac{\sqrt{\pi}\,\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}.$$

② *For all* $a,b \in \mathbb{C}$ *such that* $a,b,2a+1,a+b \notin \{-1,-2,-3,\cdots\}$ *we have*

$$\prod_{n=1}^{\infty}\Big(\frac{(n+a+b)(2n+a+2)(2n+2a+1)(2n+b)(2n+a+b+1)}{(n+2a+1)(2n+a+1)(2n+b+1)(2n+2b)(2n+a+b)}\Big)^{t_n}=\frac{2^a\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{a+b+1}{2})}.$$

(4) ① *For all* $a \in \mathbb{C}\setminus\{-1,-\frac{3}{2},-2,-\frac{5}{2},\cdots\}$ *we have*

$$\prod_{n=1}^{\infty}\Big(\frac{(n+a)(2n+a+2)(2n+2a+1)}{(n+2a+1)(2n+1)(2n+a)}\Big)^{t_n}=2^a.$$

② *For all* $a \in \mathbb{C}\setminus\{-1,-\frac{3}{2},-2,-\frac{5}{2},\cdots\}$ *we have*

$$\prod_{n=1}^{\infty}\Big(\frac{(n+1)(n+a+2)(2n+a+3)(2n+2a+1)}{(n+2)(n+2a+1)(2n+3)(2n+a+1)}\Big)^{t_n}=\frac{2^a}{a+1}.$$

③ *For all* $a \in \mathbb{C}\setminus\mathbb{Z}$ *we have*

$$\prod_{n=1}^{\infty}\Big(\frac{(2n+a+1)(2n-a+1)(2n+2a)(2n-2a)}{(2n+1)^2(2n+a)(2n-a)}\Big)^{t_n}=\cos\frac{\pi a}{2}.$$

④ *For all* $a \in \mathbb{C} \setminus (\mathbb{Z} \cup \{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots\})$ *we have*

$$\prod_{n=1}^{\infty} \Big(\frac{(2n+a+1)(2n-a+1)(2n+2a)(2n-4a+2)}{(2n+1)(2n+a)(2n-a+2)(2n-2a+1)} \Big)^{t_n} = 2^a \cos \frac{\pi a}{2}.$$

⑤ *For all* $a \in \mathbb{C} \setminus \{\pm 3, \pm 5, \pm 7, \cdots\}$ *we have*

$$\prod_{n=1}^{\infty} \Big(\frac{(2n+a+1)(2n-a+1)(4n+a+3)(4n-a+3)}{(2n+2)^2(4n+a+1)(4n-a+1)} \Big)^{t_n} = \frac{\sqrt{\pi}}{\Gamma(\frac{3+a}{4})\Gamma(\frac{3-a}{4})}.$$

⑥ *For all* $d \in \mathbb{N}$ *we have*

$$\prod_{n=1}^{\infty} \Big(\frac{(n+1)(2n+d)(2n+2)^{2d-1}}{(n+d)(2n+d+1)(2n+1)^{2d-1}} \Big)^{t_n} = \pi^{\frac{d-1}{2}} \Gamma(\frac{d+1}{2}).$$

(5) *The following concrete equalities hold.*

① $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(2n+1)(4n-1)}{(2n-1)(4n+3)} \Big)^{t_n} = \sqrt{2},$

② $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(2n+1)(4n+3)}{(2n+2)(4n+1)} \Big)^{t_n} = \frac{\Gamma(\frac{1}{4})}{\sqrt{2}\pi^{\frac{3}{4}}},$

③ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(n+1)(4n+5)}{(n+2)(4n+1)} \Big)^{t_n} = \sqrt{2},$

④ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(8n+1)(8n+7)}{(8n+3)(8n+5)} \Big)^{t_n} = \sqrt{2\sqrt{2}-2},$

⑤ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(n+1)(2n+3)^2}{(n+3)(2n+1)^2} \Big)^{t_n} = 2,$

⑥ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(3n+2)^2(6n+5)}{(3n+3)^2(6n+1)} \Big)^{t_n} = \frac{\sqrt{3}\ \Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}{4\pi^{\frac{3}{2}}},$

⑦ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(n+2)^2(2n+5)}{(n+1)(n+5)(2n+1)} \Big)^{t_n} = 4,$

⑧ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(2n+1)^2(4n-1)}{(2n-1)(2n+2)(4n+1)} \Big)^{t_n} = \frac{\Gamma(\frac{1}{4})}{\pi^{\frac{3}{4}}},$

⑨ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(2n+3)^2(4n-1)}{(2n-1)(2n+6)(4n+1)} \Big)^{t_n} = \frac{2\Gamma(\frac{1}{4})}{\pi^{\frac{3}{4}}},$

⑩ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(2n-1)(4n+3)^2}{(2n+2)(4n+1)(4n-1)} \Big)^{t_n} = \frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{3}{4}}},$

⑪ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(3n-1)^2(6n+3)}{(3n+2)(3n-2)(6n-1)} \Big)^{t_n} = 2^{\frac{2}{3}},$

⑫ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(4n+2)^2(8n-1)}{(4n-1)(4n+1)(8n+7)} \Big)^{t_n} = 2^{\frac{1}{4}},$

⑬ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(n+1)(2n+7)(4n+9)}{(n+4)(2n+3)(4n+5)} \Big)^{t_n} = \frac{4\sqrt{2}}{5},$

⑭ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(n+1)(3n+7)(6n+5)}{(n+2)(3n+2)(6n+9)} \Big)^{t_n} = 3 \cdot 2^{-\frac{5}{3}},$

⑮ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(3n+1)(6n-1)(6n+3)}{(3n-1)(6n+1)(6n+5)} \Big)^{t_n} = 2^{\frac{1}{3}},$

⑯ $\displaystyle\prod_{n=0}^{\infty} \Big(\frac{(5n+4)(10n+1)(10n+5)}{(5n+2)(10n+3)(10n+7)} \Big)^{t_n} = \frac{\sqrt{5}-1}{2^{\frac{2}{5}}}.$

3.3.3 Notation and preliminaries

Let $\{0,1\}^* := \cup_{n=0}^{\infty} \{0,1\}^n$. A map $\phi : \{0,1\}^* \to \{0,1\}^*$ is called a *morphism* if for all words $u, v \in \{0,1\}^*$, we have

$$\phi(uv) = \phi(u)\phi(v).$$

Besides, we need the following concept.

Definition 3.3.12 ([18, 118]). *Let* $q \geq 2$ *be an integer. A sequence* $u = (u_n)_{n \geq 0} \in \mathbb{C}^{\mathbb{N}_0}$ *is*

called strongly q-multiplicative if $u_0 = 1$ and

$$u_{nq+k} = u_n u_k$$

for all $k \in \{0, 1, \cdots, q-1\}$ and $n \in \mathbb{N}_0$.

The following theorem is a classical result on the Gamma function Γ (see for examples [38, Theorem 1.1] and [123, Section 12.13]).

Theorem 3.3.13. *Let $d \in \mathbb{N}$ and $a_1, a_2, \cdots, a_d, b_1, b_2, \cdots, b_d \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$. If $a_1 + a_2 + \cdots + a_d = b_1 + b_2 + \cdots + b_d$, then*

$$\prod_{n=0}^{\infty} \frac{(n+a_1)(n+a_2)\cdots(n+a_d)}{(n+b_1)(n+b_2)\cdots(n+b_d)} = \frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_d)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_d)}.$$

Besides, we need the properties on the Gamma function gathered in the following proposition.

Proposition 3.3.14 ([19, 30, 122])**.**
(1) *For all $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \{0, -\frac{1}{n}, -\frac{2}{n}, -\frac{3}{n}, -\frac{4}{n}, \cdots\}$ we have*

$$\Gamma(z)\Gamma(z+\frac{1}{n})\Gamma(z+\frac{2}{n})\cdots\Gamma(z+\frac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-nz}\Gamma(nz).$$

(2) *For all $z \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$ we have*

$$\Gamma(z+1) = z\Gamma(z)$$

and

$$\Gamma(\frac{z}{2})\Gamma(\frac{z+1}{2}) = 2^{1-z}\sqrt{\pi}\ \Gamma(z).$$

(3) *For all $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

(4) *We have*

$$\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi} \quad \text{and} \quad \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}.$$

3.3.4 Proofs of the results

Let $q \geq 2$ be an integer, $\theta_0 = 0$, $\theta_1, \cdots, \theta_{q-1} \in \{0, 1\}$ and $(\theta_n)_{n\geq 0}$ be the $(0, \theta_1, \cdots, \theta_{q-1})$-Thue-Morse sequence. Recall that $(\delta_n)_{n\geq 0}$ is defined by $\delta_n = (-1)^{\theta_n}$ for all $n \in \mathbb{N}_0$. At

the same time $(\delta_n)_{n\geq 0}$ can be view as the unique fixed point of the morphism

$$\begin{aligned} +1 &\mapsto +1, +\delta_1, \cdots, +\delta_{q-1} \\ -1 &\mapsto -1, -\delta_1, \cdots, -\delta_{q-1} \end{aligned} \tag{3.29}$$

beginning with $\delta_0 = +1$. Define the sequence of partial sums of $(\delta_n)_{n\geq 0}$ by

$$\Delta_0 := 0 \quad \text{and} \quad \Delta_n := \delta_0 + \delta_1 + \cdots + \delta_{n-1} \quad \text{for all } n \geq 1.$$

Note that $(\Delta_n)_{n\geq 0}$ depends on the choice of $(\delta_1, \cdots, \delta_{q-1}) \in \{+1, -1\}^{q-1}$. Before proving Theorem 3.3.1, we need the following proposition on $(\Delta_n)_{n\geq 0}$, which is itself valuable.

Proposition 3.3.15. *Let $q \geq 2$ be an integer.*

(1) *For all $k, s \in \mathbb{N}_0$ and $t \in \{0, 1, \cdots, q^k - 1\}$ we have*

$$\delta_{sq^k+t} = \delta_s \delta_t \quad \textit{and} \quad \Delta_{sq^k+t} = \Delta_s \Delta_{q^k} + \delta_s \Delta_t.$$

(2) *With the convention $0^0 := 1$, for all $k \in \mathbb{N}_0$ we have $\Delta_{q^k} = \Delta_q^k$,*

$$\max\Big\{|\Delta_{q^k}| : (\delta_1, \cdots, \delta_{q-1}) \in \{+1, -1\}^{q-1} \setminus \{(+1)^{q-1}\}\Big\} = (q-2)^k,$$

$$\max\Big\{|\Delta_n| : 0 \leq n \leq q^k, (\delta_1, \cdots, \delta_{q-1}) \in \{+1, -1\}^{q-1} \setminus \{(+1)^{q-1}\}\Big\} = 1 + (q-2) + \cdots + (q-2)^k.$$

(3) *If $(\delta_1, \cdots, \delta_{q-1}) \neq (+1)^{q-1}$, then for all n large enough we have*

$$|\Delta_n| \leq n^{\log_q(q-1)}.$$

Proof. (1) ① Prove $\delta_{sq^k+t} = \delta_s \delta_t$ for all $k, s \in \mathbb{N}_0$ and $t \in \{0, 1, \cdots, q^k - 1\}$.

i) Prove that $(\delta_n)_{n\geq 0}$ is strongly q-multiplicative, i.e.,

$$\delta_{sq+t} = \delta_s \delta_t \quad \text{for all } s \in \mathbb{N}_0 \text{ and } t \in \{0, 1, \cdots, q-1\}.$$

Let ψ denote the morphism (3.29). Then by $\psi((\delta_0, \delta_1, \delta_2, \cdots)) = (\delta_0, \delta_1, \delta_2, \cdots)$ we get $\psi(\delta_s) = (\delta_{sq}, \delta_{sq+1}, \cdots, \delta_{(s+1)q-1})$ for all $s \in \mathbb{N}_0$. It follows from $\psi(+1) = (+1, +\delta_1, \cdots, +\delta_{q-1})$ and $\psi(-1) = (-1, -\delta_1, \cdots, -\delta_{q-1})$ that $\delta_{sq+t} = \delta_s \delta_t$ for all $t \in \{0, \cdots, q-1\}$.

ii) Let $k \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $t \in \{0, \cdots, q^k - 1\}$. Then there exist $l \in \mathbb{N}_0$ and $s_l, \cdots, s_1, s_0, t_{k-1}, \cdots, t_1, t_0 \in \{0, 1, \cdots, q-1\}$ such that

$$s = s_l q^l + \cdots + s_1 q + s_0 \quad \text{and} \quad t = t_{k-1} q^{k-1} + \cdots + t_1 q + t_0.$$

By i) and [18, Proposition 1] we get

$$\delta_{sq^k+t} = \delta_{s_l} \cdots \delta_{s_1}\delta_{s_0}\delta_{t_{k-1}} \cdots \delta_{t_1}\delta_{t_0},$$

$$\delta_s = \delta_{s_l} \cdots \delta_{s_1}\delta_{s_0} \quad \text{and} \quad \delta_t = \delta_{t_{k-1}} \cdots \delta_{t_1}\delta_{t_0}.$$

Thus $\delta_{sq^k+t} = \delta_s\delta_t$.

② Prove $\Delta_{sq^k+t} = \Delta_s\Delta_{q^k} + \delta_s\Delta_t$ for all $k, s \in \mathbb{N}_0$ and $t \in \{0, 1, \cdots, q^k - 1\}$. In fact, we have

$$\begin{aligned}
\Delta_{sq^k+t} &= (\delta_0 + \delta_1 + \cdots + \delta_{q^k-1}) + (\delta_{q^k} + \delta_{q^k+1} + \cdots + \delta_{q^k+(q^k-1)}) \\
&\quad + \cdots + (\delta_{(s-1)q^k} + \delta_{(s-1)q^k+1} + \cdots + \delta_{(s-1)q^k+(q^k-1)}) \\
&\quad + (\delta_{sq^k} + \delta_{sq^k+1} + \cdots + \delta_{sq^k+t-1}) \\
&= \delta_0(\delta_0 + \delta_1 + \cdots + \delta_{q^k-1}) + \delta_1(\delta_0 + \delta_1 + \cdots + \delta_{q^k-1}) \\
&\quad + \cdots + \delta_{s-1}(\delta_0 + \delta_1 + \cdots + \delta_{q^k-1}) \\
&\quad + \delta_s(\delta_0 + \delta_1 + \cdots + \delta_{t-1}) \\
&= \Delta_s\Delta_{q^k} + \delta_s\Delta_t
\end{aligned}$$

where the second equality follows from ①.
(2) ① We have $\Delta_{q^k} = \Delta_q^k$ for all $k \in \mathbb{N}_0$ since (1) ② implies $\Delta_{q\cdot q^l} = \Delta_q\Delta_{q^l}$ for all $l \in \mathbb{N}_0$.
② For all $k \in \mathbb{N}_0$, the fact

$$\max\left\{|\Delta_{q^k}| : (\delta_1, \cdots, \delta_{q-1}) \in \{+1, -1\}^{q-1} \setminus \{(+1)^{q-1}\}\right\} = (q-2)^k$$

follows from ① and

$$\max\left\{|\Delta_q| : (\delta_1, \cdots, \delta_{q-1}) \in \{+1, -1\}^{q-1} \setminus \{(+1)^{q-1}\}\right\} = q - 2.$$

③ In order to prove the last equality in statement (2), since the case $k = 0$ is trivial and ② implies $|\Delta_{q^k}| \le 1 + (q-2) + (q-2)^2 + \cdots + (q-2)^k$, it suffices to verify that for all $k \in \mathbb{N}$, we have

$$\max\left\{|\Delta_n| : 0 \le n \le q^k - 1, (\delta_1, \cdots, \delta_{q-1}) \in \{+1, -1\}^{q-1} \setminus \{(+1)^{q-1}\}\right\} \\ = 1 + (q-2) + (q-2)^2 + \cdots + (q-2)^k.$$

$\boxed{\le}$ (By induction on k) For $k = 1$, obviously we have $|\Delta_0|, |\Delta_1|, \cdots, |\Delta_{q-1}| \le q - 1$. Suppose that for some $k \in \mathbb{N}$ and all $l \in \{0, 1, \cdots, k\}$, we have already had

$$|\Delta_0|, |\Delta_1|, \cdots, |\Delta_{q^l-1}| \le 1 + (q-2) + (q-2)^2 + \cdots + (q-2)^l.$$

Let $n \in \{0, 1, \cdots, q^{k+1} - 1\}$. It suffices to prove

$$|\Delta_n| \leq 1 + (q-2) + (q-2)^2 + \cdots + (q-2)^{k+1}. \tag{3.30}$$

If $n \leq q^k - 1$, this follows immediately from the inductive hypothesis. We only need to consider $q^k \leq n \leq q^{k+1} - 1$ in the following. Let $s \in \{1, \cdots, q-1\}$ and $t \in \{0, 1, \cdots, q^k - 1\}$ be such that $n = sq^k + t$. By (1) ② we get

$$\Delta_n = \Delta_s \Delta_{q^k} + \delta_s \Delta_t.$$

If $s \leq q-2$, then

$$\begin{aligned} |\Delta_n| &\leq |\Delta_s| \cdot |\Delta_{q^k}| + |\Delta_t| \\ &\leq s(q-2)^k + (1 + (q-2) + (q-2)^2 + \cdots + (q-2)^k) \\ &\leq 1 + (q-2) + (q-2)^2 + \cdots + (q-2)^{k+1} \end{aligned}$$

where the second inequality follows from ② and the inductive hypothesis. In the following we only need to consider $s = q-1$. It means that

$$\Delta_n = \Delta_{q-1} \Delta_{q^k} + \delta_{q-1} \Delta_t.$$

If there exists $p \in \{0, 1, \cdots, q-2\}$ such that $\delta_p = -1$, then $|\Delta_{q-1}| \leq q-3$ and

$$\begin{aligned} |\Delta_n| &\leq (q-3)|\Delta_{q^k}| + |\Delta_t| \\ &\leq (q-3)(q-2)^k + (1 + (q-2) + (q-2)^2 + \cdots + (q-2)^k) \\ &\leq 1 + (q-2) + (q-2)^2 + \cdots + (q-2)^{k+1} \end{aligned}$$

where the second inequality follows from ② and the inductive hypothesis. Thus it suffices to consider $\delta_0 = \delta_1 = \cdots = \delta_{q-2} = +1$ in the following. By $(\delta_1, \cdots, \delta_{q-1}) \neq (+1)^{q-1}$ we get $\delta_{q-1} = -1$. It follows from $\Delta_{q-1} = q-1$ and $\Delta_{q^k} = \Delta_q^k = (q-2)^k$ that

$$\Delta_n = (q-1)(q-2)^k - \Delta_t.$$

Thus proving (3.30) is equivalent to verifying

$$-1 - (q-2) - \cdots - (q-2)^{k-1} \leq \Delta_t \leq 1 + (q-2) + \cdots + (q-2)^{k+1} + (q-1)(q-2)^k.$$

Since the second inequality follows immediately from the inductive hypothesis, we only need to prove the first inequality. Let $u \in \{0, 1, \cdots, q-1\}$ and $v \in \{0, 1, \cdots, q^{k-1} -$

1} be such that $t = uq^{k-1} + v$. By (1) ② we get

$$\Delta_t = \Delta_u \Delta_{q^{k-1}} + \delta_u \Delta_v.$$

Since $\delta_0 = \delta_1 = \cdots = \delta_{q-2} = +1$, $\delta_{q-1} = -1$ and $0 \le u \le q-1$ imply $\Delta_u = u \ge 0$, $\Delta_q = q-2$ and $\Delta_{q^{k-1}} = \Delta_q^{k-1} = (q-2)^{k-1} \ge 0$, by $\delta_u \in \{+1, -1\}$ we get

$$\Delta_t \ge -|\Delta_v| \ge -1 - (q-2) - \cdots - (q-2)^{k-1}$$

where the last inequality follows from the inductive hypothesis.

$\boxed{\ge}$ Let $\delta_1 = \delta_2 = \cdots = \delta_{q-2} = +1$ and $\delta_{q-1} = -1$. It suffices to prove that for all $k \in \mathbb{N}$ we have

$$\Delta_{q^k - q^{k-1} - \cdots - q - 1} = (q-2)^k + \cdots + (q-2)^2 + (q-2) + 1. \tag{3.31}$$

(By induction) For $k = 1$ we have $\Delta_{q-1} = q-1$. Suppose that (3.31) is true for some $k \in \mathbb{N}$. Then for $k+1$, we have

$$\begin{aligned}
\Delta_{q^{k+1} - q^k - q^{k-1} - \cdots - q - 1} &= \Delta_{(q-2)q^k + (q^k - q^{k-1} - \cdots - q - 1)} \\
&= \Delta_{q-2}\Delta_{q^k} + \delta_{q-2}\Delta_{q^k - q^{k-1} - \cdots - q - 1} \\
&= (q-2)\Delta_q^k + (q-2)^k + \cdots + (q-2)^2 + (q-2) + 1 \\
&= (q-2)^{k+1} + (q-2)^k + \cdots + (q-2)^2 + (q-2) + 1
\end{aligned}$$

where the second equality follows from (1) ② and the third equality follows from ① and the inductive hypothesis.

(3) For $n \in \mathbb{N}$ large enough, there exists $k \in \mathbb{N}$ large enough such that $q^k + 1 \le n \le q^{k+1}$. By (2) ③ we get

$$|\Delta_n| \le 1 + (q-2) + \cdots + (q-2)^{k+1} \le (q-1)^k = (q^k)^{\log_q(q-1)} \le n^{\log_q(q-1)}$$

where the second inequality can be verified straightforwardly for k large enough. □

Proof of Theorem 3.3.1. Since (2) follows in the same way as in the proof of [13, Lemma 4.1] by applying (1), we only need to prove (1) in the following.

$\boxed{\Rightarrow}$ Suppose that $\prod_{n=1}^{\infty}(R(n))^{\delta_n}$ converges. Then $(R(n))^{\delta_n} \to 1$ as $n \to \infty$. Since $\delta_n \in \{+1, -1\}$ for all $n \in \mathbb{N}$, we get $R(n) \to 1$ as $n \to \infty$. Thus the numerator and the denominator of R have the same degree and the same leading coefficient.

$\boxed{\Leftarrow}$ Suppose that the numerator and the denominator of R have the same leading coefficient and the same degree. Decompose them into factors of degree 1. To prove that $\prod_{n=1}^{\infty}(R(n))^{\delta_n}$ converges, it suffices to show that $\prod_{n=1}^{\infty}(\frac{n+a}{n+b})^{\delta_n}$ converges for all $a, b \in \mathbb{C}$ satisfying $n + a \ne 0$ and $n + b \ne 0$ for all $n \in \mathbb{N}$ (that is, $a, b \in \mathbb{C} \setminus \{-1, -2, -3, \cdots\}$).

Since $(\frac{n+a}{n+b})^{\delta_n} \to 1$ as $n \to \infty$, we only need to prove that

$$\prod_{n=1}^{\infty} \Big((\frac{qn+a}{qn+b})^{\delta_{qn}} (\frac{qn+1+a}{qn+1+b})^{\delta_{qn+1}} \cdots (\frac{qn+q-1+a}{qn+q-1+b})^{\delta_{qn+q-1}} \Big)$$

converges. Since Proposition 3.3.15 (1) implies $\delta_{qn} = \delta_n\delta_0$, $\delta_{qn+1} = \delta_n\delta_1$, $\cdots$, $\delta_{qn+q-1} = \delta_n\delta_{q-1}$, it suffices to show that

$$\prod_{n=1}^{\infty} \big(r(n)\big)^{\delta_n}$$

converges, where

$$r(n) := (\frac{qn+a}{qn+b})^{\delta_0} (\frac{qn+1+a}{qn+1+b})^{\delta_1} \cdots (\frac{qn+q-1+a}{qn+q-1+b})^{\delta_{q-1}}.$$

This is equivalent to showing that

$$\sum_{n=1}^{\infty} \delta_n \ln r(n) \tag{3.32}$$

converges. Since there exist $c_0, c_1, \cdots, c_{q-1}, d_0, d_1, \cdots, d_{q-1} \in \mathbb{C}$ such that

$$r(n) = \frac{q^q n^q + c_{q-1}n^{q-1} + \cdots + c_1 n + c_0}{q^q n^q + d_{q-1}n^{q-1} + \cdots + d_1 n + d_0} = 1 + \frac{(c_{q-1}-d_{q-1})n^{q-1} + \cdots + (c_1 - d_1)n + (c_0 - d_0)}{q^q n^q + d_{q-1}n^{q-1} + \cdots + d_1 n + d_0},$$

we get

$$\ln r(n) - \frac{c_{q-1} - d_{q-1}}{q^q n} = \mathcal{O}(\frac{1}{n^2}),$$

which implies that

$$\sum_{n=1}^{\infty} \delta_n \big(\ln r(n) - \frac{c_{q-1} - d_{q-1}}{q^q n} \big)$$

converges absolutely. In order to prove that (3.32) converges, we only need to show that

$$\sum_{n=1}^{\infty} \frac{\delta_n}{n}$$

converges. Enlightened by partial summation (see for example the equality (6.5) in [28] related to the Thue-Morse sequence), we consider the following ① and ②, which complete the proof.

① Prove that

$$\sum_{n=1}^{\infty} \frac{\delta_1 + \cdots + \delta_n}{n(n+1)}$$

converges. In fact, since Proposition 3.3.15 (3) implies

$$\frac{|\Delta_n|}{n^2} \leq \frac{1}{n^{2-\log_q(q-1)}} \quad \text{for all } n \text{ large enough,}$$

where $2 - \log_q(q-1) > 1$, it follows that $\sum_{n=1}^{\infty} \frac{\Delta_n}{n^2}$ converges absolutely. So does $\sum_{n=1}^{\infty} \frac{\Delta_n}{n(n+1)}$. Thus we only need to check that $\sum_{n=1}^{\infty}(\frac{\delta_1+\cdots+\delta_n}{n(n+1)} - \frac{\Delta_n}{n(n+1)})$ converges. This follows immediately from $|\delta_1 + \cdots + \delta_n - \Delta_n| = |\delta_n - \delta_0| \leq 2$.

② Prove that

$$\sum_{n=1}^{\infty}(\frac{\delta_n}{n} - \frac{\delta_1 + \cdots + \delta_n}{n(n+1)})$$

converges to 0. In fact, for all $N \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{n=1}^{N} \frac{\delta_1 + \cdots + \delta_n}{n(n+1)} &= \sum_{n=1}^{N}(\delta_1 + \cdots + \delta_n)(\frac{1}{n} - \frac{1}{n+1}) \\
&= \delta_1 \sum_{n=1}^{N}(\frac{1}{n} - \frac{1}{n+1}) + \delta_2 \sum_{n=2}^{N}(\frac{1}{n} - \frac{1}{n+1}) + \cdots + \delta_N \sum_{n=N}^{N}(\frac{1}{n} - \frac{1}{n+1}) \\
&= \delta_1(1 - \frac{1}{N+1}) + \delta_2(\frac{1}{2} - \frac{1}{N+1}) + \cdots + \delta_N(\frac{1}{N} - \frac{1}{N+1}) \\
&= \sum_{n=1}^{N} \frac{\delta_n}{n} - \frac{\delta_1 + \delta_2 + \cdots + \delta_N}{N+1},
\end{aligned}$$

which implies

$$\sum_{n=1}^{N}(\frac{\delta_n}{n} - \frac{\delta_1 + \cdots + \delta_n}{n(n+1)}) = \frac{\delta_1 + \delta_2 + \cdots + \delta_N}{N+1} = \frac{\Delta_{N+1} - 1}{N+1}.$$

Since Proposition 3.3.15 (3) implies

$$\frac{|\Delta_{N+1}|}{N+1} \leq \frac{1}{(N+1)^{1-\log_q(q-1)}} \quad \text{for all } N \text{ large enough,}$$

where $1 - \log_q(q-1) > 0$, as $N \to \infty$ we get $\frac{\Delta_{N+1}}{N+1} \to 0$ and then $\sum_{n=1}^{N}(\frac{\delta_n}{n} - \frac{\delta_1+\cdots+\delta_n}{n(n+1)}) \to 0$.

□

Proof of Theorem 3.3.2. Since Proposition 3.3.15 (1) implies $\delta_{qn} = \delta_n\delta_0$, $\delta_{qn+1} = \delta_n\delta_1$,

$\cdots$, $\delta_{qn+q-1} = \delta_n\delta_{q-1}$ for all $n \in \mathbb{N}_0$, we get $f(a,b)$

$$
\begin{aligned}
&= \prod_{n=1}^{\infty}\Big(\frac{qn+a}{qn+b}\Big)^{\delta_{qn}} \prod_{n=0}^{\infty}\Big(\frac{qn+1+a}{qn+1+b}\Big)^{\delta_{qn+1}} \cdots \prod_{n=0}^{\infty}\Big(\frac{qn+q-1+a}{qn+q-1+b}\Big)^{\delta_{qn+q-1}} \\
&= \prod_{n=1}^{\infty}\Big(\frac{qn+a}{qn+b}\Big)^{\delta_n\delta_0} \prod_{n=0}^{\infty}\Big(\frac{qn+a+1}{qn+b+1}\Big)^{\delta_n\delta_1} \cdots \prod_{n=0}^{\infty}\Big(\frac{qn+a+q-1}{qn+b+q-1}\Big)^{\delta_n\delta_{q-1}} \\
&= \Big(\frac{a+1}{b+1}\Big)^{\delta_0\delta_1} \cdots \Big(\frac{a+q-1}{b+q-1}\Big)^{\delta_0\delta_{q-1}} \prod_{n=1}^{\infty}\Big(\frac{n+\frac{a}{q}}{n+\frac{b}{q}}\Big)^{\delta_n\delta_0} \prod_{n=1}^{\infty}\Big(\frac{n+\frac{a+1}{q}}{n+\frac{b+1}{q}}\Big)^{\delta_n\delta_1} \cdots \prod_{n=1}^{\infty}\Big(\frac{n+\frac{a+q-1}{q}}{n+\frac{b+q-1}{q}}\Big)^{\delta_n\delta_{q-1}} \\
&= \Big(\frac{a+1}{b+1}\Big)^{\delta_1} \cdots \Big(\frac{a+q-1}{b+q-1}\Big)^{\delta_{q-1}} f(\frac{a}{q},\frac{b}{q})\Big(f(\frac{a+1}{q},\frac{b+1}{q})\Big)^{\delta_1} \cdots \Big(f(\frac{a+q-1}{q},\frac{b+q-1}{q})\Big)^{\delta_{q-1}}.
\end{aligned}
$$

□

Proof of Corollary 3.3.3. (1) follows from Theorem 3.3.2 after multiplying by the factor corresponding to $n = 0$. The first equality in (2) follows from taking $b = a + 1$ in (1). The second equality in (2) follows from taking $b = 0$ in Theorem 3.3.2 and then multiplying the factor corresponding to $n = 0$. We should note that it does not follow from taking $b = 0$ in (1). Finally (3) follows immediately from taking $a = 1$ in the second equality in (2). □

Proof of Corollary 3.3.5. These two equalities follow from Corollary 3.3.3 (3) and the first equality in (2) of Corollary 3.3.3 respectively. □

Proof of Example 3.3.6. (1) follows from Corollary 3.3.3 (1).

(2) ① and ② follow from Corollary 3.3.3 (2).

③ follows from ② and the fact that the first equality in Corollary 3.3.5 implies

$$\prod_{n=0}^{\infty}\Big(\frac{3n+2}{3n+3}\Big)^{\delta_n} = \frac{1}{\sqrt{3}}. \tag{3.33}$$

④ follows from taking $b = \frac{1}{2}$ in (1).

(3) ① is the above equality (3.33).

②, ⑤, ⑨ and ⑬ follow from taking $a = -\frac{1}{2}, \frac{3}{2}, 1$ and 2 respectively in (2) ①.

③, ⑥, ⑧ and ⑯ follow from taking $a = \frac{1}{2}, \frac{2}{3}, \frac{3}{2}$ and $-\frac{1}{2}$ respectively in (2) ②.

④ follows from multiplying ③ and ①.

⑦ follows from taking $a = \frac{5}{6}$ in (2) ④.

⑩, ⑪, ⑫ and ⑭ follow respectively from ⑨, ⑩, ⑪ and ⑬ by applying ①.

⑮ follows from taking $a = 2$ and $b = \frac{4}{3}$ in (1).

□

Proof of Example 3.3.7. (1) follows from Corollary 3.3.3 (1).

(2) ① and ② follow from Corollary 3.3.3 (2).

③ follows from ② and the fact that the first equality in Corollary 3.3.5 implies

$$\prod_{n=0}^{\infty}\Big(\frac{3n+1}{3n+3}\Big)^{\delta_n}=\frac{1}{\sqrt{3}}.$$

(3) ① is the above equality.

② follows from taking $a=2$ in (2) ③.

③ and ⑧ follow from taking $a=\frac{3}{2}$ and 2 respectively in (2) ①.

④, ⑤, ⑪ and ⑯ follow from taking $a=\frac{1}{3},\frac{2}{3},3$ and $\frac{3}{2}$ respectively in (2) ②.

⑥, ⑨ and ⑩ follow respectively from ⑤, ⑧ and ⑨ by applying ①.

⑦ follows from taking $a=\frac{5}{8}$ and $b=\frac{7}{8}$ in (1).

⑫ and ⑬ follow respectively from multiplying and dividing ⑪ by ①.

⑭ follows from combining the results of taking $a=\frac{1}{2}$ and $-\frac{1}{2}$ in (2) ②.

⑮ follows from taking $a=1, b=\frac{1}{2}$ in (1) and then multiplying by ①.

□

Proof of Example 3.3.8. For odd $q\geq 3$, let $\theta_1=\theta_3=\cdots=\theta_{q-2}=1$ and $\theta_2=\theta_4=\cdots=\theta_{q-1}=0$. Then the $(0,\theta_1,\cdots,\theta_{q-1})$-Thue-Morse sequence $(\theta_n)_{n\geq 0}$ is exactly $(01)^\infty$. It follows that $\delta_n:=(-1)^{\theta_n}=(-1)^n$ for all $n\geq 0$.
(1) By the second equality in Corollary 3.3.3 (2) we get

$$\prod_{n=0}^{\infty}\Big(\frac{(qn+qa)(qn+a+1)(qn+2)(qn+a+3)(qn+4)\cdots(qn+a+q-2)(qn+q-1)}{(qn+a)(qn+1)(qn+a+2)(qn+3)(qn+a+4)\cdots(qn+q-2)(qn+a+q-1)}\Big)^{(-1)^n}=q \tag{3.34}$$

for all $a\in\mathbb{C}\setminus\{0,-1,-2,\cdots\}$. Then we conclude (1) by taking $a=1$ in (3.34).
(2) follows from (3.34) and (1).
(3) Note that for all $q\in\mathbb{N}$ and $a\in\mathbb{C}\setminus\{0,-1,-2,\cdots\}$ we have

$$\prod_{n=0}^{\infty}\Big(\frac{(qn+a)(qn+a+q)}{(qn+qa)(qn+qa+q)}\Big)^{(-1)^n}=\frac{1}{q} \tag{3.35}$$

since the left hand side is

$$\lim_{k\to\infty}\frac{a}{qa}\cdot\frac{a+q}{qa+q}\cdot\Big(\frac{a+q}{qa+q}\cdot\frac{a+2q}{qa+2q}\Big)^{-1}\cdot\frac{a+2q}{qa+2q}\cdot\frac{a+3q}{qa+3q}\cdots\Big(\frac{a+kq}{qa+kq}\cdot\frac{a+(k+1)q}{qa+(k+1)q}\Big)^{(-1)^k}$$

$$= \lim_{k\to\infty} \frac{a}{qa} \cdot \Big(\frac{a+(k+1)q}{qa+(k+1)q}\Big)^{(-1)^k} = \frac{1}{q}.$$

We prove the concrete equalities in the following.

① and ⑬ follow from taking $q = 3$ and 5 respectively in (1).

②, ③, ⑤ and ⑦ follow from taking $q = 3$, and then $a = 3, 2, \frac{2}{3}$ and $\frac{1}{3}$ respectively in (2).

④, ⑨, ⑩ and ⑫ are deduced by applying ① noting that (3.35) with $q = 3$ and then $a = 2, \frac{2}{3}, \frac{1}{3}$ and $-\frac{1}{3}$ give respectively

$$\prod_{n=0}^{\infty} \Big(\frac{(3n+2)(3n+5)}{(3n+6)(3n+9)}\Big)^{(-1)^n} = \frac{1}{3}, \quad \prod_{n=0}^{\infty} \Big(\frac{(9n+2)(9n+11)}{(9n+6)(9n+15)}\Big)^{(-1)^n} = \frac{1}{3},$$

$$\prod_{n=0}^{\infty} \Big(\frac{(9n+1)(9n+10)}{(9n+3)(9n+12)}\Big)^{(-1)^n} = \frac{1}{3} \quad \text{and} \quad \prod_{n=0}^{\infty} \Big(\frac{(9n-1)(9n+8)}{(9n-3)(9n+6)}\Big)^{(-1)^n} = \frac{1}{3}.$$

⑥, ⑧, ⑪ and ⑯ follow respectively from dividing ⑤ by ①, multiplying ⑦ by ①, dividing ⑨ by ⑥ and dividing ⑮ by ⑬.

⑭ and ⑮ follow from taking $q = 5$, and then $a = \frac{1}{2}$ and 3 respectively in (2). □

Before proving Theorem 3.3.9, we need the following proposition.

Proposition 3.3.16. *Let $q \geq 2$ be an integer, $\theta_0 = 0$, $(\theta_1, \cdots, \theta_{q-1}) \in \{0,1\}^{q-1} \setminus \{0^{q-1}\}$ and $(\theta_n)_{n\geq 0}$ be the $(0, \theta_1, \cdots, \theta_{q-1})$-Thue-Morse sequence. Then for all $n \in \mathbb{N}_0$ and $k \in \{0, 1, \cdots, q-1\}$ we have*

$$\theta_{nq+k} = \theta_n(-1)^{\theta_k} + \theta_k.$$

Proof. Let h denote the morphism

$$\begin{aligned} 0 &\mapsto 0\theta_1 \cdots \theta_{q-1} \\ 1 &\mapsto 1\overline{\theta_1} \cdots \overline{\theta_{q-1}} \end{aligned}$$

where $\overline{0} := 1$ and $\overline{1} := 0$. By $h(\theta_0\theta_1\theta_2\cdots) = \theta_0\theta_1\theta_2\cdots$ we get

$$h(\theta_n) = \theta_{nq}\theta_{nq+1}\cdots\theta_{nq+q-1}$$

for all $n \in \mathbb{N}_0$. It follows from $h(0) = \theta_0\theta_1\cdots\theta_{q-1}$ and $h(1) = \overline{\theta_0\theta_1}\cdots\overline{\theta_{q-1}}$ that

$$\theta_{nq+k} = \begin{cases} \theta_k & \text{if } \theta_n = 0 \\ \overline{\theta_k} & \text{if } \theta_n = 1 \end{cases} = \theta_n(-1)^{\theta_k} + \theta_k \quad \text{for all } k \in \{0, 1, \cdots, q-1\}.$$

□

Proof of Theorem 3.3.9. We have $\mathfrak{f}(a_1,\cdots,a_d;b_1,\cdots,b_d)$

$$
\begin{aligned}
&= \prod_{n=1}^{\infty}\Big(\prod_{i=1}^{d}\frac{n+a_i}{n+b_i}\Big)^{\theta_n}\\
&= \prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{k+a_i}{k+b_i}\Big)^{\theta_k}\cdot\prod_{n=1}^{\infty}\prod_{k=0}^{q-1}\Big(\prod_{i=1}^{d}\frac{nq+k+a_i}{nq+k+b_i}\Big)^{\theta_{nq+k}}\\
&\overset{(\star)}{=} \prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{a_i+k}{b_i+k}\Big)^{\theta_k}\cdot\prod_{n=1}^{\infty}\prod_{k=0}^{q-1}\Big(\prod_{i=1}^{d}\frac{qn+a_i+k}{qn+b_i+k}\Big)^{\theta_n(-1)^{\theta_k}+\theta_k}\\
&= \prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{a_i+k}{b_i+k}\Big)^{\theta_k}\cdot\prod_{n=1}^{\infty}\prod_{k=0}^{q-1}\Big(\prod_{i=1}^{d}\frac{qn+a_i+k}{qn+b_i+k}\Big)^{\theta_k}\cdot\prod_{n=1}^{\infty}\prod_{k=0}^{q-1}\Big(\prod_{i=1}^{d}\frac{qn+a_i+k}{qn+b_i+k}\Big)^{\theta_n(-1)^{\theta_k}}\\
&\overset{(\star\star)}{=} \prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{a_i+k}{b_i+k}\Big)^{\theta_k}\cdot\prod_{n=1}^{\infty}\prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{qn+a_i+k}{qn+b_i+k}\Big)^{\theta_k}\cdot\prod_{k=0}^{q-1}\prod_{n=1}^{\infty}\Big(\prod_{i=1}^{d}\frac{qn+a_i+k}{qn+b_i+k}\Big)^{\theta_n(-1)^{\theta_k}}\\
&= \prod_{n=0}^{\infty}\prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{qn+a_i+k}{qn+b_i+k}\Big)^{\theta_k}\cdot\prod_{k=0}^{q-1}\Big(\prod_{n=1}^{\infty}\Big(\prod_{i=1}^{d}\frac{qn+a_i+k}{qn+b_i+k}\Big)^{\theta_n}\Big)^{(-1)^{\theta_k}}\\
&= \prod_{k=1}^{q-1}\Big(\prod_{n=0}^{\infty}\prod_{i=1}^{d}\frac{n+\frac{a_i+k}{q}}{n+\frac{b_i+k}{q}}\Big)^{\theta_k}\cdot\prod_{k=0}^{q-1}\Big(\prod_{n=1}^{\infty}\Big(\prod_{i=1}^{d}\frac{n+\frac{a_i+k}{q}}{n+\frac{b_i+k}{q}}\Big)^{\theta_n}\Big)^{(-1)^{\theta_k}}\\
&\overset{(\star\star\star)}{=} \prod_{k=1}^{q-1}\Big(\prod_{i=1}^{d}\frac{\Gamma(\frac{b_i+k}{q})}{\Gamma(\frac{a_i+k}{q})}\Big)^{\theta_k}\cdot\prod_{k=0}^{q-1}\Big(\mathfrak{f}\big(\frac{a_1+k}{q},\cdots,\frac{a_d+k}{q};\frac{b_1+k}{q},\cdots,\frac{b_d+k}{q}\big)\Big)^{(-1)^{\theta_k}},
\end{aligned}
$$

where $(\star)$, $(\star\star)$ and $(\star\star\star)$ follow from Proposition 3.3.16, $\theta_0=0$ and Theorem 3.3.13 respectively. □

Proof of Corollary 3.3.10. (1) follows from taking $d=2$, $a_1=a$, $a_2=b+c$, $b_1=b$ and $b_2=a+c$ in Theorem 3.3.9.
(2) follows from taking $b_1=\cdots=b_d=0$ in Theorem 3.3.9.
(3) follows from taking $d=2$, $a_1=a$ and $a_2=-a$ in (2). □

Proof of Corollary 3.3.11. In the following proof, for calculations related to the Gamma function, we use Proposition 3.3.14 frequently without invoking it explicitly. (1) and (2) follow from Theorem 3.3.9 and Corollary 3.3.10 (1) respectively.

(3) ① follows from taking $b=0$ in (2) and then replacing all c by b.

② follows from taking $c=a-1$ in (2) and then replacing all a by $a+1$.

(4) ① follows from multiplying (3) ① and ②.

② follows from taking $b=2$ in (3) ②.

③ and ④ follow from taking $b=-a$ and $1-2a$ respectively in (3) ①.

⑤ follows from taking $d = 2, a_1 = \frac{1+a}{2}, a_2 = \frac{1-a}{2}, b_1 = 0$ and $b_2 = 1$ in (1).

⑥ follows from taking $a_1 = \cdots = a_d = 1, b_1 = d$ and $b_2 = \cdots = b_d = 0$ in (1).

(5) ① follows from taking $a = \frac{1}{2}$ in (4) ③.

② and ⑥ follow from taking $a = 0$ and $\frac{1}{3}$ respectively in (4) ⑤.

③, ⑤, ⑦, ⑪ and ⑫ follow from taking $a = \frac{1}{2}$, 1, 2, $-\frac{2}{3}$ and $-\frac{1}{4}$ respectively in (4) ①.

④, ⑮ and ⑯ follow from taking $a = \frac{1}{4}$, $\frac{2}{3}$ and $\frac{2}{5}$ respectively in (4) ④.

⑧, ⑨ and ⑩ follow respectively from multiplying ① by ②, multiplying ⑤ by ⑧ and dividing ② by ①.

⑬ and ⑭ follow from taking $a = \frac{3}{2}$ and $\frac{1}{3}$ respectively in (4) ②.

□

3.4 Generalized Koch curves and Thue-Morse sequences

Recall that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of positive integers $1, 2, 3, \cdots$, non-negative integers $0, 1, 2, \cdots$, real numbers and complex numbers respectively. Denote the base of the natural logarithm by e and the imaginary unit by i as usual. We still use $(t_n)_{n\geq 0}$ to denote the classical Thue-Morse sequence $0110100110010110\cdots$ in this section. It is well known that $t_n \equiv s(n) \bmod 2$ for all $n \in \mathbb{N}_0$ where $s(n)$ denotes the sum of binary digits of n. In the 1983 paper [40], Coquet interested in the behavior of the sum $\sum_{k<n}(-1)^{s(3k)}$, introduced $\sum_{k<n}(-1)^{t_k}e^{\frac{2k\pi i}{3}}$ and obtained the Koch curve [119] as a by-product in [40, Page 111]. In addition, Dekking found in [49, Pages 32-05 and 32-06] that the points

$$p(0) := 0, \quad p(n) := \sum_{k=0}^{n-1}(-1)^{t_k}e^{\frac{2k\pi i}{3}} \quad (n = 1, 2, 3, \cdots)$$

traverse the unscaled Koch curve on the complex plane (see also [50, Page 107] and [72, Page 304]). For more on the relation between the Koch curve and the classical Thue-Morse sequence, we refer the reader to [17, 94, 128].

Given any $m \in \mathbb{N}$ and $\theta_1, \cdots, \theta_m \in \{0, 1\}$, recall from the last section that the $(0, \theta_1, \cdots, \theta_m)$-Thue-Morse sequence $(\theta_n)_{n\geq 0}$ is the unique fixed point of the morphism

$$\begin{aligned} 0 &\mapsto 0\theta_1\cdots\theta_m \\ 1 &\mapsto 1\overline{\theta_1}\cdots\overline{\theta_m} \end{aligned}$$

beginning with $\theta_0 := 0$, where $\overline{0} := 1$ and $\overline{1} := 0$. Define $\delta_n := (-1)^{\theta_n}$ for all $n \in \mathbb{N}_0$. Then $(\delta_n)_{n\geq 0}$ is the unique fixed point of the morphism

$$\begin{aligned} +1 &\mapsto +1, +\delta_1, \cdots, +\delta_m \\ -1 &\mapsto -1, -\delta_1, \cdots, -\delta_m \end{aligned}$$

beginning with $\delta_0 = +1$ and $\delta_1, \cdots, \delta_m \in \{+1, -1\}$. We call $\delta = (\delta_n)_{n\geq 0}$ the $(+1, \delta_1, \cdots, \delta_m)$-*Thue-Morse sequence*. Let

$$p_{m,\delta}(0) := 0 \quad \text{and} \quad p_{m,\delta}(n) := \sum_{k=0}^{n-1} \delta_k e^{\frac{2k\pi i}{m}} \quad \text{for } n = 1, 2, 3, \cdots.$$

Noting that the classical ± 1 Thue-Morse sequence $((-1)^{t_n})_{n\geq 0}$ is not only the $(+1, -1)$ but also the $(+1, -1, -1, +1)$-Thue-Morse sequence in our terms, the above $p_{m,\delta}$ depends not only on δ but also on m. For $n \in \mathbb{N}_0$, let

$$P_{m,\delta}(n) := \bigcup_{k=1}^{(m+1)^n} [p_{m,\delta}(k-1), p_{m,\delta}(k)]$$

be the polygonal line connecting the points $p_{m,\delta}(0), p_{m,\delta}(1), \cdots, p_{m,\delta}((m+1)^n)$ one by one, where $[z_1, z_2] := \{cz_1 + (1-c)z_2 : c \in [0,1]\}$ is the segment connecting z_1 and z_2 on the complex plane $\mathbb{C}$. In addition, if $p_{m,\delta}(m+1) \neq 0$, for all $j \in \{0, 1, \cdots, m\}$, we define $S_{m,\delta,j} : \mathbb{C} \to \mathbb{C}$ by

$$S_{m,\delta,j}(z) := \frac{p_{m,\delta}(j) + \delta_j e^{\frac{2j\pi i}{m}} z}{p_{m,\delta}(m+1)} \quad \text{for } z \in \mathbb{C}.$$

When $|p_{m,\delta}(m+1)| > 1$, obviously $S_{m,\delta,0}, S_{m,\delta,1}, \cdots, S_{m,\delta,m}$ are all contracting similarities, and we call $\{S_{m,\delta,j}\}_{0\leq j\leq m}$ the $(+1, \delta_1, \cdots, \delta_m)$-*IFS* (*iterated function system*). We can see that the attractor of the $(+1, -1, -1, +1)$-IFS is exactly the Koch curve.

For simplification, if m and the $(+1, \delta_1, \cdots, \delta_m)$-Thue-Morse sequence δ are understood from the context, we use p, P and S_j instead of $p_{m,\delta}$, $P_{m,\delta}$ and $S_{m,\delta,j}$ respectively.

Let d_H be the Hausdorff metric and write $cZ := \{cz : z \in Z\}$ for any $c \in \mathbb{C}$ and $Z \subset \mathbb{C}$. The following is our main result.

Theorem 3.4.1. *Let $m \in \mathbb{N}$, $\delta_0 = +1$, $\delta_1, \cdots, \delta_m \in \{+1, -1\}$ and $\delta = (\delta_n)_{n\geq 0}$ be the $(+1, \delta_1, \cdots, \delta_m)$-Thue-Morse sequence. If $|p(m+1)| > 1$, then there exists a unique compact set $K \subset \mathbb{C}$ such that*

$$(p(m+1))^{-n} P(n) \xrightarrow{d_H} K \quad \text{as } n \to \infty,$$

and K is a continuous image of $[0,1]$. Moreover, K is the unique attractor of the $(+1, \delta_1,$

$\cdots, \delta_m)$-*IFS* $\{S_j\}_{0\le j\le m}$. *That is,* K *is the unique non-empty compact set such that*

$$K = \bigcup_{j=0}^{m} S_j(K).$$

Furthermore,

$$\dim_H K = \frac{\log(m+1)}{\log|p(m+1)|}$$

if and only if there exists $\varepsilon > 0$ *such that*

$$\varliminf_{n\to\infty} \frac{\mathcal{L}((P(n))^{\varepsilon})}{(m+1)^n} > 0,$$

where $\mathcal{L}$ *is the Lebesgue measure on the plane and* $A^{\varepsilon} := \{z \in \mathbb{C} : |z-a| < \varepsilon$ *for some* $a \in A\}$ *for* $A \subset \mathbb{C}$.

We call K in Theorem 3.4.1 the $(+1, \delta_1, \cdots, \delta_m)$-*Koch curve.* See the figures in the next two pages for some examples for $m = 3$ and 4. Note that the classical Koch curve is exactly the $(+1, -1, -1, +1)$-Koch curve in our terms.

It is well known that the classical Koch curve has Hausdorff, packing and box dimension $\log 4/\log 3$ since the corresponding IFS satisfies the open set condition (OSC). As a generalization, we have the following, where we recall that $\lfloor x \rfloor$ denotes the greatest integer no larger than x.

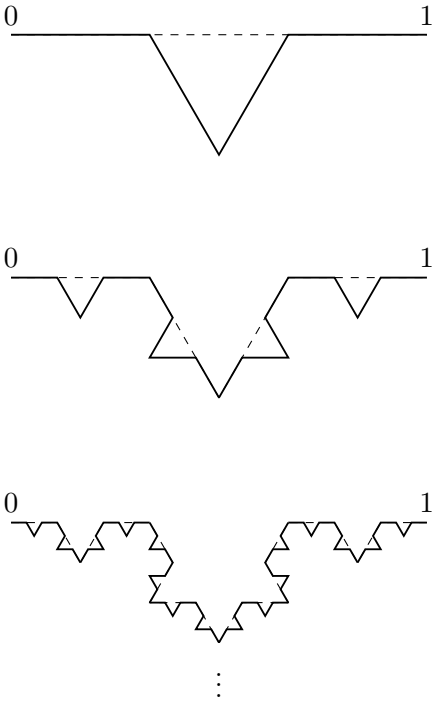
Corollary 3.4.2. *Let* $m \ge 2$ *be an integer,* $\delta_0 = \cdots = \delta_{\lfloor \frac{m}{4} \rfloor} = +1$, $\delta_{\lfloor \frac{m}{4} \rfloor + 1} = \cdots = \delta_{m - \lfloor \frac{m}{4} \rfloor - 1} = -1$, $\delta_{m - \lfloor \frac{m}{4} \rfloor} = \cdots = \delta_m = +1$ *and* $\delta = (\delta_n)_{n\ge 0}$ *be the* $(+1, \delta_1, \cdots, \delta_m)$-*Thue-Morse sequence. Then* $p(m+1)$ *is a real number in* $[3, m+1]$, *the* $(+1, \delta_1, \cdots, \delta_m)$-*IFS satisfies the OSC, and the* $(+1, \delta_1, \cdots, \delta_m)$-*Koch curve has Hausdorff, packing and box dimension* $\log(m+1)/\log p(m+1)$.

To obtain the Hausdorff dimension of the $(+1, \delta_1, \cdots, \delta_m)$-Koch curve in Corollary 3.4.2, one can try to use the last statement in Theorem 3.4.1. But here we use classical theory on IFS by verifying the OSC.

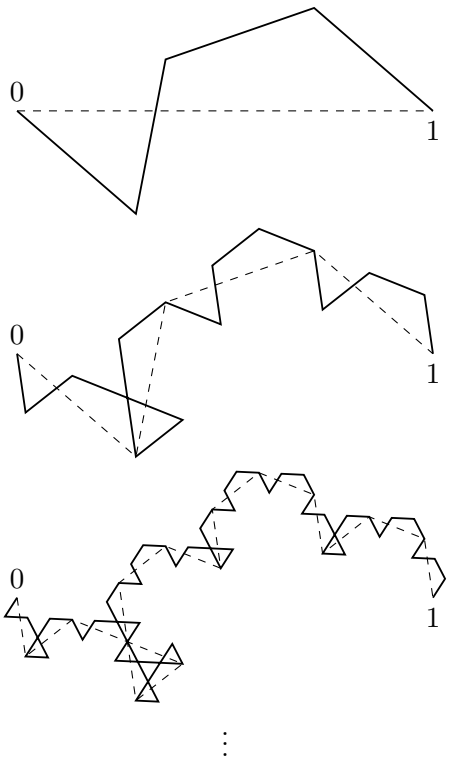
We give some notation and preliminaries in Subsection 3.4.1, and then prove Theorem 3.4.1 and Corollary 3.4.2 in Subsection 3.4.2.

3.4.1 Notation and preliminaries

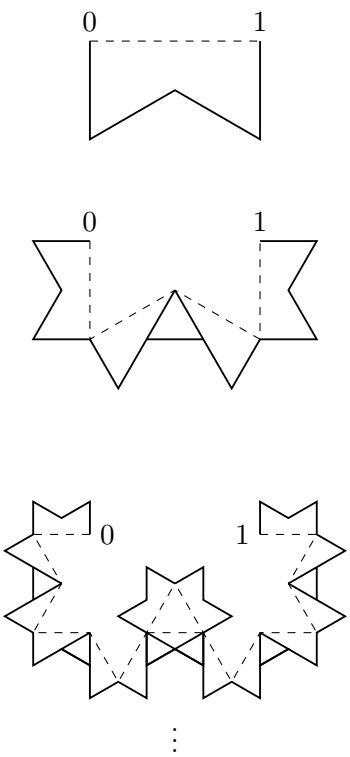
For any $z_1, z_2 \in \mathbb{C}$, we use $[z_1, z_2] := \{cz_1 + (1-c)z_2 : c \in [0,1]\}$ to denote the segment connecting z_1 and z_2. For any $c \in \mathbb{C}$ and $Z \subset \mathbb{C}$, let $cZ := \{cz : z \in Z\}$ and $c + Z := \{c + z : z \in Z\}$. Besides, for any $z \in \mathbb{C}$ we use $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ to denote respectively the real part and the imaginary part of z.



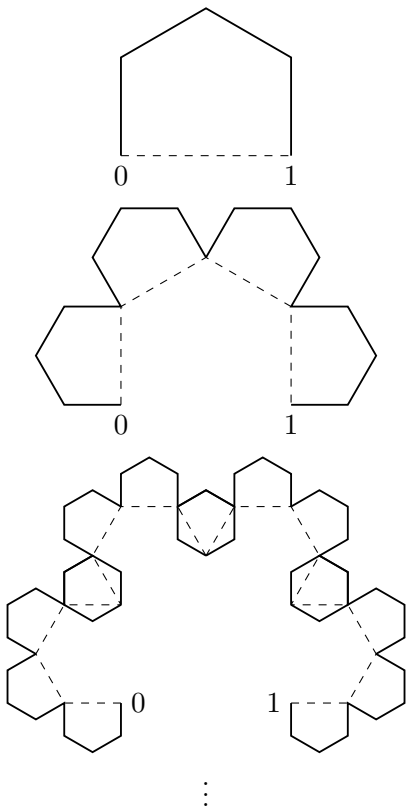
Construction of the
$(+1,-1,-1,+1)$-Koch curve.



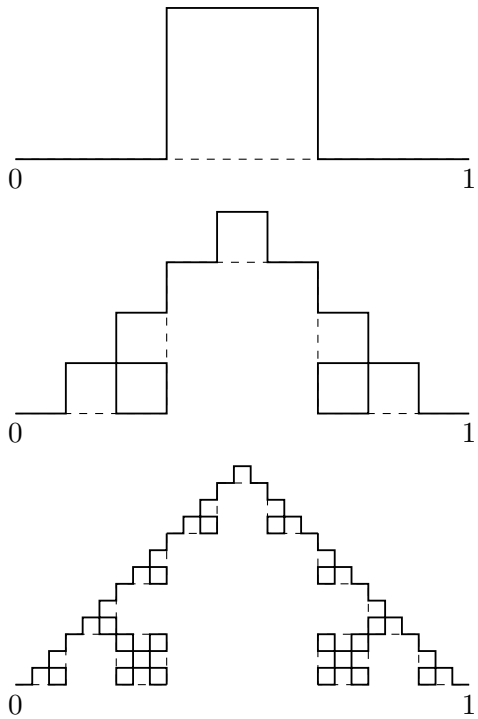
Construction of the
$(+1,+1,-1,+1)$-Koch curve.



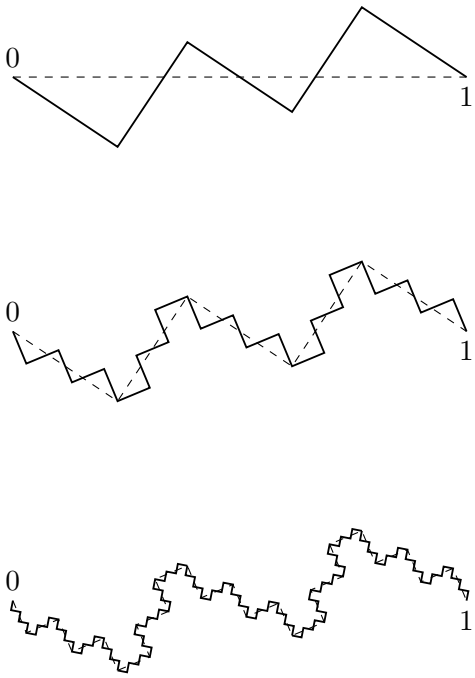
Construction of the
$(+1,+1,-1,-1)$-Koch curve.



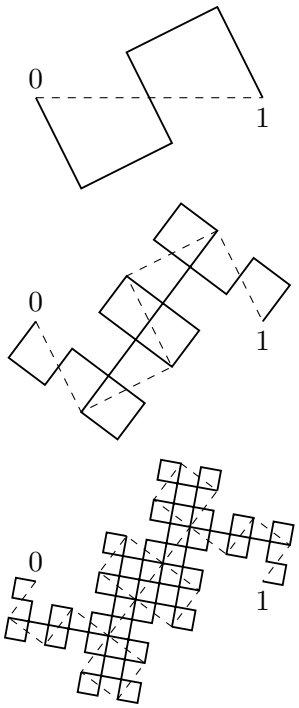
Construction of the
$(+1,-1,+1,-1)$-Koch curve.



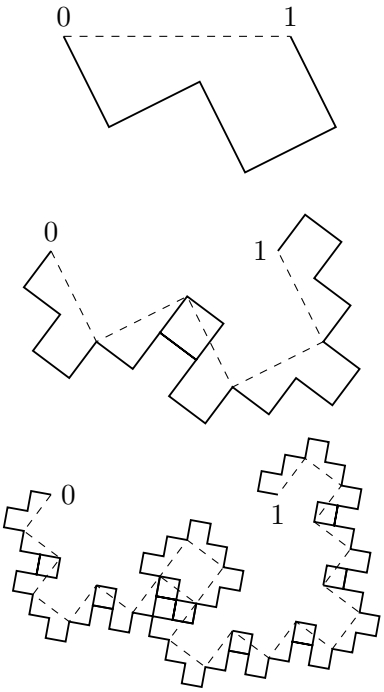
Construction of the
$(+1, +1, -1, +1, +1)$-Koch curve.



Construction of the
$(+1, +1, -1, -1, +1)$-Koch curve.



Construction of the
$(+1, +1, +1, -1, +1)$-Koch curve.



Construction of the
$(+1, +1, -1, -1, -1)$-Koch curve.

Let $\mathcal{A}$ be a finite alphabet of symbols and $\mathcal{A}^* := \cup_{n=0}^{\infty}\mathcal{A}^n$ be the free monoid generated by $\mathcal{A}$. A map $\phi : \mathcal{A}^* \to \mathcal{A}^*$ is called a *morphism* if

$$\phi(uv) = \phi(u)\phi(v)$$

for all words $u, v \in \mathcal{A}^*$. Moreover ϕ is called *null-free* if $\phi(a)$ is not the empty word for any $a \in \mathcal{A}$, and called *primitive* if there exists an $n \in \mathbb{N}$ such that $a \in \phi^n(b)$ for all $a, b \in \mathcal{A}$, where $u \in v$ denotes that u occurs in v for any words $u, v \in \mathcal{A}^*$. For a morphism $\phi : \mathcal{A}^* \to \mathcal{A}^*$, the corresponding matrix $M_\phi = (m_{a,b})_{a,b\in\mathcal{A}}$ is defined by $m_{a,b} := |\phi(a)|_b$, where $|w|_b$ denotes the number of the symbol b in the word w. In addition, recall that we use $|w|$ to denote the length of the finite word w.

A map $f : \mathcal{A}^* \to \mathbb{C}$ is called a *homomorphism* if

$$f(uv) = f(u) + f(v)$$

for all words $u, v \in \mathcal{A}^*$, and an $\mathbb{R}$-linear map $L : \mathbb{C} \to \mathbb{C}$ (regarded as $\mathbb{R}^2 \to \mathbb{R}^2$) is called *expanding* if both eigenvalues have modulus more than one.

Let $\mathcal{H}(\mathbb{C})$ be the set of all non-empty compact subsets of $\mathbb{C}$ and d_H be the Hausdorff metric on $\mathcal{H}(\mathbb{C})$ defined by

$$d_H(Z_1, Z_2) := \max\Big\{ \sup_{z_1\in Z_1} \inf_{z_2\in Z_2} |z_1 - z_2|, \sup_{z_2\in Z_2} \inf_{z_1\in Z_1} |z_1 - z_2| \Big\} \quad \text{for } Z_1, Z_2 \in \mathcal{H}(\mathbb{C}).$$

The following result was given by Dekking.

Theorem 3.4.3. *([48, Theorem 2.4]) Let $\phi : \mathcal{A}^* \to \mathcal{A}^*$ be a null-free morphism, $f : \mathcal{A}^* \to \mathbb{C}$ be a homomorphism, $L : \mathbb{C} \to \mathbb{C}$ be an expanding $\mathbb{R}$-linear map such that*

$$f \circ \phi = L \circ f,$$

and $K : \mathcal{A}^* \to \mathcal{H}(\mathbb{C})$ be a map satisfying

$$K(uv) = K(u) \cup (f(u) + K(v))$$

for all $u, v \in \mathcal{A}^*$. Then for any non-empty word $w \in \mathcal{A}^*$, there exists a unique compact set W such that

$$L^{-n}K(\phi^n(w)) \xrightarrow{d_H} W \quad \text{as } n \to \infty,$$

and W is a continuous image of $[0, 1]$.

In the following we recall some preliminaries on iterated function systems. A map $S : \mathbb{C} \to \mathbb{C}$ is called a *contraction* if there exists $c \in (0, 1)$ such that

$$|S(z_1) - S(z_2)| \le c|z_1 - z_2| \quad \text{for all } z_1, z_2 \in \mathbb{C}.$$

Moreover, if equality holds, i.e., if $|S(z_1) - S(z_2)| = c|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$, we say that S is a *contracting similarity.*

A finite family of contractions $\{S_1, S_2, \cdots, S_n\}$, with $n \geq 2$, is called an *iterated function system* (*IFS*). The following is a fundamental result. See for example [64, Theorem 9.1].

Theorem 3.4.4. *Any family of contractions $\{S_1, \cdots, S_n\}$ has a unique attractor F, i.e., a non-empty compact set such that*

$$F = \bigcup_{j=1}^{n} S_j(F).$$

We say that an IFS $\{S_1, \cdots, S_n\}$ satisfies the *open set condition* (*OSC*) if there exists a non-empty bounded open set V such that

$$\bigcup_{j=1}^{n} S_j(V) \subset V$$

with the union disjoint. The following theorem is well known. See for example [64, Theorem 9.3].

Theorem 3.4.5. *If the OSC holds for the contracting similarities $S_j : \mathbb{C} \to \mathbb{C}$ with the ratios $c_j \in (0, 1)$ for all $j \in \{1, \cdots, n\}$, then the attractor of the IFS $\{S_1, \cdots, S_n\}$ has Hausdorff, packing and box dimension s, where s is given by*

$$\sum_{j=1}^{n} c_j^s = 1.$$

To end this subsection, we present the following basic property for contractions.

Proposition 3.4.6. *Let $S_1, S_2, \cdots, S_n$ be contractions on $\mathbb{C}$. Write*

$$S(A) := \bigcup_{j=1}^{n} S_j(A) \quad \textit{for all } A \subset \mathbb{C}.$$

Then for all $F, F_1, F_2, \cdots \subset \mathbb{C}$ such that $F_k \xrightarrow{d_H} F$ as $k \to \infty$, we have $S(F_k) \xrightarrow{d_H} S(F)$.

Proof. This follows from the fact that for all $k \in \mathbb{N}$ we have

$$d_H(S(F_k), S(F)) \leq \max_{1 \leq j \leq n} d_H(S_j(F_k), S_j(F)) \leq \max_{1 \leq j \leq n} c_j d_H(F_k, F),$$

where for each $j \in \{1, \cdots, n\}$, $c_j \in (0, 1)$ satisfies $|S_j(z_1) - S_j(z_2)| \leq c_j|z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{C}$. $\square$

3.4.2 Proofs of Theorem 3.4.1 and Corollary 3.4.2

Proof of Theorem 3.4.1. Let $m \in \mathbb{N}$, $\delta_0 = +1$, $\delta_1, \cdots, \delta_m \in \{+1, -1\}$ and $\delta = (\delta_n)_{n\geq 0}$ be the $(+1, \delta_1, \cdots, \delta_m)$-Thue-Morse sequence such that $|p(m+1)| > 1$.
(1) Prove that there exists a unique compact set $K \subset \mathbb{C}$ such that

$$(p(m+1))^{-n}P(n) \xrightarrow{d_H} K \quad \text{as } n \to \infty$$

and K is a continuous image of $[0,1]$ by using Theorem 3.4.3.
① If m is odd, let $\mathcal{A} := \{0, 1, 2, \cdots, 2m-1\}$. Define the morphism $\phi : \mathcal{A}^* \to \mathcal{A}^*$ by

$$a \mapsto d_{a,0}d_{a,1}\cdots d_{a,m}$$

for all $a \in \mathcal{A}$ where

$$d_{a,k} := \begin{cases} a + 2k & \mod 2m \quad \text{if } \delta_k = +1 \\ a + 2k + m & \mod 2m \quad \text{if } \delta_k = -1 \end{cases}$$

for all $k \in \{0, 1, \cdots, m\}$. Obviously $d_{a,0} = a$ for all $a \in \mathcal{A}$ and it is straightforward to check

$$e^{\frac{d_{a,k}\pi i}{m}} = \delta_k e^{\frac{(a+2k)\pi i}{m}}$$

for all $k \in \{0, 1, \cdots, m\}$. Let ε be the empty word. Define $f(\varepsilon) := 0$ and

$$f(w_1 \cdots w_n) := \sum_{k=1}^{n} e^{\frac{w_k \pi i}{m}}$$

for any $w_1 \cdots w_n \in \mathcal{A}^*$. Then $f : \mathcal{A}^* \to \mathbb{C}$ is a homomorphism satisfying

$$f(a) = e^{\frac{a\pi i}{m}}$$

for all $a \in \mathcal{A}$ and

$$f(uv) = f(u) + f(v)$$

for all $u, v \in \mathcal{A}^*$. Let $L : \mathbb{C} \to \mathbb{C}$ be the linear map defined by

$$L(z) := p(m+1) \cdot z$$

for all $z \in \mathbb{C}$. It follows from $|p(m+1)| > 1$ that L is expanding.

We can check $f \circ \phi = L \circ f$. In fact, for the empty word we have $f \circ \phi(\varepsilon) = f(\varepsilon) = 0 = L(0) = L \circ f(\varepsilon)$, for any $a \in \mathcal{A}$ we have

$$f \circ \phi(a) \quad = \quad f(d_{a,0} \cdots d_{a,m}) = \sum_{k=0}^{m} e^{\frac{d_{a,k}\pi i}{m}} = \sum_{k=0}^{m} \delta_k e^{\frac{(a+2k)\pi i}{m}} = e^{\frac{a\pi i}{m}} \sum_{k=0}^{m} \delta_k e^{\frac{2k\pi i}{m}}$$

$$= \quad f(a)p(m+1) = L \circ f(a),$$

and for any $w_1 \cdots w_n \in \mathcal{A}^*$ we have

$$\begin{aligned} f \circ \phi(w_1 \cdots w_n) &= f(\phi(w_1) \cdots \phi(w_n)) = f(\phi(w_1)) + \cdots + f(\phi(w_n)) \\ &= L(f(w_1)) + \cdots + L(f(w_n)) = L(f(w_1) + \cdots + f(w_n)) = L \circ f(w_1 \cdots w_n). \end{aligned}$$

Define $K(\varepsilon)$ to be the singleton $\{0\}$,

$$K(a) := [0, f(a)]$$

for any $a \in \mathcal{A}$, and

$$K(w_1 \cdots w_n) := \bigcup_{k=1}^{n} \Big(f(w_1 \cdots w_{k-1}) + K(w_k) \Big)$$

for any $w_1 \cdots w_n \in \mathcal{A}^*$, where $f(w_1 \cdots w_{k-1})$ is regarded as 0 for $k = 1$. Then $K : \mathcal{A}^* \to \mathcal{H}(\mathbb{C})$ satisfies

$$K(uv) = K(u) \cup (f(u) + K(v))$$

for all $u, v \in \mathcal{A}^*$. Now applying Theorem 3.4.3, there exists a unique compact set $K \subset \mathbb{C}$ such that

$$(p(m+1))^{-n} K(\phi^n(0)) \xrightarrow{d_H} K \quad \text{as } n \to \infty,$$

and K is a continuous image of $[0, 1]$. In the following we only need to check $K(\phi^n(0)) = P(n)$ for all $n \in \mathbb{N}_0$.

i) First we prove that for all $a \in \mathcal{A}$, $j \in \{1, 2, \cdots, m\}$ and $n \in \{0, 1, 2, \cdots\}$ we have

$$f(\phi^n(d_{a,0} \cdots d_{a,j-1})) = e^{\frac{a\pi i}{m}} p(j(m+1)^n) \tag{3.36}$$

by induction on n. In fact, for $n = 0$ we have

$$f(d_{a,0} \cdots d_{a,j-1}) = \sum_{k=0}^{j-1} e^{\frac{d_{a,k}\pi i}{m}} = \sum_{k=0}^{j-1} \delta_k e^{\frac{(a+2k)\pi i}{m}} = e^{\frac{a\pi i}{m}} p(j).$$

Suppose that (3.36) is true for some $n \geq 0$. Then for $n+1$, on the one hand

$$f(\phi^{n+1}(d_{a,0} \cdots d_{a,j-1})) = L(f(\phi^n(d_{a,0} \cdots d_{a,j-1}))) = p(m+1) e^{\frac{a\pi i}{m}} \sum_{r=0}^{j(m+1)^n - 1} \delta_r e^{\frac{2r\pi i}{m}}$$

where the first equality follows from $f \circ \phi = L \circ f$ and the second equality follows

from the definition of L and the inductive hypothesis, and on the other hand

$$e^{\frac{a\pi i}{m}}p(j(m+1)^{n+1}) = e^{\frac{a\pi i}{m}}\sum_{k=0}^{j(m+1)^{n+1}-1}\delta_k e^{\frac{2k\pi i}{m}} = e^{\frac{a\pi i}{m}}\sum_{r=0}^{j(m+1)^n-1}\sum_{k=r(m+1)}^{r(m+1)+m}\delta_k e^{\frac{2k\pi i}{m}}.$$

It suffices to check

$$p(m+1)\delta_r e^{\frac{2r\pi i}{m}} = \sum_{k=r(m+1)}^{r(m+1)+m}\delta_k e^{\frac{2k\pi i}{m}}$$

for all $r \in \{0, 1, \cdots, j(m+1)^n - 1\}$. In fact we have

$$\sum_{k=r(m+1)}^{r(m+1)+m}\delta_k e^{\frac{2k\pi i}{m}} = \sum_{k=0}^{m}\delta_{r(m+1)+k}e^{\frac{2(r(m+1)+k)\pi i}{m}} = \sum_{k=0}^{m}\delta_r\delta_k e^{\frac{2r\pi i}{m}}e^{\frac{2k\pi i}{m}} = p(m+1)\delta_r e^{\frac{2r\pi i}{m}},$$

where the second equality follows from $\delta_{r(m+1)+k} = \delta_r\delta_k$ (see Proposition 3.3.15 (1)).

ii) To check $K(\phi^n(0)) = P(n)$ for all $n \in \mathbb{N}_0$, it suffices to prove

$$K(\phi^n(a)) = e^{\frac{a\pi i}{m}}P(n) \quad \text{for all } a \in \mathcal{A} \tag{3.37}$$

by induction on n. In fact, for $n = 0$ we have

$$K(a) = [0, e^{\frac{a\pi i}{m}}] = e^{\frac{a\pi i}{m}}[0, 1] = e^{\frac{a\pi i}{m}}P(0).$$

Suppose that (3.37) is true for some $n \geq 0$. Then for $n+1$, on the one hand

$$\begin{aligned}
&K(\phi^{n+1}(a)) \\
&= K(\phi^n(d_{a,0}\cdots d_{a,m})) \\
&= \bigcup_{j=0}^{m}\Big(f(\phi^n(d_{a,0}\cdots d_{a,j-1})) + K(\phi^n(d_{a,j}))\Big) \\
&\qquad \big(\text{where } f(\phi^n(d_{a,0}\cdots d_{a,j-1})) \text{ is regarded as 0 for } j = 0\big) \\
&\overset{(*)}{=} \bigcup_{j=0}^{m}\Big(e^{\frac{a\pi i}{m}}p(j(m+1)^n) + e^{\frac{d_{a,j}\pi i}{m}}P(n)\Big) \\
&= \bigcup_{j=0}^{m}\Big(e^{\frac{a\pi i}{m}}p(j(m+1)^n) + \delta_j e^{\frac{(a+2j)\pi i}{m}}P(n)\Big) \\
&= e^{\frac{a\pi i}{m}}\bigcup_{j=0}^{m}\Big(p(j(m+1)^n) + \delta_j e^{\frac{2j\pi i}{m}}P(n)\Big)
\end{aligned}$$

where $(*)$ follows from the inductive hypothesis and (3.36), and on the other hand

$$
\begin{aligned}
&P(n+1)\\
&= \bigcup_{k=1}^{(m+1)^{n+1}} [p(k-1), p(k)]\\
&= \bigcup_{j=0}^{m} \bigcup_{k=j(m+1)^n+1}^{(j+1)(m+1)^n} [p(k-1), p(k)]\\
&= \bigcup_{j=0}^{m} \bigcup_{k=1}^{(m+1)^n} [p(j(m+1)^n + k - 1), p(j(m+1)^n + k)]\\
&= \bigcup_{j=0}^{m} \bigcup_{k=1}^{(m+1)^n} \Big[\sum_{r=0}^{j(m+1)^n+k-2} \delta_r e^{\frac{2r\pi i}{m}}, \sum_{r=0}^{j(m+1)^n+k-1} \delta_r e^{\frac{2r\pi i}{m}} \Big]\\
&\qquad \Big(\text{where } \sum_{r=a}^{b} \cdot \text{ is regarded as 0 if } a > b\Big)\\
&= \bigcup_{j=0}^{m} \Big(\sum_{r=0}^{j(m+1)^n-1} \delta_r e^{\frac{2r\pi i}{m}} + \bigcup_{k=1}^{(m+1)^n} \Big[\sum_{r=j(m+1)^n}^{j(m+1)^n+k-2} \delta_r e^{\frac{2r\pi i}{m}}, \sum_{r=j(m+1)^n}^{j(m+1)^n+k-1} \delta_r e^{\frac{2r\pi i}{m}} \Big] \Big)\\
&= \bigcup_{j=0}^{m} \Big(p(j(m+1)^n) + \bigcup_{k=1}^{(m+1)^n} \Big[\sum_{r=0}^{k-2} \delta_{j(m+1)^n+r} e^{\frac{2(j(m+1)^n+r)\pi i}{m}}, \sum_{r=0}^{k-1} \delta_{j(m+1)^n+r} e^{\frac{2(j(m+1)^n+r)\pi i}{m}} \Big] \Big)\\
&\overset{(**)}{=} \bigcup_{j=0}^{m} \Big(p(j(m+1)^n) + \bigcup_{k=1}^{(m+1)^n} \Big[\sum_{r=0}^{k-2} \delta_j \delta_r e^{\frac{2(j+r)\pi i}{m}}, \sum_{r=0}^{k-1} \delta_j \delta_r e^{\frac{2(j+r)\pi i}{m}} \Big] \Big)\\
&= \bigcup_{j=0}^{m} \Big(p(j(m+1)^n) + \delta_j e^{\frac{2j\pi i}{m}} \bigcup_{k=1}^{(m+1)^n} [p(k-1), p(k)] \Big)\\
&= \bigcup_{j=0}^{m} \Big(p(j(m+1)^n) + \delta_j e^{\frac{2j\pi i}{m}} P(n) \Big)
\end{aligned}
$$

where $(**)$ follows from $\delta_{j(m+1)^n+r} = \delta_j \delta_r$ (see Proposition 3.3.15 (1)). Thus $K(\phi^{n+1}(a)) = e^{\frac{a\pi i}{m}} P(n+1)$.

② If m is even, let $\mathcal{A} := \{0, 1, 2, \cdots, m-1\}$. Define the morphism $\phi : \mathcal{A}^* \to \mathcal{A}^*$ by

$$a \mapsto d_{a,0} d_{a,1} \cdots d_{a,m}$$

for all $a \in \mathcal{A}$ where

$$d_{a,k} := \begin{cases} a + k & \mod m \quad \text{if } \delta_k = +1 \\ a + k + \frac{m}{2} & \mod m \quad \text{if } \delta_k = -1 \end{cases}$$

for all $k \in \{0, 1, \cdots, m\}$. Obviously $d_{a,0} = a$ for all $a \in \mathcal{A}$ and it is straightforward to check

$$e^{\frac{2d_{a,k}\pi i}{m}} = \delta_k e^{\frac{2(a+k)\pi i}{m}}$$

for all $k \in \{0, 1, \cdots, m\}$. Define $f(\varepsilon) := 0$ and

$$f(w_1 \cdots w_n) := \sum_{k=1}^{n} e^{\frac{2w_k\pi i}{m}}$$

for any $w_1 \cdots w_n \in \mathcal{A}^*$. Then $f : \mathcal{A}^* \to \mathbb{C}$ is a homomorphism satisfying

$$f(a) = e^{\frac{2a\pi i}{m}}$$

for all $a \in \mathcal{A}$ and

$$f(uv) = f(u) + f(v)$$

for all $u, v \in \mathcal{A}^*$. Let $L : \mathbb{C} \to \mathbb{C}$ and $K : \mathcal{A}^* \to \mathcal{H}(\mathbb{C})$ be defined in the same way as ①. Then we can prove

$$f(\phi^n(d_{a,0} \cdots d_{a,j-1})) = e^{\frac{2a\pi i}{m}} p(j(m+1)^n) \tag{3.38}$$

for all $j \in \{1, 2, \cdots, m\}$, $a \in \mathcal{A}$ and $n \in \mathbb{N}_0$, and then

$$K(\phi^n(a)) = e^{\frac{2a\pi i}{m}} P(n).$$

Thus $K(\phi^n(0)) = P(n)$ for all $n \in \mathbb{N}_0$. By applying Theorem 3.4.3, there exists a unique compact set $K \subset \mathbb{C}$ such that

$$(p(m+1))^{-n} P(n) \xrightarrow{d_H} K \quad \text{as } n \to \infty,$$

and K is a continuous image of $[0, 1]$.

(2) Prove that K is the unique attractor of the IFS $\{S_j\}_{0 \le j \le m}$.

By Theorem 3.4.4 it suffices to show $K = \cup_{j=0}^{m} S_j(K)$. Let $Q_n := (p(m+1))^{-n} P(n)$ for all $n \in \mathbb{N}_0$. Since $Q_n \xrightarrow{d_H} K$ and Proposition 3.4.6 imply $\cup_{j=0}^{m} S_j(Q_n) \xrightarrow{d_H} \cup_{j=0}^{m} S_j(K)$ as $n \to \infty$, we only need to prove $Q_{n+1} = \cup_{j=0}^{m} S_j(Q_n)$ for all $n \in \mathbb{N}_0$ in the following. In fact,

$$\begin{aligned}
Q_{n+1} &= (p(m+1))^{-(n+1)} P(n+1) \\
&\overset{(*)}{=} (p(m+1))^{-(n+1)} \bigcup_{j=0}^{m} \Big(p(j(m+1)^n) + \delta_j e^{\frac{2j\pi i}{m}} P(n) \Big) \\
&= \bigcup_{j=0}^{m} \Big((p(m+1))^{-(n+1)} p(j(m+1)^n) + (p(m+1))^{-(n+1)} \delta_j e^{\frac{2j\pi i}{m}} P(n) \Big) \\
&\overset{(**)}{=} \bigcup_{j=0}^{m} \Big((p(m+1))^{-1} p(j) + (p(m+1))^{-(n+1)} \delta_j e^{\frac{2j\pi i}{m}} P(n) \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{j=0}^{m} (p(m+1))^{-1}\Big(p(j) + \delta_j e^{\frac{2j\pi i}{m}} Q_n\Big) \\
&= \bigcup_{j=0}^{m} S_j(Q_n),
\end{aligned}
$$

where $(*)$ follows from the recurrence relation between $P(n+1)$ and $P(n)$ deduced at the end of the proof in (1) ① (noting that this relation is true no matter m is odd or even), and $(**)$ follows from $p(0) = 0$ and

$$
\begin{aligned}
p(j(m+1)^n) &\overset{\text{by (3.36)}}{\underset{\text{and (3.38)}}{=\!=\!=}} f(\phi^n(d_{0,0}\cdots d_{0,j-1})) \\
&\overset{f\circ\phi = L\circ f}{=\!=\!=} L^n(f(d_{0,0}\cdots d_{0,j-1})) \\
&\overset{\text{by (3.36)}}{\underset{\text{and (3.38)}}{=\!=\!=}} (p(m+1))^n p(j)
\end{aligned}
$$

for all $j \in \{1, 2, \cdots, m\}$ and $n \in \mathbb{N}_0$.

(3) Prove that

$$\dim_H K = \frac{\log(m+1)}{\log|p(m+1)|}$$

if and only if there exists $\varepsilon > 0$ such that

$$\varliminf_{n\to\infty} \frac{\mathcal{L}((P(n))^{\varepsilon})}{(m+1)^n} > 0.$$

Noting that $|\phi^n(0)| = (m+1)^n$, $K(\phi^n(0)) = P(n)$ is proved in (1) and $L : \mathbb{C} \to \mathbb{C}$ (regarded as $\mathbb{R}^2 \to \mathbb{R}^2$) is a similarity with eigenvalues of the same modulus $|p(m+1)| > 1$, by applying [27, Dekking's conjecture] (which was proved), we only need to check that the eigenvalue of M_ϕ (the corresponding matrix of ϕ) with greatest modulus is $m+1$ and ϕ is primitive. Note that according to whether m is odd or even, the definition of ϕ in (1) is different.

① If m is odd, recall $\mathcal{A} := \{0, 1, 2, \cdots, 2m-1\}$. On the calculation between the symbols in $\mathcal{A}$, we consider the mod $2m$ congruence class (for example $5 + (2m-3) = 2$). Recall the definition of ϕ. For any $a, b \in \mathcal{A}$, the equivalences of $d_{a,k} = b$ and $d_{a+1,k} = b+1$ for all $k \in \{0, 1, \cdots, m\}$ imply $|\phi(a)|_b = |\phi(a+1)|_{b+1}$. This means that M_ϕ is a circulant matrix, and the eigenvalue with greatest modulus is $|\phi(0)|_0 + |\phi(0)|_1 + \cdots + |\phi(0)|_{2m-1} = |\phi(0)| = m+1$.

In the following we prove that ϕ is primitive. That is, there exists $n \in \mathbb{N}$ such that $b \in \phi^n(a)$ for all $a, b \in \mathcal{A}$, where $u \in v$ means that u occurs in v for any words $u, v \in \mathcal{A}^*$. For any word $w = w_1 \cdots w_k \in \mathcal{A}^*$ and any symbol $a \in \mathcal{A}$, write

$$w + a = w_1 \cdots w_k + a := (w_1 + a) \cdots (w_k + a).$$

Then we have

$$\phi(w+a) = \phi(w_1+a)\cdots\phi(w_k+a) = (\phi(w_1)+a)\cdots(\phi(w_k)+a) = \phi(w)+a, \tag{3.39}$$

where the second equality follows from

$$\phi(b+a) = d_{b+a,0}d_{b+a,1}\cdots d_{b+a,m} = (d_{b,0}+a)(d_{b,1}+a)\cdots(d_{b,m}+a) = \phi(b)+a$$

for any $a, b \in \mathcal{A}$. By applying (3.39) consecutively, for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$ we have

$$\phi^n(a) = \phi^{n-1}(\phi(0)+a) = \phi^{n-2}(\phi^2(0)+a) = \cdots = \phi^n(0)+a, \tag{3.40}$$

and then $b \in \phi^n(a)$ is equivalent to $b-a \in \phi^n(0)$ for all $b \in \mathcal{A}$. Thus we only need to prove that there exists $n \in \mathbb{N}$ such that $a \in \phi^n(0)$ for all $a \in \mathcal{A}$.

i) Suppose $\delta_1 = +1$. Then

$$d_{0,1} = 2, d_{2,1} = 4, d_{4,1} = 6, \cdots, d_{2m-4,1} = 2m-2,$$

which imply

$$2 \in \phi(0), 4 \in \phi(2), 6 \in \phi(4), \cdots, 2m-2 \in \phi(2m-4).$$

By iterating ϕ we get

$$2 \in \phi(0), 4 \in \phi^2(0), 6 \in \phi^3(0), \cdots, 2m-2 \in \phi^{m-1}(0) \tag{3.41}$$

one by one. It follows from

$$0 \in \phi(0) \in \phi^2(0) \in \cdots \in \phi^{m-1}(0) \in \phi^m(0) \tag{3.42}$$

that $0, 2, 4, \cdots, 2m-2 \in \phi^m(0)$. It suffices to prove $1, 3, 5, \cdots, 2m-1 \in \phi^m(0)$ in the following. Since $\delta_1 = \cdots = \delta_m = +1$ will imply $p(m+1) = 1$ (which contradicts $|p(m+1)| > 1$), noting $\delta_1 = +1$, there exists $l \in \{2, 3, \cdots, m\}$ such that $\delta_l = -1$. This implies

$$d_{0,l} = 2l+m, d_{2,l} = 2l+m+2, d_{4,l} = 2l+m+4, \cdots, d_{2m-2,l} = 2l+3m-2$$

and then

$$2l+m \in \phi(0), 2l+m+2 \in \phi(2), 2l+m+4 \in \phi(4), \cdots, 2l+3m-2 \in \phi(2m-2).$$

It follows from (3.41) that

$$2l+m \in \phi(0), 2l+m+2 \in \phi^2(0), 2l+m+4 \in \phi^3(0), \cdots, 2l+3m-2 \in \phi^m(0).$$

By (3.42) we get $2l+m, 2l+m+2, 2l+m+4, \cdots, 2l+3m-2 \in \phi^m(0)$, which is equivalent to $1, 3, 5, \cdots, 2m-1 \in \phi^m(0)$. Therefore $a \in \phi^m(0)$ for all $a \in \mathcal{A}$.

ii) Suppose $\delta_1 = -1$. Then $d_{0,1} = m+2$. By $m+2 \in \phi(0)$, we get

$$2(m+2) = m+2+m+2 \in \phi(0)+m+2 \overset{\text{by (3.39)}}{=\!=\!=} \phi(m+2) \in \phi^2(0).$$

In the same way we get $3(m+2) \in \phi^3(0), 4(m+2) \in \phi^4(0), \cdots, (2m-1)(m+2) \in \phi^{2m-1}(0)$. It follows from

$$0 \in \phi(0) \in \phi^2(0) \in \cdots \in \phi^{2m-1}(0)$$

that

$$0, m+2, 2(m+2), 3(m+2), \cdots, (2m-1)(m+2) \in \phi^{2m-1}(0). \tag{3.43}$$

Since m is odd, we know that $m+2$ and $2m$ are relatively prime. This implies that $0, m+2, 2(m+2), 3(m+2), \cdots, (2m-1)(m+2)$ construct a complete residue system mod $2m$. By (3.43) we get $0, 1, 2, \cdots, 2m-1 \in \phi^{2m-1}(0)$.

② If m is even, recall $\mathcal{A} := \{0, 1, 2, \cdots, m-1\}$. On the calculation between the symbols in $\mathcal{A}$, we consider the mod m congruence class (for example $5+(m-3)=2$). Recall the definition of ϕ. In the same way as ①, we know that the eigenvalue of M_ϕ with greatest modulus is $m+1$.

In the following it suffices to prove that ϕ is primitive. In the same way as ①, we get

$$\phi^n(a) = \phi^n(0) + a \quad \text{for all } a \in \mathcal{A} \text{ and } n \in \mathbb{N}, \tag{3.44}$$

and we only need to prove that there exists $n \in \mathbb{N}$ such that $a \in \phi^n(0)$ for all $a \in \mathcal{A}$.

i) Suppose $\delta_1 = +1$. Then

$$d_{0,1} = 1, d_{1,1} = 2, d_{2,1} = 3, \cdots, d_{m-2,1} = m-1$$

which imply

$$1 \in \phi(0), 2 \in \phi(1), 3 \in \phi(2), \cdots, m-1 \in \phi(m-2).$$

By iterating ϕ we get

$$1 \in \phi(0), 2 \in \phi^2(0), 3 \in \phi^3(0), \cdots, m-1 \in \phi^{m-1}(0)$$

one by one. It follows from

$$0 \in \phi(0) \in \phi^2(0) \in \cdots \in \phi^{m-1}(0)$$

that $0, 1, 2, \cdots, m-1 \in \phi^{m-1}(0)$.

ii) Suppose $\delta_1 = -1$. Then $d_{0,1} = \frac{m}{2} + 1$ and $d_{\frac{m}{2}+1,1} = 2$, which imply $\frac{m}{2} + 1 \in \phi(0)$ and $2 \in \phi(\frac{m}{2} + 1)$. It follows from $\phi(\frac{m}{2} + 1) \in \phi^2(0)$ that $2 \in \phi^2(0)$, and then $\phi^2(2) \in \phi^4(0)$. Since (3.44) implies $\phi^2(2) = \phi^2(0) + 2$, we get $4 \in \phi^4(0)$. Repeating this process we get

$$2 \in \phi^2(0), 4 \in \phi^4(0), 6 \in \phi^6(0), \cdots, m-2 \in \phi^{m-2}(0). \tag{3.45}$$

It follows from $0 \in \phi^2(0) \in \phi^4(0) \in \cdots \in \phi^{m-2}(0)$ that

$$0, 2, 4, \cdots, m-2 \in \phi^{m-2}(0). \tag{3.46}$$

First we prove that there exits an odd $a \in \mathcal{A}$ such that $a \in \phi(0)$ by contradiction. Assume $a \notin \phi(0)$ for all odd $a \in \mathcal{A}$. By $\phi(0) = d_{0,0}d_{0,1}\cdots d_{0,m}$ we know that $d_{0,0}, d_{0,1}, \cdots, d_{0,m}$ are all even. Then $d_{0,1} = \frac{m}{2} + 1$ implies that $\frac{m}{2}$ is odd. By

$$d_{0,k} := \begin{cases} k & \text{if } \delta_k = +1 \\ k + \frac{m}{2} & \text{if } \delta_k = -1 \end{cases}$$

for all $k \in \{0, 1, \cdots, m\}$, we get

$$\delta_0 = \delta_2 = \delta_4 = \cdots = \delta_m = +1 \quad \text{and} \quad \delta_1 = \delta_3 = \cdots = \delta_{m-1} = -1.$$

It follows that

$$p(m+1) = \sum_{k=0}^{m} (-1)^k e^{\frac{2k\pi i}{m}} = 1 + \sum_{k=1}^{\frac{m}{2}} (-1)^k e^{\frac{2k\pi i}{m}} + \sum_{k=\frac{m}{2}+1}^{m} (-1)^k e^{\frac{2k\pi i}{m}},$$

where

$$\sum_{k=\frac{m}{2}+1}^{m} (-1)^k e^{\frac{2k\pi i}{m}} = \sum_{k=1}^{\frac{m}{2}} (-1)^{\frac{m}{2}+k} e^{\frac{2(\frac{m}{2}+k)\pi i}{m}} = \sum_{k=1}^{\frac{m}{2}} (-1)^{k+1} e^{\pi i} e^{\frac{2k\pi i}{m}} = \sum_{k=1}^{\frac{m}{2}} (-1)^k e^{\frac{2k\pi i}{m}}$$

and

$$\begin{aligned}
\sum_{k=1}^{\frac{m}{2}}(-1)^k e^{\frac{2k\pi i}{m}} &= \sum_{j=0}^{\frac{1}{2}(\frac{m}{2}-1)}(-1)^{2j+1}e^{\frac{2(2j+1)\pi i}{m}} + \sum_{j=1}^{\frac{1}{2}(\frac{m}{2}-1)}(-1)^{2j}e^{\frac{2(2j)\pi i}{m}} \\
&= \sum_{j=0}^{\frac{1}{2}(\frac{m}{2}-1)} e^{\frac{2(2j+1)\pi i}{m}+\pi i} + \sum_{j=1}^{\frac{1}{2}(\frac{m}{2}-1)} e^{\frac{4j\pi i}{m}} \\
&= \sum_{j=0}^{\frac{1}{2}(\frac{m}{2}-1)} e^{\frac{4(\frac{1}{2}(\frac{m}{2}+1)+j)\pi i}{m}} + \sum_{j=1}^{\frac{1}{2}(\frac{m}{2}-1)} e^{\frac{4j\pi i}{m}} \\
&= \sum_{j=\frac{1}{2}(\frac{m}{2}+1)}^{\frac{m}{2}} e^{\frac{4j\pi i}{m}} + \sum_{j=1}^{\frac{1}{2}(\frac{m}{2}-1)} e^{\frac{4j\pi i}{m}} = \sum_{j=1}^{\frac{m}{2}} e^{\frac{2j\pi i}{\frac{m}{2}}} = 0.
\end{aligned}$$

This implies $p(m+1) = 1$, which contradicts $|p(m+1)| > 1$. Thus there must exist an odd $a \in \mathcal{A}$ such that $a \in \phi(0)$, which implies

$$\phi^2(a) \in \phi^3(0), \phi^4(a) \in \phi^5(0), \cdots, \phi^{m-2}(a) \in \phi^{m-1}(0).$$

It follows from $\phi(0) \in \phi^3(0) \in \phi^5(0) \in \cdots \in \phi^{m-1}(0)$ that

$$a, \phi^2(a), \phi^4(a), \cdots, \phi^{m-2}(a) \in \phi^{m-1}(0). \tag{3.47}$$

Since (3.44) implies

$$\phi^2(a) = \phi^2(0) + a, \phi^4(a) = \phi^4(0) + a, \cdots, \phi^{m-2}(a) = \phi^{m-2}(0) + a,$$

by (3.45) we get

$$a + 2 \in \phi^2(a), a + 4 \in \phi^4(a), \cdots, a + m - 2 \in \phi^{m-2}(a).$$

It follows from (3.47) that $a, a+2, a+4, \cdots, a+m-2 \in \phi^{m-1}(0)$. Recalling that a is odd, we get $1, 3, 5, \cdots, m-1 \in \phi^{m-1}(0)$. Since $0 \in \phi(0)$ implies $\phi^{m-2}(0) \in \phi^{m-1}(0)$, by (3.46) we get $0, 2, 4, \cdots, m-2 \in \phi^{m-1}(0)$. Therefore $0, 1, 2, 3, \cdots, m-1 \in \phi^{m-1}(0)$.

□

Proof of Corollary 3.4.2. Let $m \geq 2$ be an integer, $\delta_0 = \cdots = \delta_{\lfloor \frac{m}{4} \rfloor} = +1$, $\delta_{\lfloor \frac{m}{4} \rfloor+1} = \cdots = \delta_{m-\lfloor \frac{m}{4} \rfloor-1} = -1$, $\delta_{m-\lfloor \frac{m}{4} \rfloor} = \cdots = \delta_m = +1$ and $\delta = (\delta_n)_{n\geq 0}$ be the $(+1, \delta_1, \cdots, \delta_m)$-Thue-Morse sequence.

(1) Prove $3 \le p(m+1) \le m+1$. In fact, by

$$p(m+1) = \sum_{k=0}^{m} \delta_k e^{\frac{2k\pi i}{m}} = \sum_{k=0}^{m} \delta_k \cos\frac{2k\pi}{m} + i\sum_{k=0}^{m} \delta_k \sin\frac{2k\pi}{m},$$

it suffices to consider the following ① and ②.

① We have $\sum_{k=0}^{m} \delta_k \sin\frac{2k\pi}{m} = 0$ since for all $k \in \{0, 1, \cdots, \lfloor\frac{m}{2}\rfloor\}$,

$$\delta_k \sin\frac{2k\pi}{m} + \delta_{m-k}\sin\frac{2(m-k)\pi}{m} \overset{\delta_k=\delta_{m-k}}{=\!=\!=} \delta_k(\sin\frac{2k\pi}{m} + \sin(2\pi - \frac{2k\pi}{m})) = 0.$$

② Prove $3 \le \sum_{k=0}^{m} \delta_k \cos\frac{2k\pi}{m} \le m+1$.
Since $\delta_k \cos\frac{2k\pi}{m} = 1$ for $k \in \{0, m\}$ and $\delta_k \cos\frac{2k\pi}{m} \le 1$ for $k \in \{1, 2\cdots, m-1\}$, we only need to check $\sum_{k=1}^{m-1} \delta_k \cos\frac{2k\pi}{m} \ge 1$. It suffices to consider the following i) and ii).

i) Prove $\delta_k \cos\frac{2k\pi}{m} \ge 0$ for all $k \in \{1, \cdots, m-1\}$.

ⓐ If $0 \le k \le \lfloor\frac{m}{4}\rfloor$, we have $\delta_k = +1$ and $0 \le \frac{2k\pi}{m} \le \frac{\pi}{2}$.

ⓑ If $\lfloor\frac{m}{4}\rfloor + 1 \le k \le m - \lfloor\frac{m}{4}\rfloor - 1$, we have $\delta_k = -1$ and $\frac{\pi}{2} \le \frac{2k\pi}{m} \le \frac{3\pi}{2}$.

ⓒ If $m - \lfloor\frac{m}{4}\rfloor \le k \le m$, we have $\delta_k = +1$ and $\frac{3\pi}{2} \le \frac{2k\pi}{m} \le 2\pi$.

ii) ⓐ If m is even, we have $\delta_{\frac{m}{2}} \cos\frac{2\cdot\frac{m}{2}\cdot\pi}{m} = 1$.

ⓑ If m is odd, we have

$$\begin{aligned}\delta_{\frac{m-1}{2}} \cos\frac{2\cdot\frac{m-1}{2}\cdot\pi}{m} + \delta_{\frac{m+1}{2}} \cos\frac{2\cdot\frac{m+1}{2}\cdot\pi}{m} &= -\cos(\pi - \frac{\pi}{m}) - \cos(\pi + \frac{\pi}{m}) \\ &= 2\cos\frac{\pi}{m} \ge 2\cos\frac{\pi}{3} = 1.\end{aligned}$$

(2) Since Theorem 3.4.1 says that the $(+1, \delta_1, \cdots, \delta_m)$-Koch curve is the unique attractor of the $(+1, \delta_1, \cdots, \delta_m)$-IFS $\{S_j\}_{0\le j\le m}$, to complete the proof, by applying Theorem 3.4.5, it suffices to check that $\{S_j\}_{0\le j\le m}$ satisfies the OSC.

When $m = 2$, we have $\delta_0 = +1, \delta_1 = -1, \delta_2 = +1$, $p(m+1) = 3$, $S_0(z) = \frac{z}{3}$, $S_1(z) = \frac{z}{3} + \frac{1}{3}$ and $S_2(z) = \frac{z}{3} + \frac{2}{3}$ for $z \in \mathbb{C}$, and we can take the open set $\{x + yi : x, y \in (0, 1)\}$.

When $m = 3$, we have $\delta_0 = +1, \delta_1 = \delta_2 = -1, \delta_3 = +1$, $p(m+1) = 3$, $S_0(z) = \frac{z}{3}$, $S_1(z) = \frac{1}{3} - \frac{z}{3}e^{\frac{2\pi i}{3}}$, $S_2(z) = \frac{1}{3} - \frac{1}{3}e^{\frac{2\pi i}{3}} - \frac{z}{3}e^{\frac{4\pi i}{3}}$ and $S_3(z) = \frac{z}{3} + \frac{2}{3}$ for $z \in \mathbb{C}$. The attractor of this IFS is exactly the classical Koch curve and this IFS satisfies the OSC, where the open set can be taken by the open isosceles triangle $\{x + yi : x, y \in \mathbb{R}, y < 0, x + \sqrt{3}y > 0, x - \sqrt{3}y < 1\}$.

In the following we consider $m \ge 4$. Let

$$a_m := \sum_{k=0}^{\lfloor\frac{m}{4}\rfloor} \cos\frac{2k\pi}{m} \quad \text{and} \quad b_m := \sum_{k=0}^{\lfloor\frac{m}{4}\rfloor} \sin\frac{2k\pi}{m}.$$

Then $a_m, b_m > 0$ and $p(\lfloor \frac{m}{4} \rfloor + 1) = a_m + b_m i$.
① If $m \equiv 0, 1$ or $2 \bmod 4$, define

$$V := \Big\{ x + yi : x, y \in \mathbb{R}, y > 0, b_m x - a_m y > 0, b_m x + a_m y < b_m \Big\}.$$

See Figures 3.1, 3.2 and 3.3. Obviously V is the non-empty bounded open isosceles triangle with base $[0,1]$ and vertex $\frac{1}{2} + \frac{b_m}{2a_m} i$. Note that for each $j \in \{0, 1, \cdots, m\}$, S_j is the composition of the rotation $\delta_j e^{\frac{2j\pi i}{m}} \cdot$, the scaling $(p(m+1))^{-1} \cdot$ and the translation $\cdot + \frac{p(j)}{p(m+1)}$, and S_j maps $[0,1]$ to $[\frac{p(j)}{p(m+1)}, \frac{p(j+1)}{p(m+1)}]$. It is straightforward to see that $\{S_j(V)\}_{0 \le j \le m}$ are the disjoint open isosceles triangles with bases $\{[\frac{p(j)}{p(m+1)}, \frac{p(j+1)}{p(m+1)}]\}_{0 \le j \le m}$ and vertexes $\{S_j(\frac{1}{2} + \frac{b_m}{2a_m} i)\}_{0 \le j \le m}$ all on the upper side of the polygonal line $\frac{P(1)}{p(m+1)}$. To verify $\cup_{j=0}^{m} S_j(V) \subset V$, in the following we check Im $p(\frac{m+1}{2}) \ge 0$ if m is odd and Im $p(\frac{m}{2}) \ge 0$ if m is even.

i) If m is odd, by $m \equiv 1 \bmod 4$, we have $\frac{m-1}{2} = 2\lfloor \frac{m}{4} \rfloor$ and then

$$\begin{aligned}
\text{Im } p(\frac{m+1}{2}) &= \sum_{k=0}^{\lfloor \frac{m}{4} \rfloor} \sin \frac{2k\pi}{m} - \sum_{k=\lfloor \frac{m}{4} \rfloor + 1}^{\frac{m+1}{2} - 1} \sin \frac{2k\pi}{m} \\
&= \sum_{k=1}^{\lfloor \frac{m}{4} \rfloor} \sin \frac{2k\pi}{m} - \sum_{k=\lfloor \frac{m}{4} \rfloor + 1}^{2\lfloor \frac{m}{4} \rfloor} \sin \frac{2k\pi}{m} \\
&= \sum_{k=1}^{\lfloor \frac{m}{4} \rfloor} \sin \frac{2k\pi}{m} - \sum_{k=1}^{\lfloor \frac{m}{4} \rfloor} \sin \frac{2(2\lfloor \frac{m}{4} \rfloor + 1 - k)\pi}{m} \\
&= \sum_{k=1}^{\lfloor \frac{m}{4} \rfloor} (\sin \frac{2k\pi}{m} - \sin \frac{(2k-1)\pi}{m}) \ge 0.
\end{aligned}$$

ii) If m is even and $m \equiv 2 \bmod 4$, by $\frac{m}{2} - 1 = 2\lfloor \frac{m}{4} \rfloor$, in a way similar to i) we can get Im $p(\frac{m}{2}) = 0$.

iii) If m is even and $m \equiv 0 \bmod 4$, we have

$$\begin{aligned}
\text{Im } p(\frac{m}{2}) &= \sum_{k=0}^{\frac{m}{4}} \sin \frac{2k\pi}{m} - \sum_{k=\frac{m}{4}+1}^{\frac{m}{2}-1} \sin \frac{2k\pi}{m} \\
&= \sum_{k=1}^{\frac{m}{4}-1} \sin \frac{2k\pi}{m} + \sin \frac{2 \cdot \frac{m}{4} \cdot \pi}{m} - \sum_{k=1}^{\frac{m}{4}-1} \sin \frac{2(\frac{m}{2} - k)\pi}{m} \\
&= 1 + \sum_{k=1}^{\frac{m}{4}-1} (\sin \frac{2k\pi}{m} - \sin \frac{(m-2k)\pi}{m}) = 1 \ge 0.
\end{aligned}$$

② If $m \equiv 3 \bmod 4$, we have $\frac{m+1}{2} - 1 = 2\lfloor \frac{m}{4} \rfloor + 1$ and then

$$\begin{aligned}
\text{Im } p(\frac{m+1}{2}) &= \sum_{k=0}^{\lfloor \frac{m}{4} \rfloor} \sin \frac{2k\pi}{m} - \sum_{k=\lfloor \frac{m}{4} \rfloor+1}^{2\lfloor \frac{m}{4} \rfloor+1} \sin \frac{2k\pi}{m} \\
&= \sum_{k=0}^{\lfloor \frac{m}{4} \rfloor} \sin \frac{2k\pi}{m} - \sum_{k=0}^{\lfloor \frac{m}{4} \rfloor} \sin \frac{2(2\lfloor \frac{m}{4} \rfloor + 1 - k)\pi}{m} \\
&= \sum_{k=0}^{\lfloor \frac{m}{4} \rfloor} (\sin \frac{2k\pi}{m} - \sin \frac{(2k+1)\pi}{m}) < 0.
\end{aligned}$$

Let

$$c_m := -\frac{\text{Im } p(\frac{m+1}{2})}{p(m+1)} > 0$$

and define

$$V := \Big\{ x + yi : x, y \in \mathbb{R}, b_m x - a_m y > 0, b_m x + a_m y < b_m, 2c_m x + y > 0, 2c_m x - y < 2c_m \Big\}.$$

See Figure 3.4. Obviously V is the non-empty bounded open quadrilateral containing two isosceles triangles with the same base $[0, 1]$ and one has vertex $\frac{1}{2} + \frac{b_m}{2a_m} i$ and the other has vertex $\frac{1}{2} - c_m i$. It is straightforward to see that $\{S_j(V)\}_{0 \le j \le m}$ are open quadrilaterals, each containing two isosceles triangles with the same base $[\frac{p(j)}{p(m+1)}, \frac{p(j+1)}{p(m+1)}]$ where one triangle has vertex $S_j(\frac{1}{2} + \frac{b_m}{2a_m} i)$ on the upper side of the polygon $\frac{P(1)}{p(m+1)}$ and the other has vertex $S_j(\frac{1}{2} - c_m i)$ on the lower side. By simple geometrical relation we know that $\{S_j(V)\}_{0 \le j \le m}$ are all disjoint and contained in V. □

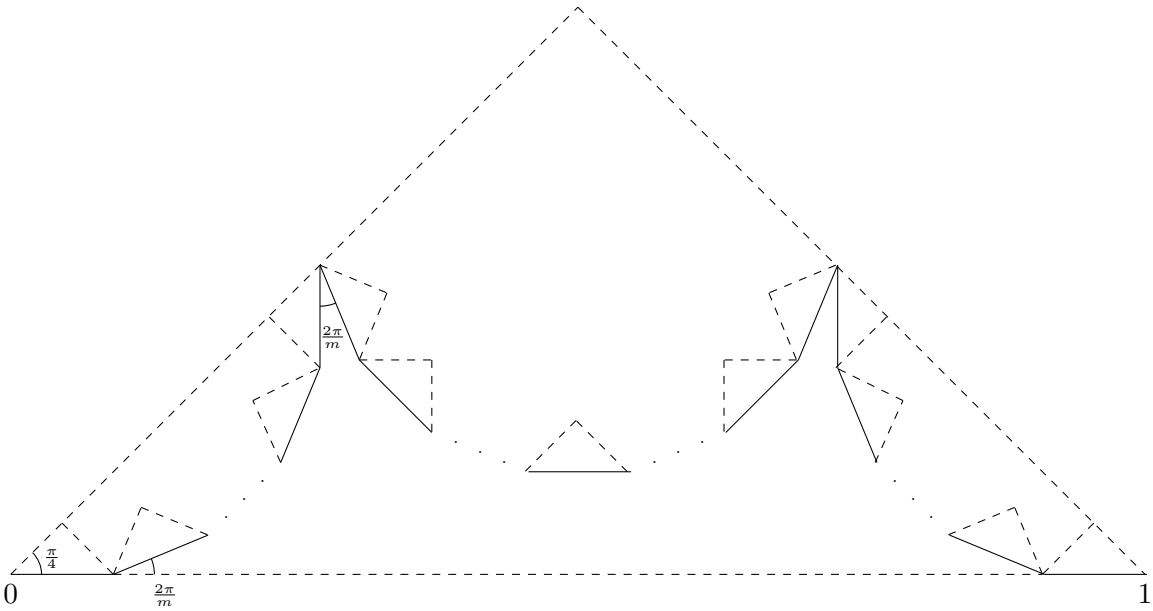


Figure 3.1: The open sets $V, S_0(V), \cdots, S_m(V)$ and geometrical relation for $m \equiv 0 \bmod 4$ where $m \geq 4$.

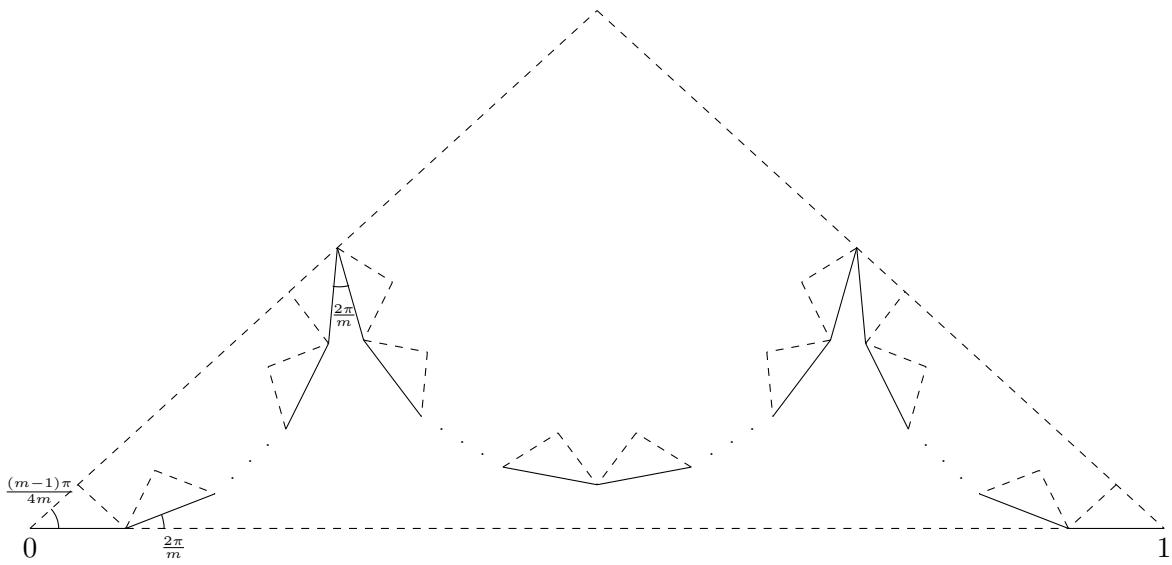


Figure 3.2: The open sets $V, S_0(V), \cdots, S_m(V)$ and geometrical relation for $m \equiv 1 \bmod 4$ where $m \geq 4$.

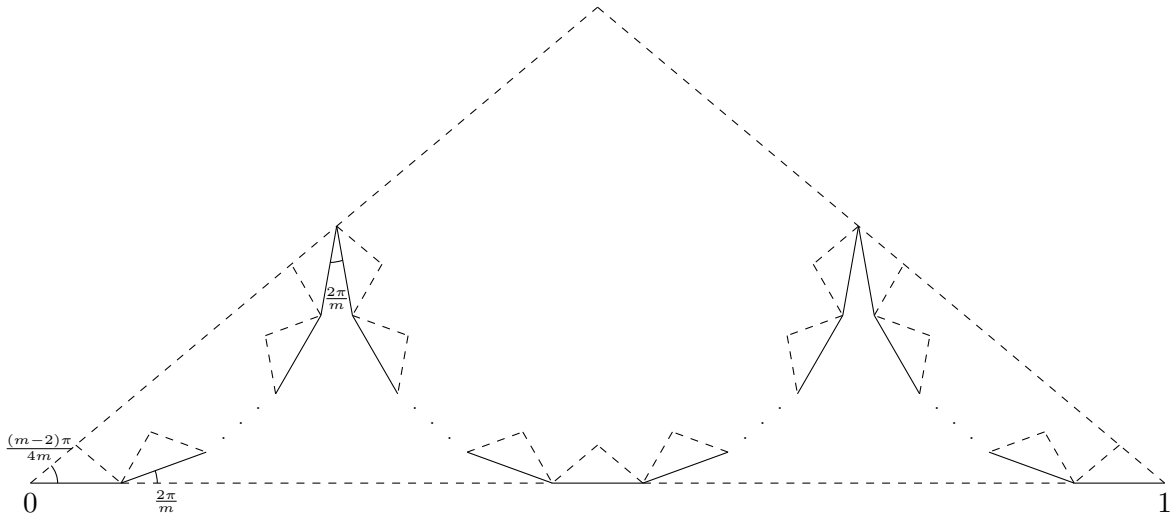


Figure 3.3: The open sets $V, S_0(V), \cdots, S_m(V)$ and geometrical relation for $m \equiv 2 \bmod 4$ where $m \geq 4$.

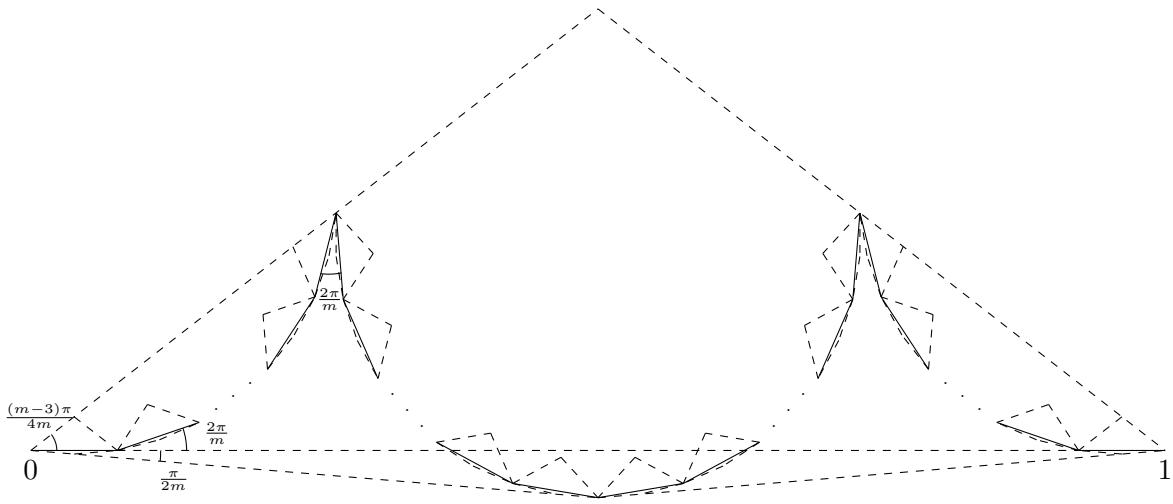


Figure 3.4: The open sets $V, S_0(V), \cdots, S_m(V)$ and geometrical relation for $m \equiv 3 \bmod 4$ where $m \geq 4$.

Research achievements during doctoral study

• Yao-Qiang Li and Bing Li, *Distributions of full and non-full words in beta-expansions*, J. Number Theory 190 (2018), 311–332.
• Yao-Qiang Li, *Digit frequencies of beta-expansions*, Acta Math. Hungar. 162 (2020), 403–418.
• Yao-Qiang Li, *Expansions in multiple bases*, Acta Math. Hungar. 163 (2021), 576–600.
• Yao-Qiang Li, *Divisibility Properties of Factors of the Discriminant of Generalized Fibonacci Numbers*, Fibonacci Quart. 59 (2021), 65–77.
• Yao-Qiang Li, *Infinite products related to generalized Thue-Morse sequences*, Monatsh. Math. 194 (2021), 577–600.
• Yao-Qiang Li, *Hausdorff dimension of frequency sets of univoque sequences*, Dyn. Syst. 36 (2021), 340–361.
• Yao-Qiang Li, *Generalized Koch curves and Thue-Morse sequences*, accepted by Fractals (2021).
• Yao-Qiang Li, *Hausdorff dimension of frequency sets in beta-expansions*, arXiv:1905.01481v3
• Bing Li, Yao-Qiang Li, and Tuomas Sahlsten, *Random walks associated to beta-shifts*, arXiv:1910.13006

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