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Relative Calabi-Yau structures in representation theory

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Résumé

La catégorie amassée généralisée (supérieure) associée à une dg-algèbre $(n+1)$-Calabi–Yau appropriée a été introduite par Claire Amiot et Lingyan Guo. Elle est Hom-finie, n-Calabi-Yau et admet un objet canonique object n-amas-basculant. Notre premier objectif dans cette thèse est de généraliser leur construction au contexte relatif. Nous prouvons l'existence d'un object n-amas-basculant dans une catégorie extriangulée de Frobenius qui est stablement n-Calabi–Yau et Hom-finie, associée à un morphisme $(n+1)$-Calabi–Yau. Nos résultats s'appliquent en particulier aux dg-algèbres relatives de Ginzburg provenant de carquois glacé à potentiel. Ils sont étroitement associées par Iyama–Oppermann aux algeébres de n-représentation finie.

En 2009, Keller et Yang ont catégorifié les mutations du carquois en l'interprétant en termes d'équivalences entre catégories dérivées. Leur approche était basée sur les algèbres de Calabi-Yau de Ginzburg et sur la mutation de Derksen-Weyman-Zelevinsky des carquois à potentiel. Récemment, Matthew Pressland a généralisé la mutation des carquois à potentiel à celle des carquois glacés à potentiel. Notre deuxième objectif dans cette thèse est de catégorifié la mutation de Pressland. Nous montrons que sa règle produit des équivalences dérivées entre les algèbres de Ginzburg relatives associées, qui sont des cas particuliers des complétions relatives Calabi-Yau déformées de Yeung. Nous donnons également une catégorifiation de la mutation aux sommets gelés telle qu'elle apparaît dans les travaux récents de Fraser–Sherman-Bennett sur les structures d'amas sur les variétés positroïdes ouvertes.

Mots-clefs. Structures relatives de Calabi–Yau, complétions relatives de Calabi–Yau, catégories relatives amassées, mutations de carquois glacés, carquois glacés à potentiel, algèbres de Ginzburg relatives, équivalences dérivées.

Abstract

The generalized (higher) cluster category arising from a suitable $(n+1)$-Calabi–Yau differential graded algebra was introduced by Claire Amiot and Lingyan Guo. It is Hom-finite, n-Calabi-Yau and admits a canonical n-cluster-tilting object. Our first aim in this thesis is to generalize their construction to the relative context. We prove the existence of an n-cluster tilting object in a Frobenius extriangulated category which is stably n-Calabi–Yau and Hom-finite, arising from a left $(n+1)$-Calabi–Yau morphism. Our results apply in particular to relative Ginzburg dg algebras coming from ice quivers with potential. They are closely linked to Iyama–Oppermann's theory of $(n+1)$-preprojective algebras of n-representation-finite algebras.

In 2009, Keller and Yang categorified quiver mutations by interpreting it in terms of equivalences between derived categories. Their approach was based on Ginzburg's Calabi–Yau algebras and on Derksen–Weyman–Zelevinsky's mutation of quivers with potential. Recently, Matthew Pressland has generalized mutation of quivers with potential to that of ice quivers with potential. Our second aim in this thesis is to categorify Pressland's mutation. We show that his rule yields derived equivalences between the associated relative Ginzburg algebras, which are special cases of Yeung's deformed relative Calabi–Yau completions. We also give a categorification of mutation at frozen vertices as it appears in recent work of Fraser–Sherman-Bennett on positroid cluster structures.

Keywords. Relative Calabi–Yau structures, relative Calabi–Yau completions, relative Cluster categories, ice quiver mutations, ice quivers with potential, relative Ginzburg algebras, derived equivalences.

À mes parents

Contents

Introduction

0.0.1 Version française

Cette thèse est consacrée à l'étude de la catégorification des algèbres amassées à coefficients en utilisant le formalisme des structures de Calabi–Yau relatives qui a été développé par Bertrand Toën [85, pp. 227-228] et Brav–Dyckerhoff [16].

Il y a à peu près 20 ans, Fomin et Zelevinsky [27] ont inventé les algèbres amassées afin de créer un cadre combinatoire pour l'étude des bases canoniques [52, 69] dans les groupes quantiques et l'étude de la positivité totale dans les groupes algébriques [70]. Parmi ces algèbres, il y a les algèbres de coordonnées homogènes sur les Grassmanniennes, sur les variétés de drapeaux et sur de nombreuses autres variétés qui jouent un rôle important dans la géométrie et la théorie des représentations. Il s'est rapidement avéré que la combinatoire des algèbres amassées apparaît également dans de nombreux autres sujets, par exemple dans la géométrie de Poisson [34, 35, 36, 37], les espaces de Teichmüller supérieurs [24, 25, 26], et dans la théorie des représentations des carquois et des algèbres de dimension finie [11, 12].

Une algèbre amassées est une algèbre commutative avec une famille distinguée de générateurs, appelées variables d'amas, qui ont des propriétés combinatoires spéciales. Pour construire une algèbre amassée, on part d'une graine

$$(X = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{m]}, B)$$

constituée, par définition, d'un ensemble X qui engendre librement un corps ambiant $\mathcal{F} = \mathbb{Q}(x_1 \ldots x_m)$ et d'une matrice $B = (b_{ij})$ de taille $m \times n$, à coefficients entiers dont la partie principale $B' = (b_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant n}$ est antisymétrique. De façon équivalente, au lieu de la matrice B, nous pouvons utiliser un carquois fini Q avec des sommets $1, 2, \ldots, m$, et sans cycles orientés de longueur 1 ou 2. Pour chaque $i = 1, \ldots, n$, la mutation $\mu_i(X, Q) = (X', Q')$ est définie en remplaçant d'abord x_i par un autre élément x_i^* dans $\mathcal{F}$ selon une règle spécifique qui dépend à la fois de $(x_1, \ldots, x_m)$ et de Q. Ensuite, nous obtenons un nouvelle partie génératrice libre $X' = (x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_n, x_{n+1}, \ldots, x_m)$. Le carquois muté $\mu_i(Q) = Q'$ est obtenu à partir de Q en appliquant une certaine règle combinatoire dépendant de i aux flèches de Q. Cela donne la nouvelle graine (X', Q'). Nous continuons à appliquer $\mu_1, \ldots, \mu_n$ à la nouvelle graine pour obtenir d'autres graines. Les ensembles à m éléments X'' présents dans les graines (X'', Q'') sont appelés *amas*, et les éléments des amas sont appelés *variables d'amas*. Les variables d'amas $x_{n+1}, \ldots, x_m$ ne peuvent pas être mutées, elles sont appelées *variables gelées*. L'algèbre amassées associée est la sous-algèbre du corps de fonctions $\mathcal{F}$ engendrée par toutes les variables d'amas.

Étant donné que la combinatoire des algèbres amassées est très compliquée, il est utile de les modéliser catégoriquement, car au niveau catégorique, davantage d'outils conceptuels sont disponibles. Considérons une algèbre amassée $\mathcal{A}$ sans variables gelées et telle que l'un des amas ait un carquois acyclique Q. Dans ce cas particulier, Buan–Marsh-Reineke–Reiten–Todorov [12] ont introduit la catégorie amassée $\mathcal{C}_Q$, définie comme la catégorie d'orbites

$$\mathcal{C}_Q = \mathcal{D}^b(kQ)/\tau\Sigma^{-1},$$

où τ désigne la translation d'Auslander-Reiten de la catégorie dérivée $\mathcal{D}^b(kQ)$ et Σ le foncteur suspension de $\mathcal{D}^b(kQ)$. Il a été démontré qu'elle était triangulée par un résultat de Keller [59]. La catégorie amassée possède un ensemble distingué $\mathcal{T}$ d'objets T, appelés *objets amas-basculants*. Tout objet T dans $\mathcal{T}$ satisfait

$$\mathrm{add}T = \{X \in \mathcal{C}_Q : \mathrm{Ext}^1_{\mathcal{C}_Q}(X,T) = 0\} = \{X \in \mathcal{C}_Q : \mathrm{Ext}^1_{\mathcal{C}_Q}(T,X) = 0\}.$$

Alors $\mathcal{C}_Q$, avec T, a les propriétés agréables suivantes :

- Tout objet T dans $\mathcal{T}$ a la forme $T = T_1 \oplus \cdots \oplus T_n$ où les T_i sont indécomposables et $T_i \not\cong T_j$ pour $i \neq j$. Les T_i correspondent alors aux variables d'amas. Le carquois de chaque amas est donné par le carquois de l'algèbre d'endomorphismes $\mathrm{End}_{\mathcal{C}_Q}(T)$ de l'objet amas-basculant correspondant.

- Pour chaque $i = 1, \ldots, n$, nous avons un objet indécomposable unique $T_i^* \not\cong T_i$, où T_i est un facteur direct indécomposable de T tel que $T/T_i \oplus T_i^*$ est dans $\mathcal{T}$. Le passage de T à $T/T_i \oplus T_i^*$ correspond à la mutation des graines dans la définition de l'algèbre amassée. Ce type de mutation est bien défini dans toute catégorie triangulée 2-Calabi–Yau dans laquelle les carquois des algèbres d'endomorphismes des objets amas-basculants n'ont pas de boucles [50].

- La catégorie amassée $\mathcal{C}_Q$ est équipée d'un caractère d'amas $M \mapsto \varphi_M \in \mathcal{A}$ [19], qui envoie les objets sur des éléments de l'algèbre amassée. Les variables d'amas sont données par φ_M pour M un objet rigide indécomposable de $\mathcal{C}_Q$, c'est-à-dire un M indécomposable tel que $\mathrm{Ext}^1_{\mathcal{C}_Q}(M,M) = 0$.

Claire Amiot [4] a généralisé la construction de la catégorie amassée à certaines algèbres de dimension finie A_0 de dimension globale $\leqslant 2$. Dans son approche, afin de montrer qu'il existe une équivalence triangulée entre $\mathcal{C}_{A_0}$, construit comme une enveloppe triangulée [59], et la catégorie quotient $\mathrm{per}\,\mathbf{\Pi}_3(A_0)/\mathrm{pvd}\,\mathbf{\Pi}_3(A_0))$, où $\mathbf{\Pi}_3(A_0)$ est la complétion 3-Calabi-Yau [62] de A_0, elle a d'abord étudié la catégorie $\mathcal{C}_{\mathbf{\Pi}} = \mathrm{per}(\mathbf{\Pi})/\mathrm{pvd}(\mathbf{\Pi})$ associée à une dg-algèbre $\mathbf{\Pi}$ avec les quatre propriétés suivantes :

- $\mathbf{\Pi}$ est homologiquement lisse,

- $\mathbf{\Pi}$ est connective, c'est-à-dire que la cohomologie de $\mathbf{\Pi}$ s'annule en degrés > 0,

- $\mathbf{\Pi}$ est 3-Calabi-Yau en tant que bimodule,

- $H^0(\mathbf{\Pi})$ est de dimension finie.

Théorème 0.0.1. *[4] Soit $\mathbf{\Pi}$ une dg-algèbre avec les propriétés ci-dessus. Alors la catégorie triangulée*

$$\mathcal{C}_{\mathbf{\Pi}} = \mathrm{per}(\mathbf{\Pi})/\mathrm{pvd}(\mathbf{\Pi})$$

est Hom-finie, 2-Calabi–Yau et l'objet $\mathbf{\Pi}$ est un objet amas-basculant avec

$$\mathrm{End}_{\mathcal{C}_n(\mathbf{\Pi})}(\mathbf{\Pi}) \simeq H^0(\mathbf{\Pi}).$$

En particulier, si nous prenons pour A_0 l'algèbre des chemins kQ d'un carquois acyclique Q, la catégorie amassée $\mathcal{C}_Q$ est équivalente à $\mathcal{C}_{\mathbf{\Pi}_3(kQ)}$. Plus tard, Lingyan Guo [39] a généralisé la construction d'Amiot aux algèbres de dimension finie A de dimension globale $\leqslant n$ et aux dg-algèbres satisfaisant $1), 2), 4)$ et n-Calabi–Yau comme bimodule.

Soit $\mathbf{\Pi}$ une dg-algèbre avec les propriétés ci-dessus. Soit $\mathcal{F}(\mathbf{\Pi})$ la sous-catégorie pleine de $\mathrm{per}(\mathbf{\Pi})$ définie par

$$\mathcal{F}(\mathbf{\Pi}) = (\mathrm{per}(\mathbf{\Pi}))_{\leqslant 0} \cap (\mathrm{per}(\mathbf{\Pi})_{\geqslant -2})^{\perp},$$

où $(\mathrm{per}(\mathbf{\Pi}))_{\leqslant p}$ (resp. $(\mathrm{per}(\mathbf{\Pi}))_{\geqslant p}$) est la sous-catégorie pleine de $\mathrm{per}(\mathbf{\Pi})$ constituée des objets ayant leur homologie concentrée en degrés $\leqslant p$ (resp. $\geqslant p$). C'est ce qu'on appelle le *domaine fondamental de* $\mathrm{per}(\mathbf{\Pi})$.

Théorème 0.0.2. *[4] La composition suivante est une équivalence de catégories k-linéaires*

$$\mathcal{F}(\mathbf{\Pi}) \hookrightarrow \mathrm{per}(\mathbf{\Pi}) \to \mathrm{per}(\mathbf{\Pi})/\mathrm{pvd}(\mathbf{\Pi}) = \mathcal{C}(\mathbf{\Pi}).$$

De plus, le diagramme suivant commute

$$\begin{tikzcd}
\mathrm{per}(\mathbf{\Pi}) \supset \mathcal{F}(\mathbf{\Pi}) \arrow[rr, "\simeq"] \arrow[dr, "H^0"'] & & \mathcal{C}(\mathbf{\Pi}) \arrow[dl, "\mathrm{Hom}_{\mathcal{C}}(\mathbf{\Pi},?)"] \\
& \mathrm{mod} H^0(\mathbf{\Pi}) &
\end{tikzcd}.$$

Soit (Q, W) un carquois à potentiel (QP). *La dg-algèbre de Ginzburg* $\mathbf{\Gamma}(Q, W)$ associée est homologiquement lisse et porte une structure canonique 3-Calabi-Yau à gauche [38, 62]. Ainsi, elle satisfait les propriétés (1), (2) et (3) du théorème 0.0.1. L'homologie en degré 0 de $\mathbf{\Gamma}(Q, W)$ est l'algèbre jacobienne $J(Q, W)$.

Théorème 0.0.3. *[4] Soit (Q, W) un carquois à potentiel Jacobi-fini, c'est-à-dire que l'algèbre jacobienne correspondante $J(Q, W)$ est de dimension finie. Alors la catégorie*

$$\mathcal{C}(Q, W) = \mathrm{per}\,\mathbf{\Gamma}(Q, W)/\mathrm{pvd}\,\mathbf{\Gamma}(Q, W)$$

est Hom-finie et possède un objet amas-basculant canonique dont l'algèbre d'endomorphismes est isomorphe à $J(Q, W)$.

La catégorie $\mathcal{C}(Q, W)$ est appelée *la catégorie amassée* associée au carquois à potentiel (Q, W). Si (Q, W) n'est pas Jacobi-fini, une généralisation de la catégorie $\mathcal{C}(Q, W)$, qui n'est pas Hom-finie, a été construite par Plamondon dans [74].

Soit Q un carquois fini et i une source de Q, c'est-à-dire un sommet sans flèches entrantes. Soit Q' la mutation de Q par rapport à i, c'est-à-dire le carquois obtenu à partir de Q en renversant toutes les flèches qui partent de i. Soient k un corps, kQ l'algèbre des chemins de Q et $\mathcal{D}(kQ)$ la catégorie dérivée de la catégorie de tous les kQ-modules à droite. Pour un sommet j de Q' respectivement de Q, soit P_j respectivement P'_j l'indécomposable projectif associé au sommet j. Par le résultat principal de Bernstein-Gelfand-Ponomarev [9] reformulé en termes de catégories dérivées suivant Happel [40], il existe une équivalence triangulée canonique

$$F : \mathcal{D}(kQ') \to \mathcal{D}(kQ)$$

qui envoie P'_j sur P_j pour $j \neq i$ et P'_i sur le cône du morphisme

$$P_i \to \bigoplus P_j$$

dont les composantes sont les multiplications à gauche par toutes les flèches qui partent de i. Cela donne une interprétation catégorique de la mutation par rapport à une *source* i.

Keller et Yang [64] ont obtenu un résultat analogue pour la mutation d'un carquois à potentiel (Q, W) par rapport à un sommet arbitraire i, où le rôle du carquois avec flèches renversées est joué par le carquois à potentiel (Q', W') obtenu à partir de (Q, W) par mutation dans le sens de Derksen-Weyman-Zelevinsky [23]. Le rôle de la catégorie dérivée $\mathcal{D}(kQ)$ est maintenant joué par la catégorie dérivée $\mathcal{D}(\mathbf{\Gamma})$ de l'algèbre différentielle graduée complète $\mathbf{\Gamma} = \mathbf{\Gamma}(Q, W)$ associée à (Q, W).

Soit (Q, W) un carquois à potentiel tel que Q n'a pas de boucles. Soit i un sommet tel que Q n'a pas 2-cycles passant par i. Soit $\mu_i(Q, W)$ la mutation de (Q, W) par rapport au sommet i [23]. Soient $\mathbf{\Gamma} = \mathbf{\Gamma}(Q, W)$ et $\mathbf{\Gamma}' = \mathbf{\Gamma}(\mu_i(Q, W))$ les algèbres complétées de Ginzburg associées respectivement à (Q, W) et $\mu_i(Q, W)$. Pour un sommet j de Q, soit $P_j = e_j\mathbf{\Gamma}$ et $P'_j = e_j\mathbf{\Gamma}'$. Keller–Yang ont prouvé les résultats suivants [64].

Théorème 0.0.4. *[64] Il existe une équivalence triangulée qui envoie P'_j sur P_j pour $j \neq i$ et sur le cône T_i^* du morphisme*

$$P_i \to \bigoplus_{\alpha} P_{t(\alpha)}$$

pour $i = j$, où nous avons un facteur $P_{t(\alpha)}$ pour chaque flèche α de source i et la composante correspondante du morphisme est la multiplication à gauche par α. Le foncteur F se restreint à des équivalences triangulées de $\mathrm{per}(\mathbf{\Pi}')$ *sur* $\mathrm{per}(\mathbf{\Pi})$ *et de* $\mathrm{pvd}(\mathbf{\Pi}')$ *sur* $\mathrm{pvd}(\mathbf{\Pi})$.

Leur résultat est un analogue mais pas une généralisation de celui de Bernstein-Gelfand-Ponomarev puisque même si le potentiel W s'annule, la catégorie dérivée $\mathcal{D}(\mathbf{\Gamma})$ n'est pas équivalente à $\mathcal{D}(kQ)$.

Remarque 0.0.1. *Il existe également une équivalence triangulée*

$$F' : \mathcal{D}(\mathbf{\Gamma}') \to \mathcal{D}(\mathbf{\Gamma})$$

qui, pour $j \neq i$, envoie P'_j à P_j et, pour $i = j$, sur le cône décalé

$$T'_i = \Sigma^{-1}(\bigoplus_{\beta} P_{s(\beta)} \to P_i),$$

où nous avons un facteur $P_{s(\beta)}$ pour chaque flèche β de cible i et la composante correspondante du morphisme est la multiplication à gauche par β. Les deux foncteurs F et F' sont liés par le foncteur de torsion t_{S_i} par rapport à l'objet 3-sphérique S_i (le dg-module simple sur $\mathbf{\Gamma}$ associé au sommet i). Plus précisément, nous avons

$$F' = t_{S_i} \circ F,$$

où $t_{S_i} : \mathcal{D}(\mathbf{\Gamma}) \to \mathcal{D}(\mathbf{\Gamma})$ est donné sur un objet X de $\mathcal{D}(\mathbf{\Gamma})$ par le triangle suivant

$$\mathbf{R}\mathrm{Hom}(S_i, X) \otimes S_i \to X \to t_{S_i}(X) \to \Sigma\mathbf{R}\mathrm{Hom}(S_i, X) \otimes S_i.$$

Les résultats suivants donnent un lien entre les dg-algèbres de Ginzburg associées à des QP liés par une mutation.

Théorème 0.0.5. *[6] Soit (Q, W) un QP sans boucles et $i \in Q_0$ un sommet qui n'est pas sur un 2-cycle dans Q. Désignons par $\Gamma = \Gamma(Q, W)$ et $\Gamma' = \Gamma(\mu_i(Q, W))$ les dg-algèbres de Ginzburg associées.*

a) Il existe des équivalences triangulées

$$\begin{tikzcd} \mathrm{per}(\mathbf{\Gamma}) \arrow[r, "\simeq"] & \mathrm{per}(\mathbf{\Gamma}') \\ \mathrm{pvd}(\mathbf{\Gamma}) \arrow[u, hook] \arrow[r, "\simeq"] & \mathrm{pvd}(\mathbf{\Gamma}') \arrow[u, hook] . \end{tikzcd}$$

Par conséquent, nous avons une équivalence triangulée $\mathcal{C}(Q, W) \simeq \mathcal{C}(\mu_i(Q, W))$.

b) Nous avons un diagramme

$$\begin{tikzcd} \mathrm{per}(\mathbf{\Gamma}) \arrow[r, "\simeq"] \arrow[d, "H^0"'] & \mathrm{per}(\mathbf{\Gamma}') \arrow[d, "H^0"] \\ \mathrm{mod}\, J(Q, W) \arrow[r, leftrightarrow, dotted, "DWZ-mutation"] & \mathrm{mod}\, J(\mu_i(Q, W)). \end{tikzcd}$$

Dans le cas des algèbres amassées avec des variables gelées, un modèle catégorique approprié devrait avoir certains objets apparaissant comme des facteurs indécomposables de chaque objet amas-basculant (ces objets correspondent à des sommets gelés). Prendre un quotient approprié de cette catégorie devrait correspondre à la suppression des variables gelées de l'algèbre amassée et la catégorie quotient devrait être la catégorie amassée habituelle.

Pour certaines algèbres amassées avec des variables gelées non inversibles, il existe un modèle naturel à cet effet, à savoir une catégorie de Frobenius $\mathcal{E}$, c'est-à-dire une catégorie exacte avec suffisamment d'objets projectifs et injectifs, et telle que les objets projectifs et injectifs coïncident. Alors par définition, chaque objet projectif-injectif I satisfait

$$\mathrm{Ext}^1_{\mathcal{E}}(I, ?) = 0 = \mathrm{Ext}^1_{\mathcal{E}}(?, I).$$

Ainsi, chaque objet projectif-injectif I est dans $\mathrm{add}\, T$ pour tout objet amas-basculant $T \in \mathcal{E}$. De plus, par un résultat de Happel [41], la catégorie stable $\underline{\mathcal{E}}$, formée en prenant le quotient par l'idéal des morphismes se factorisant à travers un objet projectif-injectif, est une catégorie triangulée. La catégorie stable correspondante $\mathcal{E}$ est 2-Calabi–Yau s'il existe une dualité bifonctorielle

$$\mathrm{Ext}^1_{\mathcal{E}}(X, Y) = D\mathrm{Ext}^1_{\mathcal{E}}(Y, X)$$

pour tous $X, Y \in \mathcal{E}$.

Remarque 0.0.2. *Pour les algèbres amassées avec des variables gelées inversibles, on peut considérer la catégorie dérivée d'une catégorie de Frobenius $\mathcal{E}$.*

Soient k un corps et Q un carquois de Dynkin. Soit J un sous-ensemble de Q_0. Nous désignons par $\widetilde{kQ}$ l'algèbre préprojective correspondante de kQ. Soit i un sommet. On désigne par S_i le $\widetilde{kQ}$-module simple supporté en i, par P_i sa couverture projective et par Q_i son enveloppe injective.

Soit $Q_J = \bigoplus_{i \in J} Q_j$. Geiß–Leclerc–Schröer [32] définissent la sous-catégorie pleine $\mathrm{Sub}\, Q_J$ de la catégorie $\mathrm{mod}\, \widetilde{kQ}$ des modules sur l'algèbre préprojective $\widetilde{kQ}$ comme ayant les objets isomorphes à des sous-modules de sommes directes finies de copies de Q_J. La catégorie $\mathrm{Sub}\, Q_J$ est stable par passage à des sous-modules. Elle a donc des noyaux qui sont en accord avec ceux de $\mathrm{mod}\, \widetilde{kQ}$, mais elle n'a pas de conoyaux en général. Cependant, elle est stable par extensions.

Ainsi, elle hérite de la structure d'une catégorie exacte dans laquelle une suite

$$X \to Y \to Z$$

en $\mathrm{Sub}\, Q_J$ est une conflation si et seulement si la suite

$$0 \to X \to Y \to Z \to 0$$

est exacte dans $\mathrm{mod}\, \widetilde{kQ}$.

Pour tout module $M \in \mathrm{mod}\, \widetilde{kQ}$, soit $\theta_J(M)$ le sous-module minimal de M tel que $M/\theta_J(M)$ est dans $\mathrm{Sub}\, Q_J$. Alors la projection canonique $M \to M/\theta_J(M)$ est une $\mathrm{Sub}\, Q_J$-approximation minimale à gauche de M. Nous définissons $F_i = I_i/\theta_J(I_i)$. Par [32, Proposition 3.2], ces F_i sont les objets projectifs-injectifs indécomposables dans $\mathrm{Sub}\, Q_J$. De plus, $\mathrm{Sub}\, Q_J$ est une catégorie de Frobenius.

Proposition 0.0.3. *[32] La catégorie $\mathrm{Sub}\, Q_J$ est une sous-catégorie fonctoriellement finie de $\mathrm{mod}\, \widetilde{kQ}$ qui est stable par extensions, de Frobenius et stablement 2-Calabi–Yau.*

En particulier, si nous prenons pour J l'ensemble des sommets Q_0, alors $\mathrm{Sub}\, Q_J$ est toute la catégorie $\mathrm{mod}\, \widetilde{kQ}$.

Une construction plus générale de $\mathrm{Sub}\, Q_J$ a été donnée par Buan–Iyama–Reiten–Scott dans [13]. Soit Q un carquois connexe fini sans cycles orientés. Nous désignons par $\{1, \ldots, n\}$ l'ensemble des sommets de Q. Pour un sommet i de Q, on désigne par I_i l'idéal $\widetilde{kQ}(1 - e_i)\widetilde{kQ}$ de $\widetilde{kQ}$. Nous désignons par W le *groupe de Coxeter* associé au carquois Q. Le groupe W est défini par les générateurs $1, \ldots, n$ et les relations:

- $i^2 = 1$ pour tous les i dans $\{1, \ldots, n\}$;

- $ij = ji$ s'il n'y a pas de flèches entre les sommets i et j;
- $iji = jij$ s'il y a exactement une flèche entre i et j.

Soit $w = i_1 i_2 \cdots i_r$ un mot réduit. Pour $m \leqslant r$, soit I_{w_m} l'idéal suivant :

$$I_{w_m} = I_{i_m} \cdots I_{i_2} I_{i_1}.$$

Pour simplifier les notations, nous écrivons I_w au lieu de I_{w_r}. Soit $\operatorname{Sub} \widetilde{kQ}/I_w$ la sous-catégorie pleine de $\operatorname{mod} \widetilde{kQ}$ dont les objets sont les $\widetilde{kQ}$-sous-modules de sommes finies de copies de $\widetilde{kQ}/I_w$. Buan, Iyama, Reiten et Scott ont prouvé les résultats suivants.

Théorème 0.0.6. *[13] La catégorie* $\operatorname{Sub} \widetilde{kQ}/I_w$ *est une catégorie de Frobenius et sa catégorie stable* $\underline{\operatorname{Sub}}\, \widetilde{kQ}/I_w$ *est* 2*-Calabi–Yau. L'objet* $T_w = \bigoplus_{m=1}^{r} e_{i_m} \widetilde{kQ}/I_w$ *est un objet amas-basculant.*

Soit (Q, F, W) un carquois glacé à potentiel, c'est-à-dire que Q est un carquois fini, F est un sous-carquois de Q et W un potentiel sur Q. Le sous-carquois F est appelé *sous-carquois gelé* de Q. Pour toute flèche α de Q, nous définissons la *dérivée cyclique* $\partial_\alpha W$ de W par rapport à α par

$$\partial_\alpha(\alpha_1 \cdots \alpha_k) = \sum_{\alpha_i = \alpha} \alpha_{i+1} \cdots \alpha_k \alpha_1 \cdots \alpha_{i-1}$$

pour n'importe quel chemin $\alpha_1 \cdots \alpha_k$, puis nous étendons linéairement cette application. Alors l'algèbre jacobienne *relative (gelée)* est définie comme le quotient

$$J(Q, F, W) = kQ/\langle \partial_\alpha W : \alpha \in Q_1 \setminus F_1 \rangle.$$

Pour chaque mot réduit w, Buan–Iyama–Reiten–Smith [14, Section 6] ont construit un carquois glacé à potentiel (Q_w, F_w, W_w) associé à w.

Theorem 0.0.4. *[14, Théorème 6.6] On a un isomorphisme d'algèbres*

$$\operatorname{End}_{\operatorname{Sub} \widetilde{kQ}/I_w}(T_w) \cong J(Q_w, F_w, W_w)$$

et donc un isomorphisme d'algèbres induit

$$\operatorname{End}_{\underline{\operatorname{Sub}}\, \widetilde{kQ}/I_w}(T_w) \cong J(\underline{Q_w}, \underline{W_w}),$$

où $\underline{Q_w}$ *est obtenu à partir de* Q_w *en supprimant les sommets gelés et les flèches incidentes avec des sommets gelés et* $\underline{W_w}$ *est le potentiel obtenu à partir de* W_w *par suppression des termes donnés par des cycles passant par des sommets gelés.*

Plus tard, Amiot–Reiten–Todorov ont démontré que la catégorie stable $\underline{\operatorname{Sub}}\, \widetilde{kQ}/I_w$ est en fait une catégorie amassée généralisée [6, Théorème 3.1].

Dans la construction d'Amiot de la catégorie amassée généralisée, l'hypothèse homologique clé est la propriété 3-Calabi-Yau de l'algèbre comme bimodule sur elle-même. Dans le contexte relatif, les structures Calabi–Yau relatives à droite ont été inventées par Bertrand Toën dans [85, pp. 227-228]. Plus tard, les structures Calabi–Yau relatives à droite et à gauche ont été étudiées par Chris Brav et Tobias Dyckerhoff dans [16]. Une structure n-Calabi–Yau relative à gauche sur un morphisme $f : B \to A$ entre des dg-algèbres lisses est la donnée d'une classe $[\xi]$ en homologie cyclique négative $HN_n(f)$ induisant certaines dualités dans $\mathcal{D}(B^e)$ et $\mathcal{D}(A^e)$. En particulier, si la dg-algèbre B est nulle, alors A est n-Calabi–Yau en tant que bimodule. Une façon canonique de produire des structures Calabi–Yau relatives à gauche est la complétion Calabi–Yau relative déformée qui a été introduite par Wai-kit Yeung [87]. Cela a généralisé la construction par Keller [62] de complétions n-Calabi–Yau déformées au contexte relatif.

L'objectif principal de cette thèse est de généraliser les constructions de Claire Amiot et Lingyan Guo au contexte relatif. Nous remplaçons les propriétés utilisées dans la construction d'Amiot par les propriétés suivantes d'un morphisme de dg-algèbres $f : B \to A$ (ne préservant pas nécessairement l'unité)

1) A et B sont homologiquement lisses,

2) A est connective, c'est-à-dire que la cohomologie de A s'annule en degrés > 0,

3) le morphisme $f : B \to A$ a une structure $(n+1)$-Calabi–Yau à gauche,

4) $H^0(A)$ est de dimension finie.

Nous introduisons la catégorie amassée relative $\mathcal{C}_n(A, B)$ associée à $f : B \to A$ et montrons qu'elle est Hom-finie sous les hypothèses ci-dessus. Nous prouvons l'existence d'un objet n-amas-basculant dans la catégorie de Higgs $\mathcal{H}$ qui est une sous-catégorie stable par extensions de $\mathcal{C}_n(A, B)$ et est stablement n-Calabi–Yau.

En 2009, Keller et Yang [64] ont catégorifié la mutation des carquois en l'interprétant en termes d'équivalences entre catégories dérivées (Théorème 0.0.8). Matthew Pressland a généralisé la mutation des carquois à potentiel à celle des carquois *glacés* à potentiel [80]. Notre deuxième objectif dans cette thèse est de catégorifier la mutation de Pressland des carquois glacés à potentiel. Nous montrons que sa règle donne des équivalences dérivées entre les algèbres de Ginzburg relatives associées, qui sont des cas particuliers de complétions de Calabi-Yau relatives déformées de Yeung [87]).

De manière inattendue, Fraser et Sherman-Bennett ont très récemment découvert une construction de mutation par rapport à certains sommets gelés dans leur étude [29] de structures amassées sur des variétés positroïdes. Soit v un sommet gelé. Supposons que v est un sommet source dans F tel qu'il n'y a pas de flèches non gelées de source v, ou que v est un sommet puits dans F tel qu'il n'y a pas de flèches non gelées de cible v. Notre résultat indique que la mutation au sommet gelé v est 'catégorifiée' par le foncteur twist (respectivement twist inverse) t_{S_v} (respectivement $t_{S_v}^{-1}$) par rapport à l'objet 2-sphérique S_v (le module simple au sommet v) dans la catégorie dérivée de l'algèbre préprojective dérivée complétée $\mathbf{\Pi}_2(F)$. Dans [86], nous montrerons comment des compositions appropriées de mutations par rapport à des sommets gelés peuvent être décatégorifiées en des isomorphismes quasi-amassés au sens de Fraser [28].

0.0.2 English version

This thesis is devoted to the study of the categorification of cluster algebras with coefficients by using the relative Calabi–Yau formalism which was developed by Bertrand Toën (see [85, pp. 227-228]) and Brav–Dyckerhoff (see [16]).

Almost 20 years ago, Fomin and Zelevinsky [27] invented cluster algebras, in order to create a combinatorial framework for the study of canonical bases [52, 69] in quantum groups and the study of total positivity in algebraic groups [70]. Among these algebras, there are the algebras of homogeneous coordinates on the Grassmannians, on the flag varieties and on many other varieties which play an important role in geometry and representation theory. It has rapidly turned out that the combinatorics of cluster algebras also appear in many other subjects, for example in Poisson geometry [34, 35, 36, 37], higher Teichmüller spaces [24, 25, 26], and in the representation theory of quivers and finite-dimensional algebras [11, 12].

A cluster algebra is a commutative algebra with a distinguished family of generators, called cluster variables, displaying special combinatorial properties. To construct it, we start with a seed

$$(X = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_m), B),$$

consisting, by definition, of a set X which freely generates an ambient field

$$\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_m),$$

and an integer $m \times n$ matrix $B = (b_{ij})$ such that the principal part $B' = (b_{ij})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant n}$ is skew symmetric. Or, we can instead of the matrix B use a finite quiver Q with vertices $1, 2, \ldots, m$, and without oriented cycles of length 1 or 2. For each $i = 1, \ldots, n$, the mutation $\mu_i(X, Q) = (X', Q')$ is defined by first replacing x_i with another element x_i^* in $\mathcal{F}$ according to a specific rule which depends upon both $(x_1, \ldots, x_m)$ and Q. Then we get a new free generating set $X' = (x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_n, x_{n+1}, \ldots, x_m)$. The mutated quiver $\mu_i(Q) = Q'$ is obtained from Q by applying a certain combinatorial rule depending on i to the arrows of Q. This yields the new seed (X', Q'). We continue applying $\mu_1, \ldots, \mu_n$ to the new seed to get further seeds. The m-element sets X'' occurring in seeds (X'', Q'') are called *clusters*, and the elements in the clusters are called *cluster variables*. The cluster variables $x_{n+1}, \ldots, x_m$ cannot be mutated, these are called *frozen variables*. The associated cluster algebra is the subalgebra of the function field $\mathcal{F}$ generated by all cluster variables.

Since the combinatorics of cluster algebras are very complicated, it is useful to model them categorically, where more conceptual tools become available. Consider a cluster algebra $\mathcal{A}$ without frozen variables and such that one of the clusters has an acyclic quiver Q. In this special case, Buan–Marsh–Reineke–Reiten–Todorov [12] introduced the cluster category $\mathcal{C}_Q$, given by the orbit category

$$\mathcal{C}_Q = \mathcal{D}^b(kQ)/\tau\Sigma^{-1},$$

where τ denotes the Auslander–Reiten translation of the derived category $\mathcal{D}^b(kQ)$ and Σ the shift functor on $\mathcal{D}^b(kQ)$. It was shown to be triangulated by a result of Keller [59]. It is a 2-Calabi–Yau triangulated category by construction. The cluster category has a distinguished set of objects $\mathcal{T}$, called *(basic) cluster tilting objects.* Any object T in $\mathcal{T}$ satisfies

$$\mathrm{add}T = \{X \in \mathcal{C}_Q : \mathrm{Ext}^1_{\mathcal{C}_Q}(X,T) = 0\} = \{X \in \mathcal{C}_Q : \mathrm{Ext}^1_{\mathcal{C}_Q}(T,X) = 0\}.$$

Then $\mathcal{C}_Q$, together with $\mathcal{T}$, has the following nice properties:

- Any object T in $\mathcal{T}$ has the form $T = T_1 \oplus \cdots \oplus T_n$ where the T_i are indecomposable and $T_i \not\simeq T_j$ for $i \neq j$. The T_i would then be the analogs of cluster variables. The quiver of each cluster is given by the quiver of the endomorphism algebra $\mathrm{End}_{\mathcal{C}_Q}(T)$ of the corresponding cluster-tilting object.
- For each $i = 1, \ldots, n$ we have a unique indecomposable object $T_i^* \not\simeq T_i$, where T_i^* is a summand of an object in $\mathcal{T}$, such that $T/T_i \oplus T_i^*$ is in $\mathcal{T}$. This would be the analogs of mutations in the definition of cluster algebra. This mutation property for cluster-tilting objects holds in any 2-Calabi–Yau triangulated category in which the quivers of endomorphism algebras of such objects have no loops (see [50]).
- The cluster category $\mathcal{C}_Q$ is equipped with a cluster character $M \mapsto \varphi_M \in \mathcal{A}$ (see [19]), mapping objects to elements of the cluster algebra. Under this assignment, the cluster variables are given by φ_M for M an indecomposable rigid object of $\mathcal{C}_Q$, i.e. an indecomposable M such that $\mathrm{Ext}^1_{\mathcal{C}_Q}(M,M) = 0$.

Claire Amiot [4] generalized the construction of the cluster category to finite-dimensional algebras A_0 of global dimension $\leqslant 2$. In her approach, in order to show that there is a triangle equivalence between $\mathcal{C}_{A_0}$, constructed as a triangulated hull [59], and the quotient category $\mathrm{per}\,\mathbf{\Pi}_3(A_0)/\mathrm{pvd}\,\mathbf{\Pi}_3(A_0))$, where $\mathbf{\Pi}_3(A_0)$ is the 3-Calabi–Yau completion [62] of A_0. She first studied the category $\mathcal{C}_{\mathbf{\Pi}} = \mathrm{per}(\mathbf{\Pi})/\mathrm{pvd}(\mathbf{\Pi})$ associated to a dg algebra $\mathbf{\Pi}$ with the following four properties:

1) $\mathbf{\Pi}$ is homologically smooth,
2) $\mathbf{\Pi}$ is connective, i.e. the cohomology of $\mathbf{\Pi}$ vanishes in degrees > 0,
3) $\mathbf{\Pi}$ is 3-Calabi–Yau as a bimodule,
4) $H^0(\mathbf{\Pi})$ is finite-dimensional.

Theorem 0.0.5. *[4] Let $\mathbf{\Pi}$ be a dg algebra with the above properties. Then the triangulated category*

$$\mathcal{C}_{\mathbf{\Pi}} = \mathrm{per}(\mathbf{\Pi})/\mathrm{pvd}(\mathbf{\Pi})$$

is Hom-finite, 2-Calabi-Yau and the object $\mathbf{\Pi}$ is a cluster-tilting object with $\mathrm{End}_{\mathcal{C}_n(\mathbf{\Pi})} \simeq H^0(\mathbf{\Pi})$.

In particular, if we take A_0 to be the path algebra kQ of an acyclic quiver Q, the cluster category $\mathcal{C}_Q$ is equivalent to $\mathcal{C}_{\mathbf{\Pi}_3(kQ)}$. Later, Lingyan Guo [39] generalized Amiot's construction to finite-dimensional algebras A of global dimension $\leqslant n$ and to dg algebras satisfying 1), 2), 4) and n-Calabi–Yau as a bimodule.

Let $\mathbf{\Pi}$ be a dg algebra with the above properties. Let $\mathcal{F}(\mathbf{\Pi})$ be the full subcategory of $\mathrm{per}(\mathbf{\Pi})$ defined by

$$\mathcal{F}(\mathbf{\Pi}) = (\mathrm{per}(\mathbf{\Pi}))_{\leqslant 0} \cap ((\mathrm{per}(\mathbf{\Pi})_{\geqslant -2}))^{\perp},$$

where $(\mathrm{per}(\mathbf{\Pi}))_{\leqslant p}$ (resp. $(\mathrm{per}(\mathbf{\Pi}))_{\geqslant p}$) is the full subcategory of $\mathrm{per}(\mathbf{\Pi})$ consisting of objects having their homology concentrated in degrees $\leqslant p$ (resp. $\geqslant p$). It is called the *fundamental domain* of $\mathrm{per}(\mathbf{\Pi})$.

Theorem 0.0.6. *[4] The following composition is an equivalence of k-linear categories*

$$\mathcal{F}(\mathbf{\Pi}) \hookrightarrow \mathrm{per}(\mathbf{\Pi}) \to \mathrm{per}(\mathbf{\Pi})/\mathrm{pvd}(\mathbf{\Pi}) = \mathcal{C}(\mathbf{\Pi}).$$

Moreover, the following diagram commutes

$$\begin{tikzcd}
\mathrm{per}(\mathbf{\Pi}) \supset \mathcal{F}(\mathbf{\Pi}) \arrow[rr, "\simeq"] \arrow[dr, "H^0"'] & & \mathcal{C}(\mathbf{\Pi}) \arrow[dl, "\mathrm{Hom}_{\mathcal{C}}(\mathbf{\Pi},?)"] \\
& \mathrm{mod} H^0(\mathbf{\Pi}) &
\end{tikzcd}.$$

Let (Q, W) be a quiver with potential (QP). The associated *completed Ginzburg dg algebra* $\mathbf{\Gamma}(Q, W)$ is homologically smooth and it carries a canonical left 3-Calabi-Yau structure (see [38, 62]). Thus, it satisfies the properties (1),(2) and (3) in Theorem 0.0.5. The zero-th homology of $\mathbf{\Gamma}(Q, W)$ is the Jacobian algebra $J(Q, W)$.

Theorem 0.0.7. *[4] Let (Q, W) be a Jacobian-finite quiver with potential, i.e. the corresponding Jacobian algebra $J(Q, W)$ is finite dimensional. Then the category*

$$\mathcal{C}(Q, W) = \mathrm{per}\,\mathbf{\Gamma}(Q, W)/\mathrm{pvd}\,\mathbf{\Gamma}(Q, W)$$

is Hom-finite and has a canonical cluster-tilting object whose endomorphism algebra is isomorphic to $J(Q, W)$.

The category $\mathcal{C}(Q, W)$ is called the *cluster category* associated with a quiver with potential (Q, W). If (Q, W) is not Jacobian-finite, a generalization of the category $\mathcal{C}(Q, W)$, which is not Hom-finite, was constructed by Plamondon in [74].

Let Q be a finite quiver and i a source of Q, i.e. a vertex without incoming arrows. Let Q' be the mutation of Q at i, i.e. the quiver obtained from Q by reversing all the arrows going out from i. Let k be a field, kQ the path algebra of Q and $\mathcal{D}(kQ)$ the derived category of the category of all right kQ-modules. For a vertex j of Q' respectively Q, let P'_j respectively P_j be the projective indecomposable associated with the vertex j. Then Bernstein–Gelfand–Ponomarev's [9] main result reformulated in terms of derived categories following Happel [40] says that there is a canonical triangle equivalence

$$F : \mathcal{D}(kQ') \to \mathcal{D}(kQ)$$

which takes P'_j to P_j for $j \neq i$ and P'_i to the cone over the morphism

$$P_i \to \bigoplus P_j$$

whose components are the left multiplications by all arrows going out from i. This gives a categorical interpretation of the mutation at a *source* i.

Keller and Yang [64] obtained an analogous result for the mutation of a quiver with potential (Q, W) at an *arbitrary* vertex i, where the role of the quiver with reversed arrows is played by the quiver with potential (Q', W') obtained from (Q, W) by mutation at i in the sense of Derksen-Weyman-Zelevinsky [23]. The role of the derived category $\mathcal{D}(kQ)$ is now played by the derived category $\mathcal{D}(\mathbf{\Gamma})$ of the complete differential graded algebra $\mathbf{\Gamma} = \mathbf{\Gamma}(Q, W)$ associated with (Q, W).

Let (Q, W) be a quiver with potential such that Q has no loops. Let i be a vertex such that Q does not have 2-cycles at i. Let $\mu_i(Q, W)$ be the mutation of (Q, W) at the vertex i (see [23]). Let $\mathbf{\Gamma} = \mathbf{\Gamma}(Q, W)$ and $\mathbf{\Gamma}' = \mathbf{\Gamma}(\mu_i(Q, W))$ be the completed Ginzburg dg algebras associated to (Q, W) and $\mu_i(Q, W)$ respectively. For a vertex j of Q, let $P_j = e_j\mathbf{\Gamma}$ and $P'_j = e_j\mathbf{\Gamma}'$. They proved the following results [64].

Theorem 0.0.8. *[64] There is a triangle equivalence*

$$F : \mathcal{D}(\mathbf{\Gamma}') \to \mathcal{D}(\mathbf{\Gamma}),$$

which sends the P'_j to P_j for $j \neq i$ and to the cone T_i over the morphism

$$P_i \to \bigoplus_{\alpha} P_{t(\alpha)}$$

for $i = j$, where we have a summand $P_{t(\alpha)}$ for each arrow α of Q with source i and the corresponding component of the map is the left multiplication by α. The functor F restricts to triangle equivalences from $\mathrm{per}(\mathbf{\Pi}')$ *to* $\mathrm{per}(\mathbf{\Pi})$ *and from* $\mathrm{pvd}(\mathbf{\Pi}')$ *to* $\mathrm{pvd}(\mathbf{\Pi})$*.*

Their result is analogous to but not a generalization of Bernstein-Gelfand-Ponomarev's since even if the potential W vanishes, the derived category $\mathcal{D}(\mathbf{\Gamma})$ is not equivalent to $\mathcal{D}(kQ)$.

Remark 0.0.9. There is also a triangle equivalence

$$F' : \mathcal{D}(\mathbf{\Gamma}') \to \mathcal{D}(\mathbf{\Gamma})$$

which, for $j \neq i$, sends the P'_j to P_j and, for $i = j$, to the shifted cone

$$T'_i = \Sigma^{-1}(\bigoplus_{\beta} P_{s(\beta)} \to P_i),$$

where we have a summand $P_{s(\beta)}$ for each arrow β of Q with target i and the corresponding component of the morphism is left multiplication by β. The two functors F and F' are related by the twist functor t_{S_i} with respect to the 3-spherical object S_i (the simple dg $\mathbf{\Gamma}$-module associated with the vertex i). More precisely, we have

$$F' = t_{S_i} \circ F,$$

where $t_{S_i} : \mathcal{D}(\mathbf{\Gamma}) \to \mathcal{D}(\mathbf{\Gamma})$ is given by the following triangle

$$\boldsymbol{R}\mathrm{Hom}(S_i, X) \otimes S_i \to X \to t_{S_i}(X) \to \Sigma\boldsymbol{R}\mathrm{Hom}(S_i, X) \otimes S_i$$

for each object X of $\mathcal{D}(\mathbf{\Gamma})$.

The following results give a link between Ginzburg dg algebras associated with QPs linked by a mutation.

Theorem 0.0.10. *[6] Let (Q, W) be a QP without loops and $i \in Q_0$ a vertex which is not on a 2-cycle in Q. Denote by $\Gamma = \Gamma(Q, W)$ and $\Gamma' = \Gamma(\mu_i(Q, W))$ the completed Ginzburg dg algebras.*

a) There are triangle equivalences

$$\begin{tikzcd} \mathrm{per}(\mathbf{\Gamma}) \arrow[r, "\simeq"] & \mathrm{per}(\mathbf{\Gamma'}) \\ \mathrm{pvd}(\mathbf{\Gamma}) \arrow[r, "\simeq"] \arrow[u, hook] & \mathrm{pvd}(\mathbf{\Gamma'}) . \arrow[u, hook] \end{tikzcd}$$

Hence we have a triangle equivalence $\mathcal{C}(Q, W) \simeq \mathcal{C}(\mu_i(Q, W))$.

b) We have a diagram

$$\begin{tikzcd} \mathrm{per}(\mathbf{\Gamma}) \arrow[r, "\simeq"] \arrow[d, "H^0"'] & \mathrm{per}(\mathbf{\Gamma'}) \arrow[d, "H^0"] \\ \mathrm{mod} J(Q, W) \arrow[r, leftrightarrow, dotted, "DWZ-mutation"] & \mathrm{mod} J(\mu_i(Q, W)). \end{tikzcd}$$

In the case of cluster algebras with frozen variables, a suitable categorical model should have certain objects occurring as summands of every cluster-tilting object (those objects correspond to frozen vertices). Taking a suitable quotient of this category should correspond to removing the frozen variables from the cluster algebra and the quotient category should be the usual cluster category.

For cluster algebras with non invertible frozen variables, there is a natural model for this purpose, a Frobenius category $\mathcal{E}$, i.e. an exact category with enough projective and injective objects, and such that the projective and injective objects coincide. Then by definition, each projective-injective object I satisfies

$$\mathrm{Ext}^1_{\mathcal{E}}(I, ?) = 0 = \mathrm{Ext}^1_{\mathcal{E}}(?, I).$$

Thus, each projective-injective object I is in add$T$ for any cluster-tilting object $T \in \mathcal{E}$. Moreover, by a result of Happel [41], the stable category $\underline{\mathcal{E}}$, formed by taking the quotient by the ideal of morphisms factoring through a projective-injective object, is a triangulated category. The corresponding stable category $\underline{\mathcal{E}}$ is 2-Calabi–Yau if there is a bifunctorial duality

$$\mathrm{Ext}^1_{\mathcal{E}}(X, Y) = D\mathrm{Ext}^1_{\mathcal{E}}(Y, X)$$

for all $X, Y \in \mathcal{E}$.

Remark 0.0.11. For cluster algebras with invertible frozen variables, we can consider the derived category of a Frobenius category $\mathcal{E}$.

Let k be a field and Q a Dynkin quiver. Let J be a subset of Q_0. We denote by $\widetilde{kQ}$ the corresponding preprojective algebra of kQ. Let i be a vertex. We denote by S_i the simple $\widetilde{kQ}$-module supported at i, by P_i its projective cover and by Q_i its injective hull.

Let $Q_J = \bigoplus_{i \in J} Q_j$. Geiß–Leclerc–Schröer (see [32]) constructed a subcategory $\operatorname{Sub} Q_J$ of the category $\operatorname{mod} \widetilde{kQ}$ of modules for the projective algebra $\widetilde{kQ}$ whose objects are isomorphic to a submodule of a direct sum of copies of Q_J. The category $\operatorname{Sub} Q_J$ is closed under submodules so it has kernels which agree with those in $\operatorname{mod} \widetilde{kQ}$, but it does not have cokernels in general. However, it is extension-closed.

Thus, it inherits the structure of an exact category in which a sequence

$$X \to Y \to Z$$

in $\operatorname{Sub} Q_J$ is a conflation if and only if the sequence

$$0 \to X \to Y \to Z \to 0$$

is exact in $\operatorname{mod} \widetilde{kQ}$.

For any module $M \in \operatorname{mod} \widetilde{kQ}$. Let $\theta_J(M)$ be the minimal submodule of M such that $M/\theta_J(M)$ is in $\operatorname{Sub} Q_J$. Then the canonical projection $M \to M/\theta_J(M)$ is a minimal left $\operatorname{Sub} Q_J$-approximation of M. We set $F_i = I_i/\theta_J(I_i)$. By [32, Proposition 3.2], these F_i are the indecomposable projective-injective objects in $\operatorname{Sub} Q_J$. Moreover, $\operatorname{Sub} Q_J$ is a Frobenius category.

Proposition 0.0.12. *[32] The category* $\operatorname{Sub} Q_J$ *is a functorially finite, extension closed, stably 2-Calabi–Yau Frobenius subcategory of* $\operatorname{mod} \widetilde{kQ}$.

In particular, if we take J to be the whole vertex set Q_0, then $\operatorname{Sub} Q_J$ is the whole category $\operatorname{mod} \widetilde{kQ}$.

A more general construction of $\operatorname{Sub} Q_J$ was given by Buan–Iyama–Reiten–Scott in [13]. Let Q be a finite connected quiver without oriented cycles. We denote by $\{1, \ldots, n\}$ the set of vertices of Q. For a vertex i of Q, we denote by I_i the ideal $\widetilde{kQ}(1 - e_i)\widetilde{kQ}$ of $\widetilde{kQ}$. We denote by W the *Coxeter group* associated to the quiver Q. The group W is defined by the generators $1, \ldots, n$ and the relations:

- $i^2 = 1$ for all i in $\{1, \ldots, n\}$;
- $ij = ji$ if there are no arrows between the vertices i and j;
- $iji = jij$ if there is exactly one arrow between i and j.

Let $w = i_1 i_2 \cdots i_r$ be a reduced word. For $m \leqslant r$, let I_{w_m} be the the following ideal:

$$I_{w_m} = I_{i_m} \cdots I_{i_2} I_{i_1}.$$

For simplicity of notation, we write I_w instead of I_{w_r}. Let $\operatorname{Sub} \widetilde{kQ}/I_w$ be the subcategory of $\operatorname{mod} \widetilde{kQ}$ generated by the $\widetilde{kQ}$-sub-modules of $\widetilde{kQ}/I_w$. Buan, Iyama, Reiten and Scott proved the following results.

Theorem 0.0.13. *[13] The category* $\operatorname{Sub} \widetilde{kQ}/I_w$ *is a Frobenius category and its stable category* $\underline{\operatorname{Sub}}\, \widetilde{kQ}/I_w$ *is* 2*-Calabi-Yau. The object* $T_w = \bigoplus_{m=1}^{r} e_{i_m} \widetilde{kQ}/I_w$ *is a cluster-tilting object.*

Let (Q, F, W) be an ice quiver with potential, i.e. Q is a finite quiver, F is a subquiver of Q and W a potential on Q. The subquiver F is called the *ice subquiver* of Q. For any arrow α in Q, we define the *cyclic derivative* $\partial_\alpha W$ of W with respect to α by

$$\partial_\alpha(\alpha_1 \cdots \alpha_k) = \sum_{\alpha_i=\alpha} \alpha_{i+1} \cdots \alpha_k \alpha_1 \cdots \alpha_{i-1}$$

on any cycle $\alpha_1 \cdots \alpha_k$, and then extending linearly. Then the *relative (frozen) Jacobian algebra* is defined as the quotient

$$J(Q, F, W) = kQ/\langle \partial_\alpha W : \alpha \in Q_1 \setminus F_1 \rangle.$$

For each reduced word w, Buan–Iyama–Reiten–Smith [14, Section 6] constructed an associated ice quiver with potential (Q_w, F_w, W_w).

Theorem 0.0.14. *[14, Theorem 6.6] The algebra* $\mathrm{End}_{\mathrm{Sub}\, \widetilde{kQ}/I_w}(T_w)$ *is isomorphic to* $J(Q_w, F_w, W_w)$ *and hence* $\mathrm{End}_{\underline{\mathrm{Sub}}\, \widetilde{kQ}/I_w}(T_w) = J(\underline{Q_w}, \underline{W_w})$*, where* $\underline{Q_w}$ *is the full subquiver of* Q_w *on the vertices corresponding to* $(Q_w)_0 \setminus (F_w)_0$ *and* $\underline{W_w}$ *is the potential obtained from* W_w *by deleting terms given by cycles passing through frozen vertices* $(F_w)_0$*.*

Later, Amiot–Reiten–Todorov proved that the stable category $\underline{\mathrm{Sub}}\, \widetilde{kQ}/I_w$ is actually a generalized cluster category (see [6, Theorem 3.1]).

In Amiot's construction of the generalized cluster category, the homological assumption is the 3-Calabi-Yau bimodule property. In the relative context, relative right Calabi–Yau structures were invented by Bertrand Toën in [85, pp. 227-228]. Later, relative right and left Calabi–Yau structures were studied by Chris Brav and Tobias Dyckerhoff in [16]. A relative left n-Calabi–Yau structure on a morphism $f : B \to A$ between smooth dg algebras is the datum of a class $[\xi]$ in negative cyclic homology $HN_n(f)$ inducing certain dualities in $\mathcal{D}(B^e)$ and $\mathcal{D}(A^e)$. In particular, if the dg algebra B is zero, then A is n-Calabi–Yau as a bimodule. A canonical way to produce relative left Calabi–Yau structures is the deformed relative Calabi–Yau completion which was introduced by Wai-kit Yeung [87]. This generalized Keller's construction [62] of deformed n-Calabi–Yau completions to the relative context.

The main aim of this thesis is to generalize the construction of Claire Amiot and Lingyan Guo to the relative context. We change the properties used in Amiot's construction to the following properties on a dg algebra morphism $f : B \to A$ (not necessarily preserving the unit)

1) A and B are homologically smooth,
2) A is connective, i.e. the cohomology of A vanishes in degrees > 0,
3) the morphism $f : B \to A$ has a left $(n + 1)$-Calabi–Yau structure,
4) $H^0(A)$ is finite-dimensional.

We introduce relative cluster category $\mathcal{C}_n(A, B)$ associated with $f : B \to A$ and show that it is Hom-finite under the above assumptions. We prove the existence of an n-cluster tilting object in the Higgs category $\mathcal{H}$ which is an extension closed subcategory of $\mathcal{C}_n(A, B)$ and is stably n-Calabi–Yau.

In 2009, Keller and Yang (see [64]) categorified quiver mutation by interpreting it in terms of equivalences between derived categories (see Theorem 0.0.8). Matthew Pressland has generalized mutation of quivers with potential to that of *ice* quivers with potential (see [80]). Our second aim of this thesis is to categorify Pressland's mutation of ice quivers with potential. We show that his rule yields derived equivalences between the associated *relative* Ginzburg algebras, which are special cases of Yeung's deformed relative Calabi–Yau completions (see [87]).

Unexpectedly, Fraser and Sherman-Bennett have very recently discovered a construction of mutation at frozen vertices in their study [29] of cluster structures on positroid varieties. Let v be a frozen vertex. Suppose that v is a source vertex in F such that there are no unfrozen arrows with source v, or v is a sink vertex in F such that there are no unfrozen arrows with target v. Our result says that the mutation at the frozen vertex v is 'categorified' by the twist (respectively inverse twist) functor t_{S_v} (respectively $t_{S_v}^{-1}$) with respect to the 2-spherical object S_v (the simple module at the vertex v) in the derived category of the complete derived preprojective algebra $\mathbf{\Pi}_2(F)$. In [86], we will show how suitable compositions of mutations at frozen vertices can be decategorified into quasi cluster isomorphisms (see [28]).

0.0.3 Organization of the thesis

The structure of the thesis is as follows. In Chapter 2, we recall the definitions of relative left Calabi–Yau structures and relative Calabi–Yau completions, and proving Proposition 2.6.2, where we obtain a reduced version of the deformed relative Calabi–Yau completion for a dg functor between finitely cellular type dg categories. We also discuss the relation between relative Calabi–Yau completions and absolute Calabi–Yau completions, see Proposition 2.7.1.

Let $f : B \to A$ be a morphism (not necessarily preserving the unit element) between dg k-algebras and let $e = f(1_A)$. Under the above assumptions on f, we define the *relative n-cluster category* $\mathcal{C}_n(A, B)$ as

$$\mathcal{C}_n(A, B) = \mathrm{per}A/\mathrm{pvd}_B(A),$$

where $\mathrm{pvd}_B(A)$ is the full subcategory of $\mathrm{pvd}(A)$ whose objects are the perfectly valued derived category $\mathrm{pvd}(A)$ formed by the dg modules whose restriction to B is acyclic (see Definition 3.0.1). The relation between the triangulated categories involved can be summarized by the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{per}(eAe) \arrow[r, equal] \arrow[d, hook] & \mathrm{per}(eAe) \arrow[d, hook] \\
\mathrm{pvd}_B(A) \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{per}(A) \arrow[r, "\pi^{rel}"] \arrow[d, "p^*"] & \mathcal{C}_n(A, B) \arrow[d, "p^*"] \\
\mathrm{pvd}(\overline{A}) \arrow[r, hook] & \mathrm{per}(\overline{A}) \arrow[r, "\pi"] & \mathcal{C}_n(\overline{A}),
\end{tikzcd}$$

where $\overline{A}$ is the homotopy cofiber of $f : B \to A$, and the rows and columns are exact sequences of triangulated categories.

Then we define the *relative fundamental domain* $\mathcal{F}^{rel}$ as a certain extension closed full subcategory of $\mathrm{per}A$ (see Definition 3.6.11). Similarly as in [4] and [39], the canonical quotient functor $\pi^{rel} : \mathrm{per}A \to \mathcal{C}_n(A, B)$ induces a fully faithful embedding $\pi^{rel} : \mathcal{F}^{rel} \hookrightarrow \mathcal{C}_n(A, B)$ (see Proposition 3.6.17). Then the *Higgs category* $\mathcal{H}$ is defined as the image of $\mathcal{F}^{rel}$ in $\mathcal{C}_n(A, B)$ (see Definition 3.6.19). We show that it is closed under extensions in $\mathcal{C}_n(A, B)$ (see Proposition 3.6.35) and thus becomes an extriangulated categories in the sense of [71]. More precisely, we prove the following theorem.

Theorem 0.0.15. (Theorem 3.6.42 and Proposition 3.6.46) *The Higgs category $\mathcal{H}$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(eA)$ and $\mathrm{add}A$ is an n-cluster-tilting subcategory of $\mathcal{H}$ with $\mathrm{End}_{\mathcal{H}}(A) = H^0(A)$. Moreover, the quotient functor $p^* : \mathcal{C}_n(A, B) \to \mathcal{C}_n(\overline{A})$ induces an equivalence of triangulated categories*

$$\mathcal{H}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{C}_n(\overline{A}),$$

where $[\mathcal{P}]$ denotes the ideal of morphisms of $\mathcal{H}$ which factor through objects in add(eA).

In [86], for $n = 3$, we will define and study cluster characters on the Higgs category and the relative cluster category.

We have the follwong results related to n-angulated categories.

Theorem 0.0.16. (Theorem 3.6.49) *Suppose that the n-cluster tilting category $\mathrm{add}\overline{A}$ satisfies*

$$\Sigma^n \mathrm{add}\overline{A} = \mathrm{add}\overline{A}$$

in $\mathcal{C}_n(\overline{A})$. Then the n-cluster-tilting subcategory $\mathrm{add}\overline{A}$ of $\mathcal{C}_n(\overline{A})$ carries a canonical $(n+2)$-angulated structure. The n-cluster-tilting subcategory $\mathrm{add}A$ of $\mathcal{H}$ carries a canonical structure of Frobenius n-exangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(eA)$. The quotient functor $p^* : \mathcal{C}_n(A, B) \to \mathcal{C}_n(\overline{A})$ induces an equivalence of $(n + 2)$-angulated categories

$$\mathrm{add}A/[\mathcal{P}] \xrightarrow{\sim} \mathrm{add}(\overline{A}).$$

In Section 3.7, under the hypotheses 1)-4), when the dg algebra A is concentrated in degree 0, we show that $H^0(A)$ is of global dimension at most $n + 1$ so that we have the equivalence

$$\mathcal{D}^b(\mathrm{mod}H^0A) \xrightarrow{\sim} \mathrm{per}A.$$

Moreover, A is internally bimodule $(n + 1)$-Calabi–Yau respect to the idempotent $e = f(1_B)$ in the sense of Matthew Pressland (see [77]) and restriction induces an equivalence from the Higgs category $\mathcal{H}$ to the category of Gorenstein projective modules over $B' = eH^0(A)e$. More precisely, we have the following theorem.

Theorem 0.0.17. (Theorem 3.7.2)

a) *The algebra $B' = eH^0(A)e$ is Iwanaga-Gorenstein of injective dimension at most $g \leqslant n + 1$ as a B'-module.*

b) *Under the equivalence $\mathcal{D}^b(\mathrm{mod}H^0A) \simeq \mathrm{per}A$, the subcategory $\mathcal{F}^{rel}$ corresponds to the subcategory $\mathrm{mod}_{n-1}(H^0A)$ of H^0A-modules of projective dimension at most $n - 1$.*

c) Via the equivalence $\mathrm{res} : \mathcal{D}^b(\mathrm{mod}H^0A) \xrightarrow{\sim} \mathrm{per}A$*, the localization* $\pi^{rel} : \mathrm{per}A \to \mathcal{C}_n(A,B)$ *identifies with the restriction functor* $\mathcal{D}^b(\mathrm{mod}H^0A) \to \mathcal{D}^b(\mathrm{mod}B')$*, i.e. we have a commutative square*

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{mod}H^0A) \arrow[r] \arrow[d, "\wr"] & \mathcal{D}^b(\mathrm{mod}B') \arrow[d, "\wr"] \\
\mathrm{per}A \arrow[r] & \mathcal{C}_n(A,B).
\end{tikzcd}$$

d) Under the equivalence $\mathcal{D}^b(\mathrm{mod}B') \xrightarrow{\sim} \mathcal{C}_n(A,B)$*, the Higgs category* $\mathcal{H} \subseteq \mathcal{C}_n(A,B)$ *corresponds to the subcategory* $\mathrm{gpr}B'$ *of Gorenstein projective modules over* $B' = eH^0(A)e$*. In particular, when* B' *is self injective, we have* $\mathcal{H} \cong \mathrm{mod}B'$*.*

We summarize the situation in the following commutative diagram

$$\begin{tikzcd}
\mathrm{per}A \arrow[rrr, "\pi^{rel}"] & & & \mathcal{C}_n(A,B) \\
& \mathcal{D}^b(\mathrm{mod}H^0A) \arrow[ul, "\sim"'] \arrow[r, "res"] & \mathcal{D}^b(\mathrm{mod}B') \arrow[ur, "\sim"] & \\
& \mathrm{mod}H^0A \arrow[u, hook] \arrow[r, "res"] & \mathrm{mod}B' \arrow[u, hook] & \\
& \mathrm{mod}_{n-1}(H^0A) \arrow[u, hook] \arrow[r, "\sim"] \arrow[dl, "\sim"'] & \mathrm{gpr}B' \arrow[u, hook] \arrow[dr, "\sim"] & \\
\mathcal{F}^{rel} \arrow[uuuu, hook] \arrow[rrr, "\sim"] & & & \mathcal{H}\,. \arrow[uuuu, hook]
\end{tikzcd}$$

The paradigmatic example for A is the relative 3-Calabi–Yau completion of the Auslander algebra of a Dynkin quiver Q (cf.below). Then B' is the preprojective algebra of Q and $\mathcal{H}$ is equivalent to the module category of B'. This motivates the terminology "Higgs category" because Higgs bundles [43, 82] are the geometric version of modules over preprojective algebras.

In Section 3.8, we apply this general approach to Jacobian-finite relative Ginzburg dg algebras associated with ice quivers with potential. In this way, with each Jacobian-finite ice quiver with potential (Q, F, W), we associate a Frobenius extriangulated categories $\mathcal{H}$ endowed with a canonical cluster-tilting object (see Theorem 3.8.9).

In Chapter 4, we apply our main result to the higher Auslander-Reiten theory. Let B_0 be an n-representation-finite algebra in the sense of Iyama-Oppermann [48]. Let τ_n^{-1} be the higher inverse Auslander-Reiten translation of B_0 and let $A_0 := \mathrm{End}_{B_0}(\oplus_{i \geqslant 0} \tau_n^{-i} B_0)$ be the higher Auslander algebra of B_0.

Then there is a natural fully faithful morphism

$$f_0 : B_0 \hookrightarrow A_0 \,.$$

The relative $(n+2)$-Calabi–Yau completion of f_0

$$f : B = \mathbf{\Pi}_{n+1}(B_0) \longrightarrow A = \mathbf{\Pi}_{n+2}(A_0, B_0)$$

satisfies the assumptions 1)-4) and A is concentrated in degree 0. Moreover, $H^0(f)$ is fully faithful (see Proposition 4.2.7). Let $\widetilde{B_0}$ denote the higher preprojective algebra of B_0 in the sense of Iyama-Oppermann [48]. We give a new proof (see Lemma 4.2.8) of the fact, first proved in [48], that $\widetilde{B_0}$ is a self-injective algebra. By our main results in Section 3.6 and Section 3.7, we have the following theorem.

Theorem 0.0.18. (Theorem 4.2.9) *Consider the relative cluster category $\mathcal{C}_{n+1}(A, B)$ associated with $f : B \to A$.*

a) The Higgs category $\mathcal{H} \subseteq \mathcal{C}_{n+1}(A, B)$ is equivalent to $\mathrm{mod}(\widetilde{B_0})$ and the image of A in $\mathcal{H}$ is an $(n+1)$-cluster-tilting object.

b) We have a triangle equivalence $\underline{\mathrm{mod}}(\widetilde{B_0}) \xrightarrow{\sim} \mathcal{C}_{n+1}(A_0/A_0eA_0)$, where $e = f(1_{B_0})$. In particular, $\underline{\mathrm{mod}}(\widetilde{B_0})$ contains a canonical $(n+1)$-cluster-tilting object.

Notice that $b)$ is the main result of [48]. We deduce it from $a)$ thereby giving a new proof which is essentially different from that of [48].

In the last Chapter, we study the categorification of mutations of ice quivers with potential which is defined by Pressland in [80]. Let (Q, F, W) be an ice quiver with potential. Let v be a unfrozen vertex of Q such that no loops or 2-cycles are incident with v. Let $\mu_v(Q, F, W) = (Q', F', W')$ be the Pressland's mutation of (Q, F, W) at vertex v. We have the following theorem which generalizes Keller-Yang's results in [64].

Theorem 0.0.19. (Theorem 5.3.3) *Let $\mathbf{\Gamma}_{rel} = \mathbf{\Gamma}_{rel}(Q, F, W)$ and $\mathbf{\Gamma}'_{rel} = \mathbf{\Gamma}_{rel}(Q', F', W')$ be the complete relative Ginzburg dg algebras associated to (Q, F, W) and $\mu_v(Q, F, W)$ respectively. For a vertex i, let $\mathbf{\Gamma}_i = e_i\mathbf{\Gamma}_{rel}$ and $\mathbf{\Gamma}'_i = e_i\mathbf{\Gamma}'_{rel}$.*

a) There is a triangle equivalence

$$\Phi_+ := J^* : \mathcal{D}(\mathbf{\Gamma}'_{rel}) \longrightarrow \mathcal{D}(\mathbf{\Gamma}_{rel}),$$

which sends the the $\mathbf{\Gamma}'_j$ to $\mathbf{\Gamma}_j$ for $j \neq v$ and to the cone over the morphism

$$\mathbf{\Gamma}_v \to \bigoplus_\alpha \mathbf{\Gamma}_{t(\alpha)}$$

for $j = v$, where we have a summand $\mathbf{\Gamma}_{t(\alpha)}$ for each arrow α of Q with source v and the corresponding component of the map is the left multiplication by α. The functor $(G')^*$ restricts to triangle equivalences from $\mathrm{per}(\mathbf{\Gamma}'_{rel})$ to $\mathrm{per}(\mathbf{\Gamma}_{rel})$ and from $\mathrm{pvd}(\mathbf{\Gamma}'_{rel})$ to $\mathrm{pvd}(\mathbf{\Gamma}_{rel})$.

b) The following diagram commutes up to isomorphism

$$\begin{tikzcd}
 & & \mathcal{D}(\mathbf{\Gamma}'_{rel}) \arrow[dd, "\Phi_+"] \\
\mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[urr, "\mathbf{L}(\boldsymbol{G}'_{rel})^*"] \arrow[drr, "\mathbf{L}(\boldsymbol{G}_{rel})^*"'] \arrow[r, phantom, "\cong"] & {} & \\
 & & \mathcal{D}(\mathbf{\Gamma}_{rel}).
\end{tikzcd}$$

We then define the associated *boundary dg algebra* $\mathrm{Bd}(Q, F, W)$ (see Definition 5.3.5) as

$$\mathrm{Bd}(Q, F, W) = \mathrm{REnd}_{\mathbf{\Gamma}_{rel}(Q,F,W)}((\boldsymbol{G}_{rel})^*(\mathbf{\Pi}_2(F))) \simeq e_F \mathbf{\Gamma}_{rel}(Q, F, W) e_F,$$

where $e_F = \sum_{i \in F} e_i$ is the sum of idempotents corresponding to the frozen vertices. Corollary 5.3.6 shows that the boundary dg algebra is invariant under mutations at unfrozen vertices. By using this Corollary, we illustrate our results on examples arising in the work of Baur–King–Marsh on dimer models and cluster categories of Grassmannians (see Example 5.3.10).

In the last section, we study the categorification of mutation at a frozen vertex. Let (Q, F, W) be an ice quiver with potential. Let v be a frozen vertex. Suppose that v satisfies the following conditions

1) v is a source vertex in F such that there are no unfrozen arrows with source v, or

2) v is a sink vertex in F such that there are no unfrozen arrows with target v.

In this situation, we define (see Definition 5.4.2) the mutation of (Q, F) at the frozen vertex v by using the same mutation rule as that defined by Pressland for mutation at unfrozen vertices. Then we also give the definition of the mutation of an ice quiver with potential at the frozen vertex v (see Definition 5.4.5).

If v is a source vertex in F, our result shows that the mutation at v is 'categorified' by the inverse twist functor $t_{S_v}^{-1}$ with respect to the 2-spherical object S_v (the simple module at vertex v) in $\mathcal{D}(\mathbf{\Pi}_2(F))$. Let us make this more precise. Write $(Q', F', W') = \tilde{\mu}_v(Q, F, W)$. Let $\mathbf{\Gamma}_{rel} = \mathbf{\Gamma}_{rel}(Q, F, W)$ and $\mathbf{\Gamma}'_{rel} = \mathbf{\Gamma}_{rel}(Q', F', W')$ be the complete relative Ginzburg dg algebras associated to (Q, F, W) and (Q', F', W') respectively.

Theorem 0.0.20. *(Theorem 5.4.8) We have a triangle equivalence*

$$\Psi_+ : \mathcal{D}(\mathbf{\Gamma}'_{rel}) \to \mathcal{D}(\mathbf{\Gamma}_{rel}),$$

which sends the $\mathbf{\Gamma}'_i$ to $\mathbf{\Gamma}_i$ for $i \neq v$ and to the cone

$$\mathrm{Cone}(\mathbf{\Gamma}_v \to \bigoplus_{\alpha} \mathbf{\Gamma}_{t(\alpha)}),$$

where we have a summand $\mathbf{\Gamma}_{t(\alpha)}$ for each arrow α of F with source v and the corresponding component of the map is the left multiplication by α. The functor Ψ_+ restricts to triangle *equivalences from* $\mathrm{per}(\mathbf{\Gamma}'_{rel})$ *to* $\mathrm{per}(\mathbf{\Gamma}_{rel})$ *and from* $\mathrm{pvd}(\mathbf{\Gamma}'_{rel})$ *to* $\mathrm{pvd}(\mathbf{\Gamma}_{rel})$. *Moreover, the following square commutes up to isomorphism*

$$\begin{tikzcd} \mathcal{D}(\mathbf{\Pi}_2(F')) \arrow[r, "G'^*"] \arrow[d, "\mathrm{can}"'] & \mathcal{D}(\mathbf{\Gamma}'_{rel}) \arrow[d, "\Psi_+"] \\ \mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[d, "t_{S_v}^{-1}"'] & \mathcal{D}(\mathbf{\Gamma}_{rel}) \arrow[d, equal] \\ \mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[r, "G^*"'] & \mathcal{D}(\mathbf{\Gamma}_{rel}), \end{tikzcd} \tag{1}$$

where can *is the canonical functor induced by an identification between* $\mathbf{\Pi}_2(F')$ *and* $\mathbf{\Pi}_2(F)$ *and* $t_{S_v}^{-1}$ *is the twist inverse functor with respect to the* 2*-spherical object* S_v*, which gives rise to a triangle*

$$t_{S_v}^{-1}(X) \to X \to \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_{\mathbf{\Pi}_2(F)}(X, S_v), S_v) \to \Sigma t_{S_v}^{-1}(X)$$

for each object X *of* $\mathcal{D}(\mathbf{\Pi}_2(F))$.

Dually, if v is a sink frozen, the mutation at v is 'categorified' by the twist functor t_{S_v} with respect to the 2-spherical object S_v in $\mathcal{D}(\mathbf{\Pi}_2(F))$.

0.0.4 Notations

Throughout this thesis, k will denote an algebraically closed field. We denote by $D = \mathrm{Hom}_k(-, k)$ the k-linear dual. All modules are right modules unless stated otherwise. We say an algebra A is Noetherian if it is Noetherian as both a left and right module over itself. We denote by $g \circ f$ or gf the composition of morphisms (or arrows) $f : X \to Y$ and $g : Y \to Z$.

Chapter 1

Preliminaries

1.1 Triangulated categories

In this section, we recall some basic definitions and properties of triangulated categories as well the facts on t-structure and co-t-structure. Our main references for this section are [41], [72], [8] and [2].

1.1.1 Definitions and basic properties

Let $\mathcal{T}$ be an additive category endowed with an autoequivalence Σ, which is usually called the suspension functor. The quasi-inverse of Σ is denoted by Σ^{-1}. A *sextuple* (X, Y, Z, u, v, w) is given by three objects $X, Y, Z \in \mathcal{T}$ and three morphisms $u : X \to Y, v : Y \to Z, w : Z \to \Sigma X$. We will note such a sextuplet by

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

A morphism of sextuples from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a tuple (f, g, h) of morphisms such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'.
\end{tikzcd}$$

If f, g and h are isomorphisms in $\mathcal{T}$, then (f, g, h) is called an *isomorphism of sextuples.*

Definition 1.1.1. An additive category $\mathcal{T}$ with suspension functor Σ is called a *triangulated category* if it is endowed with a class $\mathcal{U}$ of sextuples (called *triangles*) which satisfies the following axioms (TR1) to (TR4):

(TR1) Every sextuple isomorphic to a triangle is a triangle. Every morphism $u : X \to Y$ in $\mathcal{T}$ can be embedded into a triangle $X \to Y \to Z \to \Sigma X$. For every object X of $\mathcal{T}$, the sextuple $X \xrightarrow{\mathbf{1}_X} X \to 0 \to \Sigma X$ is a triangle.

(TR2) If $X \to Y \to Z \to \Sigma X$ is a triangle, then $Y \to Z \to \Sigma X \to \Sigma Y$ is a triangle.

(TR3) Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w'), and morphisms f and g satisfying $u' \circ f = g \circ u$, there exists a morphism (f, g, h) of triangles.

(TR4) Let

$$X \xrightarrow{u} Y \xrightarrow{i} Z' \xrightarrow{j} \Sigma X$$
$$Y \xrightarrow{v} Z \xrightarrow{i'} X' \xrightarrow{j'} \Sigma Y$$
$$X \xrightarrow{v \circ u} Z \xrightarrow{i''} Y' \xrightarrow{j''} \Sigma X$$

be three triangles. There exist two morphisms $f : Z' \to Y'$ and $g : Y' \to X'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \Sigma^{-1}X' \arrow[r, equal] \arrow[d, "-\Sigma^{-1}j'"] & \Sigma^{-1}X' \arrow[d] & \\
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "j"] \arrow[d, "f"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "v \circ u"] & Z \arrow[r, "i''"] \arrow[d, "i'"] & Y' \arrow[r, "j''"] \arrow[d, "g"] & \Sigma X \arrow[d, "\Sigma u"] \\
 & X' \arrow[r, equal] & X' \arrow[r, "j'"] & \Sigma Y
\end{tikzcd}$$

where the two middle rows and the two middle columns are triangles.

Let $(\mathcal{T}, \Sigma, \mathcal{U})$ and $(\mathcal{T}', \Sigma', \mathcal{U}')$ be two triangulated categories. A triangle functor $\mathcal{T} \to \mathcal{T}'$ is a pair (F, α) consisting an addictive functor F and an isomorphism of functors $\alpha : F\Sigma \to \Sigma' F$ such that

$$\begin{tikzcd}
FX \arrow[r, "Fu"] & FY \arrow[r, "Fv"] & FZ \arrow[rr, "(\alpha_X) \circ (Fw)"] \arrow[dr, "Fw"'] & & \Sigma FX \\
 & & & F\Sigma X \arrow[ur, "\alpha_X"'] &
\end{tikzcd}$$

is a triangle of $\mathcal{T}'$ for each triangle (X, Y, Z, u, v, w) of $\mathcal{T}$.

Proposition 1.1.2. *[41] Let $\mathcal{T}$ be a triangulated category. Let (X, Y, Z, u, v, w) be a triangle and M an object of $\mathcal{T}$. Then we have*

1) $v \circ u = w \circ v = 0$.

2) The following long exact sequences are exact:

$$\cdots \to \mathcal{T}(M, \Sigma^i X) \to \mathcal{T}(M, \Sigma^i Y) \to \mathcal{T}(M, \Sigma^i Z) \to \mathcal{T}(M, \Sigma^{i+1} X) \to \cdots$$

$$\cdots \to \mathcal{T}(\Sigma^{i+1} X, M) \to \mathcal{T}(\Sigma^i Z, M) \to \mathcal{T}(\Sigma^i Y, M) \to \mathcal{T}(\Sigma^i X, M) \to \cdots$$

3) Let (f, g, h) be a morphism of triangles. If two of the three morphisms are isomorphisms, then so is the third.

Proposition 1.1.3. *[41] Let (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w') be two triangles in a triangulated category $\mathcal{T}$. Let $g : Y \to Y'$ be a morphism. Then the following are equivalent:*

1) $v' \circ g \circ u = 0$.

2) There exists a morphism (f, g, h) from the first triangle to the second.

Proposition 1.1.4. *[72, Lemma 1.4.3] Consider the following commutative diagram whose rows are triangles*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] & \Sigma X \arrow[d, "1"] \\ X \arrow[r, "gf"] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X \end{tikzcd}$$

It may be completed to a morphism of triangles

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[r, "h"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "j"] & \Sigma X \arrow[d, "1"] \\ X \arrow[r, "gf"] & Y' \arrow[r, "h'"] & Z' \arrow[r, "\omega'"] & \Sigma X \end{tikzcd}$$

so that

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "g"] & Z \arrow[d, "j"] \\ Y' \arrow[r, "h'"] & Z' \end{tikzcd}$$

is homotopy cartesian, i.e. there is a canonical triangle in $\mathcal{T}$

$$Y \xrightarrow{\binom{g}{-h}} Y' \oplus Z \xrightarrow{(h'\ j)} Z' \xrightarrow{\partial} \Sigma Y$$

In fact, the differential $\partial : Z \to \Sigma Y$ can be chosen to be the composition

$$Z' \xrightarrow{\omega'} \Sigma X \xrightarrow{\Sigma f} \Sigma Y.$$

1.1.2 Triangulated quotients

Let $\mathcal{T}$ be a triangulated category.

Definition 1.1.5. An additive subcategory $\mathcal{N}$ of $\mathcal{T}$ is a *thick subcategory* if $\mathcal{N}$ is a full triangulated subcategory (i.e. $\mathcal{N}$ is stable under Σ and Σ^{-1} and $\mathcal{N}$ is closed under extensions) of $\mathcal{T}$ which is stable under taking direct factors.

Given a thick subcategory $\mathcal{N}$ of $\mathcal{T}$, the *triangle quotient* (denoted as $\mathcal{T}/\mathcal{N}$) is the category constructed as follows:

- The objects of $\mathcal{T}/\mathcal{N}$ are the objects of $\mathcal{T}$.

- The morphisms in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X,Y)$ are the equivalence classes $f \circ s^{-1}$ of diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y, \end{tikzcd}$$

where s and f are morphisms in $\mathcal{T}$, and s is contained in a triangle

$$Z \xrightarrow{s} X \to N \to \Sigma Z$$

with N an object of $\mathcal{N}$, while the equivalence relation is given by:

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and

$$\begin{tikzcd} & Z' \arrow[dl, "s'"'] \arrow[dr, "f'"] & \\ X & & Y \end{tikzcd}$$

are equivalent if there exist another such diagram

$$\begin{tikzcd} & Z'' \arrow[dl, "s''"'] \arrow[dr, "f''"] & \\ X & & Y \end{tikzcd}$$

and a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & Z'' \arrow[l, "s''"'] \arrow[r, "f''"] \arrow[u, "t"] \arrow[d, "t'"'] & Y \\ & Z' \arrow[ul, "s'"] \arrow[ur, "f'"'] & \end{tikzcd} .$$

Let $f \circ s^{-1}$ be in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X,Y)$ and $g \circ t^{-1}$ in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(Y,Z)$. Suppose that f is in $\mathrm{Hom}_{\mathcal{T}}(X',Y)$ and t is in $\mathrm{Hom}_{\mathcal{T}}(Y',Y)$, and the morphism t is contained in a triangle

$$Y' \xrightarrow{t} Y \xrightarrow{q} N \to \Sigma Y'$$

with $N \in \mathcal{N}$. The morphism $qf \in \mathrm{Hom}_{\mathcal{T}}(X',N)$ can be embedded into a triangle

$$W \to X' \xrightarrow{qf} N \to \Sigma W.$$

The commutative diagram

$$\begin{tikzcd} X' \arrow[r, "qf"] \arrow[d, "f"] & N \arrow[d, equal] \\ Y \arrow[r, "q"] & N \end{tikzcd}$$

can be completed to the following commutative diagram

$$\begin{tikzcd} W \arrow[r, "r"] \arrow[d, "h"] & X' \arrow[r, "qf"] \arrow[d, "f"] & N \arrow[r] \arrow[d, equal] & \Sigma W \arrow[d, "\Sigma h"] \\ Y' \arrow[r, "t"] & Y \arrow[r, "q"] & N \arrow[r] & \Sigma Y'. \end{tikzcd}$$

Then there is a new diagram

$$\begin{tikzcd} & & W \arrow[dl, "r"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

with $fr = th$. The octahedral axiom $(TR4)$ ensures that $(gh) \circ (sr)^{-1}$ lies in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X, Z)$. The composition of $f \circ s^{-1}$ and $g \circ t^{-1}$ is defined as the morphism $(gh) \circ (sr)^{-1}$. It is well-defined.

For each morphism $s \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ which is contained in a triangle

$$X \xrightarrow{s} Y \to N \to \Sigma X$$

with $N \in \mathcal{N}$, the morphism $(1_X)^{-1} \circ s$ is an isomorphism in $\mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X, Y)$ whose inverse is $s^{-1} \circ (1_X)$. The canonical functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ sends each object to itself and sends each morphism $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ to the morphism $f \circ (1_X)^{-1} \in \mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(X, Y)$. The images of the objects in $\mathcal{N}$ under Q are zero objects in $\mathcal{T}/\mathcal{N}$. The functor Q induces a triangulated structure on $\mathcal{T}/\mathcal{N}$. Moreover, the canonical functor Q has the following universal property.

Proposition 1.1.6. *[72] For any triangle functor $F : \mathcal{T} \to \mathcal{T}'$ which sends the objects of a thick subcategory $\mathcal{N}$ of $\mathcal{T}$ to zero objects of $\mathcal{T}'$, there exists a unique triangle functor $F' : \mathcal{T}/\mathcal{N} \to \mathcal{T}'$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "Q"] \arrow[dr, "F"'] & & \mathcal{T}/\mathcal{N} \arrow[dl, dashed, "F'"] \\ & \mathcal{T}'. & \end{tikzcd}$$

Let $\mathcal{T}$ be an additive category. We say that $\mathcal{T}$ is *idempotent complete* if any idempotent morphism $e : X \to X$ has a kernel. We say that a morphism $f : X \to Y$ is *right minimal* if any morphism $g : X \to X$ satisfying $fg = f$ is an isomorphism. Dually we define a *left minimal* morphism. For a collection $\mathcal{X}$ of objects in $\mathcal{T}$, we denote by $\mathrm{add}_{\mathcal{T}}\mathcal{X}$ (or simply $\mathrm{add}\mathcal{X}$) the smallest full subcategory of $\mathcal{T}$ which is closed under finite coproducts, summands and isomorphisms and contains $\mathcal{X}$.

Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$. We say that a morphism $f : X \to Y$ is a *right $\mathcal{X}$-approximation* of Y if $X \in \mathcal{X}$ and $\mathrm{Hom}_{\mathcal{T}}(X', f)$ is surjective for any $X' \in \mathcal{X}$. We say that $\mathcal{X}$ is *contravariantly finite* if any object in $\mathcal{T}$ has a right $\mathcal{X}$-approximation. Dually, we define a *left $\mathcal{X}$-approximation* and a *covariantly finite* subcategory. We say that $\mathcal{X}$ is *functorially finite* if it is contravariantly and covariantly finite. For example, if $\mathcal{T}$ satisfies the following finiteness condition $(\star)$, then $\mathrm{add}X$ is a functorially finite subcategory of $\mathcal{T}$ for any $X \in \mathcal{T}$.

($\star$) $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ is finitely generated as an $\mathrm{End}_{\mathcal{T}}(X)$-module and as an $\mathrm{End}_{\mathcal{T}}(Y)^{op}$-module.

This condition $(\star)$ is satisfied if $\mathcal{T}$ is k-linear and Hom-finite for a commutative ring k.

Denote by $[\mathcal{X}]$ the ideal of $\mathcal{T}$ consisting of morphisms which factor through an object of $\mathrm{add}_{\mathcal{T}}\mathcal{X}$ and denote by $\mathcal{T}/[\mathcal{X}]$ the corresponding additive quotient of $\mathcal{T}$ by $\mathcal{X}$. Define full subcategories

$$\mathcal{X}^{\perp_{\mathcal{T}}} := \{T \in \mathcal{T} \mid \mathrm{Hom}(\mathcal{X}, T) = 0\},$$
$$^{\perp_{\mathcal{T}}}\mathcal{X} := \{T \in \mathcal{T} \mid \mathrm{Hom}(T, \mathcal{X}) = 0\}.$$

When it does not cause confusion, we will simply write $\mathcal{X}^{\perp}$ and $^{\perp}\mathcal{X}$.

Let $\mathcal{T}$ be a triangulated category. For two objects X and Y of $\mathcal{T}$ and an integer n, by $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{>n}Y) = 0$ (respectively, $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{\geqslant n}Y) = 0$, $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{<n}Y) = 0$, $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{\leqslant n}Y) = 0$, we mean $\mathrm{Hom}_{\mathcal{T}}X, \Sigma^{i}Y) = 0$ for all $i > n$ (respectively, for all $i \geqslant n$, $i < n$, $i \leqslant n$). Let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$. We say that $\mathcal{X}$ is a thick subcategory of $\mathcal{T}$ if it is a triangulated subcategory of $\mathcal{T}$ which is closed under taking direct summands. In this case we denote by $\mathcal{T}/\mathcal{X}$ the triangle quotient of $\mathcal{T}$ by $\mathcal{X}$. In general, we denote by $\mathrm{thick}_{\mathcal{T}}\mathcal{X}$ (or simply $\mathrm{thick}\,\mathcal{X}$) the smallest thick subcategory of $\mathcal{T}$ which contains $\mathcal{X}$.

For collections $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{T}$, we denote by $\mathcal{X} * \mathcal{Y}$ the collection of objects $Z \in \mathcal{T}$ appearing in a triangle $X \to Z \to Y \to \Sigma X$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

1.1.3 Presilting and silting subcategories

Definition 1.1.7. A full subcategory $\mathcal{P}$ of $\mathcal{T}$ is *presilting* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \Sigma^{i}\mathcal{P}) = 0$ for any $i > 0$. It is *silting* if in addition $\mathcal{T} = \mathrm{thick}\,\mathcal{P}$.

We denote by $\mathrm{silt}\,\mathcal{T}$ (respectively, $\mathrm{presilt}\,\mathcal{T}$) the class of silting (respectively, presilting) subcategories of $\mathcal{T}$. As usual we identify two (pre)silting subcategories $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{T}$ when $\mathrm{add}\,\mathcal{M} = \mathrm{add}\,\mathcal{N}$. The class $\mathrm{silt}\,\mathcal{T}$ has a natural partial order:

for $\mathcal{M}, \mathcal{N} \in \mathrm{silt}\,\mathcal{T}$, we say $\mathcal{M} \geqslant \mathcal{N}$ if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \Sigma^{>0}\mathcal{N}) = 0$.

Theorem 1.1.8. *[2, Theorem 2.11] The relation $\geqslant$ gives a partial order on* $\mathrm{silt}\,\mathcal{T}$*.*

Proposition 1.1.9. *[2, Proposition 2.4] Let $\mathcal{T}$ be a triangulated category with a silting subcategory $\mathcal{M}$.*

(a) For any $X, Y \in \mathcal{T}$, there exists $i \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{\geqslant i} Y) = 0$.

(b) For any $X \in \mathcal{T}$, there exist $i, j \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \Sigma^{\geqslant i}) = 0$ *and* $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{\geqslant i}\mathcal{M}) = 0$.

1.1.4 t-structures and co-t-structures

Definition 1.1.10. [8] A *t-structure* on $\mathcal{T}$ is given by two strictly (i.e. stable under isomorphisms) full subcategories $\mathcal{T}^{\leqslant 0}$ and $\mathcal{T}^{\geqslant 0}$ which satisfy the following three conditions:

a) for $X \in \mathcal{T}^{\leqslant 0}$ and $Y \in \mathcal{T}^{\geqslant 1}$, we have that $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$,

b) $\mathcal{T}^{\leqslant 0} \subset \mathcal{T}^{\leqslant 1}$ and $\mathcal{T}^{\geqslant 1} \subset \mathcal{T}^{\geqslant 0}$,

c) for any object $X \in \mathcal{T}$, there exists a triangle $X' \to X \to X'' \to \Sigma X'$ such that $X' \in \mathcal{T}^{\leqslant 0}$ and $X'' \in \mathcal{T}^{\geqslant 1}$, where $\mathcal{T}^{\leqslant n}$ denotes $\Sigma^{-n}(\mathcal{T}^{\leqslant 0})$ and $\mathcal{T}^{\geqslant n}$ denotes $\Sigma^{-n}(\mathcal{T}^{\geqslant 0})$ for any $n \in \mathbb{Z}$.

We denote by $\heartsuit_{\mathcal{T}}$ the full subcategory $\mathcal{T}^{\leqslant 0} \cap \mathcal{T}^{\geqslant 0}$ of $\mathcal{T}$. It is called the *heart of the t-structure* $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$. The heart $\heartsuit_{\mathcal{T}}$ is an abelian category(see [8]).

The t-structure $(\mathcal{T}^{\leqslant 0}, \mathcal{T}^{\geqslant 0})$ is said to be bounded if

$$\bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\leqslant n} = \mathcal{T} = \bigcup_{n \in \mathbb{Z}} \mathcal{T}^{\geqslant n},$$

equivalently, if $\mathcal{T} = \mathrm{thick}\, \heartsuit_{\mathcal{T}}$.

There is an alternative description of t-structures which was given in the work on aisles of Keller-Vossieck (see [56]). A strictly full subcategory $\mathcal{A}$ of $\mathcal{T}$ is called an *aisle* if it is stable under shifts Σ^l ($l \in \mathbb{N}$) and extensions, and the inclusion $\mathcal{A} \to \mathcal{T}$ admits a right adjoint.

Proposition 1.1.11. *[56] A strictly full subcategory $\mathcal{A}$ is an aisle if and only if $(\mathcal{A}, (\Sigma\mathcal{A})^{\perp})$ is a t-structure.*

Definition 1.1.12. [73] A *co-t-structure* on a triangulated category $\mathcal{T}$ is a pair of full subcategories $\mathcal{T}_{\geqslant 0}$ and $\mathcal{T}_{\leqslant 0}$ of $\mathcal{T}$ such that

a) both $\mathcal{T}_{\geqslant 0}$ and $\mathcal{T}_{\leqslant 0}$ are additive and stable under taking direct factors;

b) the subcategory $\mathcal{T}_{\geqslant 0}$ is stable under Σ^{-1} and the subcategory $\mathcal{T}_{\leqslant 0}$ is stable under Σ;

c) we have $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for all $X \in \mathcal{T}^{>0}$ and all $Y \in \mathcal{T}^{\leqslant 0}$;

d) for each object $X \in \mathcal{T}$, there is a triangle

$$X' \to X \to X'' \to \Sigma X'$$

such that $X' \in \mathcal{T}_{\geqslant 0}$ and $X'' \in \Sigma\mathcal{T}^{\leqslant 0}$.

The *co-heart* $\mathcal{P}$ is defined as the intersection $\mathcal{T}_{\geqslant 0} \cap \mathcal{T}_{\leqslant 0}$. This is usually not an abelian category. For any two objects M and N in the co-heart, the morphism space

$\mathrm{Hom}_{\mathcal{T}}(M, \Sigma^m N)$ vanishes for any $m > 0$. The co-t-structure $(\mathcal{T}_{\geqslant 0}, \mathcal{T}_{\leqslant 0})$ is said to be *bounded* if

$$\bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{T}_{\geqslant 0} = \mathcal{T} \bigcup_{n\in\mathbb{Z}} \Sigma^n \mathcal{T}_{\leqslant 0},$$

equivalently, if $\mathcal{T} = \mathrm{thick}\,\mathcal{P}$. The co-heart of a bounded co-t-structure is a silting subcategory of $\mathcal{T}$.

1.2 The category of dg categories

In this section, we recall some basic definitions related to dg categories and their invariants. We refer to Keller's ICM address [60] and [22] for the details.

Let k be a commutative ring. A *differential graded or dg category* is a k-category $\mathcal{A}$ whose morphism spaces are dg k-modules and whose compositions $\mathcal{A}(y,z) \otimes \mathcal{A}(x,y) \to \mathcal{A}(x,z)$ are morphisms of dg k-modules. We denote the category of all (small) dg categories over k by dgcat_k. In particular, dg categories with one object can be identified with dg algebras A, i.e. graded k-algebras endowed with a differential d such that the Leibniz rule holds:

$$d(f \circ g) = d(f) \circ g + (-1)^p f \circ d(g)$$

for all $f \in A^p$ and all g.

Let $\mathcal{A}$ be a dg category. The *opposite dg category* $\mathcal{A}^{op}$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$\mathcal{A}^{op}(X,Y) = \mathcal{A}(Y,X);$$

the composition of $f \in \mathcal{A}^{op}(Y,X)^p$ with $g \in \mathcal{A}^{op}(Z,Y)^q$ is given by $(-1)^{pq} gf$.

The category $Z^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(Z^0\mathcal{A})(X,Y) = Z^0(\mathcal{A}(X,Y)),$$

where Z^0 is the kernel of $d^0 : \mathcal{A}(X,Y)^0 \to \mathcal{A}(X,Y)^1$. The category $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(H^0(\mathcal{A}))(X,Y) = H^0(\mathcal{A}(X,Y)),$$

where H^0 denotes the 0-th homology of the complex $\mathcal{A}(X,Y)$. We say that a morphism $f : x \to y$ in $Z^0(\mathcal{A})$ is a *homotopy equivalence* if it becomes invertible in $H^0(\mathcal{A})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories. A *dg functor* $G : \mathcal{B} \to \mathcal{A}$ is given by a map $G : \mathrm{obj}(\mathcal{B}) \to \mathrm{obj}(\mathcal{A})$ and by morphisms of dg k-modules $G(x,y) : \mathcal{B}(x,y) \to \mathcal{A}(Gx, Gy)$, $x, y \in \mathrm{obj}(\mathcal{B})$, compatible with the composition and the units. The category of small dg categories dgcat_k has the small dg categories as objects and the dg functors as morphisms.

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ has the class of objects $\mathrm{obj}(\mathcal{A}) \times \mathrm{obj}(\mathcal{B})$ and the morphism spaces $\mathcal{A}\otimes\mathcal{B}((x,y),(x',y')) = \mathcal{A}(x,x') \otimes \mathcal{B}(y,y')$ with the natural compositions and units. The *enveloping dg category* of $\mathcal{A}$ is defined as $\mathcal{A} \otimes \mathcal{A}^{op}$ and we denote it by $\mathcal{A}^e$.

Let $G, G' : \mathcal{B} \to \mathcal{A}$ be two dg functors. We define $\mathcal{H}om(G, G')^n$ to be the k-module formed by the families of morphisms

$$\phi_x \in \mathcal{A}(Gx, G'x)^n$$

such that $G'(f) \circ \phi_x = \phi_y \circ G(f)$ for all $f \in \mathcal{B}(x,y)$. We define $\mathcal{H}om(G,G')$ to be the graded k-module with components $\mathcal{H}om(G,G')^n$ and whose differential is induced by the differential of $\mathcal{A}(Gx, G'x)$. The set of morphisms $G \to G'$ is by definition in bijection with $Z^0(\mathcal{H}om(G,G'))$. Thus, we can form a dg category $\mathcal{H}om(\mathcal{B},\mathcal{A})$, which has the dg functors as objects and the morphism space $\mathcal{H}om(G,G')$ for two dg functors G and G'.

Endowed with the tensor product, the category dgcat_k becomes a symmetric tensor category which admits an internal Hom-functor, i.e.

$$\mathrm{Hom}(\mathcal{A}\otimes\mathcal{B},\mathcal{C}) = \mathrm{Hom}(\mathcal{A},\mathcal{H}om(\mathcal{B},\mathcal{C})),$$

for $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathrm{dgcat}_k$.

Definition 1.2.1. A *quasi-equivalence* is a dg functor $G : \mathcal{B} \to \mathcal{A}$ such that

(1) $G(x,y) : \mathcal{B}(x,y) \to \mathcal{A}(G(x),G(y))$ is a quasi-isomorphism for all objects x,y of $\mathcal{A}$;

(2) the induced functor $H^0(G) : H^0(\mathcal{B}) \to H^0(\mathcal{A})$ is an equivalence.

By [83], there is a model structure on dgcat_k with weak equivalences being quasi-equivalences. We denote by $\mathrm{Ho}(\mathrm{dgcat_k})$ the corresponding homotopy category.

Theorem 1.2.2. *[83, Theorem 0.1] There is a cofibrantly generated model structure (Dwyer-Kan model structure) on* dgcat_k *where a dg functor* $G : \mathcal{B} \to \mathcal{A}$ *is*

- *a weak equivalence if G is quasi-equivalence;*
- *a fibration if*
  1. *for all objects $x,y \in \mathcal{B}$ the component $G(x,y)$ is a degreewise surjection of chain complexes;*
  2. *for each isomorphism $G(x) \to z$ in $H^0(\mathcal{A})$ there is a lift to an isomorphism in $H^0(\mathcal{B})$.*

1.2.1 Drinfeld dg quotients

Suppose that k is a field. Let $\mathcal{A}$ be a dg category and $\mathcal{B} \subseteq \mathcal{A}$ a full dg subcategory. Denote by $j : \mathcal{B} \to \mathcal{A}$ the inclusion.

Definition 1.2.3. [22] The *dg quotient category* $\mathcal{A}/\mathcal{B}$ is defined as follows:

- $\mathrm{obj}(\mathcal{A}/\mathcal{B}) = \mathrm{obj}(\mathcal{A})$;
- freely add new morphisms $\epsilon_U : U \to U$ of degree -1 for each $U \in \mathrm{obj}(\mathcal{B})$, and set $d(\epsilon_U) = 1_U$.

We denote by $p : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ the canonical functor. For any objects x and y, we have a decomposition of graded k-modules

$$\mathcal{A}/\mathcal{B}(x,y) = \bigoplus_{n\geqslant 0} \bigoplus_{U_i \in \mathrm{obj}(\mathcal{B})} \mathcal{A}(U_n,y)\otimes_k k\epsilon_{U_n}\otimes_k \cdots \otimes_k k\epsilon_{U_2}\otimes_k \mathcal{A}(U_1,U_2)\otimes_k k\epsilon_{U_1}\otimes_k \mathcal{A}(x,U_1).$$

Using the formula $d(\epsilon_U) = 1_U$, one can easily find the differential on $\mathcal{A}/\mathcal{B}(x,y)$.

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor. The *homotopy cofiber* $\mathcal{A}/\mathcal{B}$ of G is defined by the following homotopy push-out diagram in dgcat_k with respect to the Dwyer-Kan model structure

$$\begin{tikzcd} \mathcal{B} \arrow[r, "G"] \arrow[d] & \mathcal{A} \arrow[d, "p"] \\ 0 \arrow[r] & \mathcal{A}/\mathcal{B} \end{tikzcd}$$

and we will call $\mathcal{B} \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ a *homotopy cofiber sequence* in dgcat_k.

The homotopy cofiber $\mathcal{A}/\mathcal{B}$ can be computed as the Drinfeld dg quotient of $\mathcal{A}$ by its full dg subcategory $\mathrm{Im}(G)$, where $\mathrm{Im}(G)$ is the full dg subcategory of $\mathcal{A}$ whose objects are the $y \in \mathcal{A}$ such that there exist an object x in $\mathcal{B}$ and an isomorphism $G(x) \cong y$ in $\mathrm{H}^0(\mathcal{A})$.

Definition 1.2.4. A dg category $\mathcal{A}$ is called *strictly pretriangulated (=spt)* if it satisfies the following:

- each object has a suspension, and $\Sigma : \mathcal{A} \to \mathcal{A}$ is dg dense (i.e. every object in the target category is dg isomorphic to some object in the image);
- each closed morphism of degree zero has a cone.

Proposition 1.2.5. *[58, Lemma 2.3] Let $\mathcal{A}$ be a spt dg category. Then $Z^0(\mathcal{A})$ has a canonical Frobenius exact structure, whose stable category coincides with $H^0(\mathcal{A})$. Therefore, $H^0(\mathcal{A})$ is canonically triangulated.*

Definition 1.2.6. The *pretriangulated hull* $\mathcal{A}^{pretr}$ is the smallest dg subcategory of $\mathcal{C}_{dg}(\mathcal{A})$ containing $\mathcal{A}$, closed under $\Sigma^{\pm}$ and cones. As $\mathcal{A}^{pertr}$ is spt, the *triangulated hull* $\mathcal{A}^{tr}$ of $\mathcal{A}$ is defined to be $H^0(\mathcal{A}^{pretr})$.

Theorem 1.2.7. *[22, Theorem 3.4] Let $\mathcal{A}$ be a dg category and $\mathcal{B} \subseteq \mathcal{A}$ a full dg subcategory. Then the canonical functor*

$$\mathcal{A}^{tr}/\mathcal{B}^{tr} \xrightarrow{\sim} (\mathcal{A}/\mathcal{B})^{tr}$$

is a triangle equivalence.

1.2.2 Homotopy between dg functors

Let $\mathcal{B}$ be a small dg category. The *dg category* $P(\mathcal{B})$ is defined as follows: its *objects* are the homotopy equivalences $f : x \to y$. The complexes of *morphisms* are defined (as $\mathbb{Z}$-graded k-modules) by:

$$P(\mathcal{B})(f, g) = \mathcal{B}(x, w) \oplus \mathcal{B}(y, z) \oplus \mathcal{B}(x, z)[-1],$$

where $f : x \to y, g : w \to z$ are in $P(\mathcal{B})$.

A homogeneous element of degree r of this graded k-module can be represented by a matrix

$$\begin{bmatrix} m_1 & 0 \\ h & m_2 \end{bmatrix},$$

where $m_1 \in \mathcal{B}(x,w)^r$, $m_2 \in \mathcal{B}(y,z)^r$ and $h \in \mathcal{B}(x,z)^{r-1}$.

The differential is given by

$$d\left\{\begin{bmatrix} m_1 & 0 \\ h & m_2 \end{bmatrix}\right\} = \begin{bmatrix} d(m_1) & 0 \\ d(h) + gm_1 - (-1)^r(m_2 f) & d(m_2) \end{bmatrix}.$$

The Composition in $P(\mathcal{B})$ corresponds to matrix multiplication and the units to the identity matrices.

Then we have a dg inclusion functor

$$I : \mathcal{B} \longrightarrow P(\mathcal{B})$$

which sends an object x in $\mathcal{B}$ to $(x = x)$ and $I(f) = \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}$.

Moreover we have two projection functors

$$P_0, P_1 : P(\mathcal{B}) \to \mathcal{B}$$

which are defined by as follows

$$P_0(f : x \to y) = x \qquad P_0\left\{\begin{bmatrix} m_1 & 0 \\ h & m_2 \end{bmatrix}\right\} = m_1;$$

$$P_1(f : x \to y) = y \qquad P_1\left\{\begin{bmatrix} m_1 & 0 \\ h & m_2 \end{bmatrix}\right\} = m_2.$$

Then we obtain the following commutative diagram in dgcat_k.

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "\triangle=(id_{\mathcal{B}}, id_{\mathcal{B}})"] \arrow[dr, "I"'] & & \mathcal{B} \times \mathcal{B} \\ & P(\mathcal{B}) \arrow[ur, "P_0 \times P_1"'] & \end{tikzcd},$$

where I is a quasi-equivalence, $P_0 \times P_1$ is a fibration, with respect to the Dwyer-Kan model structure on dgcat_k (see Theorem 1.2.2). Then the dg category $P(\mathcal{B})$ is a path object for $\mathcal{B}$ [84, Proposition 2.0.11]. Moreover P_0 and P_1 are quasi-equivalences.

Definition 1.2.8. [84, Remark 2.0.12] Let $G, G' : \mathcal{B} \to \mathcal{A}$ be two dg functors, where $\mathcal{B}$ is a cofibrant dg category. Two dg functors G and G' are homotopic if there exists a dg functor $H : \mathcal{B} \to P(\mathcal{A})$ that makes the following diagram commutes

$$\begin{tikzcd} & \mathcal{A} \\ \mathcal{B} \arrow[ur, "G"] \arrow[r, "H"] \arrow[dr, "G'"'] & P(\mathcal{A}) \arrow[u, "P_0"'] \arrow[d, "P_1"] \\ & \mathcal{A}. \end{tikzcd}$$

The dg functor H corresponds exactly to

- a homotopy equivalence $\alpha(x) : G(x) \to G'(x)$ in $\mathcal{A}$ for every object x in $\mathcal{B}$, and
- a degree -1 morphism $h = h(x,y) : \mathcal{B}(x,y) \to \mathcal{A}(G(x), G'(y))$, for all objects x and y in $\mathcal{B}$, such that

$$\alpha(y)G(f) - G'(f)\alpha(x) = d(h(f)) + h(d(f))$$

and

$$h(fg) = h(f)G(g) + (-1)^n G'(f)h(g),$$

where f and g are composable morphisms in $\mathcal{B}$ with f of degree n.

1.2.3 The derived category of a dg category

Let $\mathcal{A}$, $\mathcal{B}$ be small dg categories. A *left dg $\mathcal{A}$-module* is a dg functor $L : \mathcal{A} \to \mathcal{C}_{dg}(k)$. A *right dg $\mathcal{A}$-module* is a dg functor $M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(k)$. A *dg $\mathcal{A}$-$\mathcal{B}$-bimodule* is a dg functor $N : \mathcal{B}^{op} \otimes \mathcal{A} \to \mathcal{C}_{dg}(k)$. For each object X of $\mathcal{A}$, we have the right module represented by X

$$X^\wedge = \mathcal{A}(?, X).$$

The *category of right dg modules* $\mathcal{C}(\mathcal{A})$ has as objects the right dg $\mathcal{A}$-modules and as morphisms $L \to M$ the morphisms of dg functors.

We identify $\mathcal{A}$ bimodule with right $\mathcal{A}^e$ module via the morphism

$$M \otimes \mathcal{A}^e = M \otimes \mathcal{A} \otimes \mathcal{A}^{op} \xrightarrow{\sim} \mathcal{A}^{op} \otimes M \otimes \mathcal{A}$$

taking $m \otimes a \otimes b$ to $(-1)^{|b|(|m|+|a|)} b \otimes m \otimes a$. And we denote by $\mathcal{C}(\mathcal{A}^e)$ the category of $\mathcal{A}$ bimodules. There is a distinguished $\mathcal{A}$-bimodule $\mathcal{A}_\triangle$ given by morphisms in the category $\mathcal{A}$, i.e. $\mathcal{A}_\triangle(x,y) = \mathcal{A}(x,y)$. We call it *diagonal bimodule of* $\mathcal{A}$ and still denote it by $\mathcal{A}$.

A bimodule $M \in \mathcal{C}(\mathcal{A}^e)$ is said to be *semi-free* if there is a set of homogeneous elements $\xi_i \in M(x_i, y_i) i \in S$, called a basis of M, such that, for any pair $(x, y) \in \mathrm{obj}(\mathcal{A}) \times \mathrm{obj}(\mathcal{A})$, every object $\eta \in M(x,y)$ can be written uniquely as a finite sum

$$\eta = \Sigma_{i\in S} f_i \circ \xi_i \circ g_i,$$

where $g_i \in \mathcal{A}(x, x_i)$ and $f_i \in \mathcal{A}(y_i, y)$, and only finitely many of them are nonzero. When the basis set is finite, its cardinality is called the *rank* of the semi-free module M.

The dg category $\mathcal{C}_{dg}(\mathcal{A})$ is defined by $\mathcal{C}_{dg}(\mathcal{A}) = \mathcal{H}om(\mathcal{A}^{op}, \mathcal{C}_{dg}(k))$. We write $\mathcal{H}om(L, M)$ for the complex of morphisms from L to M in $\mathcal{C}_{dg}(\mathcal{A})$. For each $X \in \mathcal{A}$, we have a natural isomorphism

$$\mathcal{H}om(X^\wedge, M) \xrightarrow{\sim} M(X).$$

The *category up to homotopy of dg $\mathcal{A}$-modules* is

$$\mathcal{H}(\mathcal{A}) = H^0(\mathcal{C}_{dg}(\mathcal{A})).$$

A morphism $f : L \to M$ is a *quasi-isomorphism* if it induces an isomorphism in homology. Then the *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of the category $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. The *category of perfect objects* $\mathrm{per}(\mathcal{A})$ associated with $\mathcal{A}$ is the closure in $\mathcal{D}(\mathcal{A})$ of the set of representable functors $X^\wedge = \mathcal{A}(?, X)$, $X \in \mathcal{A}$, under shifts in both directions, extensions and taking direct factors. The *category of perfectly valued modules* $\mathrm{pvd}(\mathcal{A})$ is the full subcategory of $\mathcal{D}(\mathcal{A})$ formed by the dg modules M such that each dg k-module $M(X)$, $X \in \mathcal{A}$, is perfect, i.e. $\sum_p \dim H^p(M(X))$ is finite.

Definition 1.2.9. A dg category $\mathcal{A}$ is said to be *(homologically) smooth* if the diagonal bimodule $\mathcal{A}$ is perfect as a module over $\mathcal{A}^e$, i.e. $\mathcal{A}$ is in $\mathrm{per}(\mathcal{A}^e)$.

Definition 1.2.10. A dg category $\mathcal{A}$ is said to be *proper* if $\mathcal{A}(X, Y) \in \mathrm{per}(k)$ for all objects $(X, Y) \in \mathcal{A}^e$.

Definition 1.2.11. For any right $\mathcal{A}^e$ module M, we define its *derived dual* Θ_M in the derived category $\mathcal{D}(\mathcal{A}^e)$ as

$$M^\vee = \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(M, \mathcal{A}^e).$$

In particular, the *inverse dualizing bimodule* of $\mathcal{A}$ is defined as $\mathcal{A}^\vee$.

Definition 1.2.12. Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor. We define the *inverse dualizing bimodule of* G Θ_f as

$$\Theta_f = \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathrm{Cone}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \to \mathcal{A}), \mathcal{A}^e).$$

1.2.4 Derived functors

Let $\mathcal{A}$ and $\mathcal{B}$ be two dg categories. An $\mathcal{A}$-$\mathcal{B}$-bimodule N defines a functor between the categories of dg modules

$$? \otimes_{\mathcal{A}} N : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B}),$$

given by

$$\begin{aligned} M \otimes_{\mathcal{A}} N(X) =& M \otimes_{\mathcal{A}} N(-, X) \\ =& \mathrm{coker}\Big(\bigoplus_{X,Y \in \mathcal{A}} M(Y) \otimes \mathcal{A}(X, Y) \otimes N(Y) \xrightarrow{\nu} \bigoplus_{X \in \mathcal{A}} M(X) \otimes N(X)\Big), \end{aligned}$$

where $\nu(m \otimes f \otimes n) = mf \otimes n - m \otimes fn$.

This gives an adjoint pair $(? \otimes_{\mathcal{A}} N, \mathcal{H}om_{\mathcal{B}}(N, ?))$ between the categories of dg modules:

$$\begin{tikzcd} \mathcal{C}(\mathcal{A}) \arrow[r, bend left, "? \otimes_{\mathcal{A}} N"] & \mathcal{C}(\mathcal{B}). \arrow[l, bend left, "\mathcal{H}om_{\mathcal{B}}(N,?)"] \end{tikzcd}$$

This adjoint pair induces an adjoint pair between the homotopy categories:

$$\begin{tikzcd} \mathcal{H}(\mathcal{A}) \arrow[r, bend left, "? \otimes_{\mathcal{A}} N"] & \mathcal{H}(\mathcal{B}). \arrow[l, bend left, "\mathcal{H}om_{\mathcal{B}}(N,?)"] \end{tikzcd}$$

However, in general these two functors are not well-defined triangle functors between the derived categories.

Definition 1.2.13. a) A dg $\mathcal{A}$-module P is *cofibrant* if

$$\mathrm{hom}_{\mathcal{C}(\mathcal{A})}(P, L) \xrightarrow{s_*} \mathrm{hom}_{\mathcal{C}(\mathcal{A})}(P, N)$$

is surjective for each quasi-isomorphism $s : L \to N$ which is surjective in each component.

b) A dg $\mathcal{A}$-module P is *fibrant* if

$$\mathrm{hom}_{\mathcal{C}(\mathcal{A})}(N, I) \xrightarrow{i^*} \mathrm{hom}_{\mathcal{C}(\mathcal{A})}(L, I)$$

is surjective for each quasi-isomorphism $i : L \to N$ which is injective in each component.

Proposition 1.2.14. *[57] A dg $\mathcal{A}$-module is cofibrant if and only if it is a direct summand of a dg $\mathcal{A}$-module P which admits a filtration*

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_p \subset F_{p+1} \subset \cdots \subset P\,, p \in \mathbb{N}$$

in $\mathcal{C}(\mathcal{A})$ such that

a) P is the union of the F_p, $p \in \mathbb{N}$;

b) as graded $\mathcal{A}$-module, for each p, F_p is a direct summand of F_{p+1};

c) for each p, the subquotient F_{p+1}/F_p is isomorphic in $\mathcal{C}(\mathcal{A})$ to a direct summand of a direct sum of the form $\Sigma^n A^\wedge$, $A \in \mathcal{A}$, $n \in \mathbb{Z}$.

Proposition 1.2.15. *[57] A dg $\mathcal{A}$-module is fibrant if and only if it is a direct summand of a dg $\mathcal{A}$-module I which admits a filtration*

$$I = F_0 \supset F_1 \supset \cdots$$

such that

a) the canonical morphism

$$I \to \varprojlim I/F_i$$

is an isomorphism;

b) the inclusion $F_{i+1} \hookrightarrow F_i$ splits as a morphism of graded modules;

c) each quotient F_i/F_{i+1} is isomorphic in $\mathcal{C}(\mathcal{A})$ to a direct summand of a direct sum of the form $\Sigma^n \mathrm{Hom}_k(A^\wedge, k)$, $A \in \mathcal{A}$, $n \in \mathbb{Z}$.

Proposition 1.2.16. *[57] The canonical triangle functor $\pi_{\mathcal{A}} : \mathcal{H}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ admits a left adjoint $\mathbf{p}$ and a right adjoint $\mathbf{i}$ such that for each object X of $\mathcal{D}(\mathcal{A})$,*

a) the object $\mathbf{p}X$ is cofibrant and the object $\mathbf{i}X$ is fibrant, and

b) there exist quasi-isomorphisms $\mathbf{p}X \to X$ and $X \to \mathbf{i}X$.

We call $\mathbf{p}X$ a *cofibrant resolution* of X and $\mathbf{i}X$ a *fibrant resolution* of X. We have the following diagram of triangle functors:

$$\begin{tikzcd}
\mathcal{H}(\mathcal{A}) \arrow[rr, bend left, "?\otimes_{\mathcal{A}} N"] \arrow[d, bend right, "\mathbf{p}"'] \arrow[d, bend left, "\pi_{\mathcal{A}}"] & & \mathcal{H}(\mathcal{B}) \arrow[ll, bend left, "\mathcal{H}om_{\mathcal{B}}(N,?)"] \arrow[d, bend right, "\pi_{\mathcal{B}}"'] \arrow[d, bend left, "\mathbf{i}"] \\
\mathcal{D}(\mathcal{A}) & & \mathcal{D}(\mathcal{B})
\end{tikzcd}$$

Definition 1.2.17. Let N be a dg $\mathcal{A}$-$\mathcal{B}$-bimodule. The *left derived functor* $? \overset{L}{\otimes}_{\mathcal{A}} N : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is defined as the composition $\pi_{\mathcal{B}} \circ (? \otimes_{\mathcal{A}} N) \circ \mathbf{p}$. The *right derived functor* $\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(N, ?) : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is defined as the composition $\pi_{\mathcal{A}} \circ \mathcal{H}om_{\mathcal{B}}(N, ?) \circ \mathbf{i}$.

The left derived functor $? \overset{L}{\otimes}_{\mathcal{A}} N : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ and the right derived functor $\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(N, ?) : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ form an adjoint pair, i.e. there is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(L \overset{L}{\otimes}_{\mathcal{A}} N, M) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(L, \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(N, M))$$

for each dg $\mathcal{A}$-module L and dg $\mathcal{B}$-module M.

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor. Then G induces the restriction functor $G_* : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ which is given by $G_*(M) = M \circ G$. It fits into the usual triple of adjoint functors $(G^*, G_*, G^!)$ between $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(\mathcal{B})$. We denote the corresponding adjoint functors between $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{B})$ by $(\mathbf{L}G^*, fG*, \mathbf{R}G^!)$.

1.2.5 Hochschild homology

Let $\mathcal{A}$ be a dg category. The *bar resolution* $(C^{bar}(\mathcal{A}), b')$ of the diagonal bimodule $\mathcal{A}$ is the dg $\mathcal{A}$-bimodule whose value at (x, y) is given by the total complex of the bicomplex whose (n, j)-th entry is

$$C_n^{bar}(x, y)^{(j)} := \oplus_{x_0, \cdots, x_{n-1}} \{\mathcal{A}(x_{n-1}, y) \otimes_k \mathcal{A}(x_{n-2}, x_{n-1}) \otimes_k \cdots \otimes_k \mathcal{A}(x_0, x_1) \otimes_k \mathcal{A}(x, x_0)\}^{(j)} ,$$

where the horizontal arrows are given by the Hochschild differential

$$d_0(a_0 \otimes \cdots \otimes a_n) := \Sigma_{i=0}^{n-1} (-1)^{\epsilon_i} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n .$$

and the vertical arrows are the differentials of the tensor products.

The augmentation is the morphism bimodules

$$\epsilon_{\mathcal{A}} : C^{bar}(\mathcal{A}) \longrightarrow \mathcal{A}$$

which is

$$\epsilon_{x,y} : \oplus_{z \in Obj(\mathcal{A})} \mathcal{A}(z, y) \otimes_k \mathcal{A}(x, z) \longrightarrow \mathcal{A}(x, y) : f \otimes f' \longmapsto f \circ f'$$

and 0 everywhere else. $C^{bar}(\mathcal{A})$ is a cofibrant replacement of $\mathcal{A}$ in the category of $\mathcal{A}$-bimodules.

Definition 1.2.18. Let $\mathcal{A}$ be a small dg category. Then the *Hochschild complex* of $\mathcal{A}$ is defined as

$$HH(\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A}^e} C^{bar}(\mathcal{A})$$

and the *Hochschild homology* $HH_\bullet(\mathcal{A})$ of $\mathcal{A}$ is the homology of this complex. More precisely,

$$HH(\mathcal{A}) = \bigoplus_{m\geqslant 0}\{\bigoplus_{(x_0,x_1,\cdots,x_m)\in Obj(\mathcal{A})} \mathcal{A}(x_m, x_0) \otimes (\Sigma\mathcal{A}(x_{m-1}, x_m)) \otimes \cdots \otimes (\Sigma\mathcal{A}(x_0, x_1))\}$$

We denote by b the differential of $HH(\mathcal{A})$.

Let $G : \mathcal{B} \to \mathcal{A}$ be dg functor. Then G induces a canonical morphism of $\mathcal{B}$-bimodules $G_{\mathcal{B},\mathcal{A}} : C^{bar}(\mathcal{B}) \to C^{bar}(\mathcal{A})$ and we have the following commutative diagram of $\mathcal{B}$-bimodules

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "G"] & \mathcal{A} \\
C^{bar}(\mathcal{B}) \arrow[u, "\epsilon_{\mathcal{B}}"] \arrow[r, "G_{\mathcal{B},\mathcal{A}}"] & C^{bar}(\mathcal{A}). \arrow[u, "\epsilon_{\mathcal{A}}"']
\end{tikzcd}$$

Thus, we have a canonical morphism of Hochschild complexes

$$\gamma_G : HH(\mathcal{B}) = \mathcal{B} \otimes_{\mathcal{B}^e} C^{bar}(\mathcal{B}) \xrightarrow{G\otimes G_{\mathcal{B},\mathcal{A}}} \mathcal{A} \otimes_{\mathcal{B}^e} C^{bar}(\mathcal{A}) \xrightarrow{can} HH(\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A}^e} C^{bar}(\mathcal{A})$$

Definition 1.2.19. The *Hochschild homology* $HH_\bullet(G)$ of the dg functor $G : \mathcal{B} \to \mathcal{A}$ is the homology of the *relative Hochschild complex* which is defined as follows

$$HH(G) = \mathrm{Cone}(\gamma_G : HH(\mathcal{B}) \to HH(\mathcal{A}))\ .$$

1.2.6 Mixed complexes and (negative) cyclic homology

Let Λ be the dg algebra generated by an indeterminate ϵ of chain degree -1 with $\epsilon^2 = 0$ and $d\epsilon = 0$. The underlying complex of Λ is

$$\cdots \to k\epsilon \to k \to 0 \cdots .$$

Then a *mixed complex* over k is a dg right Λ-module whose underlying dg k-module is (M, b) and where ϵ acts by B. Suppose that $M = (M, b, B)$ is a mixed complex. Then the *shifted mixed complex* ΣM is the mixed complex such that $(\Sigma M)_p = M_{p-1}$ for all p, $b_{\Sigma M} = -b$ and $B_{\Sigma M} = -B$. Let $f : M \to M'$ be a morphism of mixed complexes. Then the *mapping cone* over f is the mixed complex

$$\left(M' \oplus M, \begin{bmatrix} b_{M'} & f \\ 0 & -b_M \end{bmatrix}, \begin{bmatrix} B_{M'} & 0 \\ 0 & -B_M \end{bmatrix}\right).$$

We denote by $\mathcal{M}ix$ the category of mixed complexes and by $\mathcal{DM}ix$, i.e. the derived category of the dg algebra Λ.

Let $\mathcal{A}$ be a dg category. We associate a precyclic chain complex $C(\mathcal{A})$ (see [68]) with $\mathcal{A}$ as follows: For each $n \in \mathbb{N}$, its n-th term is

$$\coprod \mathcal{A}(x_n, x_0) \otimes \mathcal{A}(x_{n-1}, x_n) \otimes \mathcal{A}(x_{n-2}, x_{n-1}) \otimes \cdots \otimes \mathcal{A}(x_0, x_1),$$

where the sum runs over all sequences $x_0, \ldots, x_n$ of objects of $\mathcal{A}$. The degeneracy maps are given by

$$d_i(f_n, \ldots, f_i, f_{i-1}, \ldots, f_0) = \begin{cases} (f_n, \ldots, f_i f_{i-1}, \ldots, f_0) & \text{if } i > 0, \\ (-1)^{n+\sigma}(f_0 f_n, \ldots, f_1) & \text{if } i = 0, \end{cases}$$

where $\sigma = (\deg f_0)(\deg f_1 + \cdots + \deg f_{n-1})$. The cyclic operator is given by

$$t(f_{n-1}, \ldots, f_0) = (-1)^{n+\sigma}(f_0, f_{n-1}, f_{n-2}, \cdots, f_1).$$

Then we associate a mixed complex $(M(\mathcal{A}), b, B)$ with this precyclic chain complex as follows: The underlying dg module of $M(\mathcal{A})$ is the mapping cone over $(1-t)$ viewed as a morphism of complexes

$$1 - t : (C(\mathcal{A}), b') \to (C(\mathcal{A}), b),$$

where $b = \sum_{i=0}^{n} (-1)^i d_i$ and $b' = \sum_{i=0}^{n-1} (-1)^i d_i$. Its underlying module is $C(\mathcal{A}) \oplus C(\mathcal{A})$; it is endowed with the grading whose n-th component is $C(\mathcal{A})_n \oplus C(\mathcal{A})_{n-1}$ and the differential is

$$\begin{bmatrix} b & 1-t \\ 0 & -b' \end{bmatrix}.$$

The operator $B : M(\mathcal{A}) \to M(\mathcal{A})$ is

$$\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix},$$

where $N = \sum_{i=0}^{n} t^i$.

Definition 1.2.20. The *cyclic homology* $HC_\bullet(\mathcal{A})$ of $\mathcal{A}$ is defined to be the homology of the *cyclic chain complex* of $\mathcal{A}$

$$HC(\mathcal{A}) = M(\mathcal{A}) \overset{L}{\otimes}_\Lambda k.$$

The *negative cyclic homology* $HN_\bullet(\mathcal{A})$ of $\mathcal{A}$ is defined to be the homology of the *negative cyclic chain complex* of $\mathcal{A}$

$$HN(\mathcal{A}) = \boldsymbol{R}\mathrm{Hom}_\Lambda(k, M(\mathcal{A})).$$

Remark 1.2.21. The dg algebra Λ is the singular homology with coefficients in k of the circle S^1. The circle action is captured algebraically in terms of the structure of a mixed complex so that the above constructions can be explained as homotopy orbit and homotopy fixed points of the Hochschild complex $C_\bullet(\mathcal{A})$ with the algebraic circle action (see [53, 68, 44]).

The augmentation morphism $\Lambda \to k$ induces natural morphisms in $\mathcal{D}(k)$

$$HN(\mathcal{A}) \to HH(\mathcal{A}) \to HC(\mathcal{A}).$$

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor. Then it induces a canonical morphism $\gamma_G : M(\mathcal{B}) \to M(\mathcal{A})$ between their mixed complexes. We denote by $M(G)$ the mapping cone over γ_G.

Definition 1.2.22. The *cyclic homology* $HC_\bullet(G)$ of $G : \mathcal{B} \to \mathcal{A}$ is defined to be the homology of the *cyclic chain complex* of G

$$HC(G) = M(G) \overset{L}{\otimes}_\Lambda k.$$

The *negative cyclic homology* $HN_\bullet(G)$ of $G : \mathcal{B} \to \mathcal{A}$ is defined to be the homology of the *negative cyclic chain complex* of G

$$HN(G) = \boldsymbol{R}\mathrm{Hom}_\Lambda(k, M(G)).$$

Similarly, the augmentation morphism $\Lambda \to k$ induces natural morphisms in $\mathcal{D}(k)$

$$HN(G) \to HH(G) \to HC(G).$$

Theorem 1.2.23. *[58, Theorem 1.5] Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories and $G : \mathcal{B} \to \mathcal{A}$ be a* Morita functor*, i.e. a dg functor such that $G_* : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is an equivalence. Then $\gamma_G : M(\mathcal{B}) \to M(\mathcal{A})$ is an isomorphism in $\mathcal{D}(\Lambda)$.*

1.3 The pseudocompact derived categories

Let k be a field and R a finite dimensional separable k-algebra (i.e. R is projective as a bimodule over itself). By definition, an R-algebra is an algebra in the monoidal category of R-bimodules.

Definition 1.3.1. [30, Chaper IV] An R-algebra A is *pseudocompact* if it is endowed with a linear topology for which it is complete and separated and admits a basis of neighborhoods of zero formed by left ideals I such that A/I is finite dimensional over k.

Remark 1.3.2. Recall [10, Lemma 4.1] that the notion of pseudocompact R-algebra can be defined equivalently using right respectively two-side ideals.

Let A be a pseudocompact R-algebra. A right A-module M is *pseudocompact* if it is endowed with a linear topology admitting a basis of neighborhoods of zero formed by submodules N of M such that M/N is finite-dimensional over k. A *morphism* between pseudocompact modules is a continuous A-linear map. We denote by $\mathrm{Pcm}(A)$ the category of pseudocompact right A-modules.

The common annihilator of the simple pseudocompact A-modules is called the *radical* of A and is denoted by $\mathrm{rad}A$.

Proposition 1.3.3. *[10, Proposition 4.3] The radical of A coincides with the ordinary Jacobson radical of A.*

Definition 1.3.4. [64, section 7.11] Let k and R be as above. Let A be an algebra in the category of graded R-bimodules. We say that A is *pseudocompact* if it is endowed with a complete separated topology admitting a basis of neighborhoods of 0 formed by graded ideals I of finite total codimension in A. We denote by PcGr(R) the category of pseudocompact graded R-algebras.

We say that a dg R-algebra A is *pseudocompact* if it is pseudocompact as a graded R-algebra and the differential d_A is continuous. We denote by PcDg(R) the category of pseudocompact dg R-algebras. We say that a morphism $f : A \to A'$ in PcDg(R) is *quasi-isomorphism* if it induces an isomorphism $H^*(A) \to H^*(A')$.

Definition 1.3.5. An *augmented R-algebra* is an algebra A equipped with k-algebra homomorphisms

$$R \xrightarrow{\eta} A \xrightarrow{\epsilon} R$$

such that $\epsilon \circ \eta$ is the identity. We use the notation $\mathrm{PcAlg}(R)$ for the category of augmented pseudocompact dg R-algebras. We say that a pseudocompact augmented dg R-algebra A is *complete* if $\overline{A} := \ker \epsilon = \mathrm{rad} A$. We denote by $\mathrm{PcAlgc}(R)$ the full subcategory of $\mathrm{PcAlg}(R)$ consisting of complete algebras.

Let S be another finite dimensional separable k-algebra and B a pseudocompact dg S-algebra. A *morphism* f from B to A consists of a k-algebra morphism (not necessarily unital) $f_0 : S \to R$ and a dg k-algebra morphism (not necessarily unital) $f_1 : B \to A$ such that the following square commutes in the category of k-algebras

$$\begin{tikzcd} S \arrow[r, "f_0"] \arrow[d, "\eta_B"'] & R \arrow[d, "\eta_A"] \\ B \arrow[r, "f_1"] & A. \end{tikzcd}$$

Similarly, we define morphisms from an object in $\mathrm{PcAlg}(S)$ (respectively $\mathrm{PcAlgc}(S)$) to an object in PcAlg(R) (respectively PcAlgc(R)).

Example 1.3.6. [64, Section 7.11] Let Q be a finite graded quiver. We take R to be the product of copies of k indexed by the vertices of Q and A to be the completed path algebra, i.e. for each integer n, the component A^n is the product of the spaces kp, where p ranges over the paths in Q of total degree n. We endow A with a continuous differential sending each arrow to a possibly infinite linear combination of paths of length $\geqslant 2$. For each n, we define I_n to be the ideal generated by the paths of length $\geq n$ and we define the topology on A to have the I_n as a basis of neighborhoods of 0. Then A is an augmented pseudocompact complete dg R-algebra.

Let A be a pseudocompact dg R-algebra. A right dg A-module M is *pseudocompact* if it is endowed with a topology for which it is complete and separated (in the category of graded A-modules) and which admits a basis of neighborhoods of 0 formed by dg submodules of finite total codimension. It is clear that A is a pseudocompact dg module over itself.

A *morphism* between pseudocompact dg A-modules is a continuous dg A-module morphism. We denote by $\mathcal{C}_{pc}(A)$ the the category of pseudocompact dg right A-modules. A morphism $f : L \to M$ between pseudocompact dg right A-modules is a *quasi-isomorphism* if it induces an isomorphism $H^*(L) \to H^*(M)$.

Proposition 1.3.7. *[64, Lemma 7.12.]*

a) The homology $H^*(A)$ is pseudocompact graded R-algebra. In particular, $H^0(A)$ is a pseudocompact R-algebra.

b) For each pseudocompact dg module M, the homology $H^*(M)$ is a pseudocompact graded module over $H^*(A)$.

The category $\mathcal{C}_{pc}(A)$ has a natural dg enhancement $\mathcal{C}_{pc}^{dg}(A)$. It has the same objects as $\mathcal{C}_{pc}(A)$. For $L, M \in \mathcal{C}_{pc}^{dg}(A)$, the *Hom-complex* $\mathcal{H}om_{\mathcal{C}_{pc}^{dg}(A)}(L, M)$ is the complex of R-modules with degree p component $\mathcal{H}om^p_{\mathcal{C}_{pc}^{dg}(A)}(L, M)$ being the space of continuous homogeneous A-linear maps of degree p form L to M (here we consider A as a pseudocompact graded algebra and L, M as pseudocompact graded A-modules) and with its differential defined by

$$d(f) = d_M \circ f - (-1)^p f \circ d_L$$

for $f \in \mathrm{Hom}^p_{\mathcal{C}_{pc}^{dg}(A)}(L, M)$.

For a dg category $\mathcal{A}$, the category $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(H^0(\mathcal{A}))(X, Y) = H^0(\mathcal{A}(X, Y)),$$

where H^0 denotes the 0-th homology of the complex $\mathcal{A}(X, Y)$. We denote by $\mathcal{H}_{pc}(A)$ the zeroth homology category of $\mathcal{C}_{pc}^{dg}(A)$, i.e. $\mathcal{H}_{pc}(A) = H^0(\mathcal{C}_{pc}^{dg}(A))$.

Theorem 1.3.8. *[75, Section 8.2] Let A be an object in* PcAlgc(R)*. There exits a model structure on $\mathcal{C}_{pc}(A)$ which is given as follows:*

(1) The weak equivalences are the morphisms with an acyclic cone. Here, an object is acyclic if it is in the smallest subcategory of the homotopy category of A which contains the total complexes of short exact sequences and is closed under arbitrary products.

(2) The cofibrations are the injective morphisms with cokernel which is projective when forgetting the differential.

(3) The fibrations are the surjective morphisms.

The corresponding homotopy category is called the pseudocompact derived category *of A. We denote it by $\mathcal{D}_{pc}(A)$.*

Remark 1.3.9. [10, pp.10] Under suitable boundedness assumptions (algebras concentrated in degrees $\leqslant 0$ and modules concentrated in degrees $\leqslant N$), weak equivalence is the same as quasi-isomorphism.

Let A be an object in PcAlgc(R). We define the *perfect derived category* $\mathrm{per}_{pc}(A)$ to be the thick subcategory of $\mathcal{D}_{pc}(A)$ generated by the free A-module of rank 1. The *perfectly valued derived category* $\mathrm{pvd}_{pc}(A)$ is defined to be the full subcategory of $\mathcal{D}_{pc}(A)$ whose objects are the pseudocompact dg modules M such that $\mathrm{Hom}_{\mathcal{D}_{pc}(A)}(P, M)$ is finite-dimensional for each perfect P.

Let A and A' be two pseudocompact dg R-algebras. Their *complete tensor product* is defined by

$$A\widehat{\otimes}_k A' = \varprojlim_{U,V} A/U \otimes_k A'/V,$$

where U, V run through the system of open neighborhoods of zero in A and A' respectively. Then $A\widehat{\otimes}_k A'$ is also pseudocompact. We define the *enveloping algebra* A^e of A to be the complete tensor product $A\widehat{\otimes}_k A^{op}$.

The dg algebra A is a pseudocompact dg module over the enveloping algebra $A\widehat{\otimes}_k A^{op}$. We say that A is *(topologically homologically) smooth* if the module A considered as a pseudocompact dg module over A^e lies in $\mathrm{per}_{pc}(A^e)$.

Proposition 1.3.10. *[64, Lemma 7.13] If A is the completed path algebra of a finite graded quiver endowed with a continuous differential sending each arrow to a possibly infinite linear combination of paths of length ≥ 2, then A is smooth.*

Proposition 1.3.11. *[64, Proposition 7.14] Let A be a pseudocompact dg R-algebra. Assume that A is smooth and connective.*

a) The canonical functor $\mathcal{H}_{pc}(A) \to \mathcal{D}_{pc}(A)$ has a left adjoint $M \mapsto \mathbf{p}M$.

b) The triangulated category $\mathrm{pvd}_{pc}A$ is generated by the dg modules of finite dimension concentrated in degree 0.

c) The full subcategory $\mathrm{pvd}_{pc}A$ of $\mathcal{D}_{pc}(A)$ is contained in the perfect derived category $\mathrm{per}_{pc}A$.

d) The opposite category $\mathcal{D}_{pc}(A)^{op}$ is compactly generated by $\mathrm{pvd}_{pc}A$.

e) Let $A \to A'$ be a quasi-isomorphism of pseudocompact, smooth dg algebras whose homology is concentrated in non positive degrees. Then the restriction functor $\mathcal{D}_{pc}(A') \to \mathcal{D}_{pc}(A)$ is an equivalence. In particular, if the homology of A is concentrated in degree 0, there is an equivalence $\mathcal{D}_{pc}(A) \to \mathcal{D}_{pc}(H^0(A))$. Moreover, in this case $\mathcal{D}_{pc}(H^0A)$ is equivalent to the derived category of the abelian category $\mathrm{Pcm}(H^0A)$.

f) Assume that A is a complete path algebra. There is an equivalence between $\mathcal{D}_{pc}(A)^{op}$ and the localizing subcategory $\mathcal{D}_0(A)$ of the ordinary derived category $\mathcal{D}(A)$ generated by the finite-dimensional dg A-modules.

Remark 1.3.12. In a) of the above Proposition, we only need the condition 'smooth'. In b), we need the condition 'connective'.

For two objects L and M of $\mathcal{D}_{pc}(A)$, define

$$\boldsymbol{R}\mathrm{Hom}_A(L, M) = \mathcal{H}om_{\mathcal{C}^{dg}_{pc}(A)}(\mathbf{p}L, M).$$

Then we have $\operatorname{Hom}_{\mathcal{D}_{pc}(A)}(L,M) = \mathcal{H}_{pc}(A)(\mathbf{p}L,M) = H^0(\mathcal{H}om_{\mathcal{C}^{dg}_{pc}(A)}(\mathbf{p}L,M))$. Moreover, we have the tensor-Hom adjunction

$$\boldsymbol{R}\operatorname{Hom}_A(X\widehat{\otimes}^L_A Y, Z) \simeq \boldsymbol{R}\operatorname{Hom}_A(X, \boldsymbol{R}\operatorname{Hom}_A(Y,Z))$$

for objects X, Y and Z in $\mathcal{D}_{pc}(A)$.

Definition 1.3.13. Let A be an object in PcAlgc(R). For a pseudocompact dg A-bimodule M, we define its *derived dual* $M^\vee$ as

$$M^\vee = \boldsymbol{R}\operatorname{Hom}_{A^e}(M, A^e).$$

In particular, the *inverse dualizing bimodule* of A is defined as $A^\vee$.

Let S be an other finite dimensional separable k-algebra and B an object in PcAlgc(S). Let f be a morphism from B to A. The *inverse dualizing bimodule* Θ_f of f is defined as

$$\Theta_f = \boldsymbol{R}\operatorname{Hom}_{A^e}(\operatorname{Cone}(A\widehat{\otimes}^L_B A \to A), A^e).$$

The morphism f induces the restriction functor $f_* : \mathcal{C}_{pc}(A) \to \mathcal{C}_{pc}(B)$. It fits into the usual triple of adjoint functors (f^*, f_*) between $\mathcal{D}_{pc}(A)$ and $\mathcal{D}_{pc}(B)$.

Proposition 1.3.14. *[64, 1] Let A be a pseudocompact dg R-algebra. The forgetful functor $\mathcal{D}_{pc}(A) \to \mathcal{D}(A)$ restricts to a triangle equivalence* $\operatorname{per}_{pc}(A) \to \operatorname{per}(A)$*. If A is also smooth, then it restricts to a triangle equivalence* $\operatorname{pvd}_{pc}(A) \to \operatorname{pvd}(A)$*.*

Corollary 1.3.15. *Let $f : A \to A'$ be a morphism in* PcAlg(R)*. Assume that A and A' are connective and smooth. Then $f_* : \mathcal{D}(A') \to \mathcal{D}(A)$ is a triangle equivalence if and only if $f_* : \mathcal{D}_{pc}(A') \to \mathcal{D}_{pc}(A)$ is.*

1.3.1 Hochschild/cyclic homology in the pseudocompact setting

Let Λ be the dg algebra generated by an indeterminate ϵ of cohomological degree -1 with $\epsilon^2 = 0$ and $d\epsilon = 0$. The underlying complex of Λ is

$$\cdots \to k\epsilon \to k \to 0 \to \cdots .$$

Then a *mixed complex* over k is a dg right Λ-module whose underlying dg k-module is (M, b) and where ϵ acts by a closed endomorphism B. Suppose that $M = (M, b, B)$ is a mixed complex. Then the *shifted mixed complex* ΣM is the mixed complex such that $(\Sigma M)^p = M^{p-1}$ for all p, $b_{\Sigma M} = -b$ and $B_{\Sigma M} = -B$. Let $f : M \to M'$ be a morphism of mixed complexes. Then the *mapping cone* over f is the mixed complex

$$\left(M' \oplus M, \begin{bmatrix} b_{M'} & f \\ 0 & -b_M \end{bmatrix}, \begin{bmatrix} B_{M'} & 0 \\ 0 & -B_M \end{bmatrix}\right).$$

We denote by $\mathcal{M}ix$ the category of mixed complexes and by $\mathcal{DM}ix$ the derived category of the dg algebra Λ.

Let A be an object in PcAlgc(R) (see Definition 1.3.5). For an R-bimodule U, we define $U_R = U/[R, U]$ and we let U^R be the R-centralizer in U. We associate a precyclic chain complex $C(A)$ (see [?, Definition 2.5.1]) with A as follows: For each $n \in \mathbb{N}$, its n-th term is

$$C_p(A) = (A \otimes_R \overline{A}^{\otimes_R^p})_R.$$

The degeneracy maps are given by

$$d_i(a_n, \ldots, a_i, a_{i-1}, \ldots, a_0) = \begin{cases} (a_n, \ldots, a_i a_{i-1}, \ldots, a_0) & \text{if } i > 0, \\ (-1)^{n+\sigma}(a_0 a_n, \ldots, a_1) & \text{if } i = 0, \end{cases}$$

where $\sigma = (\deg a_0)(\deg a_1 + \cdots + \deg a_{n-1})$. The cyclic operator is given by

$$t(a_{n-1}, \ldots, a_0) = (-1)^{n+\sigma}(a_0, a_{n-1}, a_{n-2}, \cdots, a_1).$$

Then the corresponding *product total complex* $(HH(A), b)$ of $(C(A), b = \sum_{i=0}^{n}(-1)^i d_i)$ is called the *normalized Hochschild complex* of A. The homology of this complex is denoted by $HH_\bullet(A)$ and called the *(continuous) Hochschild homology* of A. By [10, Proposition B.1], the normalized Hochschild complex is quasi-isomorphic to $A \widehat{\otimes}^L_{A^e} A$ in $\mathcal{D}(k)$.

We associate a mixed complex $(M(A), b, B)$ with this precyclic chain complex as follows: Consider the *product total complex* $(HH(A), b')$ of $(C(A), b' = \sum_{i=0}^{n-1}(-1)^i d_i)$. The underlying dg module of $M(A)$ is the mapping cone over $(1-t)$ viewed as a morphism of complexes

$$1 - t : (HH(A), b') \to (HH(A), b),$$

where $b = \sum_{i=0}^{n}(-1)^i d_i$ and $b' = \sum_{i=0}^{n-1}(-1)^i d_i$. Its underlying module is $HH(A) \oplus HH(A)$; it is endowed with the grading whose n-th component is $HH(A)_n \oplus HH(A)_{n-1}$ and the differential is

$$\begin{bmatrix} b & 1-t \\ 0 & -b' \end{bmatrix}.$$

The operator $B : M \to M$ is

$$\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix},$$

where $N = \sum_{i=0}^{n} t^i$.

Let S be an other finite dimensional separable k-algebra and B a pseudocompact dg S-algebra. Let f be a morphism from B to A. Then f induces a canonical morphism between their Hochschild complexes

$$\gamma_f : HH(B) \to HH(A).$$

Definition 1.3.16. The *(continuous) Hochschild homology* $HH_\bullet(f)$ of f is the homology of the *relative Hochschild complex* which is defined as follows

$$HH(f) = \mathrm{Cone}(\gamma_f : HH(B) \to HH(A)).$$

Definition 1.3.17. The *(continuous) cyclic homology* $HC_\bullet(A)$ of A is defined to be the homology of the *cyclic chain complex* of $\mathcal{A}$

$$HC(A) = M(A) \overset{L}{\otimes}_\Lambda k.$$

The *(continuous) negative cyclic homology* $HN_\bullet(A)$ of A is defined to be the homology of the *negative cyclic chain complex* of A

$$HN(A) = \boldsymbol{R}\mathrm{Hom}_\Lambda(k, M(A)).$$

The augmentation morphism $\Lambda \to k$ induces natural morphisms in $\mathcal{D}(k)$

$$HN(A) \to HH(A) \to HC(A).$$

The morphism f also induces a canonical morphism between their mixed complexes

$$\gamma_f : M(B) \to M(A).$$

We denote by $M(f)$ the mapping cone over γ_f.

Definition 1.3.18. The *(continuous) cyclic homology* $HC_\bullet(f)$ of $f : B \to A$ is defined to be the homology of the *cyclic chain complex group* of f

$$HC(f) = M(f) \overset{L}{\otimes}_\Lambda k.$$

The *(continuous) negative cyclic homology* $HN_\bullet(f)$ of $f : B \to A$ is defined to be the homology of the *negative cyclic chain complex* of f

$$HN(f) = \boldsymbol{R}\mathrm{Hom}_\Lambda(k, M(f)).$$

Similarly, the augmentation morphism $\Lambda \to k$ induces natural morphisms in $\mathcal{D}(k)$

$$HN(f) \to HH(f) \to HC(f).$$

Chapter 2

Relative Calabi–Yau structures

2.1 Reminder on the derived category of morphisms

Let I be the path k-category of the quiver $1 \to 2$. The letter I stands for 'interval'. Let $\mathcal{A}$ be a dg k-category. The objects of the derived category $\mathcal{D}(I^{op} \otimes \mathcal{A})$ identify with morphisms $f : M_1 \to M_2$ of dg $\mathcal{A}$-modules. Each such object gives rise to a triangle

$$M_1 \xrightarrow{f} M_2 \longrightarrow \operatorname{cof}(f) \longrightarrow \Sigma M_1$$

of $\mathcal{D}\mathcal{A}$ which is *functorial* in the object f of $\mathcal{D}(I^{op} \otimes \mathcal{A})$. Here, we write cof for the homotopy cofiber, i.e. the cone of a morphism.

For two objects $f : M_1 \to M_2$ and $f' : M_1' \to M_2'$, consider a morphism of triangles

$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "a"'] & M_2 \arrow[r] \arrow[d, "b"] & \operatorname{cof}(f) \arrow[r] \arrow[d, "c"] & \Sigma M_1 \arrow[d, "\Sigma a"] \\ M_1' \arrow[r, "f'"] & M_2' \arrow[r, "g'"] & \operatorname{cof}(f') \arrow[r] & \Sigma M_1' \end{tikzcd}$$

in the derived category $\mathcal{D}\mathcal{A}$. It is well-known that a given morphism $b : M_2 \to M_2'$ extends to such a morphism of triangles (a, b, c) if and only if we have $g' \circ b \circ f = 0$ and that in this case, the pair (a, b) lifts to a morphism of $\mathcal{D}(I^{op} \otimes \mathcal{A})$. The following easy lemma makes this more precise. Here, we write fib for the homotopy fiber, i.e. the desuspension of the cone of a morphism.

Lemma 2.1.1. *We have a canonical isomorphism bifunctorial in the objects f and f' of $\mathcal{D}(I^{op} \otimes \mathcal{A})$*

$$\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(f, f') \xrightarrow{\sim} \operatorname{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_2, M_2') \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_1, \operatorname{cof}(f'))).$$

More precisely, let $g : N_1 \to N_2$ and $g' : N_1' \to N_2'$ be objects in $\mathcal{D}(I^{op} \otimes \mathcal{A})$ and let $\alpha : f \to g$ and $\beta : g' \to g$ be morphisms in $\mathcal{D}(I^{op} \otimes \mathcal{A})$, then we have the following *commutative diagrams*

$$
\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(g,f') \arrow[r, "\sim"] \arrow[d, "\alpha^*"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(N_2,M_2') \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(N_1,\mathrm{cof}(f'))) \arrow[d, "\alpha^*"] \\
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(f,f') \arrow[r, "\sim"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_2,M_2') \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_1,\mathrm{cof}(f'))),
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(f,g') \arrow[r, "\sim"] \arrow[d, "\beta_*"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_2,N_2') \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_1,\mathrm{cof}(g'))) \arrow[d, "\beta_*"] \\
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(f,g) \arrow[r, "\sim"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_2,N_2) \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_1,\mathrm{cof}(g))).
\end{tikzcd}
$$

Proof. We have isomorphisms of dg categories

$$
\begin{aligned}
\mathcal{C}_{dg}(I^{op}\otimes\mathcal{A}) =& \mathcal{H}om(I\otimes\mathcal{A}^{op},\mathcal{C}_{dg}(k)) \\
\simeq& \mathcal{H}om(I,\mathcal{H}om(\mathcal{A}^{op},\mathcal{C}_{dg}(k))) \\
=& \mathcal{H}om(I,\mathcal{C}_{dg}(\mathcal{A})).
\end{aligned}
$$

In this way, $\mathcal{C}_{dg}(\mathcal{A})$ identifies with the category of morphisms $M_1 \to M_2$ of dg $\mathcal{A}$-modules with the dg enhancement given by

$$
\begin{tikzcd}
\mathcal{H}om(f,f') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{H}om_{\mathcal{A}}(M_2,M_2') \arrow[d] \\
\mathcal{H}om_{\mathcal{A}}(M_1,M_1') \arrow[r] & \mathcal{H}om_{\mathcal{A}}(M_1,M_2').
\end{tikzcd}
$$

The model structure on $\mathcal{C}(I^{op}\otimes\mathcal{A})$ translates into a model structure on $\mathcal{H}om(I,\mathcal{C}_{dg}(\mathcal{A}))$ whose weak equivalences are the componentwise quasi-isomorphisms and whose cofibrant objects are the graded split monomorphisms $M_1 \rightarrowtail M_2$ with cofibrant M_1 and M_2. Therefore, we can assume that f and f' are graded split injective morphisms between cofibrant dg $\mathcal{A}$-modules. Then we have the following commutative diagram in $\mathcal{C}(\mathcal{A})$

$$
\begin{tikzcd}
M_1 \arrow[r, tail] \arrow[d] & M_2 \arrow[r, two heads] \arrow[d] & \mathrm{coker}(f) \arrow[d] \\
M_1' \arrow[r, tail] & M_2' \arrow[r, two heads] & \mathrm{coker}(f'),
\end{tikzcd}
$$

where the first row and second row are graded split exact sequences. It induces the the following commutative diagram of complexes

$$
\begin{tikzcd}
\mathcal{H}om(\mathrm{coker}(f),M_2') \arrow[r, tail] \arrow[d, equal] & \mathcal{H}om(f,f') \arrow[r, two heads] \arrow[d, tail] \arrow[dr, phantom, "\square"] & \mathcal{H}om(M_1,M_1') \arrow[d, tail] \\
\mathcal{H}om(\mathrm{coker}(f),M_2') \arrow[r, tail] & \mathcal{H}om(M_2,M_2') \arrow[r, two heads] \arrow[d, two heads] & \mathcal{H}om(M_1,M_2') \arrow[d, two heads] \\
 & \mathcal{H}om(M_1,\mathrm{coker}(f')) \arrow[r, equal] & \mathcal{H}om(M_1,\mathrm{coker}(f')),
\end{tikzcd}
$$

where the upper right square is a bicartesian. Thus we have the exact sequence

$$\mathcal{H}om(f,f') \hookrightarrow \mathcal{H}om(M_2,M_2') \twoheadrightarrow \mathcal{H}om(M_1,\mathrm{coker}(f'))$$

and the canonical isomorphism

$$\boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(f,f') \xrightarrow{\sim} \mathrm{fib}(\boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(M_2,M_2') \to \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(M_1,\mathrm{cof}(f')).$$

$\surd$

Let $G:\mathcal{B}\to\mathcal{A}$ be a dg functor. It induces the dg functor

$$\mathbf{1}\otimes G : I^{op}\otimes\mathcal{B} \longrightarrow I^{op}\otimes\mathcal{A},$$

which we still denote by G. It yields the adjunction

$$\boldsymbol{L}G^* : \mathcal{D}(I^{op}\otimes\mathcal{B}) \leftrightarrows \mathcal{D}(I^{op}\otimes\mathcal{A}) : G_*.$$

Lemma 2.1.2. *Let $f:M_1\to M_2$ and $f':M_1'\to M_2'$ be objects in $\mathcal{D}(I^{op}\otimes\mathcal{B})$. We have the following commutative diagram*

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{B}}(f,f') \arrow[r, "\sim"] \arrow[d, "\mathbf{L}G^*"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(M_2,M_2') \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M_1,\mathrm{cof}(f')) \arrow[d, "\mathbf{L}G^*"] \\
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(\mathbf{L}G^*(f),\mathbf{L}G^*(f')) \arrow[r, "\sim"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{L}G^*(M_2),\mathbf{L}G^*(M_2')) \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathbf{L}G^*(M_1),\mathrm{cof}(\mathbf{L}G^*(f'
\end{tikzcd}$$

Similarly, let $g:N_1\to N_2$ and $g':N_1'\to N_2'$ be objects in $\mathcal{D}(I^{op}\otimes\mathcal{A})$. We have the following commutative diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}}(g,g') \arrow[r, "\sim"] \arrow[d, "G_*"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(N_2,N_2') \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(N_1,\mathrm{cof}(g')) \arrow[d, "G_*"] \\
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{B}}(G_*(g),G_*(g')) \arrow[r, "\sim"] & \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(G_*(N_2),G_*(N_2')) \to \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(G_*(N_1),\mathrm{cof}(G_*(g')).
\end{tikzcd}$$

Proof. We only show the first statement since the second one can be shown similarly. We can assume that f, f' are graded split injective morphisms between cofibrant dg $\mathcal{A}$-modules. Then it is easy to see that the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}om_{I^{op}\otimes\mathcal{B}}(f,f') \arrow[r, tail] \arrow[d] & \mathcal{H}om_{\mathcal{B}}(M_2,M_2') \arrow[r, two heads] \arrow[d] & \mathcal{H}om_{\mathcal{B}}(M_1,\mathrm{coker}(f')) \arrow[d] \\
\mathcal{H}om_{I^{op}\otimes\mathcal{A}}(G^*(f),G^*(f')) \arrow[r, tail] & \mathcal{H}om_{\mathcal{A}}(G^*(M_2),G^*(M_2')) \arrow[r, two heads] & \mathcal{H}om_{\mathcal{A}}(G^*(M_1),\mathrm{coker}(G^*(f'))).
\end{tikzcd}$$

Thus we get the first commutative diagram.

$\surd$

Relative right Calabi–Yau structures were invented by Bertrand Toën in [85, pp. 227-228]. Later, the theory of relative right and left Calabi–Yau structures was developed by Chris Brav and Tobias Dyckerhoff in [16].

2.2 Relative right Calabi–Yau structures

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor [1]. We denote by $D\mathcal{A}^{op}$ the dg $\mathcal{A}$-bimodule defined as follows:

$$D\mathcal{A}^{op}(X,Y) = D\mathcal{A}(Y,X), \quad \forall (X,Y) \in \mathcal{A}^e,$$

where D is the k-linear dual $\mathrm{Hom}_k(?,k)$. We call it the *linear dual bimodule* of $\mathcal{A}$. Similarly, we define the dg $\mathcal{B}$-bimodule $D\mathcal{B}^{op}$. The natural $\mathcal{B}$-bimodule morphism $u_G : \mathcal{B} \to G_*\mathcal{A}$ induces a morphism between their linear dual bimodules

$$G_*(D\mathcal{A}^{op}) \to D\mathcal{B}^{op}.$$

It canonically lifts to an object u_G^* of $\mathcal{D}(I^{op} \otimes \mathcal{B}^e)$. Similarly, its homotopy fiber

$$\mathrm{fib}(u_G^*) \to G_*(D\mathcal{A}^{op})$$

lifts to an object δ_G of $\mathcal{D}(I^{op} \otimes \mathcal{B}^e)$. Each morphism $\Sigma^{n-1}(u_G) \to \delta_G$ gives rise to a morphism of triangles in $\mathcal{D}(\mathcal{B}^e)$

$$\begin{tikzcd}
\Sigma^{n-1}\mathcal{B} \arrow[r, "\Sigma^{n-1}u_G"] \arrow[d] & \Sigma^{n-1}G_*\mathcal{A} \arrow[r] \arrow[d] & \mathrm{cof}(u) \arrow[r] \arrow[d] & \Sigma^n\mathcal{B} \arrow[d] \\
\mathrm{fib}(u_G^*) \arrow[r, "\delta_G"] & G_*(D\mathcal{A}^{op}) \arrow[r, "u_G^*"] & D\mathcal{B}^{op} \arrow[r] & \Sigma\mathrm{fib}(u_G^*)
\end{tikzcd} \tag{2.1}$$

We are therefore interested in morphisms $\Sigma^{n-1}u_G \to \delta_G$ in $\mathcal{D}(I^{op} \otimes \mathcal{B}^e)$.

Lemma 2.2.1. *We have a canonical isomorphism*

$$\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{B}^e}(u_G, \delta_G) \xrightarrow{\sim} \mathrm{fib}(\mathrm{Hom}_k(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}, k) \to \mathrm{Hom}_k(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}, k)).$$

Moreover this isomorphism is compatible with the composition of dg functors, i.e. if $Q : \mathcal{A} \to \mathcal{C}$ be another dg functor, then we have the following commutative diagram

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{B}^e}(u_{Q\circ G}, \delta_{Q\circ G}) \arrow[r, "\sim"] \arrow[d] & \mathrm{fib}(\mathrm{Hom}_k(\mathcal{C} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{C}, k) \to \mathrm{Hom}_k(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}, k)) \arrow[d] \\
\mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{B}^e}(u_G, \delta_G) \arrow[r, "\sim"] & \mathrm{fib}(\mathrm{Hom}_k(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}, k) \to \mathrm{Hom}_k(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}, k))
\end{tikzcd}$$

Proof. Using the standard adjunctions, we get

$$\boldsymbol{R}\mathrm{Hom}_{\mathcal{B}^e}(\mathcal{B}, D\mathcal{B}^{op}) \simeq \mathrm{Hom}_k(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}, k)$$

and

$$\begin{aligned}
\boldsymbol{R}\mathrm{Hom}_{\mathcal{B}^e}(G_*\mathcal{A}, G_*(D\mathcal{A}^{op})) &\simeq \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}^e}(\boldsymbol{L}G^*(G_*\mathcal{A}), D(\mathcal{A}^{op})) \\
&\simeq \mathrm{Hom}_k(G_*(\mathcal{A}) \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}^e, k) \\
&\simeq \mathrm{Hom}_k(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}, k).
\end{aligned}$$

[1] The definition we will give actually makes sense even if we do not assume $\mathcal{A}$ and $\mathcal{B}$ to be proper.

We know that the composition $Q \circ G : \mathcal{B} \to \mathcal{A} \to \mathcal{C}$ induces the following morphisms in $\mathcal{D}(I^{op} \otimes \mathcal{B}^e)$

$$u_G \to u_{Q\circ G}, \quad \delta_{Q\circ G} \to \delta_G.$$

Then the claim follows by Lemma 2.1.1.

$\surd$

We therefore obtain the following chain of morphisms

$$\begin{array}{c} \mathrm{Hom}(HC(G), k) \\ \downarrow \\ \mathrm{Hom}(HH(G), k) \cong \mathrm{fib}(\mathrm{Hom}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}, k) \to \mathrm{Hom}(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}, k)) \\ \downarrow \\ \mathrm{fib}(\mathrm{Hom}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}, k) \to \mathrm{Hom}(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}, k)) \xrightarrow{\sim} \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{B}^e}(u_G, \delta_G) \end{array} \tag{2.2}$$

Definition 2.2.2. *[16, Definition 4.7] A right n-Calabi–Yau structure* on the dg functor $G : \mathcal{B} \to \mathcal{A}$ is a class $[\omega]$ in $\mathrm{Hom}(HC_{n-1}(G), k)$ such that the associated morphism $\Sigma^{n-1}u_G \to \delta_G$ is invertible, i.e. its associated morphism of triangles (2.1) is invertible.

2.3 Relative left Calabi–Yau structures

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor. We assume that $\mathcal{B}$ is smooth. This ensures that the canonical morphism

$$\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{B}^\vee \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A} \to (\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{B} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A})^\vee$$

is invertible in $\mathcal{D}(\mathcal{A}^e)$. The composition of $\mathcal{A}$ induces the morphism

$$\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A} \to \mathcal{A}$$

of $\mathcal{D}(\mathcal{A}^e)$. It canonically lifts to an object μ_G of $\mathcal{D}(I^{op} \otimes \mathcal{A}^e)$. Similarly, its homotopy fiber

$$\mathrm{fib}(\mu_G) \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A}$$

lifts to an object ν_G of $\mathcal{D}(I^{op} \otimes \mathcal{A}^e)$. Notice that each morphism $\Sigma^{n-1}\mu_G^\vee \to \nu_G$ gives rise to a morphism of triangles in $\mathcal{D}(\mathcal{A}^e)$

$$\begin{tikzcd} \Sigma^{n-1}\mathcal{A}^\vee \arrow[r, "\Sigma^{n-1}\mu_G^\vee"] \arrow[d] & \Sigma^{n-1}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A})^\vee \arrow[r] \arrow[d] & \Sigma^{n-1}\mathrm{cof}(\mu_G^\vee) \arrow[r] \arrow[d] & \Sigma^n \mathcal{A}^\vee \arrow[d] \\ \mathrm{fib}(\mu_G) \arrow[r, "\nu_G"] & \mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A} \arrow[r, "\mu_G"] & \mathcal{A} \arrow[r] & \Sigma\mathrm{fib}(\mu_G). \end{tikzcd} \tag{2.3}$$

We are therefore interested in morphisms $\Sigma^{n-1}\mu_G^\vee \to \nu_G$ in $\mathcal{D}(I^{op} \otimes \mathcal{A}^e)$.

Lemma 2.3.1. *We have a canonical morphism*

$$\mathrm{fib}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \to \mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^\vee, \nu_G).$$

It is invertible if $\mathcal{A}$ is smooth. Moreover this canonical morphism is compatible with the composition of dg functors, i.e. if $Q : \mathcal{A} \to \mathcal{C}$ is another dg functor between smooth dg categories, then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \arrow[r] \arrow[d] & \mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^\vee, \nu_G) \arrow[d] \\
\mathrm{fib}(\mathcal{C} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{C} \to \mathcal{C} \overset{L}{\otimes}_{\mathcal{C}^e} \mathcal{C}) \arrow[r] & \mathbf{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{C}^e}(\mu_{Q\circ G}^\vee, \nu_{Q\circ G}).
\end{tikzcd}$$

Proof. By Lemma 2.1.1, we have

$$\boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^\vee, \nu_G) \xrightarrow{\sim} \mathrm{fib}(\boldsymbol{R}\mathrm{Hom}_{\mathcal{A}^e}((\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A})^\vee, \mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A}) \to \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^\vee, \mathcal{A})).$$

We have a canonical morphism

$$\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A} \to \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}^\vee, \mathcal{A}),$$

which is invertible if $\mathcal{A}$ is smooth. Moreover, we have the isomorphisms

$$\begin{aligned}
\boldsymbol{R}\mathrm{Hom}_{\mathcal{A}^e}((\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A})^\vee, \mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A}) &\simeq \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}^e}(\boldsymbol{L}G^*(\mathcal{B}^\vee), \mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A}) \\
&\simeq \boldsymbol{R}\mathrm{Hom}_{\mathcal{B}^e}(\mathcal{B}^\vee, G_*(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A})) \\
&\simeq \mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} (\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A}) \\
&\simeq \mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A},
\end{aligned}$$

where we use the smoothness of $\mathcal{B}$ for the first and the 3rd isomorphism. Thus, we have a canonical morphism

$$\mathrm{fib}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \to \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^\wedge, \nu_G),$$

which is invertible if $\mathcal{A}$ is smooth.

By Lemma 2.1.2, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \arrow[r] \arrow[d] & \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^\vee, \nu_G) \arrow[d] \\
\mathrm{fib}(\boldsymbol{L}Q^*(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}) \to \boldsymbol{L}Q^*(\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A})) \arrow[r] & \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{C}^e}(\boldsymbol{L}Q^*(\mu_G^\vee), \boldsymbol{L}Q^*(\nu_G)),
\end{tikzcd}$$

where $\boldsymbol{L}Q^*(\mu_G^\vee)$ is given by $\mathcal{A}^\vee \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{C}^e \to \mathcal{B}^\vee \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{C}^e$ and $\boldsymbol{L}Q^*(\nu_G)$ is given by $\boldsymbol{L}G^*(\mathrm{fib}(\mu_G)) \to \mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{C}^e$.

It is easy to see that we have natural morphisms $\mu_{Q\circ G}^{\vee} \to \boldsymbol{L}Q^*(\mu_G^{\vee})$ and $\boldsymbol{L}Q^*(\nu_G) \to \nu_{Q\circ G}$ in $\mathcal{D}(I^{op}\otimes\mathcal{C})$. Then by Lemma 2.1.1, we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{fib}(\mathcal{A}\overset{L}{\otimes}_{\mathcal{B}^e}\mathcal{A}\to\mathcal{A}\overset{L}{\otimes}_{\mathcal{A}^e}\mathcal{A}) \arrow[r] \arrow[d] & \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^{\vee},\nu_G) \arrow[d] \\
\mathrm{fib}(\boldsymbol{L}Q^*(\mathcal{A}\overset{L}{\otimes}_{\mathcal{B}^e}\mathcal{A})\to\boldsymbol{L}Q^*(\mathcal{A}\overset{L}{\otimes}_{\mathcal{A}^e}\mathcal{A})) \arrow[r] \arrow[d] & \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{C}^e}(\boldsymbol{L}Q^*(\mu_G^{\vee}),\boldsymbol{L}Q^*(\nu_G)) \arrow[d] \\
\mathrm{fib}(\mathcal{C}\overset{L}{\otimes}_{\mathcal{B}^e}\mathcal{C}\to\mathcal{C}\overset{L}{\otimes}_{\mathcal{A}^e}\mathcal{A}) \arrow[r] \arrow[d] & \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{C}^e}(\mu_{Q\circ G}^{\vee},\boldsymbol{L}Q^*(\nu_G)) \arrow[d] \\
\mathrm{fib}(\mathcal{C}\overset{L}{\otimes}_{\mathcal{B}^e}\mathcal{C}\to\mathcal{C}\overset{L}{\otimes}_{\mathcal{C}^e}\mathcal{C}) \arrow[r] & \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{C}^e}(\mu_{Q\circ G}^{\vee},\nu_{Q\circ G}).
\end{tikzcd}$$

$\surd$

We therefore obtain the following chain of morphisms

$$\begin{tikzcd}
HN(G) \arrow[d] \\
HH(G)\cong\Sigma\mathrm{fib}(\mathcal{B}\overset{L}{\otimes}_{\mathcal{B}^e}\mathcal{B}\to\mathcal{A}\overset{L}{\otimes}_{\mathcal{A}^e}\mathcal{A}) \arrow[d] \\
\Sigma\mathrm{fib}(\mathcal{A}\overset{L}{\otimes}_{\mathcal{B}^e}\mathcal{A}\to\mathcal{A}\overset{L}{\otimes}_{\mathcal{A}^e}\mathcal{A}) \arrow[r] & \Sigma\boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^{\vee},\nu_G).
\end{tikzcd} \tag{2.4}$$

Definition 2.3.2. *[16, Definition 4.11][87, Definition 1.13]* A *left n-Calabi–Yau structure* on the dg functor $G:\mathcal{B}\to\mathcal{A}$ is a relative negative cyclic class $[\xi]$ in $HN_n(G)$ such that

a) the associated morphism $\Sigma^{n-1}\mu_G^{\vee}\to\nu_G$ is invertible and

b) the morphism $\Sigma^{n-1}\mathcal{B}^{\vee}\to\mathcal{B}$ corresponding to the image of $[\xi]$ in $HH_{n-1}(\mathcal{B})$ is invertible in $\mathcal{D}(\mathcal{B}^e)$.

Notice that the morphism $\mu_G^{\vee}\to\Sigma^{n-1}\nu_G$ is invertible if and only if its associated morphism of triangles (2.3) is invertible. We point out that condition b) is not imposed by Brav–Dyckerhoff [16] but is imposed by Yeung [87].

Remark 2.3.3. If we take the dg category $\mathcal{B}$ to be the empty dg category $\emptyset$, which is the initial object in the category of small dg categories dgcat_k, then the above definition coincides with the definition of an *absolute left n-Calabi–Yau structure* on $\mathcal{A}$.

Proposition 2.3.4. *[16, Corollary 7.1] Let $f:\mathcal{B}\to\mathcal{A}$ be a dg functor between homologically smooth dg categories which carries a left n-Calabi–Yau structure. Then there is a canonical left n-Calabi–Yau structure on the cofiber $\mathcal{A}/\mathcal{B}$.*

Proposition 2.3.5. *Let $\mathcal{B}, \mathcal{A}, \mathcal{A}'$ be smooth dg categories. Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor and $Q : \mathcal{A} \to \mathcal{A}'$ be a quasi-equivalence. The isomorphism*

$$HN_n(G) \to HN_n(Q \circ G)$$

induced by Q yields a bijection between the left n-Calabi–Yau structures on G and on $Q \circ G$.

Proof. By Theorem 1.2.23, the functor Q induces the following quasi-isomorphism of triangles in $\mathcal{D}(\mathcal{M}ix)$

$$\begin{tikzcd}
M(\mathcal{B}) \arrow[r] \arrow[d, equal] & M(\mathcal{A}) \arrow[r] \arrow[d] & M(G) \arrow[r] \arrow[d] & \Sigma M(\mathcal{B}) \arrow[d, equal] \\
M(\mathcal{B}) \arrow[r] & M(\mathcal{C}) \arrow[r] & M(Q \circ G) \arrow[r] & \Sigma M(\mathcal{B}).
\end{tikzcd}$$

Combining with Lemma 2.3.1, the above diagram yields the following commutative diagram in $\mathcal{D}(k)$

$$\begin{tikzcd}
HN(G) \arrow[r, "\sim"] \arrow[d] & HN(Q \circ G) \arrow[d] \\
HH(G) \arrow[r, "\sim"] \arrow[d] & HH(Q \circ G) \arrow[d] \\
\Sigma \mathrm{fib}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \arrow[r, "\sim"] \arrow[d, "\simeq"] & \Sigma \mathrm{fib}(\mathcal{A}' \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}' \to \mathcal{A}' \overset{L}{\otimes}_{\mathcal{A}'^e} \mathcal{A}') \arrow[d, "\simeq"'] \\
\Sigma \boldsymbol{R}\mathrm{Hom}_{I^{op} \otimes \mathcal{A}^e}(\mu_G^\vee, \nu_G) \arrow[r, "\Theta"] & \Sigma \boldsymbol{R}\mathrm{Hom}_{I^{op} \otimes \mathcal{A}'^e}(\mu_{Q \circ G}^\vee, \nu_{Q \circ G}).
\end{tikzcd}$$

The map Θ admits the following description. The quasi-equivalence functor Q induces a quasi-equivalence

$$\mathbf{1} \otimes Q^e : I^{op} \otimes \mathcal{A}^e \to I^{op} \otimes \mathcal{A}'^e,$$

which we still denote by Q. Then the extension along Q yields an equivalence

$$\boldsymbol{L}Q^* : \mathcal{D}(I^{op} \otimes \mathcal{A}^e) \xrightarrow{\sim} \mathcal{D}(I^{op} \otimes \mathcal{A}^e).$$

The functor $\boldsymbol{L}Q^*$ maps $\mu_G^\vee$ to $\mu_{Q \circ G}^\vee$ and ν_G to $\nu_{Q \circ G}$. Then the map Θ is the map induced by $\boldsymbol{L}Q^*$ on mapping complexes. In particular, Θ preserves equivalences. Thus each left n-Calabi–Yau structure on G induces a left n-Calabi–Yau structure on $Q \circ G$. Similarly, we can use the restriction functor $Q_* : \mathcal{D}(I^{op} \otimes \mathcal{A}'^e) \xrightarrow{\sim} \mathcal{D}(I^{op} \otimes \mathcal{A}'^e)$ to show that each left n-Calabi–Yau structure on $Q \circ G$ induces a left n-Calabi–Yau structure on G. $\checkmark$

Corollary 2.3.6. *Let $\mathcal{B}, \mathcal{A}$ be two homologically smooth dg categories and moreover $\mathcal{B}$ is cofibrant with respect to the Dwyer-Kan model structure (see Theorem 1.2.2). Let $G, G' : \mathcal{B} \to \mathcal{A}$ be two homotopic dg functors. The canonical isomorphism*

$$HN_n(G) \xrightarrow{\sim} HN_n(G')$$

induces a bijection between the relative left n-Calabi–Yau structures on G and on G'.

Proof. Since G and G' are homotopic, there exists a dg functor $H : \mathcal{B} \to P(\mathcal{A})$ that makes the following diagram commutative (see Definition 1.2.8)

$$\begin{tikzcd} & \mathcal{A} \\ \mathcal{B} \arrow[ur, "G"] \arrow[r, "H"] \arrow[dr, "G'"'] & P(\mathcal{A}) \arrow[u, "P_0"'] \arrow[d, "P_1"] \\ & \mathcal{A}. \end{tikzcd}$$

We know that P_0 and P_1 are quasi-equivalences. They induce isomorphisms $HN_n(G) \xleftarrow{\sim} HN_n(H) \xrightarrow{\sim} HN_n(G')$. Now the claim follows from the above Proposition 2.3.5. $\surd$

2.4 From left to right

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor between smooth dg categories. Suppose that G carries a left n-Calabi–Yau structure. We define $\mathrm{per}_{dg}(\mathcal{A})$ to be the dg subcategory of $\mathcal{C}_{dg}(\mathcal{A})$ whose objects are the perfect cofibrant dg $\mathcal{A}$ modules and $\mathrm{pvd}_{dg}(\mathcal{A})$ to be the dg subcategory of $\mathcal{C}_{dg}(\mathcal{A})$ whose objects are the perfectly valued cofibrant dg $\mathcal{A}$ modules. Similarly, we define $\mathrm{per}_{dg}(\mathcal{B})$ and $\mathrm{pvd}_{dg}(\mathcal{B})$. The restriction along $G : \mathcal{B} \to \mathcal{A}$ induces a dg functor $R : \mathcal{E} = \mathrm{pvd}_{dg}(\mathcal{A}) \to \mathcal{F} = \mathrm{pvd}_{dg}(\mathcal{B})$.

Theorem 2.4.1. *[16, pp. 389] The functor $R : \mathcal{E} \to \mathcal{F}$ inherits a canonical right n-Calabi–Yau structure, i.e. we have a class $[\omega]$ in $\mathrm{Hom}_k(HC_{n-1}(R), k)$ which yields an isomorphism of triangles in $\mathcal{D}(\mathcal{E}^e)$*

$$\begin{tikzcd} \Sigma^{n-1}\mathcal{E} \arrow[r, "\Sigma^{n-1}u_R"] \arrow[d] & \Sigma^{n-1}R_*\mathcal{F} \arrow[r] \arrow[d] & \Sigma^{n-1}\mathrm{cof}(u_R) \arrow[r] \arrow[d] & \Sigma^n\mathcal{E} \arrow[d] \\ \mathrm{fib}(u_R^*) \arrow[r, "\delta_R"] & R_*(D\mathcal{F}^{op}) \arrow[r, "u_R^*"] & D\mathcal{E}^{op} \arrow[r] & \Sigma\mathrm{fib}(u_R^*), \end{tikzcd} \tag{2.5}$$

where R_* is the restriction along $R^e : \mathcal{E}^e \to \mathcal{F}^e$.

Proof. By the definition of $\mathrm{pvd}\mathcal{B}$, we have a dg functor

$$(\mathrm{per}_{dg}\mathcal{B})^{op} \otimes \mathcal{F} \to \mathrm{per}_{dg}(k), \quad (P, M) \longmapsto \mathcal{H}om_{\mathcal{B}}(P, M).$$

It yields a morphism in $\mathcal{DM}ix$

$$M((\mathrm{per}_{dg}\mathcal{B})^{op}) \otimes M(\mathcal{F}) \to M(\mathrm{per}_{dg}(k)) \xleftarrow{\sim} M(k) \simeq k.$$

By the adjunction between $? \otimes_\Lambda M(\mathcal{F})$ and $\mathrm{Hom}_k(M(\mathcal{F}), ?)$, we get a morphism in $\mathcal{DM}ix$

$$M(\mathcal{B}) \xrightarrow{\sim} M(\mathrm{per}_{dg}\mathcal{B}) \to \mathrm{Hom}(M(\mathcal{F}), k).$$

Similarly, we get another morphism in $\mathcal{DM}ix$

$$M(\mathcal{A}) \xrightarrow{\sim} M(\mathrm{per}_{dg}\mathcal{A}) \to \mathrm{Hom}(M(\mathcal{E}), k).$$

And those two maps fit into the following commutative diagram in $\mathcal{DM}ix$

$$\begin{tikzcd}
M(\mathcal{B}) \arrow[r, "\sim"] \arrow[d, "\gamma_G"] & M(\mathrm{per}_{dg}\mathcal{B}) \arrow[r] \arrow[d, "\gamma_{G^*}"'] & \mathrm{Hom}_k(M(\mathcal{F}), k) \arrow[d, "\gamma_R^*"] \\
M(\mathcal{A}) \arrow[r, "\sim"] & M(\mathrm{per}_{dg}\mathcal{A}) \arrow[r] & \mathrm{Hom}_k(M(\mathcal{E}), k).
\end{tikzcd}$$

It yields the following commutative diagram in $\mathcal{DM}ix$

$$\begin{tikzcd}
HN(\mathcal{B}) \arrow[r] \arrow[d] & \boldsymbol{R}\mathrm{Hom}_\Lambda(k, \mathrm{Hom}_k(M(\mathcal{E}), k)) \simeq \mathrm{Hom}_k(HC(\mathcal{F}), k) \arrow[d] \\
HN(\mathcal{A}) \arrow[r] & \boldsymbol{R}\mathrm{Hom}_\Lambda(k, \mathrm{Hom}_k(M(\mathcal{F}), k)) \simeq \mathrm{Hom}_k(HC(\mathcal{E}), k)
\end{tikzcd}$$

Therefore we get the following commutative diagram in $\mathcal{D}(k)$

$$\begin{tikzcd}
HN(G) \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}_k(\Sigma^{-1}HC(R), k) \arrow[d] \\
HH(G) \arrow[r] \arrow[d] & \mathrm{Hom}_k(\Sigma^{-1}HH(R), k) \arrow[d] \\
\Sigma\mathrm{fib}(\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A}) \arrow[r] \arrow[d] & \mathrm{cof}(\mathrm{Hom}_k(\mathcal{F} \overset{L}{\otimes}_{\mathcal{E}^e} \mathcal{F}, k) \to \mathrm{Hom}_k(\mathcal{E} \overset{L}{\otimes}_{\mathcal{E}^e} \mathcal{E}), k) \arrow[d] \\
\Sigma\boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{A}^e}(\mu_G^\vee, \nu_G) \arrow[r] & \boldsymbol{R}\mathrm{Hom}_{I^{op}\otimes\mathcal{E}^e}(\Sigma^{-1}u_R, \delta_R).
\end{tikzcd}$$

Consider the functor Ψ given by the composite

$$\mathcal{C}(\mathcal{A}^e) \to \mathcal{C}(\mathrm{per}_{dg}(\mathcal{A})^e) \to \mathcal{C}_{dg}(\mathcal{E}^{op} \otimes \mathrm{per}_{dg}\mathcal{A}) \to \mathcal{C}(\mathcal{E} \otimes (\mathrm{per}_{dg}\mathcal{A})^{op}) \to \mathcal{C}(\mathcal{E}^e)^{op},$$

where the second and last functors are given by restriction along $\mathcal{E} \subseteq \mathrm{per}_{dg}\mathcal{A}$, the first functor is given by the extension along Yoneda embedding and the third functor is given by

$$M \mapsto M^*,\ (a,p) \mapsto \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(M(?,p), \mathcal{A}(?,a)).$$

Then we obtain an induced functor

$$\boldsymbol{L}\Psi : \mathcal{D}(I^{op} \otimes \mathcal{A}^e) \longrightarrow \mathcal{D}(I^{op} \otimes \mathcal{E}^e)^{op}.$$

Explicitly, this functor associates to a graded split monomorphism of $\mathcal{A}$-bimodules $f : M_1 \rightarrowtail M_2$ with cofibrant M_1 and M_2, the morphism of $\mathcal{E}$-bimodules given by

$$(p,p') \longmapsto \begin{array}{c} \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(M_2 \otimes_{\mathcal{E}^e} \mathrm{per}_{dg}(\mathcal{A})^e(?,p'), \mathcal{A}(?,p)) \\ \big\downarrow \\ \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(M_1 \otimes_{\mathcal{E}^e} \mathrm{per}_{dg}(\mathcal{A})^e(?,p'), \mathcal{A}(?,p)) \end{array}$$

Therefore, the functor $\boldsymbol{L}\Psi$ maps $\mu_G : \mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A} \to \mathcal{A}$ to $\mathcal{E} \to R_*(\mathcal{F})$, and $\mu_G^\vee : \mathcal{A}^\vee \to (\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A})^\vee$ to $R_*(D\mathcal{F}^{op}) \to D\mathcal{E}^{op}$. Suppose that the left n-Calabi–Yau structure on $G : \mathcal{B} \to \mathcal{A}$ is induced by $[\xi] \in HN_n(G)$. Then we have an isomorphism of triangles 2.3 in $\mathcal{D}(\mathcal{A}^e)$. After applying the functor $\boldsymbol{L}\Psi$ to this diagram 2.3, we get an isomorphism of triangles 2.5 in $\mathcal{D}(\mathcal{E}^e)$ and this isomorphism is induced by the class $\alpha([\xi])$.

$\surd$

Proposition 2.4.2. *Let $G : \mathcal{B} \to \mathcal{A}$ satisfy the above assumption. For $L, M \in \mathcal{D}(\mathcal{A})$. We put*

$$\mathcal{C}(L, M) := \mathrm{Cone}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(L, M)) \to \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(G_*(L), G_*(M)).$$

Suppose that $L \in \mathrm{pvd}(\mathcal{A})$ and $M \in \mathcal{D}(\mathcal{A})$. Then there is a bifunctorial isomorphism of triangles

$$\begin{tikzcd}
D\mathcal{C}(L,M) \arrow[r] & D\mathbf{R}\mathrm{Hom}_{\mathcal{B}}(G_*(L), G_*(M)) \arrow[r, "DG_*"] & D\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(L, M) \arrow[r] & \Sigma D\mathcal{C}(L,M) \\
\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, \Sigma^{n-1}L) \arrow[r, "G_*"] \arrow[u, "\simeq"'] & \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(G_*(M), \Sigma^{n-1}G_*(L)) \arrow[r] \arrow[u, "\simeq"'] & \mathcal{C}(M, \Sigma^{n-1}L) \arrow[r] \arrow[u, "\simeq"'] & \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, \Sigma^n L). \arrow[u, "\simeq"']
\end{tikzcd} \tag{2.6}$$

If $G_*(L) = 0$ or $G_*(M) = 0$, then $D\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(L, M) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, L[n])$. In particular, the full subcategory $\mathrm{pvd}_{\mathcal{B}}(\mathcal{A})$ defined as the kernel of the restriction functor $G_* : \mathrm{pvd}\mathcal{A} \to \mathrm{pvd}\mathcal{B}$ is n-Calabi–Yau as a triangulated category.

Proof. Since $G : \mathcal{B} \to \mathcal{A}$ has a relative n-Calabi–Yau structure, we have an isomorphism in $\mathcal{D}(\mathcal{A}^e)$

$$\Sigma^n \mathcal{A}^\vee \simeq \mathrm{Cone}(\mathbf{L}G^*(\mathcal{B}) \to \mathcal{A}),$$

and an isomorphism in $\mathcal{D}(\mathcal{B}^e)$

$$\Sigma^{n-1}\mathcal{B}^\vee \simeq \mathcal{B}.$$

Let P_L and P_M be cofibrant resolutions of L and M respectively. By [61, Lemma 4.1], we have

$$D\boldsymbol{R}\mathrm{Hom}_{\mathcal{B}}(G_*(L), G_*(M)) \simeq \boldsymbol{R}\mathrm{Hom}_{\mathcal{B}}(G_*(M), \Sigma^{n-1}G_*(L))$$

and

$$\begin{aligned}
D\boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(L, M) &\simeq D\mathcal{H}om_{\mathcal{A}}(P_L, P_M) \\
&\simeq \mathcal{H}om_{\mathcal{A}}(P_M \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{A}^\vee, P_L) \\
&\simeq \mathrm{fib}(\mathcal{H}om_{\mathcal{A}}(P_M, \Sigma^n P_L) \to \mathrm{Hom}_{\mathcal{D}(A)}(P_M \otimes^{\mathbf{L}}_{\mathcal{A}} \mathbf{L}G^*(\mathcal{B}), \Sigma^n P_L)) \\
&\simeq \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, \Sigma^n L) \to \mathcal{H}om_{\mathcal{A}}(P_M \otimes^{\mathbf{L}}_{\mathcal{A}} (\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}), \Sigma^n P_L)) \\
&\simeq \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, \Sigma^n L) \to \mathcal{H}om_{\mathcal{A}}(P_M \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}, \Sigma^n P_L)) \\
&\simeq \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, \Sigma^n L) \to \mathcal{H}om_{\mathcal{B}}(G_*(P_M), \Sigma^n G_*(P_L))) \\
&\simeq \mathrm{fib}(\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, \Sigma^n L) \to \mathbf{R}\mathrm{Hom}_{\mathcal{B}}(G_*(M), \Sigma^n G_*(L))) \\
&\simeq \mathcal{C}(M, \Sigma^{n-1}L).
\end{aligned}$$

Thus, we get the bifunctorial isomorphism of triangles (2.6). If $G_*(L) = 0$ or $G_*(M) = 0$, then we have the following functorial duality

$$D\boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(L, M) \simeq \boldsymbol{R}\mathrm{Hom}_{\mathcal{A}}(M, \Sigma^n L) .$$

In particular, the kernel $\mathrm{pvd}_{\mathcal{B}}(\mathcal{A})$ of $G_* : \mathrm{pvd}(\mathcal{A}) \to \mathrm{pvd}(\mathcal{B})$ is n-Calabi–Yau as a triangulated category.

$\surd$

Let $\mathcal{B} \xrightarrow{G} \mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{B}$ be a homotopy cofiber sequence of small dg categories. By construction, the dg category $\mathcal{A}/\mathcal{B}$ is the Drinfeld dg quotient of $\mathcal{A}$ by its full dg subcategory $\mathrm{Im}(G)$, where $\mathrm{Im}(G)$ is the full dg subcategory of $\mathcal{A}$ whose objects are the y in $\mathcal{A}$ such that there exists an object x in $\mathcal{B}$ and an isomorphism $F(x) \cong y$ in $\mathrm{H}^0(\mathcal{A})$. We denote by i the dg inclusion $\mathrm{Im}(G) \hookrightarrow \mathcal{A}$.

Corollary 2.4.3. *For any dg module N and any dg module M in* $\mathrm{pvd}(\mathcal{A})$ *whose restriction to* $\mathrm{Im}G$ *is acyclic, there is a canonical isomorphism*

$$D\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(M, N) \simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(N, \Sigma^n M) .$$

Proof. Since the restriction of M to $\mathrm{Im}G$ is acyclic, we have $G_*(M) = 0$. Then the claim follows from the above Proposition 2.4.2. $\surd$

2.5 Relative Calabi–Yau completions

Given a dg category $\mathcal{B}$, the forgetful functor $(\mathrm{dgcat}_k)_{\mathcal{B}/} \to \mathcal{C}(\mathcal{B}^e)$, sending a dg functor $G : \mathcal{B} \to \mathcal{A}$ to the $\mathcal{B}$-bimodule given by $(a, a') \mapsto \mathcal{A}(G(a'), G(a))$, has a left adjoint $T_{\mathcal{B}}$, that can be described as follows:

Given a $\mathcal{B}$-bimodule M, the *tensor category* $T_{\mathcal{B}}(M)$ is defined as follows:

$$T_{\mathcal{B}}(M) = \mathcal{B} \oplus M \oplus (M \otimes_{\mathcal{B}} M) \oplus (M \otimes_{\mathcal{B}} M \otimes_{\mathcal{B}} M) \oplus \cdots$$

Thus, the dg category $T_{\mathcal{B}}(M)$ has the same objects as $\mathcal{B}$ and morphism complexes

$$\begin{aligned} T_{\mathcal{B}}(M)(x, y) = \mathcal{B}(x, y) \oplus M(x, y) \oplus \{\oplus_{z \in \mathcal{B}} M(z, y) \otimes_k M(x, z)\} \oplus \\ \{\oplus_{z_1, z_2 \in \mathcal{B}} M(z_2, y) \otimes_k M(z_1, z_2) \otimes_k M(x, z_1)\} \oplus \cdots \end{aligned}$$

The dg structure on $T_{\mathcal{B}}(M)$ is given by the differentials of $\mathcal{B}$ and M and the multiplication is given by the concatenation product. This adjunction is Quillen, and thus induces an adjunction between their homotopy categories. We will denote by $\mathbf{L}T_{\mathcal{B}}$ the left derived functor of $T_{\mathcal{B}} : \mathcal{C}(\mathcal{B}^e) \to (\mathrm{dgcat}_k)_{\mathcal{B}/}$.

An $\mathcal{B}$-bilinear (super-)derivation D of degree 1 on $\mathbf{L}T_{\mathcal{B}}(M)$ is determined by its restriction to the generating bimodule M. Then it is easy to see that each morphism $c : M \to \Sigma\mathcal{B}$ in $\mathcal{D}(\mathcal{B}^e)$ gives rise to a 'deformation'

$$(\mathbf{L}T_{\mathcal{B}}(M), d_c)$$

of $\mathbf{L}T_{\mathcal{B}}(M)$, obtained by adding the $\mathcal{A}$-bilinear (super-)derivation D_c determined by c to the differential of $\mathbf{L}T_{\mathcal{B}}(M)$.

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor between smooth dg categories and let $[\xi]$ be an element in $HH_{n-2}(G)$. Our objective is to define the *deformed relative n-Calabi–Yau completion* of $G : \mathcal{B} \to \mathcal{A}$ with respect to the Hochschild homology class $[\xi] \in HH_{n-2}(G)$.

The dg functor $G : \mathcal{B} \to \mathcal{A}$ induces a morphism of dg $\mathcal{A}$-bimodules $\mathcal{B} \otimes_{\mathcal{B}^e} \mathcal{A}^e \to \mathcal{A}$. Let Ξ be the cofiber of its bimodule dual, i.e. $\Xi = \mathrm{Cone}(\mathcal{A}^\vee \to (\mathcal{B} \otimes_{\mathcal{B}^e} \mathcal{A}^e)^\vee)$. Clearly, the the dualizing bimodule $\Theta_G = (\mathrm{Cone}(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A}^e \to \mathcal{A}))^\vee$ of G is quasi-isomorphic to $\Sigma^{-1}\Xi$.

By the definition of Hochschild homology of G, we have the following long exact sequence

$$\cdots \to HH_{n-2}(\mathcal{B}) \to HH_{n-2}(\mathcal{A}) \to HH_{n-2}(G) \to HH_{n-3}(\mathcal{B}) \to \cdots .$$

Thus, the Hochschild homology class $[\xi] \in HH_{n-2}(G)$ induces an element $[\xi_{\mathcal{B}}]$ in $HH_{n-3}(\mathcal{B})$.

Notice that since $\mathcal{B}, \mathcal{A}$ are smooth, we have the following isomorphisms

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B}^e)}(\Sigma^{n-2}\mathcal{B}^\vee, \Sigma\mathcal{B}) \simeq H^{3-n}(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}) = HH_{n-3}(\mathcal{B}),$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(\mathcal{A}^e)}(\Sigma^{n-2}\Xi, \Sigma\mathcal{A}) &\simeq \mathrm{Hom}_{\mathcal{D}(\mathcal{A}^e)}(\mathrm{Cone}(\mathcal{A}^\vee \to (\mathcal{A} \overset{L}{\otimes}_{\mathcal{B}} \mathcal{A})^\vee), \Sigma^{3-n}\mathcal{A}) \\ &\simeq H^{2-n}(\mathrm{Cone}(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A})).\end{aligned}$$

Thus, via the canonical morphism

$$HH_{n-2}(G) = H^{2-n}(\mathrm{Cone}(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A})) \to H^{2-n}(\mathrm{Cone}(\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{A} \to \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{A})),$$

the homology class $[\xi]$ induces a morphisms in $\mathcal{D}(\mathcal{A}^e)$

$$\xi : \Sigma^{n-2}\Xi \to \Sigma\mathcal{A}$$

and the homology class $[\xi_{\mathcal{B}}]$ induces a morphism in $\mathcal{D}(\mathcal{B}^e)$

$$\xi_{\mathcal{B}} : \Sigma^{n-2}\mathcal{B}^\vee \to \Sigma\mathcal{B}.$$

Moreover, we have the following commutative diagram in $\mathcal{D}(\mathcal{A}^e)$

$$\begin{tikzcd} \boldsymbol{L}G^*(\Sigma^{n-2}\mathcal{B}^\vee) \arrow[r] \arrow[d, "\xi_{\mathcal{B}}"'] & \Sigma^{n-2}\Xi \arrow[d, "\xi"] \\ \boldsymbol{L}G^*(\Sigma\mathcal{B}) \arrow[r] & \Sigma\mathcal{A}. \end{tikzcd}$$

Therefore the morphism $\xi_{\mathcal{B}}$ gives rise to a 'deformation'

$$\mathbf{\Pi}_{n-1}(\mathcal{B}, \xi_{\mathcal{B}})$$

of $\mathbf{\Pi}_{n-1}(\mathcal{B}) = \mathbf{L}T_{\mathcal{B}}(\Sigma^{n-2}\mathcal{B}^\vee)$, obtained by adding $\xi_{\mathcal{B}}$ to the differential of $\mathbf{\Pi}_{n-1}(\mathcal{B})$; the morphism ξ gives rise to a 'deformation'

$$\mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}, \xi)$$

of $\mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}) = \mathbf{L}T_{\mathcal{A}}(\Sigma^{n-2}\Xi)$, obtained by adding ξ to the differential of $\mathbf{L}T_{\mathcal{A}}(\Sigma^{n-2}\Xi)$; and the commutative diagram above gives rise to a dg functor

$$\widetilde{G} : \mathbf{\Pi}_{n-1}(\mathcal{B}, \xi_{\mathcal{B}}) \to \mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}, \xi). \tag{2.7}$$

A standard argument shows that up to quasi-isomorphism, the dg functor $\widetilde{G}$ and the deformations $\mathbf{\Pi}_{n-1}(\mathcal{B}, \xi_{\mathcal{B}})$, $\mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}, \xi)$ only depend on the class $[\xi]$.

Definition 2.5.1. [87, Definition 3.22] Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor between smooth dg categories. The dg functor $\widetilde{G}$ (2.7) defined above is called the *deformed relative n-Calabi–Yau completion* of $G : \mathcal{B} \to \mathcal{A}$ with respect to the Hochschild homology class $[\xi] \in HH_{n-2}(G)$.

Remark 2.5.2. If we take the class $[\xi]$ be 0, then the above definition coincides with the definition of the *relative n-Calabi–Yau completion* of $G : \mathcal{B} \to \mathcal{A}$. If we take $\mathcal{B}$ to be the empty category, then deformed relative n-Calabi–Yau completion is the deformed n-Calabi–Yau completion of [62].

Theorem 2.5.3. *[87, Theorem 3.23][15, Proposition 5.28.] Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor between smooth dg categories and let $[\xi]$ be an element in $HH_{n-2}(G)$. If $[\xi]$ has a negative cyclic lift, then each choice of such a lift gives rise to a canonical left n-Calabi–Yau structure on the dg functor*

$$\widetilde{G} : \mathbf{\Pi}_{n-1}(\mathcal{B}, \xi_{\mathcal{B}}) \longrightarrow \mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}, \xi) .$$

2.6 Reduced relative Calabi–Yau completions

Recall that a dg category $\mathcal{A}$ over k is said to be *semi-free* if there is a graded quiver $Q = (Q_0, Q_1)$ such that the underlying graded k-category of $\mathcal{A}$ is freely generated by the arrows of Q over the vertex set Q_0. We write this as $\mathcal{A} = T_{kQ_0}(kQ_1)$.

Definition 2.6.1. *[87, Definition 2.52]* A dg category $\mathcal{A}$ is said to be *cellular* if it is semi-free over some graded quiver $Q = (Q_0, Q_1)$ that admits a filtration

$$Q^{(1)} \subset Q^{(2)} \subset \cdots$$

such that every generating arrow $f \in Q^{(i)}$ has differential $d(f)$ contained in the graded category $T_{kQ_0}(kQ^{(i-1)})$.

We say that $\mathcal{A}$ is *finitely cellular* if the graded quiver (Q_0, Q_1) is finite (i.e. both Q_0 and Q_1 are finite).

We say that $\mathcal{A}$ is of *finite cellular type* if it is quasi-equivalent to a finitely cellular dg category.

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor between finitely cellular type dg categories. By [87, Remark 4.19], we can assume that $\mathcal{B}$ and $\mathcal{A}$ are finitely cellular and $G : \mathcal{B} \to \mathcal{A}$ is a semi-free extension, i.e. there is a finite graded quiver Q and a subquiver $F \subseteq Q$ such that the underlying graded k-category of $\mathcal{B}$ and $\mathcal{A}$ are isomorphic to $T_{kF_0}(kF_1)$ and

$T_{kQ_0}(kQ_1)$, respectively. We abbreviate $R_2 = kF_0$ and $R_1 = kQ_0$. Then we have a short exact sequence of $\mathcal{B}$-bimodules

$$0 \longrightarrow \Omega^1(\mathcal{B}) \xrightarrow{\alpha} \mathcal{B} \otimes_{R_2} \mathcal{B} \xrightarrow{m} \mathcal{B} \longrightarrow 0$$

where *the bimodule of differentials* $\Omega^1(\mathcal{B})$ is generated by $\{D(f)|f \in F_1\}$, the map α is given by $D(f) \mapsto f \otimes 1_x - 1_y \otimes f$ where $f : x \to y$ and the map m is the composition map in $\mathcal{B}$.

Similarly, we have an exact sequence of $\mathcal{A}$-bimodules for $\mathcal{A}$. We put $\mathcal{P}_B = \mathrm{Cone}(\Omega^1(\mathcal{B}) \xrightarrow{\alpha} \mathcal{B} \otimes_{R_2} \mathcal{B})$ and $\mathcal{P}_A = \mathrm{Cone}(\Omega^1(\mathcal{A}) \xrightarrow{\alpha} \mathcal{A} \otimes_{R_1} \mathcal{A})$. Then $P_{\mathcal{B}}$ and $P_{\mathcal{A}}$ are cofibrant replacements of the bimodules $\mathcal{B}$ and $\mathcal{A}$ respectively. The $\mathcal{B}$-bimodule $\mathcal{P}_{\mathcal{B}}^{\vee}$ is cellular of finite rank, with basis $\{f_{\mathcal{B}}^{\vee}|f \in F_1\}$ and $\{c_{x,\mathcal{B}}|x \in F_0\}$ where the arrow $f_{\mathcal{B}}^{\vee}$ has degree $|f_{\mathcal{B}}^{\vee}| = 1 - |f|$, and points in the opposite direction to f; the loop $c_{x,\mathcal{B}}$ has degree $|c_{x,\mathcal{B}}| = 0$, and is based at x. Similarly, the $\mathcal{A}$-bimodule $\mathcal{P}_{\mathcal{A}}^{\vee}$ is also cellular of finite rank, with basis $\{g_{\mathcal{A}}^{\vee}|g \in Q_1\}$ and $\{c_{y,\mathcal{A}}|y \in Q_0\}$ where the arrow $g_{\mathcal{A}}^{\vee}$ has degree $|g_{\mathcal{A}}^{\vee}| = 1 - |g|$, and points in the opposite direction to g; the loop $c_{y,\mathcal{A}}$ has degree $|c_{y,\mathcal{A}}| = 0$, and is based at y.

The natural map $\alpha_G : G^*(P_{\mathcal{B}}) \to P_{\mathcal{A}}$ in $\mathcal{C}(\mathcal{A}^e)$ induces the dual map $\alpha_G^{\vee} : P_{\mathcal{A}}^{\vee} \to G^*(P_{\mathcal{B}})^{\vee}$ in $\mathcal{C}(\mathcal{A}^e)$. This $\alpha_G^{\vee}$ is given as follows:

- $\alpha_G^{\vee}(c_{y,\mathcal{A}}) = c_{y,\mathcal{B}}$ if y belongs to F_0; otherwise, $\alpha_G^{\vee}(c_{y,\mathcal{A}}) = 0$,
- $\alpha_G^{\vee}(g_{\mathcal{A}}^{\vee}) = g_{\mathcal{B}}^{\vee}$ if g belongs to F_1; otherwise, $\alpha_G^{\vee}(g_{\mathcal{A}}) = 0$.

Clearly, the morphism $\alpha_G^{\vee}$ is a graded split surjection of $\mathcal{A}$-bimodules. Let $\mathcal{K}$ be the kernel of $\alpha_G^{\vee}$. Then $\mathcal{K}$ is cellular of finite rank, with basis $\{g_{\mathcal{A}}^{\vee}, c_{y,\mathcal{A}} \mid g \in N_1 = Q_1 \setminus F_1,\ y \in N_0 = Q_0 \setminus F_0\}$. We have a split exact sequence in the category of graded $\mathcal{A}$-bimodules, i.e. there exist two graded bimodule morphisms $s_G : G^*(P_{\mathcal{B}})^{\vee} \to P_{\mathcal{A}}^{\vee}$, $r_{\mathcal{K}} : P_{\mathcal{A}}^{\vee} \to \mathcal{K}$ such that $\alpha_G^{\vee} \circ s_G = \mathbf{1}_{G^*(P_{\mathcal{B}})^{\vee}}$, $r_{\mathcal{K}} \circ i_{\mathcal{K}} = \mathbf{1}_{\mathcal{K}}$, $s_G \circ \alpha_G^{\vee} + i_{\mathcal{K}} \circ r_{\mathcal{K}} = \mathbf{1}_{P_{\mathcal{A}}^{\vee}}$. We summarize the notations in the diagram

$$0 \longrightarrow \mathcal{K} \underset{r_{\mathcal{K}}}{\overset{i_{\mathcal{K}}}{\rightleftarrows}} P_{\mathcal{A}}^{\vee} \underset{s_G}{\overset{\alpha_G^{\vee}}{\rightleftarrows}} G^*(P_{\mathcal{B}})^{\vee} \longrightarrow 0. \tag{2.8}$$

We choose the graded morphisms $r_{\mathcal{K}}$ and s_G are given as follows:

- The graded morphism s_G maps $g_{\mathcal{B}}^{\vee}$ to $g_{\mathcal{A}}^{\vee}$ and maps $c_{x,\mathcal{B}}$ to $c_{x,\mathcal{A}}$.
- The graded morphism $r_{\mathcal{K}}$ maps $g_{\mathcal{A}}^{\vee}$ to $g_{\mathcal{A}}^{\vee}$ if g is in N_1; otherwise, we put $r_{\mathcal{K}}(g_{\mathcal{A}}^{\vee}) = 0$. Moreover, it maps $c_{y,\mathcal{A}}$ to $c_{y,\mathcal{A}}$ if y is in N_0; otherwise, we put $r_{\mathcal{K}}(c_{y,\mathcal{A}}) = 0$.

The above exact sequence yields a triangle in $D(\mathcal{A}^e)$

$$P_{\mathcal{A}}^{\vee} \xrightarrow{\alpha_G^{\vee}} G^*(P_{\mathcal{B}})^{\vee} \xrightarrow{u} \Sigma\mathcal{K} \longrightarrow, \tag{2.9}$$

where u is equal to $r_{\mathcal{K}} \circ d_{P_{\mathcal{A}}^{\vee}} \circ s_G$. Thus, we get the following isomorphism of triangles in $\mathcal{D}(\mathcal{A}^e)$

$$\begin{tikzcd} P_{\mathcal{A}}^{\vee} \arrow[r, "\alpha_G^{\vee}"] \arrow[d, "\mathbf{1}"] & G^*(P_{\mathcal{B}})^{\vee} \arrow[r, "u"] \arrow[d, "\mathbf{1}"] & \Sigma\mathcal{K} \arrow[r] \arrow[d, hook, "v"] & {} \\ P_{\mathcal{A}}^{\vee} \arrow[r, "\alpha_G^{\vee}"] & G^*(P_{\mathcal{B}})^{\vee} \arrow[r, "l"] & \Xi \arrow[r] & {}, \end{tikzcd} \tag{2.10}$$

where $\Xi = \mathrm{Cone}(P_{\mathcal{A}}^{\vee} \to G^*(P_{\mathcal{B}})^{\vee})$ and v is the quasi-isomorphism induced by the inclusion of $\mathcal{K}$ into $P_{\mathcal{A}}^{\vee}$.

Now we consider the derived tensor category $\mathbf{L}T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K})$. Since the $\mathcal{A}$-bimodule $\mathcal{K}$ is cofibrant, we have $\mathbf{L}T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K}) = T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K})$.

The following morphism induced by ξ

$$\xi_{\mathcal{K}} : \Sigma^{n-1}\mathcal{K} \xhookrightarrow{v} \Sigma^{n-2}\Xi \xrightarrow{\xi} \Sigma\mathcal{A}$$

determines an $\mathcal{A}$-bilinear derivation $d'_{\mathcal{K}}$ on $T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K})$. Then we get a 'deformation'

$$T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K}, \xi_{\mathcal{K}})$$

of $T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K})$, obtained by adding $d'_{\mathcal{K}}$ to the differential of $T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K})$.

Then the canonical inclusion of dg $\mathcal{A}$-bimodules $\Sigma^{n-1}\mathcal{K} \xhookrightarrow{v} \Sigma^{n-2}\Xi$ induces a fully faithful dg functor

$$\Psi : T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K}, \xi_{\mathcal{K}}) \hookrightarrow \mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}, \xi) .$$

Next we will construct a dg functor from $\mathbf{\Pi}_{n-1}(\mathcal{B}, \xi_{\mathcal{B}})$ to $T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K}, \xi_{\mathcal{K}})$.

Firstly, we have the following diagram

$$\begin{tikzcd} \Sigma^{n-2}G^*(P_{\mathcal{B}})^{\vee} \arrow[r, "u"] \arrow[d, "\mathbf{1}"'] & \Sigma^{n-1}\mathcal{K} \arrow[d, hook, "v"] \\ \Sigma^{n-2}G^*(P_{\mathcal{B}})^{\vee} \arrow[r, "l"] \arrow[d, "G^*(\xi_{\mathcal{B}})"'] & \Sigma^{n-2}\Xi \arrow[d, "\xi"] \\ \Sigma G^*(\mathcal{B}) \arrow[r, "j_G"] & \Sigma\mathcal{A}, \end{tikzcd}$$

where the upper square is commutative up to homotopy and the lower square is commutative. The homotopy is given by

$$H' : \Sigma^{n-2}G^*(P_{\mathcal{B}})^{\vee} \xrightarrow{\Sigma^{n-2}s_G^{\vee}} \Sigma^{n-2}P_{\mathcal{A}}^{\vee} \xhookrightarrow{\textit{inclusion}} \Sigma^{n-3}\Xi ,$$

where $s_G^{\vee}$ is the map defined in 2.8.

Combining those two diagrams, we get the following diagram commutative up to homotopy

$$\begin{tikzcd} \Sigma^{n-2}G^*(P_{\mathcal{B}})^{\vee} \arrow[r, "u"] \arrow[d, "G^*(\xi_{\mathcal{B}})"] & \Sigma^{n-1}\mathcal{K} \arrow[d, "\xi\circ v"] \\ \Sigma G^*(\mathcal{B}) \arrow[r, "j_G"] & \Sigma\mathcal{A} \end{tikzcd}$$

where the homotopy is given by

$$H : \Sigma^{n-2}G^*(P_{\mathcal{B}})^\vee \xrightarrow{\Sigma^{n-2}s_G^\vee} \Sigma^{n-2}P_{\mathcal{A}}^\vee \xhookrightarrow{inclusion} \Sigma^{n-3}\Xi \xrightarrow{-\xi} \mathcal{A}\,.$$

Then the following diagram commutes strictly

$$\begin{tikzcd}
\Sigma^{n-2}G^*(P_{\mathcal{B}})^\vee \arrow[r, "(-H,u)^T"] \arrow[d, "(d_{G^*(P_{\mathcal{B}})^\vee},G^*(\xi_{\mathcal{B}}))^T"'] & \mathcal{A}\oplus\Sigma^{n-1}\mathcal{K} \arrow[d, "(d_{\mathcal{A}},\xi\circ v)"] \\
\Sigma^{n-1}G^*(P_{\mathcal{B}})^\vee\oplus\Sigma G^*(\mathcal{B}) \arrow[r, "(H,j_G)"] & \Sigma\mathcal{A}.
\end{tikzcd}$$

Thus, the above commutative diagram induces a dg functor

$$\boldsymbol{G}_{rel} : \mathbf{\Pi}_{n-1}(\mathcal{B},\xi_{\mathcal{B}}) \longrightarrow \mathbf{\Pi}_n^{red}(\mathcal{A},\mathcal{B},\xi) \tag{2.11}$$

where we put $\mathbf{\Pi}_n^{red}(\mathcal{A},\mathcal{B},\xi) = T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{K},\xi_{\mathcal{K}})$. A standard argument shows that up to quasi-isomorphism, the dg functor $\boldsymbol{G}_{rel}$ and the deformed dg category $\mathbf{\Pi}_n^{red}(\mathcal{A},\mathcal{B},\xi)$ only depend on the class $[\xi]$ and the dg functor $G:\mathcal{B}\to\mathcal{A}$.

We call the dg functor $\boldsymbol{G}_{rel}$ *reduced deformed relative n-Calabi–Yau completion* of $G:\mathcal{B}\to\mathcal{A}$ with respect to the Hochschild homology class $[\xi]\in HH_{n-2}(G)$.

Proposition 2.6.2. *Let $G:\mathcal{B}\to\mathcal{A}$ be a dg functor between finitely cellular type dg categories and let $[\xi]=[(s\xi_{\mathcal{B}},\xi_{\mathcal{A}})]$ be an element in $HH_{n-2}(G)$ which has a negative cyclic lift. Then we have the following diagram which is commutative up to homotopy and where Ψ is a quasi-equivalence.*

$$\begin{tikzcd}
\mathbf{\Pi}_{n-1}(\mathcal{B},\xi_{\mathcal{B}}) \arrow[r, "\widetilde{G}"] \arrow[dr, "\boldsymbol{G}_{rel}"'] & \mathbf{\Pi}_n(\mathcal{A},\mathcal{B},\xi) \\
 & \mathbf{\Pi}_n^{red}(\mathcal{A},\mathcal{B},\xi) \arrow[u, hook, "\Psi"']
\end{tikzcd} \tag{2.12}$$

Thus, the dg functor $\boldsymbol{G}_{rel}:\mathbf{\Pi}_{n-1}(\mathcal{B},\xi_{\mathcal{B}})\to\mathbf{\Pi}_n^{red}(\mathcal{A},\mathcal{B},\xi)$ has a canonical left n-Calabi–Yau structure.

Proof. Since the map v in diagram (2.10) is a quasi-isomorphism between cofibrant dg $\mathcal{A}^e$-modules, the map v is a homotopy equivalence. Then we can construct a homotopy inverse of Ψ. Thus the dg functor Ψ is a quasi-equivalence.

Suppose that $\mathcal{B}$ and $\mathcal{A}$ are finitely cellular and $G:\mathcal{B}\to\mathcal{A}$ is a semi-free extension, i.e. there is a finite graded quiver Q and a subquiver $F\subseteq Q$, cf. above. We know that the bimodules

$$\mathcal{P}_B = \mathrm{Cone}(\Omega^1(\mathcal{B}) \xrightarrow{\alpha} \mathcal{B}\otimes_{R_2}\mathcal{B}\,)$$

and

$$\mathcal{P}_A = \mathrm{Cone}(\Omega^1(\mathcal{A}) \xrightarrow{\alpha} \mathcal{A}\otimes_{R_1}\mathcal{A}\,)$$

are cofibrant replacements of the bimodules $\mathcal{B}$ and $\mathcal{A}$ respectively. Therefore, the $\mathcal{B}$-bimodule $\Sigma^{n-2}\mathcal{P}_{\mathcal{B}}^\vee$ is cellular of finite rank, with basis $\{f_{\mathcal{B}}^\vee \,|\, f\in F_1\}$ and $\{c_{x,\mathcal{B}}\,|\, x\in F_0\}$

where the arrow $f_{\mathcal{B}}^{\vee}$ has degree $|f_{\mathcal{B}}^{\vee}| = 3 - n - |f|$, and points in the opposite direction to f; the loop $c_{x,\mathcal{B}}$ has degree $|c_{x,\mathcal{B}}| = 2 - n$, and points from x to x.

Similarly, the $\mathcal{A}$-bimodule $\Sigma^{n-1}\mathcal{P}_{\mathcal{A}}^{\vee}$ is also cellular of finite rank, with basis $\{g_{\mathcal{A}}^{\vee}|g \in Q_1\}$ and $\{c_{y,\mathcal{A}}|y \in Q_0\}$ where the arrow $g_{\mathcal{A}}^{\vee}$ has degree $|g_{\mathcal{A}}^{\vee}| = 2 - n - |g|$, and points in the opposite direction to g; the loop $c_{y,\mathcal{A}}$ has degree $|c_{y,\mathcal{A}}| = 1 - n$, and points from y to y.

Then the homotopy (see Definition 1.2.8) between $\Psi \circ \boldsymbol{G}_{rel}$ and $\widetilde{G}$ is given as follows:

- For each object x in R_1, we have $\Psi \circ \boldsymbol{G}_{rel}(x) = \widetilde{G}(x) = x$, i.e, $\alpha(x)$ is the identity map in $\mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}, \xi)$.
- For all objects x and y in R_1, the degree -1 map

$$h = h(x, y) : \mathbf{\Pi}_{n-1}(\mathcal{B}; \xi_{\mathcal{B}})(x, y) \to \mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}, \xi)(x, y)$$

  is obtained from the following map of degree -1,

$$h_2 : \Sigma^{n-2}G^*(\mathcal{P}_{\mathcal{B}}^{\vee}) \longrightarrow \Sigma^{n-1}\mathcal{P}_{\mathcal{A}}^{\vee}$$

  where h_2 is given by $f_{\mathcal{B}}^{\vee} \to f_{\mathcal{A}}^{\vee}$, and $c_{x,\mathcal{B}} \to c_{x,\mathcal{A}}$.

By Proposition 2.3.5 and Corollary 2.3.6, the dg functor $\boldsymbol{G}_{rel} : \mathbf{\Pi}_{n-1}(\mathcal{B}, \xi_{\mathcal{B}}) \to \mathbf{\Pi}_n^{red}(\mathcal{A}, \mathcal{B}, \xi)$ has a canonical left n-Calabi–Yau structure.

$\surd$

2.7 Relation with the absolute Calabi–Yau completion

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor between smooth dg categories. In [15, Section 5.2.3], Bozec–Calaque–Scherotzke defined the following tensor category over $\mathcal{A}$

$$\mathbf{\Pi}_n(G) = T_{\mathcal{A}}(\Sigma^{n-1}\mathcal{B}^{\vee} \otimes^{\mathbf{L}}_{\mathcal{B}^e} \mathcal{A}^e).$$

Let $\mathcal{A}/\mathcal{B}$ be the homotopy cofiber of G, i.e. we have the following homotopy push-out diagram in dgcat_k with Dwyer-Kan model structure [83]

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ 0 \arrow[r] & \mathcal{A}/\mathcal{B}. \end{tikzcd}$$

Proposition 2.7.1. *The following sequence is a homotopy cofiber sequence in $dgcat_k$*

$$\mathbf{\Pi}_{n-1}(\mathcal{B}) \to \mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}) \to \mathbf{\Pi}_n(\mathcal{A}/\mathcal{B}).$$

Proof. By [15, Remark 5.32], the dg functor $\mathbf{\Pi}_{n-1}(\mathcal{B}) \to \mathbf{\Pi}_n(\mathcal{A}, \mathcal{B})$ is the following composition

$$\mathbf{\Pi}_{n-1}(\mathcal{B}) \to \mathbf{\Pi}_{n-1}(G) \to \mathbf{\Pi}_n(\mathcal{A}, \mathcal{B}).$$

Consider the diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "G"] \arrow[d] \arrow[dr, phantom, "①"] & \mathcal{A} \arrow[d] & \mathbf{\Pi}_{n-1}(\mathcal{A}) \arrow[r] \arrow[dl] \arrow[d, phantom, "③"] & \mathcal{A} \arrow[dl] \\
\mathbf{\Pi}_{n-1}(\mathcal{B}) \arrow[r] \arrow[d] \arrow[dr, phantom, "②"] & \mathbf{\Pi}_{n-1}(G) \arrow[r] \arrow[d] & \mathbf{\Pi}_{n-1}(\mathcal{A},\mathcal{B}) & \\
0 \arrow[r] & \mathcal{A}/\mathcal{B}. & &
\end{tikzcd}$$

The square ① is a homotopy push-out by [15, Lemma 5.24]. Since the rectangle around ① and ② is a homotopy push-out, it follows that so is ②. By [15, Section 5.2.4], the square ③ is also a homotopy push-out.

Therefore the homotopy cofiber of $\mathbf{\Pi}_{n-1}(\mathcal{B}) \to \mathbf{\Pi}_n(\mathcal{A},\mathcal{B})$ is the homotopy push-out of the following diagram

$$\begin{tikzcd}
\mathbf{\Pi}_{n-1}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{A} \\
\mathbf{\Pi}_{n-1}(G) \arrow[d] & \\
\mathcal{A}/\mathcal{B} & .
\end{tikzcd}$$

It is easy to see that the composition $\mathbf{\Pi}_{n-1}(\mathcal{A}) \to \mathbf{\Pi}_{n-1}(G) \to \mathcal{A}/\mathcal{B}$ is equal to $\mathbf{\Pi}_{n-1}(\mathcal{A}) \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$. Consider the diagram

$$\begin{tikzcd}
\mathbf{\Pi}_{n-1}(\mathcal{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "④"] & \mathcal{A} \arrow[d] \\
\mathcal{A} \arrow[r] \arrow[d] & \mathbf{\Pi}_n(\mathcal{A}) \\
\mathcal{A}/\mathcal{B} & .
\end{tikzcd}$$

The square ④ is a homotopy push-out by [62, Proposition 5.6]. By [62, Theorem 4.6], the following diagram is a homotopy push-out

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathbf{\Pi}_n(\mathcal{A}) \arrow[d] \\
\mathcal{A}/\mathcal{B} \arrow[r] & \mathbf{\Pi}_n(\mathcal{A}/\mathcal{B}).
\end{tikzcd}$$

Thus, the sequence

$$\mathbf{\Pi}_{n-1}(\mathcal{B}) \to \mathbf{\Pi}_n(\mathcal{A},\mathcal{B}) \to \mathbf{\Pi}_n(\mathcal{A}/\mathcal{B})$$

is a homotopy cofiber sequence in dgcat_k.

$\surd$

Chapter 3

Relative cluster categories

Let $f : B \to A$ a morphism (not necessarily unital) between differential graded (=dg) k-algebras. We consider the following assumptions.

Assumption 1. Suppose that the morphism $f : B \to A$ satisfies the following properties:

1) A and B are smooth,

2) A is connective, i.e. the cohomology of A vanish in degrees > 0,

3) the morphism $f : B \to A$ has a left $(n+1)$-Calabi–Yau structure,

4) $H^0(A)$ is finite-dimensional.

Let $\mathrm{pvd}(A)$ be the perfectly valued derived category of A, i.e. $\mathrm{pvd}(A)$ is the full subcategory of $\mathcal{D}(A)$ whose objects are the perfectly valued dg A-modules. Since A is homologically smooth, $\mathrm{pvd}(A)$ is a full subcategory of $\mathrm{per}A$ (see [61, Lemma 4.1]). We denote by e the idempotent $f(1_B)$ and by $i : eAe \hookrightarrow A$ the canonical inclusion of dg algebras.

Definition 3.0.1. Let $\mathrm{pvd}_B(A)$ be the full triangulated subcategory of $\mathrm{pvd}(A)$ defined as the kernel of the restriction functor $i_* : \mathcal{D}(A) \to \mathcal{D}(eAe)$. The *relative n-cluster category* $\mathcal{C}_n(A, B)$ is defined as the following Verdier quotient

$$\mathcal{C}_n(A, B) = \mathrm{per}A/\mathrm{pvd}_B(A).$$

We denote by π^{rel} the canonical quotient functor $\mathrm{per}A \to \mathcal{C}_n(A, B)$.

3.1 Gluing t-structures

Let $G : \mathcal{B} \to \mathcal{A}$ be a dg functor. Let $\mathcal{A}/\mathcal{B}$ be the homotopy cofiber of G in dgcat_k. Then the dg category $\mathcal{A}/\mathcal{B}$ can be computed as the Drinfeld dg quotient of $\mathcal{A}$ by its full dg subcategory $\mathrm{Im}(G)$, where $\mathrm{Im}(G)$ is the full dg subcategory of $\mathcal{A}$ whose objects are the $y \in \mathcal{A}$ such that there exists $x \in \mathcal{B}$ and an isomorphism $G(x) \cong y$ in $\mathrm{H}^0(\mathcal{A})$. We denote by i the dg inclusion functor $\mathrm{Im}(G) \hookrightarrow \mathcal{A}$ and by p the quotient functor $\mathcal{A} \twoheadrightarrow \mathcal{A}/\mathcal{B}$.

Proposition 3.1.1. *[20, Theorem 5.1.3] We have the following recollement of derived categories*

$$\begin{tikzcd} \mathcal{D}(\mathcal{A}/\mathcal{B}) \arrow[r, "p_*=p_!"] & \mathcal{D}(\mathcal{A}) \arrow[l, bend right, "p^*"'] \arrow[l, bend left, "p^!"] \arrow[r, "i_*=i_!"] & \mathcal{D}(\mathrm{Im}(G)). \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \end{tikzcd} \tag{3.1}$$

The respective triangle functors are explicitly given as follows:

$$p^* = ? \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{A}/\mathcal{B} \qquad p_* = \mathbf{R}\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(\mathcal{A}/\mathcal{B}, ?) \simeq ? \otimes^{\mathbf{L}}_{\mathcal{A}/\mathcal{B}} \mathcal{A}/\mathcal{B} = p_! \qquad p^! = \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}/\mathcal{B}, ?)$$

$$i^* = ? \otimes^{\mathbf{L}}_{\mathrm{Im(G)}} \mathcal{A} \qquad i_* = \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, ?) \simeq ? \otimes^{\mathbf{L}}_{\mathcal{A}} \mathcal{A} = i_! \qquad i^! = \mathbf{R}\mathrm{Hom}_{\mathrm{Im}(G)}(\mathcal{A}, ?)$$

Consequently, we have a triangle equivalence up to direct summands

$$\mathrm{per}(\mathcal{A})/\mathrm{per}(\mathrm{Im}(G)) \xrightarrow{p^*} \mathrm{per}(\mathcal{A}/\mathcal{B}).$$

Theorem 3.1.2. *[8, Gluing t-structures] Suppose that we have the following recollement of triangulated categories*

$$\begin{tikzcd} \mathcal{U} \arrow[r, "i"] & \mathcal{T} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & \mathcal{V} . \arrow[l, bend right, "j"'] \arrow[l, bend left, "r"] \end{tikzcd}$$

Let $(\mathcal{U}^{\leqslant 0}, \mathcal{U}^{\geqslant 0})$ be a t-structure in $\mathcal{U}$ and $(\mathcal{V}^{\leqslant 0}, \mathcal{V}^{\geqslant 0})$ be a t-structure in $\mathcal{V}$. Then we have a canonical t-structure in $\mathcal{T}$ defined as follows:

$$\mathcal{T}^{\leqslant n} = \{X \in \mathcal{T} | e(X) \in \mathcal{V}^{\leqslant n} \text{ and } q(X) \in \mathcal{U}^{\leqslant n}\}$$

$$\mathcal{T}^{\geqslant n} = \{X \in \mathcal{T} | e(X) \in \mathcal{V}^{\geqslant n} \text{ and } p(X) \in \mathcal{U}^{\geqslant n}\}.$$

We say that the t-structure $(\mathcal{T}^{\leqslant n}, \mathcal{T}^{\geqslant n})$ on $\mathcal{T}$ is *glued* from the given t-structure on $\mathcal{U}$ and $\mathcal{V}$.

For any object X in $\mathcal{T}$, the canonical distinguished triangle for X with respect to the glued t-structure can be constructed as follows: Let X be an object in $\mathcal{T}$. We have a distinguished triangle in $\mathcal{V}$,

$$\tau^{\mathcal{V}}_{\leqslant 0}(e(X)) \to e(X) \to \tau^{\geqslant 1}_{\mathcal{V}}(e(X)) \to \Sigma\tau^{\mathcal{V}}_{\leqslant 0}(e(X)).$$

Hence we obtain a distinguished triangle

$$Y \xrightarrow{f} X \to r(\tau^{\mathcal{V}}_{\geqslant 1} e(X)) \to \Sigma Y,$$

where $X \to r(\tau^{\mathcal{V}}_{\geqslant 1}e(X))$ is the composition $X \to r(e(X)) \to r(\tau^{\mathcal{V}}_{\geqslant 1}e(X))$.

Similarly, we have a distinguished triangle in $\mathcal{U}$,

$$\tau^{\mathcal{U}}_{\leqslant 0}(q(Y)) \to q(Y) \to \tau^{\mathcal{U}}_{\geqslant 1}(q(Y)) \to \Sigma\tau^{\mathcal{U}}_{\leqslant 0}(q(Y)).$$

Hence we obtain a distinguished triangle

$$Z \xrightarrow{g} Y \to i(\tau^{\mathcal{U}}_{\geqslant 1}q(Y)) \to \Sigma Z,$$

where $Y \to i(\tau^{\geqslant 1}_{\mathcal{U}}q(Y))$ is the composition $Y \to i(q(Y)) \to i(\tau^{\mathcal{U}}_{\geqslant 1}q(Y))$. Thus, we have the following octahedron

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "f"] & i(\tau^{\mathcal{U}}_{\geqslant 1}q(Y)) \arrow[r] \arrow[d] & \Sigma Z \arrow[d, equal] \\
Z \arrow[r, "f\circ g"] & X \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & \Sigma Z \arrow[d, "\Sigma g"] \\
 & r(\tau^{\mathcal{V}}_{\geqslant 1}e(X)) \arrow[r, equal] \arrow[d] & r(\tau^{\mathcal{V}}_{\geqslant 1}e(X)) \arrow[r] \arrow[d] & \Sigma Y \\
 & \Sigma Y \arrow[r] & \Sigma i(\tau^{\mathcal{U}}_{\geqslant 1}q(Y)) &
\end{tikzcd} .$$

Then one can show that we have $Z \in \mathcal{T}^{\leqslant 0}$ and $U \in \mathcal{T}^{\geqslant 1}$. Thus, for any $X \in \mathcal{T}$, the canonical distinguished triangle for X with respect to the glued t-structure is given by

$$Z \to X \to U \to \Sigma Z.$$

Corollary 3.1.3. *Let $f : B \to A$ be a morphism between dg k-algebras. Suppose that A is connective and $H^0(A)$ is finite-dimensional. Let $e = f(1_B)$. We denote by $\overline{A}$ the homotopy cofiber of f. We have the following recollement*

$$\begin{tikzcd}
\mathcal{D}(\overline{A}) \arrow[r, "p_*"] & \mathcal{D}(A) \arrow[r, "i_*"] \arrow[l, bend right, "p^*"'] \arrow[l, bend left, "p^!"] & \mathcal{D}(eAe) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"]
\end{tikzcd} , \tag{3.2}$$

where the respective triangle functors are explicitly given as follows

$$p^* = ? \otimes^{\mathbf{L}}_A \overline{A} \qquad p_* = \mathbf{R}\mathrm{Hom}_{\overline{A}}(\overline{A}, ?) \simeq ? \otimes^{\mathbf{L}}_{\overline{A}} \overline{A} = p_! \qquad p^! = \mathbf{R}\mathrm{Hom}_A(\overline{A}, ?)$$

$$i^* = ? \otimes^{\mathbf{L}}_{eAe} eA \qquad i_* = \mathbf{R}\mathrm{Hom}_A(eA, ?) \simeq ? \otimes^{\mathbf{L}}_A Ae = i_! \qquad i^! = \mathbf{R}\mathrm{Hom}_{eAe}(Ae, ?).$$

Consequently, we have the following triangle equivalences

$$i^* : \mathrm{per}(eAe) \xrightarrow{\sim} \langle eA \rangle_{\mathrm{per}A},$$

$$p^* : \mathrm{per}(A)/\mathrm{per}(eAe) \xrightarrow{\sim} \mathrm{per}(\overline{A}),$$

where $\langle eA \rangle_{\mathrm{per}A}$ *is the thick subcategory of* $\mathrm{per}A$ *generated by* eA*.*

Proof. This is a special case of Proposition 3.1.1. We have a triangle equivalence up to direct summands

$$p^* : \mathrm{per}(A)/\langle eA \rangle \to \mathrm{per}(\overline{A}).$$

By Proposition 3.6.15, the functor p^* is dense. Thus, we get an equivalence

$$p^* : \mathrm{per}(A)/\langle eA \rangle \xrightarrow{\sim} \mathrm{per}(\overline{A}).$$

$\surd$

Definition 3.1.4. Let $\mathcal{A}$ be an abelian k-category. For $i \in \mathbb{Z}$ and for a complex M of objects in $\mathcal{A}$, we define the *standard truncations* $\tau_{\leqslant i}M$ and $\tau_{>i}M$ by

$$(\tau_{\leqslant i}M)^j = \begin{cases} M^j & \text{if } \; j < i \\ \ker(d_M^i) & \text{if } \; j = i \\ 0 & \text{if } \; j > i \end{cases} \qquad (\tau_{>i}M)^j = \begin{cases} 0 & \text{if } \; j < i \\ \dfrac{M}{\ker(d_M^i)} & \text{if } \; j = i \\ M^j & \text{if } \; j > i \end{cases}$$

Their respective differentials are inherited from M. Notice that $\tau_{\leqslant i}(M)$ is a subcomplex of M and $\tau_{>i}(M)$ is the corresponding quotient complex. Thus we have a sequence, which is componentwise short exact,

$$0 \to \tau_{\leqslant i}(M) \to M \to \tau_{>i}(M) \to 0.$$

Moreover, taking standard truncations behaves well with respect to cohomology, i.e. we have

$$H^j(\tau_{\leqslant i}M) = \begin{cases} H^j(M) & \text{if } \; j \leqslant i, \\ 0 & \text{if } \; j > i. \end{cases}$$

3.2 Relative t-structure

Let $f : B \to A$ be a dg k-algebra morphism satisfying the Assumptions 1. Then the map of complexes $\tau_{\leqslant 0}A \to A$ is a quasi-isomorphism of dg algebras. Thus, we may assume that the components A^p vanish for all $p > 0$. Then the canonical projection $A \to H^0(A)$ is a homomorphism of dg algebras. We view a module over $H^0(A)$ as a dg module over A via this homomorphism. This defines a natural functor $\mathrm{Mod}H^0(A) \to \mathcal{D}(A)$ which induces an equivalence from $\mathrm{Mod}H^0(A)$ onto the heart of the canonical t-structure on $\mathcal{D}(A)$ (whose left aisle (see [56]) is the full subcategory on the dg modules M such that $H^pM = 0$ for all $p > 0$).

Let $\mathrm{Mod}_B H^0(A)$ be the full subcategory of $\mathrm{Mod}H^0(A)$ whose objects are the right $H^0(A)$-modules X such that the restriction of X to $H^0(eAe)$ vanishes. Thus, we get i a natural functor $\mathrm{Mod}_B H^0(A) \to \mathcal{D}(A)$.

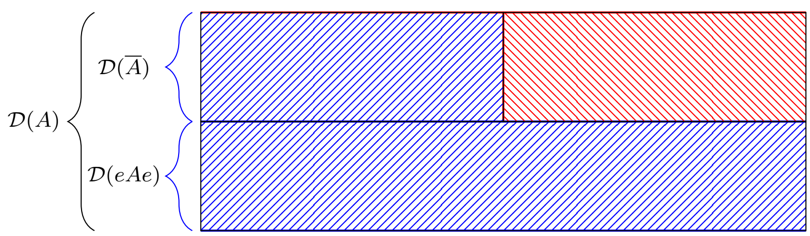
On $\mathcal{D}(\overline{A})$, we take the canonical t-structure with heart $\heartsuit = \mathrm{Mod}H^0(\overline{A})$ and on $\mathcal{D}(eAe)$, we take the trivial t-structure whose left aisle is $\mathcal{D}(eAe)$. We deduce the following corollary from Theorem 3.1.2.

Corollary 3.2.1. *There is a t-structure in $\mathcal{D}(A)$ obtained by gluing the canonical t-structure on $\mathcal{D}(\overline{A})$ with the trivial t-structure on $\mathcal{D}(eAe)$ through the recollement diagram (3.2). We denote by $(\mathcal{D}(A)^{\leqslant n}_{rel}, \mathcal{D}(A)^{\geqslant n}_{rel})$ the glued t-structure on $\mathcal{D}(A)$. That is, for any $n \in \mathbb{Z}$,*

$$\mathcal{D}(A)^{\leqslant n}_{rel} = \{X \in \mathcal{D}(A) | H^l(p^*X) = 0, \forall l > n\},$$

$$\mathcal{D}(A)^{\geqslant n}_{rel} = \{X \in \mathcal{D}(A) | i_*(X) = 0, H^l(p^!X) \cong H^l(X) = 0, \forall l < n\}$$

and the heart $\heartsuit^{rel}$ of this glued t-structure is equivalent to $Mod_B H^0(A)$. We will call $(\mathcal{D}(A)^{\leqslant n}_{rel}, \mathcal{D}(A)^{\geqslant n}_{rel})$ the relative t-structure *on $\mathcal{D}(A)$. We illustrate this glued t-structure in the following picture*



,

where the blue region represents the subcategory $\mathcal{D}(A)^{\leqslant 0}_{rel}$ and the red region represents the subcategory $\mathcal{D}(A)^{\geqslant 0}_{rel}$.

Proof. The functor $p_* : \mathcal{D}(\overline{A}) \to \ker(i_*)$ is an equivalence of triangulated categories. So the restrictions of the adjoints p^* and $p^!$ to $\ker(i_*)$ give quasi-inverses of $p_* : \mathcal{D}(\overline{A}) \to Ker(i_*)$. Thus, we have

$$\begin{aligned}\heartsuit^{rel} = \mathcal{D}(A)^{\leqslant 0}_{rel} \cap \mathcal{D}(A)^{\geqslant 0}_{rel} &= \{X \in \mathcal{D}(A) | i_*(X) = 0, H^l(p^!X) = H^l(p^*X) = 0, \ \forall l \neq 0\} \\ &= \{X \in \mathcal{D}(A) | i_*(X) = 0, H^l(X) = 0, \ \forall l \neq 0\}.\end{aligned} \tag{3.3}$$

The morphism of dg algebras $A \to H^0(A)$ induces a natural functor $i : \mathrm{Mod}_B H^0(A) \to \heartsuit^{rel}$. Let X be an object in $\heartsuit^{rel} \subseteq Ker(i_*)$. Then X is concentrated in degree 0 and X is isomorphic to an object X' in $\mathrm{Mod}H^0(A)$. Since we know that $i_*(X)$ is acyclic, X' is also in $\mathrm{Mod}_B H^0(A)$. This shows the denseness of i. The full faithfulness follows from the following commutative square with three fully faithful functors

$$
\begin{tikzcd}
\mathrm{Mod}_B H^0(A) \arrow[r, hook] \arrow[d] & \mathrm{Mod} H^0(A) \arrow[d, "\simeq"] \\
\heartsuit^{rel} \arrow[r, hook] & \heartsuit.
\end{tikzcd}
$$

$\surd$

By Corollary 3.1.3, the canonical triangle for an object $X \in \mathcal{D}(A)$ with respect to the glued t-structure can be constructed as follows: Let X be an object in $\mathcal{D}(A)$. We have the following canonical triangle triangle

$$i^*(i_*X) \longrightarrow X \longrightarrow p_*(p^*X) \longrightarrow \Sigma i^*(i_*X)\ .$$

For the object $p^*X \in \mathcal{D}(\overline{A})$, we have the following canonical triangle triangle

$$\tau_{\leqslant n}(p^*X) \longrightarrow p^*X \longrightarrow \tau_{>n}(p^*X) \longrightarrow \Sigma\tau_{\leqslant n}(p^*X).$$

Then we get a triangle in $\mathcal{D}(A)$

$$p_*(\tau_{\leqslant n}(p^*X)) \longrightarrow p_*(p^*X) \longrightarrow p_*(\tau_{>n}(p^*X)) \longrightarrow \Sigma p_*(\tau_{\leqslant n}(p^*X))\ .$$

Thus, by the octahedral axiom, there exists an object $\tau^{rel}_{\leqslant n}X$ in $\mathcal{D}^{\leqslant n}_{rel}(A)$ such that we have an isomorphism $p^*(\tau^{rel}_{\leqslant n}X) \cong \tau_{\leqslant n}(p^*X)$ and the following morphism of distinguished triangles

$$
\begin{tikzcd}
p_*(\tau_{\leqslant n}(p^*X)) \arrow[r] & 0 \arrow[r] & \Sigma p_*(\tau_{\leqslant n}(p^*X)) \arrow[r] & {} \\
\tau^{rel}_{\leqslant n}X \arrow[r] \arrow[u] & X \arrow[r] \arrow[u] & p_*(\tau_{>n}(p^*X)) \arrow[r] \arrow[u] & {} \\
i^*(i_*X) \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, "1_X"] & p_*(p^*X) \arrow[r] \arrow[u] & {} \\
p_*(\tau_{\leqslant n}(p^*X))[-1] \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & p_*(\tau_{\leqslant n}(p^*X)) \arrow[r] \arrow[u] & {} .
\end{tikzcd}
$$

Definition 3.2.2. We define the *relative truncation functor* $\tau^{rel}_{>n}$ to be the following composition

$$\tau^{rel}_{>n} : \mathcal{D}(A) \xrightarrow{p^*} \mathcal{D}(\overline{A}) \xrightarrow{\tau_{>n}} \mathcal{D}(\overline{A}) \xrightarrow{p_*} \mathcal{D}(A).$$

Thus, for any $X \in \mathcal{D}(A)$, we have a canonical triangle in $\mathcal{D}(A)$

$$\tau^{rel}_{\leqslant n}X \to X \to \tau^{rel}_{>n} \to \Sigma\tau^{rel}_{\leqslant n}X$$

such that $\tau^{rel}_{\leqslant n}X$ belongs to $\mathcal{D}(A)^{\leqslant n}_{rel}$ and $\tau^{rel}_{>n}(X) = p_*(\tau_{>n}(p^*X))$ belongs to $\mathcal{D}(A)^{\geqslant n+1}_{rel}$.

3.3 The restriction of the relative t-structure

Proposition 3.3.1. *[55, Proposition 2.5] For each $p \in \mathbb{Z}$, the space $H^p(A)$ is finite dimensional. Consequently, the category* $\mathrm{per}A$ *is Hom-finite.*

Proposition 3.3.2. *The relative t-structure on $\mathcal{D}(A)$ restricts to* $\mathrm{per}A$.

Proof. Let X be in $\mathrm{per}A$ and look at the canonical triangle respect to the relative t-structure on $\mathcal{D}(A)$

$$\tau^{rel}_{\leqslant 0}X \to X \to \tau^{rel}_{>0}X \to \Sigma\tau^{rel}_{\leqslant 0}X,$$

where $\tau^{rel}_{>0}X = p_*(\tau_{>0}p^*(X))$. By Proposition 3.5.3, the algebra $H^0(\overline{A})$ is finite-dimensional. Then by [55, Proposition 2.5], the category $\mathrm{per}(\overline{A})$ is also Home-finite. Thus, the space

$$H^l(\tau^{rel}_{>0}X) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A, \Sigma^l p_*\tau_{>0}p^*X) \simeq \mathrm{Hom}_{\mathcal{D}(\overline{A})}(\overline{A}, \Sigma^l\tau_{>0}p^*X)$$

equals zero or $H^l(p^*X)$ which is finite-dimensional. Thus the object $\tau^{rel}_{>0}X$ is in $\mathrm{pvd}(A)$ and so in $\mathrm{per}A$. Since $\mathrm{per}A$ is a triangulated subcategory, it follows that $\tau^{rel}_{\leqslant 0}X$ also lies in $\mathrm{per}A$.

$\surd$

Proposition 3.3.3. *Let $\mathrm{pvd}_B(A)^{\leqslant 0}_{rel}$ be the full subcategory of $\mathcal{D}(A)^{\leqslant 0}_{rel}$ whose objects are the $M \in \mathrm{pvd}(A)$ whose restriction along $i : eAe \hookrightarrow A$ is acyclic. Then $(\mathrm{pvd}_B(A)^{\leqslant 0}_{rel}, \mathcal{D}(A)^{\geqslant 0}_{rel})$ is a t-structure on $\mathrm{pvd}_B(A)$ and the corresponding heart is equivalent to $\mathrm{mod}_B H^0(A)$, where $\mathrm{mod}_B H^0(A)$ is the full subcategory of $\mathrm{Mod}_B H^0(A)$ whose objects are the finite-dimensional $H^0(A)$-modules. Moreover, the triangulated category $\mathrm{pvd}_B(A)$ is generated by its heart.*

Proof. Let $n \in \mathbb{Z}$. For any object $X \in \mathrm{pvd}_B(A)$, we have the following triangle

$$\tau^{rel}_{\leqslant 0}X \longrightarrow X \longrightarrow \tau^{rel}_{>0}X \longrightarrow$$

with $\tau^{rel}_{\leqslant 0}X \in \mathcal{D}(A)^{\leqslant 0}_{rel}$ and $\tau^{rel}_{>0}X \in \mathcal{D}(A)^{>0}_{rel} \subseteq \mathrm{pvd}_B(A)$. So the object $\tau^{rel}_{\leqslant 0}X$ is also in $\mathrm{pvd}_B(A)$. This is the triangle required to show that $(\mathrm{pvd}_B(A)^{\leqslant 0}_{rel}, \mathcal{D}(A)^{\geqslant 0}_{rel})$ is a t-structure.

To show the second statement, let M be an object in $\mathrm{pvd}_B(M)$. Let $n \leqslant m$ be integers such that $H^l(M) \neq 0$ only for $l \in [n, m]$. We use induction on $m - n$. If $m - n = 0$, then a shift of M is in the heart. Now suppose $m - n > 0$. Then the relative truncations yield a triangle in $\mathrm{pvd}_B(A)$

$$\tau^{rel}_{\leqslant n}M \to M \to \tau^{rel}_{>n}M \to \Sigma\tau^{rel}_{\leqslant n}M.$$

The homology of $\tau^{rel}_{\leqslant n}M$ is concentrated in degree n. Thus, the object $\tau^{rel}_{\leqslant n}M$ belongs to a shifted copy of the heart. Moreover, the homology of $\tau^{rel}_{>n}M$ is bounded between degrees $n + 1$ and m. By induction hypothesis, the object $\tau^{rel}_{>n}M$ contains in the triangulated subcategory generated by the heart. Therefore the same holds for M. $\surd$

Recall that we have defined $\mathcal{C}_n(A, B) = \mathrm{per}A/\mathrm{pvd}_B(A)$.

Proposition 3.3.4. *[4, Proposition 7.1.4] Under π^{rel} the projection functor $\mathrm{per}A \to \mathcal{C}_n(A, B)$, for any X and Y in* $\mathrm{per}A$, *we have*

$$\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\pi^{rel}X, \pi^{rel}Y) = \varinjlim_{n \leqslant 0} \mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant n}X, \tau^{rel}_{\leqslant n}Y).$$

Proof. Let X and Y be in per$A$. An element of $\varinjlim_{n\leqslant 0}\mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant n}X,\tau^{rel}_{\leqslant n}Y)$ is an equivalence class of morphisms $\tau^{rel}_{\leqslant n}X\to\tau^{rel}_{\leqslant n}Y$. Two morphisms $f:\tau^{rel}_{\leqslant n}X\to\tau^{rel}_{\leqslant n}Y$ and $g:\tau^{rel}_{\leqslant m}X\to\tau^{rel}_{\leqslant m}Y$ with $m\geqslant n$ are equivalent if there is a commutative square

$$\begin{tikzcd}
\tau^{rel}_{\leqslant n}X \arrow[r, "f"] \arrow[d] & \tau^{rel}_{\leqslant n}Y \arrow[d] \\
\tau^{rel}_{\leqslant m}X \arrow[r, "g"] & \tau^{rel}_{\leqslant m}Y,
\end{tikzcd}$$

where the vertical arrows are the canonical morphisms.

Suppose that f is a morphism $f:\tau^{rel}_{\leqslant n}X\to\tau^{rel}_{\leqslant n}Y$. We can form the following morphism from X to Y in $\mathcal{C}^{rel}_n(A,B)$

$$\begin{tikzcd}
 & \tau^{rel}_{\leqslant n}X \arrow[dl] \arrow[dr] \arrow[r, dotted, "f"] & \tau^{rel}_{\leqslant n}Y \arrow[d, dotted] \\
X & & Y,
\end{tikzcd}$$

where the morphisms $\tau^{rel}_{\leqslant n}X\to X$ and $\tau^{rel}_{\leqslant n}Y\to Y$ are the canonical morphisms.

If $f:\tau^{rel}_{\leqslant n}X\to\tau^{rel}_{\leqslant n}Y$ and $g:\tau^{rel}_{\leqslant m}X\to\tau^{rel}_{\leqslant m}Y$ with $m\geqslant n$ are equivalent, there is an equivalence of diagrams

$$\begin{tikzcd}
 & \tau^{rel}_{\leqslant n}X \arrow[dl] \arrow[dr] \arrow[r, dotted, "f"] & \tau^{rel}_{\leqslant n}Y \arrow[d, dotted] \\
X & & Y \\
 & \tau^{rel}_{\leqslant m}X \arrow[ul] \arrow[ur] \arrow[r, dotted, "g"] & \tau^{rel}_{\leqslant m}Y\ . \arrow[u, dotted]
\end{tikzcd}$$

Thus, we have a well-defined map from $\varinjlim_{n\leqslant 0}\mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant n}X,\tau^{rel}_{\leqslant n}Y)$ to $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\pi^{rel}X,\pi^{rel}Y)$ which is injective.

Let $h:X\to Y$ be a morphism in $\mathrm{Hom}_{\mathcal{C}^{rel}_n(A,B)}(\pi^{rel}X,\pi^{rel}Y)$. Suppose that h can be represented by the following right fraction

$$\begin{tikzcd}
 & X' \arrow[dl, Rightarrow, "s"'] \arrow[dr, "h'"] & \\
X & & Y.
\end{tikzcd}$$

Let X'' be the cone of s. It is an object of $\mathrm{pvd}_B(A)$ and therefore lies in $\mathcal{D}^{rel}_{>n}$ for some $n\ll 0$. Therefore there are no morphisms from $\tau^{rel}_{\leqslant n}X$ to X'' and we have the following factorization

$$\begin{tikzcd}
 & \tau^{rel}_{\leqslant n}X \arrow[dl, dotted, "s'"'] \arrow[d] \arrow[dr, "0"] & & \\
X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & \Sigma X'.
\end{tikzcd}$$

We obtain an isomorphism of diagrams

$$\begin{tikzcd}
& X' \arrow[dl, Rightarrow, "s"'] \arrow[dr, "h'"] & \\
X & & Y \\
& \tau^{rel}_{\leqslant n}X \arrow[ul, Rightarrow] \arrow[uu, "s'"] \arrow[ur, "f'=h's'"'] & .
\end{tikzcd}$$

Since $\tau^{rel}_{\leqslant n}X$ is in $\mathcal{D}(A)^{\leqslant n}_{rel}$ and $\tau^{rel}_{>n}Y$ is in $\mathcal{D}(A)^{>n}_{rel}$, the morphism $f' : \tau^{rel}_{\leqslant n}X \to Y$ induces a morphism $f : \tau^{rel}_{\leqslant n}X \to \tau^{rel}_{\leqslant n}Y$ which lifts the given morphism. Thus the map from $\varinjlim_{n\leqslant 0} \mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant n}X, \tau^{rel}_{\leqslant n}Y)$ to $\mathrm{Hom}_{\mathcal{C}^{rel}_n(A,B)}(\pi^{rel}X, \pi^{rel}Y)$ is surjective.

$\surd$

3.4 SMC Reduction

Let $\mathcal{T}$ be a Krull-Schmidt triangulated category and $\mathcal{S}$ a subcategory of $\mathcal{T}$.

Definition 3.4.1. [51, Definition 2.4.] We call $\mathcal{S}$ a *pre-simple-minded collection (pre-SMC)* if for any $X, Y \in \mathcal{S}$, the following conditions hold.

(1) $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{<0}Y) = 0$;

(2) $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}$.

We call $\mathcal{S}$ a *simple-minded collection (SMC)* if $\mathcal{S}$ is a pre-SMC and moreover, $\mathrm{thick}\,(\mathcal{S}) = \mathcal{T}$.

Let $\mathcal{S}$ be a pre-SMC. The *SMC reduction* of $\mathcal{T}$ with respect to $\mathcal{S}$ is defined as the following Verdier quotient [51, Section 3.1]

$$\mathcal{U} := \mathcal{T}/\mathrm{thick}(\mathcal{S}).$$

The subcategory $\mathrm{thick}\,(\mathcal{S})$ admits a natural t-structure $(\mathcal{X}_{\mathcal{S}}, \mathcal{Y}_{\mathcal{S}})$, where $\mathcal{X}_{\mathcal{S}}$ is the smallest extension closed subcategory of $\mathcal{T}$ containing any non-negative shift of $\mathcal{S}$ and $\mathcal{Y}_{\mathcal{S}}$ is the smallest extension closed subcategory of $\mathcal{T}$ containing any non-positive shift of $\mathcal{S}$ (see [3, Corollary 3 and Proposition 4],[65, Proposition 5.4] or [81]). Then the corresponding hear is denoted by $\mathcal{H}_{\mathcal{S}}$. It equals to the smallest extension closed subcategory of $\mathcal{T}$ containing $\mathcal{S}$.

Consider the following mild conditions:

$(R1)$ The heart $\mathcal{H}_{\mathcal{S}}$ is contravariantly finite in the Hom-orthogonal subcategory $(\Sigma^{>0}\mathcal{S})^{\perp}$ and covariantly finite in ${}^{\perp}(\Sigma^{<0}\mathcal{S})$.

$(R2)$ For any $X \in \mathcal{T}$, we have $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i\mathcal{H}_{\mathcal{S}}) = 0 = \mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_{\mathcal{S}}, \Sigma^i X)$ for $i \ll 0$.

Proposition 3.4.2. *[51, Proposition 3.2.] The following are equivalent.*

(1) $(\mathcal{X}_{\mathcal{S}}, \mathcal{X}_{\mathcal{S}}^{\perp})$ *and* $({}^{\perp}\mathcal{Y}_{\mathcal{S}}, \mathcal{Y}_{\mathcal{S}})$ *are two t-structures on* $\mathcal{T}$*;*

(2) $\mathcal{H}_\mathcal{S}$ satisfies the conditions $(R1)$ and $(R2)$.

Let $\mathcal{W}$ be the following subcategory of $\mathcal{T}$

$$\mathcal{W} := (\Sigma^{\geqslant 0}\mathcal{S})^{\perp} \cap^{\perp} (\Sigma^{\leqslant 0}\mathcal{S}).$$

Theorem 3.4.3. *[51, Theorem 3.1.] Assume the assumptions $(R1)$ and $(R2)$ hold. Then the composition*

$$\mathcal{W} \hookrightarrow \mathcal{T} \to \mathcal{U}$$

is a k-linear equivalence $\mathcal{W} \xrightarrow{\sim} \mathcal{U}$.

In our case, since the k-algebra $H^0(A)$ is a finite-dimensional k-algebra, we can suppose that 1_A has a decomposition

$$1_A = e_1 + \cdots + e_n$$

into primitive orthogonal idempotents e_i such that $e = f(1_B) = e_1 + \cdots + e_k$ for some $0 \leqslant k \leqslant n$. Then $\mathrm{mod}_B H^0(A)$ is generated by $\mathcal{S} = \{S_{k+1}, S_{k+2}, \cdots S_n\}$, where S_i is the simple $H^0(A)$ module associated to the idempotent e_i.

Then it is easy to see that $\mathcal{S}$ is a simple-minded collection of $\mathrm{pvd}_B(A)$ and is a pre-simple-minded collection of $\mathrm{per}A$.

Corollary 3.4.4. *The composition $\mathcal{W} \hookrightarrow \mathrm{per}A \to \mathcal{C}_n(A,B) = \mathrm{per}A/\mathrm{pvd}_B(A)$ is a k-linear equivalence $\mathcal{W} \xrightarrow{\sim} \mathcal{C}_n(A,B)$, where $\mathcal{W}$ is the following subcategory of* $\mathrm{per}A$

$$\mathcal{W} = (\Sigma^{\geqslant 0}\mathcal{S})^{\perp} \cap {}^{\perp}(\Sigma^{\leqslant 0}\mathcal{S}).$$

In particular, the category is idempotent complete.

Proof. It suffices to check the conditions (R1) and (R2). For any $X \in \mathrm{per}A$, it is easy to see that $\mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^i \mathcal{H}_S)$ vanishes for $i \ll 0$. By the relative Calabi–Yau duality (Corollary 2.4.3), the space $\mathrm{Hom}_{\mathcal{T}}(\mathcal{H}_\mathcal{S}, \Sigma^i X)$ also vanishes for $i \ll 0$. Therefore $\mathcal{H}_S$ satisfies the condition (R2). By the Lemma 3.4.5 below, the category $\mathrm{mod}_B H^0(A)$ is functorially finite in $\mathrm{per}A$. So $\mathcal{H}_S$ satisfies the condition (R1). Then the claim follows from Theorem 3.4.3. $\surd$

Lemma 3.4.5. *Let $B \to A$ be morphism between dg k-algebras which satisfies the assumptions 1. Then $\mathrm{mod}_B H^0(A)$ is functorially finite in* $\mathrm{per}A$.

Proof. Let P be an object in $\mathrm{per}A$. Since A is connective, there is a canonical co-t-structure (see Subsection 3.6.2) $((\mathrm{per}A)_{\geqslant 0}, (\mathrm{per}A)_{\leqslant 0})$ on $\mathrm{per}A$, where

$$(\mathrm{per}A)_{\geqslant 0} := \bigcup_{n\geqslant 0} \Sigma^{-n}\mathrm{add}A * \cdots * \Sigma^{-1}\mathrm{add}A * \mathrm{add}A \quad \text{and} \quad (\mathrm{per}A)_{\leqslant 0} := \bigcup_{n\geqslant 0} \mathrm{add}A * \Sigma\mathrm{add}A * \cdots * \Sigma^{n}\mathrm{add}A.$$

Then we have a canonical triangle in $\mathrm{per}A$

$$\sigma_{>0}P \to P \xrightarrow{t} \sigma_{\leqslant 0}P \to \Sigma\sigma_{>0}P$$

such that $\sigma_{>0}P \in (\mathrm{per}A)_{>0}$ and $\sigma_{\leqslant 0}P \in (\mathrm{per}A)_{\leqslant 0}$. Consider the object $X = \tau_{\geqslant 0}(\sigma_{\leqslant 0}P) = H^0(\sigma_{\leqslant 0}P)$. It is easy to see that $\tau_{\geqslant 0}(\sigma_{\leqslant 0}P)$ is in $\mathrm{mod}H^0(A)$ and we have a canonical morphism $f : P \xrightarrow{t} \sigma_{\leqslant 0}P \to X$. Let M be an object in $\mathrm{mod}H^0(A)$ and $g : P \to M$ be a morphism. Since the space $\mathrm{Hom}_{\mathrm{per}A}(\sigma_{>0}P, M)$ vanishes, we have $\mathrm{Hom}_{\mathrm{per}A}(P, M) \simeq \mathrm{Hom}_{\mathrm{per}A}(\sigma_{\leqslant 0}P, M)$. Then there exists a morphism $h : X \to M$ such that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "t"] \arrow[d, "g"'] & \sigma_{\leqslant 0}P \arrow[r] \arrow[dl] & X \arrow[dll, "h"] \\ M & & \end{tikzcd}.$$

This shows that $\mathrm{mod}H^0(A)$ is covariantly finite in $\mathrm{per}A$. By [51, Lemma 3.8], the subcategory $\mathrm{mod}_B H^0(A)$ is functorially finite in $\mathrm{mod}H^0(A)$. Thus, the subcategory $\mathrm{mod}_B H^0(A)$ is also covariantly finite in $\mathrm{per}A$. It remains to show $\mathrm{mod}_B H^0(A)$ is contravariantly finite in $\mathrm{per}A$.

Let N ba an object in $\mathrm{mod}_B H^0(A)$. Let $g' : N \to P$ be a morphism of dg A-modules. By the relative Calabi–Yau duality (see Corollary 2.4.3), the spaces

$$\mathrm{Hom}_{\mathrm{per}A}(N, \tau_{\leqslant -n-2}P) \simeq D\mathrm{Hom}_{\mathrm{per}A}(\tau_{\leqslant -n-2}P, \Sigma^{n+1}N)$$

and

$$\mathrm{Hom}_{\mathrm{per}A}(N, \Sigma\tau_{\leqslant -n-2}P) \simeq D\mathrm{Hom}_{\mathrm{per}A}(\tau_{\leqslant -n-2}P, \Sigma^{n}N)$$

vanish. Thus, we have $\mathrm{Hom}_{\mathrm{per}A}(N, P) \simeq \mathrm{Hom}_{\mathrm{per}A}(N, \tau_{\geqslant -n-1}P)$. We denote by g'' the composition $N \xrightarrow{g'} P \to \tau_{\geqslant -n-1}P$. Let I_P be a fibrant replacement of $\tau_{\geqslant -n-1}P$. Then we have $\mathrm{Hom}_{\mathrm{per}A}(N, P) \simeq \mathrm{Hom}_{\mathrm{per}A}(N, \tau_{\geqslant -n-1}P) \simeq \mathrm{Hom}_{\mathcal{H}(A)}(N, I_P)$.

Since $\tau_{\geqslant -n-1}P$ has finite total homology dimension, the dg module I_P also has finite total homology dimension. We write I_P as a k-complex and consider the following diagram

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, "g^0"] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & I^{-1} \arrow[r, "d^{-1}"] & I^0 \arrow[r, "d^0"] & I^1 \arrow[r] & \cdots \end{tikzcd}.$$

We put $N_P = \{x \in I^0 \,|\, d^0(x) = 0, xa = 0, \forall a \in A^p, p < 0\}$. Then N_P is in $\mathrm{mod}H^0A$ and $g''(N)$ is contained in N_P. Thus, we have the following commutative diagram

$$\begin{tikzcd} & N_P \arrow[d, hook, "i"] \\ N \arrow[ur, "g''"] \arrow[r, "g''"'] & I_P. \end{tikzcd}$$

Since the subcategory $\mathrm{mod}_B H^0(A)$ is functorially finite in $\mathrm{mod}H^0(A)$, there exists an object Y in $\mathrm{mod}_B H^0(A)$ with a right $\mathrm{mod}_B(H^0A)$-approximation $j : Y \to N_P$. Then there exists a morphism $k : N \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
 & Y \arrow[d, "j"] \\
 & N_P \arrow[d, hook, "i"] \\
N \arrow[uur, "k"] \arrow[ur] \arrow[r, "g''"'] & I_P .
\end{tikzcd}$$

This shows that $\mathrm{mod}_B H^0(A)$ is contravariantly finite in $\mathrm{per}A$.

$\surd$

Corollary 3.4.6. *The relative cluster category $\mathcal{C}_n(A, B)$ is Hom-finite.*

3.5 Relation with generalized cluster categories

In [39], Lingyan Guo generalized Claire Amiot's construction [4] of the generalized cluster categories to finite-dimensional algebras with global dimension $\leqslant n$. She studied the category $\mathcal{C}_n(\Gamma) = \mathrm{per}\Gamma/\mathrm{pvd}(\Gamma)$ associated with a dg algebra Γ under the following assumptions:

Assumption 2. 1) Γ is homologically smooth.

2) Γ is connective, i.e, $H^p(\Gamma)$ is zero for each $p > 0$.

3) Γ is $(n+1)$-Calabi–Yau as a bimodule, i.e. there is an isomorphism in $\mathcal{D}(\Gamma^e)$

$$\Sigma^{n+1}\mathbf{R}\mathrm{Hom}_{\mathcal{D}(\Gamma^e)}(\Gamma, \Gamma^e) \cong \Gamma.$$

4) The space $H^0(\Gamma)$ is finite-dimensional.

Theorem 3.5.1. *[39, Chapter 3] Let Γ be a dg k-algebra with the four properties above. Then*

(1) the category $\mathcal{C}_n(\Gamma) = \mathrm{per}\,\Gamma/\mathrm{pvd}(\Gamma)$ is Hom-finite and n-Calabi–Yau;

(2) the object $T = \pi\Gamma$ is an n-cluster tilting object in $\mathcal{C}_n(\Gamma)$ where $\pi : \mathrm{per}\Gamma \longrightarrow \mathcal{C}_n(\Gamma)$ is the canonical quotient functor, i.e. we have

$$\mathrm{Hom}_{\mathcal{C}_n(\Gamma)}(T, \Sigma^r T) = 0 \, for \quad r = 1, \cdots, n-1,$$

and for each object L in $\mathcal{C}_n(A)$, if the space $\mathrm{Hom}_{\mathcal{C}_n(\Gamma)}(T, \Sigma^r L)$ vanishes for each $r = 1, \cdots, n-1$, then L belongs to $\mathrm{add}\,T$, the full subcategory of $\mathcal{C}_n(\Gamma)$ consisting of direct summands of finite direct sums of copies of πA;

(3) the endomorphism algebra of T over $\mathcal{C}_n(\Gamma)$ is isomorphic to $H^0(\Gamma)$.

We consider the following homotopy cofiber sequence in dgcat_k

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d] & A \arrow[d, "p"] \\ 0 \arrow[r] & \overline{A}. \end{tikzcd}$$

Then we have the following immediate Proposition.

Proposition 3.5.2. *[16, Corollary 7.1] The homotopy cofiber $\overline{A}$ is homologically smooth and it has a canonical $(n+1)$-Calabi–Yau structure.*

Proposition 3.5.3. *The homotopy cofiber $\overline{A}$ is connective and $H^0(\overline{A})$ is finite-dimensional.*

Proof. By the construction of the Drinfeld dg quotient and the assumption that A is connective, the dg algebra $\overline{A}$ is also connective. By [17, Theorem 5.8], the 0-th cohomology $H^0(\overline{A})$ is isomorphic to $H^0(A)/\langle e\rangle$. Thus, the algebra $H^0(\overline{A})$ is finite-dimensional.

$\surd$

Therefore, the dg algebra $\overline{A}$ satisfies the assumptions 2. We consider the associated generalized n-cluster category $\mathcal{C}_n(\overline{A}) = \mathrm{per}\overline{A}/\mathrm{pvd}(\overline{A})$.

Proposition 3.5.4. *We have the following equivalence of triangulated categories*

$$p^* : \mathcal{C}_n(A, B)/\mathrm{per}(eAe) \xrightarrow{\sim} \mathcal{C}_n(\overline{A}).$$

Proof. We know that p^* induces an equivalence between $\mathrm{per}(A)/\mathrm{per}(eAe)$ and $\mathrm{per}(\overline{A})$. Thus, it is enough to show that we have an equivalence of triangulated categories $p^* : \mathrm{pvd}_B(A) \xrightarrow{\sim} \mathrm{pvd}(\overline{A})$ and the two subcategories $\mathrm{pvd}_B(A)$ and $\mathrm{per}(eAe)$ are left and right orthogonal to each other.

It is clear that the functor $p_* : \mathcal{D}(\overline{A}) \to \ker(i_*)$ is an equivalence of triangulated categories. Then the restriction of p^* and $p^!$ to $\ker(i_*)$ gives quasi-inverse of $p_* : \mathrm{pvd}(\overline{A}) \to \mathrm{pvd}_B(A)$.

Let X be an object in $\mathrm{pvd}_B(A)$ and let Y be an object in $\mathrm{per}(eAe)$. Then $i_*(X)$ is acyclic. Thus, we have

$$\mathrm{Hom}_{\mathcal{D}(A)}(X, i^*(Y)) \cong \mathrm{Hom}_{\mathcal{D}(eAe)}(i_*(X), Y) = 0$$

and

$$\mathrm{Hom}_{\mathcal{D}(A)}(i^*(Y), X) \cong D\mathrm{Hom}_{\mathcal{D}(A)}(X, \Sigma^{n+1} i^*(Y)) = 0,$$

where the second isomorphism is due to the relative Calabi–Yau property 2.4.3. Thus, the categories $\mathrm{pvd}_B(A)$ and $\mathrm{per}(eAe)$ are left and right orthogonal to each other.

$\surd$

Corollary 3.5.5. *We have the following commutative diagram*

$$\begin{tikzcd}
 & \mathrm{per}(eAe) \arrow[r, equal] \arrow[d, hook] & \mathrm{per}(eAe) \arrow[d, hook] \\
\mathrm{pvd}_B(A) \arrow[r, hook] \arrow[d, "\simeq"'] & \mathrm{per}(A) \arrow[r] \arrow[d] & \mathcal{C}_n(A,B) \arrow[d] \\
\mathrm{pvd}(\overline{A}) \arrow[r, hook] & \mathrm{per}(\overline{A}) \arrow[r] & \mathcal{C}_n(\overline{A})
\end{tikzcd}$$

and the rows and columns are exact sequences of triangulated categories.

3.6 Silting Reduction and relative Fundamental domain

3.6.1 Silting Reduction

Let $\mathcal{T}$ be a triangulated category. A full subcategory $\mathcal{P}$ of $\mathcal{T}$ is *presilting* if $\mathrm{Hom}_{\mathcal{T}}(P, \Sigma^i P) = 0$ for any $i > 0$. It is *silting* if in addition $\mathcal{T} = \mathrm{thick}\,\mathcal{P}$. An object P of $\mathcal{T}$ is *presilting* if $\mathrm{add}P$ is a presilting subcategory and *silting* if $\mathrm{add}P$ is a silting subcategory.

Let $\mathcal{P}$ be a presilting subcategory of $\mathcal{T}$. Let $\mathcal{S}$ be the thick subcategory $\mathrm{thick}_{\mathcal{T}}\mathcal{P}$ of $\mathcal{T}$ and $\mathcal{U}$ the quotient category $\mathcal{T}/\mathcal{S}$. We call $\mathcal{U}$ the *silting reduction* of $\mathcal{T}$ with respect to $\mathcal{P}$ (see [2]). For an integer l, there is a bounded co-t-structure $(\mathcal{S}_{\geqslant l}, \mathcal{S}_{\leqslant l})$ on $\mathcal{S}$ (see [49, Proposition 2.8.]), where

$$\mathcal{S}_{\geqslant l} = \mathcal{S}_{>l-1} := \bigcup_{i\geqslant 0} \Sigma^{-l-i}\mathcal{P} * \cdots * \Sigma^{-l-1}\mathcal{P} * \Sigma^{-l}\mathcal{P},$$

$$\mathcal{S}_{\leqslant l} = \mathcal{S}_{<l+1} := \bigcup_{i\geqslant 0} \Sigma^{-l}\mathcal{P} * \Sigma^{-l+1}\mathcal{P} \cdots * \Sigma^{-l+i}\mathcal{P}.$$

Let $\mathcal{Z}$ be the following subcategory of $\mathcal{T}$

$$\mathcal{Z} = ({}^{\perp_{\mathcal{T}}}\mathcal{S}_{<0}) \cap (\mathcal{S}_{>0}^{\perp_{\mathcal{T}}}) = {}^{\perp_{\mathcal{T}}} (\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathcal{T}}}.$$

Example 3.6.1. Let $\mathcal{E}$ be a Frobenius category. Let $\mathcal{T} = \mathcal{D}^b(\mathcal{E})$ be its bounded derived category and $\mathcal{P}$ the projective-injective subcategory of $\mathcal{E}$. Then $\mathcal{Z}$ is equal to $\mathcal{E} \subseteq \mathcal{D}^b(\mathcal{E})$.

We consider the following mild technical conditions:

(P1) $\mathcal{P}$ is covariantly finite in ${}^{\perp_{\mathcal{T}}}(\Sigma^{>0}\mathcal{P})$ and contravariantly finite in $(\Sigma^{<0}\mathcal{P})^{\perp_{\mathcal{T}}}$.

(P2) For any $X \in \mathcal{T}$, we have $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma^l\mathcal{P}) = 0 = \mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \Sigma^l X)$ for $l \gg 0$.

Proposition 3.6.2. *[49, Proposition 3.2.] The following conditions are equivalent.*

(a) The conditions (P1) and (P2) are satisfied.

(b) The two pairs $({}^{\perp_{\mathcal{T}}}\mathcal{S}_{<0}, \mathcal{S}_{\leqslant 0})$ and $(\mathcal{S}_{\geqslant 0}, \mathcal{S}_{>0}^{\perp_{\mathcal{T}}})$ are co-t-structures on $\mathcal{T}$.

In this case, the co-hearts of these co-t-structures are $\mathcal{P}$.

Theorem 3.6.3. *[49, Theorem 3.1.] Under the conditions* $(P1)$ *and* $(P2)$*, the composition* $\mathcal{Z} \subset \mathcal{T} \xrightarrow{\rho} \mathcal{U}$ *of natural functors induces an equivalence of additive categories:*

$$\overline{\rho} : \mathcal{Z}/[\mathcal{P}] \longrightarrow \mathcal{U}.$$

Moreover, we have the following.

Theorem 3.6.4. *[50, Theorem 4.2.] The category* $\mathcal{Z}/[\mathcal{P}]$ *has the structure of a triangulated category with respect to the following shift functor and triangles:*

(a) For $X \in \mathcal{Z}$*, we take a triangle*

$$X \xrightarrow{l_X} P_X \longrightarrow X\langle 1\rangle \longrightarrow \Sigma X$$

with a (fixed) left $\mathcal{P}$*-approximation* l_X*. Then* $\langle 1\rangle$ *gives a well-defined auto-equivalence of* $\mathcal{Z}/[\mathcal{P}]$*, which is the shift functor of* $\mathcal{Z}/[\mathcal{P}]$*.*

(b) For a triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ *with* $X, Y, Z \in \mathcal{Z}$*, take the following commutative diagram of triangles*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "g"] \arrow[d, "a"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "l_X"] & P_X \arrow[r] & X\langle 1\rangle \arrow[r] & \Sigma X.
\end{tikzcd}$$

Then we have a complex $X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z \xrightarrow{\overline{a}} X\langle 1\rangle$*. We define triangles in* $\mathcal{Z}/[\mathcal{P}]$ *as the complexes which are isomorphic to complexes obtained in this way.*

Theorem 3.6.5. *[49, Theorem 3.6.] The functor* $\overline{\rho} : \mathcal{Z}/[\mathcal{P}] \longrightarrow \mathcal{U}$ *in Theorem 3.6.3 is a triangle equivalence where the triangulated structure of* $\mathcal{Z}/[\mathcal{P}]$ *is given by Theorem 3.6.4.*

In our case, we put $\mathcal{T} = \mathrm{per}A$, $\mathcal{P} = \mathrm{add}(eA)$, and $\mathcal{S} = \mathrm{thick}_{\mathcal{T}}\mathcal{P} \cong \mathrm{per}(eAe)$. Then it is clear that the categories $\mathcal{T}$, $\mathcal{P}$ and $\mathcal{S}$ satisfy the conditions (P1) and (P2).

Corollary 3.6.6. *We have the following equivalence of triangulated categories*

$$p^* : \mathcal{Z}/[\mathcal{P}] \xrightarrow{\sim} \mathrm{per}A/\langle eA\rangle \xrightarrow{\sim} \mathrm{per}(\overline{A}),$$

where $\mathcal{Z} = {}^{\perp_{\mathrm{per}A}}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathrm{per}A}}$*.*

3.6.2 The standard co-t-structure on $\mathrm{per}A$

Proposition 3.6.7. *[49, Proposition 2.8.] Let* $\mathcal{T}$ *be a triangulated category and* $\mathcal{M}$ *a silting subcategory of* $\mathcal{T}$ *with* $\mathcal{M} = \mathrm{add}\mathcal{M}$*.*

(a) Then $(\mathcal{T}_{\geqslant 0}, \mathcal{T}_{\leqslant 0})$ *is a bounded co-t-structure on* $\mathcal{T}$ *, where*

$$\mathcal{T}_{\geqslant 0} := \bigcup_{n\geqslant 0} \Sigma^{-n}\mathcal{M} * \cdots * \Sigma^{-1}\mathcal{M} * \mathcal{M} \quad \text{and} \quad \mathcal{T}_{\leqslant 0} := \bigcup_{n\geqslant 0} \mathcal{M} * \Sigma\mathcal{M} * \cdots * \Sigma^{n}\mathcal{M}.$$

(b) For any integers m and n, we have

$$\mathcal{T}_{\geqslant n} \cap \mathcal{T}_{\leqslant m} = \begin{cases} \Sigma^{-m}\mathcal{M} * \Sigma^{-m+1}\mathcal{M} * \cdots * \Sigma^{-n}\mathcal{M} & \text{if} \quad n \leqslant m, \\ 0 & \text{if} \quad n > m. \end{cases}$$

Let A be a connective dg algebra. Then A is a silting object in $\mathrm{per}A$. By the above proposition, the pair $((\mathrm{per}A)_{\geqslant 0}, (\mathrm{per}A)_{\leqslant 0})$ is a co-t-structure on $\mathrm{per}A$, where

$$(\mathrm{per}A)_{\geqslant 0} := \bigcup_{n \geqslant 0} \Sigma^{-n}\mathrm{add}A * \cdots * \Sigma^{-1}\mathrm{add}A * \mathrm{add}A \quad \text{and} \quad (\mathrm{per}A)_{\leqslant 0} := \bigcup_{n \geqslant 0} \mathrm{add}A * \Sigma\mathrm{add}A * \cdots * \Sigma^{n}\mathrm{add}A.$$

The corresponding co-heart is $\mathrm{add}A$.

3.6.3 Fundamental domain for generalized cluster categories

Let $\mathcal{F}$ be the full subcategory $\mathcal{D}(\overline{A})^{\leqslant 0} \cap {}^{\perp}\mathcal{D}(\overline{A})^{\leqslant -n} \cap \mathrm{per}(\overline{A})$. In the paper [4], it is called the *fundamental domain* of $\mathrm{per}\overline{A}$. We denote by $\pi : \mathrm{per}\overline{A} \to \mathcal{C}_n(\overline{A})$ the canonical projection functor.

Lemma 3.6.8. *[39, Lemma 3.2.8] For each object X of $\mathcal{F}$, there exist $n-1$ triangles (which are not unique in general)*

$$P_1 \to Q_0 \to X \to \Sigma P_1,$$

$$P_2 \to Q_1 \to P_1 \to \Sigma P_2,$$

$$\cdots$$

$$P_{n-1} \to Q_{n-2} \to P_{n-2} \to \Sigma P_{n-1},$$

where Q_0, Q_1, $\cdots$, Q_{n-2} and P_{n-1} are in $\mathrm{add}(\overline{A})$.

Remark 3.6.9. In fact, the fundamental domain $\mathcal{F}$ is equal to

$$(\mathrm{per}\overline{A})_{\geqslant 1-n} \cap (\mathrm{per}\overline{A})_{\leqslant 0} = \mathrm{add}\overline{A} * \Sigma\mathrm{add}\overline{A} * \cdots * \Sigma^{n-1}\mathrm{add}\overline{A},$$

where $((\mathrm{per}\overline{A})_{\geqslant 0}, (\mathrm{per}\overline{A})_{\leqslant 0})$ is the canonical co-t-structure on $\mathrm{per}\overline{A}$.

Proposition 3.6.10. [39, Proposition 4.3.1] The projection functor $\pi : \mathrm{per}\overline{A} \to \mathcal{C}_n(\overline{A})$ induces a k-linear equivalence between $\mathcal{F}$ and $\mathcal{C}_n(\overline{A})$.

3.6.4 Relative Fundamental domain and Higgs category

Definition 3.6.11. We define the *relative fundamental domain* $\mathcal{F}^{rel}$ of $\mathrm{per}A$ to be the following full subcategory

$$\mathcal{Z} \cap (\mathrm{per}A)_{\geqslant 1-n} \cap (\mathrm{per}A)_{\leqslant 0} = \mathcal{Z} \cap (\mathrm{add}A * \Sigma\mathrm{add}A * \cdots * \Sigma^{n-1}\mathrm{add}A),$$

where $((\mathrm{per}A)_{\geqslant 0}, (\mathrm{per}A)_{\leqslant 0})$ is the canonical co-t-structure on $\mathrm{per}A$ and $\mathcal{Z}$ is the subcategory

$${}^{\perp_{\mathrm{per}A}}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathrm{per}A}}$$

with $\mathcal{P} = \mathrm{add}(eA)$.

By the proof of [4, Lemma 7.2.1] (or [39, Lemma 3.2.8]), we can see that the subcategory $\mathrm{add}A * \Sigma\mathrm{add}A * \cdots * \Sigma^{n-1}\mathrm{add}A$ is equal to $\mathcal{D}(A)^{\leqslant 0} \cap {}^{\perp}(\mathcal{D}(A)^{\leqslant -n}) \cap \mathrm{per}(A)$. Thus, the relative fundamental domain $\mathcal{F}^{rel}$ is also equal to $\mathcal{Z} \cap \mathcal{D}(A)^{\leqslant 0} \cap {}^{\perp}(\mathcal{D}(A)^{\leqslant -n}) \cap \mathrm{per}(A)$.

Remark 3.6.12. The relative fundamental domain $\mathcal{F}^{rel}$ is also equivalent to the full subcategory of $\mathcal{Z} \subseteq \mathrm{per}(A)$ whose objects are the $X \in \mathcal{Z}$ such that X fits into the following $n-1$ triangles in $\mathrm{per}A$

$$M_1 \to N_0 \to X \to \Sigma M_1,$$

$$M_2 \to N_1 \to M_1 \to \Sigma M_2,$$

$$\cdots$$

$$M_{n-1} \to N_{n-2} \to M_{n-2} \to \Sigma M_{n-1}$$

with N_0, N_1, $\cdots$, N_{n-2} and M_{n-1} are in $\mathrm{add}(A)$.

Proposition 3.6.13. *The relative fundamental domain $\mathcal{F}^{rel}$ is contained in*

$$\mathcal{D}(A)^{\leqslant 0}_{rel} \cap {}^{\perp}(\mathcal{D}_B(A)^{\leqslant -n}_{rel}) \cap \mathrm{per}(A),$$

where $\mathcal{D}_B(A)^{\leqslant -n}_{rel}$ is the full subcategory of $\mathcal{D}(A)^{\leqslant -n}_{rel}$ whose objects are the objects X in $\mathcal{D}(A)^{\leqslant -n}_{rel}$ whose restriction $i_*(X)$ to eAe is acyclic.

Proof. Let X be an object in $\mathcal{F}^{rel} = \mathcal{Z} \cap (\mathrm{add}A * \mathrm{add}A[1] * \cdots * \mathrm{add}A[n-1])$. Since $A, \Sigma A, \cdots, \Sigma^{n-1}A$ are in $\mathcal{D}(A)^{\leqslant 0}_{rel} \cap {}^{\perp}(\mathcal{D}_B(A)^{\leqslant -n}_{rel}) \cap \mathrm{per}(A)$, by using the triangles in Remark 3.6.12, we see that X also lies in $\mathcal{D}(A)^{\leqslant 0}_{rel} \cap {}^{\perp}(\mathcal{D}_B(A)^{\leqslant -n}_{rel}) \cap \mathrm{per}(A)$.

$\checkmark$

We still denote by p^* the restriction of $p^* : \mathrm{per}(A) \to \mathrm{per}(\overline{A})$ to $\mathcal{F}^{rel}$.

Proposition 3.6.14. *The functor $p^* : \mathcal{F}^{rel} \to \mathcal{F}$ is dense.*

Proof. It is easy to see that p^* is well defined. Let Y be an object in $\mathcal{F} \subseteq \mathrm{per}\overline{A}$. By Lemma 3.6.8, there exist $n-1$ triangles in $\mathrm{per}\overline{A}$

$$P_1 \xrightarrow{b_0} Q_0 \to Y \to \Sigma P_1,$$

$$P_2 \xrightarrow{b_1} Q_1 \to P_1 \to \Sigma P_2,$$

$$\cdots$$

$$P_{n-2} \xrightarrow{b_{n-3}} Q_{n-3} \to P_{n-3} \to \Sigma P_{n-2},$$

$$P_{n-1} \xrightarrow{b_{n-2}} Q_{n-2} \to P_{n-2} \to \Sigma P_{n-1},$$

with Q_0, Q_1, $\cdots$, Q_{n-2} and P_{n-1} are in $\mathrm{add}(\overline{A})$.

We start from the last triangle. Since the functor $p^* : \mathrm{add}A \subseteq \mathcal{Z} \longrightarrow \mathrm{add}\overline{A}$ is dense, there exist two objects M'_{n-1}, N'_{n-2} in $\mathrm{add}A$ such that $p^*(M'_{n-1}) \cong P_{n-1}$ and $p^*(N'_{n-2}) \cong Q_{n-2}$. We know that $p^* : \mathcal{Z}/[\mathcal{P}] \xrightarrow{\sim} \mathrm{per}A/\langle eA \rangle \to \mathrm{per}(\overline{A})$ is fully faithful, where $\mathcal{Z} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp}$ and $\mathcal{P} = \mathrm{add}(eA)$. Thus we have the following surjective map

$$\mathrm{Hom}_{\mathcal{Z}}(M'_{n-1}, N'_{n-2}) \twoheadrightarrow \mathrm{Hom}_{\mathrm{per}(\overline{A})}(P_{n-1}, Q_{n-2}).$$

We lift the map $b_{n-2} : P_{n-1} \to Q_{n-2}$ from $\mathrm{add}(\overline{A})$ to $\mathrm{add}(A) \subseteq \mathcal{Z}$. Then we get $g'_{n-2} : M'_{n-1} \to N'_{n-1}$ such that $p^*(g'_{n-2}) \cong b_{n-2}$.

Since $\mathcal{P}$ is covariantly finite and contravariantly finite in $\mathcal{Z}$, we can find $h_{n-2} : M'_{n-1} \to W_{n-2}$ a left $\mathrm{add}(eA)$-approximation of M'_{n-1}. We define

$$(M_{n-1} \xrightarrow{g_{n-2}} N_{n-2}) := (M'_{n-1} \xrightarrow{[g'_{n-2},\ h_{n-2}]^t} N'_{n-2} \oplus W_{n-2}).$$

Then we can see that $p^*(g_{n-2}) \cong b_{n-2}$ and the following map is surjective

$$g^*_{n-2} : \mathrm{Hom}_{\mathrm{per}A}(N_{n-2}, \mathrm{add}(eA)) \to \mathrm{Hom}_{perA}(M_{n-1}, \mathrm{add}(eA)).$$

We form a triangle in $\mathrm{per}A$

$$M_{n-1} \xrightarrow{g_{n-2}} N_{n-2} \to M_{n-2} \to \Sigma M_{n-1}.$$

Then $p^*(M_{n-2})$ is isomorphic to P_{n-2}.

Since the map $g^*_{n-2} : \mathrm{Hom}_{perA}(N_{n-2}, \mathrm{add}(eA)) \to \mathrm{Hom}_{perA}(M_{n-1}, \mathrm{add}(eA))$ is surjective, we can see that M_{n-2} is an object in $\mathcal{Z} = {}^{\perp}(\mathcal{P}[>0]) \cap (\mathcal{P}[<0])^{\perp}$.

Next, we consider the penultimate triangle. Repeating the above argument, we get a triangle in $\mathrm{per}A$

$$M_{n-2} \xrightarrow{g_{n-3}} N_{n-3} \to M_{n-3} \to \Sigma M_{n-2}$$

such that $N_{n-3} \in \mathrm{add}A$, $p^*(N_{n-3}) \cong Q_{n-3}$, $p^*(g_{n-3}) \cong b_{n-3}$, $p^*(M_{n-3}) \cong P_{n-3}$ and

$$M_{n-3} \in \mathcal{Z} = {}^{\perp}(\mathcal{P}[>0]) \cap (\mathcal{P}[<0])^{\perp}.$$

Then, we keep repeating this argument until the first triangle. We get the following $n-1$ triangles in $\mathrm{per}A$

$$M_1 \xrightarrow{g_0} N_0 \to X \to \Sigma M_1,$$
$$M_2 \xrightarrow{g_1} N_1 \to M_1 \to \Sigma M_2,$$
$$\cdots$$
$$M_{n-2} \xrightarrow{g_{n-3}} N_{n-3} \to M_{n-3} \to \Sigma M_{n-2},$$
$$M_{n-1} \xrightarrow{g_{n-2}} N_{n-2} \to M_{n-2} \to \Sigma M_{n-1}$$

such that $M_{n-1}, N_{n-2}, \cdots, N_0 \in \mathrm{add}A$, $X \in \mathcal{Z}$ and $p^*(X) \cong Y$. Thus, the object X belongs to $\mathcal{F}^{rel}$ and therefore $p^* : \mathcal{F}^{rel} \to \mathcal{F}$ is dense.

$\surd$

Proposition 3.6.15. *The functor $p^* : \mathrm{per}A \to \mathrm{per}\overline{A}$ is dense. Thus, we have equivalences*

$$\mathcal{Z}/[\mathcal{P}] \simeq \mathrm{per}A/\langle eA\rangle \simeq \mathrm{per}\overline{A},$$

where $\mathcal{Z} = {}^{\perp}(\mathcal{P}[>0]) \cap (\mathcal{P}[<0])^{\perp}$ with $\mathcal{P} = \mathrm{add}(eA)$.

Proof. There is a canonical co-t-structure $(\mathrm{per}(\overline{A})_{\geqslant 0}, \mathrm{per}(\overline{A}))_{\leqslant 0}$ on $\mathrm{per}(\overline{A})$, where

$$\mathrm{per}(\overline{A})_{\geqslant 0} = \bigcup_{n\geqslant 0} \Sigma^{-n}\mathrm{add}(\overline{A}) * \cdots * \Sigma^{-1}\mathrm{add}(\overline{A}) * \mathrm{add}(\overline{A}),$$

$$\mathrm{per}(\overline{A})_{\leqslant 0} = \bigcup_{n\geqslant 0} \mathrm{add}(\overline{A}) * \Sigma\mathrm{add}(\overline{A}) * \cdots * \Sigma^{n}\mathrm{add}(\overline{A}).$$

Let Z be an object in $\mathrm{per}(\overline{A})$. By using the canonical co-t-structure on $\mathrm{per}(\overline{A})$, we have a triangle in $\mathrm{per}(\overline{A})$

$$X \to Z \to Y \xrightarrow{h} \Sigma X,$$

with $X \in \mathrm{per}(\overline{A})_{\geqslant 0}$ and $Y \in \mathrm{per}(\overline{A})_{\leqslant 0}$.

We will find objects $U, V \in \mathcal{Z} \subseteq \mathrm{per}A$ such that $p^*(U) \cong X$ and $p^*(V) \cong Y$. Suppose that X is in $\Sigma^{-n_0}\mathrm{add}(\overline{A}) * \cdots * \Sigma^{-1}\mathrm{add}(\overline{A}) * \mathrm{add}(\overline{A})$ and Y is in $\mathrm{add}(\overline{A}) * \Sigma\mathrm{add}(\overline{A}) * \cdots * \Sigma^{n_1}\mathrm{add}(\overline{A})$. If $n_0 = 0$ or $n_1 = 0$, we are done. So we can assume that $n_0 \geqslant 1$ and $n_1 \geqslant 1$.

For the object Y, there are n_1 triangles in $\mathrm{per}(\overline{A})$

$$P_1 \to Q_0 \to Y \to \Sigma P_1,$$

$$P_2 \to Q_1 \to P_1 \to \Sigma P_2,$$

$$\cdots$$

$$P_{n_1} \to Q_{n_1-1} \to P_{n_1-1} \to \Sigma P_{n_1},$$

with $P_{n_1}, Q_{n_1-1}, \cdots, Q_0 \in \mathrm{add}(\overline{A})$.

Similarly, by the same argument in Proposition 3.6.14, there is an object $V \in \mathcal{Z} \subseteq \mathrm{per}A$ such that $p^*(V) \cong Y$.

For the object $X \in \Sigma^{-n_0}\mathrm{add}(\overline{A}) * \cdots * \Sigma^{-1}\mathrm{add}(\overline{A}) * \mathrm{add}(\overline{A})$, we have $\Sigma^{n_0}X$ is in $\mathrm{add}(\overline{A}) * \Sigma\mathrm{add}(\overline{A}) * \cdots * \Sigma^{n_0}\mathrm{add}(\overline{A})$. Thus there exist an object $U \in \mathcal{Z}$ such that $p^*(U) \cong \Sigma^{n_0}X$.

Since $\mathcal{P} = \mathrm{add}(eA)$ is covariantly finite and contravariantly finite in $\mathcal{Z}$, we can take the following $n_0 + 1$ triangles in $\mathrm{per}A$

$$U\langle -1\rangle \to R_0 \xrightarrow{f_0} U \to \Sigma U\langle -1\rangle,$$

$$U\langle -2\rangle \to R_{-1} \xrightarrow{f_{-1}} U\langle -1\rangle \to \Sigma U\langle -2\rangle,$$

$$\cdots$$

$$U\langle -n_0-1\rangle \to R_{-n_0} \xrightarrow{f_{-n_0}} U\langle -n_0\rangle \to \Sigma U\langle -n_0-1\rangle$$

with f_i is a right $\mathrm{add}(eA)$-approximation for any $-n_0 \leqslant i \leqslant 0$. Then the object $p^*(U) \cong X[n_0]$ is isomorphic to $p^*(U\langle -n_0\rangle)[n_0]$. Thus, we have $p^*(U\langle -n_0\rangle)$ is isomorphic to X.

Since $\mathcal{Z}/[\mathcal{P}] \cong \mathrm{per}A/\langle eA\rangle \to \mathrm{per}(\overline{A})$ is fully faithful, the following map is a surjection (see Proposition 3.6.25)

$$\mathrm{Hom}_{\mathcal{Z}}(V, \Sigma U\langle -n_0\rangle) \twoheadrightarrow \mathrm{Hom}_{\mathcal{Z}/[\mathcal{P}]}(Y, \Sigma X) = \mathrm{Hom}_{\mathrm{per}(\overline{A})}(Y, \Sigma X).$$

We can lift the following triangle in $\mathrm{per}(\overline{A})$

$$X \to Z \to Y \xrightarrow{h} \Sigma X,$$

to a triangle in $\mathrm{per}A$

$$U\langle -n_0\rangle \to W \to V \xrightarrow{h'} \Sigma U\langle -n_0\rangle.$$

Therefore, the object $p^*(W)$ is isomorphic to Z. Hence the functor $p^* : \mathrm{per}A \to \mathrm{per}(\overline{A})$ is dense. $\surd$

Corollary 3.6.16. *We have the following equivalence of k-categories*

$$p^* : \mathcal{F}^{rel}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{F}.$$

Proof. By Proposition 3.6.14, we know that the quotient functor $\mathcal{F}^{rel}/[\mathcal{P}] \to \mathcal{F}$ is dense. Since we have an equivalence $\mathcal{Z}/[\mathcal{P}] \xrightarrow{\sim} \mathrm{per}\overline{A}$, this quotient functor $\mathcal{F}^{rel}/[\mathcal{P}] \to \mathcal{F}$ is also fully faithful. Thus the quotient functor $\mathcal{F}^{rel}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{F}$ is an equivalence of k-categories. $\surd$

Proposition 3.6.17. *[4, proposition 7.2.1] The restriction of the quotient functor π^{rel} : $\mathrm{per}A \to \mathcal{C}_n(A,B)$ to $\mathcal{F}^{rel}$ is fully faithful.*

Proof. Let X and Y be objects in $\mathcal{F}^{rel} \subseteq \mathcal{D}(A)^{\leqslant 0}_{rel} \cap {}^{\perp}(\mathcal{D}_B(A)^{\leqslant -n}_{rel}) \cap \mathrm{per}(A)$. By Proposition 3.3.4, the space $\mathrm{Hom}_{\mathcal{C}^{rel}_n(A,B)}(\pi^{rel}X, \pi^{rel}Y)$ is isomorphic to the direct limit $\varinjlim_{l<0} \mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant l}X, \tau^{rel}_{\leqslant l}Y)$. A morphism between X and Y in $\mathcal{C}_n(A,B)$ is a diagram of the form

$$\begin{tikzcd} & \tau^{rel}_{\leqslant l}X \arrow[dl] \arrow[dr] & \\ X & & Y. \end{tikzcd}$$

The canonical triangle

$$\Sigma^{-1}(\tau^{rel}_{>l}X) \to \tau^{rel}_{\leqslant l}X \to X \to \tau^{rel}_{>l}X$$

yields a long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{>l}X, Y) \to \mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant l}X, Y) \to \mathrm{Hom}_{\mathcal{D}(A)}(X, Y) \to \mathrm{Hom}_{\mathcal{D}(A)}((\Sigma^{-1}\tau^{rel}_{>l}X), Y) \to \cdots.$$

Since $i_*(\tau^{rel}_{>l}X) = 0$, it satisfies the conditions of relative Calabi–Yau duality 2.4.3, the space

$$\mathrm{Hom}_{\mathcal{D}(A)}(\Sigma^{-1}(\tau^{rel}_{>l}X), Y)$$

is isomorphic to the space $D\mathrm{Hom}_{\mathcal{D}(A)}(Y, \Sigma^n\tau^{rel}_{>l}X)$. The object X is in $\mathcal{D}(A)^{\leqslant 0}_{rel}$, hence we have

$$(\tau^{rel}_{>l}X) \in \mathcal{D}_B(A)^{\leqslant 0}_{rel}$$

and then the space

$$\mathrm{Hom}_{\mathcal{D}(A)}(Y, \Sigma^n\tau^{rel}_{>l})$$

vanishes. For the same reasons, the space $\mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{>l}X, Y)$ vanishes. Thus there are bijections

$$\mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant l}X, \tau^{rel}_{\leqslant l}Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(A)}(\tau^{rel}_{\leqslant l}X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(A)}(X, Y)$$

Thus, the functor $\pi^{rel} : \mathcal{F}^{rel} \to \mathcal{C}_n(A, B)$ is fully faithful. $\surd$

Corollary 3.6.18. *We have an isomorphism* $\mathrm{End}_{\mathcal{C}_n(A,B)}(A) \simeq \mathrm{End}_{\mathrm{per}(A)}(A) = H^0(A)$.

Proof. This follows from Lemma 3.6.17 and the fact that A itself is in $\mathcal{F}^{rel}$. $\surd$

Definition 3.6.19. The *Higgs category* $\mathcal{H}$ is the image of $\mathcal{F}^{rel}$ in $\mathcal{C}_n(A, B)$ under the quotient functor $\pi^{rel} : \mathrm{per}A \to \mathcal{C}_n(A, B)$.

Remark 3.6.20. The reason for the name "Higgs category" is that this category generalizes the category of modules over the preprojective algebra of a Dynkin quiver and a module over the preprojective algebra can be called a "Higgs category" (in analogy with a "Higgs bundle", which is the same object in a geometric context, see [43, 82]).

3.6.5 Equivalence between the shifts of $\mathcal{F}^{rel}$

Definition 3.6.21. Let $l > 0$ be an integer. We define the *relative l-shifted fundamental domain* $\mathcal{F}^{rel}\langle l\rangle$ to be the following full subcategory of $\mathcal{Z}$

$$\mathcal{F}^{rel}\langle l\rangle = \{X \in \mathcal{Z} \mid p^*(X) \in \Sigma^l\mathcal{F} \subseteq \mathrm{per}(\overline{A})\},$$

where $\mathcal{Z} = {}^{\perp_{\mathrm{per}A}}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathrm{per}A}}$ with $\mathcal{P} = \mathrm{add}(eA)$.

Remark 3.6.22. If $l = 0$, then $\mathcal{F}^{rel}\langle 0\rangle = \{X \in \mathcal{Z} \mid p^*(X) \in \mathcal{F} \subseteq \mathrm{per}(\overline{A})\}$ is equal to $\mathcal{F}^{rel}$.

Our aim is to show that the functor $\tau^{rel}_{\leqslant -l}$ induces an equivalence

$$\mathcal{F}^{rel}\langle l-1\rangle \to \mathcal{F}^{rel}\langle l\rangle,$$

cf Proposition 3.6.30.

Let l be a positive integer and X an object of $\mathcal{F}^{rel}\langle l\rangle$. Then the object $p^*(X)$ lies in $\Sigma^l\mathcal{F} \subseteq \mathrm{per}(\overline{A})$. Hence $\Sigma^{1-l}p^*(X)$ is in $\Sigma\mathcal{F}$. By definition, there are $n-1$ triangles related to the object $\Sigma^{1-l}p^*(X)$, i.e. $\Sigma^{1-l}p^*(X)$ fits into the following $n-1$ triangles in $\mathrm{per}(\overline{A})$

$$P_1 \to \Sigma Q_0 \to \Sigma^{1-l}p^*(X) \xrightarrow{h_0} \Sigma P_1,$$

$$P_2 \to \Sigma Q_1 \to P_1 \xrightarrow{h_1} \Sigma P_2,$$

$$\cdots$$

$$P_{n-2} \to \Sigma Q_{n-3} \to P_{n-3} \xrightarrow{h_{n-3}} \Sigma P_{n-2},$$

$$\Sigma P_{n-1} \to \Sigma Q_{n-2} \to P_{n-2} \xrightarrow{h_{n-2}} \Sigma^2 P_{n-1},$$

where Q_0, Q_1, $\cdots$, Q_{n-2} and P_{n-1} are in $\mathrm{add}(\overline{A})$.

We denote by $\nu = ? \otimes_{H^0(\overline{A})} D(H^0(\overline{A}))$ the Nakayama functor on $\mathrm{mod}H^0(A)$. Then $\nu H^0(P_{n-1})$ and $\nu H^0(Q_{n-2})$ are injective $H^0(\overline{A})$-modules. Let M' be the kernel of the morphism $\nu H^0(P_{n-1}) \to \nu H^0(Q_{n-2})$. We define M to be $\Sigma^{l-1}p_*(M')$. Then it is clear that M belongs to

$$\mathcal{D}(A)_{rel}^{\geqslant -l+1} = \{X \in \mathcal{D}(A) | \, i_*(X) = 0, H^i(p^! X) \cong H^i(X) = 0, \forall i < -l\}.$$

Lemma 3.6.23. *The object $M = \Sigma^{l-1}p_*(M')$ is in $\mathcal{Z}$.*

Proof. It is clear that M belongs to $\mathrm{pvd}_B(A)$. Then M is an object in $\mathcal{Z}$ since $\mathrm{pvd}_B(A)$ is a full subcategory of $\mathcal{Z}$.

$\checkmark$

Lemma 3.6.24. *Let l be an integer. Then the subcategory $\mathcal{Z}$ of $\mathrm{per}A$ is stable under the relative truncation functors $\tau_{\leqslant l}^{rel}, \tau_{>l}^{rel} : \mathrm{per}A \to \mathrm{per}A$, i.e. $\tau_{\leqslant l}^{rel}(\mathcal{Z}) \subseteq \mathcal{Z}$ and $\tau_{>l}^{rel}(\mathcal{Z}) \subseteq \mathcal{Z}$.*

Proof. Let l be an integer and let X be an object in $\mathcal{Z}$. We have a triangle in $\mathrm{per}A$

$$\tau_{\leqslant l}^{rel} X \to X \to \tau_{>l}^{rel} X \to \Sigma \tau_{\leqslant l}^{rel} X.$$

Let L_1 be an object in $\Sigma^{>0}\mathcal{P}$. By the relative Calabi–Yau property 2.4.3 and $i_*(\tau_{>l}^{rel} X) = 0$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{per}A}(\tau_{>l}^{rel} X, eA) &\simeq D\mathrm{Hom}_{\mathrm{per}A}(eA, \Sigma^{n+1}\tau_{>l}^{rel} X) \\
&\simeq D\mathrm{Hom}_{\mathrm{per}A}(i^*(eAe), \Sigma^{n+1}\tau_{>l}^{rel} X) \\
&\simeq D\mathrm{Hom}_{\mathrm{per}(eAe)}(eAe, \Sigma^{n+1} i_*(\tau_{>l}^{rel} X)) \\
&= 0.
\end{aligned}$$

Thus, we have $\mathrm{Hom}_{\mathrm{per}A}(\tau_{>l}^{rel} X, L_1) = 0$, i.e. $\tau_{>l}^{rel} X$ is in ${}^{\perp}(\Sigma^{>0}\mathcal{P})$. And it is easy to see that $\tau_{>l}^{rel} X$ is in $(\Sigma^{<0}\mathcal{P})^{\perp}$. Thus, the object $\tau_{>l}^{rel} X$ is in $\mathcal{Z}$.

By the following exact sequence

$$\cdots \to \mathrm{Hom}_{\mathrm{per}A}(X, L_1) \to \mathrm{Hom}_{\mathrm{per}A}(\tau_{\leqslant l}^{rel} X, L_1) \to \mathrm{Hom}_{\mathrm{per}A}(\Sigma^{-1}\tau_{>l}^{rel} X, L_1) \to \cdots,$$

we can see that $\mathrm{Hom}_{\mathrm{per}A}(\tau_{\leqslant l}^{rel} X, L_1) = 0$, i.e. $\tau_{\leqslant l}^{rel} X$ is in ${}^{\perp}\Sigma^{>0}\mathcal{P}$.

Let L_2 be an object in $\Sigma^{<0}\mathcal{P}$. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{per}A}(eA, \tau_{\leqslant n}^{rel} X) &\simeq \mathrm{Hom}_{\mathrm{per}(eAe)}(eAe, i_*(\tau_{\leqslant n}^{rel} X)) \\
&\simeq \mathrm{Hom}_{\mathrm{per}(eAe)}(eAe, i_*(X)) \\
&\simeq \mathrm{Hom}_{\mathrm{per}(eAe)}(eAe, Xe).
\end{aligned}$$

Since X is in $(\Sigma^{<0}\mathcal{P})^{\perp}$, the space $\mathrm{Hom}_{\mathrm{per}A}(eA, \Sigma^k X) = H^k(Xe)$ vanishes for any positive integer k. Then we can see that $\mathrm{Hom}_{\mathrm{per}A}(L_2, \tau_{\leqslant n}^{rel} X)$ vanishes, i.e. $\tau_{\leqslant n}^{rel} X$ is in $(\Sigma^{<0}\mathcal{P})^{\perp}$. Thus $\tau_{\leqslant n}^{rel} X$ is in $\mathcal{Z}$.

$\checkmark$

Lemma 3.6.25. *Let X and Y be two objects in $\mathcal{Z}$. Let $l > 0$ be an integer. Then we have*

$$\mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^l Y) \cong \mathrm{Hom}_{\mathrm{per}(\overline{A})}(p^*(X), \Sigma^l p^*(Y)).$$

Proof. Let X and Y be two objects in $\mathcal{Z}$. For the object Y, we have the following triangle in per$A$

$$Y \xrightarrow{f_1} P_{Y_1} \to Y_1 \to \Sigma Y,$$

where $f_1 : Y \to P_{Y_1}$ is a left add(eA)-approximation and $Y_1 \in \mathcal{F}^{rel}\langle 1\rangle$.

Similarly, for the object Y_1, we have the following triangle in per$A$

$$Y_1 \xrightarrow{f_2} P_{Y_2} \to Y_2 \to \Sigma Y_1,$$

where $f_2 : Y_1 \to P_{Y_2}$ is a left add(eA)-approximation and $Y_2 \in \mathcal{F}^{rel}\langle 2\rangle$.

Repeating this process, we can get the following l triangles in per$A$

$$\begin{gathered} Y \xrightarrow{f_1} P_{Y_1} \to Y_1 \to \Sigma Y, \\ Y_1 \xrightarrow{f_2} P_{Y_2} \to Y_2 \to \Sigma Y_1, \\ \cdots \\ Y_{l-1} \xrightarrow{f_l} P_{Y_l} \to Y_l \to \Sigma Y_{l-1}, \end{gathered}$$

where for each $1 \leqslant i \leqslant l$, f_i is left add(eA)-approximation.

By the first triangle, we can see that

$$\mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^{l-1} Y_1) \cong \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^l Y).$$

Similarly, by the second triangle, we can see that

$$\mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^{l-2} Y_2) \cong \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^{l-1} Y_1).$$

Repeating this argument, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^l Y) &\cong \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^{l-1} Y_1) \\ &\cong \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^{l-2} Y_2) \\ &\cdots \\ &\cong \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma Y_{l-1}). \end{aligned}$$

By the last triangle, it induces a long exact sequence

$$\to \mathrm{Hom}_{\mathrm{per}A}(X, P_{Y_l}) \xrightarrow{\Phi} \mathrm{Hom}_{\mathrm{per}A}(X, Y_l) \to \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma Y_{l-1}) \to 0.$$

Thus we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{per}A}(X, Y[l]) &\cong \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma Y_{l-1}) \\ &\cong \mathrm{Hom}_{\mathrm{per}A}(X, Y_l)/\mathrm{Im}(\Phi) \\ &\cong \mathrm{Hom}_{\mathcal{Z}/[\mathcal{P}]}(X, Y\langle l\rangle) \\ &\cong \mathrm{Hom}_{\mathrm{per}(\overline{A})}(p^*(X), \Sigma^l p^*(Y)). \end{aligned}$$

$\surd$

Lemma 3.6.26. *[39, Lemma 3.2.9.]*
(1) There are isomorphisms of functors:

$$\mathrm{Hom}_{\mathcal{D}(\overline{A})}(?, \Sigma^{2-l}p^*(X))|_{\heartsuit(\overline{A})} \cong \mathrm{Hom}_{\mathcal{D}(\overline{A})}(?, \Sigma^2 P_1)|_{\heartsuit(\overline{A})} \cong \cdots$$
$$\cdots \cong \mathrm{Hom}_{\mathcal{D}(\overline{A})}(?, \Sigma^{n-1}P_{n-2})|_{\heartsuit(\overline{A})} \cong \mathrm{Hom}_{\heartsuit(\overline{A})}(?, M').$$

(2) There is a monomorphism of functors: $\mathrm{Ext}^1_{\heartsuit(\overline{A})}(?, M') \hookrightarrow \mathrm{Hom}_{\mathcal{D}(\overline{A})}(?, \Sigma^n P_{n-2})|_{\heartsuit(\overline{A})}$, *where* $\heartsuit(\overline{A}) = \mathrm{mod}H^0(\overline{A})$.

By the above Lemma, the following two spaces are isomorphic

$$\mathrm{Hom}_{\mathrm{per}(\overline{A})}(M', \Sigma^{2-l}p^*(X)) \cong \mathrm{Hom}_{\mathrm{per}(\overline{A})}(M', M').$$

By Lemma 3.6.25, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{per}A}(M, \Sigma X) &\simeq \mathrm{Hom}_{\mathrm{per}(\overline{A})}(p^*(M), \Sigma p^*(X)) \\
&\simeq \mathrm{Hom}_{\mathrm{per}(\overline{A})}(\Sigma^{l-1}M', \Sigma p^*(X)) \\
&\simeq \mathrm{Hom}_{\mathrm{per}(\overline{A})}(M', \Sigma^{2-l}p^*(X)) \\
&\simeq \mathrm{Hom}_{\mathrm{per}(\overline{A})}(M', M').
\end{aligned}$$

Let ϵ be the preimage of the identity map on M' under the isomorphism

$$\mathrm{Hom}_{\mathcal{D}(A)}(M, \Sigma X) \cong \mathrm{Hom}_{\mathrm{per}(\overline{A})}(M', M').$$

Then we form the corresponding triangle in $\mathrm{per}A$

$$X \longrightarrow Y \longrightarrow M \xrightarrow{\varepsilon} \Sigma X. \tag{3.4}$$

Similarly, let ε' be the the preimage of the identity map on M' under the isomorphism

$$\mathrm{Hom}_{\mathcal{D}(\overline{A})}(M', \Sigma^{2-l}p^*(X)) \cong \mathrm{Hom}_{\heartsuit(\overline{A})}(M', M')$$

Then we form the corresponding triangle in $per(\overline{A})$

$$\Sigma^{1-l}p^*(X) \to Y' \to M' \xrightarrow{\varepsilon'} \Sigma^{2-l}p^*(X).$$

Then we can see that $p^*(Y)$ is isomorphic to $\Sigma^{l-1}Y'$.

Lemma 3.6.27. *[39, Lemma 3.2.11.] The object Y' is in the fundamental domain $\mathcal{F} \subseteq$* $\mathrm{per}(\overline{A})$*.*

Lemma 3.6.28. *The object Y is in $\mathcal{F}^{rel}\langle l-1\rangle$ and $\tau^{rel}_{\leqslant l}Y$ is isomorphic to X.*

Proof. *Step 1: Y is an object in $\mathcal{F}^{rel}\langle l-1\rangle$.*
By Lemma 3.6.23, the object M is in $\mathcal{Z}$. By the triangle (3.4), we can see that Y is in $\mathcal{Z}$. Then by Lemma 3.6.27, $p^*(Y) \cong \Sigma^{l-1}Y'$ belongs to $\Sigma^{l-1}\mathcal{F}$. Thus, the object Y is in $\mathcal{F}^{rel}\langle l-1\rangle$.
Step 2: $\tau^{rel}_{\leqslant -l}Y$ is isomorphic to X.

Since $X \in \mathcal{D}(A)_{rel}^{\leqslant -l}$ and $\tau_{>-l}^{rel}(Y) = p_*\Sigma^{l-1}H^{-l+1}(p^*(Y)) \in \mathcal{D}(A)_{rel}^{\geqslant -l+1}$, the space $\mathrm{Hom}_{\mathcal{D}(A)}(X, \tau_{>-l}^{rel}(Y))$ is zero. Hence, we can obtain a commutative diagram of triangles

$$\begin{tikzcd}
\tau_{\leqslant -l}^{rel}Y \arrow[r] & Y \arrow[r] & \tau_{>-l}^{rel}(Y) \arrow[r] & \Sigma\tau_{\leqslant -l}^{rel}Y \\
X \arrow[r] \arrow[u, dashed, "\delta_2"] & Y \arrow[r] \arrow[u, equal] & M \arrow[r] \arrow[u, dashed, "\delta_1"] & \Sigma X . \arrow[u, dashed]
\end{tikzcd}$$

By the octahedral axiom, we have the following commutative diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\delta_1"] & \Sigma X \arrow[r] \arrow[d, "\delta_2[1]"] & \Sigma Y \arrow[d, equal] \\
Y \arrow[r] & \tau_{>-l}^{rel}(Y) \arrow[r] \arrow[d] & \Sigma\tau_{\leqslant -l}^{rel}Y \arrow[r] \arrow[d] & \Sigma Y \arrow[d] \\
& \mathrm{Cone}(\delta_1) \arrow[r, dashed] \arrow[d] & \Sigma\mathrm{Cone}(\delta_2) \arrow[r] \arrow[d] & \Sigma M \\
& \Sigma M \arrow[r] & \Sigma^2 X &
\end{tikzcd}$$

and the object $\mathrm{Cone}(\delta_1)$ is isomorphic to $\Sigma\mathrm{Cone}(\delta_2)$ in per$A$.

Since $\tau_{\leqslant -l}^{rel}Y \in \mathcal{D}(A)_{rel}^{\leqslant -l}$ and $X \in \mathcal{D}(A)_{rel}^{\leqslant -l}$, $\mathrm{Cone}(\delta_2)$ is also in $\mathcal{D}(A)_{rel}^{\leqslant -l}$. Thus $\Sigma\mathrm{Cone}(\delta_2)$ is in $\mathcal{D}(A)_{rel}^{\leqslant -l-1}$. On the other hand, M and $\tau_{>-l}^{rel}(Y)$ are in $\mathcal{D}_{rel}^{\geqslant -l+1}(A)$. Thus $\mathrm{Cone}(\delta_1)$ is in $\mathcal{D}_{rel}^{\geqslant -l}$. Hence we can conclude that $\mathrm{Cone}(\delta_1) \cong \Sigma\mathrm{Cone}(\delta_2)$ is zero. Thus, the relative truncation $\tau_{\leqslant -l}^{rel}Y$ of Y is isomorphic to X. $\surd$

Lemma 3.6.29. *Let $l > 0$ be an integer. The image of the functor $\tau_{\leqslant -l}^{rel}$ restricted to $\mathcal{F}^{rel}\langle l-1\rangle$ is in $\mathcal{F}^{rel}\langle l\rangle$ and the functor $\tau_{\leqslant -l}^{rel} : \mathcal{F}^{rel}\langle l-1\rangle \to \mathcal{F}^{rel}\langle l\rangle$ is fully faithful.*

Proof. *Step 1: The image of the functor $\tau_{\leqslant -l}^{rel}$ restricted to $\mathcal{F}^{rel}\langle l-1\rangle$ is in $\mathcal{F}^{rel}\langle l\rangle$.*

Let X be an object in $\mathcal{F}^{rel}\langle l-1\rangle \subseteq \mathcal{Z}$. By Lemma 3.6.24, $\tau_{\leqslant -l}^{rel}X$ is still in $\mathcal{Z}$. It is clear that $p^*(\tau_{\leqslant -l}^{rel}X) \cong \tau_{\leqslant -l}(p^*(X))$ is in $\mathcal{D}(\overline{A})^{\leqslant -l}$.

We have a triangle in per$(\overline{A})$

$$\Sigma^{-1}\tau_{>-l}(p^*(X)) \to \tau_{\leqslant -l}(p^*X) \to p^*(X) \to \tau_{>-l}(p^*(X)).$$

Let W be an object in $\mathcal{D}(\overline{A})^{\leqslant -n-l}$. The space $Hom_{\mathcal{D}(\overline{A})}(p^*(X), W)$ is zero since $p^*(X) \in ^{\perp}(\mathcal{D}(\overline{A})^{\leqslant -l-n+1})$. By the Calabi–Yau property, we have

$$\mathrm{Hom}_{\mathcal{D}(\overline{A})}(\Sigma^{-1}\tau_{>-l}(p^*(X)), W) \cong D\mathrm{Hom}_{\mathcal{D}(\overline{A})}(W, \Sigma^n\tau_{>-l}(p^*(X))).$$

The space $\mathrm{Hom}_{\mathcal{D}(\overline{A})}(W, \Sigma^n\tau_{>-l}(p^*(X)))$ vanishes because $\Sigma^n\tau_{>-l}(p^*(X)) \in \mathcal{D}(\overline{A})^{\geqslant -l-n+1}$. Thus $p^*(\tau_{\leqslant -l}^{rel}X)$ is in $\mathcal{F}[l] \subseteq per(\overline{A})$ and then $\tau_{\leqslant -l}^{rel}X$ belongs to $\mathcal{F}^{rel}\langle l\rangle$.

Step 2: The functor $\tau_{\leqslant -l}^{rel} : \mathcal{F}^{rel}\langle l-1\rangle \to \mathcal{F}^{rel}\langle l\rangle$ is fully faithful.

Let X and Y be two objects in $\mathcal{F}^{rel}\langle l-1\rangle$ and $f : \tau^{rel}_{\leqslant -l}X \to \tau^{rel}_{\leqslant -l}Y$ be a morphism.

$$\begin{tikzcd}
\Sigma^{-1}\tau^{rel}_{>-l}X \arrow[r] & \tau^{rel}_{\leqslant -l}X \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d, dotted] & \tau^{rel}_{>-l}X \\
\Sigma^{-1}\tau^{rel}_{>-l}Y \arrow[r] & \tau^{rel}_{\leqslant -l}Y \arrow[r, "g"] & Y \arrow[r] & \tau^{rel}_{>-l}Y.
\end{tikzcd}$$

By the relative Calabi–Yau property, the space $\mathrm{Hom}_{\mathcal{D}(A)}(\Sigma^{-1}\tau^{rel}_{>-l}X, Y)$ is isomorphic to $D\mathrm{Hom}_{\mathcal{D}(A)}(Y, \Sigma^{n}\tau^{rel}_{>-l}X)$. Since $Y \in^{\perp} (\mathcal{D}(A)^{\leqslant -n-l+1})$ and $\Sigma^{n}\tau^{rel}_{>-l}X \in \mathcal{D}(A)^{\leqslant -n-l+1}$, this space is zero. Then the composition gf factorizes through the canonical morphism $\tau^{rel}_{\leqslant -l}X \to X$. Thus the functor $\tau^{rel}_{\leqslant -l} : \mathcal{F}\langle l-1\rangle \to \mathcal{F}\langle l\rangle$ is full.

Now let X and Y be objects of $\mathcal{F}^{rel}\langle l-1\rangle$ and $f : X \to Y$ a morphism satisfying $\tau^{rel}_{\leqslant -l}f = 0$. Then it induces a morphism of triangles:

$$\begin{tikzcd}
\Sigma^{-1}\tau^{rel}_{>-l}X \arrow[r] \arrow[d] & \tau^{rel}_{\leqslant -l}X \arrow[r, "h"] \arrow[d, "0"] & X \arrow[r] \arrow[d, "f"] & \tau^{rel}_{>-l}X \arrow[d] \arrow[dl, dotted] \\
\Sigma^{-1}\tau^{rel}_{>-l}Y \arrow[r] & \tau^{rel}_{\leqslant -l}Y \arrow[r] & Y \arrow[r] & \tau^{rel}_{>-l}Y.
\end{tikzcd}$$

The composition $f \circ h$ vanishes, so f factorizes through $\tau^{rel}_{>-l}X$. By the relative Calabi–Yau property, the space $\mathrm{Hom}_{(D)(A)}(\tau^{rel}_{>-l}X, Y)$ is isomorphic to $D\mathrm{Hom}_{(D)(A)}(Y, \Sigma^{n+1}\tau^{rel}_{>-l}X)$ which is zero because Y lies in ${}^{\perp}(\mathcal{D}(A)^{\leqslant -n-l+1})$ and $\Sigma^{n+1}\tau^{rel}_{>-l}X \in \mathcal{D}(A)^{\leqslant -l-n}$. Thus $f = 0$, i.e, the functor

$$\tau^{rel}_{\leqslant -l} : \mathcal{F}^{rel}\langle l-1\rangle \to \mathcal{F}^{rel}\langle l\rangle$$

is faithful.

$\surd$

Proposition 3.6.30. *For any positive integer l, the functor $\tau^{rel}_{\leqslant -l}$ induces an equivalence from $\mathcal{F}^{rel}\langle l-1\rangle$ to $\mathcal{F}^{rel}\langle l\rangle$.*

Proof. This follows from Lemma 3.6.28 and Lemma 3.6.29. $\surd$

Proposition 3.6.31. *Let X and Y be two objects in the relative fundamental domain $\mathcal{F}^{rel}$. Let $l > 0$ be an integer. Then we have*

$$\mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^{l}Y) \cong \mathrm{Hom}_{\mathrm{per}(\overline{A})}(p^{*}(X), \Sigma^{l}p^{*}(Y))$$

Proof. This follows from Lemma 3.6.25. $\surd$

Proposition 3.6.32. *Let X and Y be two objects in the Higgs category $\mathcal{H}$. Let $l > 0$ be an integer. Then we have*

$$\mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma^{l}Y) \cong \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^{*}(X), \Sigma^{l}p^{*}(Y))$$

Proof. For the object Y, we have the following triangle in $\mathrm{per}A$

$$Y \xrightarrow{f_1} P_{Y_1} \xrightarrow{g_1} Y_1 \to \Sigma Y,$$

where $f_1 : Y \to P_{Y_1}$ is a left add(eA)-approximation and $Y_1 \in \mathcal{F}^{rel}\langle 1\rangle$. Since $\tau^{rel}_{\leqslant -1} : \mathcal{F}^{rel} \to \mathcal{F}^{rel}\langle 1\rangle$ is an equivalence, there is an object $W_1 \in \mathcal{F}^{rel}$ such that $\tau^{rel}_{\leqslant -1} W_1 \cong Y_1$. Thus we get a triangle in $\mathcal{C}_n(A, B)$

$$Y \xrightarrow{\pi^{rel}(f_1)} P_{Y_1} \xrightarrow{\pi^{rel}(g_1)} W_1 \to \Sigma Y.$$

For the object W_1, we have the following triangle in per$A$

$$W_1 \xrightarrow{f_2} P_{Y_2} \xrightarrow{g_2} Y_2 \to \Sigma W_1,$$

where $f_2 : W_1 \to P_{Y_2}$ is a left add(eA)-approximation and $Y_2 \in \mathcal{F}^{rel}\langle 1\rangle$. By the same reason, there is an object $W_2 \in \mathcal{F}^{rel}$ such that $\tau^{rel}_{\leqslant -1} W_2 \cong Y_2$. Thus we get a triangle in $\mathcal{C}_n(A, B)$

$$W_1 \xrightarrow{\pi^{rel}(f_2)} P_{Y_2} \xrightarrow{\pi^{rel}(g_2)} W_2 \to \Sigma W_1.$$

Repeating this process, we can get the following l triangles in per$A$

$$Y \xrightarrow{f_1} P_{Y_1} \xrightarrow{g_1} Y_1 \to \Sigma Y,$$

$$W_1 \xrightarrow{f_2} P_{Y_2} \xrightarrow{g_1} Y_2 \to \Sigma W_1,$$

$$\cdots$$

$$W_{l-2} \xrightarrow{f_{l-1}} P_{Y_{l-1}} \xrightarrow{g_{l-1}} Y_{l-1} \to \Sigma W_{l-2},$$

$$W_{l-1} \xrightarrow{f_l} P_{Y_l} \xrightarrow{g_l} Y_l \to \Sigma W_{l-1}$$

where for each $1 \leqslant i \leqslant l$, f_i is a left add(eA)-approximation, Y_i is in $\mathcal{F}^{rel}\langle 1\rangle$ and $\tau^{rel}_{\leqslant -1} W_i \cong Y_i$.

Thus we get l triangles in $\mathcal{C}_n(A, B)$

$$Y \xrightarrow{\pi^{rel}(f_1)} P_{Y_1} \xrightarrow{\pi^{rel}(g_1)} W_1 \to \Sigma Y,$$

$$W_1 \xrightarrow{\pi^{rel}(f_2)} P_{Y_2} \xrightarrow{\pi^{rel}(g_1)} W_2 \to \Sigma W_1,$$

$$\cdots$$

$$W_{l-2} \xrightarrow{\pi^{rel}(f_{l-1})} P_{Y_{l-1}} \xrightarrow{\pi^{rel}(g_{l-1})} W_{l-1} \to \Sigma W_{l-2},$$

$$W_{l-1} \xrightarrow{\pi^{rel}(f_l)} P_{Y_l} \xrightarrow{\pi^{rel}(g_l)} W_l \to \Sigma W_{l-1}.$$

Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, Y[l]) &\simeq \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma^{l-1} W_1) \\ &\simeq \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma^{l-2} W_2) \\ &\cdots \\ &\simeq \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma W_{l-1}) \end{aligned}$$

By the last triangle, we have the following exact sequence

$$\to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, P_{Y_l}) \xrightarrow{\Phi} \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, W_l) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma W_{l-1}) \to 0.$$

Thus, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_n}(A,B)(X, Y[l]) &\simeq \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma W_{l-1}) \\
&\simeq \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, W_l)/Im(\Phi) \\
&\simeq \mathrm{Hom}_{\mathrm{per}A}(X, W_l)/Im(\Phi) \\
&\simeq \mathrm{Hom}_{\mathrm{per}(\overline{A})}(p^*(X), p^*(W_l)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(X), p^*(W_l)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(X), \tau_{\leqslant -1} p^*(W_l)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(X), p^*(\tau^{rel}_{\leqslant -1}(W_l))) \\
&\simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(X), p^*(Y_l)) \\
&\simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(X), \Sigma^l p^*(Y)).
\end{aligned}$$

$\surd$

Proposition 3.6.33. *[39, Proposition 4.8.1.] Suppose that X and Y are two objects in $\mathcal{F} \subseteq \mathcal{C}_n(\overline{A})$. Then there is a long exact sequence*

$$\begin{gathered}
0 \to \mathrm{Ext}^1_{\mathcal{D}(\overline{A})}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}_n(\overline{A})}(X,Y) \to D\mathrm{Ext}^{n-1}_{\mathcal{D}(\overline{A})}(X,Y) \\
\to \mathrm{Ext}^2_{\mathcal{D}(\overline{A})}(X,Y) \to \mathrm{Ext}^2_{\mathcal{C}_n(\overline{A})}(X,Y) \to D\mathrm{Ext}^{n-2}_{\mathcal{D}(\overline{A})}(X,Y) \\
\to \quad \cdots \quad \to \\
\to \mathrm{Ext}^{n-1}_{\mathcal{D}(\overline{A})}(X,Y) \to \mathrm{Ext}^{n-1}_{\mathcal{C}_n(\overline{A})}(X,Y) \to D\mathrm{Ext}^1_{\mathcal{D}(\overline{A})}(X,Y) \to 0.
\end{gathered}$$

Corollary 3.6.34. *Suppose that X and Y are two objects in the Higgs category $\mathcal{H} \subseteq \mathcal{C}_n(A,B)$. Then there is a long exact sequence*

$$\begin{gathered}
0 \to \mathrm{Ext}^1_{\mathcal{D}(A)}(X,Y) \to \mathrm{Ext}^1_{\mathcal{C}_n(A,B)}(X,Y) \to D\mathrm{Ext}^{n-1}_{\mathcal{D}(A)}(X,Y) \\
\to \mathrm{Ext}^2_{\mathcal{D}(A)}(X,Y) \to \mathrm{Ext}^2_{\mathcal{C}_n(A,B)}(X,Y) \to D\mathrm{Ext}^{n-2}_{\mathcal{D}(\overline{A})}(X,Y) \\
\to \quad \cdots \quad \to \\
\to \mathrm{Ext}^{n-1}_{\mathcal{D}(A)}(X,Y) \to \mathrm{Ext}^{n-1}_{\mathcal{C}_n(A,B)}(X,Y) \to D\mathrm{Ext}^1_{\mathcal{D}(A)}(X,Y) \to 0.
\end{gathered}$$

Proof. This follows from Proposition 3.6.31, Proposition 3.6.32 and Proposition 3.6.33.

$\surd$

Proposition 3.6.35. *The Higgs category $\mathcal{H}$ is an extension closed subcategory of $\mathcal{C}_n(A,B)$.*

Proof. Let X and Y be two objects in $\mathcal{H} \subseteq \mathcal{C}_n(A,B)$. For the object Y, we take a triangle in per$A$

$$Y \xrightarrow{f_1} P_{Y_1} \to Y_1 \xrightarrow{\phi_1} \Sigma Y,$$

where $f_1 : Y \to P_{Y_1}$ is a fixed left $\mathrm{add}(eA)$-approximation and $Y_1 \in \mathcal{F}^{rel}\langle 1\rangle$. Then we can get a triangle in $\mathcal{C}_n(A,B)$

$$Y \xrightarrow{\pi^{rel}(f_1)} P_{Y_1} \to Y_1 \xrightarrow{\pi^{rel}(\phi_1)} \Sigma Y.$$

This induces a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, Y_1) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma Y) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma P_{Y_1}) \to \cdots .$$

Since $P_{Y_1} \in \mathrm{pvd}_B(A)^{\perp}$, we have

$$\mathrm{Hom}_{\mathcal{C}_n^{rel}}(X, \Sigma P_{Y_1}) \cong \mathrm{Hom}_{\mathrm{per}A}(X, \Sigma P_{Y_1}) = 0.$$

Thus we get the following surjective map

$$\cdots \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, Y_1) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma Y) \to 0.$$

For the object X, we have a canonical triangle in $\mathrm{per}A$

$$\tau^{rel}_{\leqslant -1} X \to X \to \tau^{rel}_{\geqslant 0} X \to \Sigma \tau^{rel}_{\leqslant -1} X.$$

Hence, $\tau^{rel}_{\leqslant -1} X$ is isomorphic to X in $\mathcal{C}_n(A,B)$. Then we get the following exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(\tau^{rel}_{\leqslant -1} X, Y_1) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma Y) \to 0.$$

It is clear that $\tau^{rel}_{\leqslant -1} X$ and Y_1 are in $\mathcal{F}^{rel}\langle 1\rangle$. Since $\pi^{rel} : \mathcal{F}^{rel} \cong \mathcal{F}^{rel}\langle 1\rangle \to \mathcal{C}_n(A,B)$ is also fully faithful, we have that the space $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\tau^{rel}_{\leqslant -1} X, Y_1)$ is isomorphic to $\mathrm{Hom}_{perA}(\tau^{rel}_{\leqslant -1} X, Y_1)$ and the following sequence is exact

$$\cdots \to \mathrm{Hom}_{perA}(\tau^{rel}_{\leqslant -1} X, Y_1) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma Y) \to 0.$$

Let ϵ be an element in $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma Y)$. We suppose that the corresponding triangle in $\mathcal{C}_n(A,B)$ is given by

$$Y \to W \to X \xrightarrow{\varepsilon} \Sigma Y.$$

We need to show that W is also in $\mathcal{H}$.

Since the map $\mathrm{Hom}_{\mathrm{per}A}(\tau^{rel}_{\leqslant -1} X, Y_1) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma Y)$ is surjective, there is a morphism $\epsilon' : \tau^{rel}_{\leqslant -1} X \to Y_1$ in $\mathrm{per}A$ such that $\pi^{rel}(\phi_1 \circ \varepsilon') \cong \varepsilon$ in $\mathcal{C}_n(A,B)$.

We take a triangle in $\mathrm{per}A$

$$Y \to W_1 \to \tau^{rel}_{\leqslant -1} X \xrightarrow{\phi_1 \circ \varepsilon'} \Sigma Y.$$

Then, the following morphism of triangles in $\mathcal{C}_n(A,B)$ is an isomorphism

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "1_Y"'] & W_1 \arrow[r] \arrow[d] & \tau^{rel}_{\leqslant -1} X \arrow[r, "\pi^{rel}(\phi_1 \circ \varepsilon')"] \arrow[d] & \Sigma Y \arrow[d, "1_Y"] \\
Y \arrow[r] & W \arrow[r] & X \arrow[r, "\varepsilon"] & \Sigma Y.
\end{tikzcd}$$

In particular, W_1 is isomorphic to W in $\mathcal{C}_n(A,B)$.

Since Y and $\tau^{rel}_{\leqslant -1}X$ are in $\mathcal{Z} \subseteq \mathrm{per}A$, W_1 is also in $\mathcal{Z}$. It is easy to see that

$$p^*(Y) \in \mathcal{F} = \mathcal{D}(\overline{A})^{\leqslant 0} \cap^{\perp} (\mathcal{D}(\overline{A})^{\leqslant -n}) \cap \mathrm{per}(\overline{A})$$

and

$$p^*(\tau^{rel}_{\leqslant -1}X) \cong \tau_{\leqslant -1}(p^*(X)) \in \Sigma\mathcal{F} = \mathcal{D}(\overline{A})^{\leqslant -1} \cap^{\perp} (\mathcal{D}(\overline{A})^{\leqslant -n-1}) \cap \mathrm{per}(\overline{A}).$$

Then by the triangle in $\mathrm{per}A$

$$Y \to W_1 \to \tau^{rel}_{\leqslant -1}X \xrightarrow{\phi_1 \circ \varepsilon'} \Sigma Y,$$

we can see that $p^*(W_1)$ is in $\mathcal{D}(\overline{A})^{\leqslant 0} \cap^{\perp} (\mathcal{D}(\overline{A})^{\leqslant -n-1}) \cap \mathrm{per}(\overline{A})$.

Next we consider the object $\tau^{rel}_{\leqslant -1}W_1 \in \mathrm{per}A$. Since W_1 is in $\mathcal{Z}$, $\tau^{rel}_{\leqslant -1}W_1$ is still in $\mathcal{Z}$. And we have a canonical triangle in $\mathrm{per}(\overline{A})$

$$\tau_{\leqslant -1}(p^*(W_1)) \to p^*(W_1) \to \tau_{\geqslant 0}(p^*(W_1)) \to \Sigma\tau_{\leqslant -1}(p^*(W_1)).$$

Because that $p^*(W_1)$ is in $\mathcal{D}(\overline{A})^{\leqslant 0} \cap^{\perp} (\mathcal{D}(\overline{A})^{\leqslant -n-1}) \cap per(\overline{A})$, we have

$$\tau_{\leqslant -1}(p^*(W_1)) \in \mathcal{D}(\overline{A})^{\leqslant -1} \cap^{\perp} (\mathcal{D}(\overline{A})^{\leqslant -n-1}) \cap per(\overline{A}) = \Sigma\mathcal{F}.$$

Thus the object $\tau^{rel}_{\leqslant -1}W_1$ is in $\mathcal{F}^{rel}\langle 1\rangle$. By the equivalence $\tau^{rel}_{\leqslant -1} : \mathcal{F}^{rel} \to \mathcal{F}^{rel}\langle 1\rangle$, there exist an object $W_2 \in \mathcal{F}^{rel}$ such that $\tau^{rel}_{\leqslant -1}W_2 \cong \tau^{rel}_{\leqslant -1}W_1$.

Since W_2 and W_1 are isomorphic in $\mathcal{C}_n(A,B)$, W_2 is isomorphic to W in $\mathcal{C}_n(A,B)$. Thus W is an object in $\mathcal{H} \subseteq \mathcal{C}_n(A,B)$. Therefore, $\mathcal{H}$ is an extension closed subcategory of $\mathcal{C}_n(A,B)$.

$\surd$

Recall that a full subcategory $\mathcal{P}$ of a triangulated category $\mathcal{T}$ is presilting if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \Sigma^{>0}\mathcal{P}) = 0$.

Proposition 3.6.36. *(1) $\mathcal{P} = \mathrm{add}(eA)$ is a presilting subcategory of $\mathcal{C}_n(A,B) = \mathrm{per}A/\mathrm{pvd}_B(A)$.*

(2) $\mathcal{P}$ is covariantly finite in ${}^{\perp_{\mathcal{C}_n(A,B)}}(\Sigma^{>0}\mathcal{P})$ and contravariantly finite in $(\Sigma^{<0}\mathcal{P})^{\perp_{\mathcal{C}_n(A,B)}}$.

(3) For any $X \in \mathcal{C}_n(A,B)$, we have $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma^l P) = 0 = \mathrm{Hom}_{\mathcal{C}_n(A,B)}(P, \Sigma^l X)$ for $l \gg 0$.

Proof. For $P \in \mathcal{P}$, $X \in \mathrm{per}A$ and $m \in \mathbb{Z}$, we have isomorphisms

$$\mathrm{Hom}_{\mathrm{per}A}(P, \Sigma^m X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_n(A,B)}(P, \Sigma^m X)$$

and

$$\mathrm{Hom}_{\mathrm{per}A}(X, \Sigma^m P) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma^m P)$$

because $\mathcal{P}$ is left orthogonal and right orthogonal to $\mathrm{pvd}_B(A)$. This implies (1), (2) and (3). $\surd$

Corollary 3.6.37. *Let $\mathcal{E}$ be the following additive subcategory of $\mathcal{C}_n(A,B)$*

$$\mathcal{E} = {}^{\perp_{\mathcal{C}_n(A,B)}}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathcal{C}_n(A,B)}}.$$

Then the composition $\mathcal{E} \subseteq \mathcal{C}_n(A,B) \xrightarrow{p^*} \mathcal{C}_n(\overline{A})$ induces a triangle equivalence

$$\mathcal{E}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{C}_n(\overline{A}).$$

Proof. This follows from Proposition 3.6.36 and Theorem 3.6.3. $\surd$

Theorem 3.6.38. *The Higgs category $\mathcal{H} \subseteq \mathcal{C}_n(A,B)$ is equal to $\mathcal{E} = {}^{\perp_{\mathcal{C}_n(A,B)}}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathcal{C}_n(A,B)}}$. In particular, the Higgs category $\mathcal{H}$ is idempotent complete.*

Proof. It is clear that we have the inclusion $\mathcal{H} \subseteq \mathcal{E}$. Let X be an object in $\mathcal{E} = {}^{\perp_{\mathcal{C}_n(A,B)}}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathcal{C}_n(A,B)}}$.

Since $\mathrm{per}(eAe)$ and $\mathrm{pvd}_B(A)$ are left orthogonal and right orthogonal to each other, we see that X is in $\mathcal{Z} = {}^{\perp_{\mathrm{per}A}}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp_{\mathrm{per}A}} \subseteq \mathrm{per}A$.

For the object $p^*(X) \in \mathrm{per}(\overline{A})$, there exists a non-negative integer r such that $p^*(X)$ is in ${}^{\perp}(\mathcal{D}(\overline{A})^{\leqslant -n-r})$. We consider the object $X' = \tau^{rel}_{\leqslant -r}X$. Then X' becomes isomorphic to X in $\mathcal{C}_n(A,B)$ and X' belongs to $\mathcal{F}^{rel}\langle r\rangle$. By Proposition 3.6.30, there exists an object Y in $\mathcal{F}^{rel}$ such that Y is isomorphic to X' in $\mathcal{C}_n(A,B)$. Thus, X is in the image of $\mathcal{F}^{rel}$, i.e. X belongs to $\mathcal{H}$. Hence $\mathcal{H}$ is equal to $\mathcal{E}$.

By Corollary 3.4.4, $\mathcal{H}$ is idempotent complete.

$\surd$

Theorem 3.6.39. *For any object $X \in \mathcal{C}_n(A,B)$, there exists $l \in \mathbb{Z}$, $F \in \mathcal{H}$ and $P \in \mathrm{per}(eAe)$, such that we have a triangle in $\mathcal{C}_n(A,B)$*

$$\Sigma^l F \longrightarrow X \longrightarrow P \longrightarrow \Sigma^{l+1}F\,.$$

Dually, there exist $m \in \mathbb{Z}$, $F' \in \mathcal{H}$ and $P' \in \mathrm{per}(eAe)$, such that we have a triangle in $\mathcal{C}_n(A,B)$

$$P' \longrightarrow X \longrightarrow \Sigma^m F' \longrightarrow \Sigma P'\,.$$

Proof. We only show the first statement since the second statement can be shown dually. Let X be an object in $\mathcal{C}_n(A,B)$. We view it as an object in $\mathrm{per}A$. There exists a positive integer r_1 such that the object X is in $\mathcal{D}(A)^{\leqslant r_1}$. We set $Y = \Sigma^{r_1}X$. Then Y is in $\mathcal{D}(A)^{\leqslant 0}$.

By Proposition 3.6.2, the pairs $({}^{\perp_{\mathcal{T}}}\mathcal{S}_{<0}, \mathcal{S}_{\leqslant 0})$ and $(\mathcal{S}_{\geqslant 0}, \mathcal{S}_{>0}^{\perp_{\mathcal{T}}})$ are co-t-structures on $\mathcal{T} = \mathrm{per}A$, where

$$\mathcal{S}_{\geqslant l} = \mathcal{S}_{>l-1} := \bigcup_{i\geqslant 0} \Sigma^{-l-i}\mathcal{P} * \cdots * \Sigma^{-l-1}\mathcal{P} * \Sigma^{-l}\mathcal{P},$$

$$\mathcal{S}_{\leqslant l} = \mathcal{S}_{<l+1} := \bigcup_{i\geqslant 0} \Sigma^{-l}\mathcal{P} * \Sigma^{-l+1}\mathcal{P} \cdots * \Sigma^{-l+i}\mathcal{P}$$

and $\mathcal{P} = \mathrm{add}(eA)$. Hence we have a triangle

$$X' \to Y \to S \to \Sigma X'$$

where $X' \in {}^{\perp}(\mathcal{S}_{<0})$ and $S \in \mathcal{S}_{<0} \subseteq \mathcal{D}(A)^{\leqslant -1}$. And we can see that X' belongs to $\mathcal{D}(A)^{\leqslant 0}$.

Step 1: The object X' is in $\mathcal{Z} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp}$.

Since $X' \in {}^{\perp}(\mathcal{S}_{<0})$, it is enough to show that X' is also in $(\Sigma^{<0}\mathcal{P})^{\perp}$. For any positive integer k, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(A)}(eA, \Sigma^k X') &\cong \mathrm{Hom}_{\mathcal{D}(A)}(i^*(eAe), \Sigma^k X')\\ &\cong \mathrm{Hom}_{\mathcal{D}(eAe)}(eAe, \Sigma^k i_*(X')).\end{aligned}$$

The space $\mathrm{Hom}_{\mathcal{D}(A)}(eA, \Sigma^k X')$ vanishes for any positive integer k. Thus the object X' is in $\mathcal{Z}$.

Step 2: There exists an object $W \in \mathcal{F}^{rel}$ such that W is isomorphic to X' in $\mathcal{C}_n(A,B)$.

By Step 1, the object X' is in $\mathcal{Z} \subseteq \mathrm{per}A$. Thus, there is a non-negative integer r_2 such that $p^*(X') \in^{\perp} (\mathcal{D}(\overline{A})^{\leqslant -n-r_2})$. We consider the object $W' = \tau^{rel}_{\leqslant -r_2} W$. Then W' is isomorphic to W in $\mathcal{C}_n(A,B)$ and W' belongs to $\mathcal{F}^{rel}\langle r\rangle$. By Proposition 3.6.30, there exists an object W'' in $\mathcal{F}^{rel}$ such that W'' is isomorphic to W' in $\mathcal{C}^{rel}_n(A,B)$. Thus, we get the following triangle in $\mathcal{C}_n(A,B)$

$$W'' \to \Sigma^{r_1} X \to S \to \Sigma W'',$$

where W'' is in $\mathcal{H}$ and S is in $\mathrm{per}(eAe)$.

$\surd$

3.6.6 Frobenius n-exangulated categories

In this subsection, we describe our results using the framework of n-exangulated categories. We refer to the readers to [71], [42] and [67] for the relevant Definitions and facts concerning n-exangulated categories.

Definition 3.6.40. [67, Definition 3.2.] Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an n-exangulated category.

(1) An object $P \in \mathcal{C}$ is called *projective* if, for any distinguished n-exangle

$$A_0 \xrightarrow{\alpha_0} A_1 \to \cdots \to A_n \xrightarrow{\alpha_n} A_{n+1} \overset{\delta}{\dashrightarrow}$$

and any morphism c in $\mathcal{C}(P, A_{n+1})$, there exists a morphism $b \in \mathcal{C}(P, A_n)$ satisfying $\alpha_n b = c$. We denote the full subcategory of projective objects in $\mathcal{C}$ by $\mathcal{P}$. Dually, the full subcategory of injective objects in $\mathcal{C}$ is denoted by $\mathcal{I}$.

(2) We say that $\mathcal{C}$ has *enough projectives* if for any object $C \in \mathcal{C}$, there exists a distinguished n-exangle

$$B \xrightarrow{\alpha_0} P_1 \to \cdots \to P_n \xrightarrow{\alpha_n} C \overset{\delta}{\dashrightarrow}$$

satisfying $P_1, P_2, \cdots, P_n \in \mathcal{P}$. We can define the notion of *having enough injectives* dually.

(3) $\mathcal{C}$ is said to be *Frobenius* if $\mathcal{C}$ has enough projectives and enough injectives and if moreover the projectives coincide with the injectives.

Remark 3.6.41. In the case $n = 1$, these agree with the usual definitions [71, Definition 3.23, Definition 3.25 and Definition 7.1].

Theorem 3.6.42. *The Higgs category $\mathcal{H}$ carries a canonical structure of Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(eA)$. The functor $p^* : \mathcal{C}_n(A,B) \to \mathcal{C}_n(\overline{A})$ induces an equivalence of triangulated categories*

$$\mathcal{H}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{C}_n(\overline{A}).$$

Proof. *Step 1: $\mathcal{H}$ is an extriangulated category.*

By Proposition 3.6.35, the Higgs category $\mathcal{H}$ is an extension closed subcategory of $\mathcal{C}_n(A,B)$. Then by [71, Remark 2.18.], $\mathcal{H}$ is an extriangulated category and $(\mathcal{H},\mathbb{E},\mathfrak{s})$ can be described as follows:

(1) For any two objects $X,Y\in\mathcal{H}\subseteq\mathcal{C}_n^{rel}(A,B)$, the $\mathbb{E}$-extension space $\mathbb{E}(X,Y)$ is given by $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(X,\Sigma Y)$.

(2) For any $\delta\in\mathbb{E}(X,Y)=\mathrm{Hom}_{\mathcal{C}_n(A,B)}(Z,\Sigma X)$, take a distinguished triangle

$$X\xrightarrow{f}Y\xrightarrow{g}Z\xrightarrow{\delta}\Sigma X$$

and define $\mathfrak{s}(\delta)=[X\xrightarrow{f}Y\xrightarrow{g}Z]$. This $\mathfrak{s}(\delta)$ does not depend on the choice of the distinguished triangle above.

Step 2: $\mathcal{H}$ has has enough injectives and the full subcategory of injective objects in $\mathcal{H}$ is $\mathcal{P}=\mathrm{add}(eA)$.

Let I be an object in $\mathrm{add}(eA)$. For any distinguished triangle in $\mathcal{H}$

$$X\rightarrow Y\rightarrow Z\overset{\delta}{-\rightarrow}.$$

The space $\mathrm{Hom}_{\mathcal{C}_n(A,B}(\Sigma^{-1}Z,I)\cong\mathrm{Hom}_{\mathrm{per}A}(Z,\Sigma I)$ vanishes since $Z\in\mathcal{Z}={}^{\perp}(\mathcal{P}[>0])\cap\mathcal{P}[<0]^{\perp}\subseteq\mathrm{per}A$. Thus, we have the following exact sequence

$$\mathrm{Hom}_{\mathcal{C}_n(A,B)}(Y,I)\rightarrow\mathrm{Hom}_{\mathcal{C}_n(A,B)}(X,I)\rightarrow 0.$$

Thus, any object in $\mathrm{add}(eA)$ is injective.

Now let X be an object in $\mathcal{H}\subseteq\mathcal{C}_n(A,B)$. Then X is an object in $\mathcal{Z}\subseteq\mathrm{per}A$. We take a triangle in $\mathrm{per}A$

$$X\xrightarrow{l_X}P_X\rightarrow X_1\rightarrow\Sigma X$$

with a left $\mathcal{P}=\mathrm{add}(eA)$-approximation l_X and $X_1\in\mathcal{Z}$. It is easy to see that X_1 is in $\mathcal{F}^{rel}\langle 1\rangle$. By Proposition 3.6.30, there is an object $X_2\in\mathcal{F}^{rel}$ such that $\tau^{rel}_{\leqslant -1}X_2\cong X_1$. Thus, we have a triangle in $\mathcal{C}_n(A,B)$

$$X\xrightarrow{l_X}P_X\rightarrow X_2\rightarrow\Sigma X$$

with P_X in $\mathrm{add}(eA)$ and l_X an inflation. Therefore, $\mathcal{H}$ has has enough injectives.

It remains to show that any injective object is in $\mathrm{add}(eA)$. Let J be an injective object in $\mathcal{H}$. We take a triangle in $\mathrm{per}A$

$$J\xrightarrow{l_J}P_J\rightarrow J_1\rightarrow\Sigma J$$

with a left $\mathcal{P}=\mathrm{add}(eA)$-approximation l_J and $J_1\in\mathcal{Z}$. Since J is injective, the morphism $l_J:J\rightarrow P_J$ is split in $\mathcal{H}\subseteq\mathcal{C}_n^{rel}(A,B)$. Thus l_J is also split in $\mathcal{F}^{rel}\subseteq\mathcal{Z}\subseteq\mathrm{per}A$.

Therefore, J belongs to $\mathrm{add}(eA)$ and the subcategory of injective objects in $\mathcal{H}$ is $\mathcal{P}=\mathrm{add}(eA)$.

Step 3: $\mathcal{H}$ has has enough projectives and the full subcategory of projective objects in $\mathcal{H}$ is $\mathcal{P}=\mathrm{add}(eA)$.

This follows from the dual of the argument in Step 2.

Step 4: $\mathcal{H}$ is a Frobenius extriangulated category.

By Steps 1, 2, and 3, the Higgs category $\mathcal{H}$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(eA)$. By Corollary 3.6.16, we have the equivalence between triangulated categories

$$\mathcal{H}/[\mathcal{P}] \cong \mathcal{F} \cong \mathcal{C}_n(\overline{A}).$$

$\surd$

3.6.7 Higher extensions in an extriangulated category

Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Assume that it has enough projectives and injectives, and let $\mathcal{P} \subseteq \mathcal{C}$ (respectively, $\mathcal{I} \subseteq \mathcal{C}$) denote the full subcategory of projectives (resp. injectives). We denote the ideal quotients $\mathcal{C}/[\mathcal{P}]$ and $\mathcal{C}/[\mathcal{I}]$ by $\underline{\mathcal{C}}$ and $\overline{\mathcal{C}}$, respectively. The extension group bifunctor $\mathbb{E} : \mathcal{C}^{op} \times \mathcal{C} \to Ab$ induces $\mathbb{E} : \underline{\mathcal{C}^{op}} \times \overline{\mathcal{C}} \to Ab$, which we denote by the same symbol. To define the higher extension groups, we need the following assumptions

Assumption 3. Each object $A \in \mathcal{C}$ is assigned the following data (i) and (ii).

(i) A pair $(\Sigma A, l^A)$ of an object $\Sigma A \in \mathcal{C}$ and an extension $l^A \in \mathbb{E}(\Sigma A, A)$, for which $\mathfrak{s}(l^A) = [A \to I \to \Sigma A]$ satisfies $I \in \mathcal{I}$.

(ii) A pair $(\Omega A, \omega^A)$ of an object $\Omega A \in \mathcal{C}$ and an extension $\omega^A \in \mathbb{E}(A, \Omega A)$, for which $\mathfrak{s}(\omega^A) = [\Omega A \to P \to A]$ satisfies $P \in \mathcal{P}$.

Definition 3.6.43. [42, Definition 5.4.] Let $i \geqslant 1$ be any integer. Define a biadditive functor $\mathbb{E}^i : \mathcal{C}^{op} \times \mathcal{C} \to Ab$ to be the composition of

$$\mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathcal{C}^{op}} \times \overline{\mathcal{C}} \xrightarrow{\mathrm{Id} \times \Sigma^{i-1}} \underline{\mathcal{C}^{op}} \times \overline{\mathcal{C}} \xrightarrow{\mathbb{E}} Ab,$$

where Σ^{i-1} is the $(i-1)$-times iteration of the endfunctor Σ.

Dually, define $\mathbb{E}^i_{\dagger} : \mathcal{C}^{op} \times \mathcal{C} \to Ab$ to be the composition of

$$\mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathcal{C}^{op}} \times \overline{\mathcal{C}} \xrightarrow{\Omega^{i-1} \times Id} \underline{\mathcal{C}^{op}} \times \overline{\mathcal{C}} \xrightarrow{\mathbb{E}} Ab,$$

where Ω^{i-1} is the $(i-1)$-times iteration of the endfunctor Ω.

Proposition 3.6.44. *[42, Proposition 5.9.] Let i be a positive integer. we have natural isomorphism*

$$\mathbb{E}^i_{\dagger} \xrightarrow{\cong} \mathbb{E}^i.$$

Thus, for any pair of objects $X, Y \in \mathcal{C}$, we have $\mathbb{E}^i_{\dagger}(X, Y) \cong \mathbb{E}^i(X, Y)$.

By Theorem 3.6.42, the Higgs category $\mathcal{H}$ is a Frobenius extriangulated category (or Frobenius 1-exangulated category) with projective-injective objects $\mathcal{P} = \mathrm{add}eA$. Thus the higher extension can be computed as follows:

Let X and Y be two objects in $\mathcal{H}$. Let $l > 0$ be an integer. We have

$$\mathbb{E}^l(X, Y) = \mathrm{Hom}_{\mathcal{Z}/[\mathcal{P}]}(X, Y\langle l \rangle) \cong \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(A), \Sigma^l p^*(Y)) \cong \mathrm{Hom}_{\mathcal{C}_n(A,B)}(X, \Sigma^l Y).$$

Definition 3.6.45. [42, Definition 5.19.] Let $\mathcal{T} \subseteq \mathcal{C}$ be a full additive subcategory closed under isomorphisms and direct summands. Then $\mathcal{T}$ is called an *n-cluster tilting subcategory* of $\mathcal{C}$, if it satisfies the following conditions.

(1) $\mathcal{T} \subseteq \mathcal{C}$ is functorially finite.

(2) For any $C \in \mathcal{C}$, the following are equivalent.
   (i) $C \in \mathcal{T}$,
   (ii) $\mathbb{E}^i(C, \mathcal{T}) = 0$ for any $1 \leqslant i \leqslant n-1$,
   (iii) $\mathbb{E}^i(\mathcal{T}, C) = 0$ for any $1 \leqslant i \leqslant n-1$.

Proposition 3.6.46. *The category* add$A$ *is an n-cluster-tilting subcategory of* $\mathcal{H}$.

Proof. Since $\mathcal{H}$ is Hom-finite, it is clear that add(A) is functorially finite in $\mathcal{H}$.
Step 1: $\pi^{rel}(A)$ is an n-rigid object in $\mathcal{H}$.
By Proposition 3.6.32, we have that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\pi^{rel}(A), \Sigma^i \pi^{rel}(A)) &\simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(\overline{A}, \Sigma^i \overline{A}) \\ &\simeq 0\end{aligned}$$

for any $1 \leqslant i \leqslant n-1$. Therefore, the endomorphism algebra of $\pi^{rel}(A)$ is isomorphic to the zeroth homology $H^0(A)$ of A and

$$\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\pi^{rel}(A), \Sigma^i \pi^{rel}(A)) = 0, \quad i = 1, \cdots, n-1.$$

Step 2: Let X be an object in $\mathcal{H}$ satisfying $\mathbb{E}^i(X, \mathrm{add}A) = 0$ for $1 \leqslant i \leqslant n-1$. Then X is in add$A$.

Since $\mathbb{E}^i(X, \mathrm{add}A) = 0$ for $1 \leqslant i \leqslant n-1$, we have $\mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(X), \mathrm{add}(\overline{A})) = 0$ for $1 \leqslant i \leqslant n-1$. We know that add$(\overline{A})$ is an n-cluster tilting subcategory of $\mathcal{F} \cong \mathcal{C}_n(\overline{A})$ (see [4, 39]). Hence $p^*(X)$ is in add$(\overline{A})$. By the equivalence $p^* : \mathcal{Z}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{F} \xrightarrow{\sim} \mathcal{C}_n(\overline{A})$, the object X is in add$A$.

Step 3: Let X be an object in $\mathcal{H}$ satisfying $\mathbb{E}^i(\mathrm{add}A, X) = 0$ for $1 \leqslant i \leqslant n-1$. Then X is in add$A$.

This is due to Step 2.

Thus, the category add$A$ is an n-cluster tilting subcategory of $\mathcal{H}$. $\checkmark$

Proposition 3.6.47. *Suppose that the n-cluster tilting category* add$\overline{A}$ *of* $\mathcal{C}_n(\overline{A})$ *satisfies* $\Sigma^n(\mathrm{add}\overline{A}) = \mathrm{add}\overline{A}$. *Then we have:*

(1) If $X \in \mathcal{H}$ satisfies $\mathbb{E}^{n-1}(\mathrm{add}A, X) = 0$, then there is $\mathfrak{s}$-triangle

$$Y \xrightarrow{f} P \to X \dashrightarrow \quad (P \in \mathcal{P} = \mathrm{add}(eA))$$

for which,

$$\mathcal{H}(T, f) : \mathrm{Hom}_{\mathcal{H}}(T, Y) \to \mathrm{Hom}_{\mathcal{H}}(T, P)$$

is injective for any $T \in \mathrm{add}(A)$.

(2) Dually, if $Z \in \mathcal{H}$ satisfies $\mathbb{E}^{n-1}(Z, \mathrm{add}A) = 0$, then there is $\mathfrak{s}$-triangle

$$Z \to I \xrightarrow{g} W \dashrightarrow \quad (I \in \mathcal{P} = \mathrm{add}(eA))$$

for which,

$$\mathcal{H}(g, T) : \mathrm{Hom}_{\mathcal{H}}(W, T) \to \mathrm{Hom}_{\mathcal{H}}(I, T)$$

is injective for any $T \in \mathrm{add}(A)$.

Proof. We only show the first statement since the second statement can be shown dually. Let X be an object in $\mathcal{H}$ which satisfies

$$\mathbb{E}^{n-1}(\mathrm{add}A, X) = \mathrm{Hom}_{\mathcal{C}_n(A,B)}(\mathrm{add}(A), \Sigma^{n-1}X) \simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(\mathrm{add}(\overline{A}), \Sigma^{n-1}p^*X) = 0.$$

Since $\mathcal{H}$ is a Frobenius extriangulated category, there is a $\mathfrak{s}$-triangle

$$Y \xrightarrow{f} P \to X \dashrightarrow \quad (P \in \mathcal{P} = \mathrm{add}(eA))$$

with Y in $\mathcal{H}$ and P in $\mathcal{P} = \mathrm{add}(eA)$. Then it is enough to show that $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\Sigma\mathrm{add}(eA), X) = 0$.

By Proposition 3.6.32, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\Sigma\mathrm{add}(A), X) \simeq & \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(\Sigma\mathrm{add}(\overline{A}), p^*X) \\ \simeq & \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(\Sigma^n\mathrm{add}(\overline{A}), \Sigma^{n-1}p^*X) \\ \simeq & \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(\mathrm{add}(\overline{A}), \Sigma^{n-1}p^*X) \\ = & 0.\end{aligned}$$

Thus, the space $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(\Sigma\mathrm{add}(eA), X)$ vanishes.

$\surd$

Proposition 3.6.48. *By Theorem 3.6.42, $(\mathcal{H}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category. We have*

(1) Let $f \in \mathcal{H}(X, Y)$, $g \in \mathcal{H}(Y, Z)$ be any pair of morphisms. If $g \circ f$ is an $\mathfrak{s}$-inflation (see [42, Definition 2.23]), then so is f.

(2) Let $f \in \mathcal{H}(X, Y)$, $g \in \mathcal{H}(Y, Z)$ be any pair of morphisms. If $g \circ f$ is an $\mathfrak{s}$-deflation (see [42, Definition 2.23]), then so is g.

Proof. We only show the first statement since the second statement can be shown dually. Let $f \in \mathcal{H}(X, Y)$, $g \in \mathcal{H}(Y, Z)$ be any pair of morphisms. Suppose that gf is an $\mathfrak{s}$-inflation, i.e. there is a triangle in $\mathcal{C}_n(A, B)$

$$X \xrightarrow{gf} Z \to W \to \Sigma X$$

such that W is also in $\mathcal{H}$. By the octahedral axiom, we have the following commutative diagram in $\mathcal{C}_n(A, B)$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & A \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "gf"] & Z \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \Sigma X \\
& B \arrow[r, equal] \arrow[d] & B \arrow[d] & \\
& \Sigma Y \arrow[r] & \Sigma A &
\end{tikzcd}$$

and the upper middle commutative diagram is a homotopy bi-cartesian square. Thus, there is a triangle in $\mathcal{C}_n(A, B)$

$$Y \to A \oplus Z \to W \to \Sigma Y.$$

Since $\mathcal{H}$ is an extension closed subcategory of $\mathcal{C}_n(A, B)$, $A \oplus Z$ is in $\mathcal{H}$. By Theorem 3.6.38, $\mathcal{H}$ is closed under taking direct summand. Thus A is in $\mathcal{H}$. We can conclude that $f : X \to Y$ is an $\mathfrak{s}$-inflation.

$\surd$

Theorem 3.6.49. *Suppose that the n-cluster tilting category $\mathrm{add}\overline{A}$ of $\mathcal{C}_n(\overline{A})$ satisfies*

$$\Sigma^n \mathrm{add}\overline{A} = \mathrm{add}\overline{A}.$$

Then the n-cluster-tilting subcategory $\mathrm{add}\overline{A}$ of $\mathcal{C}_n(\overline{A})$ carries a canonical $(n+2)$-angulated structure. Moreover, the n-cluster-tilting subcategory $\mathrm{add}A$ of $\mathcal{H}$ carries a canonical structure of Frobenius n-exangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(eA)$. The quotient functor $p^* : \mathcal{C}_n(A, B) \to \mathcal{C}_n(\overline{A})$ induces an equivalence of $(n+2)$-angulated categories

$$\mathrm{add}A/[\mathcal{P}] \xrightarrow{\sim} \mathrm{add}(\overline{A}).$$

Proof. Since $\mathrm{add}\overline{A}$ is closed under the n-th power of the shift functor in $\mathcal{C}_n(\overline{A})$, by [33, Theorem 1], the n-cluster-tilting subcategory $\mathrm{add}\overline{A}$ carries a canonical $(n+2)$-angulated structure $(\mathrm{add}\overline{A}, \Sigma^n, \pentagon)$, where $\pentagon$ is the class of all $(n+2)$-sequences in $\mathrm{add}\overline{A}$

$$M \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} N \xrightarrow{\delta} \Sigma^n M$$

such that there exists a diagram

$$\begin{tikzcd}
& X_1 \arrow[rr, "\alpha_2"] \arrow[dr] & & X_2 \arrow[dr] & & \cdots & & X_n \arrow[dr, "\alpha_n"] & \\
M \arrow[ur, "\alpha_0"] & & X_{1.5} \arrow[ll, "|" marking] \arrow[ur] & & X_{2.5} \arrow[ll, "|" marking] & \cdots & X_{n-0.5} \arrow[ur] & & N \arrow[ll, "|" marking]
\end{tikzcd}$$

with $X_i \in \mathcal{C}_n(\overline{A})$ for $i \notin \mathbb{Z}$, such that all oriented triangles are triangles in $\mathcal{C}_n(\overline{A})$, all non-oriented triangles commute, and δ is the composition along the lower edge of the diagram.

For any two objects M, N in $\mathrm{add}A$, the category $\mathbf{T}^{n+2}_{M,N}$ (see [42]) is defined as follows:

(a) An object in $\mathbf{T}^{n+2}_{M,N}$ is a complex $X^\bullet = (X^i, d^i_X)$ of the form

$$X^0 \xrightarrow{d^0_X} X^1 \xrightarrow{d^1_X} \cdots \xrightarrow{d^{n-1}_X} X^n \xrightarrow{d^n_X} X^{n+1}$$

with all X_i in $\mathrm{add}A$ and $X^0 = M$, $X^{n+1} = N$.

(b) For any $X^\bullet, Y^\bullet \in \mathbf{T}^{n+2}_{M,N}$, a morphism f between $X^\bullet$ and $Y^\bullet$ is a chain map $f = (f^0, \cdots, f^{n+1})$ such that $f^0 = 1_M$ and $f^{n+1} = 1_N$. Two morphisms $f^\bullet$ and $g^\bullet \in \mathbf{T}^{n+2}_{M,N}(X^\bullet, Y^\bullet)$ are *homotopic* if there is a sequence of morphisms $h^\bullet = (h^1, \cdots, h^n)$ satisfying

$$\begin{aligned} 0 =& h^1 \circ d^0_X, \\ g^i - f^i =& d^{i-1}_Y \circ h^i + h^{i+1} \circ d^i_X \quad (1 \leqslant i \leqslant n), \\ 0 =& d^n_Y \circ h^{n+1}. \end{aligned}$$

By Propositions 3.6.47 and 3.6.48, the n-cluster-tilting subcategory $\mathrm{add}A \subseteq \mathcal{H}$ satisfies the conditions in [42, Theorem 5.39]. Thus, it carries a canonical n-exangulated structure $(\mathrm{add}A, \mathbb{E}^n, \mathfrak{s}^n)$ which is given by

(1) For any two objects M, N in $\mathrm{add}A$, the group $\mathbb{E}^n(M, N)$ is the higher extension group defined in Definition 3.6.43, i.e. $\mathbb{E}^n(N, M) = \mathrm{Hom}_{\mathcal{C}_n(A,B)}(N, \Sigma^n M) \simeq \mathrm{Hom}_{\mathcal{C}_n(\overline{A})}(p^*(N), \Sigma^n p^*(M))$;

(2) For any M, N in $\mathrm{add}A$ and any $\delta \in \mathbb{E}^n(N, M)$, define

$$\mathfrak{s}^n(\delta) = [X^\bullet]$$

to be the homotopy equivalence class of $X^\bullet$ in $\mathbf{T}^{n+2}_{M,N}$, where $X^\bullet$ is given by an $(n+2)$-sequence in $\mathrm{add}A$

$$M \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} N \xrightarrow{\delta} \Sigma^n M$$

such that there exists a diagram

$$\begin{tikzcd}
 & X_1 \arrow[rr, "\alpha_2"] \arrow[dr] & & X_2 \arrow[dr] & & \cdots & & X_n \arrow[dr, "\alpha_n"] & \\
M \arrow[ur, "\alpha_0"] & & X_{1.5} \arrow[ll, "|"] \arrow[ur] & & X_{2.5} \arrow[ll, "|"] & \cdots & X_{n-0.5} \arrow[ur] & & N \arrow[ll, "|"]
\end{tikzcd}$$

with $X_i \in \mathcal{H}$ for $i \notin \mathbb{Z}$, such that all oriented triangles are triangles in $\mathcal{C}_n(A, B)$, all non-oriented triangles commute, and δ is the composition along the lower edge of the diagram.

Next, we will show that $\mathrm{add}A$ carries a canonical structure of Frobenius n-exangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(eA)$.

Firstly, we show that $\mathcal{P} = \mathrm{add}(eA)$ consist of projective-injective objects in $\mathrm{add}A$. Let P be an object in $\mathrm{add}(eA)$. We take a distinguished n-exangle in $\mathrm{add}A$

$$Y_0 \xrightarrow{\alpha_0} Y_1 \to \cdots \to Y_n \xrightarrow{\alpha_n} Y_{n+1} \xrightarrow{\delta} \Sigma^n Y_0.$$

Then we have a distinguished triangle in $\mathcal{C}_n(A,B)$

$$X \to Y_n \xrightarrow{\alpha_n} Y_{n+1} \to \Sigma X$$

such that X is in $\mathcal{H}$. Let $c : P \to Y_{n+1}$ be a morphism in add$A$. It induces the following long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(P, Y_n) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(P, Y_{n+1}) \to \mathrm{Hom}_{\mathcal{C}_n(A,B)}(P, \Sigma X) \to \cdots .$$

Since X is in $\mathcal{H}$, the space $\mathrm{Hom}_{\mathcal{C}_n(A,B)}(P, \Sigma X) \simeq \mathrm{Hom}_{\mathrm{per}A}(P, \Sigma X)$ vanishes. Thus, there exists a morphism $b : P \to Y_n$ in add$A$ satisfying $\alpha_n \circ b = c$. This shows that P is projective. Dually, we can show that P is injective.

Let N be an object in add$A$. Since add(eA) is functorially finite in $\mathcal{C}_n(A,B)$, there exists a distinguished triangle in $\mathcal{C}_n(A,B)$

$$Q_n \xrightarrow{a_n} P_n \xrightarrow{b_n} N \xrightarrow{c_n} \Sigma Q_n$$

with P_n in add(eA). We see that $p^*(N) \simeq \Sigma p^*(Q_n) \in \mathrm{add}(\overline{A})$ in $\mathcal{C}_n(\overline{A})$.

For the object Q_n, we also have a distinguished triangle in $\mathcal{C}_n(A,B)$

$$Q_{n-1} \xrightarrow{a_{n-1}} P_{n-1} \xrightarrow{b_{n-1}} Q_n \xrightarrow{c_{n-1}} \Sigma Q_{n-1}$$

with P_{n-1} in add(eA). And we see that $p^*(N) \simeq \Sigma^2 p^*(Q_{n-1}) \in \mathrm{add}(\overline{A})$ in $\mathcal{C}_n(\overline{A})$.

Repeating the process, we get the following triangles in $\mathcal{C}_n(A,B)$

$$Q_n \xrightarrow{a_n} P_n \xrightarrow{b_n} N \xrightarrow{c_n} \Sigma Q_n,$$

$$Q_{n-1} \xrightarrow{a_{n-1}} P_{n-1} \xrightarrow{b_{n-1}} Q_n \xrightarrow{c_{n-1}} \Sigma Q_{n-1},$$

$$\cdots$$

$$Q_0 \xrightarrow{a_0} P_0 \xrightarrow{b_0} Q_1 \xrightarrow{c_0} \Sigma Q_0$$

such that all P_i, $0 \leqslant i \leqslant n$, are in add$(eA)$ and $p^*(N) \simeq \Sigma^n p^*(Q_0) \in \mathrm{add}(\overline{A})$.

By our assumption $\Sigma^n \mathrm{add}\overline{A} = \mathrm{add}\overline{A}$, we see that $p^*(Q_0)$ is in add$\overline{A}$. Thus, the object Q_0 is in add$A$. Then we get a distinguished n-exangle in add$A$

$$Q_0 \xrightarrow{a_0} P_0 \xrightarrow{a_1 \circ b_0} P_1 \to \cdots \to P_n \xrightarrow{b_n} N \xrightarrow{\delta} \Sigma^n Q_0,$$

where δ is the composition

$$N \xrightarrow{c_n} \Sigma Q_n \xrightarrow{\Sigma c_{n-1}} \Sigma^2 Q_{n-1} \to \cdots \to Q_1 \xrightarrow{\Sigma^{n-1} c_0} \Sigma^n Q_0.$$

Thus, this shows that add$A$ has enough projectives. Dually, we can show that add$A$ has enough injectives. Moreover, projective-injective objects form exactly the subcategory $\mathcal{P} = \mathrm{add}(eA)$. Therefore, we have shown that add$A$ carries a canonical structure of Frobenius n-exangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(eA)$.

The stable category add$A/[\mathcal{P}]$ has the same objects as add$A$. For any two objects M and N, the morphism space is given by the quotient group

$$\mathrm{Hom}_{\mathrm{add}A}(M,N)/[\mathcal{P}](M,N),$$

where $[\mathcal{P}](M,N)$ is the subgroup of $\mathrm{Hom}_{\mathrm{add}A}(M,N)$ consisting of those morphisms which factor through an object in $\mathcal{P} = \mathrm{add}eA$.

For any object M in $\mathrm{add}A$, we have the following triangles in $\mathcal{C}_n(A,B)$

$$M \xrightarrow{a_0} I_0 \xrightarrow{b_0} Q_0 \xrightarrow{c_0} \Sigma M,$$

$$Q_0 \xrightarrow{a_1} I_1 \xrightarrow{b_1} Q_1 \xrightarrow{c_1} \Sigma Q_0,$$

$$\cdots$$

$$Q_{n-1} \xrightarrow{a_n} I_n \xrightarrow{b_n} Q_n \xrightarrow{c_n} \Sigma Q_{n-1}$$

such that all I_i, $0 \leqslant i \leqslant n$, are in $\mathrm{add}(eA)$ and Q_n is in $\mathrm{add}A$. Those triangles induce a distinguished n-exangle in $\mathrm{add}A$

$$M \xrightarrow{a_0} I_0 \xrightarrow{a_1 \circ b_0} I_1 \to \cdots \to I_n \xrightarrow{c_n} Q_n \xrightarrow{\delta} \Sigma^n M,$$

where δ is the composition

$$Q_n \xrightarrow{c_n} \Sigma Q_{n-1} \to \cdots \to \Sigma^{n-1} Q_0 \xrightarrow{\Sigma^{n-1} c_0} \Sigma^n M.$$

We define the functor $S : \mathrm{add}A/[\mathcal{P}] \to \mathrm{add}A/[\mathcal{P}]$ such that it takes M to Q_n. By [67, Proposition 3.7], the S functor is well defined and it is an auto-equivalence. It is easy to see that $S(M)$ is isomorphic to $\Sigma^n p^*(M)$ in $\mathcal{C}_n(\overline{A})$.

Thus, by [67, Theorem 3.13], the stable category $\mathrm{add}A/[\mathcal{P}]$ carries a canonical $(n+2)$-angulated structure $(\mathrm{add}A/[\mathcal{P}], S, \square_S)$ which is given by

(1) The functor S defined as above.

(2) For any two objects M, N in $\mathrm{add}A$, there is a one-to-one correspondence between $\mathbb{E}^n(N,M) = \mathrm{Hom}_{\mathcal{C}_n(A,B)}(N, \Sigma^n M)$ and $\mathrm{Hom}_{\mathrm{add}/[\mathcal{P}]}(N, S(M)) \simeq \mathbb{E}^n(N,M)$ (see [67, Lemma 3.12]). Any distinguished n-exangle

$$M \xrightarrow{\alpha_0} X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \xrightarrow{\alpha_n} N \xrightarrow{\delta} \Sigma^n M$$

in $\mathrm{add}A$ induces an $(n+2)$-sequence

$$M \xrightarrow{\overline{\alpha}_0} X_1 \xrightarrow{\overline{\alpha}_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\overline{\alpha}_{n-1}} X_n \xrightarrow{\overline{\alpha}_n} N \xrightarrow{\overline{\delta}} S(M)$$

in $\mathrm{add}A/[\mathcal{P}]$. We call such sequence an $(n+2)$-*S-sequence*. We denote by $\square_S$ the class of $(n+2)$-S-sequences.

Then it is clear that we have an equivalence of $(n+2)$-angulated categories

$$\mathrm{add}A/[\mathcal{P}] \xrightarrow{\sim} \mathrm{add}(\overline{A}).$$

$\checkmark$

Remark 3.6.50. If $\mathrm{add}A$ is stable under Σ^n in $\mathcal{C}_n(A,B)$, then the algebra B is zero. So the n-cluster-tilting subcategory $\mathrm{add}A \subseteq \mathcal{C}_n(A,B)$ can only carry an n-angulated structure with higher suspension Σ^n if $B = 0$ (see [33, Theorem 1]).

3.7 The case when A is concentrated in degree 0

Let $f : B \to A$ be a morphism (not necessarily preserving the identity element) between two differential graded (=dg) k-algebras. We assume that f satisfies the assumptions 1 and moreover, A is concentrated in degree 0. In particular, f carries a relative $(n+1)$-Calabi–Yau structure.

Proposition 3.7.1. *Under the assumption above, the k-algebra $H^0(A)$ is a finite-dimensional with* $\mathrm{gldim}\, H^0(A) \leqslant n+1$.

Proof. By assumptions 1, the algebra $H^0(A)$ is finite-dimensional. Suppose that $1_{H^0(A)}$ has decomposition

$$1_{H^0(A)} = e_1 + e_2 + \cdots + e_n$$

into primitive orthogonal idempotents such that

$$e = f(1_B) = e_1 + \cdots + e_k$$

for an integer $0 \leqslant k \leqslant n$. Here we regard e as an element of $H^0(A)$. By Proposition 2.4.2, $\mathrm{pvd}_B(A)$ is an $(n+1)$-Calabi–Yau triangulated category. Thus for each simple module S_i, $k+1 \leqslant i \leqslant n$, we have $\mathrm{pdim}\, S_i \leqslant n+1$.

Let M be a finite-dimensional $H^0(A)$-module. For each simple module S_i, $1 \leqslant i \leqslant k$, by Proposition 2.4.2, we have the following isomorphism of triangles

$$\begin{tikzcd}
\mathcal{C}(S_i, \Sigma^{-1}M) \arrow[r] \arrow[d, "\simeq"] & \mathbf{R}\mathrm{Hom}_A(S_i, M) \arrow[r] \arrow[d, "\simeq"] & \mathbf{R}\mathrm{Hom}_B(S_i|_B, M|_B) \arrow[r] \arrow[d, "\simeq"] & {} \\
D\mathbf{R}\mathrm{Hom}_A(M, \Sigma^{n+1}S_i) \arrow[r] & D\mathcal{C}(M, \Sigma^n S_i) \arrow[r] & D\mathbf{R}\mathrm{Hom}_B(M|_B, \Sigma^n S_i|_B) \arrow[r] & {} .
\end{tikzcd}$$

For each integer $p \geqslant n+2$, we have

$$\mathrm{Hom}_{\mathcal{D}(B)}(S_i|_B, \Sigma^p(M|_B)) = 0$$

because B is n-Calabi–Yau.

Thus, we have

$$\begin{tikzcd}
H^0(\mathcal{C}(S_i, \Sigma^{p-1}M)) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}^p_{H^0(A)}(S_i, M) \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 = D\mathrm{Ext}^{n+1-p}_{H^0(A)}(M, S_i) \arrow[r] & DH^0(\mathcal{C}(M, \Sigma^{n-1-p}S_i)) \arrow[r] & 0.
\end{tikzcd}$$

We see that the space $\mathrm{Ext}^p_{H^0(A)}(S_i, M)$ vanishes for each $p \geqslant n+2$. Then $\mathrm{pdim}\, S_i \leqslant n+1$ for each $1 \leqslant i \leqslant k$. Therefore, we have $\mathrm{gldim}\, H^0(A) \leqslant n+1$. $\surd$

Theorem 3.7.2. *a) The algebra $B' = eH^0(A)e$ is Iwanaga-Gorenstein of injective dimension at most $g \leqslant n+1$ as a B'-module.*

b) Under the equivalence $\mathcal{D}^b(\mathrm{mod} H^0A) \simeq \mathrm{per} A$, the subcategory $\mathcal{F}^{rel}$ corresponds to the subcategory $\mathrm{mod}_{n-1}(H^0A)$ of H^0A-modules of projective dimension at most $n-1$.

c) Via the equivalence $\mathrm{res} : \mathcal{D}^b(\mathrm{mod}H^0A) \xrightarrow{\sim} \mathrm{per}A$*, the localization* $\pi^{rel} : \mathrm{per}A \to \mathcal{C}_n(A,B)$ *identifies with the restriction functor* $\mathcal{D}^b(\mathrm{mod}H^0A) \to \mathcal{D}^b(\mathrm{mod}B')$*, i.e. we have a commutative square*

$$\begin{tikzcd}
\mathcal{D}^b(\mathrm{mod}H^0A) \arrow[r] \arrow[d, "\wr"] & \mathcal{D}^b(\mathrm{mod}B') \arrow[d, "\wr"] \\
\mathrm{per}A \arrow[r] & \mathcal{C}_n(A,B).
\end{tikzcd}$$

d) Under the equivalence $\mathcal{D}^b(\mathrm{mod}B') \xrightarrow{\sim} \mathcal{C}_n(A,B)$*, the Higgs category* $\mathcal{H} \subseteq \mathcal{C}_n(A,B)$ *corresponds to the subcategory* $\mathrm{gpr}B'$ *of Gorenstein projective modules over* $B' = eH^0(A)e$*. In particular, when* B' *is self injective, we have* $\mathcal{H} \cong \mathrm{mod}B'$*.*

Proof. Since $H^0(A)$ is of finite global dimension, the restriction along the quasi isomorphism

$$A \to H^0(A)$$

induces a triangle equivalence

$$\mathcal{D}^b(\mathrm{mod}H^0(A)) \xrightarrow{\sim} \mathrm{per}A.$$

Under this equivalence, $\mathrm{pvd}_B(A)$ identifies with

$$\begin{aligned}\mathcal{D}^b_{\mathcal{N}}(\mathrm{mod}H^0(A)) =& \{X \in \mathcal{D}^b(\mathrm{mod}H^0(A)) \mid H^l(X)|_{B'} = 0, \forall l \in \mathbb{Z}\} \\ =& \{X \in \mathcal{D}^b(\mathrm{mod}H^0(A)) \mid H^l(A) \in \mathcal{N}, \forall l \in \mathbb{Z}\},\end{aligned}$$

where $\mathcal{N} = \{M \in \mathrm{mod}H^0(A) \mid M|_{B'} = 0\}$. Clearly, the category $\mathcal{N}$ is a Serre subcategory of $\mathrm{mod}H^0(A)$ and the restriction $\mathrm{mod}H^0(A) \to \mathrm{mod}B'$ induces an exact sequence of abelian categories

$$0 \to \mathcal{N} \to \mathrm{mod}H^0(A) \to \mathrm{mod}B' \to 0.$$

This exact sequence induces an exact sequence of triangulated categories

$$0 \to \mathcal{D}^b_{\mathcal{N}}(\mathrm{mod}H^0A) \to \mathcal{D}^b(\mathrm{mod}H^0(A)) \to \mathcal{D}^b(\mathrm{mod}B') \to 0.$$

Thus, the restriction $\mathrm{per}A \cong \mathcal{D}^b(\mathrm{mod}H^0(A)) \to \mathcal{D}^b(\mathrm{mod}B')$ induces an equivalence

$$\mathcal{C}_n(A,B) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}B').$$

By inspecting the definition of $\mathcal{F}^{rel}$, it is equivalent to the following subcategory

$$\mathrm{mod}_{n-1}(H^0A) = \{M \in \mathrm{mod}H^0(A) \mid \mathrm{pdim}M \leqslant n-1\}.$$

The Higgs category $\mathcal{H}$ is contained in $\mathrm{mod}B'$ and stable under extensions in $\mathcal{C}_n(A,B) \simeq \mathcal{D}(B')$. Thus, it is a fully exact subcategory of $\mathrm{mod}B'$ with the induced exact structure. Moreover, $\mathcal{H}$ is a Frobenius exact category with projective-injective objects $\mathcal{P} = \mathrm{proj}(B')$ and $\mathcal{H}$ contains an n-cluster-tilting object T, namely the image of A, such that $\mathrm{End}_{\mathcal{H}}(T) \cong H^0(A)$ with $\mathrm{gldim}\, H^0(A) \leqslant n+1$.

By Theorem 3.6.38, the Higgs category is idempotent complete. Thus, we can apply Iyama–Kalck–Wemyss–Yang's structure theorem for Frobenius category with an n-cluster-tilting object (see [54, Theorem 2.7]), to conclude that B' is Iwanaga-Gorenstein of injective dimension at most $g \leqslant n+1$ as a B'-module and that restriction to B' is an equivalence from $\mathcal{H} \subseteq \mathrm{mod}H^0(A)$ to the category $\mathrm{gpr}(B')$ of Gorenstein projective B'-modules, i.e. we have an equivalence

$$\mathcal{H} \xrightarrow{\sim} \mathrm{gpr}(B') = \{M \in \mathrm{mod}B' \mid \mathrm{Ext}^i_{B'}(M, B') = 0,\ \forall i > 0\}.$$

$\surd$

3.7.1 Relation with Matthew Pressland's works

Definition 3.7.3. [76] Let A be a k-algebra, e an idempotent of A, and d a non-negative integer. We say that A is *internally d-Calabi–Yau* with respect to e if

(1) $\mathrm{gldim}A \leqslant d$, and

(2) for each $i \in \mathbb{Z}$, there is a functorial duality

$$D\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^{d-i}_A(N, M)$$

where M and N are perfect A-modules such that M is also a finite dimensional A/AeA-module.

Let A be an algebra and e an idempotent of A. We denote the corresponding quotient algebra by $\overline{A} := A/\langle e\rangle$. Let $\mathcal{D}(A)$ be the unbounded derived category of A, $\mathcal{D}_e(A)$ the full subcategory of $\mathcal{D}(A)$ consisting of complexes with homology groups in $\mathrm{Mod}(\overline{A})$, and $\mathrm{pvd}_e(A)$ the full subcategory of $\mathcal{D}_e(A)$ consisting of objects with finite dimensional total cohomology.

Definition 3.7.4. [77] An algebra A is *bimodule internally n-Calabi–Yau* with respect to an idempotent $e \in A$ if

- $\mathrm{pdim}_{A^e} A \leqslant n$,
- $A \in \mathrm{per}A^e$, and
- there exists a triangle

$$A \longrightarrow \Sigma^n \Theta_A \longrightarrow C \longrightarrow \Sigma A$$

in $\mathcal{D}(A^e)$, such that $\mathbf{R}\mathrm{Hom}_A(C, M) = 0 = \mathbf{R}\mathrm{Hom}_{A^{op}}(C, N)$ for any $M \in \mathrm{pvd}_e(A)$ and $N \in \mathrm{pvd}_e(A^{op})$.

Proposition 3.7.5. *[76, Corollary 5.12] If A is internally bimodule n-Calabi–Yau with respect to an idempotent e of A, then it is internally n-Calabi–Yau with respect to e.*

Proposition 3.7.6. *Let $f : B \to A$ be a morphism between dg k-algebras. Suppose that f satisfies the assumptions 1 and moreover, A is concentrated in degree* 0*. Then A and A^{op} are internally bimodule $(n+1)$-Calabi–Yau with respect to $e = f(1_B)$. Hence, the algebras A and A^{op} are internally $(n+1)$-Calabi–Yau with respect to $e = f(1_B)$.*

Proof. By the definition of a relative $(n+1)$-Calabi–Yau structure, we have the following triangle in $\mathcal{D}(A^e)$

$$A \longrightarrow \Sigma^{n+1}A^\vee \longrightarrow \Sigma^{n+1}\boldsymbol{L}f^*(B^\vee) \longrightarrow \Sigma A,$$

where $A^\vee = \mathbf{R}\mathrm{Hom}_{A^e}(A, A^e)$, $B^\vee = \mathbf{R}\mathrm{Hom}_{B^e}(B, B^e)$ and $\mathbf{L}f^*(B^\vee) \cong A \otimes^{\mathbf{L}}_B B^\vee \otimes^{\mathbf{L}}_B A$.

Let M be an object in $\mathrm{pvd}_e(A)$. We have

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_A(\mathbf{L}f^*(B^\vee), M) &= \mathbf{R}\mathrm{Hom}_A(A \overset{L}{\otimes}_B B^\vee \overset{L}{\otimes}_B A, M) \\
&\simeq \mathbf{R}\mathrm{Hom}_A(A \overset{L}{\otimes}_B B^\vee \overset{L}{\otimes}_B, \boldsymbol{R}\mathrm{Hom}_B(A, M|_B)) \\
&= 0
\end{aligned}$$

Similarly, we have $\mathbf{R}\mathrm{Hom}_{A^{op}}(\mathbf{L}f^*(B^\vee), N) = 0$ for any $N \in \mathrm{pvd}_e(A^{op})$. Thus, the algebra A is bimodule internally $(n+1)$-Calabi–Yau with respect to the idempotent $e = f(1_B)$. By the same way, we can show that A^{op} is bimodule internally $(n+1)$-Calabi–Yau with respect to the idempotent e.

$\surd$

3.8 Relative cluster categories for Jacobi-finite ice quivers with potential

3.8.1 Ice Quivers with potential

Definition 3.8.1. A *quiver* is a tuple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are sets, and $s, t : Q_1 \to Q_0$ are functions. Each $\alpha \in Q_1$ is realised as an arrow $\alpha : s(\alpha) \to t(\alpha)$. We call Q finite if Q_0 and Q_1 are finite sets.

Definition 3.8.2. Let Q be a quiver. A quiver $F = (F_0, F_1, s', t')$ is called a *subquiver* of Q if $F_0 \subseteq Q_0$, $F_1 \subseteq Q_1$ and s', t' are the restrictions of s, t to F_1. We call F is a *full subquiver* of Q if F a subquiver and $F_1 = \{\alpha \in Q_1 : s(\alpha), t(\alpha) \in F_0\}$.

Definition 3.8.3. An *ice quiver* is a pair (Q, F), where Q is a quiver, and F is a subquiver of Q.

Let Q be a finite quiver. For each arrow a of Q, we define the cyclic derivative with respect to a as the unique linear map

$$\partial_a : kQ/[kQ, kQ] \longrightarrow kQ$$

which takes the class of a path p to the sum $\sum_{p=uav} vu$ taken over all decompositions of the path p.

Definition 3.8.4. An element of $kQ/[kQ, kQ]$ is called a *potential* on Q. It is given by a linear combination of cycles in Q. An ice quiver with potential is a tuple (Q, F, W) in which (Q, F) is a finite ice quiver, and W is a potential on Q. If F is the empty quiver $\emptyset$, then $(Q, \emptyset, W) := (Q, W)$ is simply called a quiver with potential.

3.8.2 Relative Ginzburg algebras and relative Jacobian algebras

Definition 3.8.5. Let (Q, F, W) be a finite ice quiver with potential. Let $\widetilde{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q,
- an arrow $a^* : j \to i$ of degree -1 for each arrow a of Q not belonging to F,
- a loop $t_i : i \to i$ of degree -2 for each vertex i of Q not belonging to F.

The *relative Ginzburg dg algebra* $\mathbf{\Gamma}_{rel}(Q, F, W)$ is the dg algebra whose underlying graded space is the graded path algebra $k\widetilde{Q}$. Its differential is the unique linear endomorphism of degree 1 which satisfies the Leibniz rule

$$d(u \circ v) = d(u) \circ v + (-1)^p u \circ d(v)$$

for all homogeneous u of degree p and all v, and takes the following values on the arrows of $\widetilde{Q}$:

- $d(a) = 0$ for each arrow a of Q,
- $d(a^*) = \partial_a W$ for each arrow a of Q not belonging to F,
- $d(t_i) = e_i(\sum_{a\in Q_1}[a, a^*])e_i$ for each vertex i of Q not belonging to F, where e_i is the lazy path corresponding to the vertex i.

Definition 3.8.6. Let (Q, F, W) be a finite ice quiver with potential. The *relative (or frozen) Jacobian algebra* $J(Q, F, W)$ is the zeroth cohomology of the relative Ginzburg algebra $\mathbf{\Gamma}_{rel}(Q, F, W)$. It is the quotient algebra

$$kQ/\langle \partial_a W, a \in Q_1 \setminus F_1\rangle$$

where $\langle \partial_a W, a \in Q_1 \setminus F_1\rangle$ is the two-sided ideal generated by $\partial_a W$ with $a \in Q_1 \setminus F_1$.

Let (Q, F, W) be a finite ice quiver with potential. Since W can be viewed as an element in $HC_0(Q)$, $c = B(W)$ is the element in $HH_1(Q)$, where B is the Connes'connecting map (see [62, Section 6.1])

$$B : HC_n(kQ) \to HH_{n+1}(kQ).$$

Then $\xi = (0, c)$ is an element of $HH_0(G)$ which provides the deformation parameter for the relative 3-Calabi–Yau completion of $G : kF \hookrightarrow kQ$, namely the functor

$$\boldsymbol{G}_{rel} : \mathbf{\Pi}_2(kF) \to \mathbf{\Pi}_3^{red}(kQ, kF, \xi)$$

defined in Proposition 2.6.2. An easy check shows that the dg algebra $\mathbf{\Pi}_3^{red}(kQ, kF, \xi)$ is isomorphic to $\mathbf{\Gamma}_{rel}(Q, F, W)$ and that the dg functor $\boldsymbol{G}_{rel}$ takes the following values as follows:

- $\boldsymbol{G}_{rel}(i) = i$ for each frozen vertex $i \in F_0$,
- $\boldsymbol{G}_{rel}(a) = a$ for each arrow $a \in F_1$,

- $\boldsymbol{G}_{rel}(\tilde{a}) = -\partial_a W$ for each arrow $a \in F_1$,
- $\boldsymbol{G}_{rel}(r_i) = e_i(\sum_{a\in Q_1\setminus F_1}[a, a^*])e_i$ for each frozen vertex $i \in F_0$.

Proposition 3.8.7. *Let (Q, F, W) be a finite ice quiver with potential. Let $\overline{Q}$ be the quiver obtained from Q by deleting all vertices in F and all arrows incident with vertices in F. Let $\overline{W}$ be the potential on $\overline{Q}$ obtaining bu deleting all cycles passing through vertices of F in W. Then*

$$\mathbf{\Pi}_2(F) \xrightarrow{\boldsymbol{G}_{rel}} \mathbf{\Gamma}_{rel}(Q, F, W) \to \mathbf{\Gamma}(\overline{Q}, \overline{W})$$

is a homotopy cofiber sequence of dg categories, where $\mathbf{\Gamma}(\overline{Q}, \overline{W})$ is the Ginzburg algebra (see [62]) associated with quiver with potential $(\overline{Q}, \overline{W})$.

Proof. By Proposition 2.6.2, the homotopy cofiber of $\boldsymbol{G}_{rel}$ is isomorphic to that of $\tilde{G}$. Since $\tilde{G}$ is a cofibration, the dg quotient identifies with the quotient of $\mathbf{\Pi}_3(kQ, kF, \xi)$ by the 2-side ideal generated by the image of $\tilde{G}$. This quotient is isomorphic to $\mathbf{\Gamma}(\overline{Q}, \overline{W})$ as a dg category. $\surd$

3.8.3 Jacobi-finite ice quivers with potential

An ice quiver with potential (Q, F, W) is called *Jacobi-finite* if the relative Jacobian algebra $J(Q, F, W)$ is finite-dimensional.

Definition 3.8.8. Let (Q, F, W) be a Jacobi-finite ice quiver with potential. Denote by $\mathbf{\Gamma}_{rel}$ the relative Ginzburg dg algebra $\mathbf{\Gamma}_{rel}(Q, F, W)$. Let $e = \sum_{i\in F} e_i$ be the idempotent associated with all frozen vertices. Let $\mathrm{pvd}_e(\mathbf{\Gamma}_{rel})$ the full subcategory of $\mathrm{pvd}(\mathbf{\Gamma}_{rel})$ of the dg $\mathbf{\Gamma}_{rel}$-modules whose restriction to frozen vertices is acyclic.

Then the *relative cluster category* $\mathcal{C}(Q, F, W)$ associated to (Q, F, W) is defined as the Verdier quotient of triangulated categories

$$\mathrm{per}(\mathbf{\Gamma}_{rel})/\mathrm{pvd}_e(\mathbf{\Gamma}_{rel}),$$

The relative fundamental domain $\mathcal{F}^{rel}$ associated to (Q, F, W) is defined as the following subcategory of $\mathrm{per}\,\mathbf{\Gamma}_{rel}$

$$\mathcal{F}^{rel} := \{\mathrm{Cone}(X_1 \xrightarrow{f} X_0) \mid X_i \in \mathrm{add}(\mathbf{\Gamma}_{rel}) \text{ and } \mathrm{Hom}(f, I) \text{ is surjective, } \forall\, I \in \mathcal{P} = \mathrm{add}(e\mathbf{\Gamma}_{rel})\}.$$

We have a fully faithful embedding $\pi^{rel} : \mathcal{F}^{rel} \subseteq \mathrm{per}\,\mathbf{\Gamma}_{rel} \to \mathcal{C}(Q, F, W)$. Then the Higgs category $\mathcal{H}$ associated to (Q, F, W) is the image of $\mathcal{F}^{rel}$ in $\mathcal{C}(Q, F, W)$ under the functor π^{rel}.

Combining Theorem 3.6.42 and Proposition 3.6.46, we get the result.

Theorem 3.8.9. *Let (Q, F, W) be a Jacobi-finite ice quiver with potential. Then the relative cluster category $\mathcal{C}(Q, F, W)$ is Hom-finite, the Higgs category $\mathcal{H}$ is a Frobenius 2-Calabi–Yau extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma}_{rel})$. The free module $\mathbf{\Gamma}_{rel}$ in $\mathcal{H}$ is a cluster-tilting object. Its endomorphism algebra is isomorphic to the relative Jacobian algebra $J(Q, F, W)$.*

Moreover, the stable category of $\mathcal{H}$ is equivalent to the usual cluster category

$$\underline{\mathcal{H}} = \mathcal{H}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{C}(\overline{Q}, \overline{W}) = \mathrm{per}(\mathbf{\Gamma}(\overline{Q}, \overline{W}))/\mathrm{pvd}(\mathbf{\Gamma}(\overline{Q}, \overline{W}))$$

and the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{per}(e\mathbf{\Gamma}_{rel}e) \arrow[r, equal] \arrow[d, hook] & \mathrm{per}(e\mathbf{\Gamma}_{rel}e) \arrow[d, hook] \\
\mathrm{pvd}_e(\mathbf{\Gamma}_{rel}) \arrow[r, hook] \arrow[d, "\cong"'] & \mathrm{per}(\mathbf{\Gamma}_{rel}) \arrow[r] \arrow[d, two heads] & \mathcal{C}(Q, F, W) \arrow[d, two heads] \\
\mathrm{pvd}(\mathbf{\Gamma}(\overline{Q}, \overline{W})) \arrow[r, hook] & \mathrm{per}(\mathbf{\Gamma}(\overline{Q}, \overline{W})) \arrow[r] & \mathcal{C}(\overline{Q}, \overline{W})
\end{tikzcd}$$

where the rows and columns are exact sequences of triangulated categories.

Chapter 4

Relative Calabi–Yau structures in higher Auslander-Reiten theory

4.1 For algebras of finite global dimension

Let n be a non-negative integer. Let B_0 be a finite dimensional algebra with global dimension at most n. Let $\mathbb{S}_{B_0} = ? \otimes^{\mathbf{L}}_{B_0} DB$ be the Serre functor of $\mathcal{D}^b(\mathrm{mod}B_0)$. The corresponding inverse Serre functor is given by $\mathbb{S}_{B_0}^{-1} = ? \otimes^{\mathbf{L}}_{B_0} \mathbf{R}\mathrm{Hom}_{B^e}(B_0, B_0^e)$. Moreover, the *Nakayama functor* ν_{B_0} for $\mathrm{mod}B_0$ is given by $\nu_{B_0} = D\mathrm{Hom}_{B_0}(?, B_0)$.

Definition 4.1.1. [45] The *higher inverse Auslander-Reiten translation* τ_n^{-1} of $\mathrm{mod}B_0$ is defined to be the following composition

$$\tau_n^{-1} : \mathrm{mod}B_0 \hookrightarrow \mathcal{D}^b(B_0) \xrightarrow{\Sigma^n \mathbb{S}_{B_0}^{-1}} \mathcal{D}^b(B_0) \xrightarrow{H^0} \mathrm{mod}B_0.$$

Definition 4.1.2. Let $f : \mathcal{B} \to \mathcal{A}$ be a dg functor. The *relative inverse Serre functor* for $\mathcal{D}(\mathcal{A})$ is defined as

$$\mathbb{S}_{\mathcal{A},\mathcal{B}}^{-1} = ? \otimes^{\mathbf{L}}_{\mathcal{A}} \Theta_f : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) ,$$

where $\Theta_f = \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathrm{Cone}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \to \mathcal{A}), \mathcal{A}^e) \in \mathcal{D}(\mathcal{A}^e)$.

Remark 4.1.3. It is clear that we have an isomorphism $\mathbf{\Pi}_{n+2}(\mathcal{A}, \mathcal{B}) \simeq \bigoplus_{i \geqslant 0} (\Sigma^{n+1}\Theta_f)^{\otimes^{\boldsymbol{L}}_i}$ in $\mathcal{D}(\mathcal{A})$.

Definition 4.1.4. [48] Let B_0 be an algebra of global dimension at most n. Then the $(n+1)$-*preprojective algebra* of B_0 is defined as

$$\widetilde{B_0} = T_{B_0}(\mathrm{Ext}^n_{B_0}(DB_0, B_0)),$$

i.e, the tensor algebra of the B_0-bimodule $\mathrm{Ext}^n_{B_0}(DB_0, B_0)$ over B_0. Then $\widetilde{B_0}$ is isomorphic to $\oplus_{i \geqslant 0} \tau_n^{-i} B_0$ as a B_0-module.

Remark 4.1.5. In [62, Section 4], Keller introduced the notion of derived $(n+1)$-preprojective algebras $\mathbf{\Pi}_{n+1}(B_0)$ (also called $(n+1)$-Calabi–Yau completion of B_0). The $(n+1)$-preprojective algebras are the 0-th homology of his derived $(n+1)$ preprojective algebras.

We denote by $\mathcal{B} := \mathrm{proj}B_0 \subseteq \mathrm{mod}B_0$ the projective modules. Let $\mathcal{A}$ be a subcategory of $\mathrm{mod}B_0$ which contains $\mathcal{B}$ as a full subcategory. Then there is a natural dg inclusion functor

$$f_0 : \mathcal{B} \hookrightarrow \mathcal{A}.$$

For any $X \in \mathcal{A}$, we put $X^\wedge := \mathrm{Hom}_{B_0}(?, X)|_{\mathcal{A}} \in \mathrm{proj}\mathcal{A}$.

Proposition 4.1.6. *Assume that $\mathcal{A}$ is homologically smooth and is an n-rigid subcategory of $\mathrm{mod}B_0$, i.e. $\mathrm{Ext}^k_{B_0}(\mathcal{A}, \mathcal{A}) = 0$ for $1 \leqslant k \leqslant n-1$. Then for $X \in \mathcal{A}$, we have a functorial isomorphism $X^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^{n+1}\Theta_{f_0} \cong (\tau_n^{-1}X)^\wedge$.*

Proof. Let X be an object in $\mathcal{A}$. We will show that

$$X^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^{n+1}\Theta_{f_0} \cong (\tau_n^{-1}X)^\wedge,$$

where $\Theta_{f_0} = \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathrm{Cone}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} \to \mathcal{A}), \mathcal{A}^e)$.

Step 1. We compute the image of $X^\wedge$ under the functor $? \otimes^{\mathbf{L}}_{\mathcal{A}} \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}, \mathcal{A}^e) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.

Since $\mathcal{B}$ and $\mathcal{A}$ are smooth as dg categories, we have

$$\mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}, \mathcal{A}^e) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{B} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}, \mathcal{A}^e) \cong \mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \Theta_{\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A},$$

where $\Theta_{\mathcal{B}} = \mathbf{R}\mathrm{Hom}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e)$.

Then we have

$$\begin{aligned} X^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^{n+1}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \Theta_{\mathcal{B}} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}) &\cong (X^\wedge \otimes^{\mathbf{L}}_{\mathcal{B}} \Theta_{\mathcal{B}}) \otimes^{\mathbf{L}}_{\mathcal{B}} \Sigma^{n+1}\mathcal{A} \\ &\cong \mathbb{S}^{-1}_{B_0}(\Sigma^{n+1}X)^\wedge \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}. \end{aligned}$$

Fix a minimal injective resolution of X

$$0 \to X \to I^0 \to I^1 \cdots \to I^n \to 0.$$

Then $\mathbb{S}^{-1}_{B_0}(\Sigma^{n+1}X) = \nu^{-1}_{B_0}(\Sigma^{n+1}X)$ is the following complex

$$0 \to P_0 \to P_1 \cdots \to P_n \to 0,$$

where P_i is in degree $i-n-1$ and $P_i = \nu^{-1}_{B_0}(I^i) \in \mathrm{proj}B_0$. After applying the functor

$$? \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A} : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A}),$$

we get

$$0 \to (?, P_0) \to (?, P_1) \to \cdots \to (?, P_n) \to 0,$$

where $(?, P_i) = \mathrm{Hom}_{\mathrm{mod}B_0}(?, P_i)|_{\mathcal{A}} \in \mathrm{proj}(\mathcal{A})$.

Thus the image of $X^\wedge$ under the functor $? \otimes^{\mathbf{L}}_{\mathcal{A}} \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A} \otimes^{\mathbf{L}}_{\mathcal{B}} \mathcal{A}, \mathcal{A}^e) : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is

$$0 \to (?, P_0) \to (?, P_1) \to \cdots \to (?, P_n) \to 0,$$

where $(?, P_i) = \mathrm{Hom}_{\mathrm{mod}B_0}(?, P_i)|_{\mathcal{A}} \in \mathrm{proj}(\mathcal{A})$.

Step 2. We compute the image of $X^\wedge$ under the functor $? \otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^{n+1}\mathcal{A}^\wedge = \Sigma^{n+1}\mathbb{S}^{-1}_{\mathcal{A}} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$.

We have the minimal injective resolution in Step 1

$$0 \to X \to I^0 \to I^1 \cdots \to I^n \to 0.$$

Then $\Sigma^{n+1}\mathbb{S}_{\mathcal{A}}^{-1}(X^\wedge)$ is the following complex

$$0 \to (?, X) \to (?, I^0) \to (?, I^1) \cdots \to (?, I^n).$$

For any $1 \leqslant i \leqslant n-1$, the cohomology at $(?, I^i)$ is $\mathrm{Ext}^i_{B_0}(?, X) = 0$ because $\mathcal{A}$ is an n-rigid subcategory of $\mathrm{mod}B_0$. The cohomology at $(?, I^n)$ is $\mathrm{Ext}^n_{B_0}(?, X)$. For any object L in $\mathrm{mod}B_0$, we have

$$\begin{aligned} D\mathrm{Ext}^n_{B_0}(L, X) &\simeq D\mathrm{Hom}_{\mathcal{D}(B_0)}(L, \Sigma^n X) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(B_0)}(\Sigma^n X, \mathbb{S}_{B_0}(L)) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(B_0)}(\Sigma^n(\mathbb{S}_{B_0}^{-1}X), L) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(B_0)}(H^0(\Sigma^n(\mathbb{S}_{B_0}^{-1}X)), L) \\ &\simeq \mathrm{Hom}_{\mathcal{D}(B_0)}(\tau_n^{-1}X, L) \\ &\simeq \mathrm{Hom}_{B_0}(\tau_n^{-1}X, L), \end{aligned}$$

where the fourth equivalence follows from $\mathbb{S}_{B_0}^{-1} = \mathbf{R}\mathrm{Hom}_{B_0}(DB_0, ?)$ and $\mathrm{gldim}(B_0) \leqslant n$. Then the cohomology at $(?, I^n)$ is isomorphic to $D\mathrm{Hom}_{B_0}(\tau_n^{-1}X, ?)$.

Since we have isomorphisms $(?, I^i) \simeq D(P_i, ?)$ for all $1 \leqslant i \leqslant n-1$, we get the following injective resolution of $(?, X)$

$$0 \to (?, X) \to D(P_0, ?) \to D(P_1, ?) \cdots \to D(P_n, ?) \to D(\tau_n^{-1}X, ?) \to 0.$$

Applying the functor $\Sigma^{n+1}\mathbb{S}_{\mathcal{A}}^{-1} : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ to the above complex, we get a complex

$$0 \to (?, P_0) \to (?, P_1) \to \cdots \to (?, P_n) \to (?, \tau_{B_0,n}^{-1}X) \to 0,$$

where $(?, P_i)$ is in degree $i-n-1$ and $(?, \tau_n^{-1}X)$ is in degree 0. This is because $\mathbb{S}_{\mathcal{A}}^{-1}(D(?, M)) = \nu_{\mathrm{mod}\mathcal{A}}^{-1}(D(M, ?)) = (?, M)$ for any object M in $\mathcal{A}$.

Step 3. From the computations in step 1 and step 2, the object $X^\wedge \otimes^{\mathbf{L}}_{\mathcal{A}} \Sigma^{n+1}\Theta_{f_0}$ is equal to the homotopy fiber of the following morphism of complexes

$$\begin{tikzcd} 0 \arrow[r] & (?, P_0) \arrow[r] \arrow[d, equal] & (?, P_1) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & (?, P_n) \arrow[r] \arrow[d, equal] & (?, \tau_n^{-1}X) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (?, P_0) \arrow[r] & (?, P_1) \arrow[r] & \cdots \arrow[r] & (?, P_n) \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

Thus, the object $\Sigma^{n+1}(\mathbb{S}_{\mathcal{A},\mathcal{B}})(X^\wedge)$ is quasi-isomorphic to $\tau_n^{-1}(X)^\wedge$. $\surd$

Corollary 4.1.7. *Let $\mathcal{B} = \mathrm{proj}B_0 \subseteq \mathrm{mod}B_0$ be the subcategory of projectives and let $\mathcal{A}$ be a subcategory of $\mathrm{mod}B_0$ which contains $\mathcal{B}$ as a full subcategory. Suppose that $\mathcal{A}$ is homologically smooth and is n-rigid in $\mathrm{mod}B$. Then the relative $(n+2)$-Calabi–Yau completion of $f_0 : \mathcal{B} \hookrightarrow \mathcal{A}$*

$$f : \mathbf{\Pi}_{n+1}(\mathcal{B}) \longrightarrow \mathbf{\Pi}_{n+2}(\mathcal{A}, \mathcal{B})$$

can be described as follows:

- *The objects in $\mathbf{\Pi}_{n+2}(\mathcal{A},\mathcal{B})$ are the same as those of $\mathcal{A}$;*
- *For any two objects L, M in $\mathcal{A}$, the space $\mathrm{Hom}_{\mathbf{\Pi}_{n+2}(\mathcal{A},\mathcal{B})}(L,M)$ is given by*

$$\begin{aligned}\mathrm{Hom}_{\mathbf{\Pi}_{n+2}(\mathcal{A},\mathcal{B})}(L,M) &\cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(L^{\wedge}, \oplus_{i\geqslant 0} M^{\wedge}\otimes^{\mathbf{L}}_{\mathcal{A}} (\Sigma^{n+1}\Theta_{f_0})^{\otimes^{\boldsymbol{L}}_i})\\ &\cong \mathbf{R}\mathrm{Hom}_{\mathcal{A}}(L^{\wedge}, \oplus_{i\geqslant 0}(\tau_n^{-i}M)^{\wedge})\\ &\cong \mathrm{Hom}_{B_0}(L, \oplus_{i\geqslant 0}(\tau_n^{-i}M)).\end{aligned}$$

In particular, the dg category $\mathbf{\Pi}_{n+2}(\mathcal{A},\mathcal{B})$ is concentrated in degree 0 and we have a fully faithful functor

$$H^0(f) : H^0(\mathbf{\Pi}_{n+1}(\mathcal{B})) \cong \widetilde{B_0} \hookrightarrow \mathbf{\Pi}_{n+2}(\mathcal{A},\mathcal{B}).$$

4.2 n-representation-finite algebras

Let $n\geqslant 0$ be an integer. Let B_0 be a finite dimensional algebra with global dimension at most n.

Definition 4.2.1. [47] We say that B_0 is *τ_n-finite* if $\tau_n^{-i}B_0 = 0$ for sufficiently large i. We say that B_0 is *n-representation-finite* if $\mathrm{mod}B_0$ contains an n-cluster tilting object.

Remark 4.2.2. If B_0 is n-representation-finite, then it is τ_n-finite.

Theorem 4.2.3. *[47, Proposition 1.3] Suppose that B_0 is an n-representation-finite algebra. Then $\widetilde{B_0} \cong \oplus_{i\geqslant 0}\tau^{-i}_{B_0,n}B_0$ is the unique basic n-cluster tilting object in* $\mathrm{mod}B_0$.

Theorem 4.2.4. *[46, Theorem 0.2] Let B_0 be n-representation-finite. Then*

$$\mathrm{gldim}\,\mathrm{End}_{B_0}(\widetilde{B_0}) \leqslant n+1.$$

Let B_0 be an n-representation-finite algebra. The corresponding *n-Auslander algebra* is given by $\mathrm{End}_{B_0}(\oplus_{i\geqslant 0}\tau_n^{-i}B_0)$. We denote it by A_0. Then there is a natural fully faithful morphism

$$f_0 : B_0 \hookrightarrow A_0 = \mathrm{End}_{B_0}(\oplus_{i\geqslant 0}\tau_n^{-i}B_0).$$

Proposition 4.2.5. *Let $e = f_0(1_{B_0})$. The homotopy cofiber of $f_0 : B_0 \to A_0$ is equal to the usual quotient A_0/A_0eA_0, i.e. the* stable Auslander algebra *of B_0.*

Proof. Let $\mathcal{A}$, $\mathcal{P}$ and $\mathcal{B}$ be the following full subcategories of $\mathrm{mod}B_0$

$$\begin{aligned}\mathcal{A} &= \mathrm{ind}(\mathrm{add}(\{\tau_n^{-i}(B_0) \mid i\geqslant 0\})),\\ \mathcal{P} &= \mathrm{ind}(\mathrm{proj}B_0),\\ \mathcal{B} &= \{M\in\mathcal{A} \mid M\notin\mathcal{P}\}.\end{aligned}$$

Foe $P\in\mathcal{P}$ and $M\in\mathcal{B}$, we have

$$\begin{aligned}D\mathrm{Hom}_{B_0}(M,P) \cong& \mathrm{Hom}_{\mathcal{D}(B_0)}(P, \mathbb{S}_{B_0}(M))\\ =& \mathrm{Hom}_{\mathcal{D}(B_0)}(P, \Sigma^n(\tau_n(M))).\end{aligned}$$

The above space vanishes since $P\in\mathcal{P}$ and $\tau_n(M)\in\mathrm{mod}B_0$. Then, by Lemma 4.2.6 below, the homotopy cofiber of $f_0 : B_0\to A_0$ is equal to the usual quotient A_0/A_0eA_0.

$\surd$

Lemma 4.2.6. *Let $\mathcal{A}$ be a dg category, $\mathcal{P} \subseteq \mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{A}$ two full dg subcategories such that* $\mathrm{obj}(\mathcal{A}) = \mathrm{obj}(\mathcal{P}) \sqcup \mathrm{obj}(\mathcal{B})$ *and* $\mathrm{Hom}_{\mathcal{A}}(B, P)$ *is acyclic for all $B \in \mathcal{B}$, $P \in \mathcal{P}$. Then the Drinfeld dg quotient $\mathcal{A}/\mathcal{P}$ is Morita equivalent to $\mathcal{B}$.*

Proof. The restriction functor $f_* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ induced by $f : \mathcal{B} \hookrightarrow \mathcal{A}$ is a localization functor. Moreover, its kernel is generated (as a localizing subcategory) by its intersection with per$\mathcal{A}$. Since the space $\mathrm{Hom}_{\mathcal{A}}(B, P)$ is acyclic for all $B \in \mathcal{B}$ and $P \in \mathcal{P}$, the induction functor

$$\mathcal{D}(\mathcal{P}) \hookrightarrow \mathcal{D}(\mathcal{A})$$

induces an equivalence between $\ker(f_*)$ and $\mathcal{D}(\mathcal{P})$. Thus, we have an exact sequence of triangulated categories

$$0 \to \mathcal{D}(\mathcal{P}) \to \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B}) \to 0.$$

It follows that the Drinfeld dg quotient $\mathcal{A}/\mathcal{P}$ is Morita equivalent to $\mathcal{B}$.

$\surd$

Proposition 4.2.7. *Via the relative $(n+2)$-Calabi–Yau completion of $f_0 : B_0 \hookrightarrow A_0$, we get the following dg functor which has a canonical left $(n+2)$-Calabi–Yau structure*

$$f : B = \mathbf{\Pi}_{n+1}(B_0) \longrightarrow A = \mathbf{\Pi}_{n+2}(A_0, B_0).$$

Then

1) *The dg algebra A is concentrated in degree 0.*
2) *$H^0(A)$ is a finite-dimensional algebra with finite global dimension at most $n+2$.*
3) *The homotopy cofiber of f is equal to $\mathbf{\Pi}_{n+2}(A_0/A_0eA_0)$ where $e = f(1_{B_0})$.*
4) *The functor $H^0(f) : H^0(B) = \widetilde{B_0} \to H^0(A)$ is fully faithful.*
5) *A is internally bimodule $(n+2)$-Calabi–Yau with respect to $e = f(1_{B_0})$.*

Proof. The first and fourth statement follows from Corollary 4.1.7. The third statement follows from Proposition 2.7.1 and the last statement follows from Proposition 3.7.6. It remains to show the second one.

By Corollary 4.1.7 and the fact that B_0 is τ_n-finite, the algebra $H^0(\mathbf{\Pi}_{n+2}(A_0, B_0))$ is finite-dimensional. By Proposition 3.7.1, the algebra $H^0(\mathbf{\Pi}_{n+2}(A_0, B_0))$ has finite global dimension at most $n+2$.

$\surd$

Suppose that 1_{B_0} has decomposition

$$1_{B_0} = e_1 + e_2 + \cdots + e_n$$

into primitive orthogonal idempotents. We denote by $P_i = e_iB_0$ the projective B_0-module associated with the idempotent e_i. Let $\mathcal{U}$ be the following full subcategory of $\mathcal{D}^b(\mathrm{mod}B_0)$ (see [48, Theorem 2.16])

$$\mathcal{U} = \mathrm{add}\{\mathbb{S}_n^i B_0 \,|\, i \in \mathbb{Z}\} \subseteq \mathcal{D}^b(\mathrm{mod}B_0),$$

where $\mathbb{S}_n = \Sigma^{-n}\mathbb{S}_{B_0}$ and $\mathbb{S}_{B_0}$ is the Serre functor of $\mathcal{D}^b(\mathrm{mod}B_0)$. By [48, Theorem 2.16], $\mathcal{U}$ is an n-cluster tilting subcategory of $\mathcal{D}^b(\mathrm{mod}B_0)$.

Let $\Sigma^{\mathbb{Z}}\mathcal{U} = \mathrm{add}\{\Sigma^k\mathbb{S}_n^i B_0 \,|\, i,k \in \mathbb{Z}\}$ be the Σ closure of $\mathcal{U}$ in $\mathcal{D}^b(\mathrm{mod}B_0)$. It is a bigraded category where the gradings are given by $\mathbb{S}_n$ and Σ.

The dg category

$$\mathbf{\Pi}_{n+1}(B_0) = T_{B_0}(\Sigma^n B_0^\vee)$$

is Adams graded with $|\Sigma^n B_0^\vee|_a = 1$. Let $P_i^{\mathbf{\Pi}} = e_i\mathbf{\Pi}_{n+1}(B_0)$ be the cofibrant dg $\mathbf{\Pi}_{n+1}(B_0)$-module associated with e_i. For any integer k, let $T^k P_i^{\mathbf{\Pi}}$ be the shift of $P_i^{\mathbf{\Pi}}$ by degree k with respect the Adams grading.

We denote by $\mathcal{C}_{dg}^{\mathbb{Z}}(\mathbf{\Pi}_{n+1}(B_0))$ the category of Adams graded dg $\mathbf{\Pi}_{n+1}(B_0)$-modules with morphisms of bigraded $(0,0)$. The corresponding derived category is denoted by $\mathcal{D}^{\mathbb{Z}}(\mathbf{\Pi}_{n+1}(B_0))$.

For any two objects $P_i^{\mathbf{\Pi}}$, $P_j^{\mathbf{\Pi}}$ in $\mathcal{D}^{\mathbb{Z}}(\mathbf{\Pi}_{n+1}(B_0))$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}^{\mathbb{Z}}(\mathbf{\Pi}_{n+1}(B_0))}(P_i^{\mathbf{\Pi})}, T^k\Sigma^p P_j^{\mathbf{\Pi}}) &\simeq \mathrm{Hom}_{\mathcal{D}(\mathrm{mod}B_0)}(P_i, \mathbb{S}_n^{-k}\Sigma^i P_j)\\ &= (k,p)\text{-component of } e_j H^*(\mathbf{\Pi}_{n+1}(B_0))e_i,\end{aligned}$$

where $H^*(\mathbf{\Pi}_{n+1}(B_0))$ is the graded algebra whose i-th component is $H^i(\mathbf{\Pi}_{n+1}(B_0))$.

We have an equivalence of bigraded categories

$$\mathcal{D}^{\mathbb{Z}}(\mathbf{\Pi}_{n+1}(B_0)) \supseteq \mathrm{add}(T^k\Sigma^p P_i^{\mathbf{\Pi}} \,|\, i,k \in \mathbb{Z}) \xrightarrow{\sim} \Sigma^{\mathbb{Z}}\mathcal{U}$$

which maps $T^k\Sigma^p P_i^{\mathbf{\Pi}}$ to $\mathbb{S}_n^{-k}\Sigma^p P_i$.

Via taking the orbit categories with respect to T and $\mathbb{S}_n$ respectively, we have the following equivalence of graded categories

$$\mathrm{add}(T^k\Sigma^p P_i^{\mathbf{\Pi}} \,|\, i,k \in \mathbb{Z})/T \xrightarrow{\sim} \Sigma^{\mathbb{Z}}\mathcal{U}/\mathbb{S}_n.$$

We denote by $\mathrm{Gpr}(H^*\mathbf{\Pi}_{n+1}(B_0))$ the category of graded projective modules over $\mathbf{\Pi}_{n+1}(B_0)$. There is an equivalence of graded categories

$$\mathrm{add}(T^k\Sigma^p P_i^{\mathbf{\Pi}} \,|\, i,k \in \mathbb{Z})/T \simeq \mathrm{Gpr}(H^*\mathbf{\Pi}_{n+1}(B_0)).$$

Thus, we have an equivalence of graded categories

$$\mathrm{Gpr}(H^*\mathbf{\Pi}_{n+1}(B_0)) \simeq \Sigma^{\mathbb{Z}}\mathcal{U}/\mathbb{S}_n.$$

Since B_0 is n-representation-finite, by [48, Theorem 3.1 and Proposition 3.6], we have

$$\mathrm{Hom}_{\mathcal{D}(\mathrm{mod}B_0)}(\Sigma^i\mathcal{U}, \mathcal{U}) = 0$$

for $1 \leqslant i \leqslant n-1$. Thus, the i-th homology $H^i(B)$ of $B = \mathbf{\Pi}_{n+1}(B_0)$ vanishes for $i = -1, \ldots, -n+1$.

Lemma 4.2.8. *The higher preprojective algebra $\widetilde{B_0}$ is self-injective.*

Proof. It is enough to show that $\widetilde{B_0}$ is injective as a right $\widetilde{B_0}$-module. The category $\mathrm{pvd}(B)$ has a canonical t-structure

$$(\mathrm{pvd}(B)_{\leqslant 0}, \mathrm{pvd}(B)_{\geqslant 0}),$$

where $\mathrm{pvd}(B)_{\leqslant 0}$ is the full subcategory of $\mathrm{pvd}(B)$ whose objects are the dg modules X such that $H^p(X)$ vanishes for all $p > 0$ and $\mathrm{pvd}(B)_{\geqslant 0}$ is the full subcategory of $\mathrm{pvd}(B)$ whose objects are the dg modules X such that $H^p(X)$ vanishes for all $p < 0$. The corresponding heart is equivalent to $\mathrm{mod}\widetilde{B_0}$. Moreover, by Section 3.1.7 of [8], for all X and Y in $\mathrm{mod}\widetilde{B_0}$, we have an isomorphism

$$\mathrm{Ext}^1_{B_0}(X, Y) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(X, \Sigma Y).$$

Let M be an object in $\mathrm{mod}\widetilde{B_0}$. By the $(n+1)$-Calabi–Yau property of $\mathrm{pvd}(B)$ and the above isomorphism, we have

$$\begin{aligned}\mathrm{Ext}^1_{\widetilde{B_0}}(M, \widetilde{B_0}) &\simeq \mathrm{Hom}_{\mathcal{D}(B)}(M, \Sigma\widetilde{B_0}) \\ &\simeq D\mathrm{Hom}_{\mathcal{D}(B)}(\widetilde{B_0}, \Sigma^n M).\end{aligned}$$

If $n = 1$, we have $\mathrm{Hom}_{\mathcal{D}(B)}(\widetilde{B_0}, \Sigma M) = \mathrm{Hom}_{\mathcal{D}(B)}(\widetilde{B_0}, \Sigma M) \simeq \mathrm{Ext}^1_{\widetilde{B_0}}(\widetilde{B_0}, \Sigma M) = 0$. Suppose that $n > 1$. There exists a canonical triangle in $\mathcal{D}(B)$

$$\tau_{\leqslant -1} B \to B \to \widetilde{B_0} \to \Sigma\tau_{\leqq -1} B.$$

Since the spaces $\mathrm{Hom}_{\mathcal{D}(B)}(B, \Sigma^n M)$ and $\mathrm{Hom}_{\mathcal{D}(B)}(\Sigma B, \Sigma^n M)$ vanish, we have

$$\mathrm{Hom}_{\mathcal{D}(B)}(\widetilde{B_0}, \Sigma^n M) \simeq \mathrm{Hom}_{\mathcal{D}(B)}(\tau_{\leqslant -1} B, \Sigma^{n-1} M).$$

We see that $\tau_{\leqslant -1} B$ is in $\mathcal{D}(B)_{\leqslant -n}$. Thus, the space $\mathrm{Hom}_{\mathcal{D}(B)}(\tau_{\leqslant -1} B, \Sigma^{n-1} M)$ vanishes. Therefore, $\mathrm{Ext}^1_{B_0}(M, \tilde{B}_0)$ vanishes. It follows that $\widetilde{B_0}$ is injective.

$\surd$

By Propositions 3.7.2, 4.2.7 and the above Lemma which first proved in [48, Corollary 3.4], we get the following Theorem.

Theorem 4.2.9. *[48, Theorem 1.1] Consider the relative cluster category $\mathcal{C}_{n+1}(A, B)$ associated with*

$$f : B = \mathbf{\Pi}_{n+1}(B_0) \longrightarrow A = \mathbf{\Pi}_{n+2}(A_0, B_0).$$

a) The Higgs category $\mathcal{H} \subseteq \mathcal{C}_{n+1}(A, B)$ is equivalent to $\mathrm{mod}(\widetilde{B_0})$ *and the image of A in $\mathcal{H}$ is an $(n+1)$-cluster-tilting object.*

b) We have a triangle equivalence $\underline{\mathrm{mod}}(\widetilde{B_0}) \cong \mathcal{C}_{n+1}(A_0/A_0eA_0)$*, where $e = f_0(1_{B_0})$.*

Remark 4.2.10. Above, we have used a different method to reprove Iyama–Oppermann's results in [48]. Notice that the algebra $H^0(\mathbf{\Pi}_{n+2}(A_0, B_0))$, which is quasi-isomorphic to $\mathbf{\Pi}_{n+2}(A_0, B_0)$, is isomorphic to the (non stable) endomorphism algebra of the $(n+1)$-cluster-titling object T given by the image of A in $\mathcal{H}$. This algebra does not appear explicitly in [48].

Example 4.2.11. Let Q be a Dynkin quiver and let $Aus(kQ)$ be the Auslander algebra of the path algebra kQ. We consider the following canonical dg inclusion

$$f_0 : kQ \hookrightarrow Aus(kQ).$$

We know that $\mathrm{gl.dim}(kQ) = 1$ and $\mathrm{gl.dim}(Aus(kQ)) \leqslant 2$. Moreover, the homotopy cofiber of f is the stable Auslander algebra $\overline{Aus}(kQ) = Aus(kQ)/\langle e \rangle$, where $e = f(1_{kQ})$ (see Proposition 4.2.5).

Applying the relative 3-Calabi–Yau completion to the functor f_0, we get the following dg functor f which has a canonical left 3-Calabi–Yau structure and $\mathbf{\Pi}_3(Aus(kQ), kQ)$ is concentrated in degree 0

$$f : \mathbf{\Pi}_2(kQ) \to \mathbf{\Pi}_3(Aus(kQ), kQ).$$

On the level of H^0, we get a fully faithful inclusion

$$H^0(f) : \widetilde{kQ} \hookrightarrow \mathbf{\Pi}_3(Aus(kQ), kQ),$$

where $\widetilde{kQ}$ is the preprojective algebra of Q and hence is self-injective. So the Higgs category $\mathcal{H} = \mathrm{gpr}(\widetilde{kQ})$ is equivalent to $\mathrm{mod}(\widetilde{kQ})$. By Theorem 4.2.9, we get a triangle equivalence

$$\underline{\mathrm{mod}}(\widetilde{kQ}) \xrightarrow{\sim} \mathcal{C}_{\overline{A_0}} = \mathrm{per}\mathbf{\Pi}_3(\overline{A_0})/\mathrm{pvd}(\mathbf{\Pi}_3(\overline{A_0})),$$

where $\overline{A}_0$ is the stable Auslander algebra $\overline{Aus}(kQ)$. Thus, we have reproved that $\mathrm{mod}(\widetilde{kQ})$ contains a canonical cluster-tilting object (see [31]) and that $\underline{\mathrm{mod}}(\widetilde{kQ})$ is triangle equivalent to $\mathcal{C}_{\overline{A}_0}$ (see [6]).

Chapter 5

Derived equivalences from mutations of ice quivers with potential

5.1 Relative Calabi–Yau completions in the pseudocompact setting

Let k be a field and R a finite dimensional separable k-algebra. Let A be an object in PcAlgc(R).

Definition 5.1.1. Let M be a pseudocompact dg A-bimodule. The *completed tensor algebra* $T_A(M)$ is defined as

$$T_A(M) = \prod_{n=0}^{\infty} M^{\widehat{\otimes}^n},$$

where $M^{\widehat{\otimes}^0} = A$ and $M^{\widehat{\otimes}^n} = \underbrace{M\widehat{\otimes}_A M\widehat{\otimes}_A \cdots \widehat{\otimes}_A M}_{n\text{-times}}$ for $n \geqslant 1$. The dg algebra structure on $T_A(M)$ is given by the differentials of A and M and the multiplication is given by the concatenation product.

The *derived completed tensor algebra* is defined as

$$\mathbf{L}T_A(M) = T_A(\mathbf{p}M)$$

where $\mathbf{p}M$ is a cofibrant replacement of M as a pseudocompact dg A-bimodule. Up to weak equivalence, it is well defined.

The ideals $T_A(M)_{\geqslant s} = \prod_{n\geqslant s} M^{\widehat{\otimes}^n}$ are clearly closed in $T_A(M)$ and we have

$$T_A(M) = \varprojlim_{s\in\mathbb{N}} T_A(M)/T_A(M)_{\geqslant s}.$$

Thus, the completed tensor algebra is again in PcAlgc(R).

5.1.1 Relative deformed Calabi–Yau completions

Let S be another finite dimensional separable k-algebra and B an object in PcAlgc(S). Let f be a morphism from B to A. We assume that B and A are topologically homologically smooth and connective. Let $[\xi]$ be an element in $HH_{n-2}(f)$. Our objective is to define the deformed relative n-Calabi–Yau completion of $f : B \to A$ with respect to the Hochschild homology class $[\xi] \in HH_{n-2}(f)$.

The morphism f induces a morphism in $\mathcal{D}_{pc}(A^e)$

$$m_f : B\widehat{\otimes}^L_{B^e} A^e \to A.$$

After taking the bimodule dual, using the smoothness of B, we get a morphism

$$m_f^\vee : A^\vee \to B^\vee \widehat{\otimes}^L_{B^e} A^e.$$

Let Ξ be the cofiber of $m_f^\vee$. The dualizing bimodule $\Theta_f = (\mathrm{cof}(B\widehat{\otimes}^L_{B^e} A^e \to A))^\vee$ of f is quasi-isomorphic to $\Sigma^{-1}\Xi$.

By the definition of Hochschild homology of f, we have the following long exact sequence

$$\cdots \to HH_{n-2}(B) \to HH_{n-2}(A) \to HH_{n-2}(f) \to HH_{n-3}(B) \to \cdots .$$

Thus, the Hochschild homology class $[\xi] \in HH_{n-2}(f)$ induces an element $[\xi_B]$ in $HH_{n-3}(B)$.

Notice that since B, A are smooth, we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}_{pc}(B^e)}(\Sigma^{n-2}B^\vee, \Sigma B) \simeq & H^{3-n}(B\widehat{\otimes}^L_{B^e} B) = HH_{n-3}(B),\\
\mathrm{Hom}_{\mathcal{D}_{pc}(A^e)}(\Sigma^{n-2}\Xi, \Sigma A) \simeq & H^{2-n}(\mathrm{Cone}(B\widehat{\otimes}^L_{B^e} A \to A\widehat{\otimes}^L_{A^e} A))\\
\leftarrow & H^{2-n}(\mathrm{Cone}(B\widehat{\otimes}^L_{B^e} B \to A\widehat{\otimes}^L_{A^e} A))\\
\simeq & HH_{n-2}(f).
\end{aligned}$$

Thus, the homology class $[\xi]$ induces a morphism in $\mathcal{D}_{pc}(A^e)$

$$\xi : \Sigma^{n-2}\Xi \to \Sigma A$$

and the homology class $[\xi_B]$ induces a morphism in $\mathcal{D}_{pc}(B^e)$

$$\xi_B : \Sigma^{n-2}B^\vee \to \Sigma B.$$

Moreover, we have the following commutative diagram in $\mathcal{D}_{pc}(A^e)$

$$\begin{tikzcd}
Lf^*(\Sigma^{n-1}B^\vee) \arrow[r] \arrow[d, "\xi_B"] & \Sigma^{n-2}\Xi \arrow[d, "\xi"] \\
Lf^*(\Sigma B) \arrow[r] & \Sigma A.
\end{tikzcd}$$

Therefore, the morphism ξ_B gives rise to a 'deformation'

$$\mathbf{\Pi}_{n-1}(B, \xi_B)$$

of $\mathbf{\Pi}_{n-1}(B) = \boldsymbol{L}T_B(\Xi^{n-2}B^\vee)$, obtained by adding ξ_B to the differential of $\mathbf{\Pi}_{n-1}(B)$; the morphism ξ gives rise to a 'deformation'

$$\mathbf{\Pi}_n(A, B, \xi)$$

of $\mathbf{\Pi}_n(A, B) = \boldsymbol{L}T_A(\Sigma^{n-2}\Xi)$, obtained by adding ξ to the differential of $\boldsymbol{L}T_A(\Sigma^{n-2}\Xi)$; and the commutative diagram above gives rise to a morphism

$$\tilde{f} : \mathbf{\Pi}_{n-1}(B, \xi_B) \to \mathbf{\Pi}_n(A, B, \xi).$$

A standard argument shows that up to weak equivalence, the morphism $\tilde{f}$ and the deformations $\mathbf{\Pi}_{n-1}(B, \xi_B)$, $\mathbf{\Pi}_n(A, B, \xi)$ only depend on the class $[\xi]$.

Definition 5.1.2. [87, Definition 3.14] The dg functor $\tilde{f}$ defined above is called the *deformed relative n-Calabi–Yau completion* of $f : B \to A$ with respect to the Hochschild homology class $[\xi] \in HH_{n-2}(f)$.

Theorem 5.1.3. *[87, Theorem 3.23][15, Proposition 5.28] If $[\xi]$ has a negative cyclic lift, then each choice of such a lift gives rise to a canonical left n-Calabi–Yau structure on the morphism*

$$\tilde{f} : \mathbf{\Pi}_{n-1}(B, \xi_B) \to \mathbf{\Pi}_n(A, B, \xi).$$

5.2 Ice quiver mutations and complete relative Ginzburg algebras

5.2.1 Ice quivers

Definition 5.2.1. A *quiver* is a tuple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are sets, and $s, t : Q_1 \to Q_0$ are functions. We think of the elements of Q_0 as vertices and those of Q_1 as arrows, so that each $\alpha \in Q_1$ is realised as an arrow $\alpha : s(\alpha) \to t(\alpha)$. We call Q *finite* if Q_0 and Q_1 are finite sets.

Definition 5.2.2. Let Q be a quiver. A quiver $F = (F_0, F_1, s', t')$ is a *subquiver* of Q if it is a quiver such that $F_0 \subseteq Q_0$, $F_1 \subseteq Q_1$ and the functions s' and t' are the restrictions of s and t to F_1 . We say F is a *full subquiver* if $F_1 = \{\alpha \in Q_1 : s(\alpha), t(\alpha) \in F_0\}$, so that a full subquiver of Q is completely determined by its set of vertices.

Definition 5.2.3. An *ice quiver* is a pair (Q, F), where Q is a finite quiver, F is a (not necessarily full) subquiver of Q. We call F_0, F_1 and F the frozen vertices, arrows and subquiver respectively. We also call $Q_0 \setminus F_0$ and $Q_1 \setminus F_1$ the unfrozen vertices and arrows respectively.

5.2.2 Combinatorial mutations

Definition 5.2.4. (Pressland [80, Definition 4.4]) Let (Q, F) be an ice quiver and let v be an unfrozen vertex such that no loops or 2-cycles of Q are incident with v. Then the *extended mutation* $\mu_v^P(Q, F) = (\mu_v^P(Q), \mu_2^P(F))$ of (Q, F) at v is defined to be the output of the following procedure.

(1) For each pair of arrows $\alpha : u \to v$ and $\beta : v \to w$, add an unfrozen arrow $[\beta\alpha] : u \to w$ to Q.

(2) Replace each arrow $\alpha : u \to v$ by an arrow $\alpha^* : v \to u$, and each arrow $\beta : v \to w$ by an arrow $\beta^* : w \to v$.

(3) Remove a maximal collection of unfrozen 2-cycles, i.e. 2-cycles avoiding the subquiver F.

(4) Choose a maximal collection of half-frozen 2-cycles, i.e. 2-cycles in which precisely one arrow is frozen. Replace each 2-cycle in this collection by a frozen arrow, in the direction of the unfrozen arrow in the 2-cycle.

Remark 5.2.5. Because of the choices involved in steps (3) and (4), this operation is only defined up to quiver isomorphism. If we ignore all arrows between frozen vertices, we obtain the usual definition of Fomin–Zelevinsky mutation (see [27, Definition 4.2]).

Example 5.2.6. Consider the ice quiver (Q, F) given by

$$\begin{tikzcd} & \boxed{2} \arrow[dr, "c"] & \\ \boxed{1} \arrow[ur, "b"] & & 3 \arrow[ll, "a"] \end{tikzcd},$$

where the blue part is the subquiver F.

We perform the extended Fomin–Zelevinsky mutation at the vertex 3. Then we get the following ice quiver $\mu_3^P(Q, F) = (Q', F')$

$$\begin{tikzcd} & \boxed{2} \arrow[dl, "{[ac]}"'] & \\ \boxed{1} \arrow[rr, "a^*"'] & & 3 \arrow[ul, "c^*"'] \end{tikzcd}.$$

If we preform the usual Fomin–Zelevinsky mutation at the vertex 3, we get a different final quiver

$$\begin{tikzcd} & 2 & \\ 1 \arrow[rr, "a^*"'] & & 3 \arrow[ul, "c^*"'] \end{tikzcd}.$$

5.2.3 Ice quivers with potential

Let k be a field. Let Q be a finite quiver.

Definition 5.2.7. Let S be the semisimple k-algebra $\prod_{i \in Q_0} ke_i$. The vector space kQ_1 naturally becomes an S-bimodule. Then the *complete path algebra* of Q is the completed tensor algebra

$$\widehat{kQ} = T_S(kQ_1).$$

It has underlying vector space

$$\prod_{d=0}^{\infty} (kQ_1)^{\otimes_S^d},$$

and multiplication given by concatenation. The algebra $\widehat{kQ}$ becomes a graded pseudo-compact S-algebra.

Definition 5.2.8. [80, Definition 2.8] The natural grading on $\widehat{kQ}$ induces a grading on the continuous Hochschild homology $HH_0(\widehat{kQ}) = \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$. A *potential* on Q is an element W in $HH_0(\widehat{kQ})$ expressible as a (possibly infinite) linear combination of homogeneous elements of degree at least 2, such that any term involving a loop has degree at least 3. An *ice quiver with potential* is a tuple (Q, F, W) in which (Q, F) is a finite ice quiver and W is a potential on Q. If $F = \emptyset$ is the empty quiver, then $(Q, \emptyset, W) = (Q, W)$ is called simply a *quiver with potential*. We say that W is *irredundant* if each term of W includes at least one unfrozen arrow.

A potential can be thought of as a formal linear combination of cyclic paths in Q (of length at least 2), considered up to the equivalence relation on such cycles induced by

$$\alpha_n \cdots \alpha_1 \sim \alpha_{n-1} \cdots \alpha_1 \alpha_n.$$

Definition 5.2.9. Let $p = \alpha_n \cdots \alpha_1$ be a cyclic path, with each $\alpha_i \in Q_1$, and let $\alpha \in Q_1$ be any arrow. Then the *cyclic derivative* of p with respect to α is

$$\partial_\alpha p = \Sigma_{\alpha_i = \alpha} \alpha_{i-1} \cdots \alpha_1 \cdots \alpha_{i+1}.$$

We extend ∂_α by linearity and continuity. Then it determines a map $HH_0(\widehat{kQ}) \to \widehat{kQ}$. For an ice quiver with potential (Q, F, W), we define the *relative Jacobian algebra*

$$J(Q, F, W) = \widehat{kQ}/\overline{\langle \partial_\alpha W : \alpha \in Q_1 \setminus F_1 \rangle}.$$

If $F = \emptyset$, we call $J(Q, W) = J(Q, \emptyset, W)$ the *Jacobian algebra* of the quiver with potential (Q, W).

Definition 5.2.10. Let Q be a quiver. An ideal of $\widehat{kQ}$ is called *admissible* if it is contained in the square of the closed ideal generated by the arrows of Q. We call an ice quiver with potential (Q, F, W) *reduced* if W is irredundant and the Jacobian ideal of $\widehat{kQ}$ determined by F and W is admissible. An ice quiver with potential (Q, F, W) is *trivial* if its relative Jacobian algebra $J(Q, F, W)$ is a product of copies of the base field k.

Definition 5.2.11. [80, Definition 3.7] Let (Q, F, W) and(Q', F', W') be ice quivers with potential such that $Q_0 = Q'_0$ and $F_0 = F'_0$. In particular, this means that $\widehat{kQ}$ and $\widehat{kQ'}$ are complete tensor algebras over the same semisimple algebra $S = kQ_0$. An isomorphism $\varphi : \widehat{kQ} \to \widehat{kQ'}$ is said to be a *right equivalence* of the ice quivers with potential if

(1) $\varphi|_S = \mathbf{1}_S$,

(2) $\varphi(\widehat{kF}) = \widehat{kF'}$, where $\widehat{kF}$ and $\widehat{kF'}$ are treated in the the natural way as subalgebras of $\widehat{kQ}$ and $\widehat{kQ'}$ respectively, and

(3) $\varphi(W)$ equals W' in $HH_0(\widehat{kQ'})$.

The following lemma provides a normal form for irredundant potentials, up to right equivalence.

Lemma 5.2.12. *[80, Lemma 3.14] Let (Q, F, W) be an ice quiver with potential such that W is irredundant. Then W admits a representative of the form*

$$\widetilde{W} = \sum_{i=1}^{M} \alpha_i\beta_i + \sum_{i=M+1}^{N}(\alpha_i\beta_i + \alpha_i p_i) + W_1 \tag{5.1}$$

for some arrows α_i and β_i and elements $p_i \in \mathfrak{m}^2$, where

(i) α_i is unfrozen for all $1 \leqslant i \leqslant N$, and β_i is frozen if and only if $i > M$. Then for $1 \leqslant i \leqslant M$, the $\alpha_i\beta_i$ are unfrozen 2-cycles and for $M + 1 \leqslant i \leqslant N$, they are half frozen 2-cycles,

(ii) the arrows α_i and β_i with $1 \leqslant i \leqslant M$ each appear exactly once in the expression (5.1),

(iii) the arrows β_i for $1 \leqslant i \leqslant N$, do not appear in any of the p_j, and

(iv) the arrow α_i and β_i, for $1 \leqslant i \leqslant N$, do not appear in the term W_1, and this term does not contain any 2-cycles.

The following result allows us to replace any ice quiver with potential by a reduced one, without affecting the isomorphism class of the Jacobian algebra.

Theorem 5.2.13. *[80, Theorem 3.6] Let (Q, F, W) be an ice quiver with potential. Then there exists a reduced ice quiver with potential $(Q_{red}, F_{red}, W_{red})$ such that $J(Q, F, W) \cong J(Q_{red}, F_{red}, W_{red})$.*

Proposition 5.2.14. *[80, Proposition 3.15] Let (Q, F, W) be an irredundant ice quiver with potential. Then the ice quiver with potential $(Q_{red}, F_{red}, W_{red})$ from Theorem 5.2.13 is uniquely determined up to right equivalence by the right equivalence class of (Q, F, W).*

Definition 5.2.15. [80, Definition 3.16] Let (Q, F, W) is an irredundant ice quiver with potential. We call $(Q_{red}, F_{red}, W_{red})$ from Theorem 5.2.13 the *reduction* of (Q, F, W).

5.2.4 Algebraic mutations

Let (Q, F, W) be an ice quiver with an irredundant potential. Let v be an unfrozen vertex of Q such that no loops or 2-cycles of Q are incident with v.

Definition 5.2.16. [80, Definition 4.1] The ice quiver with potential $\tilde{\mu}_v(Q, F, W)$, called the *pre-mutation* of (Q, F, W) at v, is the output of the following procedure.

(1) For each pair of arrows $\alpha : u \to v$ and $\beta : v \to w$, add an unfrozen 'composite' arrow $[\beta\alpha] : u \to w$ to Q.

(2) Reverse each arrow incident with v.

(3) Pick a representative $\widetilde{W}$ of W in kQ such that no term of W begins at v (which is possible since there are no loops at v). For each pair of arrows α, β as in (1), replace each occurrence of $\beta\alpha$ in $\widetilde{W}$ by $[\beta\alpha]$, and add the term $[\beta\alpha]\alpha^*\beta^*$.

Let us write (Q', F', W') for $\tilde{\mu}_v(Q, F, W)$. It is clear that F' equals F and the new potential W' is also irredundant, since the arrows $[\beta\alpha]$ are unfrozen, but it need not be reduced even if (Q, F, W) is. We define $\mu_v(Q, F, W)$ as replace the resulting ice quiver with potential $\tilde{\mu}_v(Q, F, W)$ by its reduction, as in Theorem 5.2.13, this being unique up to right equivalence by Proposition 5.2.14. We call μ_v the *mutation* at the vertex v.

Theorem 5.2.17. *a) [80, Proposition 4.2] The right equivalence class of $\mu_v(Q, F, W)$ is determined by the right equivalence class of (Q, F, W).*

b) [80, Theorem 4.3] Let (Q, F, W) be a reduced ice quiver with potential and $v \in Q_0 \setminus F_0$ an unfrozen vertex. Then $\mu_v^2(Q, F, W)$ is right equivalent to (Q, F, W).

Example 5.2.18. Consider the following ice quiver with potential (Q, F, W)

$$Q = \begin{tikzcd} \boxed{1} \arrow[dr, "\alpha"'] & & \boxed{3} \arrow[ll, dashed, "\gamma"'] \\ & 2 \arrow[ur, "\beta"'] & \end{tikzcd},$$

where F is the full subquiver on $\{1, 3\} \subseteq Q_0$ and the potential $W = \gamma\beta\alpha$. Then the pre-mutation of (Q, F, W) at vertex 2 is given by the following ice quiver with potential

$$\tilde{\mu}_2(Q) = \begin{tikzcd} \boxed{1} \arrow[rr, bend left, "{[\beta\alpha]}"] & & \boxed{3} \arrow[ll, dashed, bend left, "\gamma"] \arrow[dl, "\beta^*"] \\ & 2 \arrow[ul, "\alpha^*"] & \end{tikzcd},$$

where F is the subquiver with vertex set $F_0 = \{1, 2\}$ and arrow set $F_1 = \{\gamma\}$ and the new potential is $\tilde{\mu}_2(W) = \alpha^*\beta^*[\beta\alpha] + \gamma[\beta\alpha]$. This ice quiver with potential is not reduced. The mutation $\mu_2(Q, F, W)$ is given by its reduction, which is the ice quiver

$$(\mu_2(Q), \mu_2(F)) = \begin{tikzcd} \boxed{1} \arrow[rr, dashed, "{[\beta\alpha]}"] & & \boxed{3} \arrow[dl, "\beta^*"] \\ & 2 \arrow[ul, "\alpha^*"] & \end{tikzcd}$$

with potential $\mu_2 W = \beta^*[\beta\alpha]\alpha^*$.

Theorem 5.2.19. *[80, Proposition 4.6] Let (Q, F, W) be an ice quiver with an irredundant potential and v an unfrozen vertex. If $(\mu_v Q, \mu_v F)$ has no 2-cycles containing unfrozen arrows, then the underlying ice quiver of $\mu_v(Q, F, W)$ agrees with $\mu_v^P(Q, F)$ defined in Definition 5.2.4.*

5.2.5 The complete relative Ginzburg algebra and the Ginzburg functor

Definition 5.2.20. Let (Q, F, W) be a finite ice quiver with potential. Let $\widetilde{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q,
- an arrow $a^\vee : j \to i$ of degree -1 for each unfrozen arrow a,
- a loop $t_i : i \to i$ of degree -2 for each unfrozen vertex i.

Define the *complete relative Ginzburg dg algebra* $\mathbf{\Gamma}_{rel}(Q, F, W)$ as the dg algebra whose underlying graded space is the completed graded path algebra $\widehat{k\widetilde{Q}}$. Its differential is the unique k-linear continuous endomorphism of degree 1 which satisfies the Leibniz rule

$$d(u \circ v) = d(u) \circ v + (-1)^p u \circ d(v)$$

for all homogeneous u of degree p and all v and takes the following values on the arrows of $\widetilde{Q}$:

- $d(a) = 0$ for each arrow a of Q,
- $d(a^\vee) = \partial_a W$ for each unfrozen arrow a,
- $d(t_i) = e_i(\sum_{a \in Q_1}[a, a^*])e_i$ for each unfrozen vertex i, where e_i is the lazy path corresponding to the vertex i.

Similarly, we define the *complete derived preprojective algebra* $\mathbf{\Pi}_2(F)$ for any finite quiver F. Let $\widetilde{F}$ be the graded quiver with the same vertices as F and whose arrows are

- the arrows of F,
- an arrow $\tilde{a} : j \to i$ of degree 0 for each arrow a of F,
- a loop $r_i : i \to i$ of degree -1 for each vertex i of F.

Define the complete derived preprojective algebra $\mathbf{\Pi}_2(F)$ as the dg algebra whose underlying graded space is the completed graded path algebra $\widehat{k\widetilde{F}}$. Its differential is the unique k-linear continuous endomorphism of degree 1 which satisfies the Leibniz rule

$$d(u \circ v) = d(u) \circ v + (-1)^p u \circ d(v)$$

for all homogeneous u of degree p and all v, and takes the following values on the arrows of $\widetilde{F}$:

- $d(a) = 0$ for each arrow a of F,
- $d(\tilde{a}) = 0$ for each arrow a in F,
- $d(r_i) = e_i(\sum_{a \in F_1}[a, \tilde{a}])e_i$ for each vertex i of F, where e_i is the lazy path corresponding to the vertex i.

Via the deformed relative 3-Calabi–Yau completion of $G : \widehat{kF} \hookrightarrow \widehat{kQ}$ with respect to the potential W, we get a dg functor

$$\boldsymbol{G}_{rel} : \boldsymbol{\Pi}_2(F) \to \boldsymbol{\Gamma}_{rel}(Q, F, W).$$

We call it *Ginzburg functor*. It is given explicitly as follows:

- $\boldsymbol{G}_{rel}(i) = i$ for each frozen vertex $i \in F_0$,
- $\boldsymbol{G}_{rel}(a) = a$ for each arrow $a \in F_1$,
- $\boldsymbol{G}_{rel}(\tilde{a}) = -\partial_a W$ for each arrow $a \in F_1$,
- $\boldsymbol{G}_{rel}(r_i) = e_i(\sum_{a \in Q_1 \setminus F_1}[a, a^\vee])e_i$ for each frozen vertex $i \in F_0$.

By Theorem 5.1.3, we get the following Proposition.

Proposition 5.2.21. *The Ginzburg functor $\boldsymbol{G}_{rel} : \boldsymbol{\Pi}_2(F) \to \boldsymbol{\Gamma}_{rel}(Q, F, W)$ has a canonical left 3-Calabi–Yau structure.*

The following lemma is an easy consequence of the definition.

Lemma 5.2.22. *Let (Q, F, W) be an ice quiver with potential. Then the Jacobian algebra $J(Q, F, W)$ is the 0-th homology of the complete Ginzburg dg algebra $\boldsymbol{\Gamma}_{rel}(Q, F, W)$, i.e.*

$$J(Q, F, W) = H^0(\boldsymbol{\Gamma}_{rel}(Q, F, W)).$$

Moreover, the complete preprojective algebra $\widetilde{kF}$ is the 0-th homology of the complete derived preprojective algebra $\boldsymbol{\Pi}_2(F)$, i.e.

$$\widetilde{kF} = H^0(\boldsymbol{\Pi}_2(F)).$$

Let (Q, F, W) be an ice quivers with potential. Let $\boldsymbol{\Gamma}_{rel} = \boldsymbol{\Gamma}_{rel}(Q, F, W)$ be the associated complete relative Ginzburg dg algebra. Let $(Q_{red}, F_{red}, W_{red})$ be the reduction of (Q, F, W) as in Theorem 5.2.13.

Lemma 5.2.23. *There is an irredundant potential W' such that $\boldsymbol{\Gamma}_{rel}(Q, F, W) \cong \boldsymbol{\Gamma}_{rel}(Q, F, W')$.*

Proof. We collect all terms containing only frozen arrows. Then there is a unique decomposition $W = W' + W_F$ in which W' is irredundant and W_F is a potential on F. Since $\partial_a W_F$ is 0 for any arrow $a \in Q_1 \setminus F_1$, it is clear that $\boldsymbol{\Gamma}_{rel}(Q, F, W)$ is isomorphic to $\boldsymbol{\Gamma}_{rel}(Q, F, W')$. $\surd$

Lemma 5.2.24. *Let (Q', F', W') be another ice quivers with potential. We denote by $\boldsymbol{\Gamma}'_{rel} = \boldsymbol{\Gamma}_{rel}(Q', F', W')$ the associated complete relative Ginzburg dg algebra. Assume that (Q, F, W) and (Q', F', W') are right-equivalent. Then $\boldsymbol{\Gamma}_{rel}$ and $\boldsymbol{\Gamma}'_{rel}$ are isomorphic to each other.*

Proof. The proof is the same as that of [64, Lemma 2.9]. $\surd$

Lemma 5.2.25. *There exists a quasi-isomorphism between the complete relative Ginzburg algebras*

$$\boldsymbol{\Gamma}_{rel}(Q, F, W) \to \boldsymbol{\Gamma}_{rel}(Q_{red}, F_{red}, W_{red}).$$

Proof. By Lemma 5.2.23, we can assume that W is irredundant. By Lemmas 5.2.12 and 5.2.24, we can assume that W has an expression of the form (5.1) satisfying conditions (i)–(iv), i.e. the potential W can be written as

$$W = \sum_{i=1}^{M} \alpha_i\beta_i + \sum_{i=M+1}^{N}(\alpha_i\beta_i + \alpha_i p_i) + W_1$$

as in (5.1). Take Q' to be the subquiver of Q consisting of all vertices and the arrows α_i, β_i for $i \leqslant M$ and $W' = \sum_{i=1}^{M} \alpha_i\beta_i$. Then the quiver with potential (Q', W') is trivial.

Let Q'' be the subquiver of Q consisting of all vertices and those arrows that are not in Q'_1. Let $W'' = W - W' = \sum_{i=M+1}^{N}(\alpha_i\beta_i + \alpha_i p_i) + W_1$. We see that W'' does not involve any arrows of Q'. Thus, this defines a potential on Q''. As in [64, Lemma 2.10], we see that the canonical projection $\Gamma_{rel}(Q, F, W) \to \Gamma_{rel}(Q'', F, W'')$ is a quasi-isomorphism. Simplifying the expression for W'' and relabeling arrows for simplicity, we have

$$W'' = \sum_{i=1}^{K} \alpha_i\beta_i + W_{red},$$

where each α_i is unfrozen and each β_i is frozen and does not appear in any term of W_{red}. This is ensured by the condition (iii) in Lemma 5.2.12.

By the proof of [80, Theorem 3.6], the ice quiver (Q^{red}, F^{red}) is obtained from (Q'', F) by deleting β_i and freezing α_i for each $1 \leqslant i \leqslant K$. Then the ice quiver with potential $(Q_{red}, F_{red}, W_{red})$ is the reduction of (Q, F, W).

Let M_{red} be the pseudocompact dg $\widehat{kQ''_0}$-bimodule generated by

$$S_{red} = \{\gamma, \delta^\vee, t_i \,|\, \gamma \in (Q_{red})_1,\ \delta \in (Q_{red})_1 \setminus (F_{red})_1, i \in (Q_{red})_0 \setminus (F_{red})_0\}.$$

Let M'' be the pseudocompact dg $\widehat{kQ''_0}$-bimodule generated by

$$S'' = S_{red} \cup \{\beta_i, \alpha_i^\vee \,|\, 1 \leqslant i \leqslant K\}.$$

By the construction of relative Ginzburg dg algebras, the underlying graded categories have the forms

$$\mathbf{\Gamma}_{rel}(Q_{red}, F_{red}, W_{red}) = T_{\widehat{kQ''_0}}(M_{red})$$

and

$$\mathbf{\Gamma}_{rel}(Q'', F, W'') = T_{\widehat{kQ''_0}}(M'').$$

It is easy to see that we have an inclusion $i : M_{red} \hookrightarrow M''$ of dg $\widehat{kQ''_0}$-bimodules. Then i induces a fully faithful morphism $i : \Gamma_{rel}(Q_{red}, F_{red}, W_{red}) \hookrightarrow \Gamma_{rel}(Q'', F, W'')$ in $\mathrm{PcAlgc}(\widehat{kQ''_0})$. We define another dg $\widehat{kQ''_0}$-bimodule morphism $\varphi : M'' \to M_{red}$ as follows:

- $\varphi(\gamma) = \gamma$ for any arrow γ in Q''_1 such that $\gamma \neq \beta_i$ with $1 \leqslant i \leqslant K$,
- $\varphi(\gamma^\vee) = \gamma^\vee$ for any arrow γ in $Q''_1 \setminus F_1$ such that $\gamma \neq \alpha_i$ with $1 \leqslant i \leqslant K$,
- $\varphi(\beta_i) = -\partial_{\alpha_i} W_{red}$ for each $1 \leqslant i \leqslant K$,
- $\varphi(\alpha_i^\vee) = 0$ for each $1 \leqslant i \leqslant K$,

- $\varphi(t_i) = t_i$ for each unfrozen vertex i.

Then φ induces a morphism $f : \Gamma_{rel}(Q'', F, W'') \to \Gamma_{rel}(Q_{red}, F_{red}, W_{red})$ in $\mathrm{PcAlgc}(\widehat{kQ''_0})$. To see that f is a quasi-isomorphism, it is enough to show that f is a homotopy inverse of i.

It is clear that we have $f \circ i = \mathbf{1}$. We define a continuous morphism $h : \Gamma_{rel}(Q'', F, W'') \to \Gamma_{rel}(Q'', F, W'')$ of graded k-modules which is homogeneous of degree -1, satisfies

$$h(xy) = h(x)f(y) + (-1)^{-n} x h(y)$$

for all $x \in \mathbf{\Gamma}^{(n)}_{rel}(Q'', F, W'')$, $y \in \mathbf{\Gamma}_{rel}(Q'', F, W'')$ and

- $h(\beta_i) = \alpha_i^{\vee}$ for each $1 \leqslant i \leqslant K$,
- $h(\delta) = 0$ for all other arrows δ.

Then we have $\mathbf{1} - i \circ f = d(h)$. Thus, the morphism f is a homotopy inverse of i. $\surd$

5.2.6 Cofibrant resolutions of simples over a tensor algebra

Let Q be a finite graded quiver and $\widehat{kQ}$ the complete path algebra. Let $\mathfrak{m}$ be the two-sided ideal of $\widehat{kQ}$ generated by arrows of Q. Let $A = (\widehat{kQ}, d)$ be a pseudocompact dg algebra whose differential takes each arrow of Q to an element of $\mathfrak{m}$.

For a vertex i of Q, let $P_i = e_i A$, and let S_i be the simple module corresponding to i. Then we have a short exact sequence in $\mathcal{C}(A)$

$$0 \to \ker(\pi) \to P_i \xrightarrow{\pi} S_i \to 0,$$

where π is the canonical projection from P_i to S_i. More explicitly, the graded A-module $\ker(\pi)$ decomposes as

$$\ker(\pi) = \bigoplus_{\rho \in Q_1 : t(\rho) = i} \rho P_{s(\rho)}$$

with the induced differential. Thus, the simple module S_i is quasi-isomorphic to

$$P = \mathrm{Cone}(\ker(\pi) \to P_i),$$

whose underlying graded space is

$$\bigoplus_{\rho \in Q_1 : t(\rho) = i} \Sigma \rho P_{s(\rho)} \bigoplus P_i.$$

By [64, Section 2.14], the dg module P is a cofibrant dg A-module and hence it is a cofibrant resolution of S_i. In particular, the simple dg module S_i belongs to the perfect derived category $\mathrm{per}(A)$.

5.3 Main results

5.3.1 Compatibility with Morita functors and localizations

Let R be a finite dimensional separable k-algebra. Let $J : A \to A'$ a morphism in PcAlgc(R). We say that J is a *localization functor* if the derived functor J^* induces an equivalence

$$J^* : \mathcal{D}(A)/\mathcal{N} \xrightarrow{\sim} \mathcal{D}(A')$$

for some localizing subcategory $\mathcal{N}$ of $\mathcal{D}(B)$. Equivalently, the restriction functor $J_* : \mathcal{D}(A') \to \mathcal{D}(A)$ is fully faithful.

We say that $J : A \to A'$ is a *Morita functor* if restriction functor J_* is an equivalence from $\mathcal{D}(A')$ to $\mathcal{D}(A)$. Equivalently, the derived functor J^* is an equivalence.

Theorem 5.3.1. *Let S be another finite dimensional separable k-algebra and B an object in* PcAlgc(S)*. Let $I : B \to B'$ be a localization functor in* PcAlgc(S) *and $J : A \to A'$ a localization functor in* PcAlgc(R)*. Suppose that we have morphisms $f : B \to A$ and $f' : B' \to A'$ such that the following square commutes in the category of dg k-algebras*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "I"'] & A \arrow[d, "J"] \\ B' \arrow[r, "f'"] & A'. \end{tikzcd} \tag{5.2}$$

Assume that B, A, B' and A' are smooth and connective. Let $[\xi] = [(s\xi_B, \xi_A)]$ be an element of $HH_{n-2}(f)$ and $[\xi'] = [(s\xi_{B'}, \xi_{A'})]$ the element of $HH_{n-2}(f')$ obtained as the image of $[\xi]$ under the map induced by I and J. Then we have the following commutative diagram in the category of dg k-algebras

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, "I"'] & \mathbf{\Pi}_{n-1}(B, \xi_B) \arrow[r, "\widetilde{f}"] \arrow[d, "I'"'] & \mathbf{\Pi}_n(A, B, \xi) \arrow[d, "J'"'] \\ B' \arrow[r, hook] & \mathbf{\Pi}_{n-1}(B', \xi_{B'}) \arrow[r, "\widetilde{f'}"] & \mathbf{\Pi}_n(A', B', \xi') \end{tikzcd}$$

where I', J' are also localization functors. Moreover, I' (respectively, J') is a Morita functors if I (respectively, J) is.

Proof. By [62, Thm 5.8], there exists a canonical localization functor I' such that the leftmost square above commutes and I' is a Morita functor if I is. We use the same method as in [62, Thm 5.8] to show the existence of J' and the commutativity of the rightmost square.

Let P_B, P_A, $P_{A'}$ and $P_{B'}$ be the canonical bar resolutions of B, A, B' and A' as pseudocompact bimodules over themselves respectively (see [10, Lemma B.2]). We denote by $j_f : f^*(P_B) \to P_A$, $j_{f'} : f'^*(P_{B'}) \to P_{A'}$, $j_I : I^*(P_B) \to P_{B'}$ and $j_J : J^*(P_A) \to P_{A'}$ the canonical morphisms induced by f, f', I and J respectively. We denote by k_I :

$P_B \to I_*(P_{B'})$ the canonical morphism induced by the adjunction (I^*, I_*). Then we have a commutative diagram in $\mathcal{C}_{pc}(A^e)$

$$\begin{tikzcd} P_A^\vee \arrow[r, "j_f^\vee"] \arrow[d, "j_J^\vee"] & f^*(P_B^\vee) \arrow[r, "l_f"] \arrow[d, "f^*(k_I)^\vee"] & \Xi_f \arrow[d, "r"] \\ J_*(P_{A'}^\vee) \arrow[r, "J_*(j_{f'})^\vee"] & J_*f'^*(P_{B'}^\vee) = f^*I_*(P_{B'}^\vee) \arrow[r, "J_*(l_{f'})"] & J_*(\Xi_{f'}), \end{tikzcd} \tag{5.3}$$

where Ξ_f is the mapping cone of $j_f^\vee$ and $\Xi_{f'}$ is the mapping cone of $j_{f'}^\vee$.

By the commutative diagram (5.2), we have the following commutative diagram of Hochschild complexes

$$\begin{tikzcd} HH(B) \arrow[r, "f"] \arrow[d, "I"] & HH(A) \arrow[r] \arrow[d, "J"] & HH(f) \arrow[d] \\ HH(B') \arrow[r, "f'"] & HH(A') \arrow[r] & HH(f'). \end{tikzcd}$$

Thus, we have $\xi_{B'} = I(\xi_B)$ and $\xi_{A'} = J(\xi_A)$.

The Hochschild homology classes $\xi_{B'}$ and $\xi_{A'}$ yield the following two commutative squares in $\mathcal{C}_{pc}(B^e)$ and $\mathcal{C}_{pc}(A^e)$ respectively

$$\begin{tikzcd} P_B^\vee \arrow[r, "\xi_B"] \arrow[d] & B \arrow[d] \\ I_*(P_{B'}^\vee) \arrow[r, "I(\xi_{B'})"] & I_*(B'), \end{tikzcd} \tag{5.4}$$

$$\begin{tikzcd} P_A^\vee \arrow[r, "\xi_A"] \arrow[d] & A \arrow[d] \\ J_*(P_{A'}^\vee) \arrow[r, "J(\xi_{A'})"] & J_*(A'). \end{tikzcd} \tag{5.5}$$

Then we get the following commutative diagram in $\mathcal{C}_{pc}(A^e)$

$$\begin{tikzcd} \Xi_f \arrow[r, "\xi"] \arrow[d] & \mathcal{A} \arrow[d] \\ J_*(\Xi_{f'}) \arrow[r, "\xi'"] & J_*(A'). \end{tikzcd} \tag{5.6}$$

Combining the commutative diagrams (5.3), (5.4), (5.6) and the proofs in [62, Theorem 4.6, Theorem 5.8], we obtain the following commutative diagram in the category of dg

k-algebras.

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, "I"'] & \mathbf{\Pi}_{n-1}(B, \xi_B) \arrow[r, "\widetilde{f}"] \arrow[d, "I'"'] & \mathbf{\Pi}_n(A, B, \xi) \arrow[d, "J'"'] \\ B' \arrow[r, hook] & \mathbf{\Pi}_{n-1}(B', \xi_{B'}) \arrow[r, "\widetilde{f'}"] & \mathbf{\Pi}_n(A', B', \xi'). \end{tikzcd}$$

It remains to be shown that the restriction functor $J'_* : \mathcal{D}(\mathbf{\Pi}_n(A', B', \xi')) \to \mathcal{D}(\mathbf{\Pi}_n(A, B, \xi))$ induced by J' is fully faithful.

Let M be a right $\mathbf{\Pi}_n(A', B', \xi')$-module and suppose that M is cofibrant. It is given by its underlying right A'-module and a morphism of graded modules homogeneous of degree 0

$$\lambda : M \otimes \Xi_{f'}[n-2] \longrightarrow M$$

such that $(d\lambda)(m \otimes x) = m\xi'(x)$ for all $m \in M$ and $x \in \Xi_{f'}[n-2]$. Since $\mathbf{\Pi}_n(A', B', \xi')$ is also cofibrant as right A'-module, the underlying A'-module of M is also cofibrant. Then we have an exact sequence of cofibrant $\mathbf{\Pi}_n(A', B', \xi')$-modules

$$0 \longrightarrow M \otimes_{A'} \Xi_{f'}[n-2] \otimes_{A'} T_{A'}(\Xi_{f'}[n-2]) \xrightarrow{\alpha} M \otimes_{A'} T_{A'}(\Xi_{f'}[n-2]) \longrightarrow M \longrightarrow 0$$

where $\alpha(m \otimes x \otimes u) = mx \otimes u - m \otimes xu$.

This shows that the cone over the following morphism

$$M \otimes_{A'} \Xi_{f'}[n-2] \otimes_{A'} T_{A'}(\Xi_{f'}[n-2]) \xrightarrow{\alpha} M \otimes_{A'} T_{A'}(\Xi_{f'}[n-2])$$

is quasi-equivalent to M. Let N be another right $\mathbf{\Pi}_n(A', B', \xi')$-module. Then $\mathrm{Hom}_{\mathcal{D}(\mathbf{\Pi}_n(A',B',\xi'))}(M, L)$ can be computed as the cone of the following morphism of dg k-modules

$$\mathrm{Hom}_{A'}(M, N) \longrightarrow \mathrm{Hom}_{A'}(M \otimes_{A'} \Xi_{f'}[n-2], N)$$

Similarly, $\mathrm{Hom}_{\mathcal{D}(\mathbf{\Pi}_n(A,B,\xi))}(J'_*(M), J'_*(M))$ can be computed analogously. Thus, it suffices to check that for all N, the dg functor J induces the following bijections

$$\mathrm{Hom}_{\mathcal{D}(A')}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(J_*(M), J_*(N))$$

and

$$\mathrm{Hom}_{\mathcal{D}(A')}(M \otimes^{\boldsymbol{L}}_{A'} \Xi_{f'}[n-2], N) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(J_*(M) \otimes^{\boldsymbol{L}}_{A} \Xi_{f}[n-2], J_*(N)).$$

The first bijection follows from the full faithfulness of J_*. For the second one, it is enough to show that the following two morphisms are bijections

$$\mathrm{Hom}_{\mathcal{D}(A')}(M \otimes^{\boldsymbol{L}}_{A'} A'^{\vee}, N) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(J_*(M) \otimes^{\boldsymbol{L}}_{A} A^{\vee}, J_*(N))$$

$$\mathrm{Hom}_{\mathcal{D}(A')}(M \otimes^{\boldsymbol{L}}_{A'} f'^*(B^{\vee}), N) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(J_*(M) \otimes^{\boldsymbol{L}}_{A} f^*(B^{\vee}), J_*(N)) .$$

In deed, this is a consequence of the full faithfulness of J_* and of the following formulas (see [62, Proposition 3.10])

$$J^*(J_*(M)\otimes^{\boldsymbol{L}}_A A^\vee) \xrightarrow{\sim} M\otimes^{\boldsymbol{L}}_{A'} A'^\vee\,,\; B'^\vee = I^*(B^\vee)$$

and

$$J^*(J_*M\otimes^{\boldsymbol{L}}_A f^*(B^\vee)) \xrightarrow{\sim} M\otimes^{\boldsymbol{L}}_{A'} J^*(f^*(B^\vee)).$$

If J is a Morita functor, by part (e) of [62, Proposition 3.10], the morphisms $j_J^\vee$ and $k_I^\vee$ in diagram 5.3 are quasi-isomorphisms. Then we see that $r:\Xi_f\to J_*(\Xi_{f'})$ is a quasi-isomorphism in $\mathcal{C}_{pc}(A^e)$. It induces quasi-isomorphisms between the tensor powers

$$\Xi_f^{\widehat{\otimes}^n} \longrightarrow J_*(\Xi_{f'}^{\widehat{\otimes}^n})$$

for all $n\geqslant 1$. Thus, r and J yield a morphism

$$J' : \mathbf{\Pi}_n(A,B,\xi)\longrightarrow \mathbf{\Pi}_n(A',B',\xi')$$

which is a quasi-equivalence. Thus, the morphism J' is a Morita functor.

$\surd$

5.3.2 Derived equivalences

Let (Q,F,W) be a finite ice quiver with potential. For a vertex i of Q, let $P_i=e_i\widehat{kQ}$. Let v be an unfrozen vertex of Q such that no loops or 2-cycles are incident with v. Let $T=\oplus_{j\neq v}P_j\oplus T_v$, where T_v is the mapping cone of the following morphism in $\mathcal{D}(\widehat{kQ})$

$$P_v\to \bigoplus_{\alpha\in Q_1:s(\alpha)=v} P_j\,,$$

where the sum is taken over all arrows $\alpha:v\to j$ and the corresponding component of the map from P_v to the sum is the left multiplication by α. If there are no arrows starting at v, the direct sum is zero. It is easy to see that T is a silting object in $\mathcal{D}(\widehat{kQ})$. When v is the source of at least one arrow, the object T_v is quasi-isomorphic to $\operatorname{coker}(P_v\to\bigoplus_{\alpha\in Q_1:s(\alpha)=v}P_j)$. Then T is a tilting module over $\widehat{kQ}$.

Theorem 5.3.2. *[57] Let A be a dg algebra and T an object of $\mathcal{D}(A)$. Denote by A' the dg algebra $R\mathrm{Hom}^\bullet_A(T,T)$. Denote by $\langle T\rangle_A$ the thick subcategory of $\mathcal{D}(A)$ generated by T. Then the functor $R\mathrm{Hom}^\bullet_A(T,?):\mathcal{D}(A)\to\mathcal{D}(A')$ induces a triangle equivalence*

$$\mathbf{R}\mathrm{Hom}^\bullet_A(T,?):\langle T\rangle_A\to \mathrm{per}A'.$$

Now let A' be the pseudocompact dg algebra $R\mathrm{Hom}_{\widehat{kQ}}(T,T)$. Since T is a silting object in $\mathcal{D}(\widehat{kQ})$, by the above Theorem 5.3.2, we have a triangle equivalence

$$?\overset{L}{\otimes}_{A'}T:\mathrm{per}A'\to\langle T\rangle_{\widehat{kQ}}\simeq \mathrm{per}\,\widehat{kQ}.$$

Thus, the pseudocompact dg algebras A' and $\widehat{kQ}$ are Morita equivalent.

Let $\mathcal{A}$ be the full dg subcategory of $\mathcal{C}_{pc}^{dg}(\widehat{kQ})$ whose objects are T_v and the P_i, $i \neq v$. Then $\mathcal{A}$ is Morita equivalent to $\widehat{kQ}$. Let $\mathcal{A}'$ be the full dg subcategory of $\mathcal{C}_{pc}^{dg}(A')$ whose objects are the $P_i' = e_i A'$, $i \in Q_0$. Then $\mathcal{A}'$ is equivalent to A'. We define a dg functor

$$J : \mathcal{A}' \to \mathcal{A}$$

as follows:

- For $k \neq v$, we put $J(P_k') = P_k$,
- For $k = v$, we put $J(P_v') = T_v$.

Then J is a Morita functor. It induces an isomorphism $HC_0(A') \simeq HC_0(\widehat{kQ})$. We denote by $W' \in HC_0(A')$ the element corresponding to $W \in HC_0(\widehat{kQ})$.

Let $\mathcal{B}$ be the full dg subcategory of $\mathcal{C}_{pc}^{dg}(\widehat{kF})$ whose objects are $P_i = e_i\widehat{kF}$, $i \in F_0$. Since the vertex v is unfrozen, we have dg inclusions $G : \mathcal{B} \hookrightarrow \mathcal{A}$ and $G' : \mathcal{B} \hookrightarrow \mathcal{A}'$. Moreover, the following diagram commutes

$$\begin{tikzcd}
 & \mathcal{A}' \arrow[dd, "J"] \\
\mathcal{B} \arrow[ur, hook, "G'"] \arrow[dr, hook, "G"'] & \\
 & \mathcal{A}\,.
\end{tikzcd}$$

Applying the deformed relative 3-Calabi-Yau completion to the above diagram with respect to the potentials W' and $W = J(W')$, we get the following commutative diagram of pseudocompact dg algebras

$$\begin{tikzcd}
 & \mathbf{\Pi}_3(\mathcal{A}', \mathcal{B}, W') \arrow[dd, "J"] \\
\mathbf{\Pi}_2(F) \arrow[ur, "\boldsymbol{G}'_{rel}"] \arrow[dr, "\boldsymbol{G}_{rel}"'] & \\
 & \mathbf{\Gamma}_{rel}(Q, F, W).
\end{tikzcd}$$

By Theorem 5.3.1, the dg functor J is a Morita functor.

We define the quiver Q' to be obtained from the quiver of A' by adding a new arrow $\rho_r : j \to i$ for each minimal relation $r : i \to j$. We defined a potential W_1 on Q' by $W_1 = \Sigma \rho_r r$ and a potential W_2 by lifting W' along the surjection $kQ' \to A'$ taking all arrows ρ_r to zero (see [62, Section 7.6]). Thus, the ice quiver with potential $(Q', F, W_1 + W_2)$ is the pre-mutation of (Q, F, W) at the vertex v, i.e. $\tilde{\mu}_v(Q, F, W) = (Q', F, W_1 + W_2)$. The pseudocompact dg algebra $\mathbf{\Pi}_3(\mathcal{A}', \mathcal{B}, W')$ is quasi-isomorphic to the complete relative Ginzburg algebra $\mathbf{\Gamma}_{rel}(Q', F, W_1 + W_2)$ (see [62, Theorem 6.10]). Therefore, we have the following theorem which generalizes the result in [64, Theorem 3.2].

Theorem 5.3.3. *Let $\mathbf{\Gamma}_{rel} = \mathbf{\Gamma}_{rel}(Q, F, W)$ and $\mathbf{\Gamma}'_{rel} = \mathbf{\Gamma}_{rel}(Q', F, W' = W_1 + W_2)$ be the complete Ginzburg dg algebras associated to (Q, F, W) and $\tilde{\mu}_v(Q, F, W)$ respectively. For a vertex i, let $\mathbf{\Gamma}_i = e_i\mathbf{\Gamma}_{rel}$ and $\mathbf{\Gamma}'_i = e_i\mathbf{\Gamma}'_{rel}$.*

a) There is a triangle equivalence

$$\Phi_+ := J^* : \mathcal{D}(\mathbf{\Gamma}'_{rel}) \longrightarrow \mathcal{D}(\mathbf{\Gamma}_{rel}),$$

which sends the the $\mathbf{\Gamma}'_j$ to $\mathbf{\Gamma}_j$ for $j \neq v$ and to the cone over the morphism

$$\mathbf{\Gamma}_v \to \bigoplus_{\alpha} \mathbf{\Gamma}_{t(\alpha)}$$

for $j = v$, where we have a summand $\mathbf{\Gamma}_{t(\alpha)}$ for each arrow α of Q with source v and the corresponding component of the map is the left multiplication by α. The functor Φ_+ restricts to triangle equivalences from $\operatorname{per}(\mathbf{\Gamma}'_{rel})$ to $\operatorname{per}(\mathbf{\Gamma}_{rel})$ and from $\operatorname{pvd}(\mathbf{\Gamma}'_{rel})$ to $\operatorname{pvd}(\mathbf{\Gamma}_{rel})$.

b) Let $\mathbf{\Gamma}^{red}_{rel}$ respectively $\mathbf{\Gamma}'^{red}_{rel}$ be the complete Ginzburg dg algebra associated with the reduction of (Q, F, W) respectively the reduction $\mu_v(Q, F, W) = (Q'', F'', W'')$ of $\tilde{\mu}_v(Q, F, W)$. Then functor Φ_+ yields a triangle equivalence

$$\Phi^{red}_+ : \mathcal{D}(\mathbf{\Gamma}'^{red}_{rel}) \longrightarrow \mathcal{D}(\mathbf{\Gamma}^{red}_{rel}),$$

which restricts to triangle equivalences from $\operatorname{per}(\mathbf{\Gamma}'^{red}_{rel})$ to $\operatorname{per}(\mathbf{\Gamma}^{red}_{rel})$ and from $\operatorname{pvd}(\mathbf{\Gamma}'^{red}_{rel})$ to $\operatorname{pvd}(\mathbf{\Gamma}^{red}_{rel})$.

c) The following diagram commutes

$$\begin{tikzcd}
 & \mathcal{D}(\mathbf{\Gamma}'_{rel}) \arrow[dd, "\Phi_+"] \\
\mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[ur, "(\boldsymbol{G}'_{rel})^*"] \arrow[dr, "(\boldsymbol{G}_{rel})^*"'] & \\
 & \mathcal{D}(\mathbf{\Gamma}_{rel}).
\end{tikzcd}$$

d) Since the frozen parts of $\mu_v(Q, F, W) = (Q'', F'', W'')$ and of $\tilde{\mu}_v(Q, F, W) = (Q', F, W')$ only differ in the directions of the frozen arrows, we have a canonical isomorphism between $\Pi_2(kF)$ and $\Pi_2(kF'')$. It induces a canonical triangle equivalence

$$\operatorname{can} : \mathcal{D}(\mathbf{\Pi}_2(F)) \to \mathcal{D}(\mathbf{\Pi}_2(F'')).$$

Moreover, the following diagram commutes up to isomorphism

$$\begin{tikzcd} & & \mathcal{D}(\mathbf{\Gamma}'^{red}_{rel}) \arrow[dd, "\Phi^{red}_{+}"] \\ \mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[urr, "(\boldsymbol{G}'_{rel})^*"] \arrow[drr, "(\boldsymbol{G}_{rel})^*"'] & \cong & \\ & & \mathcal{D}(\mathbf{\Gamma}^{red}_{rel}). \end{tikzcd}$$

Remark 5.3.4. Instead of using the silting object $T = \oplus_{j\neq v} P_j \oplus T_v$, we can use $T' = \oplus_{j\neq v} P_j \oplus T'_v$, where T'_v is the shifted cone

$$T'_v = \Sigma^{-1}(\bigoplus_{\beta\in Q_1; t(\beta)=v} P_{s(\beta)} \to P_v),$$

where the sum is taken over all arrows $\beta : j \to v$ and the corresponding component of the map from P_j to P_v is the left multiplication by β. Then T' is also a silting object in $\mathcal{D}(\widehat{kQ})$. When v is the target of at least one arrow, the object T'_v is isomorphic to $\ker(P_v \to \bigoplus_{\alpha\in Q_1: s(\alpha)=v} P_j)$. Then T is a tilting module over $\widehat{kQ}$.

There is also a triangle equivalence $\Phi_- : \mathcal{D}(\mathbf{\Gamma}'_{rel}) \to \mathcal{D}(\mathbf{\Gamma}_{rel})$ which, for $j \neq v$, sends the $\mathbf{\Gamma}'_j$ to $\mathbf{\Gamma}_j$ and for $j = v$, to the shifted cone

$$\Sigma^{-1}(\bigoplus_{\beta\in Q_1; t(\beta)=v} \mathbf{\Gamma}_{s(\beta)} \to \mathbf{\Gamma}_v),$$

where we have a summand $\mathbf{\Gamma}_{s(\beta)}$ for each arrow β of Q with target i and the corresponding component of the morphism is left multiplication by β. Moreover, the two equivalences Φ_+ and Φ_- are related by the twist functor t_{S_v} with with respect to the 3-spherical object S_v, i.e. $\Phi_- = t_{S_v} \circ \Phi_+$. For each object X in $\mathcal{D}(\mathbf{\Gamma}_{rel})$, the object $t_{S_v}(X)$ is given by the following triangle

$$\boldsymbol{R}\mathrm{Hom}(S_v, X) \otimes_k S_v \to X \to t_{S_v}(X) \to \Sigma\boldsymbol{R}\mathrm{Hom}(S_v, X) \otimes_k S_v.$$

Definition 5.3.5. Let (Q, F, W) be a ice quiver with potential. The *boundary dg algebra* is defined to be the dg subalgebra of $\mathbf{\Gamma}_{rel}(Q, F, W)$

$$\mathrm{Bd}(Q, F, W) = \mathrm{REnd}_{\mathbf{\Gamma}_{rel}(Q,F,W)}(\boldsymbol{G}^*_{rel}(\mathbf{\Pi}_2(F))) \simeq e_F \mathbf{\Gamma}_{rel}(Q, F, W) e_F,$$

where $e_F = \sum_{i\in F} e_i$ is the sum of idempotents corresponding to the frozen vertices.

Corollary 5.3.6. The boundary dg algebra is invariant under the mutations at unfrozen vertices. Moreover, if $\mathbf{\Gamma}_{rel}(Q, F, W)$ is concentrated in degree 0, then the boundary dg algebra $\mathrm{Bd}(Q, F, W)$ is also concentrated in degree 0.

Proof. This follows from the Definition of boundary dg algebra and part (c) of Theorem 5.3.3.

$\surd$

5.3.3 Stability under mutation of relative Ginzburg algebras concentrated in degree 0

Let (Q, F, W) be an ice quiver with potential. Let $\mathbf{\Gamma}_{rel}$ be the complete Ginzburg algebra associated with (Q, F, W). For each vertex i of Q, we denote by $\mathbf{\Gamma}_i$ the cofibrant dg $\mathbf{\Gamma}_{rel}$-module associated with i. Recall that the relative Jacobian algebra $J_{rel} = J(Q, F, W)$ is the 0-th homology of $\mathbf{\Gamma}_{rel}(Q, F, W)$.

For each vertex i of Q, we denote by S_i the associated simple module and by $P_i = e_i J_{rel}$ its projective cover. Let v be an unfrozen vertex. Consider the complex T'_{rel} which is the sum of the P_j, $j \neq v$, concentrated in degree 0 and of the complex

$$0 \to P_v \xrightarrow{c} \bigoplus_{\alpha \in Q_1 : s(\alpha) = v} P_{t(\alpha)} \to 0,$$

where P_v is in degree -1 and the components of c are the left multiplications by the corresponding arrows.

Theorem 5.3.7. *[64, Theorem 6.2] Suppose that the complete Ginzburg algebra $\mathbf{\Gamma}_{rel} = \mathbf{\Gamma}_{rel}(Q, F, W)$ has its homology concentrated in degree 0. Then T'_{rel} is a tilting object in the perfect derived category $\mathrm{per}(J_{rel}) \simeq \mathrm{per}(\mathbf{\Gamma}_{rel})$. Thus, the complete relative Ginzburg algebra $\mathbf{\Gamma}'_{rel}$ associated with $\mu_v(Q, F, W)$ still has its homology concentrated in degree 0 and then $\mathrm{Bd}(\mu_v(Q, F, W))$ is concentrated in degree 0.*

Proof. The proof follows the lines of that of [64, Theorem 6.2]. There is a decomposition of J_{rel} as right J_{rel}-module

$$J_{rel} = P_v \bigoplus \bigoplus_{i \in Q_0 : i \neq v} P_i.$$

By the construction of T'_{rel}, we have a map $c : P_v \to \bigoplus_{\alpha \in Q_1 : s(\alpha) = v} P_{t(\alpha)}$. We set $B = \bigoplus_{\alpha \in Q_1 ; s(\alpha) = v} P_{t(\alpha)}$ and $T_1 = \bigoplus_{i \in Q_0 : i \neq v} P_i$. Then by [64, Proposition 6.5], we have to check that the map $c : P_v \to B$ satisfies the following conditions

1) B belongs to $\mathrm{add}(T_1)$;

2) the map $c^* : \mathrm{per}(\mathbf{\Gamma}_{rel})(B, T_1) \to \mathrm{per}(\mathbf{\Gamma}_{rel})(P_v, T_1)$ is surjective and

3) the map $c_* : \mathrm{per}(\mathbf{\Gamma}_{rel})(T_1, P_v) \to \mathrm{per}(\mathbf{\Gamma}_{rel})(T_1, B)$ is injective.

Condition 1) holds since B belongs to $\mathrm{add}(T_1)$. Condition 2) holds since $c : P_v \to B$ is a left $\mathrm{add}(T_1)$-approximation. Finally, in order to show condition 3), it is enough to show that c is injective. Since the homology of Γ_{rel} is concentrated in degree 0, the functor $? \overset{L}{\otimes}_{\mathbf{\Gamma}_{rel}} J_{rel}$ is an equivalence from $\mathrm{per}(\mathbf{\Gamma}_{rel})$ to $\mathrm{per}(J_{rel})$ whose inverse is given by the restriction along the projection morphism $\mathbf{\Gamma}_{rel} \to J_{rel}$. Applying the equivalence $? \overset{L}{\otimes}_{\mathbf{\Gamma}_{rel}} J_{rel}$ to the cofibrant resolution $P_{S_v} \to S_v$ constructed in subsection 5.2.6, we obtain a projective resolution of S_v

$$0 \to P_v \xrightarrow{c} B \to B' \to P_v \to S_v \to 0.$$

Thus, the map c is injective. $\checkmark$

Let $Q_0^m = Q_0 \setminus F_0$ and $Q_1^m = Q_1 \setminus F_1$. Let S be the semisimple k-algebra $\prod_{i\in Q_0} ke_i$. We denote by S^m, V and V^m the S-bimodules generated by Q_0^m, Q_1 and Q_1^m respectively. Let V^{m*} be the dual bimodule $\mathrm{Hom}_{S^e}(V^m, S^e)$.

We have a canonical short exact sequence of $\mathbf{\Gamma}_{rel}$-bimodules

$$0 \to \ker(m) \xrightarrow{\rho} \mathbf{\Gamma}_{rel} \otimes_S \mathbf{\Gamma}_{rel} \xrightarrow{m} \mathbf{\Gamma}_{rel} \to 0,$$

where the map m is induced by the multiplication of $\mathbf{\Gamma}_{rel}$. The mapping cone $\mathrm{Cone}(\rho)$ of ρ is a cofibrant resolution of $\mathbf{\Gamma}_{rel}$ as a bimodule over itself.

Then $J_{rel} \otimes_{\mathbf{\Gamma}_{rel}} \mathrm{Cone}(\rho) \otimes_{\mathbf{\Gamma}_{rel}} J_{rel}$ is the following complex and we denote it by $P(J_{rel})$ (see [79, pp. 10])

$$0 \longrightarrow J_{rel} \otimes_S \otimes R^m \otimes_S J_{rel} \xrightarrow{m_3} J_{rel} \otimes_S \otimes V^{m*} \otimes_S J_{rel} \xrightarrow{m_2} J \otimes_S \otimes V \otimes_S J_{rel} \xrightarrow{m_1} J \otimes_S J_{rel} \longrightarrow 0,$$

where m_3, m_2 and m_1 are given by as follows:

$$m_1(x \otimes a \otimes y) = xa \otimes y - x \otimes ay$$

and

$$m_3(x \otimes t_i \otimes y) = \sum_{a, t(a)=t_i} xa \otimes a^* \otimes y - \sum_{b, s(b)=t_i} x \otimes b^* \otimes by.$$

For any path $p = a_m \cdots a_1$ of kQ, we define

$$\triangle_a(p) = \sum_{a_i = a} a_m \ldots a_{i+1} \otimes a_i \otimes a_{i-1} \cdots a_1,$$

and extend by linearity to obtain a map $\triangle_a : kQ \to J_{rel} \otimes_S kQ_1 \otimes_R J_{rel}$. Then m_2 is given by:

$$m_2(x \otimes a^* \otimes y) = \sum_{b \in Q_1} x \triangle_b(\partial_a W) y.$$

There is a canonical morphism $P(J_{rel}) \to J_{rel}$, which is induced by the multiplication map m in J_{rel}

$$\begin{tikzcd}
0 \arrow[r] & J_{rel} \otimes_S \otimes R^m \otimes_S J_{rel} \arrow[r, "m_3"] \arrow[d] & J_{rel} \otimes_S \otimes V^{m*} \otimes_S J_{rel} \arrow[r, "m_2"] \arrow[d] & J_{rel} \otimes_S \otimes V \otimes_S J_{rel} \arrow[r, "m_1"] \arrow[d] & J_{rel} \otimes_S J_{rel} \arrow[r] \arrow[d, "m"] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & J_{rel}. &
\end{tikzcd} \tag{5.7}$$

Remark 5.3.8. When $F = \emptyset$, the complex (5.7) defined above is the complex associated to (Q, W) defined by Ginzburg in [38, Section 5]. In general, it is exactly the complex $P(J_{rel})$ defined by Pressland in [79, pp. 10]. And it also has already appeared in work of Amiot–Reiten–Todorov [6, Propostion 2.2].

Lemma 5.3.9. *If the complex (5.7) is exact, then $\mathbf{\Gamma}_{rel}$ is concentrated in degree 0.*

Proof. Let $\mathcal{D}_{pc}(\boldsymbol{\Gamma}_{rel})$ be the pseudocompact derived category of $\boldsymbol{\Gamma}_{rel}$. For each vertex i, we denote by S_i the simple $\boldsymbol{\Gamma}_{rel}$-module (or J_{rel}-module) associated with i. By Proposition 1.3.11, the opposite category $\mathcal{D}_{pc}(\boldsymbol{\Gamma}_{rel})^{op}$ is compactly generated by $\{S_i \,|\, i \in Q_0\}$ and similarly for $\mathcal{D}_{pc}(J_{rel})^{op}$. The restriction functor

$$R : \mathcal{D}_{pc}(J_{rel}) \to \mathcal{D}_{pc}(\boldsymbol{\Gamma}_{rel})$$

takes S_i to S_i. Thus, we can conclude that R is an equivalence if it induces isomorphisms

$$\mathrm{Ext}^*_{J_{rel}}(S_i, S_j) \xrightarrow{\sim} \mathrm{Ext}^*_{\boldsymbol{\Gamma}_{rel}}(S_i, S_j),\ \forall i, j \in Q_0.$$

If the complex (5.7) is exact, then $P(J_{rel})$ is a projective resolution of J_{rel} as a bimodule over itself. Thus, for each vertex $i \in Q_0$, $S_i \otimes_{J_{rel}} P_{J_{rel}}$ is a cofibrant resolution of S_i as a right J_{rel}-module. So we have:

$$\begin{aligned}
\boldsymbol{R}\mathrm{Hom}_{J_{rel}}(S_i, S_j) &= \mathrm{Hom}_{J_{rel}}(S_i \otimes_{J_{rel}} P_{J_{rel}}, S_j)\\
&= \mathrm{Hom}_{J_{rel}}(S_i \otimes_{\boldsymbol{\Gamma}_{rel}} \mathrm{Cone}(\rho) \otimes_{\boldsymbol{\Gamma}_{rel}} J_{rel}, S_j)\\
&= \mathrm{Hom}_{\boldsymbol{\Gamma}_{rel}}(S_i \otimes_{\boldsymbol{\Gamma}_{rel}} \mathrm{Cone}(\rho), \mathrm{Hom}_{J_{rel}}(J_{rel}, S_j))\\
&= \mathrm{Hom}_{\boldsymbol{\Gamma}_{rel}}(S_i \otimes_{\boldsymbol{\Gamma}_{rel}} \mathrm{Cone}(\rho), S_j)\\
&= \boldsymbol{R}\mathrm{Hom}_{\boldsymbol{\Gamma}_{rel}}(S_i, S_j).
\end{aligned}$$

Thus, the restriction functor R is an equivalence. It follows that $\boldsymbol{\Gamma}_{rel}$ is concentrated in degree 0. $\checkmark$

Example 5.3.10. Let D be a Postnikov diagram in the disc (see [79]). We can associate to D an ice quiver with potential (Q_D, F_D, W_D) (see [79, Definition 2.3]). By [79, Proposition 3.6] and Lemma 5.3.9, the corresponding complete relative Ginzburg algebra $\boldsymbol{\Gamma}_{rel}(Q_D, F_D, W_D)$ is concentrated in degree 0. Thus, the associated boundary dg algebra $\mathrm{Bd}(Q_D, F_D, W_D)$ is also concentrated in degree 0. Hence the boundary dg algebra is invariant under the mutations at the unfrozen vertices.

If D has the property that every strand has exactly k boundary regions on its right, then each strand must terminate at a marked point k steps clockwise from its source. Such D is called a *(k,n)-diagram* (see [7]). In this case, Corollary 5.3.6 gives a different proof of Baur–King–Marsh's result [7, Corollary 10.4] which says that the boundary algebra is independent of the choice of Postnikov diagram D, up to isomorphism.

5.4 Mutation at frozen vertices

Let (Q, F) be an ice quiver. Let v be a frozen vertex.

Definition 5.4.1. We say that v is a *frozen source* of Q if v is a source vertex of F an no unfrozen arrows with source v. Similarly, We say that v is a *frozen sink* of Q if v is a sink vertex of F and no unfrozen arrows with target v. For two vertices i and j, we say that they have the *same state* if they are both in F_0 or $Q_0 \setminus F_0$. Otherwise, we say that they have *different state*. Similarly, for two arrows in Q, we say that they have the *same state* if they are both in F_1 or $Q_1 \setminus F_1$. Otherwise, we say that they have *different state*.

5.4.1 Combinatorial mutations

Mutation at frozen vertices first appears in recent work of Fraser–Sherman-Bennett on positroid cluster structures [29].

Definition 5.4.2. Let v be a frozen source or a frozen sink of Q. The *mutation* $\mu_v^P(Q,F) = (\mu_v^P(Q), \mu_v^P(F))$ of (Q,F) at v is defined to be the output of the following procedure.

(1) For each pair of arrows $\alpha : u \to v$ and $\beta : v \to w$, add an unfrozen arrow $[\beta\alpha] : u \to w$ to Q.

(2) Replace each arrow $\alpha : u \to v$ by an arrow $\alpha^* : v \to u$ of the same state as α and each arrow $\beta : v \to w$ by an arrow $\beta^* : w \to v$ of the same state as β.

(3) Remove a maximal collection of unfrozen 2-cycles, i.e. 2-cycles avoiding the subquiver F.

(4) Choose a maximal collection of half-frozen 2-cycles, i.e. 2-cycles in which precisely one arrow is frozen. Replace each 2-cycle in this collection by a frozen arrow, in the direction of the unfrozen arrow in the 2-cycle.

Remark 5.4.3. The procedure in the Definition above is the same as in Pressland's Definition 5.2.4.

Example 5.4.4. Consider the following ice quiver (Q,F)

$$(Q,F) = \begin{tikzcd} \boxed{1} \arrow[dr] & & \boxed{2} \arrow[ll] \arrow[dd] \\ & 5 \arrow[ur] \arrow[dl] & \\ \boxed{3} \arrow[uu] \arrow[rr] & & \boxed{4} \arrow[ul] \end{tikzcd},$$

where the frozen subquiver F is drawn in blue. The frozen vertex 3 is a frozen source. Performing the mutation at vertex 3, we get the following ice quiver

$$(Q',F') = \begin{tikzcd} \boxed{1} \arrow[dd] & & \boxed{2} \arrow[ll] \arrow[dd] \\ & 5 \arrow[ur] & \\ \boxed{3} \arrow[ur] & & \boxed{4} \arrow[ll] \end{tikzcd}.$$

The frozen vertex 2 is a frozen source. Performing the mutation at vertex 2, we get the following ice quiver

$$(Q'', F'') = \begin{tikzcd} \boxed{1} \arrow[rr] \arrow[dd] & & \boxed{2} \arrow[dl] \\ & 5 \arrow[ul] \arrow[dr] & \\ \boxed{3} \arrow[ur] & & \boxed{4} \arrow[ll] \arrow[uu] \end{tikzcd}.$$

Surprisingly, after performing the mutation $\mu_5^P(Q, F)$ at the unfrozen vertex 5, we get the same ice quiver (Q'', F''), i.e. $\mu_5^P(Q, F) = \mu_2^P(\mu_3^P(Q, F))$.

5.4.2 Algebraic mutations

Let (Q, F, W) be an ice quiver with an irredundant potential. Let v be a frozen source or a frozen sink.

Definition 5.4.5. [80, Definition 4.1] The ice quiver with potential $\tilde{\mu}_v(Q, F, W)$, called the *pre-mutation* of (Q, F, W) at v, is the output of the following procedure.

(1) For each pair of arrows $\alpha : u \to v$ and $\beta : v \to w$, add an unfrozen 'composite' arrow $[\beta\alpha] : u \to w$ to Q.

(2) Replace each arrow $\alpha : u \to v$ by an arrow by an arrow $\alpha^* : v \to u$ of the same state as α and each arrow $\beta : v \to w$ by an arrow $\beta^* : w \to v$ of the same state as β.

(3) Pick a representative $\widetilde{W}$ of W in kQ such that no term of W begins at v (which is possible since there are no loops at v). For each pair of arrows α, β as in (1), replace each occurrence of $\beta\alpha$ in $\widetilde{W}$ by $[\beta\alpha]$, and add the term $[\beta\alpha]\alpha^*\beta^*$.

Let us write (Q', F', W') for $\tilde{\mu}_v(Q, F, W)$. It is clear that the new potential W' is also irredundant, since the arrows $[\beta\alpha]$ are unfrozen, but it need not be reduced even if (Q, F, W) is. We define $\mu_v(Q, F, W)$ by replacing the resulting ice quiver with potential $\tilde{\mu}_v(Q, F, W)$ by its reduction, as in Theorem 5.2.13, this being unique up to right equivalence by Proposition 5.2.14. We call μ_v the *mutation* at the vertex v.

Example 5.4.6. Consider the following ice quiver (Q, F)

$$(Q, F) = \begin{tikzcd} \boxed{1} \arrow[dr, "a"] & & \boxed{2} \arrow[ll, "c"'] \arrow[dd, "f"] \\ & 5 \arrow[ur, "b"'] \arrow[dl, "e"] & \\ \boxed{3} \arrow[uu, "g"] \arrow[rr, "i"'] & & \boxed{4} \arrow[ul, "h"] \end{tikzcd},$$

where the blue subquiver represents the frozen subquiver F. We consider the potential

$$W = cba - gea + hie - fbh.$$

The vertex 3 is a frozen source. The pre-mutation at vertex 3 is the following ice quiver with potential

$$(\mu_3'(Q), \mu_3'(F)) = \qquad \begin{tikzcd} \boxed{1} \arrow[dd, "g^*"'] \arrow[dr, "a"] & & \boxed{2} \arrow[ll, "c"'] \arrow[dd, "f"] \\ & 5 \arrow[ul, bend right, "{[ge]}"'] \arrow[ur, "b"] \arrow[dr, bend right, "{[ie]}"'] & \\ \boxed{3} \arrow[ur, "e^*"] & & \boxed{4} \arrow[ll, "i^*"] \arrow[ul, "h"'] \end{tikzcd},$$

where the blue subquiver represents the frozen subquiver $\mu_3'(F)$. The new potential is given by $\mu_3'(W)$ is given by

$$\mu_3'(W) = cba - [ge]a + h[ie] - fbh + [ge]e^*g^* + [ie]e^*i^*.$$

This ice quiver with potential is not reduced. Then $\mu_3(Q, F, W)$ is given by its reduction, which is the following ice quiver with potential

$$(\mu_3(Q), \mu_3(F), \mu_3(W)) = \qquad \begin{tikzcd} \boxed{1} \arrow[dd, "g^*"'] & & \boxed{2} \arrow[ll, "c"'] \arrow[dd, "f"] \\ & 5 \arrow[ur, "b"] & \\ \boxed{3} \arrow[ur, "e^*"] & & \boxed{4} \arrow[ll, "i^*"] \end{tikzcd},$$

where the blue subquiver represents the frozen subquiver $\mu_3(F)$ and the new potential $\mu_3(W)$ is zero. We see that the underlying ice quiver of $\mu_3(Q, F, W)$ is the same as $\mu_3^P(Q, F)$ in Example 5.4.4.

Theorem 5.4.7. *[80, Proposition 4.6] Let (Q, F, W) be an ice quiver with potential and v a frozen source or a frozen sink. If $(\mu_v Q, \mu_v F)$ has no 2-cycles containing unfrozen arrows, then the underlying ice quiver of $\mu_v(Q, F, W)$ agrees with $\mu_v^P(Q, F)$ defined in Definition 5.4.2.*

5.4.3 Categorical mutations

Let (Q, F, W) be an ice quiver with potential. Let v be a frozen source. Write $(Q', F', W') = \tilde{\mu}_v(Q, F, W)$. Let $\mathbf{\Gamma}_{rel} = \mathbf{\Gamma}_{rel}(Q, F, W)$ and $\mathbf{\Gamma}'_{rel} = \mathbf{\Gamma}_{rel}(Q', F', W')$ be the complete relative Ginzburg dg algebras associated to (Q, F, W) and (Q', F', W') respectively. For a vertex i, let $\mathbf{\Gamma}_i = e_i\mathbf{\Gamma}_{rel}$ and $\mathbf{\Gamma}'_i = e_i\mathbf{\Gamma}'_{rel}$.

Theorem 5.4.8. *We have a triangle equivalence*

$$\Psi_+ : \mathcal{D}(\mathbf{\Gamma}'_{rel}) \to \mathcal{D}(\mathbf{\Gamma}_{rel}),$$

which sends the $\mathbf{\Gamma}'_i$ to $\mathbf{\Gamma}_i$ for $i \neq v$ and $\mathbf{\Gamma}_v$ to the cone

$$\mathrm{Cone}(\mathbf{\Gamma}_v \to \bigoplus_{\alpha} \mathbf{\Gamma}_{t(\alpha)}),$$

where we have a summand $\mathbf{\Gamma}_{t(\alpha)}$ for each arrow α of F with source v and the corresponding component of the map is the left multiplication by α. The functor Ψ_+ restricts to triangle equivalences from $\mathrm{per}(\mathbf{\Gamma}'_{rel})$ *to* $\mathrm{per}(\mathbf{\Gamma}_{rel})$ *and from* $\mathrm{pvd}(\mathbf{\Gamma}'_{rel})$ *to* $\mathrm{pvd}(\mathbf{\Gamma}_{rel})$. *Moreover, the following square commutes up to isomorphism*

$$\begin{tikzcd} \mathcal{D}(\mathbf{\Pi}_2(F')) \arrow[r, "G'^*"] \arrow[d, "\mathrm{can}"'] & \mathcal{D}(\mathbf{\Gamma}'_{rel}) \arrow[d, "\Psi_+"] \\ \mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[d, "t^{-1}_{S_v}"'] & \mathcal{D}(\mathbf{\Gamma}_{rel}) \arrow[d, equal] \\ \mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[r, "G^*"'] & \mathcal{D}(\mathbf{\Gamma}_{rel}), \end{tikzcd} \tag{5.8}$$

where can *is the canonical functor induced by an identification between $\mathbf{\Pi}_2(F')$ and $\mathbf{\Pi}_2(F)$ and $t^{-1}_{S_v}$ is the inverse twist functor with respect to the 2-spherical object S_v, which gives rise to a triangle*

$$t^{-1}_{S_v}(X) \to X \to \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_{\mathbf{\Pi}_2(F)}(X, S_v), S_v) \to \Sigma t^{-1}_{S_v}(X)$$

for each object X of $\mathcal{D}(\mathbf{\Pi}_2(F))$.

Proof. By using the same proof as for Theorem 5.3.3, we can show the existence of Ψ_+. For a frozen vertex i, let $Q_i = e_i\mathbf{\Pi}_2(F)$ and $Q'_i = e_i\mathbf{\Pi}_2(F')$. For the commutativity of the diagram, it is enough to show that we have $(\Psi_+ \circ G'^*)(Q'_i) \simeq (G^* \circ t^{-1}_{S_v} \circ \mathrm{can})(Q'_i)$ for each dg $\mathbf{\Pi}_2(F')$-module Q'_i.

For $i \neq v$, we have $\boldsymbol{R}\mathrm{Hom}_{\mathbf{\Pi}_2(F)}(Q_i, S_v) = 0$. Then $t^{-1}_{S_k}(Q_i) = Q_i$ and we have

$$\begin{aligned} (\Psi_+ \circ G'^*)(Q'_i) &= \Psi_+(\mathbf{\Gamma}'_i) = \mathbf{\Gamma}_i \\ (G^* \circ t^{-1}_{S_k} \circ \mathrm{can})(Q'_i) &= (G^* \circ t^{-1}_{S_k})(Q_i) \\ &= G^*(Q_i) \\ &= \mathbf{\Gamma}_i. \end{aligned}$$

Thus, for each $i \neq v$, we have $(\Psi_+ \circ G'^*)(Q'_i) = (G^* \circ t^{-1}_{S_k} \circ \mathrm{can})(Q'_i)$.

For $i = v$, we have $\boldsymbol{R}\mathrm{Hom}_{\mathbf{\Pi}_2(F)}(Q_v, S_v) \cong k$. Then $t^{-1}_{S_v}(Q_v)$ is computed by the following triangle

$$t^{-1}_{S_v}(Q_v) \to Q_v \to S_v \to \Sigma t^{-1}_{S_v}(Q_v).$$

By Subsection 5.2.6, we have a short exact sequence in $\mathcal{C}(\mathbf{\Pi}_2(F))$

$$0 \to \ker(\pi_1) \to Q_v \xrightarrow{\pi_1} S_v \to 0,$$

where π_1 is the canonical projection from Q_v to S_v. Explicitly, we have

$$\ker(\pi_1) = r_v Q_v + \bigoplus_{a \in F_1 : t(a) = v} \tilde{a} Q_{s(\tilde{a})}$$

with the induced differential. Then it is easy to see that $\ker(\pi_1)$ is isomorphic to the mapping cone of the morphism below

$$Q_v \to \bigoplus_{a\in F_1 : s(a)=v} Q_{t(a)},$$

where the corresponding component of the map is the left multiplication by a. Thus, we see that

$$t_{S_v}^{-1}(Q_v) = \mathrm{Cone}(Q_v \to \bigoplus_{a\in F_1 : s(a)=v} Q_{t(a)}).$$

We have

$$\begin{aligned}
(\Psi_+ \circ G'^*)(Q'_v) &= \Psi_+(\mathbf{\Gamma}'_v) = \mathrm{Cone}(\mathbf{\Gamma}_v \to \bigoplus_{a\in F_1 : s(a)=v} \mathbf{\Gamma}_{t(a)}) \\
(G^* \circ t_{S_v}^{-1} \circ \mathrm{can})(Q'_v) &= (G^* \circ t_{S_v}^{-1})(Q_v) \\
&= G^*(\mathrm{Cone}(Q_v \to \bigoplus_{a\in F_1 : s(a)=v} Q_{t(a)})) \\
&\simeq \mathrm{Cone}(\mathbf{\Gamma}_v \to \bigoplus_{a\in F_1 : s(a)=v} \mathbf{\Gamma}_{t(a)}).
\end{aligned}$$

Thus, for each $i \in F_0$, we have $(\Psi_+ \circ G'^*)(Q'_i) \simeq (G^* \circ t_{S_v}^{-1} \circ \mathrm{can})(Q'_i)$. The commutativity of diagram 5.8 is now easy.

$\surd$

Similarly, if v is a frozen sink, we have the following dual of Theorem 5.4.8.

Theorem 5.4.9. *Suppose that v is a frozen sink in Q. Write $(Q', F', W') = \tilde{\mu}_v(Q, F, W)$. Let $\mathbf{\Gamma}_{rel} = \mathbf{\Gamma}_{rel}(Q, F, W)$ and $\mathbf{\Gamma}'_{rel} = \mathbf{\Gamma}_{rel}(Q', F', W')$ be the complete relative Ginzburg dg algebras associated to (Q, F, W) and (Q', F', W') respectively. We have a triangle equivalence*

$$\Psi_- : \mathcal{D}(\mathbf{\Gamma}'_{rel}) \to \mathcal{D}(\mathbf{\Gamma}_{rel}),$$

which sends the $\mathbf{\Gamma}'_i$ to $\mathbf{\Gamma}_i$ for $i \neq v$ and to the shifted cone

$$\Sigma^{-1}\mathrm{Cone}(\bigoplus_{\alpha} \mathbf{\Gamma}_{s(\alpha)} \to \mathbf{\Gamma}_v),$$

where we have a summand $\mathbf{\Gamma}_{s(\alpha)}$ for each arrow α of F with target v and the corresponding component of the map is the left multiplication by α. The functor Ψ restricts to a triangle equivalence from $\mathrm{per}(\mathbf{\Gamma}'_{rel})$ to $\mathrm{per}(\mathbf{\Gamma}_{rel})$ and from $\mathrm{pvd}(\mathbf{\Gamma}'_{rel})$ to $\mathrm{pvd}(\mathbf{\Gamma}_{rel})$. Moreover, the following square commutes up to isomorphism

$$\begin{tikzcd}
\mathcal{D}(\mathbf{\Pi}_2(F')) \arrow[r, "G'^*"] \arrow[d, "\mathrm{can}"'] & \mathcal{D}(\mathbf{\Gamma}'_{rel}) \arrow[d, "\Psi_-"] \\
\mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[d, "t_{S_v}"'] & \mathcal{D}(\mathbf{\Gamma}_{rel}) \arrow[d, equal] \\
\mathcal{D}(\mathbf{\Pi}_2(F)) \arrow[r, "G^*"'] & \mathcal{D}(\mathbf{\Gamma}_{rel}),
\end{tikzcd}$$

where can *is the canonical functor which identifies* $\mathbf{\Pi}_2(F')$ *with* $\mathbf{\Pi}_2(F)$ *and* t_{S_k} *is the twist functor with respect to the* 2*-spherical object* S_v*, which give rise to a triangle*

$$\mathbf{R}\mathrm{Hom}_{\mathbf{\Pi}_2(F)}(S_v, X) \otimes_k S_v \to X \to t_{S_v}(X) \to \Sigma\mathbf{R}\mathrm{Hom}_{\mathbf{\Pi}_2(F)}(S_v, X) \otimes_k S_v$$

for each object X *of* $\mathcal{D}(\mathbf{\Pi}_2(F))$*.*

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