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Zhouhang Mao

On derived crystalline cohomology and perfectoid rings as Thom spectra

dirigée par Matthew Morrow

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M. Christian Ausoni	Université Paris 13	examinateur
M. Kęstutis Česnavičius	Institut de Mathématique d'Orsay	rapporteur
M. Matthew Morrow	Institut de Mathématiques de Jussieu-PRG	directeur
M$^{\text{me}}$ Wiesława Nizioł	Institut de Mathématiques de Jussieu-PRG	examinatrice

Institut de mathématiques de Jussieu-
Paris Rive gauche. UMR 7586.
Boîte courrier 247
4 place Jussieu
75 252 Paris Cedex 05

Université de Paris.
École doctorale de sciences
mathématiques de Paris centre.
Boîte courrier 290
4 place Jussieu
75 252 Paris Cedex 05

Abstract

This thesis[1] contains two independent parts. The first part is devoted to the realization of perfectoid rings as Thom spectra, which generalizes the classical Hopkins-Mahowald theorem for finite fields $\mathbb{F}_p$. As a byproduct, we construct a spherical version of Fontaine's pro-infinitesimal thickening for perfectoid rings. Furthermore, we also realize the ring of integers $\mathcal{O}_K$ for any totally ramified extensions $K/\mathbb{Q}_p$, and complete regular local rings as Thom spectra.

In the second part, we first develop an animated analogue of ring-ideal pairs, PD-pairs and PD-envelopes. This allows us to generalize classical results to non-flat, non-finitely-generated settings. Next, we develop several approaches to derived crystalline cohomology, and establish comparison theorems. In particular, we show that the derived crystalline cohomology coincides with the classical crystalline cohomology when, roughly speaking, the (affine) scheme in question is quasisyntomic, which generalizes B. Bhatt's result for syntomic schemes. We also develop a non-completed animated analogue of prisms and prismatic envelopes. We prove a variant of the Hodge-Tate comparison for animated prismatic envelopes from which we deduce a result about flat cover of the final object for quasisyntomic algebras, which generalizes several known results under smoothness and finiteness conditions.

Keywords: perfectoid ring, Thom spectrum, prism, (derived) crystalline cohomology, animation, prismatic cohomology

Résumé

Il y a deux parties indépendantes de cette thèse[1]. La première partie est consacrée à la réalisation des anneaux perfectoïdes comme spectres de Thom, ce qui généralise le théorème classique de Hopkins-Mahowald pour les corps finis $\mathbb{F}_p$. Comme conséquence, nous construisons une version sphérique d'épaississement pro-infinitésimal de Fontaine. De plus, nous réalisons l'anneau des entiers $\mathcal{O}_K$, pour tout extension totalement ramifiées $K/\mathbb{Q}_p$ et les anneaux locaux, reguliers et complets.

Dans la seconde partie, nous développons tout d'abord un analogue animé de paires d'anneau-idéal, de PD-paires (c'est-à-dire, de paires à puissances divisées) et de PD-enveloppes. Cela nous permet de généraliser des résultats classiques aux situations non-plates et non-finies. Ensuite, nous developpons quelques approches à la cohomologie cristalline dérivée et montrer des théorèmes de comparaison. En particulier, nous démonstrons que la cohomologie cristalline dérivée coïncide avec la cohomologie cristalline classique quand le schéma (affine) est, *grosso modo*, quasi-syntomique, ce qui généralise le résultat de B. Bhatt pour les schémas syntomiques. Nous developpons aussi un analogue animé et non-complété des prismes et de l'enveloppe prismatique. Nous prouvons une variante du théorème de comparison de Hodge-Tate et en déduisons un résultat sur des recouvrement plat de l'objet final pour les algèbres quasi-syntomiques, ce qui généralise quelques résultats connus sous des hypothèses de lissité et de finitude.

Mots-clés: anneau perfectoïde, spectre de Thom, prisme, cohomologie cristalline (dérivée), animation, cohomologie prismatique

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Introduction

Here we give a short introduction to the background of the topics covered in the current thesis, which is made up of two independent chapters. More detailed introductions can be found at the beginning of each chapter.

Let X be a smooth projective variety over $\mathbb{C}$ and let X^{an} denote the complex manifold associated to X. Then we know that there is a canonical isomorphism $H^i(X^{\mathrm{an}}, \mathbb{C}) \cong H^i_{\mathrm{dR}}(X)$ where the left hand side is the singular cohomology and the right hand side is the de Rham cohomology. Note that the de Rham cohomology could be computed algebraically by defining equations of X, so we see that the cohomology of X^{an} with $\mathbb{C}$-coefficients could be computed algebraically.

Question. Could we determine the cohomology $H^i(X^{\mathrm{an}}, \mathbb{Z})$ of X^{an} with $\mathbb{Z}$-coefficients "algebraically"? Or more simply, how much do we know about $H^i(X^{\mathrm{an}}, \mathbb{F}_p)$ via algebraic means?

By spreading out, we know that X comes from a smooth projective variety over a finitely generated ring. We restrict ourselves to the special case that X comes from a smooth projective scheme over $\mathbb{Z}[N^{-1}]$ for some $N \in \mathbb{N}$. By abuse of notation, we still denote by X the smooth projective scheme over $\mathbb{Z}[N^{-1}]$. Let $p \neq \ell$ be two distinct primes which not divide N. Then by proper base change and étale-singular comparison, we have $H^i(X^{\mathrm{an}}, \mathbb{F}_\ell) = H^i_{\mathrm{et}}(X, \mathbb{F}_\ell) = H^i_{\mathrm{et}}(X_p \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{F}_\ell)$ where $X_p := X \times_{\mathrm{Spec}(\mathbb{Z}[N^{-1}])} \mathrm{Spec}(\mathbb{F}_p)$ is the fiber of X over the closed point $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z}[N^{-1}])$ (more explicitly, it is given by the defining equations of X modulo p). Unfortunately, the étale cohomology $H^i_{\mathrm{et}}(X_p \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{F}_\ell)$ is not obviously algebraically computable.

We remark that there is a cohomology theory which is algebraically computable "when $\ell = p$", namely the de Rham cohomology. Then the following question is natural:

Question. How to relate $H^i_{\mathrm{et}}(X_p \times_{\mathbb{F}_p} \overline{\mathbb{F}}_p, \mathbb{F}_\ell)$ for $\ell \neq p$ and $H^i_{\mathrm{dR}}(X_p)$?

Before going further, let us summarize the cohomology theories "$H^i(X_p, \mathbb{F}_\ell)$" that appeared above in a table. We understand X_∞ as $X(\mathbb{C})$.

cohomology	$p=2$	$p=3$	$p=5$	...	$p=\infty$
$\ell=2$	de Rham	étale	étale		étale = singular
$\ell=3$	étale	de Rham	étale		étale = singular
$\ell=5$	étale	étale	de Rham		étale = singular
...					
$\ell=\infty$	?	?	?		de Rham = singular

We can ask a more general question: how to relate all these cohomology theories? To do so, it is natural to pass to some kind of "neighborhoods" and study the deformation. We could deform both in p-direction and ℓ-direction. For example, for the fiber X_p, we can deform ℓ both in the case that $\ell = p$ and the case that $\ell \neq p$: the former is given by the crystalline cohomology, and the later is given by the ℓ-adic cohomology. On the other hand, for deforming p, we could take the formal neighborhood $\mathrm{Spf}(\mathbb{Z}_p) \to \mathrm{Spec}(\mathbb{Z}[N^{-1}])$ of the closed point $\mathrm{Spec}(\mathbb{F}_p) \to \mathrm{Spec}(\mathbb{Z}[N^{-1}])$. The *prismatic cohomology*, introduced in [BS19], is a framework to cover various cohomology theories where both p and ℓ are deformed[1].

Now we summarize the contents of the thesis in terms of the previous discussions. Chapter 1 exhibits a further deformation: instead of the world of rings, we deform $\mathrm{Spec}(\mathbb{F}_p)$ to a formal neighborhood $\mathrm{Spf}(\mathbb{S}_p^\wedge)$, which is closely related to the computation of topological Hochschild homology. In Chapter 2, we study generalizations of the previous picture to the non-smooth cases. We also briefly discuss the analytic theory by replacing the formal neighborhood $\mathrm{Spf}(\mathbb{Z}_p) \to \mathrm{Spec}(\mathbb{Z}[N^{-1}])$ by analytic versions in the sense of condensed mathematics [Sch19a]. For example, at the generic point of $\mathbb{Z}[N^{-1}]$, we coud consider the "analytic neighborhoods" $\mathrm{AnSpec}(\mathbb{C}, \mathcal{M}_{<q}) \to \mathrm{Spec}(\mathbb{Z}[N^{-1}])$ for different q.

1. More precisely, for prismatic cohomology, we have $p = \ell$, but the two are not necessarily "deformed in the same way". We thank Prof. Kęstutis Česnavičius for this point.

Chapter 1

Perfectoid rings as Thom spectra [1]

Abstract. The Hopkins–Mahowald theorem realizes the Eilenberg–MacLane spectra $H\mathbb{F}_p$ as Thom spectra for all primes $p \in \mathbb{N}_{>0}$. In this article, we record a known proof of a generalization of Hopkins–Mahowald theorem, realizing Hk as Thom spectra for perfect rings k, and we provide a further generalization by realizing HR as Thom spectra for perfectoid rings R. We also discuss even further generalizations to prisms (A, I) and indicate how to adapt our proofs to Breuil–Kisin case.

1.1. Introduction

In this article, since most of our results are p-typical, we fix a prime $p \in \mathbb{N}_{>0}$. We first describe the classical Hopkins–Mahowald theorem. We know that $\mathbb{F}_p \cong \mathbb{Z}_p / p$, that is to say, $\mathbb{F}_p$ is the free $\mathbb{Z}_p$-algebra in which $p = 0$. For some reasons, we need to extend this kind of results to a category of "less linear" algebras in which the addition is not commutative or even associative on the nose, but only up to coherent homotopy. To be more precise, we need to understand whether the (Eilenberg–MacLane) ring spectrum $H\mathbb{F}_p$ is still the free object in the category of $\mathbb{S}_p^\wedge$-algebras satisfying certain associativity and commutativity with $p = 0$? The classical Hopkins–Mahowald theorem answers this affirmatively: they are the free object in the category of $\mathbb{E}_2$-$\mathbb{S}_p^\wedge$-algebras with $p = 0$. There are two ways to describe "free $\mathbb{E}_2$-algebras with $p = 0$". In this article, we will mainly adopt the description via Thom spectra. We will go to another, more direct and natural but technically more burdened description in Section 1.7. We start with formal definitions of Thom spectra with informal illustrations and refer to [AB19] for further discussions.

Definition 1.1.1. *Given a ring spectrum R, we define the ∞-category $\mathrm{BGL}_1(R)$ to be the full subcategory of $\mathrm{LMod}_R^{\simeq}$ spanned by left R-module spectra equivalent to R, where we denote by $\mathcal{C}^{\simeq}$ the maximal groupoid associated to an ∞-category $\mathcal{C}$.*

Remark 1.1.2. The ∞-category $\mathrm{BGL}_1(R)$ is in fact an ∞-groupoid, and if we further suppose that R is an $\mathbb{E}_{n+1}$-ring spectrum, then $\mathrm{BGL}_1(R)$ inherits an $\mathbb{E}_n$-monoidal structure from LMod_R.

We admit the following result, which could be understand as an analogue of the fact that $\pi_1(BG) = G$ for any discrete group G:

Proposition 1.1.3. $\pi_1(\mathrm{BGL}_1(R)) = \mathrm{GL}_1(\pi_0 R)$ *for any ring spectra R. Concretely, an invertible element $a \in \pi_0 R$ corresponds to a multiplication map $m_a : R \to R$ in $\mathrm{BGL}_1(R)$.*

Remark 1.1.4. In fact, $\mathrm{BGL}_1(R)$ is a delooping of the group of invertible elements in R.

Now we recall the definition of Thom spectra:

Definition 1.1.5. *Given a ring spectrum R, a space X and a map $f: X \to \mathrm{BGL}_1(R)$,* the Thom spectrum Mf associated to f *is the colimit of the composition*

$$X \to \mathrm{BGL}_1(R) \to \mathrm{LMod}_R$$

We note that by definition of colimits, we can understand the colimit as a kind of "free objects satisfying several equations". We will choose a special space X to encode the $\mathbb{E}_2$-commutativity (understood as a generalized version of classical associativity, a collection of equations) and a map $f : X \to \mathrm{BGL}_1(R)$ to encode the "equation" $p = 0$.

1. Extracted from [Mao20].

Remark 1.1.6. As a special case of [Lur09, Proposition 4.1.2.6], any homotopy equivalence of Kan complexes is cofinal, therefore the formation of the colimit does not depend on the choice of models of the space X.

Remark 1.1.7. In this article, we only consider the case that R is a connective $\mathbb{E}_\infty$-ring spectrum. As a consequence, we can replace LMod_R by Mod_R and the Thom spectrum Mf is connective.

Remark 1.1.8. In Definition 1.1.5, if X is endowed with an $\mathbb{E}_n$-algebra structure, and f is assumed to be $\mathbb{E}_n$-monoidal, then the Thom spectrum Mf naturally inherits an $\mathbb{E}_n$-R-algebra structure. In this case, we will call Mf the $\mathbb{E}_n$-Thom spectrum associated to f.

In the classical Hopkins–Mahowald theorem, we will choose $X = \Omega^2 S^3$, the free $\mathbb{E}_2$-group in the ∞-category $\mathcal{S}$ of spaces.

Remark 1.1.9. As a special case, $\pi_1(\mathrm{BGL}_1(\mathbb{S}_p^\wedge)) = \mathrm{GL}_1(\mathbb{Z}_p) = \{a \in \mathbb{Z}_p \mid a \bmod p \neq 0\}$. The invertible element $1 - p\,u$ in $\mathbb{Z}_p$ gives rise to a map $S^1 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ where $u \in \mathrm{GL}_1(\mathbb{Z}_p)$ is an invertible element in $\mathbb{Z}_p$. Since the p-adic sphere spectrum $\mathbb{S}_p^\wedge$ is an $\mathbb{E}_\infty$-ring spectrum, by Remark 1.1.2 this map extends to a double loop map $\Omega^2 S^3 \simeq \Omega^2 \Sigma^2 S^1 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$, which we denote by $f_{\mathbb{F}_p, pu}$.

We note that the choice of $1 - p\,u$ essentially imposes an equation $1 - p\,u = 1$. This could be seen by the fact that taking the colimit along $f_{\mathbb{F}_p, pu}$ is essentially taking the homotopy orbits of the $\Omega^2 S^3$-action, which is somehow "multiplying by" $1 - p\,u$.

Remark 1.1.10. In the first drafts of this article, we simply took $u = 1$. Later, we realized that it might be easier to introduce u to fix a gap in commutative algebra for technical reasons.

Now we formulate the classical Hopkins–Mahowald theorem (cf. [AB19, Theorem 5.1], where $u = 1$, but the proof works for the general case. See also [KN, Theorem A.1]):

THEOREM 1.1.11. (HOPKINS–MAHOWALD) *The Eilenberg–MacLane spectrum $H\mathbb{F}_p$ is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{\mathbb{F}_p, pu} : \Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$.*

This arouses a natural question: what other discrete rings are Thom spectra in a similar fashion? The first guess will come from the observation that $\mathbb{Z}_p \cong W(\mathbb{F}_p)$, so it would be natural to ask whether we have similar results for perfect $\mathbb{F}_p$-algebras?

In this article, our main purpose is to show that this is the case for perfectoid rings (which is inspired by computational results of topological Hochschild homology of perfectoid rings in [BMS19]), and consequently, for perfect $\mathbb{F}_p$-algebras. In order to do so, we need the concept of spherical Witt vectors $W^+(k)$ for perfect $\mathbb{F}_p$-algebras k, which we will recall in section 1.2. For the moment, we will take advantage of the fact that $\pi_0(W^+(k)) = W(k)$ where $W(k)$ is the ring of (classical) Witt vectors. One example is that $W^+(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$.

Remark 1.1.12. Given a perfectoid ring R, denote by ξ a generator of the kernel of Fontaine's pro-infinitesimal thickening $\theta : W(R^\flat) \to R$, which we will review in section 1.4. As in Remark 1.1.9, the invertible element in $W(R^\flat)$, $1 - \xi \in \mathrm{GL}_1(W(R^\flat)) = \pi_1(\mathrm{BGL}_1(W^+(R^\flat)))$ gives rise to a map $S^1 \to \mathrm{BGL}_1(W^+(R^\flat))$ which extends to a double loop map $f_{R,\xi} : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(R^\flat))$.

THEOREM 1.1.13. (MAIN THEOREM) *The Eilenberg–MacLane spectrum HR is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{R,\xi}$ for any perfectoid ring R.*

Fontaine's pro-infinitesimal thickening θ is in fact surjective. Note that $R \cong W(R^\flat)/\xi$, and our result amounts to saying that the ring spectrum HR is a free $\mathbb{E}_2$-$W^+(R^\flat)$-algebra with $\xi = 0$.

Remark 1.1.14. When R is a perfect $\mathbb{F}_p$-algebra, we can take $\xi = p\,u$ where $u \in \mathrm{GL}_1(R)$ is an invertible element in R, and we note that $R^\flat = R$. Especially, when $R = \mathbb{F}_p$, $f_{R,pu}$ coincides with $f_{\mathbb{F}_p, pu}$, hence Theorem 1.1.13 generalizes Theorem 1.1.11.

Remark 1.1.15. The composite map $W^+(R^\flat)\xrightarrow{\tau_{\le 0}} HW(R^\flat)\xrightarrow{H\theta} HR$ should be understood as a spherical analogue of Fontaine's map $\theta: W(R^\flat)\to R$. We will establish a universal property, Proposition 1.4.18, similar to Fontaine's, Proposition 1.4.16, which might be of independent interest.

The motivation to realize $H\mathbb{F}_p$ as a free $\mathbb{E}_2$-algebra with $p=0$ is that it describes a direct "generation-relation" like description with respect to the (p-completed) sphere spectrum $\mathbb{S}_p^\wedge$. Similarly, realization of HR as a free $\mathbb{E}_2$-$W^+(R)$-algebra with $\xi=0$ enables us to relate HR more directly to the ring $W^+(R^\flat)$ of spherical Witt vectors, which allows us to deduce "topological" results about these rings. For example, as a consequence, we can compute the topological Hochschild homology $\mathrm{THH}(HR)$ (of a perfectoid ring R) as an $\mathbb{E}_1$-ring spectrum and deduce Bökstedt's periodicity. By [KN, Proposition 4.7], as in the proof of Theorem 4.1 there, we have

PROPOSITION 1.1.16. *The (relative) topological Hochschild homology* $\mathrm{THH}(HR/W^+(R^\flat))\simeq HR\otimes \Omega S^3$ *as* $\mathbb{E}_1$-$W^+(R^\flat)$*-algebras for any perfectoid ring R.*

The proof is somehow technical, but essentially it is similar to the classical computation of the Hochschild homology $\mathrm{HH}(R/W(R^\flat))$, via resolving R by $W(R^\flat)$-CDGAs. We refer to first paragraphs of the proof of [HN19, Theorem 1.3.2] for this classical case. As a consequence of Proposition 1.1.16, we have (see subsection 1.5.5):

PROPOSITION 1.1.17. *The (absolute) topological Hochschild homology* $\mathrm{THH}(HR)_p^\wedge\simeq HR\otimes\Omega S^3$ *as* $\mathbb{E}_1$*-ring spectra.*

By known results on the homology of ΩS^3 (a classical reference is [Bot82]), we deduce Bökstedt's periodicity for perfectoid rings (cf. [BMS19, Theorem 6.1]).

COROLLARY 1.1.18. (BÖKSTEDT'S PERIODICITY) $\pi_*(\mathrm{THH}(HR)_p^\wedge)\cong R[u]$ *where u is any generator of* $\pi_2(\mathrm{THH}(HR)_p^\wedge)$ *as a* $\pi_0(\mathrm{THH}(HR)_p^\wedge)$*-module.*

In fact, our question was motivated by Bökstedt's periodicity for perfectoid rings: we wanted to understand why Bökstedt's periodicity holds.

Further generalizations of Theorem 1.1.13 to prisms, the concept introduced in [BS19], seem plausible. However, we are only capable to reach another special case of prisms motivated by Breuil–Kisin cohomology, parallel to the perfectoid case, proposed by Matthew Morrow:

THEOREM 1.1.19. *Let A be complete discrete valuation ring of mixed characteristic with residue field k being perfect of characteristic p. Then the Eilenberg–MacLane spectrum HA is the* $\mathbb{E}_2$*-Thom spectrum associated to a map* $f_E:\Omega^2 S^3\to \mathrm{BGL}_1(W^+(k)[[u]])$.

Inspired by [KN19, Section 9], we will also provide a version of Hopkins–Mahowald theorem for complete regular local rings:

THEOREM 1.1.20. *Let* $(A,\mathfrak{m})$ *be a complete regular local ring of mixed characteristic with residue field k being perfect of characteristic p. Let* $(a_1,\ldots,a_n)\subseteq\mathfrak{m}$ *be a regular sequence which generates the maximal ideal* $\mathfrak{m}$*. Then the Eilenberg–MacLane spectrum HA is the* $\mathbb{E}_2$*-Thom spectrum associated to a map* $f_A:\Omega^2 S^3\to\mathrm{BGL}_1(W^+(k)[[u_1,\ldots,u_n]])$.

In this article, we will first review spherical Witt vectors. We then record a known proof of perfect rings being Thom spectrum, the special case of Theorem 1.1.13 for perfect rings, which we learn from Sanath Devalapurkar, but the proof is also well-known to experts such as Achim Krause and Thomas Nikolaus, see [KN19]. This result is needed in the proof of the general case of Theorem 1.1.13. Then we start with recalling the definition and some basic properties of perfectoid rings, and prove Theorem 1.1.13. As far as we know, although this is known to several experts, the proof is not found in the literature. We will finally discuss further generalizations to prisms in Section 1.6, and especially Hopkins–Mahowald theorem for Breuil–Kisin cases, which seems also to be known by experts (see [KN19, Remark 3.4]). We take an opportunity to write down those proofs. The author thanks Matthew Morrow for various suggestions during the construction of this article.

Warning 1.1.21. For spectra M, N, we will denote the smash product of M, N by $M \otimes N$. Let R be an $\mathbb{E}_1$-ring (spectrum), M a right R-module (spectrum) and N a left R-module (spectrum), we will denote the relative tensor product by $M \otimes_R N$. In order to avoid possible ambiguities, for discrete rings A, right A-modules P and left A-modules Q, we will denote the ordinary (algebraic) tensor product by $\mathrm{Tor}_0^A(P, Q)$ (instead of $P \otimes_A Q$). It is important that in general the Eilenberg–MacLane spectrum $H\,\mathrm{Tor}_0^A(P, Q)$ do not coincide with the relative tensor product $HP \otimes_{HA} HQ$ of spectra. Rather, the relative tensor $HP \otimes_{HA} HQ$ coincides with the Eilenberg–MacLane spectrum $H(P \otimes_A^{\mathbb{L}} Q)$ of the derived tensor product. Since the concept of the derived tensor product does not play a great role in this article, we will not use the notation $\otimes_A^{\mathbb{L}}$, and we will uniformly preserve the notation $\otimes$ for smash products and relative tensor products of spectra.

NOTATION 1.1.22. *In this article, we mainly adopt notations in [Lur17], [Lur18a] and [Lur18b]. In particular, we will let* LMod_R *denote the* ∞*-category of an* $\mathbb{E}_1$*-ring* R*, let* Mod_R *denote the symmetric monoidal* ∞*-category of an* $\mathbb{E}_\infty$*-ring* R *and let* $\mathrm{Alg}_R^{\mathbb{E}_n}$ *denote the* ∞*-category of* $\mathbb{E}_n$*-*R*-algebras for an* $\mathbb{E}_\infty$*-ring* R *and a positive integer* $n \in \mathbb{N}_{>0}$*. In particular, we will denote* $\mathrm{Alg}_R^{\mathbb{E}_\infty}$ *by* CAlg_R*, referred to as the* ∞*-category of commutative* R*-algebras. On the other hand, we will denote* $\mathrm{Mod}_R^\heartsuit$ *the* ∞*-category of discrete* R*-modules, and* $\mathrm{CAlg}_R^\heartsuit$ *the* ∞*-category of discrete commutative* R*-algebras.*

1.2. RECOLLECTION OF SPHERICAL WITT VECTORS

In this section, we will review the definition and some basic properties of spherical Witt vectors. We quote some definitions and propositions from [Lur18a, Section 5.2].

DEFINITION 1.2.1. ([LUR18A, DEFINITION 5.2.1]) *Let* A *be a connective* $\mathbb{E}_\infty$*-ring, let* $I \subseteq \pi_0 A$ *be a finitely generated ideal, and set* $A_0 = \pi_0(A)/I$*. Suppose that we are given a commutative diagram of connective* $\mathbb{E}_\infty$*-rings*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

where B_0 *is a discrete commutative ring. We will say that* σ exhibits f as an A-thickening of f_0 *if the following conditions are satisfied:*

a) *The* $\mathbb{E}_\infty$*-ring* B *is* I*-complete as an* A*-module;*

b) *The diagram* σ *induces an isomorphism of commutative rings* $\pi_0(B)/I\pi_0(B) \to B_0$*;*

c) *Let* R *be any connective* $\mathbb{E}_\infty$*-algebra over* A *which is* I*-complete. Then the canonical map*

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, R) \to \mathrm{Hom}_{\mathrm{CAlg}_{A_0}^\heartsuit}(B_0, \pi_0(R)/I\pi_0(R))$$

is a homotopy equivalence. In particular, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_A}(B, R)$ *is discrete up to homotopy equivalence, that is, each connected component is contractible.*

Remark 1.2.2. (UNIQUENESS, [LUR18A, REMARK 5.2.2]) Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that we are given a homomorphism of commutative rings $f_0 : A_0 \to B_0$. It follows immediately from the definition that if there exists a diagram σ:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

which exhibits f as an A-thickening of f_0, then the morphism f (and the diagram σ) is uniquely determined up to equivalence.

Remark 1.2.3. ([Lur18a, Remark 5.2.4]) Suppose that we are given a commutative diagram of commutative $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' \\ \downarrow & & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

Assume that A_0, B_0 are discrete rings and the left vertical maps induce surjective ring morphisms $\pi_0 A \to \pi_0 A' \to A_0$ whose composition has kernel $I \subseteq \pi_0 A$. Suppose that the outer rectangle exhibits f as an A-thickening of f_0 and that the upper square exhibits B' as an I-completion of $B \otimes_A A'$. Then the lower square exhibits f' as an A'-thickening of f_0.

Theorem 1.2.4. ([Lur18a, Theorem 5.2.5]) *Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that A_0 is an $\mathbb{F}_p$-algebra such that HA_0 is almost perfect as an A-module and that the Frobenius map $\varphi_{A_0}: A_0 \to A_0$ is flat. Let $f: A_0 \to B_0$ be a morphism of commutative $\mathbb{F}_p$-algebras which is relatively perfect, then there exists a diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

which exhibits f as an A-thickening of f_0. Moreover, σ is a pushout square.

Example 1.2.5. (Classical Witt vectors, [Lur18a, Example 5.2.6]) In the statement of Theorem 1.2.4 take $A = H\mathbb{Z}_p$ and $I = p\mathbb{Z}_p$. Then $A_0 = \pi_0(A)/I$ is the finite field $\mathbb{F}_p$ and a map $f_0: A_0 \to B_0$ of discrete rings is relative perfect if and only if B_0 is a perfect $\mathbb{F}_p$-algebra. If this condition is satisfied, then Theorem 1.2.4 allows us to lift B_0 to an $\mathbb{E}_\infty$-$H\mathbb{Z}_p$-algebra which is complete with respect to the ideal $p\mathbb{Z}_p$ and for which the quotient $\pi_0(B)/p\pi_0(B)$ is isomorphic to B_0. This $\mathbb{Z}_p$-algebra is in fact the Eilenberg–MacLane spectrum of the ring of Witt vectors $W(B_0)$. See also [Ser79, Section II.5, Proposition 10] for a classical description of this universal property.

Example 1.2.6. (Spherical Witt vectors, [Lur18a, Example 5.2.7]) In the statement of Theorem 1.2.4 take $A = \mathbb{S}_p^\wedge$ and $I = (p)$. Then $A_0 = \pi_0(A)/I$ is the finite field $\mathbb{F}_p$ and a morphism $f_0: A_0 \to B_0$ is relative perfect if and only if B_0 is a perfect $\mathbb{F}_p$-algebra. If this condition is satisfied, Theorem 1.2.4 allows us to lift B_0 to an $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra which is complete with respect to the ideal (p) and the tensor product $H\mathbb{F}_p \otimes_{\mathbb{S}_p^\wedge} B \simeq \pi_0(B)/p\pi_0(B)$ is isomorphic to B_0. This is the $\mathbb{E}_\infty$-ring $W^+(B_0)$ of "spherical" Witt vectors.

Proposition 1.2.7. *$\pi_0(W^+(k))$ is isomorphic to $W(k)$, the ring of Witt vectors, and $HW(k) \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ for any perfect $\mathbb{F}_p$-algebra k.*

Proof. First, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(k) \\ \downarrow & & \\ H\mathbb{Z}_p & & \downarrow \\ \downarrow & & \\ H\mathbb{F}_p & \longrightarrow & Hk \end{array}$$

Figure 1.2.1.

where the outer square is given by Theorem 1.2.4. The right vertical map $W^+(k) \to Hk$ factors through the pushout $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ in the category of $\mathbb{E}_\infty$-rings. Note that $\mathbb{S}_p^\wedge$ is a coherent ring as in Definition A.0.10, and $H\mathbb{Z}_p \simeq H\pi_0(\mathbb{S}_p^\wedge)$ is an almost perfect $\mathbb{S}_p^\wedge$-module by Corollary A.0.12, which implies that $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ is an almost perfect $W^+(k)$-module by Proposition A.0.8. By Definition 1.2.1, $W^+(k)$ is a p-complete $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra, therefore by Proposition A.0.27, the spectrum $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ is p-complete. Now we take $A = \mathbb{S}_p^\wedge$, $A' = H\mathbb{Z}_p$, $A_0 = H\mathbb{F}_p$, $B = W^+(k)$, $B' = W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ and $B_0 = Hk$ in Remark 1.2.3, we deduce that the lower square

$$\begin{array}{ccc} H\mathbb{Z}_p & \longrightarrow & W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p \\ \downarrow & & \downarrow \\ H\mathbb{F}_p & \longrightarrow & Hk \end{array}$$

constitutes a commutative diagram of thickening as in Definition 1.2.1. Then it follows from Remark 1.2.2 and Example 1.2.5 that $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ is equivalent to $HW(k)$ as $\mathbb{E}_\infty$-$H\mathbb{Z}_p$-algebras, which implies that $W(k) \cong \pi_0(HW(k)) \cong \mathrm{Tor}_0^{\pi_0(\mathbb{S}_p^\wedge)}(\pi_0(W^+(k)), \pi_0(H\mathbb{Z}_p)) \cong \mathrm{Tor}_0^{\mathbb{Z}_p}(\pi_0(W^+(k)), \mathbb{Z}_p) \cong \pi_0(W^+(k))$. □

Proposition 1.2.8. (Recognition of Thickenings, [Lur18a, Proposition 5.2.9]) *Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that A_0 is an $\mathbb{F}_p$-algebra which is almost perfect as an A-module and that the Frobenius map $\varphi_{A_0}: A_0 \to A_0$ is flat. Suppose we are given a commutative diagram of connective $\mathbb{E}_\infty$-rings σ:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

where f_0 is a relative perfect morphism of commutative $\mathbb{F}_p$-algebras. Then σ exhibits f as an A-thickening of f_0 if and only if the following conditions are satisfied:

i. The $\mathbb{E}_\infty$-ring B is I-complete as an A-module;

ii. The diagram σ is a pushout square.

1.3. Perfect rings being Thom spectra

We first admit a (superficially) slightly stronger Hopkins–Mahowald's theorem for sake of convenience. Given a perfect $\mathbb{F}_p$-algebra k and an invertible element $u \in \mathrm{GL}_1(W(k))$, as a special case of Remark 1.1.12, we have a map $f_{k,pu}: \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k))$.

Theorem 1.3.1. (Hopkins–Mahowald for k) *The Eilenberg–MacLane spectrum Hk is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{k,pu}$.*

For technical reasons, we start with the special case that $u \in \mathrm{GL}_1(\mathbb{Z}_p) \subseteq \mathrm{GL}_1(W(k))$. In this case, it is a direct consequence of that for $\mathbb{F}_p$.

Lemma 1.3.2. *Theorem 1.3.1 is true when $u \in \mathrm{GL}_1(\mathbb{Z}_p) \subseteq \mathrm{GL}_1(W(k))$.*

Proof. We note that the image of the multiplication map $m_{1-pu}: \mathbb{S}_p^\wedge \to \mathbb{S}_p^\wedge$ given by $1 - pu \in \pi_0(\mathbb{S}_p^\wedge) \cong \mathbb{Z}_p$ under the canonical (symmetric monoidal) functor $W^+(k) \otimes_{\mathbb{S}_p^\wedge} - : \mathrm{Mod}_{\mathbb{S}_p^\wedge} \to \mathrm{Mod}_{W^+(k)}$ is still a multiplication map $m_{1-pu}: W^+(k) \to W^+(k)$ given by $1 - pu \in \pi_0(W^+(k)) \cong W(k)$, and therefore the map $f_{k,pu}$ coincides with the composition map

$$\Omega^2 S^3 \xrightarrow{f_{\mathbb{F}_p,pu}} \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \xrightarrow{W^+(k) \otimes_{\mathbb{S}_p^\wedge} -} \mathrm{BGL}_1(W^+(k))$$

Since $Mf_{\mathbb{F}_p,pu} \simeq H\mathbb{F}_p$ as $\mathbb{E}_2$-ring spectra by Theorem 1.1.11,

$$Mf_{k,pu} \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} Mf_{\mathbb{F}_p,pu} \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{F}_p \simeq Hk$$

as $\mathbb{E}_2$-ring spectra, where the first equivalence follows from the fact that the functor $W^+(k)\otimes_{\mathbb{S}_p^\wedge}-$ is a left adjoint therefore commutes with colimits and the last equivalence is given by the last claim in Theorem 1.2.4. □

In order to prove Theorem 1.3.1, it suffices to show that $Mf_{k,pu}\simeq Mf_{k,p}$ holds for all $u\in \mathrm{GL}_1(W(k))$, therefore $Mf_{k,pu}\simeq Mf_{k,p}\simeq Hk$ by Lemma 1.3.2. We will base the proof on a universal property of Thom spectra which we will not use elsewhere, and the author looks forward to an alternative proof which does not depend on this universal property.

Lemma 1.3.3. (Proposition 4.9 in [AB19] along with the discussions after Lemma 4.6) *The $\mathbb{E}_2$-Thom spectrum $Mf_{k,pu}$ satisfies the following universal property: For all $\mathbb{E}_2$-$W^+(k)$-algebras A, the mapping space $\mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,pu},A)$ could be naturally identified with the space of null-homotopies of the composite map $W^+(k)\xrightarrow{m_{pu}}W^+(k)\xrightarrow{\eta}A$ in the category of $W^+(k)$-modules where $\eta:W^+(k)\to A$ is the canonical map given by the $\mathbb{E}_2$-$W^+(k)$-algebra structure, and $m_{pu}:W^+(k)\to W^+(k)$ is the multiplication map given by $pu\in W(k)=\pi_0(W^+(k))$.*

Proof of Theorem 1.3.1. Note that the multiplication map $m_u:W^+(k)\to W^+(k)$ is an equivalence of $W^+(k)$-modules since $u\in W(k)=\pi_0(W^+(k))$ is invertible. Hence by Lemma 1.3.3, the map m_u induces an equivalence of spaces $\mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,p},A)\to\mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,pu},A)$ which is natural in A. By the Yoneda lemma, we deduce that $Mf_{k,pu}\simeq Mf_{k,p}$ as $\mathbb{E}_2$-$W^+(k)$-algebras. □

1.4. Recollection of perfectoid rings

In this section, we will review basic definitions and properties of perfectoid rings.

1.4.1. Basic definitions and properties

Definition 1.4.1. *Let A be a ring and $I\subseteq A$ be an ideal. Then the ring A is called* I-adically complete *if the canonical map from A to the (inverse) limit of the tower*

$$\cdots\to A/I^n\to\cdots\to A/I^2\to A/I$$

is an isomorphism. The ring A is called I-adically separated if the intersection $\bigcap_{n=0}^\infty I^n=0$.

Warning 1.4.2. In the literature, sometimes authors call a ring A is I-adically complete when the canonical map $A\to\lim_{n\in(\mathbb{N},>)}(A/I^n)$ is only supposed to be surjective, and our I-adic completeness is equivalent to their I-adic completeness plus I-adic separateness.

Definition 1.4.3. *Let A be an $\mathbb{F}_p$-algebra. The* direct limit perfection A_{perf} *of A is the direct limit of the telescope $A\xrightarrow{\varphi}A\xrightarrow{\varphi}A\xrightarrow{\varphi}\cdots$.*

Definition 1.4.4. *An $\mathbb{F}_p$-algebra A is called* semiperfect *if the Frobenius map $\varphi:A\to A$ is surjective.*

Remark 1.4.5. For a semiperfect $\mathbb{F}_p$-algebra A, the direct limit perfection A_{perf} coincides with $A_{\mathrm{red}}=A/\sqrt{0}$, by checking that A_{red} satisfies the universal property of A_{perf}.

Remark 1.4.6. The canonical map $R\to R_{\mathrm{perf}}$ is initial among all $\mathbb{F}_p$-algebra morphisms $R\to S$ such that S is a perfect $\mathbb{F}_p$-algebra. This follows directly from the universal property of direct limits in the definition of direct limit perfections.

Definition 1.4.7. *Let R be a commutative ring which is p-adically complete. The* tilt *of R, denoted by $R^\flat$, is a perfect $\mathbb{F}_p$-algebra defined by the limit of the tower*

$$\cdots\xrightarrow{\varphi}R/p\xrightarrow{\varphi}R/p\xrightarrow{\varphi}R/p$$

where $\varphi:R/p\to R/p$ is the Frobenius map. In particular, if R is an $\mathbb{F}_p$-algebra, then $R^\flat$ is the inverse limit perfection of R, and if furthermore R is semiperfect, then the canonical map $R^\flat\to R$ is a surjection.

We need the following classical proposition to define the Fontaine's pro-infinitesimal thickening map. We omit the proof which is routine. One can find a proof in, say, [HN19, Section 1.3].

PROPOSITION 1.4.8. *Let R be a p-adically complete commutative ring. Then there exists a multiplicative map (that is to say, a morphism of multiplicative monoids) $R^\flat \xrightarrow{(-)^\sharp} R$ that sends $a=(x_n)_{n\in\mathbb{N}}\in R^\flat$ to $a^\sharp := \lim_{n\to\infty} y_n^{p^n}$ where $(x_n)_{n\in\mathbb{N}}$ satisfies $\varphi(x_{n+1})=x_n$ for all $n\in\mathbb{N}$, and $(y_n)_{n\in\mathbb{N}}\in R^{\mathbb{N}}$ is any sequence such that for each $n\in\mathbb{N}$, y_n is a lift of $x_n\in R/p$ in R. We note that $a^\sharp$ does not depend on choice of $(y_n)_{n\in\mathbb{N}}$.*

DEFINITION 1.4.9. Fontaine's map $\theta: W(R^\flat)\to R$ *is given by* $\theta(\sum_{i=0}^{\infty}[a_i]\,p^i)=\sum_{i=0}^{\infty} a_i^\sharp\,p^i$, *where* $[-]: R^\flat \to W(R^\flat)$ *is the Teichmüller representative.*

DEFINITION 1.4.10. ([BMS18, DEFINITION 3.5]) *A commutative ring R is* perfectoid *if there exists $\pi\in R$ such that $p\in\pi^p R$, such that the ring R is (π)-adically complete, such that the $\mathbb{F}_p$-algebra R/p is semiperfect, and such that the kernel of $\theta: W(R^\flat)\to R$ is a principal ideal.*

DEFINITION 1.4.11. *Let R be a perfectoid ring. The* special fiber*, denoted by κ, is the direct limit perfection of R/p, that is to say $\kappa := (R/p)_{\mathrm{perf}} = R/\sqrt{pR}$ since R/p is semiperfect.*

NOTATION 1.4.12. *Let R be a perfectoid ring. We denote by ξ a generator of Fontaine's map $\theta: W(R^\flat)\to R$.*

PROPOSITION 1.4.13. ([BMS18, LEMMA 3.13]) *Let R be a perfectoid ring. Then the commutative diagram*

$$\begin{array}{ccc} W(R^\flat) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ W(\kappa) & \xrightarrow{\bmod p} & \kappa \end{array}$$

is a Tor*-independent pushout square.*

COROLLARY 1.4.14. *Let R be a perfectoid ring. For any generator $\xi\in\ker\theta$, there exists an invertible element $u\in\mathrm{GL}_1(W(\kappa))$ such that the image of $\xi\in W(R^\flat)$ in $W(\kappa)$ is $p\,u$.*

Proof. By Proposition 1.4.13, the image of $u\in W(R^\flat)$ in $W(\kappa)$ is a generator of the ideal $p\,W(\kappa)$. Since $p\in W(\kappa)$ is not a zero divisor, we deduce the result that we need. ☐

PROPOSITION 1.4.15. *Let R be a perfectoid ring. Then the kernel of the composition $R^\flat\to R/p\to\kappa$ is $\sqrt{\xi R^\flat}$.*

Proof. The kernel of the composition $W(R^\flat)\to R/p\to\kappa$ is $\sqrt{p\,W(R^\flat)+\xi\,W(R^\flat)}$ whose image under the canonical map $W(R^\flat)\to R^\flat$ is $\sqrt{\xi R^\flat}$. ☐

1.4.2. Universal properties of Fontaine's map (and a spherical analogue)

The results of this subsection will not be used later. However, we find it better to understand that Fontaine's map $\theta: W(R^\flat)\to R$ and its spherical analogue $W^+(R^\flat)\to\tau_{\le 0}(W^+(R^\flat))\simeq HW(R^\flat)\xrightarrow{H\theta} HR$ satisfy a universal property, which is related to the thickening defined in Definition 1.2.1. Roughly speaking, they are mixed characteristic "absolute" versions of thickenings in Definition 1.2.1. The following proposition is essentially due to Fontaine (see [Fon94], Theorem 1.2.1), rephrased in the modern language:

PROPOSITION 1.4.16. ([HN19, PROPOSITION 1.3.4]) *Let R be a perfectoid ring. Then Fontaine's map $\theta: W(R^\flat)\to R$ is initial among surjections $\theta_D: D\to R$ of rings such that the ring D is both p-adically complete and $\ker\theta_D$-adically complete.*

We will sketch the proof in [HN19] for the universal property, that is, assume that the p-adic completeness and the ξ-adic completeness of $W(R^\flat)$ is already given, we show that Fontaine's map $\theta: W(R^\flat)\to R$ is initial as claimed.

Proof. Let $\theta_D : D \to R$ be a map of rings such that D is both p-adically complete and $\ker \theta_D$-adically complete. We need to show that θ_D factors uniquely through $\theta : W(R^\flat) \to R$. In view of Example 1.2.5 and Definition 1.2.1, we have a bijection

$$\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, D/p)$$

(here everything is discrete therefore classical, but in order to avoid conflicts of notations with other parts of the article, we retain the cumbersome notations $\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}$ and $\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}$) which is given as follows: for any map $W(R^\flat) \to D$ of discrete $\mathbb{Z}_p$-algebras, we compose it with the canonical map $D \to D/p$ to get the map $W(R^\flat) \to D/p$, which factors uniquely through $W(R^\flat) \to W(R^\flat)/p \cong R^\flat$ therefore gives rise to a map $R^\flat \to D/p$. Note that $\theta_R = \mathrm{id}_R : R \to R$ serves as a special choice of θ_D since the perfectoid ring R is p-adically complete by Definition 1.4.10 and tautologically $\ker(\theta_R) = (0)$-adically complete. That is to say, we also have a bijection

$$\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p)$$

The map $\theta_D : D \to R$ gives rise to a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), D) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, D/p) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p)
\end{array}$$

So in order to show that the map $\theta_D : D \to R$ factors through the canonical map θ, or equivalently put, the preimage of the element $\theta \in \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R)$ under the induced map $\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), D) \to \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R)$ is a singleton, it suffices to show that the preimage of the element $(R^\flat \to R/p) \in \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p)$ under the map $\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, D/p) \to \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p)$ is a singleton, or equivalently put, the canonical map $\sigma : R^\flat \to R/p$ lifts uniquely through the map $\sigma_D : D/p \to R/p$ induced by the map $\theta_D : D \to R$. Note that the surjectivity of θ_D implies that of σ_D. Since the ring D is $(p, \ker \theta_D)$-adically complete, the ring D/p is $\ker \sigma_D$-adically complete.

We can conclude the existence and the uniqueness of lift of the map $\sigma : R^\flat \to R/p$ along the surjection $\sigma_D : D/p \to R/p$ simply by the fact that the $\mathbb{F}_p$-algebra $R^\flat$ is perfect and thus the cotangent complex $\mathbb{L}_{R^\flat/\mathbb{F}_p}$ is contractible, which implies the existence and the uniqueness of such lift.

However, we prefer to give a direct argument: We set the $\mathbb{F}_p$-algebra $A := R^\flat$ to stress that we only depend on the fact that A is a perfect $\mathbb{F}_p$-algebra, but not on the properties of the map $\sigma : A \to R/p$. Denote by $\varphi_B : B \to B, x \mapsto x^p$ the Frobenius map on any $\mathbb{F}_p$-algebra A. Then the Frobenius map φ_A is an isomorphism by assumption.

For each $a \in A$, we choose a sequence $(b_n)_{n=0}^{\infty} \in (D/p)^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $\sigma_D(b_n) = \sigma(\varphi_A^{-n}(a))$.

Note that the sequence $(\varphi_{D/p}^n(b_n))_{n=0}^{\infty}$ converges $\ker \sigma_D$-adically: $\sigma_D(b_n - b_{n+1}^p) = \sigma_D(b_n) - \sigma_D(b_{n+1})^p = \sigma(\varphi_A^{-n}(a)) - \sigma\big(\varphi_A^{-(n+1)}(a)\big)^p = \sigma(\varphi_A^{-n}(a)) - \sigma(\varphi_A(\varphi_A^{-n}(a))) = 0$ and therefore $\varphi_{D/p}^n(b_n) - \varphi_{D/p}^{n+1}(b_{n+1}) = \varphi_{D/p}^n(b_n - b_{n+1}^p) \in \varphi_{D/p}^n(\ker \sigma_D) \subseteq (\ker \sigma_D)^{p^n}$. Let $b := \lim_{n\to\infty} \varphi_{D/p}^n(b_n)$.

We first note that $\sigma_D(b) = \sigma(a)$, since $\sigma(\varphi_{D/p}^n(b_n)) = \sigma(b_n^{p^n}) = \sigma(b_n)^{p^n} = \sigma(\varphi_A^{-n}(a))^{p^n} = \sigma(\varphi_A^n(\varphi_A^{-n}(a))) = \sigma(a)$ for all $n \in \mathbb{N}$.

Now the value $b \in D/p$ does not depend on the choice of (b_n), since for any other choice (c_n), we have $c_n - b_n \in \ker \sigma_D$, thus $\varphi_{D/p}^n(b_n) - \varphi_{D/p}^n(c_n) = \varphi_{D/p}^n(b_n - c_n) \in \varphi_{D/p}^n(\ker \sigma_D) \subseteq (\ker \sigma_D)^{p^n}$ which implies that $\lim_{n\to\infty} \varphi_{D/p}^n(c_n) = b$.

Combining the preceding discussions, we have shown that for each $a \in A$, we can associate a $b \in D/p$ such that $\sigma_D(b) = \sigma(a)$. It is routine to check that $a \mapsto b$ defines a map $A \to D/p$ of rings which serves as a lift of $\sigma : A \to R/p$. Furthermore, the uniqueness essentially follows from the above argument that the value $b \in D/p$ does not depend on the choice of (b_n). $\square$

Remark 1.4.17. In fact, we can weaken our assumption on D to be derived p-complete and that the map $D \to R$ is Adams complete (due to [Car08] while the terminology is coined in [Bha12b]) by using some basic facts about Adams complete surjective maps of animated rings.

Now we give a spherical version of Fontaine's universal property:

PROPOSITION 1.4.18. *Let R be a perfectoid ring. We compose Fontaine's map $\theta : W(R^\flat) \to R$ with the 0th Postnikov section $W^+(R^\flat) \to \tau_{\leq 0}(W^+(R^\flat)) = HW(R^\flat)$, obtaining the map $\eta : W^+(R^\flat) \to HR$. Then we have*

1. *The $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra $W^+(R^\flat)$ of spherical Witt vectors is $(p, \ker\theta)$-complete. Furthermore, the discrete ring $\pi_0(W^+(R^\flat))/p$ is $\ker\theta/(p\,\pi_0(W^+(R^\flat)) + \ker\theta)$-adically separated.*
2. *The map $\eta : W^+(R^\flat) \to HR$ is initial among all maps $\eta_D : D \to HR$ surjective on π_0 where D is an $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra such that D is $(p, \ker\eta_D)$-complete and the discrete ring $\pi_0(D)/p$ is $\ker\theta_D/(p\,\pi_0(D) + \ker\theta_D)$-adically separated, where we denote the map $\pi_0(\eta_D) : \pi_0(D) \to R$ by θ_D.*

Remark 1.4.19. In Proposition 1.4.18, the technical conditions imposed on the $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra D are somewhat complicated. However, we can restrict to the full subcategory of η_D such that $\pi_0(D)$ is $(p, \ker\eta_D)$-adically complete, where $\eta : W^+(R^\flat) \to HR$ lives (see the proof of Proposition 1.4.18) and hence η is still an initial object in this full subcategory.

Remark 1.4.20. Using Remark 1.4.17, we can drop the adic completeness of $\pi_0(D)/p$ in Proposition 1.4.18.

Now we want to establish some computational results about homotopy groups of the ring $W^+(k)$ of spherical Witt vectors of a perfect $\mathbb{F}_p$-algebra k. First, we need the following proposition, which follows from Serre's computations of homotopy groups of spheres:

PROPOSITION 1.4.21. *The sphere spectrum $\mathbb{S}$ is connective, $\pi_0(\mathbb{S}) = \mathbb{Z}$, and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(\mathbb{S})$ is finite.*

Thus for each $n \in \mathbb{N}$, the homotopy group $\pi_n(\mathbb{S})$ has bounded p-torsion. Combined with Milnor sequence of homotopy groups, we have

COROLLARY 1.4.22. *The p-adic sphere spectrum $\mathbb{S}_p^\wedge$ is connective, $\pi_0(\mathbb{S}_p^\wedge) = \mathbb{Z}_p$ and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(\mathbb{S}_p^\wedge)$ is a finite direct sum of finite abelian groups of form $\mathbb{Z}/p^r \cong \mathbb{Z}_p/p^r$ for some positive integer $r \in \mathbb{N}_{>0}$.*

We need a result announced in [Lur18a, Example 5.2.7] the argument of which we learn from Matthew Morrow:

PROPOSITION 1.4.23. *Let k be a perfect $\mathbb{F}_p$-algebra. Then the ring of spherical Witt vectors $W^+(k)$ is a flat $\mathbb{S}_p^\wedge$-module.*

Proof. First, by Proposition 1.2.7, $\pi_0(W^+(k)) \cong W(k)$ which is a torsion-free $\mathbb{Z}_p$-module. Since $\mathbb{Z}_p$ is a valuation ring, we deduce that $W(k)$ is a flat $\mathbb{Z}_p$-module (see [Sta21, Tag 0539]). Now we consider the Postnikov tower $(\tau_{\geq n}\mathbb{S}_p^\wedge)_{n\in\mathbb{N}}$ of the p-adic sphere spectrum $\mathbb{S}_p^\wedge$, which induces a tower $X_n := (\tau_{\geq n}\mathbb{S}_p^\wedge) \otimes_{\mathbb{S}_p^\wedge} W^+(k)$. Note that $X_n/X_{n-1} \cong \Sigma^n H\pi_n(\mathbb{S}_p^\wedge) \otimes_{\mathbb{S}_p^\wedge} W^+(k)$. We have shown in Corollary 1.4.22 that $\pi_n(\mathbb{S}_p^\wedge)$ is a direct sum of finite abelian groups of form $\mathbb{Z}_p/p^r$, which allows us to realize $H\pi_n(\mathbb{S}_p^\wedge)$ as a direct sum of spectra of form $\mathrm{cofib}\Big(H\mathbb{Z}_p \xrightarrow{p^r} H\mathbb{Z}_p\Big)$. Note that the smash product $-\otimes_{\mathbb{S}_p^\wedge} W^+(k)$ commutes with taking cofibers, we deduce that $H\pi_n(\mathbb{S}_p^\wedge) \otimes_{\mathbb{S}_p^\wedge} W^+(k) \cong H\,\mathrm{Tor}_0^{\mathbb{Z}_p}(\pi_n(\mathbb{S}_p^\wedge), W(k))$. Thus the tower $(X_n)_{n\in\mathbb{N}}$ constitutes the Postnikov tower of the spectrum $W^+(k)$, therefore $\pi_n(W^+(k)) \cong \mathrm{Tor}_0^{\pi_0(\mathbb{S}_p^\wedge)}(\pi_n(\mathbb{S}_p^\wedge), W(k))$. □

COROLLARY 1.4.24. *Let k be a perfect $\mathbb{F}_p$-algebra. Then the ring of spherical Witt vectors $W^+(k)$ is connective, $\pi_0(W^+(k)) = W(k)$, and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(W^+(k))$ is a finite direct sum of $W(k)$-modules of form $W(k)/p^r$.*

We are now ready to prove Proposition 1.4.18:

Proof of Proposition 1.4.18. We check two statements one by one:

1. Proposition 1.4.16 tells us that the discrete ring $\pi_0(W^+(R^\flat)) \cong W(R^\flat)$ is $(p, \ker\theta)$-adically complete, therefore by Proposition A.0.28, it is $(p, \ker\theta)$-complete. Furthermore, we deduce from $(p, \ker\theta)$-adic completeness that $\pi_0(W^+(R^\flat))$ is $\ker\theta/(p\,\pi_0(W^+(R^\flat)) + \ker\theta)$-adically separated. In view of Theorem A.0.25, it remains to show that for each $n \in \mathbb{N}_{>0}$, the homotopy group $\pi_n(W^+(R^\flat))$ is (derived) $(p, \ker\theta)$-complete as a discrete $W(R^\flat)$-module. However, by Corollary 1.4.24, we have realized $\pi_n(W^+(R^\flat))$ as a direct sum of cofibers of $(p, \ker\theta)$-complete modules, therefore it is $(p, \ker\theta)$-complete.
2. This part is parallel to the proof of Proposition 1.4.16. We start with the following commutative diagram induced by the map $\eta_D : D \to HR$:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), D) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, \pi_0(D)/p) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), HR) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, R/p) \end{array}$$

as in the proof of Proposition 1.4.16. It follows from Definition 1.2.1 and Example 1.2.6 that the horizontal maps are homotopy equivalences, which implies that the connected components of each space on the left are all contractible. We pick the connected component of $\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), HR)$ corresponds to the map $\eta: W^+(R^\flat) \to HR$. In order to show that η is an initial object, it suffices to show that there exists one and only one connected component in $\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), D)$ which maps to the connect component corresponding to η. Note that the image of η in $\mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, R/p)$ along the bottom horizontal map coincides with $\sigma \in \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, R/p)$ defined in the proof of Proposition 1.4.16. In view of the commutative diagram, it remains to show that the preimage of $\sigma \in \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, R/p)$ in $\mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}^\heartsuit}(R^\flat, \pi_0(D)/p)$. The rest of the proof is identical to that of Proposition 1.4.16. □

1.5. Proof of the main theorem

Fix a perfectoid ring R and a generator ξ of Fontaine's map $\theta: W(R^\flat) \to R$, the goal of this section is to prove Theorem 1.1.13. We first need a much weaker version which says that the 0th homotopy group of the $\mathbb{E}_2$-Thom spectrum in question, as a ring, is isomorphic to R:

Lemma 1.5.1. *The 0th homotopy group $\pi_0(Mf_{R,\xi})$ of the Thom spectrum associated to $f_{R,\xi}$ is isomorphic to R for any perfectoid ring R.*

Proof. We mimic a segment of the proof of Theorem A.1 in [KN]:

We note that $Mf_{R,\xi}$ is connective, so we have

$$\pi_0(Mf_{R,\xi}) \cong \pi_0(W^+(R^\flat)_{h\Omega^3 S^3}) \cong \pi_0(W^+(R^\flat))_{\pi_0(\Omega^3 S^3)}$$

where the $\pi_0(\Omega^3 S^3) \cong \mathbb{Z}$-action on $\pi_0(W^+(R^\flat)) \cong W(R^\flat)$ is given by multiplication by $1-\xi$, hence

$$\pi_0(W^+(R^\flat))_{\pi_0(\Omega^3 S^3)} \cong W(R^\flat)/(1-(1-\xi)) \cong R$$ □

In view of Lemma 1.5.1, in order to prove Theorem 1.1.13, it suffices to show that

Proposition 1.5.2. *The 0th Postnikov section $t_{R,\xi}: Mf_{R,\xi} \to \tau_{\leq 0} Mf_{R,\xi} \simeq HR$, being an $\mathbb{E}_2$-map a priori, is an equivalence of spectra.*

To begin with, we first note that the special case when R is a perfect $\mathbb{F}_p$-algebra is already covered by previous considerations:

Lemma 1.5.3. *The $t_{R,\xi}$ in question is an equivalence of spectra when R is a perfect $\mathbb{F}_p$-algebra.*

Proof. Theorem 1.3.1 tells us that there is an equivalence $Mf_{R,\xi} \to HR$. The lemma follows from the fact that HR lives in $(\mathrm{Mod}_{W^+(R)})_{\leq 0}$ and that the 0th Postnikov section is essentially unique. □

We first note that both $Mf_{R,\xi}$ and HR admit canonical $W^+(R^\flat)$-module structures. Our strategy breaks up into several steps:

1. Prove some finiteness and completeness results of $Mf_{R,\xi}$ and HR as $W^+(R^\flat)$-modules;
2. Show that $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat) \to W^+(\kappa)$, and hence an equivalence after a further base change along $W^+(\kappa) \to H\kappa$ to the special fiber $H\kappa$;
3. The composition $W^+(R^\flat) \to W^+(\kappa) \to H\kappa$ coincides with the composition $W^+(R^\flat) \to HR^\flat \to H\kappa$, and a Nakayama-like argument shows that $t_{R,\xi}$ is an equivalence after base change along $W^+(R^\flat) \to HR^\flat$;
4. Deduce that $t_{R,\xi}$ is an equivalence by completeness.

To proceed, by Corollary 1.4.14, we also fix an invertible element $u \in \mathrm{GL}_1(W(\kappa))$ associated to ξ so that the image of ξ in $W(\kappa)$ is $p\,u$.

1.5.1. Finiteness and completeness of $Mf_{R,\xi}$ and HR as $W^+(R^\flat)$-modules

Lemma 1.5.4. *$HW(k)$ is an almost perfect $W^+(k)$-module for any perfect $\mathbb{F}_p$-algebra k.*

Proof. If $k = \mathbb{F}_p$, then $W^+(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$ is a coherent ring as in Definition A.0.10, and $HW(\mathbb{F}_p) \simeq H\mathbb{Z}_p \simeq H\pi_0(W^+(\mathbb{F}_p))$ is an almost perfect $\mathbb{S}_p^\wedge$-module by Corollary A.0.12.

In general, by Proposition 1.2.7, we have $HW(k) \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$, hence $HW(k)$ is almost perfect by Proposition A.0.8. □

Corollary 1.5.5. *HR is an almost perfect $W^+(R^\flat)$-module.*

Proof. HR is the cofiber of the multiplication map $m_\xi : HW(R^\flat) \to HW(R^\flat)$ where the domain and the codomain are almost perfect (Lemma 1.5.4), hence HR is also almost perfect (Proposition A.0.7). □

We need a nontrivial input from algebraic topology:

Proposition 1.5.6. *There exists a Kan complex $X_\bullet$ which is homotopy equivalent to the double loop space $\Omega^2 S^3$ of the 3-sphere such that X_n is a finite set for each $[n] \in \Delta^{\mathrm{op}}$.*

Proof. This is essentially due to [Wal65, Thm A and B] and Serre. We first note that, the loop space $\Omega^2 S^3$ is a loop space therefore simple [MP12, Cor 1.4.5]. Now we show that $\Omega^2 S^3$ is of finite type, i.e. homotopy equivalent to a CW-complex with finite skeleta. By [MP12, Thm 4.5.2], it suffices to show that $H_i(\Omega^2 S^3; \mathbb{Z})$ are finitely generated for all $i \in \mathbb{N}_{>0}$. The argument is standard (due to Serre): we know that $H_i(S^3; \mathbb{Z})$ are finitely generated for all $i \in \mathbb{N}$. Applying [tD08, Thm 20.4.1] to the fiber sequence $\Omega S^3 \to * \to S^3$ in $\mathcal{S}$, we deduce that $H_i(\Omega S^3)$ are finitely generated for all $i \in \mathbb{N}$. We apply again [tD08, Thm 20.4.1] to the fiber sequence $\Omega^2 S^3 \to * \to \Omega S^3$, we deduce that $H_i(\Omega^2 S^3)$ are finitely generated. Now the result follows from the simplicial approximation theorem. □

Lemma 1.5.7. *$Mf_{R,\xi}$ is an almost perfect $W^+(R^\flat)$-module.*

Proof. We first pick up a Kan complex $X_\bullet$ representing $\Omega^2 S^3$ where each X_n is a finite set as in Proposition 1.5.6. It follows from Bousfield-Kan formula (see, for example, Corollary 12.3 in [Sha18]) that $Mf_{R,\xi}$ could be written as the geometric realization of a simplicial object $N_\bullet$ where each N_n is a free $W^+(R^\flat)$-module of finite rank, hence almost perfect by Proposition A.0.7. □

Corollary 1.5.8. $\mathrm{cofib}(t_{R,\xi})$ *is an almost perfect* $W^+(R^\flat)$*-module.*

Proof. The subcategory of almost perfect modules are closed under taking cofibers and base changes (Proposition A.0.7). The statement then follows from Corollary 1.5.5 and Lemma 1.5.7. □

Lemma 1.5.9. *The spectrum* HR *is* p*-complete.*

Proof. By definition of perfectoid rings, R is p-adically complete, therefore HR is p-complete by Proposition A.0.28. □

Lemma 1.5.10. *The spectrum* $Mf_{R,\xi}$ *is* p*-complete.*

Proof. We note that $W^+(R^\flat)$ is p-complete by definition of spherical Witt vectors, and $Mf_{R,\xi}$ is almost perfect, therefore p-complete by Proposition A.0.27. □

Corollary 1.5.11. *The spectrum* $\mathrm{cofib}(t_{R,\xi})$ *is* p*-complete.*

Proof. It follows from Corollary 1.5.8 and Proposition A.0.27. □

1.5.2. $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat) \to W^+(\kappa)$

The proof is similar to that of Theorem 1.3.1, except that we need to be more careful to identify the maps.

Lemma 1.5.12. *There is a canonical equivalence* $Mf_{\kappa,pu} \xrightarrow{\simeq} W^+(\kappa) \otimes_{W^+(R^\flat)} Mf_{R,\xi}$ *of* $W^+(\kappa)$*-modules.*

Proof. We first note that the image of the multiplication map $m_{1-\xi}: W^+(R^\flat) \to W^+(R^\flat)$ under the base change functor $W^+(\kappa) \otimes_{W^+(R^\flat)} - : \mathrm{Mod}_{W^+(R^\flat)} \to \mathrm{Mod}_{W^+(\kappa)}$ is the multiplication map $m_{1-pu}: W^+(\kappa) \to W^+(\kappa)$.

Therefore $f_{\kappa,pu}$ coincides with the composition

$$\Omega^2 S^3 \xrightarrow{f_{R,\xi}} \mathrm{BGL}_1(W^+(R^\flat)) \xrightarrow{W^+(\kappa)\otimes_{W^+(R^\flat)} -} \mathrm{BGL}_1(W^+(\kappa))$$

Along with the fact that the functor $W^+(\kappa) \otimes_{W^+(R^\flat)} - : \mathrm{Mod}_{W^+(R^\flat)} \to \mathrm{Mod}_{W^+(\kappa)}$ commutes with small colimits, or to be more precise, that the natural transformation $\mathrm{colim}_i (W^+(\kappa) \otimes_{W^+(R^\flat)} M_i) \to W^+(\kappa) \otimes_{W^+(R^\flat)} (\mathrm{colim}_i M_i)$ is an equivalence for any diagram $(M_i)_i$ in $\mathrm{Mod}_{W^+(R^\flat)}$, we deduce that there is a canonical equivalence $Mf_{\kappa,pu} \xrightarrow{\simeq} W^+(\kappa) \otimes_{W^+(R^\flat)} Mf_{R,\xi}$ as $W^+(\kappa)$-modules. □

Lemma 1.5.13. *Given a morphism of perfect* $\mathbb{F}_p$*-algebras* $k \to K$*, the commutative diagram of* $\mathbb{E}_\infty$*-rings*

$$\begin{array}{ccc} W^+(k) & \longrightarrow & W^+(K) \\ \downarrow & & \downarrow \\ HW(k) & \longrightarrow & HW(K) \end{array}$$

is a pushout square.

Proof. Consider the commutative diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(k) & \longrightarrow & W^+(K) \\ \downarrow & & \downarrow & & \downarrow \\ H\mathbb{Z}_p & \longrightarrow & HW(k) & \longrightarrow & HW(K) \end{array}$$

By Proposition 1.2.7, we know that the left square and the outer square are pushout squares, therefore so is the right square. □

Lemma 1.5.14. *There is a canonical equivalence* $W^+(\kappa) \otimes_{W^+(R^\flat)} HR \to H\kappa$ *of* $W^+(\kappa)$*-modules.*

Proof. Combining two pushout squares in the category of $\mathbb{E}_\infty$-rings:

$$\begin{array}{ccc} W^+(R^\flat) & \longrightarrow & W^+(\kappa) \\ \downarrow & \sigma & \downarrow \\ HW(R^\flat) & \longrightarrow & HW(\kappa) \\ \downarrow & \tau & \downarrow \\ HR & \longrightarrow & H\kappa \end{array}$$

where σ is a pushout square by Lemma 1.5.13 and τ is a pushout square by Proposition 1.4.13. □

LEMMA 1.5.15. *The map $W^+(\kappa)\otimes_{W^+(R^\flat)} t_{R,\xi}: W^+(\kappa)\otimes_{W^+(R^\flat)} Mf_{R,\xi}\to W^+(\kappa)\otimes_{W^+(R^\flat)} HR$ is equivalent to $t_{\kappa,pu}: Mf_{\kappa,pu}\to H\kappa$.*

Proof. In view of Lemma 1.5.12 and Lemma 1.5.14, we only need to show that $t_{\kappa,pu}: Mf_{\kappa,pu}\to H\kappa$ coincides with the composition of the equivalences $Mf_{\kappa,pu}\to W^+(\kappa)\otimes_{W^+(R^\flat)} Mf_{R,\xi}$, $W^+(\kappa)\otimes_{W^+(R^\flat)} t_{R,\xi}$ and $W^+(\kappa)\otimes_{W^+(R^\flat)} HR\to H\kappa$. In other words, it suffices to show that the composition in question is the 0th Postnikov section. We only need to check that the composition induces an isomorphism on π_0 by basic properties of t-structures, since $\tau_{\leq 0} Mf_{\kappa,pu}\simeq H\kappa$. It suffices to show that $W^+(\kappa)\otimes_{W^+(R^\flat)} t_{R,\xi}$ induces an isomorphism on π_0, and this follows from the fact that all spectra in question are connective and that $t_{R,\xi}$ induces an isomorphism on π_0 by Lemma 1.5.1. □

COROLLARY 1.5.16. *$H\kappa\otimes_{W^+(R^\flat)} t_{R,\xi}$ is an equivalence of spectra.*

Proof. It follows from Lemma 1.5.15 and Lemma 1.5.3. □

1.5.3. $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat)\to HR^\flat$

LEMMA 1.5.17. *Let M be an $HR^\flat$-module which is bounded below and almost perfect. If there exists an $r\in\mathbb{N}$ such that $\xi^r\pi_n(M)=0$ for all $n\in\mathbb{Z}$, and $H\kappa\otimes_{HR^\flat} M\simeq 0$, then $M\simeq 0$.*

Proof. We show inductively on n that $\pi_n M=0$.

- Since M is bounded below, $\pi_n M=0$ for $n\ll 0$;
- Suppose that for $m<n$ we have $\pi_m M=0$. Then by unrolling Definition A.0.6, $\pi_n M$ is a compact object in the category of discrete $R^\flat$-modules, therefore is finitely presented and in particular finitely generated. Now we have

  $$0=\pi_n(H\kappa\otimes_{HR^\flat} M)=\mathrm{Tor}_0^{R^\flat}(\kappa,\pi_n M).$$

  By Proposition 1.4.15 and that $\xi^r\pi_n(M)=0$, we have

  $$\mathrm{Tor}_0^{R^\flat}(\kappa,\pi_n M)=\mathrm{Tor}_0^{R^\flat/\xi^r}(\kappa,\pi_n M)$$

  We note that $\ker(R^\flat/\xi^r\to\kappa)$ lies in the (nil-radical, therefore) Jacobson radical of $R^\flat/\xi^r$, thus $\pi_n M=0$, by Nakayama's lemma along with the fact that $\pi_n M$ is finitely generated. □

Remark 1.5.18. Matthew Morrow told us that in Lemma 1.5.17, the hypothesis $\xi^r\pi_n(M)=0$ is redundant, since the kernel of the map $R^\flat\to\kappa$ of $\mathbb{F}_p$-algebras lies in the radical of the ideal $\xi R^\flat\subseteq\mathrm{Rad}(R^\flat)$ where $\mathrm{Rad}(R^\flat)$ is the Jacobson radical of the $\mathbb{F}_p$-algebra $R^\flat$ and the inclusion $\xi R^\flat\subseteq\mathrm{Rad}(R^\flat)$ is deduced from the $\xi R^\flat$-adically completeness of the $\mathbb{F}_p$-algebra $R^\flat$. Since the Jacobson radical is "radical", the kernel of the map $R^\flat\to\kappa$ also lies in the Jacobson radical $\mathrm{Rad}(R^\flat)$. We decide to preserve the original version to reflect our real thoughts.

COROLLARY 1.5.19. *$HR^\flat\otimes_{W^+(R^\flat)} t_{R,\xi}$ is an equivalence of spectra.*

Proof. Note that

$$\pi_0(HR^\flat\otimes_{W^+(R^\flat)} Mf_{R,\xi})=\mathrm{Tor}_0^{W(R^\flat)}(R^\flat,\pi_0(Mf_{R,\xi}))=R^\flat/\xi R^\flat$$

and

$$\pi_0(HR^\flat \otimes_{W^+(R^\flat)} HR) = \mathrm{Tor}_0^{W(R^\flat)}(R^\flat, R) = R^\flat / \xi R^\flat$$

and that $HR^\flat \otimes_{W^+(R^\flat)} Mf_{R,\xi}$, $HR^\flat \otimes_{W^+(R^\flat)} HR$ are connective $\mathbb{E}_\infty$-rings, we conclude that the homotopy groups of these $\mathbb{E}_\infty$-rings are ξ-torsion groups, which implies that for all $n \in \mathbb{Z}$,

$$\xi^2 \pi_n(\mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi})) = 0$$

In addition, since the subcategory of almost perfect modules are closed under base changes (Proposition A.0.8), we deduce from Corollary 1.5.8 that $\mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi}) \simeq HR^\flat \otimes_{W^+(R^\flat)} \mathrm{cofib}(t_{R,\xi})$ is almost perfect. On the other hand, being the cofiber of a map of connective spectra, it is also connective. Then we invoke Lemma 1.5.17 along with Corollary 1.5.16 to conclude that $\mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi}) \simeq 0$. □

1.5.4. Conclude: $t_{R,\xi}$ is an equivalence

We are now at the final stage to conclude a proof of Proposition 1.5.2, and consequently, Theorem 1.1.13.

Proof of Proposition 1.5.2. We recall that by Theorem 1.2.4 and Example 1.2.6, there is a pushout square of $\mathbb{E}_\infty$-rings:

$$\begin{array}{ccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(R^\flat) \\ \big\downarrow & & \big\downarrow \\ H\mathbb{F}_p & \longrightarrow & HR^\flat \end{array}$$

Therefore by Corollary 1.5.19 we have

$$0 \simeq \mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi}) \simeq HR^\flat \otimes_{W^+(R^\flat)} \mathrm{cofib}(t_{R,\xi}) \simeq H\mathbb{F}_p \otimes_{\mathbb{S}_p^\wedge} \mathrm{cofib}(t_{R,\xi})$$

We then invoke Corollary A.0.32 with Corollary 1.5.11 to deduce that $\mathrm{cofib}(t_{R,\xi}) \simeq 0$. □

1.5.5. An intermezzo: Identifying $\mathrm{THH}(-/W^+(k))$ with $\mathrm{THH}(-)$ after p-completion

In this subsection, we will show that Proposition 1.1.17 follows from Proposition 1.1.16. It suffices to prove the following lemma:

Lemma 1.5.20. *Let R be an $\mathbb{E}_1$-algebra over $W^+(k)$ where k is a perfect $\mathbb{F}_p$-algebra. Then the canonical map $\mathrm{THH}(R) \to \mathrm{THH}(R/W^+(k))$ induced by $\mathbb{S} \to \mathbb{S}_p^\wedge$ is an equivalence after p-completion.*

Proof. Note that $\mathrm{THH}(R/W^+(k)) \simeq W^+(k) \otimes_{\mathrm{THH}(W^+(k))} \mathrm{THH}(R)$. We are left to show that the canonical map $\mathrm{THH}(W^+(k)) \to W^+(k)$ is an equivalence after p-completion. In view of Corollary A.0.32, we only need to check it after tensoring with $H\mathbb{F}_p$. We note that the base changed map $H\mathbb{F}_p \otimes \mathrm{THH}(W^+(k)) \to H\mathbb{F}_p \otimes W^+(k)$ fits into the commutative diagram

$$\begin{array}{ccc} H\mathbb{F}_p \otimes \mathrm{THH}(W^+(k)) & \longrightarrow & H\mathbb{F}_p \otimes W^+(k) \\ \big\downarrow & \sigma & \big\| \\ \mathrm{THH}(H\mathbb{F}_p \otimes W^+(k) / H\mathbb{F}_p) & \longrightarrow & H\mathbb{F}_p \otimes W^+(k) \\ \big\downarrow & \tau & \big\downarrow \\ \mathrm{THH}(Hk / H\mathbb{F}_p) & \longrightarrow & Hk \end{array}$$

where the commutativity of σ follows from the functoriality of the base change functor of THH, and the commutative of τ follows from the functoriality of the natural transformation $\mathrm{THH}(-/H\mathbb{F}_p) \to (-)$. All vertical maps are equivalences of spectra: the upper left map is the base change equivalence, and the lower right map is the equivalence by Proposition 1.2.7, and the lower left map is the image of this equivalence under the functor $\mathrm{THH}(-/H\mathbb{F}_p)$ and hence also an equivalence. The bottom horizontal map is an equivalence by the fact that k is a perfect $\mathbb{F}_p$-algebra. □

1.6. Analogues

It is worth to note that in Bhatt and Scholze's recent work [BS19], they introduced the concept of prisms (A, I) which serves as a "non-perfect" version of perfectoid rings. Especially, the category of perfect prisms (A, I) is equivalent to that of perfectoid rings A/I, and given a perfectoid ring R, the corresponding perfect prism is given by $(W(R^\flat), \ker\theta)$. It is interesting to know whether we can generalize our description for general orientable prisms (A, I), that is to say,

Question 1. Given an orientable prism $(A, I=(d))$. When can we find an $\mathbb{E}_\infty$-ring spectrum A^+ (which satisfies some hypotheses related to A. A naive guess would be that $\pi_0(A^+) = A$) and a map $\Omega^2 S^3 \to \mathrm{BGL}_1(A^+)$ to which the associated $\mathbb{E}_2$-Thom spectrum (possibly after p-completion) coincides with A/I.

We don't know the answer in this generality. However, we will discuss another special class of prism (related to Breuil–Kisin cohomology) for which an analogue holds. This result is more-or-less known by experts. In fact, it is essentially equivalent to Remark 3.4 in [KN19] of which no proof is presented. In this section, we will first recall some basic facts about complete discrete valuation rings, then we will indicate briefly how to adapt our proof above to this special class.

1.6.1. Preparations

Definition 1.6.1. ([Ser79, Section I.1]) *A ring A is called a* discrete valuation ring*, or a* DVR*, if it is a principal ideal domain that has a unique non-zero prime ideal $\mathfrak{m}$. In this case, since A is local, we also denote the DVR A by $(A, \mathfrak{m})$. The field $A/\mathfrak{m}$ is called the* residue field *of A. A generator of $\mathfrak{m}$, unique up to multiplication by an invertible element, is called a* uniformizer*, usually denoted by ϖ.*

Definition 1.6.2. *A DVR $(A, \mathfrak{m})$ is called* of mixed characteristics $(0, p)$ *if the field of fraction* $\mathrm{Frac}(A)$ *of A is of characteristics* 0 *while the residue field $A/\mathfrak{m}$ is of characteristics p, which implies that $0 \neq p \in \mathfrak{m}$.*

Definition 1.6.3. ([Ser79, Section I.1]) *Let $(A, \mathfrak{m})$ be a DVR. The* valuation *of an element $x \in A \setminus 0$ is defined to be the maximal integer $n \in \mathbb{N}$ such that $x \in \mathfrak{m}^n$, which always exists, denoted by $v(x) \in \mathbb{N}$.*

Definition 1.6.4. ([Ser79, Section II.5]) *Let $(A, \mathfrak{m})$ be a DVR of mixed characteristics $(0, p)$. Then the integer $e = v(p)$ is called the* absolute ramification index *of A.*

Definition 1.6.5. ([Ser79, Chapter II]) *A DVR $(A, \mathfrak{m})$ is called* complete *if it is complete with respect to the $\mathfrak{m}$-adic topology, that is to say, the canonical map from A to the limit of the tower*

$$\cdots \to A/\mathfrak{m}^n \to \cdots \to A/\mathfrak{m}^2 \to A/\mathfrak{m}$$

is an isomorphism.

Proposition 1.6.6. ([Ser79, Section II.5, Theorem 4]) *Let $(A, \mathfrak{m})$ be a complete DVR of mixed characteristics $(0, p)$ with residue field k being perfect. Let e be its absolute ramification index. Let $\varpi \in \mathfrak{m}$ be a uniformizer. Then there exists an Eisenstein $W(k)$-polynomial $E(u) \in W(k)[u]$ (that is, a $W(k)$-polynomial $E(u) = u^e + \sum_{j=0}^{e-1} a_j u^j$ such that $p \mid a_j$ for $j = 0, \ldots, e-1$ and $p^2 \nmid a_0$, where $W(k)$ is the ring of Witt vectors as before) along with an isomorphism $W(k)[u]/(E(u)) \xrightarrow{\sim} A$ which maps u to the uniformizer $\varpi \in \mathfrak{m}$.*

In the rest of this section, we will fix a complete DVR $(A, \mathfrak{m})$ of mixed characteristics $(0, p)$ with residue field k being perfect, absolute ramification index e and a uniformizer $\varpi \in \mathfrak{m}$. We also fix a choice of an Eisenstein $W(k)$-polynomial $E(u) \in W(k)[u]$ as in Proposition 1.6.6. We first note that

Proposition 1.6.7. *The element $1 - E(u) \in W(k)[[u]]$ is invertible.*

Proof. Write $E(u) = u^e + \sum_{j=0}^{e-1} a_j u^j$ as in Proposition 1.6.6. Note that $W(k)$ is p-adically complete, therefore $1 - a_0$ is invertible in $W(k)$, which implies that $1 - E(u) \in W(k)[[u]]$ is invertible. □

Let $W^+(k)[u]$ be the "single variable polynomial $W^+(k)$-algebra", that is, the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)\otimes_{\mathbb{S}}\mathbb{S}[\mathbb{N}]$. Since the space $\mathbb{N}$ is endowed with discrete topology, we have

Proposition 1.6.8. *As a $W^+(k)$-module, $W^+(k)[u]$ is equivalent to the direct sum $\bigoplus_{j=0}^{\infty} u^j W^+(k)$, a free $W^+(k)$-module. The graded homotopy group $\pi_*(W^+(k)[u])$, as a (graded-commutative) $\pi_*(W^+(k))$-algebra, is equivalent to $\pi_*(W^+(k))[u]$, where* $\deg u = 0$.

Now let $W^+(k)[[u]]$ be the (u)-completion of the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[u]$. To study $W^+(k)[[u]]$, we need some preparations.

Proposition 1.6.9. *Let $n\in\mathbb{N}$ be a natural number. Let $m_{u^n}: W^+(k)[u]\to W^+(k)[u]$ be the multiplication map given by $u^n\in\pi_0(W^+(k)[u]) = W(k)[u]$. Then the $W^+(k)[u]$-module* $\mathrm{cofib}(m_{u^n})$ *as a $W^+(k)$-module is a free $W^+(k)$-module $\bigoplus_{j=0}^{n-1} u^j W^+(k)$ of rank n, which admits an $\mathbb{E}_\infty$-$W^+(k)[u]$-algebra structure. In particular, we have the cofiber sequence*

$$W^+(k)[u]\xrightarrow{m_u} W^+(k)[u]\to W^+(k)$$

of $W^+(k)[u]$-modules.

Proof. For any space $X\in\mathcal{S}$, we let $X_+\in\mathcal{S}_*$ denote the pointed discrete space $\{*\}\cup X$. Especially, $\mathbb{N}_+ = \{*\}\cup\mathbb{N}$ and $(\mathbb{N}_{<n})_+ = \{*\}\cup\mathbb{N}_{<n}$. The addition map $\mathbb{N}\to\mathbb{N}, m\mapsto n+m$ induces a map of pointed spaces $\alpha_n: \mathbb{N}_+\to\mathbb{N}_+$. Note that in the ∞-category $\mathcal{S}$ of spaces, we have a pushout diagram

$$\begin{tikzcd} \mathbb{N}_+ \arrow[r, "\alpha_n"] \arrow[d] & \mathbb{N}_+ \arrow[d] \\ \{*\} \arrow[r] & (\mathbb{N}_{<n})_+ \end{tikzcd}$$

to which we apply the functor $\Sigma^\infty: \mathcal{S}_*\to\mathrm{Sp}$, left adjoint of the functor $\Omega^\infty_*: \mathrm{Sp}\to S_*$ therefore preserving colimits, we get a cofiber sequence $\mathbb{S}[u]\xrightarrow{u^n}\mathbb{S}[u]\to\bigoplus_{j=0}^{n-1} u^j\,\mathbb{S}$. A further base change to $W^+(k)$ gives rise to the result. In addition, the multiplication structure could be seen from the fact that the addition map $\mathbb{N}\to\mathbb{N}, m\mapsto n+m$ in fact defines a monoidal action. ☐

Corollary 1.6.10. *Let $n\in\mathbb{N}$ be a natural number. Let $m_{u^n}: W^+(k)[u]\to W^+(k)[u]$ be the multiplication map. Then homotopy groups $\pi_*(\mathrm{cofib}(m_{u^n}))$ of the cofiber as $\pi_*(W^+(k))$ could be identified with $\pi_*(W^+(k))[u]/(u^n)$, and the long exact sequence of homotopy groups associated to the cofiber sequence $W^+(k)[u]\xrightarrow{m_{u^n}} W^+(k)[u]\to\mathrm{cofib}(m_{u^n})$ decomposes as short exact sequences, which assemble to a short exact sequence of graded $\pi_*(W^+(k))[u]$-modules:*

$$0\to\pi_*(W^+(k))[u]\xrightarrow{u^n}\pi_*(W^+(k))[u]\to\pi_*(W^+(k))[u]/(u^n)\to 0$$

Furthermore, this sequence is functorial in $n\in(\mathbb{N},>)$.

Proposition 1.6.11. *The $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[[u]]$ is connective. The zeroth homotopy group of $\pi_0(W^+(k)[[u]])$ is isomorphic to the (u)-adic completion of the polynomial $W(k)$-algebra $W(k)[u]$, that is, the formal power series $W(k)$-algebra $W(k)[[u]]$, as $W(k)$-algebras.*

Our proof is incomplete: we only identify the $W(k)$-module structures on homotopy groups. A formal identification of algebra structures would require more rudiments about the symmetric monoidal structure on the completion functor than we know.

Proof. We reinterpret Proposition A.0.19 as follows: since the limit functor is exact, it commutes with cofibers, therefore we can rewrite $W^+(k)[[u]] = (W^+(k)[u])^\wedge_{(u)}$ as the limit of the tower

$$\cdots\to\mathrm{cofib}\Big(W^+(k)[u]\xrightarrow{u^2} W^+(k)[u]\Big)\to\mathrm{cofib}\big(W^+(k)[u]\xrightarrow{u} W^+(k)[u]\big)$$

After passage to homotopy groups, by Corollary 1.6.10, we get the tower of graded $\pi_*(W^+(k))[u]$-modules

$$\cdots\to\pi_*(W^+(k))[u]/(u^n)\to\cdots\to\pi_*(W^+(k))[u]/(u^2)\to\pi_*(W^+(k))[u]/(u) \tag{1.6.1}$$

which is degree-wise a tower of surjective maps. It follows from Milnor's sequence that the graded $\pi_*(W^+(k))[u]$-module $\pi_*(W^+(k)[[u]])$ is isomorphic to the (ordinary) inverse limit of the tower (1.6.1), that is, $\pi_*(W^+(k))[[u]]$. Take $*=0$, we get the result. $\square$

The following lemma serves as a key tool in our proof:

LEMMA 1.6.12. *Let M be a $W^+(k)[u]$-(or $W^+(k)[[u]]$-)module (spectrum). If the spectrum $W^+(k)\otimes_{W^+(k)[u]} M$ (or $W^+(k)\otimes_{W^+(k)[[u]]} M$ respectively) is contractible, then so is the (u)-completion of the spectrum M. In particular, if furthermore $W^+(k)[u]$-(or $W^+(k)[[u]]$-)module M is assumed to be (u)-complete, then the spectrum M is contractible.*

Proof. We first assume that the spectrum $W^+(k)\otimes_{W^+(k)[u]} M$ is contractible. In this case, we apply the exact functor $-\otimes_{W^+(k)[u]} M$ to the cofiber sequence

$$W^+(k)[u] \xrightarrow{m_u} W^+(k)[u] \to W^+(k) \tag{1.6.2}$$

indicated in Proposition 1.6.9 obtaining that the base-changed map $M \xrightarrow{m_u\otimes_{W^+(k)[u]} M} M$ is an equivalence of spectra. Note that this map is just the multiplication map, denoted by $m_{M,u}$. Now we look at Proposition A.0.19: the (u)-completion of the $W^+(k)[u]$-module M is the cofiber of the canonical map $T(M)\to M$, where $T(M)$ is the limit of the tower

$$\cdots \xrightarrow{m_{M,u}} M \xrightarrow{m_{M,u}} M \xrightarrow{m_{M,u}} M$$

Since all maps in the tower are equivalences of spectra, we deduce that the canonical map $T(M)\to M$ is an equivalence of spectra, which implies that the (u)-completion of the $W^+(k)[u]$-module M is contractible. In particular, the $W^+(k)[u]$-module M is assumed to be (u)-complete, therefore the spectrum M is contractible.

If, on the other hand, $W^+(k)\otimes_{W^+(k)[[u]]} M$ is contractible, then to adopt the proof above, it suffices to establish the cofiber sequence

$$W^+(k)[[u]] \xrightarrow{m_u} W^+(k)[[u]] \to W^+(k) \tag{1.6.3}$$

We apply the (u)-complete functor to the cofiber sequence (1.6.2), and note that the $W^+(k)[u]$-module $W^+(k)$ is (u)-nilpotent (in fact, multiplying u is the zero map on $W^+(k)$), therefore $W^+(k)$ is (u)-complete by Corollary A.0.17, which leads to the cofiber sequence (1.6.3). The rest of the proof is same as before. $\square$

1.6.2. The Breuil–Kisin case

As before, we fix a complete DVR $(A,\mathfrak{m})$ of mixed characteristics $(0,p)$ with residue field k being perfect, absolute ramification index e, a uniformizer $\varpi\in\mathfrak{m}$ and an Eisenstein $W(k)$-polynomial $E(u)\in W(k)[u]$ which induces an isomorphism $W(k)[u]/(E(u)) \xrightarrow{\sim} A$, $u\mapsto\varpi$ as in Proposition 1.6.6. As in Remark 1.1.9 and Remark 1.1.12, $1-E(u)\in W(k)[[u]]=\pi_1(\mathrm{BGL}_1(W^+(k)[[u]]))$ gives rise to a map $f_E:\Omega^2 S^3\to \mathrm{BGL}_1(W^+(k)[[u]])$. The proof of Lemma 1.5.1 results in the following analogue:

LEMMA 1.6.13. *The zeroth homotopy group of the $\mathbb{E}_2$-Thom spectrum Mf_E associated to the map f_E is isomorphic to the $W(k)$-algebra $W(k)[[u]]/(E(u))\cong W(k)[u]/(E(u))\cong A$.*

The $W(k)[u]$-module structure on A gives rise to a $W^+(k)[u]$-module structure on HA. Since A is $\mathfrak{m}=(\varpi)$-adically complete, the $W(k)[u]$-module structure on A also gives rise to a $W(k)[[u]]$-module structure on A and consequently a $W^+(k)[[u]]$-module structure on HA. We readily check that these structures are compatible, in the sense that the $W^+(k)[u]$-module structure on HA coincides with the image of the $W^+(k)[[u]]$-module HA under the forgetful functor $\mathrm{Mod}_{W^+(k)[[u]]}\to \mathrm{Mod}_{W^+(k)[u]}$. Matthew Morrow proposed the following analogue of the Hopkins–Mahowald theorem:

THEOREM 1.6.14. *The truncation map $t_E: Mf_E\to H\pi_0(Mf_E)\cong HA$ of $\mathbb{E}_2$-$W^+(k)[[u]]$-algebras is an equivalence of spectrum. Thus the Eilenberg–MacLane spectrum HA is the $\mathbb{E}_2$-Thom spectrum Mf_E associated to the map $f_E:\Omega^2 S^3\to \mathrm{BGL}_1(W^+(k)[[u]])$.*

Corollary 1.6.15. (see [KN19, Remark 3.4]) *The $\mathbb{E}_2$-HA-algebra $HA\otimes_{W^+(k)[[u]]}HA$ is a free $\mathbb{E}_2$-HA-algebra on a single generator in degree 1.*

Proof. The strategy is already covered in the proof of Lemma 1.3.2 and Lemma 1.5.12. Since this pattern will appear again soon, we find it beneficial to present again. Let's recall that the $\mathbb{E}_2$-Thom spectrum Mf_E is the colimit of the composite functor

$$\Omega^2 S^3 \xrightarrow{f_E} \mathrm{BGL}_1(W^+(k)[[u]]) \to \mathrm{Mod}_{W^+(k)[[u]]}$$

which by abuse of notation will be still denoted by f_E.

Since the base change functor $HA\otimes_{W^+(k)[[u]]} - : \mathrm{Mod}_{W^+(k)[[u]]}\to \mathrm{Mod}_{HA}$ is a left adjoint, it commutes with colimits, we deduce that $HA\otimes_{W^+(k)[[u]]} Mf_E \simeq M\,(f_E\otimes_{W^+(k)[[u]]}HA)$, where $f_E\otimes_{W^+(k)[[u]]}HA$ is the map $\Omega^2 S^3\to \mathrm{BGL}_1(HA)$.

As in the proof of Lemma 1.3.2, we can identify map as follows: we pick the image of $1-E(u)\in \mathrm{GL}_1(W(k)[[u]])$ under the map $\mathrm{GL}_1(W(k)[[u]])\to \mathrm{GL}_1(A)$, that is, the element $1\in \mathrm{GL}_1(A)\cong \pi_1(\mathrm{BGL}_1(HA))$, which gives rise to the constant map $S^1\to \mathrm{BGL}_1(HA)$ and consequently the constant map $f_A:\Omega^2 S^3\to \mathrm{BGL}_1(HA)$, as in Remark 1.1.9 and Remark 1.1.12.

In conclusion, the map $f_E\otimes_{W^+(k)}HA:\Omega^2 S^3\to \mathrm{BGL}_1(HA)$ coincides with the constant map f_A, and the $\mathbb{E}_2$-Thom spectrum Mf_A is thus the colimit of a constant map, which evaluates to $HA\otimes\Omega^2 S^3$, the free $\mathbb{E}_2$-HA-algebra on a single generator in degree 1. $\square$

Recall that $E(u)\in W(k)[u]$ is an Eisenstein $W(k)$-polynomial. Let a_0 denote the constant term of $E(u)$. By assumption, $p\mid a_0$ but $p^2\nmid a_0$. Let $a_0=p\,b_0$ where $b_0\in W(k)$. Since p is not a zero-divisor in $W(k)$, we have $p\nmid b_0$, which implies that the image of b_0 in $W(k)/p\cong k$ is invertible since k is a field. Now since $W(k)$ is p-adically complete, we have $b_0\in \mathrm{GL}_1(W(k))$.

The strategy to prove Theorem 1.6.14 is similar to the approach to attack Theorem 1.1.13. We first show that the base change of the truncation map t_E along the map $W^+(k)[[u]]\to W^+(k)$ coincides with the truncation map t_{k,a_0}, then it follows from Lemma 1.5.3 that the base changed map $W^+(k)\otimes_{W^+(k)[[u]]}t_E\simeq t_{k,a_0}$ is an equivalence of spectra, and by completeness, we deduce that the map t_E is also an equivalence of spectra by Lemma 1.6.12.

Lemma 1.6.16. *There is a canonical equivalence $Mf_{k,a_0}\xrightarrow{\simeq} W^+(k)\otimes_{W^+(k)[[u]]}Mf_E$ of $W^+(k)$-modules.*

Proof. We will duplicate the proof of Lemma 1.5.12. The image of the multiplication map $m_{1-E(u)}$: $W^+(k)[[u]]\to W^+(k)[[u]]$ under the base change functor $W^+(k)\otimes_{W^+(k)[[u]]} - : \mathrm{Mod}_{W^+(k)[[u]]}\to \mathrm{Mod}_{W^+(k)}$ is the multiplication map $m_{1-a_0}: W^+(k)\to W^+(k)$. Note also that the base change functor is symmetric monoidal. Now we conclude that the map f_{k,a_0} coincides with the composite map

$$\Omega^2 S^3 \xrightarrow{f_E} \mathrm{BGL}_1(W^+(k)[[u]]) \xrightarrow{W^+(k)\otimes_{W^+(k)[[u]]}-} \mathrm{BGL}_1(W^+(k))$$

Thus by commuting the colimit and the base-change, we obtain

$$\begin{aligned} Mf_{k,a_0} &= \mathrm{colim}(W^+(k)\otimes_{W^+(k)[[u]]}f_E)\\ &\xrightarrow{\simeq} W^+(k)\otimes_{W^+(k)[[u]]}\mathrm{colim}\, f_E\\ &= W^+(k)\otimes_{W^+(k)[[u]]}Mf_E\end{aligned}$$

where by abuse of notation, the colimit of the maps f_{k,a_0} (or f_E respectively) are understood as the colimit of the maps f_{k,a_0} (or f_E respectively) composed with the functor $\mathrm{BGL}_1(W^+(k))\to \mathrm{Mod}_{W^+(k)}$ (or $\mathrm{BGL}_1(W^+(k)[[u]])\to \mathrm{Mod}_{W^+(k)[[u]]}$ respectively) as in the definition of Thom spectra. $\square$

Lemma 1.6.17. *There is a canonical equivalence $W^+(k)\otimes_{W^+(k)[[u]]}HA\xrightarrow{\simeq} Hk$ of $W^+(k)$-modules.*

Proof. As in the proof of Lemma 1.6.12, we identify $W^+(k)$ with the cofiber of the multiplication map $m_u: W^+(k)[[u]]\to W^+(k)[[u]]$ which gives us an equivalence

$$W^+(k)\otimes_{W^+(k)[[u]]}HA\simeq \mathrm{cofib}\big(HA\xrightarrow{m_{HA,u}} HA\big)$$

Now by the definition of the $W^+(k)[[u]]$-module structure on HA and that u is not a zero-divisor in A, we have the equivalence $\mathrm{cofib}\big(HA \xrightarrow{m_{HA,u}} HA\big) \simeq H\big(\mathrm{coker}\big(A \xrightarrow{\varpi} A\big)\big) \simeq Hk$. Thus we obtain an equivalence $W^+(k) \otimes_{W^+(k)[[u]]} HA \simeq Hk$. We can readily check that this equivalence could be described as follows: consider the commutative diagram in the ∞-category of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} W^+(k)[[u]] & \longrightarrow & W^+(k) \\ \downarrow & & \downarrow \\ HA & \longrightarrow & Hk \end{array}$$

where the left vertical map is the composite map $W^+(k)[[u]] \to H(\pi_0(W^+(k)[[u]])) \simeq H(W(k)[[u]]) \xrightarrow{u \mapsto \varpi} HA$ (where the first map is the Postnikov section). The commutative diagram induces a map $W^+(k) \otimes_{W^+(k)[[u]]} HA \to Hk$ (note that the left hand side is a pushout of $\mathbb{E}_\infty$-rings), which coincides with the equivalence obtained above. $\square$

LEMMA 1.6.18. *The equivalences in Lemma 1.6.16 and Lemma 1.6.17 assembles into a commutative diagram:*

$$\begin{array}{ccc} Mf_{k,a_0} & \xrightarrow{t_{k,a_0}} & Hk \\ \downarrow \simeq & & \uparrow \simeq \\ W^+(k) \otimes_{W^+(k)[[u]]} Mf_E & \longrightarrow & W^+(k) \otimes_{W^+(k)[[u]]} HA \end{array}$$

where the top horizontal map is the 0th Postnikov section t_{k,a_0} defined in Proposition 1.5.2 and the bottom horizontal map is the base-changed 0th Postnikov section $W^+(k) \otimes_{W^+(k)[[u]]} t_E$.

Proof. As in the proof of Lemma 1.5.15, it suffices to show that the composite map on the 0th homotopy group $\pi_0(Mf_{k,a_0}) \to \pi_0(W^+(k) \otimes_{W^+(k)[[u]]} Mf_E) \to \pi_0(W^+(k) \otimes_{W^+(k)[[u]]} HA) \to \pi_0(Hk) \cong k$ is an isomorphism, which follows from an explicit element chasing. $\square$

Combined with Lemma 1.5.3, we obtain that

COROLLARY 1.6.19. *The base-changed map $W^+(k) \otimes_{W^+(k)[[u]]} t_E : W^+(k) \otimes_{W^+(k)[[u]]} Mf_E \to W^+(k) \otimes_{W^+(k)[[u]]} HA$ is an equivalence of $W^+(k)$-modules.*

Apply Lemma 1.6.12 to the cofiber $\mathrm{cofib}(t_E)$, we deduce that

COROLLARY 1.6.20. *The map $t_E : Mf_E \to HA$ is an equivalence of spectra after (u)-completion.*

As in Lemma 1.5.9, we deduce from Theorem A.0.25 that

LEMMA 1.6.21. *The $W^+(k)[[u]]$-module HA is (u)-complete.*

Now, given the nontrivial topological input Proposition 1.5.6, as in Lemma 1.5.10 and Corollary 1.5.11, we deduce that

LEMMA 1.6.22. *The $W^+(k)[[u]]$-module Mf_E is (u)-complete.*

COROLLARY 1.6.23. *The cofiber $\mathrm{cofib}(t_E)$ is a (u)-complete $W^+(k)[[u]]$-module, and thus the map t_E is an equivalence of spectra by Corollary 1.6.20.*

This completes the proof of Theorem 1.6.14.

1.6.3. Complete regular local rings

Inspired by [KN19, Section 9], we will provide a Hopkins–Mahowald theorem for complete regular local rings of mixed characteristic. We will show how to modify our proof of Theorem 1.6.14 to deduce this. Note that this is also a special case of Question 1, by [BS19, Remark 3.11].

We need some preparations in higher algebra:

Let $W^+(k)[u_1,\dots,u_n]$ be the "n-variate polynomial $W^+(k)$-algebra", that is, the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)\otimes_{\mathbb{S}}\mathbb{S}[\mathbb{N}^n]$. Since the space $\mathbb{N}^n$ is endowed with discrete topology, parallel to Proposition 1.6.8, we have

Proposition 1.6.24. *As a $W^+(k)$-module, $W^+(k)[u_1,\dots,u_n]$ is equivalent to the direct sum $\bigoplus_{\alpha\in\mathbb{N}^n} u^\alpha W^+(k)$, a free $W^+(k)$-module. The graded homotopy group $\pi_*(W^+(k)[u_1,\dots,u_n])$, as a (graded-commutative) $\pi_*(W^+(k))$-algebra, is equivalent to $\pi_*(W^+(k))[u_1,\dots,u_n]$, where $\deg u_1=\dots=\deg u_n=0$.*

Now let $W^+(k)[[u_1,\dots,u_n]]$ be the $(u_1,\dots,u_n)$-completion of the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[u_1,\dots,u_n]$. By induction on $n\in\mathbb{N}_{>0}$ and argue as in Proposition 1.6.11, we obtain:

Proposition 1.6.25. *The $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[[u_1,\dots,u_n]]$ is connective. The zeroth homotopy group of $\pi_0(W^+(k)[[u_1,\dots,u_n]])$ is isomorphic to the $(u_1,\dots,u_n)$-adic completion of the polynomial $W(k)$-algebra $W(k)[u_1,\dots,u_n]$, that is, the formal power series $W(k)$-algebra $W(k)[[u_1,\dots,u_n]]$, as $W(k)$-algebras.*

Similarly, argue inductively on $n\in\mathbb{N}_{>0}$ as in Lemma 1.6.12, we obtain:

Lemma 1.6.26. *Let M be a $W^+(k)[u_1,\dots,u_n]$-(or $W^+(k)[[u_1,\dots,u_n]]$-)module (spectrum). If the spectrum $W^+(k)\otimes_{W^+(k)[u_1,\dots,u_n]} M$ (or $W^+(k)\otimes_{W^+(k)[[u_1,\dots,u_n]]} M$ respectively) is contractible, then so is the $(u_1,\dots,u_n)$-completion of the spectrum M. In particular, if furthermore $W^+(k)[u_1,\dots,u_n]$-(or $W^+(k)[[u_1,\dots,u_n]]$-)module M is assumed to be $(u_1,\dots,u_n)$-complete, then the spectrum M is contractible.*

We note that in these inductive arguments, we heavily depend on Proposition A.0.23.

Now we are ready to formulate the Hopkins–Mahowald theorem for complete regular local rings. We fix a positive integer $n\in\mathbb{N}_{>0}$, a perfectoid ring R. As in Section 1.5, let $\theta: W(R^\flat)\to R$ be Fontaine's pro-infinitesimal thickening. Let $\phi\in W(R^\flat)[[u_1,\dots,u_n]]$ be formal power series such that $\phi(0,\dots,0)\in W(R^\flat)$ is a generator of $\ker\theta$. We recall that $\ker\theta$ is principal by definition. We note that the element $1-\phi(u_1,\dots,u_n)\in W(R^\flat)[[u_1,\dots,u_n]]$ is invertible, since $1-\phi(0,\dots,0)\in W(R^\flat)$ is invertible as the ring $W(R^\flat)$ is $\ker\theta$-adically complete. As in Remark 1.1.9 and Remark 1.1.12, the element $1-\phi(u_1,\dots,u_n)\in \mathrm{GL}_1(W(R^\flat)[[u_1,\dots,u_n]])$ gives rise to an $\mathbb{E}_2$-map $f:\Omega^2 S^3\to \mathrm{BGL}_1(W(R^\flat)[[u_1,\dots,u_n]])$. The proof of Lemma 1.5.1 results in the following analogue:

Lemma 1.6.27. *The zeroth homotopy group of the $\mathbb{E}_2$-Thom spectrum Mf associated to the map f is isomorphic to the $W(R^\flat)$-algebra $W(R^\flat)[[u_1,\dots,u_n]]/(\phi(u_1,\dots,u_n))$.*

We now phrase the following variant of the Hopkins–Mahowald theorem:

Theorem 1.6.28. *The truncation map $t: Mf\to H\pi_0(Mf)\cong HW(R^\flat)[[u_1,\dots,u_n]]/(\phi(u_1,\dots,u_n))$ of $\mathbb{E}_2$-$W^+(R^\flat)[[u_1,\dots,u_n]]$-algebras is an equivalence of spectrum. Thus the Eilenberg–MacLane spectrum $HW(R^\flat)[[u_1,\dots,u_n]]/(\phi(u_1,\dots,u_n))$ is the $\mathbb{E}_2$-Thom spectrum Mf associated to the map $f:\Omega^2 S^3\to \mathrm{BGL}_1(W^+(R^\flat)[[u_1,\dots,u_n]])$.*

The proof is parallel to that of Theorem 1.6.14, which we will omit. Now let $(A,\mathfrak{m})$ be a complete regular local ring with residue field $k=A/\mathfrak{m}$ being perfect of characteristic p. We also assume that $p\neq 0$ in A. Let $(a_1,\dots,a_n)\subseteq\mathfrak{m}$ be a regular sequence which generates the maximal ideal $\mathfrak{m}$. We need the following lemma:

Lemma 1.6.29. ([KN19, Lemma 9.2]) *There exists a map $W(k)[[u_1,\dots,u_n]]\to A$ of rings given by $u_i\mapsto a_i$ for $i=1,\dots,n$, which is surjective with kernel being principal, generated by a formal power series $\phi\in W(k)[[u_1,\dots,u_n]]$ with $\phi(0,\dots,0)=p$.*

Proof. First, the isomorphism $k \to A/\mathfrak{m}$ lifts to a map $W(k) \to A$ since A is $\mathfrak{m}$-adically complete, see Example 1.2.5 or [Ser79, Section II.5, Proposition 10]. The map $W(k)[[u_1, \ldots, u_n]] \to A$ is then well-defined since A is $\mathfrak{m}$-adic complete. Let C, K be the cokernel and the kernel of the map $W(k)[[u_1,\ldots,u_n]] \to A$ of $W(k)[[u_1,\ldots,u_n]]$-modules. By right-exactness of classical tensor products, we have

$$\mathrm{Tor}_0^{W(k)[[u_1,\ldots,u_n]]}(C, W(k)) \cong \mathrm{coker}(W(k) \to k) \cong 0$$

Now by inspecting the exact sequence $0 \to \mathfrak{m} \to A \to k \to 0$, we deduce that A is a finitely generated $W(k)[[u_1, \ldots, u_n]]$-module, therefore so is C. We deduce from Nakayama's lemma that $C \cong 0$, therefore the map $W(k)[[u_1, \ldots, u_n]]$ is surjective. Now we obtain a short exact sequence of $W(k)[[u_1, \ldots, u_n]]$-modules

$$0 \to K \to W(k)[[u_1, \ldots, u_n]] \to A \to 0$$

which gives rise to an exact sequence of $W(k)$-modules

$$\mathrm{Tor}_1^{W(k)[[u_1,\ldots,u_n]]}(A, W(k)) \to \mathrm{Tor}_0^{W(k)[[u_1,\ldots,u_n]]}(K, W(k)) \to W(k) \to k \to 0$$

Since $(a_1, \ldots, a_n)$ is a regular sequence, it is also Koszul regular [Sta21, Tag 062F], hence $\mathrm{Tor}_1^{W(k)[[u_1,\ldots,u_n]]}(A, W(k)) \cong 0$. Thus

$$\mathrm{Tor}_0^{W(k)[[u_1,\ldots,u_n]]}(K, W(k)) \cong \ker(W(k) \to k) \cong p\,W(k)$$

We pick a lift $\phi \in K$ of $p \in p\,W(k)$. By Nakayama's lemma, the $W(k)[[u_1, \ldots, u_n]]$ module (and hence the ideal) K is generated by the element $\phi \in K$. Furthermore, by multiplying an invertible element in $W(k)$, we can assume that the lift ϕ is so chosen that $\phi(0, \ldots, 0) = p$. □

Remark 1.6.30. Our proof of Lemma 1.6.29 leads to a more general result: Let A be a commutative ring with an ideal $I \subseteq A$ generated by a (Koszul) regular sequence $(a_1, \ldots, a_n) \subseteq I$. If A is both p-adically complete and I-adically complete, and $R := A/I$ is a perfectoid ring, then by Proposition 1.4.16, there exists a unique map $W(R^\flat) \to A$ such that the composite map $W(R^\flat) \to A \to R$ coincides with Fontaine's map, which allows us to view A as a $W(R^\flat)$-algebra. Now we consider the map $\varphi: W(R^\flat)[[u_1, \ldots, u_n]] \to A$ of $W(R^\flat)$-algebras given by $u_i \mapsto a_i$ for $i = 1, \ldots, n$. Our proof of Lemma 1.6.29 implies that the map φ is surjective with kernel being principal, generated by a formal power series $\phi \in W(R^\flat)[[u_1,\ldots,u_n]]$ such that $\phi(0,\ldots,0)$ generates the kernel $\ker(\theta)$ of Fontaine's map $\theta: W(R^\flat) \to R$.

COROLLARY 1.6.31. *Let $\phi \in W(k)[[u_1, \ldots, u_n]]$ be a power series as described in Lemma 1.6.29. Let $f: \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k)[[u_1, \ldots, u_n]])$ be the map given by the element $1 - \phi(u_1, \ldots, u_n) \in \mathrm{GL}_1(W(k)[[u_1, \ldots, u_n]])$. Then the $\mathbb{E}_2$-Thom spectrum Mf associated to the map f is as an $\mathbb{E}_2$-$W^+(k)[[u_1, \ldots, u_n]]$-algebra equivalent to the Eilenberg–MacLane spectrum HA of the complete regular local ring A (of mixed characteristic).*

Proof. It follows from Theorem 1.6.28 by taking $R = k$ and Lemma 1.6.29. □

1.7. CHARACTERIZING THOM SPECTRA AS QUOTIENTS OF FREE $\mathbb{E}_2$-ALGEBRAS

In this section, we will discuss an alternative characterization of Thom spectra which we learn from [AB19]. This characterization will enable us to peel off some redundant restraints in the definition of Thom spectra. We will rephrase Question 1 more broadly, and give a toy example related to the Breuil–Kisin case. We note that in fact, we have already used this characterization in Lemma 1.3.3.

We first present a theorem which we learn from Antolín-Camarena and Barthel's paper:

Remark 1.7.1. Let R be an $\mathbb{E}_\infty$-ring. Let $R[\Omega^2 S^2]$ be the free $\mathbb{E}_2$-R-algebra on a single generator in degree 0. Then for all $\mathbb{E}_2$-R-algebra S and elements $x \in \pi_0(S)$, the universal property of free $\mathbb{E}_2$-R-algebras gives rise to a map $R[\Omega^2 S^2] \to S$ which maps the generator (in fact, a connected component) to x. We will call this map the evaluation map of $R[\Omega^2 S^2]$ at x.

Theorem 1.7.2. ([AB19, Theorem 4.10]) *Let R be an $\mathbb{E}_\infty$-ring and $\alpha \in \pi_1(\mathrm{BGL}_1(R)) \cong \mathrm{GL}_1(\pi_0 R)$. Let $q: S^1 \to \mathrm{BGL}_1(R)$ a loop representing $\alpha \in \pi_1(\mathrm{BGL}_1(R))$. Let $f: \Omega^2 S^3 \to \mathrm{BGL}_1(R)$ be the double loop map associated to q (see Remark 1.1.2). Then the $\mathbb{E}_2$-Thom spectrum Mf associated to the $\mathbb{E}_2$-map f fits into a pushout diagram of $\mathbb{E}_2$-R-algebras:*

$$\begin{array}{ccc} R[\Omega^2 S^2] & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ R & \longrightarrow & Mf \end{array}$$

where $R[\Omega^2 S^2] \cong R \otimes_{\mathbb{S}} \Sigma^\infty_+ S^2$ is the free $\mathbb{E}_2$-R-algebra on a single generator in degree 0, and two maps $R[\Omega^2 S^2] \to R$ are evaluation maps of $R[\Omega^2 S^3]$ at $0 \in \pi_0 R$ and $1-\alpha \in \pi_0 R$ respectively.

Remark 1.7.3. Theorem 1.7.2 shows that the Thom spectrum description is equivalent to the pushout-diagram description. However, we note that the pushout-diagram description is more general in the sense that even if $\alpha \in \pi_0 R$ is not invertible, the pushout-diagram description is still valid while we can no longer, at least superficially, give a Thom spectrum description. We find it easier to write down proofs for Thom spectrum description so we adapted the Thom spectrum description for perfectoid rings.

We can now rephrase Question 1 as follows:

Question 2. Given an orientable prism $(A, I=(d))$. When can we find an $\mathbb{E}_\infty$-ring spectrum A^+ (which satisfies some hypotheses related to A. A naive guess would be that $\pi_0(A^+) = A$) so that the Eilenberg–MacLane spectrum $H(A/I)$ as an $\mathbb{E}_2$-A^+-algebra fits into a pushout diagram

$$\begin{array}{ccc} A^+[\Omega^2 S^2] & \longrightarrow & A^+ \\ \big\downarrow & & \big\downarrow \\ A^+ & \longrightarrow & H(A/I) \end{array}$$

such that two maps $A^+[\Omega^2 S^2] \to A^+$ are evaluation maps of the free $\mathbb{E}_2$-A^+-algebra $A^+[\Omega^2 S^2]$ at $0 \in \pi_0(A^+)$ and $d \in \pi_0(A^+)$ respectively.

Remark 1.7.4. Theorem 1.7.2 shows that Theorem 1.1.13 answers this question affirmatively when (A, I) is a perfect prism $(W(R^\flat), \ker\theta)$, with $A^+ := W^+(R^\flat)$.

Remark 1.7.5. Similarly, Theorem 1.6.14 answers this question affirmatively when (A, I) is a prism $(W(k)[[u]], (E(u)))$ associated to Breuil–Kisin cohomology where k is a perfect $\mathbb{F}_p$-algebra and $E(u) \in W(k)[u]$ is an Eisenstein polynomial.

We now announce a toy example of a variant of Theorem 1.6.14. As there, we fix a complete DVR $(A, \mathfrak{m})$ of mixed characteristics $(0, p)$ with residue field k being perfect, absolute ramification index e, a uniformizer $\varpi \in \mathfrak{m}$ and an Eisenstein $W(k)$-polynomial $E(u) \in W(k)[u]$ which induces an isomorphism $W(k)[u]/(E(u)) \xrightarrow{\sim} A, u \mapsto \varpi$ as in Proposition 1.6.6.

Theorem 1.7.6. *The (u)-completion of the total cofiber of the commutative diagram of $\mathbb{E}_2$-$W^+(k)[u]$-algebras*

$$\begin{array}{ccc} W^+(k)[u] \otimes_{\mathbb{S}} \mathbb{S}[\Omega^2 S^2] & \longrightarrow & W^+(k)[u] \\ \big\downarrow & & \big\downarrow \\ W^+(k)[u] & \longrightarrow & HA \end{array}$$

is contractible, where two maps $W^+(k)[u] \otimes_{\mathbb{S}} \mathbb{S}[\Omega^2 S^3] \to W^+(k)[u]$ are given by evaluation maps at $0 \in \pi_0(W^+(k)[u])$ and $E(u) \in \pi_0(W^+(k)[u])$ respectively. Equivalently put, the commutative diagram above induces an equivalence of $W^+(k)[u]$-modules from the $\mathbb{E}_2$-pushout of the diagram $W^+(k)[u] \leftarrow W^+(k)[u] \otimes_{\mathbb{S}} \mathbb{S}[\Omega^2 S^2] \to W^+(k)[u]$ to the Eilenberg–MacLane spectrum HA after (u)-completion.

Corollary 1.7.7. ([KN19, Remark 3.4]) *The $\mathbb{E}_2$-HA-algebra $HA \otimes_{W^+(k)[u]} HA$ is the (p)-completion of the free $\mathbb{E}_2$-HA-algebra on a single generator in degree 1.*

Proof. Note that $E(u)$ vanishes after tensoring HA, and that (u)-completion coincides with (p)-completion for HA since ϖ^e/p is an invertible element, the result follows. □

We sketch a proof of Theorem 1.7.6, which is totally parallel to that of Theorem 1.6.14.

A sketch of a proof of Theorem 1.7.6. Let X be the pushout of the diagram $W^+(k)[u] \leftarrow W^+(k)[u] \otimes_{\mathbb{S}} \mathbb{S}[\Omega^2 S^2] \rightarrow W^+(k)[u]$ in question. We first check that the induced map $X \rightarrow HA$ is the 0th Postnikov section. Then we perform a base change $W^+(k) \otimes_{W^+(k)[u]} -$. We show that after such a base change, the induced map $X \rightarrow HA$ becomes an equivalence, given by Theorem 1.7.2 and Lemma 1.5.3. We then conclude the result by Corollary 1.6.20. □

Appendix A

Recollection of Higher Algebra

This appendix is devoted to a recollection of basic facts in Higher Algebra needed in the main text. Our main reference is [Lur17], [Lur18b] and [Lur18a].

A.0.1. Finiteness properties of rings and modules

We will include some definitions and properties from [Lur17, Section 7.2.4].

Definition A.0.1. ([Lur17, Notation 7.1.1.10, Proposition 7.1.1.13]) *Given a connective $\mathbb{E}_1$-ring R, there is a canonical accessible t-structure on LMod_R determined by subcategories $(\mathrm{LMod}_R)_{\geq 0}$ and $(\mathrm{LMod}_R)_{\leq 0}$, where $(\mathrm{LMod}_R)_{\geq 0}$ is the full subcategory of LMod_R spanned by those left R-modules M for which $\pi_n M \cong 0$ for $n < 0$, and $(\mathrm{LMod}_R)_{\leq 0}$ is the full subcategory of LMod_R spanned by those left R-modules M for which $\pi_n M \cong 0$ for $n > 0$.*

Proposition A.0.2. ([Lur17, Proposition 7.1.1.13]) *Let R be a connective $\mathbb{E}_1$-ring, then the subcategories $(\mathrm{LMod}_R)_{\geq 0}, (\mathrm{LMod}_R)_{\leq 0} \subseteq \mathrm{LMod}_R$ are stable under small products and small filtered colimits.*

Definition A.0.3. ([Lur17, Proposition 7.2.2.10]) *Let M be a left module over an $\mathbb{E}_1$-ring R. We will say that M is* flat *if the following conditions are satisfied:*

1. *The homotopy group $\pi_0 M$ is flat as a left module over $\pi_0 R$ in the usual sense.*
2. *For each $n \in \mathbb{Z}$, the natural map $\mathrm{Tor}_0^{\pi_0 R}(\pi_n R, \pi_0 M) \to \pi_n M$ is an isomorphism of abelian groups.*

Definition A.0.4. ([Lur17, Definition 7.2.4.1]) *Let R be an $\mathbb{E}_1$-ring. We let $\mathrm{LMod}_R^{\mathrm{perf}}$ denote the smallest stable subcategory of LMod_R which contains R (regarded as a left module over itself) and is closed under retracts. We will say that a left R-module M is perfect if it belongs to $\mathrm{LMod}_R^{\mathrm{perf}}$.*

Definition A.0.5. ([Lur17, Definition 7.2.4.8]) *Let $\mathcal{C}$ be a compactly generated ∞-category. We will say that an object $C \in \mathcal{C}$ is* almost compact *if $\tau_{\leq n} C$ is a compact object of $\tau_{\leq n}\mathcal{C}$ for all $n \geq 0$.*

Definition A.0.6. ([Lur17, Definition 7.2.4.10]) *Let R be a connective $\mathbb{E}_1$-ring. We will say that a left R-module M is* almost perfect *if there exists an integer k such that $M \in (\mathrm{LMod}_R)_{\geq k}$ and is almost compact as an object of $(\mathrm{LMod}_R)_{\geq k}$. We let $\mathrm{LMod}_R^{\mathrm{aperf}}$ denote the full subcategory of LMod_R spanned by the almost perfect left R-modules.*

Proposition A.0.7. ([Lur17, Proposition 7.2.4.11]) *Let R be a connective $\mathbb{E}_1$-ring. Then:*

1. *The full subcategory $\mathrm{LMod}_R^{\mathrm{aperf}} \subseteq \mathrm{LMod}_R$ is closed under translation and finite colimits, and is therefore a stable subcategory of LMod_R;*
2. *The full subcategory $\mathrm{LMod}_R^{\mathrm{aperf}} \subseteq \mathrm{LMod}_R$ is closed under retracts;*
3. *Every perfect left R-module is almost perfect;*
4. *The full subcategory $(\mathrm{LMod}_R^{\mathrm{aperf}})_{\geq 0} \subseteq \mathrm{LMod}_R$ is closed under geometric realizations of simplicial objects;*
5. *Let M be a left R-module which is connective and almost perfect. Then M can be obtained as the geometric realization of a simplicial left R-module $P_\bullet$ such that each P_n is a free R-module of finite rank.*

PROPOSITION A.0.8. *Let $f: A \to A'$ be a map of connective $\mathbb{E}_1$-rings. Let M be a connective left A-module and set $M' = A' \otimes_A M$. If M is an almost perfect left A-module, then M' is an almost perfect left A'-module.*

Proof. Since M is connective and almost perfect, by Proposition A.0.7, there exists a simplicial object $P_\bullet$ in LMod_A such that each P_n is a free A-module of finite rank and M is equivalent to the geometric realization of $P_\bullet$. Therefore M' is equivalent to the geometric realization of $A' \otimes_A P_\bullet$, by the fact the tensor products commute with small colimits. On the other hand, each $A' \otimes_A P_n$ is a free A'-module of finite rank, hence perfect, thus almost perfect. Now M' is equivalent to the geometric realization of almost perfect modules, therefore M' is almost perfect by Proposition A.0.7. □

DEFINITION A.0.9. ([LUR17, DEFINITION 7.2.4.13]) *A discrete associative ring R is* left coherent *if every finitely generated left ideal of R is finitely presented as a left R-module.*

DEFINITION A.0.10. ([LUR17, DEFINITION 7.2.4.16]) *Let R be an $\mathbb{E}_1$-ring. We will say that R is* left coherent *if the following conditions are satisfied:*

1. *The $\mathbb{E}_1$-ring R is connective;*
2. *The discrete associative ring $\pi_0 R$ is left coherent;*
3. *For each $n \geq 0$, the homotopy group $\pi_n R$ is finitely presented as a left module over $\pi_0 R$.*

PROPOSITION A.0.11. ([LUR17, PROPOSITION 7.2.4.17]) *Let R be an $\mathbb{E}_1$-ring and M a left R-module. Suppose that R is left coherent. Then M is almost perfect if and only if the following conditions are satisfied:*

i. *For $m \ll 0$, $\pi_m M = 0$;*

ii. *For every integer m, $\pi_m M$ is finitely presented as a left $\pi_0 R$-module.*

COROLLARY A.0.12. *Let R be a left coherent $\mathbb{E}_1$-ring, then $H\,\pi_0(R)$ as a left R-module is almost perfect.*

A.0.2. Nilpotent, local and complete modules

We will include several definitions and propositions from [Lur18b], Chapter 7.

DEFINITION A.0.13. ([LUR18B, DEFINITION 7.1.1.1, EXAMPLE 7.1.1.2]) *Let R be a connective $\mathbb{E}_\infty$-ring and let $x \in \pi_0 R$. An R-module M is* x-nilpotent *if the localization $M[1/x]$ vanishes. Equivalently, M is x-nilpotent if and only if the action of x on $\pi_* M$ is locally nilpotent, that is, if and only if for each $y \in \pi_j M$, there exists an integer $n \gg 0$ such that $x^n y = 0$ in $\pi_j M$ for all $j \in \mathbb{Z}$.*

DEFINITION A.0.14. ([LUR18B, DEFINITION 7.1.1.6]) *Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. We say that an R-module M is* I-nilpotent *if it is x-nilpotent for each $x \in I$.*

DEFINITION A.0.15. ([LUR18B, DEFINITION 7.2.4.1]) *Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. We say that an R-module M is* I-local *if for every I-nilpotent R-module N, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible.*

DEFINITION A.0.16. ([LUR18B, DEFINITION 7.3.1.1]) *Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. We will say that an R-module M is* I-complete *if for every I-local R-module N, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible.*

COROLLARY A.0.17. *Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. If M is an I-nilpotent R-module, then it is also an I-complete R-module.*

PROPOSITION A.0.18. ([LUR18B, PROPOSITION 7.3.1.4 AND NOTATION 7.3.1.5]) *Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then every left R-module M fits into an (essentially unique) fiber sequence $M' \to M \to M''$, where M' is I-local and M'' is I-complete. Moreover, there is a functor, called the I-completion functor, $\mathrm{Mod}_R \to \mathrm{Mod}_R$, which maps M to M''. We denote by $M_I^\wedge$ the image of M under the I-completion functor.*

We can compute the I-completion functor when I is principal:

Proposition A.0.19. ([Lur18b, Proposition 7.3.2.1]) *Let R be a connective $\mathbb{E}_\infty$-ring and let $x\in\pi_0 R$ be an element. For any R-module $M\in\mathrm{Mod}_R$, let $T(M)$ denote the limit of the tower*

$$\cdots\xrightarrow{x}M\xrightarrow{x}M\xrightarrow{x}M\xrightarrow{x}M$$

Then $T(M)$ is (x)-local and the (x)-completion of M can be identified with the cofiber of the canonical map $\theta:T(M)\to M$.

Corollary A.0.20. ([Lur18b, Corollary 7.3.2.2]) *Let R be a connective $\mathbb{E}_\infty$-ring and let $x\in\pi_0 R$ be an element. The following conditions on an R-module $M\in\mathrm{Mod}_R$ are equivalent:*

1. *The module M is (x)-complete.*
2. *The limit of the tower*
$$\cdots\xrightarrow{x}M\xrightarrow{x}M\xrightarrow{x}M\xrightarrow{x}M$$
vanishes.

Corollary A.0.21. ([Lur18b, Corollary 7.3.2.3]) *Let R be a connective $\mathbb{E}_\infty$-ring, $I\subseteq\pi_0 R$ an ideal and $x\in\pi_0 R$ an element. Then the (x)-completion functor $\mathrm{Mod}_R\to\mathrm{Mod}_R, M\mapsto M^\wedge_{(x)}$ carries I-complete modules to I-complete modules.*

Corollary A.0.22. ([Lur18b, Corollary 7.3.2.4]) *Let R be a connective $\mathbb{E}_\infty$-ring, $x\in\pi_0 R$ and let M be an R-module.*

1. *If the R-module M is connective, then the (x)-completion $M^\wedge_{(x)}$ is connective.*
2. *If $M\in(\mathrm{Mod}_R)_{\le 0}$, then $M^\wedge_{(x)}\in(\mathrm{Mod}_R)_{\le 1}$.*

Proof. Let $T(M)$ be the limit of the tower $\left(\cdots\xrightarrow{x}M\xrightarrow{x}M\xrightarrow{x}M\right)$. Then by Proposition A.0.19, we have the cofiber sequence $T(M)\to M\to M^\wedge_{(x)}$ which gives rise to a long exact sequence

$$\cdots\to\pi_n(T(M))\to\pi_n(M)\to\pi_n(M^\wedge_{(x)})\to\pi_{n-1}(T(M))\to\pi_{n-1}(M)\to\pi_{n-1}(M^\wedge_{(x)})\to\cdots$$

Furthermore, let $T_n(M)_*$ be the tower

$$\cdots\xrightarrow{x}\pi_n(M)\xrightarrow{x}\pi_n(M)\xrightarrow{x}\pi_n(M)$$

Then there is a Milnor sequence

$$0\to\lim{}^1 T_{n+1}(M)_*\to\pi_n(T(M))\to\lim T_n(M)_*\to 0$$

Especially, if M is assumed to be connective, then $T_n(M)_*$ is a tower of 0 for $n<0$, which implies that $\pi_{n-1}(T(M))$ vanishes when $n<0$. We deduce from the long exact sequence that $\pi_n(M^\wedge_{(x)})$ vanishes when $n<0$. Similarly, if $M\in(\mathrm{Mod}_R)_{\le 0}$, then $T_n(M)_*$ is a tower of 0 for $n>0$, thus $\pi_n(T(M))$ vanishes when $n\ge 0$. We deduce from the long exact sequence that $\pi_n(M^\wedge_{(x)})$ vanishes when $n>0$. $\square$

Proposition A.0.23. ([Lur18b, Corollary 7.3.3.3]) *Let R be a connective $\mathbb{E}_\infty$-ring and $I\subseteq\pi_0 R$ be a finitely generated ideal. Let M be an R-module. Then the following conditions on M are equivalent:*

1. *M is I-complete;*
2. *For each $x\in I$, M is (x)-complete;*
3. *There exists a set of generators $x_1,\ldots,x_n$ for the ideal I such that M is (x_i)-complete for $i=1,\ldots,n$.*

Remark A.0.24. ([Lur18b, Corollary 7.3.3.6]) Let $\phi:R\to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings, $I\subseteq\pi_0 R$ a finitely generated ideal and $I'=\phi(I)\,\pi_0(R')$ the ideal generated by the image of I. Then

1. An R'-module M is I'-complete if and only if it is I-complete as an R-module;

2. For every R'-module M, the canonical map $M \to M_I^\wedge$ exhibits $M_I^\wedge$ as an I-completion of M, regarded as a morphism of R-modules.

THEOREM A.0.25. ([LUR18B, THEOREM 7.3.4.1]) *Let R be an $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal and let M be an R-module. The following conditions are equivalent:*

a) The R-module M is I-complete;

b) For every integer k, the homotopy group $\pi_k M$ satisfies the condition that for each $x \in I$, we have $\mathrm{Ext}^0_A(A[1/x], \pi_k M) = 0 = \mathrm{Ext}^1_A(A[1/x], \pi_k M)$ where $A = \pi_0 R$.

PROPOSITION A.0.26. ([LUR18B, PROPOSITION 7.3.4.8]) *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $x \in \pi_0 R$ be an element whose image in $(\pi_0 R)/I$ is invertible. If M is an I-complete left R-module, then multiplication by x induces an equivalence from M to itself.*

PROPOSITION A.0.27. ([LUR18B, PROPOSITION 7.3.5.7]) *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let M be an almost perfect R-module. If R is I-complete, then so is M.*

PROPOSITION A.0.28. ([LUR18B, COROLLARY 7.3.6.3]) *Let R be a discrete commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let M be a discrete R-module. The following conditions are equivalent:*

a) The module M is I-adically complete;

b) The module HM is I-complete and M is I-adically separated.

Warning A.0.29. By Proposition A.0.28, the concept of I-adic completeness does not coincide with the concept of I-completeness for discrete modules over discrete commutative rings. Rather, the former is stronger than the latter.

DEFINITION A.0.30. *A spectrum X is called* p-complete *if it is (p)-complete as an $\mathbb{S}$-module. For any spectrum X, the p-completion of X, denoted by $X_p^\wedge$, is the (p)-completion of X as an $\mathbb{S}$-module.*

Remark A.0.31. When M is an R-module for a connective $\mathbb{E}_\infty$-ring R, (p) is also an ideal of $\pi_0 R$. In this case, it follows from Remark A.0.24 that M is (p)-complete as an $\mathbb{S}$-module if and only if it is (p)-complete as an R-module, so there is completely no ambiguity to talk about p-completeness. Similarly, Remark A.0.24 implies the p-completion of an R-module M is the underlying spectrum of the (p)-completion of M as an R-module.

COROLLARY A.0.32. *Let X be a bounded below spectrum. If X is p-complete and $H\mathbb{F}_p \otimes X \simeq 0$, then $X \simeq 0$.*

Proof. We will show inductively on $n \in \mathbb{Z}$ that $\pi_n X = 0$.

1. Since X is bounded below, $\pi_n X = 0$ for $n \ll 0$.
2. Suppose now that for every $m < n$, we have $\pi_m X = 0$. We will show that $\pi_n X = 0$. In this case, we have $0 = \pi_n(H\mathbb{F}_p \otimes X) \cong \mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{F}_p, \pi_n X)$. Thus for each $x \in \pi_n X$, there exists (by axiom of choice) a sequence $(x_j)_{j \in \mathbb{N}} \in (\pi_n X)^{\mathbb{N}}$ such that $x_0 = x$ and $x_j = p x_{j+1}$ for all $j \in \mathbb{N}$, which gives rise to a map $\varphi_x : \mathbb{Z}[1/p] \to \pi_n X$ of abelian groups given by $\varphi(1/p^j) = x_j$. Theorem A.0.25 tells us that $\varphi_x = 0$, and especially, $x = 0$. In conclusion, we have proved that $x = 0$ for each $x \in \pi_n X$, thus $\pi_n X = 0$. □

Chapter 2

Revisiting derived crystalline cohomology

Abstract. We prove that the ∞-category of surjections of animated rings is projectively generated, introduce and study the notion of animated PD-pairs - surjections of animated rings with a "derived" PD-structure. This allows us to generalize classical results to non-flat and non-finitely-generated situations.

Using animated PD-pairs, we develop several approaches to derived crystalline cohomology and establish comparison theorems. As an application, we generalize the comparison between derived and classical crystalline cohomology from syntomic (affine) schemes (due to Bhatt) to quasisyntomic schemes.

We also develop a non-completed animated analogue of prisms and prismatic envelopes. We prove a variant of the Hodge–Tate comparison for animated prismatic envelopes from which we deduce a result about flat cover of the final object for quasisyntomic schemes, which generalizes several known results under smoothness and finiteness conditions.

2.1. Introduction

2.1.1. Background and main results

Regular sequences and local complete intersections play an important role in the study of Noetherian rings. However, in arithmetic geometry, Noetherianness is not preserved by operations related to perfectoids. Various generalizations to the non-Noetherian case are available. In [BMS19], it has been shown that, the *quasiregularity* (à la Quillen) is a particularly good candidate to replace the (Koszul) regularity in classical algebraic geometry: an ideal I of a ring A is called *quasiregular* (Definition 2.3.47) if the A/I-module I/I^2 is flat and the homotopy groups $\pi_i(\mathbb{L}_{(A/I)/A})$ of the cotangent complex vanish for $i>1$, or equivalently put, $\mathbb{L}_{(A/I)/A}\simeq(I/I^2)[1]$. In particular, if an ideal is generated by a Koszul-regular sequence, then it is also quasiregular.

Let us briefly review some details in the simple case of characteristic p (instead of mixed characteristic). An $\mathbb{F}_p$-algebra R is called *perfect* if the Frobenius map $R\to R, x\mapsto x^p$ is bijective. An $\mathbb{F}_p$-algebra S is called *quasiregular semiperfect* if there exists a perfect $\mathbb{F}_p$-algebra R along with a surjective map $R\twoheadrightarrow S$ of rings of which the kernel $I\subseteq R$ is quasiregular. In this case, [BMS19, Thm 8.12] shows that the derived de Rham cohomology of R with respect to the base $\mathbb{F}_p$ is concentrated in degree 0, and as a ring, it is equivalent to the PD-envelope of (R,I). Since the cotangent complex $\mathbb{L}_{R/\mathbb{F}_p}$ vanishes, the base $\mathbb{F}_p$ of the derived de Rham cohomology could be replaced by R.

This result was already known [Bha12a, Thm 3.27] when the kernel I of the map $R\twoheadrightarrow S$ in question is Koszul regular. In other words, [BMS19] generalizes the classical results about Koszul-regular ideals to quasiregular ideals.

In this article, we develop a different approach which works in greater generality: we do not need the base to be perfect, of characteristic p or even "p-local" such as $\mathbb{Z}_p$ or a perfectoid ring. We build a machinery to extrapolate results about Koszul-regular ideals to quasiregular ideals in a systematic fashion. We say that a map $R\to S$ of animated rings [CS19, §5.1] is *surjective* if the induced map $\pi_0(R)\to\pi_0(S)$ is surjective (Definition 2.3.21).

Theorem. (Theorem 2.3.23) *The ∞-category of surjective maps of animated rings is projectively generated. The set $\{\mathbb{Z}[x_1,\dots,x_m,y_1,\dots,y_n]\twoheadrightarrow\mathbb{Z}[x_1,\dots,x_m]\mid m,n\in\mathbb{N}\}$ of objects forms a set of compact projective generators.*

For technical reasons, we will introduce the ∞-category of *animated pairs*, which is equivalent to the ∞-category of surjective maps of animated rings. By the formalism of left derived functors (Proposition B.0.10), given a functor defined for "standard" Koszul-regular pairs $(\mathbb{Z}[X,Y],(Y))$ where $X=\{x_1,\ldots,x_m\}$ and $Y=\{y_1,\ldots,y_m\}$, we get a functor defined on *all* animated pairs, and in particular, on classical ring-ideal pairs (A,I), and any comparison map between such functors is determined by the restriction to these Koszul-regular pairs.

In order to formulate a reasonable generalization of the result for quasiregular semiperfect rings, just as we need animated pairs, we also need *animated PD-pairs* (Definition 2.3.15), denoted by $(A\twoheadrightarrow A'',\gamma)$ (Notation 2.3.25). There is a canonical forgetful functor from the ∞-category of animated PD-pairs to the ∞-category of animated pairs, which preserves small colimits (Proposition 2.3.34). This is remarkable since the forgetful functor from the 1-category of PD-pairs to the 1-category of ring-ideal pairs does not preserve small colimits (Remark 2.3.35). The formalism gives us the left adjoint to the forgetful functor, called the *animated PD-envelope functor*.

In general, the animated PD-envelope, considered as a kind of derived functor, is different from the PD-envelope. We will show that, there is a canonical filtration on the animated PD-envelope of $\mathbb{F}_p$-pairs[2.1.1] (i.e. pairs (A,I) where A is an $\mathbb{F}_p$-algebra), called the *conjugate filtration* (Definition 2.3.59), of which we can control the associated graded pieces:

Theorem. (Corollaries 2.3.60 and 2.3.54) *Let A be an $\mathbb{F}_p$-algebra and $I\subseteq A$ an ideal. Then*

1. *the animated PD-envelope of (A,I) admits a natural animated $\varphi_A^*(A/I)$-algebra structure.*
2. *for every $i\in\mathbb{N}$, the $(-i)$-th associated graded piece of the animated PD-envelope of A is, as a $\varphi_A^*(A/I)$-module spectrum, naturally equivalent to $\varphi_A^*(\Gamma^i_{A/I}(\mathbb{L}_{(A/I)/A}[-1]))$, where $\Gamma^i_{A/I}$ is the i-th derived divided power.*

As a corollary, the quasiregularity provides an important *acyclicity condition*: along with a mild assumption, the animated PD-envelope coincides with the classical PD-envelope:

Theorem. (Corollary 2.3.68) *Let A be an $\mathbb{F}_p$-algebra, $I\subseteq A$ a quasiregular ideal. Suppose that the (derived) Frobenius twist $(A/I)\otimes^{\mathbb{L}}_{A,\varphi_A}A$ is concentrated in degree 0, i.e., $\operatorname{Tor}^i_A(A/I,A)\cong 0$ (where the last A is viewed as an A-module via the Frobenius $\varphi_A\colon A\to A$) for all $i\in\mathbb{N}_{>0}$. Then the animated PD-envelope of (A,I) coincides with the classical PD-envelope.*

We want to point out that $(A/I)\otimes^{\mathbb{L}}_{A,\varphi_A}A$ being concentrated in degree 0 is a very mild assumption. For example, when $I\subseteq A$ is generated by a Koszul-regular sequence, then this holds automatically [Bha12a, Lem 3.41]. This also happens when (A,I) comes from a "good" PD-envelope, see Remark 2.4.62. Using this, we show that

Theorem. (Proposition 2.3.72) *Let A be a ring and $I\subseteq A$ an ideal generated by a Koszul-regular sequence. Then the animated PD-envelope of (A,I) coincides with the classical PD-envelope.*

Moreover, this mild assumption is not needed if we are only interested in associated graded pieces of the PD-filtration, which answers a question of Illusie:

Theorem. (Propositions 2.3.77 and 2.3.83) *Let A be an $\mathbb{F}_p$-algebra, $I\subseteq A$ a quasiregular ideal. Then there is a canonical comparison map from the animated PD-envelope to the classical PD-envelope of (A,I) compatible with PD-filtrations which induces equivalences on associated graded pieces. Furthermore, these associated graded pieces are given by divided powers of I/I^2 over A/I.*

The key point is that animated PD-envelopes admit natural PD-filtrations of which we can control the associated graded pieces (Proposition 2.3.77).

Based on animated PD-pairs, we develop a theory of *derived crystalline cohomology* (Definition 2.4.17) based on a technical construction called *derived de Rham cohomology of a map of animated PD-pairs* (Definition 2.4.9) which generalizes the derived de Rham cohomology of a map of animated rings. In other words, our derived crystalline cohomology should be understood as a variant of derived de Rham cohomology, not site-theoretic cohomology. These functors preserve small colimits by Proposition 2.4.19 and Lemma 2.4.12, therefore formal properties such as base change compatibility and "Künneth" formula hold (Corollaries 2.4.20, 2.4.21, and 2.4.22).

2.1.1. Or more generally, of animated $\mathbb{F}_p$-pairs.

In fact, the animated PD-envelope is, roughly speaking, a special case of derived crystalline cohomology:

Theorem. (Proposition 2.4.64) *Let $(A \twoheadrightarrow A'', \gamma_A)$ be an animated PD-pair and $A'' \twoheadrightarrow R$ be a surjective map. Let $(B \twoheadrightarrow R, \gamma_B)$ be the relative animated PD-envelope of $A \twoheadrightarrow R$ with respect to $(A \twoheadrightarrow A'', \gamma_A)$. Then the underlying $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra of B is equivalent to the derived crystalline cohomology of R with respect to $(A \twoheadrightarrow A'', \gamma_A)$.*

From this we deduce a generalization of [BMS19, Thm 8.12] under quasiregularity and the Tor-independent assumption mentioned above. To see this, similar to the animated PD-envelope, we introduce the *conjugate filtration* on the derived crystalline cohomology (Definition 2.4.41) and on the relative animated PD-envelope (Definition 2.4.58) in characteristic p, and we have a similar control of associated graded pieces for the conjugate filtration on relative animated PD-envelopes (Corollary 2.4.59) and also on the derived crystalline cohomology, which is a crystalline variant of the *Cartier isomorphism*:

Theorem. (Proposition 2.4.46) *Let (A, I, γ) be a PD-$\mathbb{F}_p$-pair. Note that the Frobenius map φ_A: $A \to A$ factors through $A \twoheadrightarrow A/I$, giving rise to a natural map $\varphi_{(A,I)}: A/I \to A$ (cf. Lemma 2.4.36). Then for every animated A/I-algebra R and $n \in \mathbb{N}$, the $(-i)$-th associated graded piece of the conjugate filtration on the derived crystalline cohomology of R relative to (A, I, γ) is, as a $\varphi^*_{(A,I)}(R)$-module spectrum, equivalent to $\varphi^*_{(A,I)}(\bigwedge^i_R \mathbb{L}_{R/(A/I)})[-i]$.*

On the other hand, similar to [Ber74], we develop an *affine crystalline site* (Definition 2.4.65) based on animated PD-pairs (Bhatt had already indicated such a possibility, see the paragraph before [Bha12a, Ex 3.21]). Recall that a map $A \to R$ of rings is called *quasisyntomic* (Definition 2.4.85) if it is flat and the cotangent complex $\mathbb{L}_{R/A}$, as an R-module spectrum, has Tor-amplitude in $[0, 1]$. We could also compare the derived crystalline cohomology to the site-theoretic cohomology:

Theorem. (Propositions 2.4.66, 2.4.87, and 2.4.90) *Let (A, I, γ_A) be a PD-pair and R an A/I-algebra.*

1. *There is a comparison map from the derived crystalline cohomology of R with respect to (A, I, γ_A) to the cohomology of the affine crystalline site, which is an equivalence when as an A/I-algebra, R is either of finite type, or quasisyntomic.*
2. *There is a comparison map from the cohomology of the affine crystalline site to the (classical) crystalline cohomology of R with respect to (A, I, γ_A). When R is a quasisyntomic A/I-algebra,*
   a. *Supposing that p is nilpotent in A, then the comparison map is an equivalence.*
   b. *Supposing that A is p-torsion free, then the comparison map becomes an equivalence after derived p-completion, or equivalently, after derived modulo p.*

The theorem above generalizes [Bha12a, Prop 3.25] which is established for syntomic algebras.

We do not know whether the derived crystalline cohomology and the cohomology of the affine crystalline site are equivalent without any assumption, we reduced this equivalence to a descent property of the derived crystalline cohomology "with respect to the base animated PD-pair" (Proposition 2.4.70).

In addition to PD-pairs and the crystalline cohomology, we also introduce *animated δ-rings* and *animated δ-pairs*, and a non-complete but animated version of prisms, the static version of which was introduced in [BS19]. Similar to animated PD-envelopes, the non-completed animated prismatic envelope, which generalizes[2.1.2] the prismatic envelope for local complete intersections [BS19, Prop 3.13], admits the *conjugate filtration* of which the associated graded pieces are easily determined by a variant of the *Hodge–Tate comparison*:

2.1.2. More precisely, it is a non-completed version.

THEOREM. (THEOREM 2.5.46) *Let (A, d) be a prism and $J \subseteq A/d$ an ideal. Then for every $i \in \mathbb{N}$, the $(-i)$-th associated graded piece of non-completed prismatic envelope, as an $A/(d, J)$-module spectrum, is equivalent*[2.1.3] *to* $\Gamma^i_{A/(d,J)}(\mathbb{L}_{(A/(d,J))/(A,d)}[-1])$.

As a corollary, similar to animated PD-envelopes, when the ideal J is p-completely quasiregular, roughly speaking, the (p, d)-completed animated prismatic envelope satisfies the universal property of the prismatic envelope in [BS19, Prop 3.13] (Remark 2.5.51). Furthermore, the non-completed prismatic envelope satisfies a faithful flatness (Proposition 2.5.49), which leads to a technical result which is essentially about the flat cover of the final object (Proposition 2.5.55), and a similar argument shows the (p, d)-completed variant:

THEOREM. (PROPOSITION 2.5.56) *Let (B, d) be a bounded oriented prism, R a derived p-complete and p-completely quasisyntomic B/d-algebra. Let P be a derived (p, d)-complete animated δ-B-algebra which is (p, d)-completely quasismooth over B, equipped with a surjection $P \twoheadrightarrow R$ of B-algebras. Then the (completed) prismatic envelope of $P \twoheadrightarrow R$ exists and is a flat cover of the final object in the prismatic site.*

We stress that our theory is non-completed. Technically, it is easier to deal with non-completed version than with p-completed version because the ∞-category of p-completed objects is usually not projectively generated. For example, $\mathbb{Z}_p \in D_{\mathrm{comp}}(\mathbb{Z}_p)$ is not a compact object. We could overcome this issue by applying the techniques developed in Subsection 2.2.5, but it would make the theory inconvenient.

However, thanks to Clausen-Scholze's condensed mathematics, the non-completed version could serve a cornerstone of an analytic version which allows us to put "topologies" and "analytic structures" on our animated rings. We will sketch a theory of *analytic pairs*, *analytic PD-pairs* and *analytic PD-envelope* in Section 2.6.

Remark. In a future work, we will develop the theory of *analytic crystalline cohomology*. We now briefly describe how it would lead to classical crystalline cohomology: An analytic PD-pair $((\mathcal{A}, \mathcal{M}) \twoheadrightarrow \mathcal{A}'', \gamma)$ consists of the datum of an analytic ring $(\mathcal{A}, \mathcal{M})$, a surjection $\mathcal{A} \twoheadrightarrow \mathcal{A}''$ of condensed ring and a condensed PD-structure γ. We recall that *Huber pairs* (A, A^+) give rise to analytic rings [Sch19a, Prop 13.16]. In particular, $(\mathbb{Z}_p, \mathbb{Z}_p)$ is a Huber pair, which gives rise to an analytic ring $\mathbb{Z}_{p,\blacksquare}$. In general, given an analytic PD-pair $((\mathcal{A}, \mathcal{M}) \twoheadrightarrow \mathcal{A}'', \gamma)$, we have a canonical analytic structure $\mathcal{M}''$ on $\mathcal{A}''$, and we would like to define the analytic crystalline cohomology for any map $(\mathcal{A}'', \mathcal{M}'') \to (\mathcal{R}, \mathcal{N})$ of analytic rings under certain condition such as nuclearity. In particular, any $\mathbb{F}_p$-algebra R gives rise to a map $\mathbb{F}_{p,\blacksquare} \to R_\blacksquare$ of analytic rings, and we expect that the analytic crystalline cohomology of $R_\blacksquare$ with respect to the analytic PD-pair $(\mathbb{Z}_{p,\blacksquare} \twoheadrightarrow \mathbb{F}_p, \gamma)$ would recover the classical crystalline cohomology of R.

2.1.2. Main techniques

We systematically adopt two techniques in this article: the *animation* and a kind of local-global principle for $\mathbb{Z}$. We briefly summarize them as follows:

There is a procedure to associate to 1-projectively generated 1-categories projectively generated ∞-categories, called the *animation*, introduced in [CS19, §5.1], and defined by the *non-abelian derived category* of a *set of compact* 1*-projective generators*.

Example. The abelian category of R-modules admits a set of compact 1-projective generators given by free R-modules of finite rank. The animation of this category is the connective part $D_{\geq 0}(R)$ of the derived category $D(R)$.

Example. The 1-category of rings admits a set of compact 1-projective generators given by polynomial rings on finitely many variables.

Remark. It is not a coincidence that the sets of compact 1-projective generators above are given by "finite free objects". Indeed, it is a corollary of Proposition B.0.31, applied to the pairs $\mathrm{Set} \rightleftarrows \mathrm{Mod}_R$ and $\mathrm{Set} \rightleftarrows \mathrm{Ring}$ of adjoint functors.

2.1.3. Here we suppress the Breuil–Kisin twists.

We review the definition of animation and summarize its main properties in Subsection B.0.3. When applying this construction to the 1-category of rings, we get *the ∞-category of animated rings*. We apply this construction to the 1-category of δ-rings, obtaining the *∞-category of animated δ-rings* (Definition 2.5.5).

The technical advantage of this construction is that, to produce a sifted-colimit-preserving functor from a projectively generated ∞-category, it suffices to produce a functor from the full subcategory spanned by a set of compact projective generators which, as we have seen, is given by "finite free objects".

Now we want to apply this procedure to the 1-category of ring-ideal pairs. Unfortunately, the 1-category of ring-ideal pairs is not 1-projectively generated. However, it is reasonable to say that "standard" Koszul-regular pairs $(\mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n], (y_1, \ldots, y_n))$ are "finite free objects". We pick the non-abelian derived category of the full category spanned by these pairs, and the 1-category of ring-ideal pairs embeds fully faithfully into it (Proposition 2.3.17). This ∞-category is equivalent to the ∞-category of surjections of animated rings (Theorem 2.3.23). Similarly, we apply this "modified animation" to the 1-category of PD-pairs, obtaining the *∞-category of animated PD-pairs*. The PD-envelope functor gives rise to the *animated PD-envelope* (Definition 2.3.15): a "good enough" pair of adjoint functors between 1-projectively generated 1-categories give rise to a pair of adjoint functors between animations (Corollary 2.2.3). However, here the story is a bit more complicated due to our "modification" of the animation.

In a similar fashion, we apply this animation formalism to δ-pairs, obtaining *animated δ-pairs* (Definition 2.5.8), and we use similar techniques to define and analyze non-completed animated prismatic envelopes. We also use the animation techniques to define the "de Rham context" dRCon, the "crystalline context" CrysCon, the derived de Rham cohomology and the derived crystalline cohomology in Subsection 2.4.1.

Now we describe the second main technique that we used: the local-global principle for $\mathbb{Z}$. Some techniques are only valid in characteristic p. For example, we do not know how to define the conjugate filtration on the derived crystalline cohomology beyond characteristic p. However, these arithmetic objects, such as PD-structures, usually degenerate in characteristic 0. In view of these, we can usually then glue the results for each prime number $p \in \mathbb{N}$ and the result after rationalization. The simplest case is the following: Let $X \in \mathrm{Sp}$ be a spectrum. Suppose that the spectrum $X /^{\mathbb{L}} p$ is equivalent to 0 for every prime number $p \in \mathbb{N}$, and that X is also contractible after rationalization. Then the spectrum X itself is contractible. We establish similar results (Lemmas 2.3.69 and 2.3.71) under connectivity assumptions. These results allow us to deduce integral results.

2.1.3. Leitfaden

Here is a Leitfaden of the article: Section 2.2 is devoted to technical preparations. We suggest the readers skip it in the first reading. Section 2.3 is devoted to the theory of animated pairs and animated PD-pairs, and to the study of the animated PD-envelope. Section 2.4 is devoted to relative animated PD-envelopes, derived crystalline cohomology, cohomology of the affine crystalline site and their comparisons. Section 2.5 is devoted to animated δ-rings, animated δ-pairs, non-complete animated prisms, non-completed animated prismatic envelope and a variant of the Hodge–Tate comparison. Appendix B is a collection of basic facts about animations and projectively generated categories (which we suggest the reader read first if they have not seen this concept before). Appendix C is about symmetric monoidal ∞-categories, which does not play an important role in this article and the reader should feel free to ignore it.

2.1.4. Notations and terminology

In this article, since we often work in the ∞-category of certain "derived" categories, we try to distinguish the "ordinary" objects and "derived" objects by choosing different words.

Given an ∞-category $\mathcal{C}$ and a diagram $Y \leftarrow X \to Z$ in $\mathcal{C}$, the pushout of the diagram is denoted by $Y \amalg_X Z$. In particular, if $\mathcal{C}$ admits an initial object, the coproduct of two objects Y, Z is denoted by $Y \amalg Z$.

We will denote by $\mathcal{S}$ the ∞-category of (small) animae, that is, the simplicial nerve of the simplicial category of (small) Kan complexes [Lur09, Def 1.2.16.1].

We say that an anima $X \in \mathcal{S}$ or a spectrum $X \in \mathrm{Sp}$ is *static*[2.1.4] if $\pi_i(X) \cong 0$ for all $i \neq 0$. For two spectra $X, Y \in \mathrm{Sp}$, we will denote by $X \otimes^{\mathbb{L}} Y$ the smash product. *Rings* are always static and commutative, while $\mathbb{E}_n$-*rings* are $\mathbb{E}_n$-algebras in the symmetric monoidal ∞-category $(\mathrm{Sp}, \otimes^{\mathbb{L}})$.

Given a ring A, we will refer to a "classical" A-module a *static A-module*. The category of static A-modules will be denoted by Mod_A. The category of *ring-module pairs* (A, M) where $M \in \mathrm{Mod}_A$ is denoted by Mod. An object in the derived ∞-category $D(A)$ an *A-module spectrum*.

Given an $\mathbb{E}_1$-ring A, the ∞-category of left (resp. right) A-module spectra will be denoted by LMod_A (resp. RMod_A). Given a right A-module spectrum M and a left A-module spectrum N, their relative tensor product is denoted by $M \otimes_A^{\mathbb{L}} N$, to avoid confusion with the ordinary tensor product of static modules.

Given an $\mathbb{E}_\infty$-ring A, the ∞-category of A-module spectra is denoted by $D(A)$. In particular, we have $\mathrm{Sp} = D(\mathbb{S})$. An $\mathbb{E}_n$-*A-algebra* is an $\mathbb{E}_n$-algebra in the symmetric monoidal ∞-category $(D(A), \otimes_A^{\mathbb{L}})$.

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2.2. CATEGORICAL PREPARATIONS

In this section, we will do some technical preparations of ∞-categories which will be used throughout this article. We try our best to refer to this section explicitly so that the reader could first skip this section and read back when needed.

2.2.1. Animation of adjoint functors

This subsection is devoted to proving that animation behaves well for certain "monadic" pairs of adjoint functors. Here is a general lemma.

LEMMA 2.2.1. *Let $n \in \mathbb{N}_{>0} \cup \{\infty\}$. Let $\mathcal{C}$ be a small n-category which admits finite coproducts and $\mathcal{D}$ a locally small n-category which admits small colimits. Let $f: \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite coproducts. Then*

1. *There is a pair of adjoint functors $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ (Notations B.0.6 and B.0.23) where F is the left derived functor (Propositions B.0.10 and B.0.27) of f and G is the functor given by $D \mapsto \mathrm{Map}_{\mathcal{D}}(f(\cdot), D) \in \mathcal{P}(\mathcal{C})$.*
2. *Suppose that for all objects $C \in \mathcal{C}$, the object $f(C) \in \mathcal{D}$ is compact and n-projective. Then the functor G preserves filtered colimits and geometric realizations. Under this assumption, if f is further assumed to be fully faithful, then so is F.*
3. *Suppose that the set $\{f(C) \mid C \in \mathcal{C}\} \subseteq \mathcal{D}$ generates $\mathcal{D}$ under small colimits. Then the functor G is conservative.*

Proof. We exhibit the proof for $n = \infty$. First, the functor $f: \mathcal{C} \to \mathcal{D}$ extends uniquely to a functor $\tilde{F}: \mathcal{P}(\mathcal{C}) \to \mathcal{D}$ which preserves small colimits by [Lur09, Thm 5.1.5.6]. Since $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is stable under sifted colimits, it follows that the functor F is equivalent to the composite $\mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C}) \xrightarrow{\tilde{F}} \mathcal{D}$. The functor $\tilde{F}$ admits a right adjoint by [Lur09, Cor 5.2.6.5] which is equivalent to the composite $\mathcal{D} \xrightarrow{G} \mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C})$, therefore (F, G) is a pair of adjoint functors.

Part 2 follows from the fact that $\mathcal{P}_\Sigma(\mathcal{C}) \subseteq \mathcal{P}(\mathcal{C})$ is stable under sifted colimits (Proposition B.0.7). The later statement follows from Proposition B.0.11.

2.1.4. This is usually called *discrete* in homotopy theory. We follow Clausen-Scholze's terminology in condensed mathematics to call them *static* to distinguish from the point-set topological discreteness. In particular, the *static* object $\mathbb{Z}_p$ might be equipped with the p-adic topology which is different from the *discrete* topology.

Suppose that $\{f(C)\,|\,C\in\mathcal{C}\}$ generates $\mathcal{D}$ under small colimits, then for any map $X\to Y$ in $\mathcal{D}$, if the induced map $G(X)\to G(Y)$ is an equivalence in $\mathcal{P}_\Sigma(\mathcal{C})$, then for all objects $C\in\mathcal{C}$, the induced map $\mathrm{Map}_{\mathcal{D}}(f(C),X)\to\mathrm{Map}_{\mathcal{D}}(f(C),Y)$ is an equivalence. Let $\mathcal{D}'\subseteq\mathcal{D}$ be the full subcategory spanned by those $D\in\mathcal{D}$ such that the induced map $\mathrm{Map}_{\mathcal{D}}(D,X)\to\mathrm{Map}_{\mathcal{D}}(D,Y)$ is an equivalence. Then $\mathcal{D}'$ is stable under colimits, and $f(C)\in\mathcal{D}'$ for all $C\in\mathcal{C}$. The result follows. □

It then follows from Lemma B.0.26 and Corollary B.0.30 that

Corollary 2.2.2. *Let $\mathcal{C},\mathcal{D}$ be two small n-categories which admit finite coproducts and $f:\mathcal{C}\to\mathcal{D}$ a functor which preserves finite coproducts. Then*

1. *There is a pair of adjoint functors $\mathcal{P}_{\Sigma,n}(\mathcal{C})\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{P}_{\Sigma,n}(\mathcal{D})$ where G is given by $\mathcal{P}_{\Sigma,n}(\mathcal{D})\ni H\mapsto H\circ F\in\mathcal{P}_{\Sigma,n}(\mathcal{C})$ and F is the left derived functor of the composite functor $\mathcal{C}\xrightarrow{f}\mathcal{D}\hookrightarrow\mathcal{P}_{\Sigma,n}(\mathcal{D})$.*
2. *The functor G preserves sifted colimits, and the canonical map $\tau_{\le m}\circ G\to G\circ\tau_{\le m}$ of functors is an equivalence for all $m\in\mathbb{N}$ (cf. [Lur09, Rem 5.5.8.26] and the discussion before Lemma B.0.39).*
3. *If f is fully faithful, then so is the functor F.*
4. *If f is essentially surjective, then the functor G is conservative.*

Now we apply this to animations:

Corollary 2.2.3. *Let $\mathcal{C}\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{D}$ be a pair of adjoint functors between 1-categories such that*

1. *The 1-category $\mathcal{D}$ admits filtered colimits and reflexive coequalizers (or equivalently, geometric realizations, by Remark B.0.21), and G preserves filtered colimits and reflexive coequalizers.*
2. *The 1-category $\mathcal{C}$ is projectively generated.*
3. *The functor G is conservative.*

Then $\mathcal{D}$ is 1-projectively generated, and we have a pair $\mathrm{Ani}(\mathcal{C})\underset{\mathrm{Ani}(G)}{\overset{\mathrm{Ani}(F)}{\rightleftarrows}}\mathrm{Ani}(\mathcal{D})$ of adjoint functors between ∞-categories after animation. Furthermore, the functor $\mathrm{Ani}(G)$ is conservative, preserves sifted colimits, and the canonical map $\tau_{\le 0}\circ\mathrm{Ani}(G)\to G\circ\tau_{\le 0}$ of functors is an equivalence. If G preserves small colimits, then so does $\mathrm{Ani}(G)$.

Proof. It follows from Proposition B.0.31 that the 1-category $\mathcal{D}$ is 1-projectively generated, therefore $\mathcal{C},\mathcal{D}$ admit small colimits which are preserved by F. Furthermore, let $\mathcal{C}^0\subseteq\mathcal{C}$ be the full subcategory spanned by finite coproducts of a chosen set of compact 1-projective generators for $\mathcal{C}$, and $\mathcal{D}^0\subseteq\mathcal{D}$ the full subcategory spanned by the images of objects of $\mathcal{C}$ under F, then there are equivalences $\mathcal{C}\simeq\mathcal{P}_{\Sigma,1}(\mathcal{C}^0)$ and $\mathcal{D}\simeq\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ of 1-categories by Proposition B.0.29 (note that F preserves finite coproducts).

Let $f:\mathcal{C}^0\to\mathcal{D}^0$ be the functor induced by F, which preserves finite coproducts and is essentially surjective. It follows from Corollary 2.2.2 with $n=1$ and the uniqueness of the right adjoint functor that the functor $G:\mathcal{D}\to\mathcal{C}$ is equivalent to $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)\to\mathcal{P}_{\Sigma,1}(\mathcal{C}^0), H\mapsto H\circ f$.

We invoke again Corollary 2.2.2 with $n=\infty$ to obtain a pair of adjoint functors $\mathcal{P}_\Sigma(\mathcal{C}^0)\rightleftarrows\mathcal{P}_\Sigma(\mathcal{D}^0)$ induced by f. It follows from the definitions that $\mathrm{Ani}(\mathcal{C})\simeq\mathcal{P}_\Sigma(\mathcal{C}^0)$, $\mathrm{Ani}(\mathcal{D})\simeq\mathcal{P}_\Sigma(\mathcal{D}^0)$ and that the functor $\mathcal{P}_\Sigma(\mathcal{C}^0)\to\mathcal{P}_\Sigma(\mathcal{D}^0)$ obtained above is equivalent to $\mathrm{Ani}(F)$. Let $G':\mathrm{Ani}(\mathcal{D})\to\mathrm{Ani}(\mathcal{C})$ be the right adjoint to $\mathrm{Ani}(F)$. Since f is essentially surjective, G' is conservative. It remains to show that G' is equivalent to $\mathrm{Ani}(G)$.

Indeed, both G' and $\mathrm{Ani}(G)$ preserve sifted colimits. Since the functor $G:\mathcal{D}\to\mathcal{C}$ is equivalent to $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)\to\mathcal{P}_{\Sigma,1}(\mathcal{C}^0)$, $H\mapsto H\circ f$, the restrictions of G' and $\mathrm{Ani}(G)$ to the full subcategory $\mathcal{D}^0\subseteq\mathrm{Ani}(\mathcal{D})$ are equivalent. It then follows from Proposition B.0.10 that G' and $\mathrm{Ani}(G)$ are equivalent. The colimit preserving properties follow from Corollary B.0.38. □

Now we look at two simple examples:

Example 2.2.4. Let $R \to S$ be a map of rings. Then there is a pair $\mathrm{Mod}_R \overset{\cdot\otimes_R S}{\rightleftarrows} \mathrm{Mod}_S$ of adjoint functors between the categories of static modules. Since the forgetful functor $\mathrm{Mod}_S \to \mathrm{Mod}_R$ is conservative, and preserves small colimits, we have the pair of adjoint functors $\mathrm{Ani}(\mathrm{Mod}_R) \overset{\mathrm{Ani}(\cdot\otimes_R S)}{\rightleftarrows} \mathrm{Ani}(\mathrm{Mod}_S)$. Under the equivalences $\mathrm{Ani}(\mathrm{Mod}_R) \simeq D_{\geq 0}(R)$ and $\mathrm{Ani}(\mathrm{Mod}_S) \simeq D_{\geq 0}(S)$, the functor $\mathrm{Ani}(\cdot \otimes_R S)$ is equivalent to the functor $\cdot \otimes_R^{\mathbb{L}} S$.

Example 2.2.5. Let Ring be the 1-category of rings and Ab the 1-category of abelian groups. Then we have a pair $\mathrm{Ab} \overset{\mathrm{Sym}_{\mathbb{Z}}}{\rightleftarrows} \mathrm{Ring}$ of adjoint functors. Since the forgetful functor $\mathrm{Ring} \to \mathrm{Ab}$ is conservative, and preserves filtered colimits and reflexive coequalizers, we get a pair $D_{\geq 0}(\mathbb{Z}) \overset{\mathbb{L}\mathrm{Sym}_{\mathbb{Z}}}{\rightleftarrows} \mathrm{Ani}(\mathrm{Ring})$ of adjoint functors.

In Corollary 2.2.3, the functor G (resp. $\mathrm{Ani}(G)$) exhibits $\mathcal{D}$ (resp. $\mathrm{Ani}(\mathcal{D})$) as monadic over $\mathcal{C}$ (resp. $\mathrm{Ani}(\mathcal{C})$). The associated endomorphism monad is given by $G \circ F$ (resp. $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \simeq \mathrm{Ani}(G \circ F)$ by Proposition B.0.40).

LEMMA 2.2.6. *Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors between ∞-categories. Let K be a small simplicial set. Then $G \circ F$ preserves K-indexed colimits if G preserves K-indexed colimits. The converse is true if G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$.*

Proof. If G preserves K-indexed colimits, since F is a left adjoint, it follows that so does $T := G \circ F$. Conversely, if G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$, then $\mathcal{D} \simeq \mathrm{LMod}_T(\mathcal{C})$ and the result follows from [Lur17, Cor 4.2.3.5]. □

2.2.2. Diagram categories and undercategories

In this subsection, we will show that diagram n-categories and undercategories of n-projectively generated categories are n-projectively generated, for which we give an explicit choice of n-projective generators. We first show the version for ∞-categories, then list the analogues for n-categories for which the proof is nearly verbatim. We start with diagram categories.

LEMMA 2.2.7. *Let $(\mathcal{C}_\alpha)_{\alpha \in T}$ be a small collection of projectively generated ∞-category. Then the ∞-category $\prod_{\alpha \in T} \mathcal{C}_\alpha$ is projectively generated. More precisely, let 1_α denote the initial objects of $\mathcal{C}_\alpha$. If the collections $S_\alpha \subseteq \mathcal{C}_\alpha$ of objects are sets of compact projective generators for $\mathcal{C}_\alpha$, then the collection $\{i_{s,\beta} \mid s \in S_\beta, \beta \in T\} \subseteq \prod_{\alpha \in T} \mathcal{C}_\alpha$ is a set of compact projective generators for $\prod_{\alpha \in T} \mathcal{C}_\alpha$, where $i_{s,\beta} \in \prod_{\alpha \in T} \mathcal{C}_\alpha$ is given by $\left(\begin{cases} s & \beta' = \beta \\ 1_{\beta'} & \beta' \neq \beta \end{cases} \right)_{\beta' \in T}$.*

Proof. Since the small colimits in $\prod \mathcal{C}_\alpha$ are computed pointwise, it follows that $\prod \mathcal{C}_\alpha$ is cocomplete. Now given S_α and $i_{s,\beta}$, let $\mathcal{D} \subseteq \prod \mathcal{C}_\alpha$ be the full subcategory generated by $\{i_{s,\beta}\}$ under colimits. For all $\beta \in T$, the fully faithful embedding $j_\beta : \mathcal{C}_\beta \to \prod \mathcal{C}_\alpha$ given by $C \mapsto \left(\begin{cases} C & \beta' = \beta \\ 1_{\beta'} & \beta' \neq \beta \end{cases} \right)_{\beta' \in T}$ preserves small colimits, and $j_\beta(s) = i_{s,t}$. Thus the "skyscraper" functor $j_\beta(C)$ is an object of $\mathcal{D}$ for $C \in \mathcal{C}_\beta$.

Finally, we can write any object $F \in \prod \mathcal{C}_\alpha$ as a small colimit $\mathrm{colim}_{\beta \in T} j_\beta(F_\beta)$, therefore $\mathcal{D} = \prod \mathcal{C}_\alpha$. □

Now let $\mathcal{C}$ be a cocomplete ∞-category, $K \in \mathrm{Set}_\Delta$ a small simplicial set and $K_0 \subseteq K$ the set of vertices. Then we have a pair of adjoint functors $\mathrm{Fun}(K_0, \mathcal{C}) \underset{(K_0 \to K)^*}{\overset{\mathrm{Lan}_{K_0 \to K}}{\rightleftarrows}} \mathrm{Fun}(K, \mathcal{C})$ where $\mathrm{Lan}_{K_0 \hookrightarrow K}$ is the functor of left Kan extension along the map $K_0 \to K$, and $(K_0 \to K)^*$ denotes the restriction along $K_0 \to K$.

Warning 2.2.8. In an early draft, we called $K_0 \to K$ an "inclusion". However, any map of simplicial sets is equivalent to a cofibration up to a trivial fibration in Joyal model structure. That is to say, the concept of "non-full subcategory" is not model-independent. We decided to suppress such model-dependent expressions.

It then follows from Proposition B.0.15 and Lemma 2.2.7 that

COROLLARY 2.2.9. *Let $\mathcal{C}$ be a projectively generated ∞-category and $K \in \mathrm{Set}_\Delta$ a small simplicial set. Then the ∞-category $\mathrm{Fun}(K, \mathcal{C})$ of functors is projectively generated.*

Next, we study undercategories.

LEMMA 2.2.10. *Let $\mathcal{C}$ be a projectively generated ∞-category and $Z \in \mathcal{C}$ an object. Then the undercategory $\mathcal{C}_{Z/}$ is projectively generated. More precisely, letting $S \subseteq \mathcal{C}$ be a set of projective generators for $\mathcal{C}$, then the set $\{Z \to X \amalg Z \mid X \in S\}$ is a set of compact projective generators for the undercategory $\mathcal{C}_{Z/}$.*

Proof. Consider the pair $\mathcal{C} \underset{Y \leftarrow\!\shortmid (Z \to Y)}{\overset{X \mapsto (Z \to X \amalg Z)}{\rightleftarrows}} \mathcal{C}_{Z/}$ of adjoint functors. The forgetful functor $\mathcal{C}_{Z/} \to \mathcal{C}$

- is conservative, since an object in $\mathcal{C}_{Z/}$ could be identified with a map $\Delta^1 \to \mathcal{C}, 0 \mapsto Z$, and a map in $\mathcal{C}_{Z/}$ between two objects could be identified with a homotopy between two maps $\Delta^1 \rightrightarrows \mathcal{C}$, then we invoke [Lur20, Tag 01DK] to conclude.
- preserves sifted colimits. It suffices to show that for every sifted simplicial set K, the inclusion $K \hookrightarrow \{*\} \star K =: L$ is cofinal. In view of a variant [Lur09, Theorem 4.1.3.1] of Quillen's Theorem A, it is equivalent to check that $M := K \times_L L_{X/}$ is weakly contractible for all $X \in L$. When X is the distinguished point $*$, $M \simeq K$ as ∞-categories. When $X \in K$, $M \simeq K_{X/}$ as ∞-categories, and [Lur17, Lemma 5.5.3.12] tells us that $K_{X/} \to K$ is cofinal, therefore a weak homotopy equivalence by [Lur09, Proposition 4.1.1.3(3)]. In all cases, M is weak equivalent to K as simplicial sets, which is weakly contractible by [Lur09, Proposition 5.5.8.7].

We then invoke Proposition B.0.15 to conclude[2.2.1]. ☐

Now we list the n-categorical analogues:

LEMMA 2.2.11. *Let $\mathcal{C}$ be an n-projectively generated n-category and $K \in \mathrm{Set}_\Delta$ a small simplicial set. Then the n-category $\mathrm{Fun}(K, \mathcal{C})$ of functors is n-projectively generated.*

LEMMA 2.2.12. *Let $\mathcal{C}$ be an n-projectively generated n-category and $Z \in \mathcal{C}$ an object. Then the undercategory $\mathcal{C}_{Z/}$ is n-projectively generated. More precisely, let $S \subseteq \mathcal{C}$ be a set of n-projective generators for $\mathcal{C}$, then the set $\{Z \to X \amalg Z \mid X \in S\}$ is a set of compact n-projective generators for the undercategory $\mathcal{C}_{Z/}$.*

Now we deduce the corollaries for animations.

COROLLARY 2.2.13. *Let $\mathcal{C}$ be an n-projectively generated n-category. Then there is a canonical equivalence $\mathrm{Ani}(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})) \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ani}(\mathcal{C}))$ of ∞-categories, or equivalently, a canonical equivalence $\mathrm{Ani}(\mathrm{Fun}(\Delta^1, \mathcal{C})) \to \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathcal{C}))$ of ∞-categories.*

Proof. Let $S \subseteq \mathcal{C}$ be a set of compact n-projective generators for $\mathcal{C}$. Spelling out the proof of Corollary 2.2.9 (more precisely, its analogue Lemma 2.2.11), we extract an explicit set of compact n-projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, namely, $T := \{X \leftarrow 0 \mid X \in S\} \cup \left\{X \xleftarrow{\mathrm{id}_X} X \mid X \in S\right\}$. Note that $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ani}(\mathcal{C}))$ is a full subcategory, and again by the proof of Corollary 2.2.9, it follows that T is a set of compact projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ani}(\mathcal{C}))$. The result follows. ☐

2.2.1. We believe that our argument could be vastly simplified. However, we point out that the map $K \hookrightarrow \{*\} \star K$ is not necessarily cofinal if the simplicial set K is not sifted. For example, take K to be a discrete set with at least two elements.

The same proof leads to the following (compare with [Rak20, Cons 4.3.4]).

COROLLARY 2.2.14. *Let $\mathcal{C}$ be an n-projectively generated n-category. Then there are canonical equivalences*

$$\begin{aligned} \mathrm{Ani}(\mathrm{Fun}((\mathbb{Z},\geq),\mathcal{C})) &\longrightarrow \mathrm{Fun}((\mathbb{Z},\geq),\mathrm{Ani}(\mathcal{C})) \\ \mathrm{Ani}(\mathrm{Fun}(\mathbb{Z},\mathcal{C})) &\longrightarrow \mathrm{Fun}(\mathbb{Z},\mathrm{Ani}(\mathcal{C})) \\ \mathrm{Ani}(\mathrm{Fun}(\{0,1\},\mathcal{C})) &\longrightarrow \mathrm{Fun}(\{0,1\},\mathrm{Ani}(\mathcal{C})) \\ \mathrm{Ani}(\mathcal{C}_{Z/}) &\longrightarrow \mathrm{Ani}(\mathcal{C})_{Z/} \end{aligned}$$

of ∞-categories. The same for replacing $\mathbb{Z}$'s by $\mathbb{N}$'s.

2.2.3. Comma categories and base change

In this subsection, we will discuss comma categories, which serves as our basic language to discuss various base changes.

DEFINITION 2.2.15. *Let $\mathcal{C},\mathcal{D}$ be ∞-categories and $F:\mathcal{C}\to\mathcal{D}$ a functor. The* comma category*, sometimes denoted by $F\downarrow\mathcal{D}$, is given by the simplicial set $\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$, where the map $\mathcal{C}\to\mathrm{Fun}(\{0\},\mathcal{D})$ is given by F and the map $\mathrm{Fun}(\Delta^1,\mathcal{D})\to\mathrm{Fun}(\{0\},\mathcal{D})$ is induced by the vertex $\{0\}\to\Delta^1$.*

Example 2.2.16. Consider the functor $\mathrm{id}_{\mathrm{Ani(Ring)}}:\mathrm{Ani(Ring)}\to\mathrm{Ani(Ring)}$. The comma category $\mathrm{Ani(Ring)}\times_{\mathrm{Fun}(\{0\},\mathrm{Ani(Ring)})}\mathrm{Fun}(\Delta^1,\mathrm{Ani(Ring)})$ is equivalent to $\mathrm{Fun}(\Delta^1,\mathrm{Ani(Ring)})$. An object is simply given by a base $A\in\mathrm{Ani(Ring)}$ and an A-algebra $A\to R$.

Example 2.2.17. Consider the functor $\mathrm{Pair}\to\mathrm{Ring}$, $(A,I)\mapsto A/I$ and the composite functor $\mathrm{PDPair}\to\mathrm{Pair}\to\mathrm{Ring}$. Concretely, the objects in the comma category $\mathrm{PDPair}\times_{\mathrm{Fun}(\{0\},\mathrm{Ring})}\mathrm{Fun}(\Delta^1,\mathrm{Ring})$ are given by a PD-pair (A,I,γ) along with an A/I-algebra $A/I\to R$. This is the non-animated version of CrysCon that will be introduced in Subsection 2.4.1.

Remark 2.2.18. A similar comma category plays an role for prismatic cohomology. We will study a non-complete version in Subsection 2.5.3.

LEMMA 2.2.19. *Let $\mathcal{C},\mathcal{D}$ be ∞-categories and $F:\mathcal{C}\to\mathcal{D}$ a functor. Then the simplicial set $\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$ is an ∞-category.*

Proof. It follows from [Lur09, Corollary 2.3.2.5] applied to the inner fibration $\mathcal{D}\to\{*\}$ that $\mathrm{Fun}(\Delta^1,\mathcal{D})\to\mathrm{Fun}(\{0\},\mathcal{D})$ is an inner fibration. Then it follows [Lur09, Corollary 2.4.6.5] that $\mathrm{Fun}(\Delta^1,\mathcal{D})\to\mathrm{Fun}(\{0\},\mathcal{D})$ is a categorical fibration. The result follows. □

Remark 2.2.20. The canonical projection $\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})\to\mathcal{C}$ admits a fully faithful section induced by $\mathcal{D}\to\mathrm{Fun}(\Delta^1,\mathcal{D}), D\mapsto\mathrm{id}_D$ which is also a left adjoint of the projection in question.

LEMMA 2.2.21. *Let $\mathcal{C},\mathcal{D}$ be ∞-categories and $F:\mathcal{C}\to\mathcal{D}$ a functor. Suppose that $\mathcal{D}$ admits finite coproducts. Then the functor $\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})\to\mathcal{C}\times\mathcal{D}$ induced by $\mathrm{Fun}(\Delta^1,\mathcal{D})\to\mathrm{Fun}(\{1\},\mathcal{D})\simeq\mathcal{D}$ admits a left adjoint informally given by $(C,D)\mapsto(C,F(C)\to F(C)\amalg D)$.*

Proof. We need the concept of relative adjunctions [Lur17, §7.3.2]. In fact, the adjunction is relative to $\mathcal{C}$.

To see this, we start with the special case that $\mathcal{C}=\mathcal{D}$ and $F=\mathrm{id}_{\mathcal{D}}$. The point is that, the pair $\mathcal{D}\times\mathcal{D}\rightleftarrows\mathrm{Fun}(\Delta^1,\mathcal{D})$ of adjoint functors satisfies [Lur17, Prop 7.3.2.1], where the functor $\mathcal{D}\times\mathcal{D}\to\mathrm{Fun}(\Delta^1,\mathcal{D})$ is given by left Kan extension along the functor $\{0,1\}\to\Delta^1$, and the functor $\mathrm{Fun}(\Delta^1,\mathcal{D})\to\mathcal{D}\times\mathcal{D}$ is simply given by the restriction along $\{0,1\}\to\Delta^1$.

The general case follows from [Lur17, Prop 7.3.2.5] by base change along $F:\mathcal{C}\to\mathcal{D}$. □

It follows from Proposition B.0.15 that

Corollary 2.2.22. *Let $\mathcal{C},\mathcal{D}$ be projectively generated ∞-categories and $F:\mathcal{C}\to\mathcal{D}$ a functor. Then the comma category $\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$ is projectively generated. More precisely, let $S\subseteq\mathcal{C}$ and $T\subseteq\mathcal{D}$ be sets of compact projective generators. Then $\{(C,F(C)\to F(C)\amalg D)\mid C\in S, D\in T\}$ is a set of compact projective generators for $\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$.*

It follows from [Lur09, Lem 5.4.5.5] that the colimits in comma categories exist and are easy to describe under the assumption that the functor in question preserves colimits:

Lemma 2.2.23. *Let $\mathcal{C},\mathcal{D}$ be ∞-categories and $F:\mathcal{C}\to\mathcal{D}$ a functor. Let K be a simplicial set. Suppose that $\mathcal{C},\mathcal{D}$ admits K-indexed colimits which are preserved by F. Then the comma category $\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$ admits K-indexed colimits which are preserved by projection to either factor.*

Remark 2.2.24. (Base change) Let $\mathcal{C},\mathcal{D}$ be ∞-categories which admit finite colimits and $F:\mathcal{C}\to\mathcal{D}$ a functor which preserves finite colimits. Given an object $(C,F(C)\to D)\in\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$, there is a unique map $(C,\mathrm{id}_{F(C)})\to(C,F(C)\to D)$ (which is in fact the unit map for the adjunction in Remark 2.2.20). For all maps $C\to C'$ in $\mathcal{C}$, we have the pushout of the diagram $(C',\mathrm{id}_{F(C')})\leftarrow(C,\mathrm{id}_{F(C)})\to(F(C)\to D)$ in $\mathcal{C}$, which is $(C',D\amalg_{F(C)}F(C'))$ by Lemma 2.2.23. At the beginning of this section, we said that the objects in $\mathcal{C}$ are considered as "bases". Thus we understand this pushout as "base change".

Example 2.2.25. In Example 2.2.16, given a map $A\to B$ of animated rings, the base change of $A\to R$ along $A\to B$ is $B\to R\otimes_A^{\mathbb{L}}B$. Since the cotangent complex functor $\mathbb{L}_{\cdot/\cdot}:\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))\to\mathrm{Ani}(\mathrm{Mod})$ preserves small colimits (Lemma 2.2.35), we get the base change property: the natural map $\mathbb{L}_{R/A}\otimes_A^{\mathbb{L}}B\to\mathbb{L}_{R\otimes_A^{\mathbb{L}}B/B}$ is an equivalence (here we implicitly used Lemma 2.2.36). Similarly, we get the base change property $\mathrm{HH}(R/A)\otimes_A^{\mathbb{L}}B\simeq\mathrm{HH}(R\otimes_A^{\mathbb{L}}B/B)$ for Hochschild homology (the reader should feel free to ignore this since it will not be used in this article).

Example 2.2.26. In Example 2.2.17, given a map $(A,I,\gamma)\to(B,J,\delta)$ of PD-pairs, the base change of $((A,I,\gamma),A/I\to R)$ along $(A,I,\gamma)\to(B,J,\delta)$ is $((B,J,\delta),B/J\to R\otimes_{A/I}(B/J))$.

Remark 2.2.27. We have a prismatic version of base change by Remark 2.2.18.

Remark 2.2.28. (Colimits over a fixed base) Let $\mathcal{C},\mathcal{D}$ be cocomplete ∞-categories and $F:\mathcal{C}\to\mathcal{D}$ a functor which preserves small colimits. Given an object $C\in\mathcal{C}$, a small simplicial set K and a diagram $q:K\to\mathcal{D}_{F(C)/}$, we associate a diagram $K\to\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$ informally given by $k\mapsto(C,F(C)\to q(k))$ (the formal description necessitates a discussion of "fat" overcategories [Lur09, §4.2.1]). By Lemma 2.2.23, the colimit of this diagram is given by $(C,\operatorname{colim}q)$. We understand this colimit as taking colimits over a fixed base.

Example 2.2.29. In Example 2.2.16, given an animated ring A and two A-algebras R,S, the map $(A\to R\otimes_A^{\mathbb{L}}S)$, seen as an object of $\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))$, is the pushout of the diagram $(A\to R)\leftarrow(A,\mathrm{id}_A)\to(A\to S)$. Since the cotangent complex functor $\mathbb{L}_{\cdot/\cdot}:\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))\to\mathrm{Ani}(\mathrm{Mod})$ (which we will review in Definition 2.2.33) preserves small colimits (Lemma 2.2.35), we get the "Künneth formula": the natural map $\mathbb{L}_{R/A}\otimes_R^{\mathbb{L}}(R\otimes_A^{\mathbb{L}}S)\oplus\mathbb{L}_{S/A}\otimes_S^{\mathbb{L}}(R\otimes_A^{\mathbb{L}}S)\to\mathbb{L}_{(R\otimes_A^{\mathbb{L}}S)/A}$ is an equivalence (again we used Lemma 2.2.36, and also the form of colimits in Ani(Mod)). Similarly, we have $\mathrm{HH}(R/A)\otimes_A^{\mathbb{L}}\mathrm{HH}(S/A)\simeq\mathrm{HH}(R\otimes_A^{\mathbb{L}}S/A)$ for Hochschild homology (again, Hochschild homology is not needed in this article).

Remark 2.2.30. In view of Remark 2.2.18, prismatic cohomology has a similar "Künneth formula" [AL19a, Prop 3.5.1].

Remark 2.2.31. (Transitivity) Let $\mathcal{C},\mathcal{D}$ be ∞-categories which admit finite colimits and $F:\mathcal{C}\to\mathcal{D}$ a functor which preserves finite colimits. Given a map $C\to C'$ in $\mathcal{C}$, any object $(C',F(C')\to D)\in\mathcal{C}\times_{\mathrm{Fun}(\{0\},\mathcal{D})}\mathrm{Fun}(\Delta^1,\mathcal{D})$ could be written as the pushout of the diagram $(C,F(C)\to D)\leftarrow(C,F(C)\to F(C'))\to(C',\mathrm{id}_{F(C')})$. This is closely related to transitivity sequence in the cohomology theory, as shown in examples below.

Example 2.2.32. In Example 2.2.16, for any maps $A \to B \to R$ of animated rings, the "relative" map $B \to R$, viewed as an object of $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$, is the pushout of the diagram $(A \to R) \leftarrow (A \to B) \to (\mathrm{id}_B : B \to B)$. Since the cotangent complex functor $\mathbb{L}_{\cdot/\cdot} : \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring})) \to \mathrm{Ani}(\mathrm{Mod})$ preserves small colimits (Lemma 2.2.35), we get the transitivity sequence

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} R \longrightarrow \mathbb{L}_{R/A} \longrightarrow \mathbb{L}_{R/B}$$

(Lemma 2.2.36 was used) Similarly, we have $\mathrm{HH}(R/A) \otimes^{\mathbb{L}}_{\mathrm{HH}(B/A)} B \simeq \mathrm{HH}(R/B)$ for Hochschild homology.

Finally, we briefly review the theory of the cotangent complex of maps of animated rings, meanwhile we explain how this "coincides" with the theory of cotangent complex of maps of animated A-algebra for some ring A. By Corollary 2.2.13, the ∞-category $\mathrm{AniArr} := \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$ is projectively generated, and the proof leads to a set $\{\mathbb{Z}[X] \to \mathbb{Z}[X, Y] \mid X, Y \in \mathrm{Fin}\}$ of compact projective generators. Let $\mathrm{AniArr}^0 \subseteq \mathrm{AniArr}$ denote the full subcategory spanned by those compact projective generators.

Definition 2.2.33. *The* cotangent complex functor $\mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod})$ *is defined to be the left derived functor (Proposition B.0.10) of the functor* $\mathrm{AniArr}^0 \to \mathrm{Ani}(\mathrm{Mod}), (A \to B) \mapsto (B, \Omega^1_{B/A})$. *The image of an object* $(A \to B) \in \mathrm{AniArr}$ *is denoted by* $(B, \mathbb{L}_{B/A})$.

Remark 2.2.34. In fact, this functor is also the animation of the functor $\mathrm{Fun}(\Delta^1, \mathrm{Ring}) \to \mathrm{Mod}$, $(A \to B) \mapsto (B, \Omega^1_{B/A})$. We do not take this as the definition since later we will apply the same idea to functors which are not defined by the animation of a functor.

Since the functor $\mathrm{AniArr}^0 \to \mathrm{Ani}(\mathrm{Mod}), B \mapsto (B, \Omega^1_{B/A})$ preserves finite coproducts, by Proposition B.0.10, we get

Lemma 2.2.35. *The cotangent complex functor* $\mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod})$ *preserves small colimits.*

Now we consider the functor $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniArr}, A \mapsto (\mathrm{id}_A : A \to A)$. This functors preserves small colimits[2.2.2], thus so does the composite functor $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniArr} \xrightarrow{\mathbb{L}_{\cdot/\cdot}} \mathrm{Ani}(\mathrm{Mod})$, concretely given by $A \mapsto (A, \mathbb{L}_{A/A})$. The next simple[2.2.3] lemma is key to the "independence of the choice of the base", which was already used in examples before:

Lemma 2.2.36. *The composite functor* $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniArr} \xrightarrow{\mathbb{L}_{\cdot/\cdot}} \mathrm{Ani}(\mathrm{Mod})$ *above coincides with the functor* $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{Ani}(\mathrm{Mod}), A \mapsto (A, 0)$.

Proof. By the colimit-preserving property above and Proposition B.0.10, it suffices to check this for polynomial rings $A = \mathbb{Z}[x_1, \ldots, x_n]$, but this follows directly from the definitions. □

We now consider the full subcategory $\mathcal{P}^0$ of $\mathrm{Fun}(\Delta^1, \mathrm{Ring})$ spanned by maps $A[X] \to A[X, Y]$ with $A \in \mathrm{Ring}$ and $X, Y \in \mathrm{Fin}$. The functor $\mathrm{Fun}(\Delta^1, \mathrm{Ring}) \to \mathrm{Ani}(\mathrm{Mod}), (A \to B) \mapsto (B, \Omega^1_{B/A})$ restricts to a functor $G : \mathcal{P}^0 \to \mathrm{Ani}(\mathrm{Mod})$. By Proposition B.0.10, the restriction F of the cotangent complex functor $\mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod})$ to the full subcategory $\mathcal{P}^0$ is left Kan extended from $\mathrm{AniArr}^0 \subseteq \mathcal{P}^0$, therefore we have a comparison map $F \to G$, which becomes an equivalence after restricting to the full subcategory AniArr^0. By Example 2.2.25, this comparison map is an equivalence since G also has the "base change property".

Now we fix a ring A, and let AniArr_A denote the ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Alg}_A))$. As before, by Corollary 2.2.13, it is projectively generated with a set $\{A[X] \to A[X, Y] \mid X, Y \in \mathrm{Fin}\}$ of compact projective generators, which spans a full subcategory $\mathrm{AniArr}^0_A \subseteq \mathrm{AniArr}_A$. Note that the functor $\mathrm{AniArr}^0_A \to \mathrm{Ani}(\mathrm{Mod}), (B \to C) \mapsto (C, \Omega^1_{C/B})$ coincides with the composite functor $\mathrm{AniArr}^0_A \to \mathcal{P}^0 \xrightarrow{G} \mathrm{Ani}(\mathrm{Mod})$, and since $F \simeq G$, this composite functor is just the cotangent complex functor applied to the underlying map of animated rings. It follows from Proposition B.0.10 that

2.2.2. In fact, this is fully faithful. However, in order to apply the same idea to later contexts, we only abstract out the colimit-preserving property.

2.2.3. We warn the reader that this lemma is not tautological.

LEMMA 2.2.37. *The composite functor* $\mathrm{AniArr}_A \to \mathrm{AniArr} \to \mathrm{Ani}(\mathrm{Mod}), (B \to C) \mapsto (C, \mathbb{L}_{C/B})$ *is equivalent to the left derived functor of* $\mathrm{AniArr}_A^0 \to \mathrm{Ani}(\mathrm{Mod}), (A[X] \to A[X, Y]) \mapsto \Omega^1_{A[X,Y]/A[X]}$.

That is to say, the definition of the cotangent complex does not depend on the choice of the base. This argument applies to similar situations, such as animated PD-envelope, and such phenomenon will appear frequently in this article.

2.2.4. ∞-category of graded and filtered objects

In this section, we recollect basic properties of the ∞-category of graded and filtered objects. Our main reference is [Rak20, §3].

The ∞-category of ($\mathbb{Z}$-)graded objects in an ∞-category $\mathcal{C}$ is the ∞-category $\mathrm{Gr}(\mathcal{C}) := \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$ of functors, where $\mathbb{Z}$ is the set of integers as an ∞-category. Given a graded object $G \in \mathrm{Gr}(\mathcal{C})$, we will denote the value of G at $i \in \mathbb{Z}$ by X^i. This defines a functor $(\cdot)^i : \mathrm{Gr}(\mathcal{C}) \to \mathcal{C}$.

When the ∞-category $\mathcal{C}$ is presentable, for all $i \in \mathbb{Z}$, the functor $(\cdot)^i$ admits a fully faithful left adjoint $\mathrm{ins}^i : \mathcal{C} \to \mathrm{Gr}(\mathcal{C})$ simply given by $X \mapsto G$ where $G^j = \begin{cases} X & j = i \\ 0_{\mathcal{C}} & \text{otherwise} \end{cases}$ where $0_{\mathcal{C}} \in \mathcal{C}$ is the initial object.

We say that a graded object $G \in \mathrm{Gr}(\mathcal{C})$ is *nonnegatively graded* (resp. *nonpositively graded*) if the restriction $F|_{\mathbb{Z}_{<0}}$ (resp. $F|_{\mathbb{Z}_{>0}}$) is constantly $0_{\mathcal{C}}$. The full subcategory spanned by nonnegatively graded (resp. nonpositively graded) objects is denoted by $\mathrm{Gr}^{\geq 0}(\mathcal{C})$ (resp. $\mathrm{Gr}^{\leq 0}(\mathcal{C})$), which is canonically equivalent to $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathcal{C})$ (resp. $\mathrm{Fun}(\mathbb{Z}_{\leq 0}, \mathcal{C})$).

Similarly, the ∞*-category of ($\mathbb{Z}$-)filtered objects* in an ∞-category $\mathcal{C}$ is the ∞-category $\mathrm{Fil}(\mathcal{C}) := \mathrm{Fun}((\mathbb{Z}, \geq), \mathcal{C})$ of functors. Given a filtered object $F \in \mathrm{Fil}(\mathcal{C})$, we will systematically denote the value of F at $i \in \mathbb{Z}$ by $\mathrm{Fil}^i F$ instead of $F(i)$ to indicate that we consider it as a filtered object. This defines a functor $\mathrm{Fil}^i : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$.

When the ∞-category $\mathcal{C}$ is presentable, for all $i \in \mathbb{Z}$, the functor Fil^i admits a fully faithful left adjoint $\mathrm{ins}^i : \mathcal{C} \to \mathrm{Fil}(\mathcal{C})$ given by the left Kan extension along $\{i\} \to (\mathbb{Z}, \geq)$. Given $X \in \mathcal{C}$, $\mathrm{Fil}^j(\mathrm{ins}^i(X)) = \begin{cases} X & j \leq i \\ 0_{\mathcal{C}} & j > i \end{cases}$ where $0_{\mathcal{C}} \in \mathcal{C}$ is the initial object.

We say that a filtered object $F \in \mathrm{Fil}(\mathcal{C})$ is *nonnegatively filtered* if the restriction $F|_{\mathbb{Z}_{\leq 0}}$ is a constant functor $(\mathbb{Z}_{\leq 0}, \geq) \to \mathcal{C}$. We denote by $\mathrm{Fil}^{\geq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$ the full subcategory spanned by nonnegatively filtered objects, which is canonically equivalent to $\mathrm{Fun}((\mathbb{Z}_{\geq 0}, \geq), \mathcal{C})$. Similarly, we say that a filtered object $F \in \mathrm{Fil}(\mathcal{C})$ is *nonpositively filtered* if the restriction $F|_{\mathbb{Z}_{>0}}$ is constantly $0_{\mathcal{C}}$. We denote by $\mathrm{Fil}^{\leq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$ the full subcategory spanned by nonpositively filtered objects, which is canonically equivalent to $\mathrm{Fun}((\mathbb{Z}_{\leq 0}, \geq), \mathcal{C})$.

Given a filtered object $F \in \mathrm{Fil}(\mathcal{C})$, the *union* $\mathrm{Fil}^{-\infty}$ is defined to be the colimit $\mathrm{colim}_{(\mathbb{Z}, \geq)} F$ (when it exists). When $\mathcal{C}$ admits all sequential colimits, this defines a functor $\mathrm{Fil}^{-\infty} : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$.

Remark 2.2.38. To avoid confusions, our filtrations are always decreasing. When we need increasing filtrations, we invert the sign to get a decreasing filtration.

Now let $(\mathcal{C}, \otimes)$ be a presentable symmetric monoidal ∞-category. Note that $\mathbb{Z}$ (resp. $(\mathbb{Z}, \geq)$) has a symmetric monoidal structure given by the addition $+$, so the ∞-category $\mathrm{Gr}(\mathcal{C})$ (resp. $\mathrm{Fil}(\mathcal{C})$) admits a presentable symmetric monoidal structure given by the *Day convolution* $\otimes^{\mathrm{Day}}$ [Nik16, §3]. Informally, given two graded (resp. filtered) objects F, G, we have $(F \otimes^{\mathrm{Day}} G)^i = \bigoplus_{j+k=i} F^j \otimes G^k$ (resp. $\mathrm{Fil}^i (F \otimes^{\mathrm{Day}} G) = \mathrm{colim}_{j+k \geq i} \mathrm{Fil}^j F \otimes \mathrm{Fil}^k G$). Under this symmetric monoidal structure, $(\cdot)^0 : \mathrm{Gr}(\mathcal{C}) \to \mathcal{C}$ (resp. $\mathrm{Fil}^0 : \mathrm{Fil}(\mathcal{C}) \to \mathcal{C}$) is lax symmetric monoidal, while the fully faithful left adjoint $\mathrm{ins}^0 : \mathcal{C} \to \mathrm{Gr}(\mathcal{C})$ (resp. $\mathcal{C} \to \mathrm{Fil}(\mathcal{C})$) is symmetric monoidal.

The stable subcategory $\mathrm{Gr}^{\geq 0}(\mathcal{C}) \subseteq \mathrm{Gr}(\mathcal{C})$ (resp. $\mathrm{Gr}^{\leq 0}(\mathcal{C}) \subseteq \mathrm{Gr}(\mathcal{C})$) inherits a presentable symmetric monoidal structure, and the 0th piece $(\cdot)^0 : \mathrm{Gr}^{\geq 0}(\mathcal{C}) \to \mathcal{C}$ (resp. $\mathrm{Gr}^{\leq 0}(\mathcal{C}) \to \mathcal{C}$) is symmetric monoidal.

Similarly, the stable subcategory $\mathrm{Fil}^{\geq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$ (resp. $\mathrm{Fil}^{\leq 0}(\mathcal{C}) \subseteq \mathrm{Fil}(\mathcal{C})$) inherits a presentable symmetric monoidal structure, and the 0th piece $\mathrm{Fil}^0 : \mathrm{Fil}^{\geq 0}(\mathcal{C}) \to \mathcal{C}$ (resp. $\mathrm{Fil}^{\leq 0}(\mathcal{C}) \to \mathcal{C}$) is symmetric monoidal.

Now we study the relation between graded objects and filtered objects. First, the symmetric monoidal functor $\mathbb{Z} \to (\mathbb{Z}, \geq)$ induces a lax symmetric monoidal functor $\mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$, which admits a symmetric monoidal left adjoint $I : \mathrm{Gr}(\mathcal{C}) \to \mathrm{Fil}(\mathcal{C})$, the left Kan extension along $\mathbb{Z} \to (\mathbb{Z}, \geq)$. Concretely, it is given by $G \mapsto F$ where $\mathrm{Fil}^i F = \coprod_{j \geq i} G^j$.

All of the functors mentioned above preserve small colimits. From now on, let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category. Then these functors are exact. Now we consider the associated graded functor $\mathrm{gr}: \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C}), F \mapsto G$ where $G^i = \mathrm{cofib}(F^{i+1} F \to \mathrm{Fil}^i F)$. It turns out that the functor gr behaves well:

PROPOSITION 2.2.39. ([LUR15, PROP 3.2.1] [GP18, PROP 2.26]) *Let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category. Then there exists a symmetric monoidal structure on the functor* $\mathrm{gr}^*: \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$. *Moreover, this symmetric monoidal structure can be chosen so that the composite functor* $\mathrm{Gr}(\mathcal{C}) \xrightarrow{I} \mathrm{Fil}(\mathcal{C}) \xrightarrow{\mathrm{gr}^*} \mathrm{Gr}(\mathcal{C})$ *is homotopic to the identity as a symmetric monoidal functor.*

We also need the *Beilinson t-structure* on the ∞-category $\mathrm{Fil}(\mathcal{C})$ of filtered objects. As before, let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a presentable stable symmetric monoidal ∞-category. Moreover, we assume that $\mathcal{C}$ admits an accessible t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that $1_{\mathcal{C}} \in \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\geq 0}$ is closed under $\otimes$.

LEMMA 2.2.40. *Under the assumptions above, the heart $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ admits a canonical symmetric monoidal structure $\otimes^{\heartsuit}$ given by $X \otimes^{\heartsuit} Y := \tau_{\leq 0}(X \otimes Y)$, and the embedding $\mathcal{C}^{\heartsuit} \to \mathcal{C}$ is then lax symmetric monoidal.*

The following is the ∞-categorical enrichment of [Bei87, App].

PROPOSITION 2.2.41. ([RAK20, PROP 3.3.11]) *Let $\mathrm{Fil}(\mathcal{C})^B_{\geq 0} \subseteq \mathrm{Fil}(\mathcal{C})$ be the full subcategory spanned by $X \in \mathrm{Fil}(\mathcal{C})$ such that $\mathrm{gr}^i(X) \in \mathcal{C}_{\geq -i}$ for all $i \in \mathbb{Z}$. Then $1_{\mathrm{ins}^0(1_{\mathcal{C}})} \in \mathrm{Fil}(\mathcal{C})^B_{\geq 0}$, $\mathrm{Fil}(\mathcal{C})^B_{\geq 0}$ is closed under $\otimes^{\mathrm{Day}}$ and is the connective part of an accessible t-structure, called the* Beilinson t-structure*, whose heart is equivalent as symmetric monoidal 1-categories to the 1-category $\mathrm{Ch}(\mathcal{C}^{\heartsuit})$ of chain complexes with "stupid" truncation $\mathrm{Fil}^i K = K_{\leq -i}$ for all $i \in \mathbb{Z}$ and $K \in \mathrm{Ch}(\mathcal{C}^{\heartsuit})$.*

In particular, when $\mathcal{C}$ is the derived ∞-category of a ring R, the *filtered derived category* $\mathrm{DF}(R)$ is the ∞-category $\mathrm{Fil}(D(R))$ of filtered objects in the derived ∞-category $D(R)$ with the symmetric monoidal structure given by the derived tensor product $\cdot \otimes^{\mathbb{L}}_R \cdot$, and $\mathrm{DF}^{\geq 0}(R)$ is the ∞-category $\mathrm{Fil}^{\geq 0}(D(R))$ of nonnegatively filtered objects in $D(R)$. In this case, we will still denote by $\cdot \otimes^{\mathbb{L}}_R \cdot$ the Day convolution.

Remark 2.2.42. ([RAK20, CONS 4.3.4]) Let R be a ring. The ∞-category $\mathrm{DF}(R)$ admits a structure of *derived algebraic context* [Rak20, Def 4.2.1], of which the derived commutative algebras are called *filtered derived (commutative) R-algebras*. When $R = \mathbb{Z}$, they are also called *filtered derived rings*. Although we will not use this fact, we might comment when a filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra admits such a structure.

We need the following lemma, which follows from the fact that left Kan extensions are pointwise colimits which preserve cofibers and filtered colimits:

LEMMA 2.2.43. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{C}^0 \subseteq \mathcal{C}$ a full subcategory, $\mathcal{E}$ a stable ∞-category which admits filtered colimits, and $\tilde{F}: \mathcal{C} \to \mathrm{Fil}(\mathcal{E})$ a functor left Kan extended along the fully faithful embedding $\mathcal{C}^0 \hookrightarrow \mathcal{C}$. Then*

1. *The composite functor $\mathrm{gr}^* \circ \tilde{F}: \mathcal{C} \to \mathrm{Fil}(\mathcal{E}) \xrightarrow{\mathrm{gr}^*} \mathrm{Gr}(\mathcal{E})$ is left Kan extended along $\mathcal{C}^0 \hookrightarrow \mathcal{C}$.*
2. *The composite functor $\mathrm{Fil}^{-\infty} \circ \tilde{F}: \mathcal{C} \to \mathrm{Fil}(\mathcal{E}) \xrightarrow{\mathrm{Fil}^{-\infty}} \mathcal{E}$ is left Kan extended along $\mathcal{C}^0 \hookrightarrow \mathcal{C}$.*

2.2.5. Reflective subcategories In this subsection, we will develop the necessary machinery to deal with the (derived) p-complete or more generally I-complete situations. We start with the general formalism of reflective subcategories.

DEFINITION 2.2.44. ([LUR09, REM 5.2.7.9 & DEF 5.2.7.2]) *Let $\mathcal{C}$ be an ∞-category. We say that a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ is* reflective *if the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ admits a left adjoint $L: \mathcal{C} \to \mathcal{D}$. In such case, we call the left adjoint $L: \mathcal{C} \to \mathcal{D}$ a* localization.

Proposition 2.2.45. ([Lur09, Prop 5.2.7.8]) *Let $\mathcal{C}$ be an ∞-category. A full subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$ is reflective if and only if for every object $C \in \mathcal{C}$, there exists an object $D \in \mathcal{C}^0$ along with a map $f: C \to D$ which induces an equivalence $\mathrm{Map}_{\mathcal{C}}(D, E) \to \mathrm{Map}_{\mathcal{C}}(C, E)$ for each object $E \in \mathcal{C}^0$ (in this case, $LC \simeq D$ where $L: \mathcal{C} \to \mathcal{C}^0$ is the localization).*

Example 2.2.46. Let $D_{\mathrm{comp}}(\mathbb{Z}_p) \subseteq D(\mathbb{Z})$ be the p-complete derived category of $\mathbb{Z}$, consisting of (derived) p-complete $\mathbb{Z}_p$-module spectra. Then $D_{\mathrm{comp}}(\mathbb{Z}_p) \subseteq D(\mathbb{Z})$ is reflective. The localization is the (derived) p-completion functor $D(\mathbb{Z}) \to D_{\mathrm{comp}}(\mathbb{Z}_p)$. Similarly, $D_{\mathrm{comp}, \geq 0}(\mathbb{Z}_p) \subseteq D_{\geq 0}(\mathbb{Z})$ is the reflective subcategory of connective p-complete $\mathbb{Z}_p$-module spectra.

Example 2.2.47. More generally, let A be an animated ring and $I \subseteq \pi_0(A)$ a finitely generated ideal. Then the I-complete derived category $D_{\mathrm{comp}}(A)$ is a reflective subcategory of the derived category $D(A)$. The same for $D_{\mathrm{comp}, \geq 0}(A) \subseteq D_{\geq 0}(A)$.

Now we study the left derived functors. Unfortunately, the localization does not in general map compact projective objects to compact projective objects. For example, $\mathbb{Z} \in D(\mathbb{Z})$ is compact and projective but $\mathbb{Z}_p \in D_{\mathrm{comp}}(\mathbb{Z}_p)$ is not. We suspect that $D_{\mathrm{comp}}(\mathbb{Z}_p)$ is not projectively generated, therefore we are probably unable to left derive “arbitrary” functors as in the projectively generated case. However, most functors in practice are good enough to have a reasonable theory of left derived functors[2.2.4]. We start with a general discussion about the interaction between localization and left Kan extension [Lur09, Def 4.3.2.2].

Setup 2.2.48. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a reflective (full) subcategory with the localization $L: \mathcal{C} \to \mathcal{D}$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory, $\mathcal{D}^0 \subseteq \mathcal{D}$ the full subcategory spanned by objects LC where C runs through all objects in $\mathcal{C}^0$. Let $\mathcal{C}^1 \subseteq \mathcal{C}$ be the full subcategory spanned by vertices of both $\mathcal{C}^0$ and $\mathcal{D}^0$.*

It follows from definitions that

Lemma 2.2.49. *In Setup 2.2.48, $\mathcal{D}^0 \subseteq \mathcal{C}^1$ is a reflective subcategory with localization $L|_{\mathcal{C}^1}: \mathcal{C}^1 \to \mathcal{D}^0$ being the restriction of $L: \mathcal{C} \to \mathcal{D}$.*

Example 2.2.50. One of the crucial example for the setup above: $\mathcal{C}$ is the ∞-category of animated rings, $\mathcal{D}$ is the full subcategory of p-complete animated $\mathbb{Z}_p$-algebras, and $\mathcal{C}^0 \subseteq \mathcal{C}$ is the full subcategory spanned by polynomial rings $\mathbb{Z}[X_1, \ldots, X_n]$. More generally, let A be an animated ring and $I \subseteq \pi_0(A)$ a finitely generated ideal. Then we can consider the case that $\mathcal{C}$ is the ∞-category of animated A-algebras and $\mathcal{D} \subseteq \mathcal{C}$ is the full subcategory of I-complete animated A-algebras, and $\mathcal{C}^0 \subseteq \mathcal{C}$ is the full subcategory spanned by polynomial A-algebras $A[X_1, \ldots, X_n] := \mathbb{Z}[X_1, \ldots, X_n] \otimes_{\mathbb{Z}}^{\mathbb{L}} A$.

Lemma 2.2.51. *In Setup 2.2.48, let $\mathcal{E}$ be an ∞-category and $\tilde{F}: \mathcal{C} \to \mathcal{E}$ a functor left Kan extended from the fully faithful embedding $\mathcal{C}^0 \hookrightarrow \mathcal{C}$. Then the restriction $\widetilde{F}|_{\mathcal{D}}$ is left Kan extended from the fully faithful embedding $\mathcal{D}^0 \hookrightarrow \mathcal{D}$.*

Proof. It follows from [Lur09, Lem 5.2.6.6] that the restriction $\tilde{F}|_{\mathcal{D}}$ is a left Kan extension of $\tilde{F}$ along $L: \mathcal{C} \to \mathcal{D}$, therefore is left Kan extended from the composite functor $\mathcal{C}^0 \hookrightarrow \mathcal{C} \xrightarrow{L} \mathcal{D}$. The composite functor $\mathcal{C}^0 \hookrightarrow \mathcal{C} \to \mathcal{D}$ could be rewritten as the composite $\mathcal{C}^0 \xrightarrow{L} \mathcal{D}^0 \hookrightarrow \mathcal{D}$, therefore $\tilde{F}|_{\mathcal{D}}$ is left Kan extended from $\mathcal{D}^0 \hookrightarrow \mathcal{D}$. $\square$

Example 2.2.52. In Example 2.2.50, the cotangent complex $\mathbb{L}_{\cdot/\mathbb{Z}}: \mathcal{C} = \mathrm{Ani}(\mathrm{Ring}) \to D(\mathbb{Z})$ is left Kan extended from $\mathrm{Poly}_{\mathbb{Z}} \subseteq \mathrm{Ring}$. Consequently, the restriction $\mathbb{L}_{\cdot/\mathbb{Z}}|_{\mathcal{D}}: \mathcal{D} \to D(\mathbb{Z})$ is left Kan extended from p-completed polynomial rings. Similarly, the p-completed cotangent complex $(\mathbb{L}_{\cdot/\mathbb{Z}})_p^\wedge: \mathrm{Ani}(\mathrm{Ring}) \to D_{\mathrm{comp}}(\mathbb{Z}_p)$ is left extended from $\mathrm{Poly}_{\mathbb{Z}} \subseteq \mathrm{Ring}$, therefore the restriction $(\mathbb{L}_{\cdot/\mathbb{Z}})_p^\wedge|_{\mathcal{D}}: \mathcal{D} \to D_{\mathrm{comp}}(\mathbb{Z}_p)$ is left extended from p-completed polynomial rings.

2.2.4. This approach is essentially depicted in the special case of p-completed rings in Bhatt's Eilenberg Lectures notes [Bha18, Lecture VII]. We are informed by Yu Min of this approach in private discussions.

SETUP 2.2.53. *In Setup 2.2.48, let $F:\mathcal{D}^0\to\mathcal{E}$ be a functor equipped with a left Kan extension $\tilde{F}:\mathcal{C}\to\mathcal{E}$ along the fully faithful inclusion $\mathcal{C}^0\hookrightarrow\mathcal{C}$ of the composite functor $\mathcal{C}^0\xrightarrow{L}\mathcal{D}^0\xrightarrow{F}\mathcal{E}$.*

Remark 2.2.54. In our applications, $\mathcal{C}$ will be a projectively generated ∞-category (Definition B.0.3) with a set S of compact projective generators. We will choose $\mathcal{C}^0\subseteq\mathcal{C}$ to be the full subcategory spanned by finite coproducts of objects in S, and $\mathcal{E}$ will be a cocomplete ∞-category. In this case, the left Kan extension in question always exists (Propositions B.0.10 and B.0.12). More generally, if $\mathcal{C}^0$ is a small subcategory and that $\mathcal{C}$ is assumed to be locally small, then the left Kan extension exists.

In Setup 2.2.53, we first assume without loss of generality that $L|_{\mathcal{D}}=\mathrm{id}_{\mathcal{D}}$ by [Lur09, Prop 5.2.7.4], then $L^2=L$. Now we let $F_1:\mathcal{C}^1\to\mathcal{E}$ denote the composite $\mathcal{C}^1\to\mathcal{D}^0\xrightarrow{F}\mathcal{E}$, which is an extension of the composite $\mathcal{C}^0\to\mathcal{D}^0\xrightarrow{F}\mathcal{E}$ along $\mathcal{C}^0\to\mathcal{C}^1$. Since $\tilde{F}|_{\mathcal{C}^1}:\mathcal{C}^1\to\mathcal{E}$ is, by definition, a left Kan extension of $\mathcal{C}^0\to\mathcal{D}^0\xrightarrow{F}\mathcal{E}$ along $\mathcal{C}^0\to\mathcal{C}^1$, there exists an essentially unique comparison map $\tilde{F}|_{\mathcal{C}^1}\to F_1$ of functors $\mathcal{C}^1\rightrightarrows\mathcal{E}$. Restricting to the full subcategory $\mathcal{D}^0\subseteq\mathcal{C}^1$, we get a comparison map $\tilde{F}|_{\mathcal{D}^0}\to F$. It follows from Lemma 2.2.51 that

COROLLARY 2.2.55. *In Setup 2.2.53, if we assume that the comparison map $\tilde{F}|_{\mathcal{D}^0}\to F$ is an equivalence, then $\tilde{F}|_{\mathcal{D}}$ is the left Kan extension of F along the fully faithful embedding $\mathcal{D}^0\hookrightarrow\mathcal{D}$.*

We need the following concept:

PROPOSITION 2.2.56. ([LUR09, PROP 5.2.7.12]) *Let $\mathcal{C}$ be an ∞-category and let $L:\mathcal{C}\to L\mathcal{C}\subseteq\mathcal{C}$ be a localization functor. Let S denote the collection of all morphisms f in $\mathcal{C}$ such that Lf is an equivalence. Then for every ∞-category $\mathcal{D}$, composition with L induces a fully faithful functor $\psi:\mathrm{Fun}(L\mathcal{C},\mathcal{D})\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$. Moreover, the essential image of ψ consists of those functors $F:\mathcal{C}\to\mathcal{D}$ such that $F(f)$ is an equivalence in $\mathcal{D}$ for each $f\in S$.*

DEFINITION 2.2.57. *Let $\mathcal{C}$ be an ∞-category, $L:\mathcal{C}\to L\mathcal{C}\subseteq\mathcal{C}$ a localization functor and $\mathcal{D}$ an ∞-category. We say that a functor $F:\mathcal{C}\to\mathcal{D}$ is* L-invariant *if for every morphism f in $\mathcal{C}$ such that Lf is an equivalence, then so is $F(f)$ in $\mathcal{D}$.*

Now we come back to our previous discussion.

LEMMA 2.2.58. *Under the above discussion, consider the following conditions:*

i. *The left Kan extension $\tilde{F}:\mathcal{C}\to\mathcal{E}$ is L-invariant.*

ii. *The comparison map $\tilde{F}|_{\mathcal{C}^1}\to F_1$ constructed above is an equivalence.*

iii. *The comparison map $\tilde{F}|_{\mathcal{D}^1}\to F$ constructed above is an equivalence.*

We have

1. *Conditions ii and iii are equivalent.*
2. *Condition i implies condition iii.*
3. *Under the assumptions in Remark 2.2.54, condition ii implies condition i.*

Proof. First, restricting the comparison map $\tilde{F}|_{\mathcal{C}^1}\to F_1$ to $\mathcal{C}^0$, we get the identity, so conditions ii and iii are equivalent.

If $\tilde{F}$ is L-invariant, then for all $X\in\mathcal{C}^1$, the unit map $X\to LX$ induces a commutative diagram

$$\begin{array}{ccc}\tilde{F}(X) & \longrightarrow & \tilde{F}(LX)\\ \downarrow & & \downarrow\\ F_1(X) & \longrightarrow & F_1(LX)\end{array}$$

with the horizontal maps being equivalences. In particular, for all $Y\in\mathcal{D}^0$, there exists $X\in\mathcal{C}^0$ such that $Y\simeq LX$. Then $\tilde{F}(X)\to F_1(X)$ is an equivalence, therefore so are $\tilde{F}(LX)\to F_1(LX)$ and $\tilde{F}(Y)\to F_1(Y)$, which proves condition ii.

We now assume that we are in the special case described in Remark 2.2.54. Suppose that condition ii holds. Note that F_1 is, by definition, L-invariant, therefore for all $X \in \mathcal{C}^0$, $\tilde{F}$ maps the unit map $X \to LX$ to an equivalence. Let $\mathcal{C}' \subseteq \mathcal{C}$ be the full subcategory spanned by those $X \in \mathcal{C}$ such that $\tilde{F}$ maps $X \to LX$ to an equivalence. Then $\mathcal{C}^0 \subseteq \mathcal{C}'$. It follows from Propositions B.0.10 and B.0.12 that $\tilde{F}$ preserves sifted colimits. Since L preserves small colimits, $\mathcal{C}'$ is closed under sifted colimits, therefore $\mathcal{C}' = \mathcal{C}$ by Lemma B.0.9. □

Remark 2.2.59. We conjecture that all conditions in Lemma 2.2.58 are equivalent under Setup 2.2.53 without the assumptions in Remark 2.2.54.

Now we describe how the setups above give rise to derived prismatic cohomology in [BS19]. Let (A, I) be a bounded prism [BS19, Def 3.2]. Let $\mathcal{C} = \mathrm{Ani}(\mathrm{Alg}_{A/I})$ be the ∞-category of A/I-algebras and $\mathcal{D} \subseteq \mathcal{C}$ the full subcategory of p-completed A/I-algebras. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory of polynomial A/I-algebras. Then $\mathcal{D}^0 \subseteq \mathcal{D}$ is the full subcategory of p-completed polynomial A/I-algebras. [BS19, §4.2] defines the functors $F := \mathbb{\Delta}_{\cdot/A} : \mathcal{D}^0 \to D_{\mathrm{comp}}(A)$ and $G := \overline{\mathbb{\Delta}}_{\cdot/A} : \mathcal{D}^0 \to D_{\mathrm{comp}}(A/I)$, where $D_{\mathrm{comp}}(A)$ is the ∞-category of (p, I)-complete A-module spectra, and $D_{\mathrm{comp}}(A/I)$ is the ∞-category of p-complete A/I-module spectra. In Setup 2.2.53 and Remark 2.2.54, we claim that the functor $\tilde{F}$ and $\tilde{G}$ are left Kan extended from $\mathcal{D}^0$ after restriction to $\mathcal{D}$. That is to say, $\tilde{F}$ and $\tilde{G}$ are left derived functors $L\,\mathbb{\Delta}_{\cdot/A}$ and $L\,\overline{\mathbb{\Delta}}_{\cdot/A}$ defined in [BS19, Cons 7.6]. Thanks to Lemma 2.2.58, it suffices to show that $\tilde{F}$ and $\tilde{G}$ are L-invariant. We will first describe our proof, then we offer the lemmas used in the proof.

We start with $\tilde{G}$. Composing G with the Postnikov tower $D_{\mathrm{comp}}(A/I) \to \mathrm{DF}_{\mathrm{comp}}(A/I), X \mapsto (\tau_{\geq n} X)_{n \in (\mathbb{Z}, \geq)}$ where $\mathrm{DF}_{\mathrm{comp}}(A/I) := \mathrm{Fil}(D_{\mathrm{comp}}(A/I))$ is the filtered derived ∞-category of p-completed A/I-module spectra, we get a functor $G^P : \mathcal{D}^0 \to \mathrm{DF}_{\mathrm{comp}}(A/I)$ such that the *union* (see Corollary 2.2.61) $\mathrm{Fil}^{-\infty} G^P : \mathcal{D}^0 \to D_{\mathrm{comp}}(A/I)$ is equivalent to G. It follows from the Hodge–Tate comparison [BS19, Prop 6.2] that the functorial comparison map $(\bigwedge^i \mathbb{L}_{\cdot/(A/I)}\{-i\}[-i])^\wedge_p \to \mathrm{gr}^{-i} \circ G^P$ is an equivalence). Now Remark 2.2.54 shows that $G^P : \mathcal{D}^0 \to \mathrm{DF}_{\mathrm{comp}}(A/I)$ gives rise to $\widetilde{G^P} : \mathcal{C} \to \mathrm{DF}_{\mathrm{comp}}(A/I)$ and the functor $(\bigwedge^i \mathbb{L}_{\cdot/(A/I)}\{-i\}[-i])^\wedge_p : \mathcal{D}^0 \to \mathrm{DF}_{\mathrm{comp}}(A/I)$ gives rise to some $\mathcal{C} \to \mathrm{DF}_{\mathrm{comp}}(A/I)$, which is $(\bigwedge^i \mathbb{L}_{\cdot/(A/I)}\{-i\}[-i])^\wedge_p$ by Example 2.2.52, and in particular, L-invariant. It follows from Lemma 2.2.43 that the associated graded pieces $\mathrm{gr}^{-i} \circ \widetilde{G^P}$ are L-invariant and therefore the L-invariance of $\tilde{G}$ follows from Corollary 2.2.61.

Note that $\tilde{F}$ coincides with $\tilde{G}$ composed with the derived modulo I, that is, the composite functor $\mathrm{Ani}(\mathrm{Alg}_{A/I}) \xrightarrow{\tilde{F}} D_{\mathrm{comp}}(A) \xrightarrow{\cdot \hat{\otimes}^{\mathbb{L}}_A (A/I)} D_{\mathrm{comp}}(A/I)$. We deduce by derived Nakayama [Sta21, Tag 0G1U] that $\tilde{F}$ is also L-invariant.

Here are the lemmas that we used in the argument above:

Lemma 2.2.60. *Let $\mathcal{C}$ be an ∞-category and $\mathcal{D} \subseteq \mathcal{C}$ a reflective subcategory with localization $L : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{E}$ be a stable ∞-category. Let $F : \mathcal{C} \to \mathrm{Fil}^{\leq 0}(\mathcal{E})$ be a functor. If the* associated graded pieces *$\mathrm{gr}^i \circ \tilde{F}$ are L-invariant for all $i \in \mathbb{Z}$, then so is $\tilde{F}$.*

Proof. For all $C \in \mathcal{C}$, we inductively show that the unit map $C \to LC$ induces an equivalence $\mathrm{Fil}^i(\tilde{F}(C)) \to \mathrm{Fil}^i(\tilde{F}(LC))$. By assumption, this is true for all $i > 0$. Now consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Fil}^{i+1}(\tilde{F}(C)) & \longrightarrow & \mathrm{Fil}^i(\tilde{F}(C)) & \longrightarrow & \mathrm{gr}^i(\tilde{F}(C)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Fil}^{i+1}(\tilde{F}(LC)) & \longrightarrow & \mathrm{Fil}^i(\tilde{F}(LC)) & \longrightarrow & \mathrm{gr}^i(\tilde{F}(LC))
\end{array}$$

where the horizontal maps are fiber sequences. Suppose that the result is true for $i + 1$. Then the leftmost and the rightmost vertical maps are equivalences, therefore so is the middle vertical maps, which shows that the result is true for i. □

It then follows from definitions that

Corollary 2.2.61. *Under the assumptions in Lemma 2.2.60, if we further assume that $\mathcal{E}$ admits filtered colimits, then the* union *$\mathrm{Fil}^{-\infty} \circ \tilde{F} : \mathcal{C} \to \mathrm{Fil}(\mathcal{E}) \xrightarrow{\mathrm{Fil}^{-\infty}} \mathcal{E}$ is also L-invariant.*

2.3. Animated ideals and PD-pairs

In this section, we will first give an informal exposition of Smith ideals introduced in [Hov14] in terms of ∞-categories. See also [WY17b, WY17a] for various generalizations. Then we will show how to apply these ideas to define and study "ideals" of animated rings and animated PD-pairs, which are the cornerstones of the animated theory of crystalline cohomology.

2.3.1. Smith ideals

We fix a presentable stable symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$. See Appendix C for this terminology. The reader should feel free to take the special case that $\mathcal{C} = \mathrm{Sp}$ is the ∞-category of spectra and $\otimes$ is the smash product of spectra.

Consider the 1-simplex Δ^1, which is simply the 1-category associated to the ordinal $[1] = \{0 < 1\}$. The opposite category $(\Delta^1)^{\mathrm{op}}$ has a symmetric monoidal structure given by $\max\{\cdot, \cdot\}$[2.3.1].

Thus the presentable stable ∞-category $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$ admits a presentable symmetric monoidal structure given by the *Day convolution* $\otimes^{\mathrm{Day}}$ [Nik16, §3].

Informally, the unit object $\mathbf{1}_{\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})}$ is given by $(\mathbf{1}_{\mathcal{C}} \leftarrow 0) \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, and given n functors $F_1, \ldots, F_n \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, the Day convolution $F_1 \otimes^{\mathrm{Day}} \cdots \otimes^{\mathrm{Day}} F_n$ is given as follows: $F_1, \ldots, F_n$ determines an n-cube $F: (\Delta^1)^{\mathrm{op}} \times \cdots \times (\Delta^1)^{\mathrm{op}} \to \mathcal{C}, (e_1, \ldots, e_n) \mapsto F_1(e_1) \otimes \cdots \otimes F_n(e_n)$. This cube, except the final vertex, determines a "cubical pushout" mapping to the final vertex: $(F(0, \ldots, 0) \leftarrow \mathrm{colim}_{(\Delta^1)^{\mathrm{op}} \times \cdots \times (\Delta^1)^{\mathrm{op}} \setminus (0, \ldots, 0)} F)$, which is $F_1 \otimes^{\mathrm{Day}} \cdots \otimes^{\mathrm{Day}} F_n$.

In particular, when $n = 2$, the Day convolution of $(X_0 \leftarrow X_1)$ and $(Y_0 \leftarrow Y_1)$ is given by $(X_0 \otimes Y_0 \leftarrow (X_0 \otimes Y_1) \amalg_{X_1 \otimes Y_1} (X_1 \otimes Y_0))$. This is essentially equivalent to the *pushout product monoidal structure* in [Hov14, Thm 1.2].

On the other hand, there exists a pointwise symmetric monoidal structure $\otimes$ on the stable ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$ where $F_1 \otimes \cdots \otimes F_n$ is given by the functor $e \mapsto F_1(e) \otimes \cdots \otimes F_n(e)$.

There is a comparison between these two stable symmetric monoidal ∞-categories:

Proposition 2.3.1. *There is an equivalence* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})$ *of presentable stable symmetric monoidal ∞-categories. The equivalence is given by* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \ni F \mapsto (F(0) \to \mathrm{cofib}(F(1) \to F(0))) \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ *of which the inverse is given by* $\mathrm{Fun}(\Delta^1, \mathcal{C}) \ni G \mapsto (G(0) \leftarrow \mathrm{fib}(G(0) \to G(1))) \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$.

Proof. The pair of inverse functors are clearly well-defined and exact. It remains to show that the functor $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \ni F \mapsto (F(0) \to \mathrm{cofib}(F(1) \to F(0))) \in \mathrm{Fun}(\Delta^1, \mathcal{C})$ is symmetric monoidal and so is its inverse. We give an informal argument for the first as follows:

Given n functors $F_1, \ldots, F_n \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, as previous, let F denote the n-cube $((\Delta^1)^{\mathrm{op}})^{\times n} \to \mathcal{C}$, $(e_1, \ldots, e_n) \mapsto F_1(e_1) \otimes \cdots \otimes F_n(e_n)$. The cofiber of the Day convolution $(F(0, \ldots, 0) \leftarrow \mathrm{colim}_{(\Delta^1)^{\mathrm{op}} \times \cdots \times (\Delta^1)^{\mathrm{op}} \setminus (0, \ldots, 0)} F)$ is the total cofiber of the cube F, which could be computed inductively in each direction, and since the tensor product $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite colimits separately in each variable, one can inductively show that the total cofiber is the tensor product of cofibers. □

Now we assume that $\mathcal{C}$ admits a symmetric monoidal t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ (see Definition C.0.5). This is the case when $\mathcal{C} = \mathrm{Sp}$ and $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ is the canonical t-structure for spectra. Then so does $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$, that is to say, $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})_{\geq 0} := \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}_{\geq 0})$ and $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})_{\leq 0} := \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}_{\leq 0})$. Transferring this t-structure along the equivalence in Proposition 2.3.1, we get a t-structure on $\mathrm{Fun}(\Delta^1, \mathcal{C})$ where $\mathrm{Fun}(\Delta^1, \mathcal{C})_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ is spanned by edges $f: X \to Y$ in $\mathcal{C}$ such that $X \in \mathcal{C}_{\geq 0}$ and $\mathrm{fib}(Y \to X) \in \mathcal{C}_{\geq 0}$, or equivalently, $X, Y \in \mathcal{C}_{\geq 0}$ and f is 1-connective (that is to say, $\pi_0(f)$ is surjective). In summary,

Corollary 2.3.2. *The equivalence in Proposition 2.3.1 induces an equivalence of presentable symmetric monoidal full subcategories* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}_{\geq 0}) \simeq \mathrm{Fun}(\Delta^1, \mathcal{C})_{\geq 0}$, *where the full subcategory* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})_{\geq 0}$ *is spanned by maps* $Y \leftarrow X$ *in* $\mathcal{C}_{\geq 0}$.

2.3.1. This is informed to us by Denis Nardin.

Passing to $\mathbb{E}_n$-algebras for any $n \in \mathbb{N} \cup \{\infty\}$, we get

Corollary 2.3.3. *There is an equivalence between the* ∞*-category of* $\mathbb{E}_n$*-algebras in* $(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}), \otimes^{\mathrm{Day}})$ *and the* ∞*-category of* $\mathbb{E}_n$*-maps between* $\mathbb{E}_n$*-algebras in* $(\mathcal{C}, \otimes)$*. This equivalence induces an equivalence between the full subcategory spanned by connective* $\mathbb{E}_n$*-algebras in* $(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}), \otimes^{\mathrm{Day}})$ *and the full subcategory spanned by* 1*-connective* $\mathbb{E}_n$*-maps between connective* $\mathbb{E}_n$*-algebras in* $(\mathcal{C}, \otimes)$*.*

Explicitly, for any $\mathbb{E}_n$-algebra in $((\mathrm{Fun}(\Delta^1)^{\mathrm{op}}, \mathcal{C}), \otimes^{\mathrm{Day}})$ of which the underlying object is $(A \leftarrow I)$, the object $A \in \mathcal{C}$, the cofiber $\mathrm{cofib}(I \to A)$ and the map $A \to \mathrm{cofib}(I \to A)$ admit canonical $\mathbb{E}_n$-algebra structures. We can then understand I as an "ideal" of $\mathbb{E}_n$-algebra A. When A is connective, the previous identification also describes connective "ideals" of A. This is the Smith ideal in [Hov14], which gives rise to a reasonable theory of ideals (resp. connective ideals) of $\mathbb{E}_n$-rings (resp. connective $\mathbb{E}_n$-rings) when $\mathcal{C}$ is the presentable stable symmetric monoidal ∞-category Sp of spectra.

In the rest of this section, we will study the animated analogue of the preceding equivalence, that is to say, "ideals" and "PD-ideals" of an animated ring. To do so, we need to exploit more structures of $D(\mathbb{Z})$.

2.3.2. Animated (PD-)pairs

In this subsection, we introduce the central object of this section: animated pairs and (absolute) animated PD-pairs.

Let Pair denote the 1-category of ring-ideal pairs (A, I), that is, a (commutative) ring A along with an ideal $I \subseteq A$. Let PDPair denote the 1-category of divided power rings (A, I, γ) [Sta21, Tag 07GU]. We recall that the forgetful functor $\mathrm{PDPair} \to \mathrm{Pair}$ admits a left adjoint, called the *(absolute) PD-envelope functor* [Sta21, Tag 07H9]. The PD-envelope of (A, I) will be denoted by $D_A(I) \in \mathrm{PDPair}$.

Let $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ be the full subcategory spanned by injective maps $M \hookleftarrow M'$. We note that there is a pair $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \rightleftarrows \mathrm{Pair}$ of adjoint functors where $\mathrm{Pair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ is the forgetful functor $(A, I) \mapsto (A \leftarrow I)$, and $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Pair}$ is the "symmetric product" $(M \hookleftarrow M') \mapsto (\mathrm{Sym}_{\mathbb{Z}}(M), M' \mathrm{Sym}_{\mathbb{Z}}(M))$ where $\mathrm{Sym}_{\mathbb{Z}}(M) \hookleftarrow M' \mathrm{Sym}_{\mathbb{Z}}(M)$ is the ideal generated by elements in M'.

Unfortunately, $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$ does not seem to be 1-projectively generated. In particular, we cannot apply Corollary 2.2.3 to deduce that Pair is 1-projectively generated (we believe that it is not), and to construct "Ani(Pair)". In fact, we need to embed Pair as a full subcategory of a 1-projectively generated 1-category and then the ∞-category of animated pairs coincides with the animation of that larger 1-category.

We begin by analyzing the full subcategory $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$. Note that $\{\mathbb{Z} \leftarrow 0, \mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z}\} \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$ is a set of compact 1-projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ by Lemma 2.2.11. Let $\mathcal{C}^0 \subseteq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ denote the finite coproducts of objects in $\{\mathbb{Z} \leftarrow 0, \mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z}\}$, taken in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$, which is effectively a full subcategory of $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$. It follows from Proposition B.0.29 that there is an equivalence $\mathcal{P}_{\Sigma,1}(\mathcal{C}^0) \xrightarrow{\simeq} \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ of ∞-categories. It then follows from Lemma 2.2.1 that the fully faithful embedding $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathcal{C}^0)$ admits a left adjoint given by the left derived functor of $\mathcal{C}^0 \hookrightarrow \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$. We claim that

Lemma 2.3.4. *The essential image of* $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathcal{C}^0)$ *is spanned by those finite-product-preserving functors* $F : (\mathcal{C}^0)^{\mathrm{op}} \to \mathrm{Set}$ *which maps the edge* $(\mathbb{Z} \leftarrow 0) \to (\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z})$ *in* $\mathcal{C}^0$ *to an injective map of sets.*

Proof. The functors $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab}) \rightrightarrows \mathrm{Set}$ corepresented by $\mathrm{id}_{\mathbb{Z}} \in \mathcal{C}^0$ and $(\mathbb{Z} \leftarrow 0) \in \mathcal{C}^0$ are given by $(A \leftarrow A') \mapsto A'$ and $(A \leftarrow A') \mapsto A$ respectively, and the edge $(\mathbb{Z} \leftarrow 0) \to (\mathrm{id}_{\mathbb{Z}} : \mathbb{Z} \leftarrow \mathbb{Z})$ gives rise to the natural map $A \leftarrow A'$ of the two functors. It follows that an object $F \in \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ lies in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$ if and only if the value of the natural map on F is an injection. The result then follows from the equivalence $\mathcal{P}_{\Sigma,1}(\mathcal{C}^0) \xrightarrow{\simeq} \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$. $\square$

Let $\mathcal{D}^0 \subseteq \mathrm{Pair}$ denote the full subcategory spanned by images of $\mathcal{C}^0$ under the functor $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \to \mathrm{Pair}$, concretely spanned by pairs of the form $(\mathbb{Z}[X,Y],(Y))$ Then by Corollary 2.2.2,

LEMMA 2.3.5. *The essentially surjective functor $\mathcal{C}^0 \to \mathcal{D}^0$ gives rise to the* forgetful functor $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ *which is conservative and preserves sifted colimits.*

Lemma 2.2.1 gives us a canonical pair of adjoint functors $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \rightleftarrows \mathrm{Pair}$, where $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \to \mathrm{Pair}$ is the left derived 1-functor (Proposition B.0.27) of the inclusion $\mathcal{D}^0 \hookrightarrow \mathrm{Pair}$, and $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ is the given by the restricted Yoneda embedding $(A,I) \mapsto \mathrm{Hom}_{\mathrm{Pair}}(\cdot,(A,I))$. We first note that the forgetful functors are compatible:

LEMMA 2.3.6. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Pair} \arrow[r] \arrow[d] & \mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \arrow[d] \\
\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \arrow[r, hook] & \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})
\end{tikzcd}$$

of 1-categories.

Proof. Given a pair $(A,I) \in \mathrm{Pair}$, the image in $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ is given by $\mathcal{D}^0 \ni (B,J) \mapsto \mathrm{Hom}_{\mathrm{Pair}}((B,J),(A,I))$, subsequently mapped to $\mathcal{C}^0 \ni (M \hookleftarrow M') \mapsto \mathrm{Hom}_{\mathrm{Pair}}((\mathrm{Sym}_{\mathbb{Z}}(M), M' \,\mathrm{Sym}_{\mathbb{Z}}(M)),(A,I)) \cong \mathrm{Hom}_{\mathrm{Fun}((\Delta^1)^{\mathrm{op}},\mathrm{Ab})}(M \hookleftarrow M', A \hookleftarrow I)$. The other composite is the same. This identification is functorial in (A,I). $\square$

Now we show that $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ is an embedding to a 1-projectively generated 1-category that we want. The trick is to talk about maps $(\mathbb{Z}[X],0) \to (A,I)$ and $(\mathbb{Z}[X],(X)) \to (A,I)$ in place of elements in A and I respectively to do certain "element chasing". We remind the reader that $\mathrm{Poly}_{\mathbb{Z}}$ is a set of compact projective objects for Ring, which gives rise to an equivalence $\mathcal{P}_{\Sigma,1}(\mathrm{Poly}_{\mathbb{Z}}) \simeq \mathrm{Ring}$ of 1-categories, where $\mathrm{Poly}_{\mathbb{Z}}$ is the 1-category of polynomial rings.

LEMMA 2.3.7. *The functor* $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ *is fully faithful.*

Proof. The faithfulness follows from Lemma 2.3.6 and the faithfulness of the forgetful functor $\mathrm{Pair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$. Given two pairs $(A,I),(B,J)$ in Pair and a natural map

$$\mathrm{Hom}_{\mathrm{Pair}}(\cdot,(A,I))|_{(\mathcal{D}^0)^{\mathrm{op}}} \to \mathrm{Hom}_{\mathrm{Pair}}(\cdot,(B,J))|_{(\mathcal{D}^0)^{\mathrm{op}}}$$

of finite-product-preserving functors $(\mathcal{D}^0)^{\mathrm{op}} \rightrightarrows \mathrm{Set}$, we need to show that this is induced by some map $(A,I) \to (B,J)$ of pairs.

By Lemma 2.3.6, there exists a unique map $(A \hookleftarrow I) \to (B \hookleftarrow J)$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$ which corresponds to the natural transform after composition $(\mathcal{C}^0)^{\mathrm{op}} \to (\mathcal{D}^0)^{\mathrm{op}} \rightrightarrows \mathrm{Set}$.

Similarly, since $\mathcal{P}_{\Sigma,1}(\mathrm{Poly}_{\mathbb{Z}}) \simeq \mathrm{Ring}$, there exists a unique map $A \to B$ of rings which corresponds to the natural transform after composition $\mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}} \to (\mathcal{D}^0)^{\mathrm{op}} \rightrightarrows \mathrm{Set}$ where $\mathrm{Poly}_{\mathbb{Z}} \to \mathcal{D}^0$ is given by $R \mapsto (R,0)$.

It then follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Free}_{\mathbb{Z}}^{\mathrm{fin}} \arrow[r, "\mathrm{Sym}"] \arrow[d] & \mathrm{Poly}_{\mathbb{Z}} \arrow[d] \\
\mathcal{C}^0 \arrow[r] & \mathcal{D}^0
\end{tikzcd}$$

Figure 2.3.1

of 1-categories, where $\mathrm{Free}_{\mathbb{Z}}^{\mathrm{fin}}$ is the 1-category of finite free abelian groups, with finite-coproduct-preserving functors that the two maps $(A \hookleftarrow I) \to (B \hookleftarrow J)$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}$ and $A \to B$ in Ring are compatible, which gives rise to a map $(A,I) \to (B,J)$ in Pair. $\square$

Now we characterize the image of this embedding:

Lemma 2.3.8. *The square in Lemma 2.3.6 is Cartesian. Equivalently by Lemma 2.3.4, the essential image of the fully faithful functor* $\mathrm{Pair}\hookrightarrow\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ *is spanned by those finite-product-preserving functors* $F:(\mathcal{D}^0)^{\mathrm{op}}\to\mathrm{Set}$ *which maps the edge* $(\mathbb{Z}[X],0)\to(\mathbb{Z}[X],(X))$ *in* $\mathcal{D}^0$ *to an injective map of sets.*

Proof. Let $F:(\mathcal{D}^0)^{\mathrm{op}}\to\mathrm{Set}$ be a functor which preserves finite products such that the composite $(\mathcal{C}^0)^{\mathrm{op}}\to(\mathcal{D}^0)^{\mathrm{op}}\xrightarrow{F}\mathrm{Set}$ belongs to the full subcategory $\mathrm{Fun}((\Delta^1)^{\mathrm{op}},\mathrm{Ab})_{\mathrm{inj}}$ under the identification $\mathcal{P}_{\Sigma}(\mathcal{C}^0)\simeq\mathrm{Fun}((\Delta^1)^{\mathrm{op}},\mathrm{Ab})$. The goal is to show that there exists a pair $(A,I)\in\mathrm{Pair}$ which represents F.

Let $(A\hookleftarrow I)\in\mathrm{Fun}((\Delta^1)^{\mathrm{op}},\mathrm{Ab})_{\mathrm{inj}}$ correspond to the composite functor $(\mathcal{C}^0)^{\mathrm{op}}\to(\mathcal{D}^0)^{\mathrm{op}}\xrightarrow{F}\mathrm{Set}$, and the map $A\hookleftarrow I$ of underlying sets is precisely induced by the map $(\mathbb{Z}[X],0)\to(\mathbb{Z}[X],(X))$ in $\mathcal{D}^0$.

The ring structure is given as follows: the functor $\mathrm{Poly}_{\mathbb{Z}}\to\mathcal{D}^0$ given by $R\mapsto(R,0)$ preserves finite coproducts, thus the composite functor $\mathrm{Poly}_{\mathbb{Z}}^{\mathrm{op}}\to(\mathcal{D}^0)^{\mathrm{op}}\xrightarrow{F}\mathrm{Set}$ preserves finite products, which corresponds to a ring structure on A. The compatibility follows from Figure 2.3.1.

Now we show that $A\hookleftarrow I$ is an ideal, that is to say, the ring multiplication $A\times A\to A$ restricts to a map $A\times I\to I$. By the above construction, $A=F(\mathbb{Z}[Y],0)$ and $I=F(\mathbb{Z}[X],(X))$, and since F preserves finite products, $A\times I=F(\mathbb{Z}[X,Y],(X))$. Consider $(\mathbb{Z}[T],(T))\in\mathcal{D}^0$. The map $(\mathbb{Z}[T],(T))\to(\mathbb{Z}[X,Y],(X)),T\mapsto XY$ in $\mathcal{D}^0$ induces a map $A\times I\to I$. The commutative diagram

$$\begin{array}{ccc}(\mathbb{Z}[X,Y],0) & \longrightarrow & (\mathbb{Z}[T],0)\\ \downarrow & & \downarrow\\ (\mathbb{Z}[X,Y],(X)) & \longrightarrow & (\mathbb{Z}[T],(T))\end{array}$$

in $\mathcal{D}^0$ shows that the preceding map $A\times I\to I$ is compatible with the ring structure and the inclusion $I\to A$.

It remains to construct an isomorphism $F\to\mathrm{Fun}_{\mathrm{Pair}}(\cdot,(A,I))|_{(\mathcal{D}^0)^{\mathrm{op}}}$ of finite-product-preserving functors $(\mathcal{D}^0)^{\mathrm{op}}\rightrightarrows\mathrm{Set}$. Composing with the functor $(\mathcal{C}^0)^{\mathrm{op}}\to(\mathcal{D}^0)^{\mathrm{op}}$ denoted by j, we get a map $F\circ j\to\mathrm{Fun}_{\mathrm{Pair}}(\cdot,(A,I))\circ j$ of functors $(\mathcal{C}^0)^{\mathrm{op}}\rightrightarrows\mathrm{Set}$ which is an equivalence by construction (and the adjunction $\mathrm{Fun}(\Delta^1,\mathrm{Ab})_{\mathrm{inj}}\rightleftarrows\mathrm{Pair}$). We need to show that this equivalence descends along the essentially surjective functor j.

First, for any $(B,J)\in\mathcal{D}^0$, by picking any lift under j, the map $F(B,J)\to\mathrm{Fun}_{\mathrm{Pair}}((B,J),(A,I))$ could be described as follows: for any $f\in F(B,J)$ and any $b\in B$, the element b corresponds uniquely to a map $\bar{b}:(\mathbb{Z}[t],0)\to(B,J)$ of pairs. Note that $\bar{b}^*(f)\in F(\mathbb{Z}[t],0)\cong A$. The image of f, as a map $(B,J)\to(A,I)$ of pairs, is concretely given by $b\mapsto\bar{b}^*(f)$, which is independent of the choice of the lift of (B,J).

Now it remains to show that, for any map $\varphi:(B,J)\to(C,K)$ of pairs, the diagram

$$\begin{array}{ccc}F(B,J) & \longrightarrow & \mathrm{Fun}_{\mathrm{Pair}}((B,J),(A,I))\\ \uparrow & & \uparrow\\ F(C,K) & \longrightarrow & \mathrm{Fun}_{\mathrm{Pair}}((C,K),(A,I))\end{array}$$

is commutative. Indeed, for any $f\in F(C,K)$, the image in $\mathrm{Fun}_{\mathrm{Pair}}((C,K),(A,I))$ is given by $c\mapsto\bar{c}^*(f)$, and the image in $\mathrm{Fun}_{\mathrm{Pair}}((B,J),(A,I))$ is given by $b\mapsto\overline{\varphi(b)}^*(f)$. On the other hand, the image of f in $F(B,J)$ is $\varphi^*(f)$, and the image in $\mathrm{Fun}_{\mathrm{Pair}}((B,J),(A,I))$ is given by $b\mapsto\bar{b}^*(\varphi^*(f))$. The result follows from the fact that $\varphi\circ\bar{b}=\overline{\varphi(b)}$ as maps $(\mathbb{Z}[t],0)\rightrightarrows(C,K)$ of pairs. ☐

Remark 2.3.9. The 1-category $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ contains more objects than Pair. They might be of independent interest. For example, let A be a ring and I an invertible A-module along with a map $j:I\to A$ of A-modules. If the map j in question is not injective, then it does not "faithfully" correspond to a ring-ideal pair such as $(A,\mathrm{im}(j))$, that is to say, it represents an object in $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ which is different from $(A,\mathrm{im}(j))$. In fact, the 1-category $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ could be identified with the 1-category of commutative algebra objects in $\mathrm{Fun}(\Delta^1,\mathrm{Ab})_{\mathrm{surj}}$ with pushout product monoidal structure, 1-categorical version of Subsection 2.3.1.

We now develop a PD analogue as follows: let $\mathcal{E}^0 \subseteq \mathrm{PDPair}$ denote the full subcategory spanned by the images of $(A, I) \in \mathcal{D}^0$ under the functor of PD-envelope [Sta21, Tag 07H9], denoted by $(D_A(I) \twoheadrightarrow A/I, \gamma)$ instead of the cumbersome notation $(D_A(I), \ker(D_A(I) \twoheadrightarrow A/I), \gamma)$. Then by Lemma 2.2.1, we get a pair $\mathcal{P}_{\Sigma,1}(\mathcal{E}^0) \rightleftarrows \mathrm{PDPair}$ of adjoint functors. Explicitly, the objects in $\mathcal{E}^0$ are of the form $D_{\mathbb{Z}[X,Y]}(Y)$.

Remark 2.3.10. The notations $\mathcal{D}^0$ and $\mathcal{E}^0$ are temporary. However, they will be occasionally used in Subsections 2.3.3, 2.3.4, and 2.3.5.

On the other hand, it follows from Corollary 2.2.2 that

LEMMA 2.3.11. *The essentially surjective functor $\mathcal{D}^0 \to \mathcal{E}^0$ gives rise to the* forgetful functor $\mathcal{P}_{\Sigma,1}(\mathcal{E}^0) \to \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ *which is conservative and preserves sifted colimits.*

There is another *forgetful functor* $\mathrm{PDPair} \to \mathrm{Pair}$. These functors are compatible:

LEMMA 2.3.12. *The diagram*

$$\begin{tikzcd}
\mathrm{PDPair} \arrow[r] \arrow[d] & \mathcal{P}_{\Sigma,1}(\mathcal{E}^0) \arrow[d] \\
\mathrm{Pair} \arrow[r, hook] & \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)
\end{tikzcd}$$

is a commutative diagram of 1*-categories.*

Proof. For any PD-pair $(A, I, \gamma) \in \mathrm{PDPair}$, the image in $\mathcal{P}_{\Sigma,1}(\mathcal{E}^0)$ is given by $\mathcal{E}^0 \ni (B, J, \delta) \mapsto \mathrm{Hom}_{\mathrm{PDPair}}((B, J, \delta), (A, I, \gamma))$, which is subsequently mapped to an object in $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ given by $\mathcal{D}^0 \ni (B, J) \mapsto \mathrm{Hom}_{\mathrm{PDPair}}(D_J(B), (A, I, \gamma))$. On the other hand, the image of (A, I, γ) in Pair is (A, I), which is subsequently mapped to an object in $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ given by $\mathcal{D}^0 \ni (B, J) \mapsto \mathrm{Hom}_{\mathrm{Pair}}((B, J), (A, I))$. It then follows from the functorial isomorphism $\mathrm{Hom}_{\mathrm{PDPair}}(D_J(B), (A, I, \gamma)) \cong \mathrm{Hom}_{\mathrm{Pair}}((B, J), (A, I))$ by adjunction. □

Similarly, we have the embedding:

LEMMA 2.3.13. *The functor* $\mathrm{PDPair} \to \mathcal{P}_{\Sigma,1}(\mathcal{E}^0)$ *is fully faithful.*

Proof. The proof is similar to that of Lemma 2.3.7. The faithfulness follows from Lemma 2.3.12 and the faithfulness of the forgetful functor $\mathrm{PDPair} \to \mathrm{Pair}$. Given two PD-pairs (A, I, γ) and (B, J, δ) in PDPair and a map

$$F := \mathrm{Hom}_{\mathrm{PDPair}}(\cdot, (A, I, \gamma))|_{(\mathcal{E}^0)^{\mathrm{op}}} \to \mathrm{Hom}_{\mathrm{PDPair}}(\cdot, (B, J, \delta))|_{(\mathcal{E}^0)^{\mathrm{op}}} =: G$$

of finite-product-preserving functors $(\mathcal{E}^0)^{\mathrm{op}} \rightrightarrows \mathrm{Set}$, we need to show that this is induced by some map $(A, I, \gamma) \to (B, J, \delta)$ of PD-pairs.

By Lemma 2.3.7, there exists a unique map $(A, I) \to (B, J)$ of pairs which correspond to the natural transform after composition $(\mathcal{D}^0)^{\mathrm{op}} \to (\mathcal{E}^0)^{\mathrm{op}} \rightrightarrows \mathrm{Set}$. It remains to show that this map preserves the PD-structure.

Indeed, any $x \in I$ corresponds to a map $(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to (A, I, \gamma)$ of PD-pairs, i.e., an element $\overline{x} \in F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$, and the image y of $x \in I$ in J is given by the image $\overline{y} \in G(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$ under the map $F \to G$. For any $n \in \mathbb{N}_{>0}$, there is a canonical endomorphism $(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0), t \mapsto \gamma_n(t)$ of PD-pairs which induces endomorphisms $F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$ and $G(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0) \to G(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma_0)$ compatible with the map $F \to G$. In particular, the image, denoted by $\overline{x}_n$, of $\overline{x}$ under the first endomorphism maps to the image, denoted by $\overline{y}_n$, of $\overline{y}$ under the second endomorphism. We note that $\overline{x}_n$ corresponds to $\gamma_n(x)$ and $\overline{y}_n$ corresponds to $\gamma_n(y)$. Thus the map $(A, I) \to (B, J)$ maps $\gamma_n(x)$ to $\gamma_n(y)$. □

With the following description of the essential image (cf. Lemma 2.3.8):

LEMMA 2.3.14. *The square in Lemma 2.3.12 is Cartesian. Equivalently by Lemma 2.3.8, the essential image of the fully faithful functor* $\mathrm{PDPair} \to \mathcal{P}_{\Sigma,1}(\mathcal{E}^0)$ *is spanned by those finite-product-preserving functors* $F : (\mathcal{E}^0)^{\mathrm{op}} \to \mathrm{Set}$ *which maps the edge* $(\mathbb{Z}[X], 0, 0) \to (\Gamma_{\mathbb{Z}}(X) \twoheadrightarrow \mathbb{Z}, \gamma)$ *in* $\mathcal{E}^0$ *to an injective map of sets.*

Proof. The proof is similar to that of Lemma 2.3.8. Let $F : (\mathcal{E}^0)^{\mathrm{op}} \to \mathrm{Set}$ be a functor such that the composite $(\mathcal{D}^0)^{\mathrm{op}} \to (\mathcal{E}^0)^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$ lies in the essential image of the fully faithful functor $\mathrm{Pair} \to \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$. We need to construct a PD-pair $(A, I, \gamma) \in \mathrm{PDPair}$ which represents the functor F.

Let (A, I) represent the composite functor $(\mathcal{D}^0)^{\mathrm{op}} \to (\mathcal{E}^0)^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$. Unrolling the definitions, we see that $A = F(\mathbb{Z}[t], 0, 0)$, $I = F(\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma)$ and the map $I \to A$ is induced by the map $(\mathbb{Z}[t], 0, 0) \to (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma)$ of PD-pairs. We endow a PD-structure (A, I) as follows: there exists a canonical endomorphism $\gamma_n : (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(t) \twoheadrightarrow \mathbb{Z}, \gamma), t \mapsto \gamma_n(t)$ of PD-pair, which induces a map $\gamma_n : I \to I$ for all $n \in \mathbb{N}_{>0}$. We need to check that $(\gamma_n)_{n \in \mathbb{N}_{>0}}$ satisfies the axioms of divided power structure [Sta21, Tag 07GL], setting $\gamma_0 = \mathrm{id}$. We spell out the verification of two of them:

$\boldsymbol{\gamma_n(x+y) = \sum_i \gamma_i(x)\,\gamma_{n-i}(y)}$ **for** $\boldsymbol{(x, y) \in I^2}$**.** First, in the PD-pair $(\Gamma_{\mathbb{Z}}(X, Y) \twoheadrightarrow \mathbb{Z}, \gamma)$, the identity $\gamma_n(X+Y) = \sum_i \gamma_i(X)\,\gamma_{n-i}(Y)$ holds. This implies that the composite

$$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(T_0, \ldots, T_n) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(X, Y) \twoheadrightarrow \mathbb{Z}, \gamma)$$

where the first map is induced by $T \mapsto \sum_i T_i$, and the second map is induced by $T_i \mapsto \gamma_i(X)\,\gamma_{n-i}(Y)$, coincides with the composite

$$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \xrightarrow{\gamma_n} (\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}}(X, Y) \twoheadrightarrow \mathbb{Z}, \gamma)$$

where the second map is induced by $T \mapsto X + Y$. Applying F to the two compositions, using the fact that F preserves finite products, and that $(x, y) \in I^2$ corresponds to an element in $F(\Gamma_{\mathbb{Z}}(X, Y) \twoheadrightarrow \mathbb{Z}, \gamma)$, we get the result (for the part $T_i \mapsto \gamma_i(X)\,\gamma_{n-i}(Y)$, one need to separate $i = 0$ and $i > 0$).

$\boldsymbol{\gamma_n(a\,x) = a^n\,\gamma_n(x)}$ **for** $\boldsymbol{(a, x) \in A \times I}$**.** In the PD-pair $(\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma)$, the identity $\gamma_n(Y X) = Y^n\,\gamma_n(X)$ holds. This implies that the composite

$$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}[T_1, \ldots, T_n]}(t) \twoheadrightarrow \mathbb{Z}[T_1, \ldots, T_n], \gamma) \to (\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma)$$

where the first map is induced by $T \mapsto T_1 \cdots T_n\,t$, and the second map is induced by $T_i \mapsto Y$ and $t \mapsto X$, coincides with the composite

$$(\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \xrightarrow{\gamma_n} (\Gamma_{\mathbb{Z}}(T) \twoheadrightarrow \mathbb{Z}, \gamma) \to (\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma)$$

where the second map is induced by $T \mapsto X Y$. Applying F to the two compositions, using the fact that F preserves finite products, and that $(a, x) \in A \times I$ corresponds to an element in $F(\Gamma_{\mathbb{Z}[Y]}(X) \twoheadrightarrow \mathbb{Z}[Y], \gamma)$, we get the result.

Finally, the proof of the fact that (A, I, γ) represents F is parallel to the corresponding part of the proof of Lemma 2.3.8. □

Now we arrive at the definition of animated pairs and animated PD-pairs:

DEFINITION 2.3.15. *The* ∞*-category* AniPair *of* animated pairs *is defined to be the* ∞*-category* $\mathcal{P}_{\Sigma}(\mathcal{D}^0)$*, and the* ∞*-category* AniPDPair *of* animated PD-pairs *is defined to be the* ∞*-category* $\mathcal{P}_{\Sigma}(\mathcal{E}^0)$*.*

The forgetful functor $\mathrm{AniPair} \to \mathrm{Fun}(\Delta^1, D(\mathbb{Z})_{\geq 0})$ *is given by the pair* $\mathcal{P}_{\Sigma}(\mathcal{C}^0) \rightleftarrows \mathcal{P}_{\Sigma}(\mathcal{D}^0)$ *obtained by applying Corollary 2.2.2 to the essentially surjective functor* $\mathcal{C}^0 \to \mathcal{D}^0$*.*

The forgetful functor $\mathrm{AniPDPair} \to \mathrm{AniPair}$ *and the* animated PD-envelope functor $\mathrm{AniPDEnv} : \mathrm{AniPair} \to \mathrm{AniPDPair}$ *are given by the pair* $\mathcal{P}_{\Sigma}(\mathcal{D}^0) \rightleftarrows \mathcal{P}_{\Sigma}(\mathcal{E}^0)$ *obtained by applying Corollary 2.2.2 to the essentially surjective functor* $\mathcal{D}^0 \to \mathcal{E}^0$ *given by PD-envelope.*

It follows from Corollary 2.2.2 that

COROLLARY 2.3.16. *The forgetful functors* $\mathrm{AniPair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ *and* $\mathrm{AniPDPair} \to \mathrm{AniPair}$ *are conservative and preserve sifted colimits.*

These forgetful functors are compatible with canonical embeddings $\mathrm{Pair} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \hookrightarrow \mathrm{AniPair}$ and $\mathrm{PDPair} \hookrightarrow \mathcal{P}_{\Sigma,1}(\mathcal{E}^0) \hookrightarrow \mathrm{AniPDPair}$:

PROPOSITION 2.3.17. *The diagram*

$$\begin{tikzcd}
\mathrm{PDPair} \arrow[r, hook] \arrow[d] & \mathrm{AniPDPair} \arrow[d] \\
\mathrm{Pair} \arrow[r, hook] \arrow[d] & \mathrm{AniPair} \arrow[d] \\
\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}} \arrow[r, hook] & \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})
\end{tikzcd}$$

is a commutative diagram of ∞-categories. Moreover, the squares are Cartesian.

Proof. The commutativity follows from Remark B.0.35 and Lemmas 2.3.6 and 2.3.12. The last claim follows from Lemmas 2.3.8 and 2.3.14. □

Remark 2.3.18. The embeddings $\mathrm{Pair} \hookrightarrow \mathrm{AniPair}$ and $\mathrm{PDPair} \hookrightarrow \mathrm{AniPDPair}$ admits left adjoints given by the composite functors $\mathrm{AniPair} \xrightarrow{\tau_{\leq 0}} \mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \to \mathrm{Pair}$ and $\mathrm{AniPDPair} \xrightarrow{\tau_{\leq 0}} \mathcal{P}_{\Sigma,1}(\mathcal{E}^0) \to \mathrm{PDPair}$ (see Remark B.0.35 for $\tau_{\leq 0}$). We will give an explicit description of the functor $\mathrm{AniPair} \to \mathrm{Pair}$ in Proposition 2.3.29.

Taking the left adjoints to the upper square of the diagram in Proposition 2.3.17, we get

COROLLARY 2.3.19. *The diagram*

$$\begin{tikzcd}
\mathrm{AniPair} \arrow[r] \arrow[d] & \mathrm{Pair} \arrow[d] \\
\mathrm{AniPDPair} \arrow[r] & \mathrm{PDPair}
\end{tikzcd}$$

is a commutative diagram of ∞-categories.

In particular, we rewrite the PD-envelope in terms of animated PD-envelope:

COROLLARY 2.3.20. *The PD-envelope functor* $\mathrm{Pair} \to \mathrm{PDPair}$ *coincides with the composite functor* $\mathrm{Pair} \hookrightarrow \mathrm{AniPair} \xrightarrow{\mathrm{AniPDEnv}} \mathrm{AniPDPair} \to \mathrm{PDPair}$.

In fact, there is a more concrete description of AniPair, given by the following:

DEFINITION 2.3.21. *The* ∞-category of surjective maps of animated rings *is the full subcategory* $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$ *of maps $A \to A''$ such that the induced map $\pi_0(A) \to \pi_0(A'')$ on the 0th homotopy groups is surjective.*

We now show that the strategy to prove Corollary 2.3.3 adapts to our case. Indeed, by Corollary 2.3.2, we have the equivalence $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) \simeq \mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$ of ∞-categories, therefore a set of compact projective generators for $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ gives rise to a set of compact projective generators for $\mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$: $\left\{\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \mathbb{Z} \to 0\right\}$. Now we study two adjunctions over these ∞-categories.

We have a pair $\mathrm{Fun}(\Delta^1, D(\mathbb{Z})_{\geq 0}) \rightleftarrows \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$ of adjoint functors induced by the pair $D(\mathbb{Z})_{\geq 0} \overset{\mathbb{L}\mathrm{Sym}_{\mathbb{Z}}}{\rightleftarrows} \mathrm{Ani}(\mathrm{Ring})$ of adjoint functors. Restricting to full subcategories, we get a pair $\mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0} \rightleftarrows \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$, the later is defined before Corollary 2.3.2. We summarize the preceding discussion by the diagram

$$\begin{array}{ccccc}
\mathcal{P}_\Sigma(\mathcal{D}^0) & & \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} & \subseteq & \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring})) \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) & \xrightarrow{\simeq} & \mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0} & \subseteq & \mathrm{Fun}(\Delta^1, D(\mathbb{Z})_{\geq 0})
\end{array} \tag{2.3.1}$$

We note that both full subcategories are stable under small colimits, therefore the forgetful functor $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \to \mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$ preserves sifted colimits. Since the forgetful functor is also conservative, it follows by Proposition B.0.15 that $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$ is projectively generated, for which $\left\{\mathbb{Z}[t] \xrightarrow{\mathrm{id}} \mathbb{Z}[t], \mathbb{Z}[t] \to \mathbb{Z}\right\}$ is a set of compact projective generators. Let $\mathcal{Z} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$ denote the full subcategory spanned by finite coproducts of these objects, which is effectively a full subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Ring})$. The following lemma is then obvious:

Lemma 2.3.22. *There is an equivalence $\mathcal{D}^0 \simeq \mathcal{Z}$ of 1-categories given by $\mathcal{D}^0 \to \mathcal{Z}, (A, I) \mapsto (A \twoheadrightarrow A/I)$ and $\mathcal{Z} \to \mathcal{D}^0, (A \twoheadrightarrow A'') \mapsto (A, \ker(A \twoheadrightarrow A''))$.*

It follows from previous discussion that

Theorem 2.3.23. *There is an equivalence $\mathrm{AniPair} = \mathcal{P}_\Sigma(\mathcal{D}^0) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$ of ∞-categories which fits into (2.3.1), making the left square a commutative square*[2.3.2].

Remark 2.3.24. Corollary 2.3.3 says that the ∞-category of $\mathbb{E}_\infty$-algebras in the symmetric monoidal ∞-category $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ is equivalent to that of $\mathbb{E}_\infty$-algebras in the symmetric monoidal ∞-category $\mathrm{Fun}(\Delta^1, D(\mathbb{Z}))_{\geq 0}$ since two symmetric monoidal ∞-categories are equivalent. Our result essentially says that both ∞-categories admits endomorphism monads which is also preserved under this equivalence, therefore the module categories over these monads are equivalent.

Notation 2.3.25. *Given the equivalence in Theorem 2.3.23, we will symbolically denote an object in* AniPDPair *by $(A \twoheadrightarrow A'', \gamma)$ where $A \twoheadrightarrow A''$ is the image under the forgetful functor* $\mathrm{AniPDPair} \to \mathrm{AniPair} \xrightarrow{\simeq} \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$. *When the PD-structure is the "obvious" one (like $\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]$), by abuse of notation, we will omit the γ in question. Under this notation, objects in $\mathcal{D}^0$ could be identified with $\mathbb{Z}[X, Y] \twoheadrightarrow \mathbb{Z}[X]$, and objects in $\mathcal{E}^0$ could be identified with $\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]$.*

Remark 2.3.26. In Theorem 2.3.23, we can replace $D(\mathbb{Z})$ by any *derived algebraic context* $\mathcal{C}$ [Rak20, Def 4.2.1] and then both $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C})$ and $\mathrm{Fun}(\Delta^1, \mathcal{C})$ admit canonical structures of derived algebraic contexts which are preserved under the equivalence $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$, and Theorem 2.3.23 essentially generalizes to the equivalence between the ∞-categories of connective maps of *derived commutative algebras* [Rak20, Rem 4.2.24] (note that $\mathrm{AniPair} \simeq \mathrm{DAlg}(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})))^{\mathrm{cn}}$ and $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \simeq \mathrm{DAlg}(\mathrm{Fun}(\Delta^1, D(\mathbb{Z})))^{\mathrm{cn}}$).

Remark 2.3.27. In our future work, we will show that Bhatt-Mathew's machinery in [Rak20, §4] allows us to define the ∞-category of *derived PD-pairs* of which the connective objects spans a full subcategory equivalent to the ∞-category of animated PD-pairs.

Warning 2.3.28. We warn the reader that the heart $\mathrm{DAlg}(\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})))^{\heartsuit}$ in [Rak20, Rem 4.2.24] is equivalent to the 1-category $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$, not the 1-category Pair.

We also identify the equivalence in Theorem 2.3.23 restricted to the full subcategory $\mathrm{Pair} \subseteq \mathrm{AniPair}$:

Proposition 2.3.29. *Let $\mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Ring})$ be the full subcategory spanned by those surjective maps $A \twoheadrightarrow A''$ of rings. Then the equivalence $\mathrm{Pair} \to \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}}, (A, I) \mapsto (A \twoheadrightarrow A/I)$ fits into a canonical commutative diagram*

$$\begin{tikzcd} \mathrm{Pair} \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}} \arrow[d, hook] \\ \mathrm{AniPair} \arrow[r, "\simeq"] & \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \end{tikzcd}$$

2.3.2. More precisely, there are two possible left squares in (2.3.1). However, by uniqueness of left/right adjoint, roughly speaking, one commutes if and only if the other commutes.

of ∞-categories. Furthermore, the localization $\mathrm{AniPair} \to \mathrm{Pair}$ *(Remark 2.3.18) could be identified with* $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \to \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}}, (A \twoheadrightarrow A'') \mapsto (\pi_0(A) \twoheadrightarrow \pi_0(A''))$.

Proof. We note that $\mathrm{Fun}(\Delta^1, \mathrm{Ring}) \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$ is the reflective subcategory (Definition 2.2.44) spanned by the 1-truncated objects, of which the localization is given by $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring})) \to \mathrm{Fun}(\Delta^1, \mathrm{Ring}), (A \to A'') \mapsto (\pi_0(A) \to \pi_0(A''))$ by Corollary 2.2.13 and Remark B.0.35. Restricting to the full subcategory $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$, we get a localization $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \to \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}}$. Consider the diagram

$$\begin{tikzcd}
\mathrm{AniPair} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \arrow[d] \\
\mathrm{Pair} \arrow[r, "\simeq"] & \mathrm{Fun}(\Delta^1, \mathrm{Ring})_{\mathrm{surj}}
\end{tikzcd}$$

of ∞-categories, where the vertical arrows are localizations (Remark 2.3.18). We claim that this is a commutative diagram. Indeed, both compositions commute with filtered colimits and geometric realizations (in fact, all small colimits, since both vertical arrows are localizations in Definition 2.2.44), and when restricting to $\mathcal{D}^0 \subseteq \mathrm{AniPair}$, both compositions are canonically equivalent. Then the claim follows from Proposition B.0.10.

Another way to show the commutativity is to show that the top right composition is ($\mathrm{AniPair} \to \mathrm{Pair}$)-invariant in Definition 2.2.57, then invoke Proposition 2.2.56.

Then the result follows by taking the right adjoints to the vertical arrows. □

COROLLARY 2.3.30. *The lower square in Proposition 2.3.17 is left-adjointable [Lur17, Def 4.7.4.13], which gives rise to a commutative diagram*

$$\begin{tikzcd}
\mathrm{AniPair} \arrow[r] \arrow[d] & \mathrm{Pair} \arrow[d] \\
\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) \arrow[r] & \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})_{\mathrm{inj}}
\end{tikzcd}$$

of 1-categories, where the vertical arrows are forgetful functors.

Warning 2.3.31. The upper square in Proposition 2.3.17 is not left-adjointable. That is to say, the localizations $\mathrm{AniPair} \to \mathrm{Pair}$ and $\mathrm{AniPDPair} \to \mathrm{PDPair}$ are not compatible with forgetful functors, otherwise the forgetful functor $\mathrm{PDPair} \to \mathrm{Pair}$ would commute with small colimits, which is false (see Remark 2.3.35).

It follows from Propositions 2.3.17 and 2.3.29 that

PROPOSITION 2.3.32. *The essential image of the fully faithful embedding* $\mathrm{PDPair} \hookrightarrow \mathrm{AniPDPair}$ *is spanned by those animated PD-pairs $(A \twoheadrightarrow A'', \gamma)$ such that both A and A'' are static.*

To understand the difference between Pair and $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \simeq \tau_{\leq 0}(\mathrm{AniPair})$ better, we compute the following example:

Example 2.3.33. Consider $(\mathbb{Z}/4\,\mathbb{Z}, (2)) \in \mathrm{Pair}$ as an animated pair. By Proposition 2.3.29, this corresponds to the surjective map $\mathbb{Z}/4\,\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$ of rings. Let us study the coproduct $(\mathbb{Z}/4\,\mathbb{Z}, (2)) \amalg (\mathbb{Z}/4\,\mathbb{Z}, (2))$ taken in AniPair. Thanks to Theorem 2.3.23, this corresponds to the surjective map $\mathbb{Z}/4\,\mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/4\,\mathbb{Z} \twoheadrightarrow \mathbb{F}_2 \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{F}_2$ of animated rings. The underlying map in $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$ is given by

$$(\mathbb{Z}/4\,\mathbb{Z})[1] \oplus \mathbb{Z}/4\,\mathbb{Z} \twoheadrightarrow \mathbb{F}_2[1] \oplus \mathbb{F}_2$$

induced by $0: (\mathbb{Z}/4\,\mathbb{Z})[1] \to \mathbb{F}_2[1]$ and the canonical projection $\mathbb{Z}/4\,\mathbb{Z} \twoheadrightarrow \mathbb{F}_2$. Under the forgetful functor $\mathrm{AniPair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$, the image of $(\mathbb{Z}/4\,\mathbb{Z}, (2)) \amalg (\mathbb{Z}/4\,\mathbb{Z}, (2))$ is thus given by

$$\begin{aligned}
(\mathbb{Z}/4\,\mathbb{Z})[1] \oplus \mathbb{Z}/4\,\mathbb{Z} \;&\longleftarrow\; \mathrm{fib}((\mathbb{Z}/4\,\mathbb{Z})[1] \oplus \mathbb{Z}/4\,\mathbb{Z} \twoheadrightarrow \mathbb{F}_2[1] \oplus \mathbb{F}_2) \\
&\xleftarrow{\simeq}\; (\mathbb{Z}/4\,\mathbb{Z})[1] \oplus \mathbb{F}_2 \oplus 2\,\mathbb{Z}/4\,\mathbb{Z}
\end{aligned}$$

induced by $\mathbb{F}_2 \to 0$ and other maps are canonical. Since the forgetful functor $\mathrm{AniPair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ commutes with $\tau_{\leq 0}$ (Remark B.0.35), we can identify the underlying object of $\tau_{\leq 0}((\mathbb{Z}/4\mathbb{Z}, (2)) \amalg (\mathbb{Z}/4\mathbb{Z}, (2)))$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$ with $(\mathbb{Z}/4\mathbb{Z} \leftarrow \mathbb{F}_2 \oplus 2\mathbb{Z}/4\mathbb{Z})$, which is not injective. Roughly speaking, the localization $\mathcal{P}_\Sigma(\mathcal{D}^0) \to \mathrm{Pair}$ will kill the kernel $\mathbb{F}_2$.

We now prove a stronger colimit-preserving property of the forgetful functor from animated PD-pairs to animated pairs, which does not seem to be obvious without this identification:

Proposition 2.3.34. *The forgetful functor* $\mathrm{AniPDPair} \to \mathrm{AniPair}$ *preserves small colimits.*

Proof. By Proposition B.0.10, it suffices to show that the composite functor $\mathcal{E}^0 \hookrightarrow \mathrm{AniPDPair} \to \mathrm{AniPair}$ preserves finite coproducts. We first "simplify" this composition, then we compute the finite coproducts by hand.

Since $\mathcal{E}^0 \hookrightarrow \mathrm{AniPDPair}$ factors as $\mathcal{E}^0 \hookrightarrow \mathrm{PDPair} \hookrightarrow \mathrm{AniPDPair}$, it follows from Proposition 2.3.17 that the composite functor $\mathcal{E}^0 \hookrightarrow \mathrm{AniPDPair} \to \mathrm{AniPair}$ is equivalent to the composite functor $\mathcal{E}^0 \hookrightarrow \mathrm{PDPair} \to \mathrm{Pair} \hookrightarrow \mathrm{AniPair}$. Under the equivalence in Theorem 2.3.23, this functor is concretely given by $\mathcal{E}^0 \ni (A, I, \gamma) \mapsto (A \twoheadrightarrow A/I) \in \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$. Since $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$ is stable under small colimits, we can take the finite coproducts in the larger ∞-category $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$.

Every object in $\mathcal{E}^0$ is the PD-envelope of a pair of form $(\mathbb{Z}[X_1,\ldots,X_m,Y_1,\ldots,Y_n],(Y_1,\ldots,Y_n))$, which we will denote by $\Gamma_{\mathbb{Z}[X_1,\ldots,X_m]}(Y_1,\ldots,Y_n) \twoheadrightarrow \mathbb{Z}[X_1,\ldots,X_m]$. Now the result follows from the fact that

$$\Gamma_{\mathbb{Z}[X]}(Y) \otimes^{\mathbb{L}}_{\mathbb{Z}} \Gamma_{\mathbb{Z}[X']}(Y') \simeq \Gamma_{\mathbb{Z}[X,X']}(Y,Y')$$

and

$$\mathbb{Z}[X] \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}[X'] \simeq \mathbb{Z}[X,X']$$

where $X = (X_1, \ldots, X_m)$, $X' = (X'_1, \ldots, X'_{m'})$, $Y = (Y_1, \ldots, Y_n)$ and $Y' = (Y'_1, \ldots, Y'_{n'})$. □

Remark 2.3.35. Proposition 2.3.34 implies that the forgetful functor $\mathcal{P}_{\Sigma,1}(\mathcal{E}^0) \to \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$ preserves small colimits, cf. Lemma 2.3.11. However, the forgetful functor $\mathrm{PDPair} \to \mathrm{Pair}$ does not preserve small colimits, even pushouts [Sta21, Tag 07GY]. The counterexample there is given by two PD-structures on the pair $(\mathbb{Z}/4\mathbb{Z}, (2))$. We explain the incompatibility of the localizations in Warning 2.3.31 by Example 2.3.33: the localization $\mathrm{AniPair} \to \mathrm{Pair}$ kills the kernel $\mathbb{F}_2$, while the localization $\mathrm{AniPDPair} \to \mathrm{PDPair}$ kills more, since the PD-structure does not necessarily pass to the quotient, so one needs to quotient out more relations.

Corollary 2.3.36. *The composite functor* $\mathrm{AniPair} \xrightarrow{\mathrm{AniPDEnv}} \mathrm{AniPDPair} \to \mathrm{AniPair}$ *preserves small colimits where* $\mathrm{AniPDPair} \to \mathrm{AniPair}$ *is the forgetful functor.*

2.3.3. Basic properties

In this subsection, we will discuss basic properties of animated pairs (resp. animated PD-pairs).

First, we recall that, given a pair (A, I), let (B, J, γ) be its PD-envelope, then there is a canonical equivalence $A/I \cong B/J$ [Sta21, Tag 07H7]. There is an analogue for animated PD-envelope:

Lemma 2.3.37. *The composite functor* $F: \mathrm{AniPair} \xrightarrow{\mathrm{AniPDEnv}} \mathrm{AniPDPair} \to \mathrm{AniPair}$ *is compatible with the evaluation* $\mathrm{ev}_{[1]}: \mathrm{AniPair} \simeq \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \to \mathrm{Ani}(\mathrm{Ring})$ *at* $[1] \in \Delta^1$*. That is to say, the composite functor* $\mathrm{AniPair} \xrightarrow{F} \mathrm{AniPair} \xrightarrow{\mathrm{ev}_{[1]}} \mathrm{Ani}(\mathrm{Ring})$ *is homotopy equivalent to the functor* $\mathrm{AniPair} \xrightarrow{\mathrm{ev}_{[1]}} \mathrm{Ani}(\mathrm{Ring})$.

Proof. Both functors are left derived functors, therefore it suffices to check on the full subcategory $\mathrm{Poly}_{\mathbb{Z}} \subseteq \mathrm{AniPair}$, which follows from a direct identification. □

We note that the functor $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$, $A \mapsto (\mathrm{id}_A: A \to A)$ is fully faithful, admits a left adjoint $\mathrm{ev}_{[1]}$ and a right adjoint $\mathrm{ev}_{[0]}$. Restricting to the fully faithful embedding $\mathrm{AniPair} \hookrightarrow \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))$, we get

LEMMA 2.3.38. *The functor* $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniPair}, A \mapsto (\mathrm{id}_A : A \to A)$ *is fully faithful and admits a left adjoint* $\mathrm{ev}_{[1]} : \mathrm{AniPair} \to \mathrm{Ani}(\mathrm{Ring}), (A \twoheadrightarrow A'') \mapsto A''$ *and a right adjoint* $\mathrm{ev}_{[0]} : \mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniPair}, (A \twoheadrightarrow A'') \mapsto A$.

This functor preserves small colimits, therefore by Proposition B.0.10, it is the left derived functor of the composite functor $\mathrm{Poly}_{\mathbb{Z}} \to \mathcal{D}^0 \hookrightarrow \mathrm{AniPair}, A \mapsto (A, 0)$. Apply Corollary 2.2.2 to the composite $\mathrm{Poly}_{\mathbb{Z}} \to \mathcal{D}^0 \to \mathcal{E}^0$, we get:

LEMMA 2.3.39. *The composite functor* $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniPair} \to \mathrm{AniPDPair}$ *is fully faithful, where the first functor is* $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniPair}, A \mapsto (\mathrm{id}_A : A \to A)$, *and the second functor* $\mathrm{AniPair} \to \mathrm{AniPDPair}$ *is the animated PD-envelope functor (Definition 2.3.15), and a further composition* $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniPDPair} \to \mathrm{AniPair}$ *is equivalent to the fully faithful functor* $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniPair}$, $A \mapsto (\mathrm{id}_A : A \to A)$ *above.*

Despite Warning 2.3.31, the image of an animated PD-pair $(A \twoheadrightarrow A'', \gamma)$ under the localization $\mathrm{AniPDPair} \to \mathrm{PDPair}$ is of the form $\cdot \to \tau_{\leq 0}(A'')$:

LEMMA 2.3.40. *There is a canonical commutative diagram*

$$\begin{tikzcd} \mathrm{AniPDPair} \arrow[r] \arrow[d] & \mathrm{AniPair} \arrow[r, "\mathrm{ev}_{[1]}"] & \mathrm{Ani}(\mathrm{Ring}) \arrow[d, "\tau_{\leq 0}"'] \\ \mathrm{PDPair} \arrow[r] & \mathrm{Pair} \arrow[r, "\mathrm{ev}_{[1]}"] & \mathrm{Ring} \end{tikzcd}$$

of ∞*-categories.*

Proof. We first note that the composite functor $\mathrm{AniPDPair} \to \mathrm{AniPair} \to \mathrm{Ani}(\mathrm{Ring}) \to \mathrm{Ring}$ preserves small colimits, therefore is a left derived functor (Proposition B.0.10), hence left Kan extended along $\mathcal{E}^0 \hookrightarrow \mathrm{AniPDPair}$. The diagram is canonically commutative on the full subcategory $\mathcal{E}^0 \subseteq \mathrm{AniPDPair}$. It remains to show the existence of the extension of the equivalence in question.

Now consider the diagram

$$\begin{tikzcd} \mathrm{Ring} \arrow[r, hook] \arrow[d, hook] & \mathrm{Pair} \arrow[r] & \mathrm{PDPair} \arrow[d] \\ \mathrm{Ani}(\mathrm{Ring}) \arrow[r, hook] & \mathrm{AniPair} \arrow[r] & \mathrm{AniPDPair} \end{tikzcd}$$

of ∞-categories where the functors $\mathrm{Ring} \to \mathrm{Pair}$ and $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{AniPair}$ are given by $A \mapsto (A, 0)$ and $A \mapsto (\mathrm{id}_A : A \to A)$ respectively, and the functor $\mathrm{Pair} \to \mathrm{PDPair}$ and the functor $\mathrm{AniPair} \to \mathrm{AniPDPair}$ are the PD-envelope (resp. animated PD-envelope) functors. This is a commutative diagram by Lemma 2.3.39. Taking the right adjoints, we get the commutativity by Lemma 2.3.38. $\square$

Next, we show that animated PD-envelope "does nothing" after rationalization. More precisely,

LEMMA 2.3.41. *Consider the unit map* η *from the functor* $\mathrm{id}_{\mathrm{AniPair}}$ *to the composite functor* $\mathrm{AniPair} \to \mathrm{AniPDPair} \to \mathrm{AniPair}$ *where the first functor is the animated PD-envelope functor and the second is the forgetful functor. Then the composite of* η *with the rationalization functor* $\mathrm{AniPair} \simeq \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0} \xrightarrow{\cdot \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{Q}} \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Alg}_{\mathbb{Q}}))_{\geq 0}$ *is an equivalence of functors.*

Proof. Since the rationalization functor preserves filtered colimits and geometric realizations, by Proposition B.0.10, it suffices to show the equivalence on $\mathcal{D}^0 \subseteq \mathrm{AniPair}$. Concretely, it is saying that the canonical map $\mathbb{Z}[X, Y] \to \Gamma_{\mathbb{Z}[X]}(Y)$ becomes an equivalence after rationalization, which follows from definitions. $\square$

Now we consider the base change. Given a surjective map $(A \twoheadrightarrow A'') \in \mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring}))_{\geq 0}$ and a map $A \to B$ of animated rings, the base changed map $B \to A'' \otimes_A^{\mathbb{L}} B$ is also surjective. The key observation is that this base change is a pushout $(A \twoheadrightarrow A'') \amalg_{(\mathrm{id}_A : A \to A)} (\mathrm{id}_B : B \to B)$. Since the animated PD-envelope functor, being a left adjoint, and the forgetful functor preserve small colimits (Proposition 2.3.34), it follows from Lemma 2.3.39 that (to compare with Remark 2.2.24).

Lemma 2.3.42. *The composite functor* $\mathrm{AniPair} \to \mathrm{AniPDPair} \to \mathrm{AniPair}$ *is compatible with base change, where the first functor is the animated PD-envelope functor and the second is the forgetful functor. More precisely, there is an equivalence from* $(C \otimes^{\mathbb{L}}_A B \twoheadrightarrow C'' \otimes^{\mathbb{L}}_A B)$ *to the animated PD-envelope of* $B \to A'' \otimes^{\mathbb{L}}_A B$ *between animated pairs, where* $(C \twoheadrightarrow C'', \gamma)$ *is the animated PD-envelope of* $(A \twoheadrightarrow A'')$*, which is functorial with respect to the diagram* $\begin{array}{ccc} A & \twoheadrightarrow & A'' \\ \downarrow & & \\ B & & \end{array}$ *in* $\mathrm{Ani}(\mathrm{Ring})$.

Remark 2.3.43. (General base) Let R be a ring. We could then replace $\mathbb{Z}$ by R in the theory of animated pairs and PD-pairs. For example, the 1-category Ab is replaced by Mod_R, the ∞-category $D(\mathbb{Z})$ is replaced by $D(R)$, the ∞-category $\mathrm{Ani}(\mathrm{Ring})$ is replaced by $\mathrm{Ani}(\mathrm{Alg}_R)$, the 1-category $\mathcal{D}^0$ is replaced by $\mathcal{D}^0_R$ consisting the pairs of the form $(R[X,Y],(Y))$, and $\mathcal{E}^0$ is replaced by $\mathcal{E}^0_R$ consisting the PD-pairs of the form $\Gamma_{R[X]}(Y) \twoheadrightarrow R[X]$, etc. We get $\mathrm{AniPair}_R$ and $\mathrm{AniPDPair}_R$. There are canonical base change functors $\mathrm{Ani}(\mathrm{Ring}) \to \mathrm{Ani}(\mathrm{Alg}_R)$, $\mathrm{AniPair} \to \mathrm{AniPair}_R$ and $\mathrm{AniPDPair} \to \mathrm{AniPDPair}_R$ essentially induced by the base change $D(\mathbb{Z}) \xrightarrow{\cdot \otimes^{\mathbb{L}}_{\mathbb{Z}} R} D(R)$.

It follows from Corollary 2.2.14 that

Lemma 2.3.44. *There are canonical equivalences of* ∞*-categories*

$$\begin{aligned} \mathrm{Ani}(\mathrm{Alg}_R) &\xrightarrow{\simeq} \mathrm{Ani}(\mathrm{Ring})_{R/} \\ \mathrm{AniPair}_R &\xrightarrow{\simeq} \mathrm{AniPair}_{(\mathrm{id}_R: R \to R)/} \\ \mathrm{AniPDPair}_R &\xrightarrow{\simeq} \mathrm{AniPDPair}_{(\mathrm{id}_R: R \to R, 0)/} \end{aligned}$$

By the proof of Lemma 2.2.37, it follows from Lemma 2.3.42 that

Lemma 2.3.45. *Let* R *be a ring. Then there is a canonical commutative diagram*

$$\begin{tikzcd} \mathrm{AniPair}_R \arrow[r] \arrow[d] & \mathrm{AniPDPair}_R \arrow[d] \\ \mathrm{AniPair} \arrow[r] & \mathrm{AniPDPair} \end{tikzcd}$$

of ∞*-categories where the vertical arrows are forgetful functors and the horizontal arrows are animated PD-envelope functors.*

Moreover, again by Lemma 2.3.42, we have

Lemma 2.3.46. *Let* R *be a ring. Then there is a canonical commutative diagram*

$$\begin{tikzcd} \mathrm{AniPair} \arrow[r] \arrow[d] & \mathrm{AniPDPair} \arrow[d] \\ \mathrm{AniPair}_R \arrow[r] & \mathrm{AniPDPair}_R \end{tikzcd}$$

of ∞*-categories, where the horizontal arrows are animated PD-envelope functors and the vertical arrows are base change functors.*

2.3.4. Quasiregular pairs

This subsection is devoted to comparison of animated theory of pairs and PD-pairs with the classical version. Quasiregularity, introduced by Quillen, play an important role:

Definition 2.3.47. ([Qui67, Thm 6.13]) *We say that a pair* $(A, I) \in \mathrm{Pair}$ *is* quasiregular *if the shifted cotangent complex* $\mathbb{L}_{(A/I)/A}[-1] \in D(A/I)$ *is a flat* A/I*-module. We will denote by* $\mathrm{QReg} \subseteq \mathrm{Pair}$ *the full subcategory spanned by quasiregular pairs.*

Example 2.3.48. Let A be a ring and $I\subseteq A$ an ideal generated by a Koszul-regular sequence. Then $\mathbb{L}_{(A/I)/A}\simeq(I/I^2)[1]$ [Sta21, Tag 08SJ], and I/I^2 is a free A/I-module [Sta21, Tag 062I]. We warn the reader that Quillen's quasiregular is different from "quasi-regular" in [Sta21, Tag 061M], and the later is not used in this article.

The first goal of this subsection is to show that there is a "derived" version of the adic filtration on animated pairs. Furthermore, for pairs, there is a natural comparison map from the "derived" version to the classical version (strictly speaking, our comparison is slightly more general), which becomes an equivalence when the pair in question is quasiregular. We refer to Subsection 2.2.4 for concepts and notations about filtrations. We need the following results, which relates the cotangent complex to powers of ideals.

LEMMA 2.3.49. ([STA21, TAG 08RA]) *There exists a map $\Sigma(I/I^2)\to\mathbb{L}_{(A/I)/A}$ in $D(A/I)$ which is functorial in $(A,I)\in\mathrm{Pair}$, such that the composite map $\Sigma(I/I^2)\to\mathbb{L}_{(A/I)/A}\to\tau_{\leq 1}\mathbb{L}_{(A/I)/A}$ is an equivalence.*

Remark 2.3.50. By abuse of terminology, by a map $M_{(A,I)}\to N_{(A,I)}$ in $D_{\geq 0}(A/I)$ being functorial in $(A,I)\in\mathrm{Pair}$, we mean that the map in question is a map between two functors $(A,I)\rightrightarrows\mathrm{Ani}(\mathrm{Mod})$ given by $(A,I)\mapsto(A/I,M_{(A,I)})$ and $(A,I)\mapsto(A/I,N_{(A,I)})$ respectively.

LEMMA 2.3.51. ([STA21, TAG 08SI]) *For any $(A,I)\in\mathcal{D}^0\subseteq\mathrm{Pair}$, the cotangent complex $\mathbb{L}_{(A/I)/A}$ is 1-truncated.*

COROLLARY 2.3.52. *There exists an equivalence $\Sigma(I/I^2)\to\mathbb{L}_{(A/I)/A}$ in $D_{\geq 0}(A/I)$ functorial in $(A,I)\in\mathcal{D}^0$.*

We now define the *adic filtration* on animated pairs. Consider the *classical adic filtration functor* $\mathrm{AdFil}:\mathrm{Pair}\to\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$, $(A,I)\mapsto(I^n)_{n\in\mathrm{N}_{\geq 0}}$. Restricting to the full subcategory $\mathcal{D}^0\subseteq\mathrm{Pair}$ and applying Proposition B.0.10, we get a functor $\mathbb{L}\,\mathrm{AdFil}:\mathrm{AniPair}\to\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$, called the *adic filtration functor*.

Remark 2.3.53. By the same argument, there is a natural structure of filtered derived ring (Remark 2.2.42) on $\mathbb{L}\,\mathrm{AdFil}$, which we will not use in this article.

By Theorem 2.3.23, we can identify $\mathrm{Fil}^0\circ\mathbb{L}\,\mathrm{AdFil}:\mathrm{AniPair}\to\mathrm{CAlg}_{\mathbb{Z}}$ with the composite functor

$$\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))\xrightarrow{\mathrm{ev}_{[0]}}\mathrm{Ani}(\mathrm{Ring})\to\mathrm{CAlg}_{\mathbb{Z}},(A\twoheadrightarrow A'')\mapsto A$$

and $\mathrm{gr}^0\circ\mathbb{L}\,\mathrm{AdFil}:\mathrm{AniPair}\to\mathrm{CAlg}_{\mathbb{Z}}$ with the composite functor

$$\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))\xrightarrow{\mathrm{ev}_{[1]}}\mathrm{Ani}(\mathrm{Ring})\to\mathrm{CAlg}_{\mathbb{Z}},(A\twoheadrightarrow A'')\mapsto A''$$

Combining Corollary 2.3.52, Proposition B.0.10, sifted-colimit-preserving properties of $\mathbb{L}\,\mathrm{Sym}^*$, and the concrete analysis of pairs in $\mathcal{D}^0\subseteq\mathrm{AniPair}$, we get

COROLLARY 2.3.54. *For every $(A\twoheadrightarrow A'')\in\mathrm{AniPair}$, the shifted cotangent complex $\mathbb{L}_{A''/A}[-1]\simeq\mathrm{gr}^1(\mathbb{L}\,\mathrm{AdFil}(A\twoheadrightarrow A''))$ is connective, and there exists an equivalence*

$$\mathbb{L}\,\mathrm{Sym}^*_{A''}(\mathrm{gr}^1(\mathbb{L}\,\mathrm{AdFil}(A\twoheadrightarrow A'')))\to\mathrm{gr}^*(\mathbb{L}\,\mathrm{AdFil}(A\twoheadrightarrow A''))$$

of graded $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras functorial in $(A\twoheadrightarrow A'')\in\mathrm{AniPair}$.

Now we construct a comparison map between the "derived" filtration $\mathbb{L}\,\mathrm{AdFil}$ and the "non-derived" filtration AdFil. We apply a trick used in the proof of Proposition 2.3.29 and Lemma B.0.39. Consider the functor $\mathbb{L}\,\mathrm{AdFil}:\mathrm{AniPair}\to\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ and the composite functor $\mathrm{AniPair}\to\mathrm{Pair}\xrightarrow{\mathrm{AdFil}}\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ where $\mathrm{AniPair}\to\mathrm{Pair}$ is the localization (Remark 2.3.18). A comparison map from the former $\mathbb{L}\,\mathrm{AdFil}$ to the later is furnished by Proposition B.0.10 and the universal property of left Kan extensions, which is essentially unique. Our next goal is to show that the comparison map is an equivalence after restriction to $\mathrm{QReg}\subseteq\mathrm{AniPair}$.

Since the forgetful functor $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \to \mathrm{DF}^{\geq 0}(\mathbb{Z})$ is conservative, we can show the equivalence after forgetting the $\mathbb{E}_\infty$-structure.

The previous discussion show that the comparison map is an equivalence after composing with $\mathrm{Fil}^0 : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z})$ and $\mathrm{gr}^0 : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z})$ on the 1-category Pair (not only for quasiregular pairs). We define the functor $\mathrm{gr}^{[0,n)} : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z}), F \mapsto \mathrm{cofib}(\mathrm{Fil}^n(F) \to \mathrm{Fil}^0(F))$. Thus it suffice to prove that the comparison map is an equivalence after composing with $\mathrm{gr}^{[0,n)} : \mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z})$ for all $n > 1$ for quasiregular pairs. Note that by definition, the essential image of $\mathrm{gr}^{[0,n)} \circ \mathrm{AdFil}$ already lies in $\mathrm{Ab} \subseteq D(\mathbb{Z})$. We show a more general statement (cf. the proof of Proposition 2.3.29 and Lemma B.0.39):

Lemma 2.3.55. *There is a commutative diagram*

$$\begin{tikzcd} \mathrm{AniPair} \arrow[r, "\mathrm{gr}^{[0,n)} \circ \mathbb{L}\,\mathrm{AdFil}"] \arrow[d] & D(\mathbb{Z})_{\geq 0} \arrow[d, "\tau_{\leq 0}"] \\ \mathrm{Pair} \arrow[r, "\mathrm{gr}^{[0,n)} \circ \mathrm{AdFil}"] & \mathrm{Ab} \end{tikzcd}$$

of ∞-categories, where the comparison from the top-right composition to the bottom-left composition is induced by the comparison map $\mathbb{L}\,\mathrm{AdFil} \to \mathrm{AdFil} \circ (\mathrm{AniPair} \to \mathrm{Pair})$ previously constructed.

Proof. The trick is to consider an auxiliary functor. Let $(A \twoheadrightarrow A'') \in \mathrm{AniPair}$ and let $I := \mathrm{fib}(A \twoheadrightarrow A'') \in D(\mathbb{Z})_{\geq 0}$. We recall that, by Theorem 2.3.23, and (2.3.1) in particular, the forgetful functor $\mathrm{AniPair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ is just $(A \twoheadrightarrow A'') \mapsto (A \leftarrow I)$.

Then the map $I \to A$ in $D(\mathbb{Z})_{\geq 0}$ induces a map $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to \mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} A$. Composing with the multiplication $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} A \to A$, we get the map $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to A$. We consider the functor $F : \mathrm{AniPair} \to D(\mathbb{Z})_{\geq 0}, (A \twoheadrightarrow A'') \mapsto \mathrm{cofib}(\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to A)$.

First, the functor F preserves filtered colimits and geometric realizations, since the functor $\mathbb{L}\,\mathrm{Sym}_{\mathbb{Z}}$ and the forgetful functor $\mathrm{AniPair} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})$ do (Lemma 2.3.5). In fact, F is the left derived functor (Proposition B.0.10) of the functor $\mathcal{D}^0 \to D(\mathbb{Z})_{\geq 0}, (A, I) \mapsto \mathrm{cofib}(\mathrm{Sym}^n_{\mathbb{Z}} I \to A)$.

Next, note that for $(A, I) \in \mathcal{D}^0$, the map $\mathrm{Sym}^n_{\mathbb{Z}} I \to A$ factors functorially as $\mathrm{Sym}^n_{\mathbb{Z}} I \to I^n \to A$ and the map $\mathrm{Sym}^n_{\mathbb{Z}} I \to I^n$ is surjective. It follows that there is a natural surjective map $\mathrm{cofib}(\mathrm{Sym}^n_{\mathbb{Z}} I \to A) \to A / I^n$, which gives rise to a map $F \to \mathrm{gr}^{[0,n)} \circ \mathbb{L}\,\mathrm{AdFil}$ of functors which becomes an equivalence after composition with $\tau_{\leq 0} : D(\mathbb{Z})_{\geq 0} \to \mathrm{Ab}$.

We now show that the functor $\tau_{\leq 0} \circ F : \mathrm{AniPair} \to \mathrm{Ab}$ factors through the localization $\mathrm{AniPair} \to \mathrm{Pair}$. First, since Ab is a 1-category, it factors through $\mathrm{AniPair} \to \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$. Given $(A \twoheadrightarrow A'') \in \mathcal{P}_{\Sigma,1}(\mathcal{D}^0)$, let $I = \mathrm{fib}(A \twoheadrightarrow A'')$ as before. Then $(A \leftarrow I) \in \mathcal{P}_{\Sigma,1}(\mathcal{C}^0) \simeq \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathrm{Ab})$, therefore A, I are static. Let $I' = \mathrm{im}(I \to A)$. It follows that the localization $\mathcal{P}_{\Sigma,1}(\mathcal{D}^0) \to \mathrm{Pair}$ maps $(A \twoheadrightarrow A'')$ to $(A, I') \in \mathrm{Pair}$. By Proposition 2.2.56, it suffices to show that F maps $(A \twoheadrightarrow A'') \to (A, I')$ to an equivalence. This simply follows from the fact that $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to \mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I'$ is a surjection on π_0, and the "multiplication" map $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to A$ factors as $\mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I \to \mathbb{L}\,\mathrm{Sym}^n_{\mathbb{Z}} I' \to A$.

In conclusion, we have already shown that there exists an equivalence of two compositions in the diagram that we need to prove. To show that this equivalence is the equivalence that we want, we note that the top right composition preserves filtered colimits and geometric realizations, then the first paragraph of the proof of Lemma 2.3.40 applies. □

In particular, when (A, I) is quasiregular, it follows from Corollary 2.3.54 that $\mathrm{gr}^n(\mathbb{L}\,\mathrm{AdFil}(A \twoheadrightarrow A/I)) \in D(\mathbb{Z})_{\geq 0}$ is static for all $n \in \mathbb{N}$, which implies that $\mathrm{gr}^{[0,n)}(\mathbb{L}\,\mathrm{AdFil}(A \twoheadrightarrow A/I))$ is static for all $n \in \mathbb{N}$. Consequently, we have

Proposition 2.3.56. *The comparison map from the functor $\mathbb{L}\,\mathrm{AdFil} : \mathrm{AniPair} \to \mathrm{CAlg}(\mathrm{DF}(\mathbb{Z}))$ to the composite functor $\mathrm{AniPair} \to \mathrm{Pair} \xrightarrow{\mathrm{AdFil}} \mathrm{CAlg}(\mathrm{DF}(\mathbb{Z}))$ becomes an equivalence after restricting to the full subcategory $\mathrm{QReg} \subseteq \mathrm{AniPair}$.*

Corollary 2.3.57. ([Qui67, 6.11]) *For every quasiregular pair (A, I), the canonical map $\mathrm{Sym}^*_{A/I}(I/I^2) \to \bigoplus I^*/I^{*+1}$ of graded rings is an equivalence.*

Proof. It suffices to show that the equivalence given by Corollary 2.3.54 coincides with the canonical map induced by the multiplicative structure on A. For any element $\overline{x_1}\cdots\overline{x_n}\in\mathrm{Sym}^n_{A/I}(I/I^2)$, we pick a lift $x_1,\ldots,x_n\in I$, which gives rise to a map $(B,J):=(\mathbb{Z}[X_1,\ldots,X_n],(X_1,\ldots,X_n))\to(A,I)$ of pairs, which induces the commutative diagram

$$\begin{tikzcd}
\mathrm{Sym}^n_{B/J}(J/J^2) \arrow[r] \arrow[d] & J^n/J^{n+1} \arrow[d] \\
\mathbb{L}\mathrm{Sym}^n_{A/I}(\mathrm{gr}^1(\mathbb{L}\mathrm{AdFil}(A\twoheadrightarrow A/I))) \arrow[r] & \mathrm{gr}^n(\mathbb{L}\mathrm{AdFil}(A\twoheadrightarrow A/I))
\end{tikzcd}$$

in the ∞-category $D(\mathbb{Z})_{\geq 0}$. Taking $\tau_{\leq 0}$ and trace the element $\overline{X_1}\cdots\overline{X_n}\in\mathrm{Sym}^n_{B/J}(J/J^2)$, we get the result. □

We are unable to answer the following question in full generality:

Question 3. Let (A,I) be a quasiregular pair. Let $(B\twoheadrightarrow B'',\gamma)$ denote its animated PD-envelope. Is it true that B,B'' are static, so by Proposition 2.3.32 and Corollary 2.3.20, it coincides with the classical PD-envelope?

However, we are able to answer it under certain flatness. First, it follows from Lemma 2.3.41 that when A is a $\mathbb{Q}$-algebra, the animated PD-envelope of (A,I) is just $A\twoheadrightarrow A/I$, which is also the classical PD-envelope.

Now we consider the characteristic $p>0$ case, switching the ground ring from $\mathbb{Z}$ to $\mathbb{F}_p$ (which is valid by Lemma 2.3.45). We will use the notations $\mathcal{D}^0$ and $\mathcal{E}^0$ in Subsection 2.3.2 but the occurrences of $\mathbb{Z}$ are replaced by $\mathbb{F}_p$. We recall that the Frobenius map $A\to A, x\mapsto x^p$ of an $\mathbb{F}_p$-algebra A gives rise to an endomorphism $\varphi:\mathrm{id}_{\mathrm{Alg}_{\mathbb{F}_p}}\to\mathrm{id}_{\mathrm{Alg}_{\mathbb{F}_p}}$ of the identity functor $\mathrm{id}_{\mathrm{Alg}_{\mathbb{F}_p}}:\mathrm{Alg}_{\mathbb{F}_p}\to\mathrm{Alg}_{\mathbb{F}_p}$, which gives rise to an endomorphism $\mathrm{id}_{\mathrm{Ani}(\mathrm{Alg}_{\mathbb{F}_p})}\to\mathrm{id}_{\mathrm{Ani}(\mathrm{Alg}_{\mathbb{F}_p})}$ still denoted by φ. We now introduce the *conjugate filtration* on the animated PD-envelope of animated $\mathbb{F}_p$-pairs that we learned from [Bha12a].

Let (A,I) be an $\mathbb{F}_p$-pair such that the Frobenius $\varphi_A:A\to A$ is flat, and let (B,J,γ) denote its PD-envelope. We first note that there is a $\varphi_A^*(A/I)$-algebra structure on B since $f^p=p\,\gamma_p(f)=0$ for all $f\in J$.

We have a filtration on B given by $\mathrm{Fil}^{-n}B$ for $n\geq 0$ to be the $\varphi_A^*(A/I)$-submodule of B generated by $\{\gamma_{k_1p}(f_1)\cdots\gamma_{k_mp}(f_m)\mid k_1+\cdots+k_m\leq n \text{ and } f_1,\ldots,f_m\in I\}$, which gives rise to a structure of nonpositively filtered $\varphi_A^*(A/I)$-algebra. We note that the filtration is exhaustive, i.e. $\mathrm{Fil}^{-\infty}B=\mathrm{colim}_{n\in(\mathbb{Z},\geq)}\mathrm{Fil}^{-n}B\to B$ is an isomorphism, and we can rephrase the nonpositively filtered $\varphi_A^*(A/I)$-algebra structure as a map $\varphi_A^*(A/I)\to\mathrm{Fil}\,B$[2.3.3] of nonpositively filtered ring. We need the following result:

LEMMA 2.3.58. ([BHA12A, LEM 3.42]) *Let (A,I) be an $\mathbb{F}_p$-pair such that I/I^2 is a flat A/I-module and the Frobenius $\varphi_A:A\to A$ is flat. The PD-envelope (B,J,γ) and the filtration* Fil^*B *are constructed above.*

Then there is a comparison map $\varphi_A^*(\Gamma^i_{A/I}(I/I^2))\to\mathrm{gr}^{-i}B$ of $\varphi_A^*(A/I)$-modules induced by the maps $(\gamma_{kp})_{k\in\mathbb{N}}$ which is functorial in (A,I). For example, when I/I^2 is free, an element in $\Gamma^i_{A/I}(I/I^2)$ represented by $\frac{f^{\otimes i}}{i!}$ will be mapped to $\gamma_{ip}(f)$ for $f\in I$.

Furthermore, if $I\subseteq A$ is generated by a Koszul-regular sequence[2.3.4]*, then the comparison map above is an isomorphism.*

Now we define the conjugate filtration on the animated PD-envelope.

DEFINITION 2.3.59. *The* conjugate filtration functor (on the animated PD-envelope) $\mathbb{L}\,\mathrm{ConjFil}:\mathrm{AniPair}_{\mathbb{F}_p}\to\mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *together with the* structure map *of functors* $\mathrm{AniPair}_{\mathbb{F}_p}\rightrightarrows\mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *from $(A\twoheadrightarrow A'',\gamma)\mapsto\varphi_A^*(A'')=A''\otimes^{\mathbb{L}}_{A,\varphi_A}A$ to $\mathbb{L}\,\mathrm{ConjFil}$ is defined to be the left derived functor (Proposition B.0.10) of $\mathcal{D}^0\ni(A,I)\mapsto(\varphi_A^*(A/I)\to\mathrm{Fil}\,B)\in\mathrm{Fun}(\Delta^1,\mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$ constructed above.*

2.3.3. We will from time to time suppress the asterisk in Fil^* to avoid confusion with φ^*.

2.3.4. We only need the special case that $(A,I)\in\mathcal{D}^0$.

We note that the conjugate filtration is exhaustive, i.e. the filtration $\mathrm{Fil}^{-\infty} \circ \mathbb{L}\,\mathrm{ConjFil}|_{\mathcal{D}^0}$ is given by the animated PD-envelope, so is $\mathbb{L}\,\mathrm{ConjFil}$, which follows either from Proposition B.0.10 and Lemma 2.2.43 or the fact that $\mathrm{AniPair} \simeq \mathcal{P}_\Sigma(\mathcal{D}^0) \subseteq \mathcal{P}(\mathcal{D}^0)$ is stable under filtered colimits (Proposition B.0.7). It follows from Lemma 2.3.58 that

Corollary 2.3.60. *For every $(A \twoheadrightarrow A'') \in \mathrm{AniPair}_{\mathbb{F}_p}$, there exists an equivalence*

$$\varphi_A^*(\Gamma^i_{A''}(\mathrm{gr}^1(\mathbb{L}\,\mathrm{AdFil}(A \twoheadrightarrow A'')))) \to \mathrm{gr}^{-i}(\mathbb{L}\,\mathrm{ConjFil}(A \twoheadrightarrow A''))$$

in $D(\varphi_A^*(A''))_{\geq 0}$ for all $i \in \mathbb{N}$ which is functorial in $(A \to A'') \in \mathrm{AniPair}_{\mathbb{F}_p}$.

Remark 2.3.61. One might wonder what precisely the functor is, since the target category $D(\varphi_A^*(A''))_{\geq 0}$ depends on $(A \twoheadrightarrow A'') \in \mathrm{AniPair}_{\mathbb{F}_p}$. One can rigorously describe this $\varphi_A^*(A'')$-algebra structure in terms of structure maps (as in Definition 2.3.59). However, this is cumbersome and we keep the current "imprecise" presentation.

We now apply this to a quasiregular pair $(A, I) \in \mathrm{QReg}_{\mathbb{F}_p}$. We first recall that

Definition 2.3.62. ([Lur17, Def 7.2.2.10]) *Let A be an $\mathbb{E}_1$-ring. We say that a right A-module spectrum M is* flat *if*

1. *The homotopy group $\pi_0(M)$ is a flat right $\pi_0(A)$-module.*
2. *For each $n \in \mathbb{Z}$, the canonical map $\pi_0(M) \otimes_{\pi_0(A)} \pi_n(A) \to \pi_n(M)$ is an isomorphism of abelian groups.*

The same concept applies to left A-module spectra.

Remark 2.3.63. ([Lur17, Rem 7.2.2.11 & 7.2.2.12]) Let R be an $\mathbb{E}_1$-ring and M a flat right R-module spectrum. By definition, if R is connective (resp. static), then so is R. In particular, when R is static, a flat R-module spectrum is simply a flat R-module, therefore we will sometimes refer to flat module spectra simply as flat modules since there is no ambiguity.

Lemma 2.3.64. *Let A be a connective $\mathbb{E}_1$-ring, and $M' \to M \to M''$ a fiber sequence of right A-module spectra. If M', M'' are flat right A-modules, then so is M.*

Proof. First, M', M'' are connective by flatness, therefore so is M. For every static left A-module N, we have a fiber sequence $N \otimes^{\mathbb{L}}_A M' \to N \otimes^{\mathbb{L}}_A M \to N \otimes^{\mathbb{L}}_A M''$. By flatness of M' and M'' and [Lur17, Prop 7.2.2.13], the spectra $N \otimes^{\mathbb{L}}_A M'$ and $N \otimes^{\mathbb{L}}_A M''$ are static, therefore so is $N \otimes^{\mathbb{L}}_A M$. The result then follows from [Lur17, Thm 7.2.2.15]. □

For future usages, we need to generalize slightly the concept of quasiregular pairs:

Definition 2.3.65. *We say that an animated pair $A \twoheadrightarrow A''$ is* quasiregular *if the shifted cotangent complex $\mathbb{L}_{A''/A}[-1] \in D(A'')$ is a flat A''-module spectrum. We will denote by $\mathrm{AniQReg} \subseteq \mathrm{AniPair}$ the full subcategory spanned by quasiregular animated pairs. The same for $\mathrm{AniQReg}_{\mathbb{F}_p} \subseteq \mathrm{AniPair}_{\mathbb{F}_p}$.*

Corollary 2.3.66. *Let $(A \twoheadrightarrow A'') \in \mathrm{AniQReg}_{\mathbb{F}_p}$ be a quasiregular animated $\mathbb{F}_p$-pair, and let $(B \twoheadrightarrow B'', \gamma)$ denote its animated PD-envelope. Then B is a flat $\varphi_A^*(A'')$-module spectrum.*

Proof. It follows from Corollary 2.3.60, Γ^* and base change preserving flatness ([Lur18b, Cor 25.2.3.3] & [Lur17, Prop 7.2.2.16]) that $\mathrm{gr}^{-i}(\mathbb{L}\,\mathrm{ConjFil}(A \twoheadrightarrow A''))$ is a flat $\varphi_A^*(A'')$-module spectrum. The result follows from the fact that the full subcategory spanned by flat modules over a connective $\mathbb{E}_1$-ring is stable under extension (Lemma 2.3.64) and under filtered colimits by [Lur17, Thm 7.2.2.14(1)]. □

Remark 2.3.67. In fact, by Lemma 2.5.48, the map $\varphi_A^*(A'') \to B$ in Corollary 2.3.66 is faithfully flat.

It follows from Proposition 2.3.32, Corollaries 2.3.20 and 2.3.66, and Remark 2.3.63 that

COROLLARY 2.3.68. *Let $(A,I)\in\mathrm{QReg}_{\mathbb{F}_p}$ be a quasiregular pair. Suppose that $\varphi_A^*(A/I)$ is static. Then the animated PD-envelope $(B\twoheadrightarrow B'',\gamma)$ of $(A\twoheadrightarrow A/I)$ belongs to $\mathrm{PDPair}_{\mathbb{F}_p}$, therefore coincides with the classical PD-envelope.*

We want to point out that such results for $\mathbb{F}_p$ will be used to deduce integral results, which is based on the following lemmas:

LEMMA 2.3.69. *Let $M\in\mathrm{Sp}_{\geq 0}$ be a connective spectrum. Suppose that the rationalization $M\otimes_{\mathbb{S}}^{\mathbb{L}}\mathbb{Q}$ is static, and for every prime $p\in\mathbb{N}$, the homotopy groups of $M/^{\mathbb{L}}p:=\mathrm{cofib}\big(M\xrightarrow{p}M\big)$ are concentrated in degree $0,1$. Then M is static.*

Proof. Since $\mathbb{Q}$ is $\mathbb{S}$-flat, $\pi_i(M)\otimes_{\mathbb{Z}}\mathbb{Q}\cong\pi_i(M\otimes_{\mathbb{S}}^{\mathbb{L}}\mathbb{Q})\cong 0$ when $i\neq 0$. On the other hand, $\pi_{i+1}(M/^{\mathbb{L}}p)\cong 0$ for $i>0$ implies that the map $\pi_i(M)\xrightarrow{p}\pi_i(M)$ is injective for every prime $p\in\mathbb{N}$ and $i>0$. It follows that $\pi_i(M)\cong 0$ for every $i>0$. □

Warning 2.3.70. Lemma 2.3.69 is false if M is not assumed to be connective. A counterexample is given by $M=(\mathbb{Q}/\mathbb{Z})[-1]$, for which $M\otimes_{\mathbb{S}}^{\mathbb{L}}\mathbb{Q}\simeq 0$ and $M/^{\mathbb{L}}p\simeq\mathbb{F}_p$ for every prime number $p\in\mathbb{N}$.

LEMMA 2.3.71. (CF. [STA21, TAG 039C]) *Let A be an animated ring and $M\in D_{\geq 0}(A)$ a connective A-module spectrum. Then the following conditions are equivalent:*

1. *M is a flat A-module.*
2. *$M\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q}$ is a flat $A\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q}$-module, and for every prime $p\in\mathbb{N}$, $M/^{\mathbb{L}}p$ is a flat $A/^{\mathbb{L}}p$-module.*

Proof. The first implies the second by the stability of flatness under base change [Lur17, Prop 7.2.2.16]. We now assume the second. By [Lur17, Thm 7.2.2.15], it suffices to show that for each static A-module N, the tensor product $M\otimes_A^{\mathbb{L}}N$ is also static. Indeed,

$$(M\otimes_A^{\mathbb{L}}N)\otimes_{\mathbb{S}}^{\mathbb{L}}\mathbb{Q}\simeq(M\otimes_A^{\mathbb{L}}N)\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q}\simeq(M\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q})\otimes_{A\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q}}^{\mathbb{L}}(N\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q})$$

is static by the $\mathbb{Z}$-flatness of $\mathbb{Q}$ and the flatness of $M\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q}$. On the other hand,

$$(M\otimes_A^{\mathbb{L}}N)/^{\mathbb{L}}p\simeq(M/^{\mathbb{L}}p)\otimes_{A/^{\mathbb{L}}p}^{\mathbb{L}}(N/^{\mathbb{L}}p)$$

for every prime $p\in\mathbb{N}$. Since $M/^{\mathbb{L}}p$ is $A/^{\mathbb{L}}p$-flat,

$$\pi_i((M/^{\mathbb{L}}p)\otimes_{A/^{\mathbb{L}}p}^{\mathbb{L}}(N/^{\mathbb{L}}p))\simeq\pi_0(M/^{\mathbb{L}}p)\otimes_{\pi_0(A/^{\mathbb{L}}p)}\pi_i(N/^{\mathbb{L}}p)\cong 0$$

for all $i>1$ by [Lur17, Prop 7.2.2.13]. It then follows from Lemma 2.3.69 that $M\otimes_A^{\mathbb{L}}N$ is static. □

We record a simple consequence (compare with [BS19, Lem 2.42]):

PROPOSITION 2.3.72. *Let A be a ring and $I\subseteq A$ an ideal generated by a Koszul-regular sequence. Then the animated PD-envelope $(B\twoheadrightarrow B'',\gamma)$ of $(A\twoheadrightarrow A/I)$ belongs to PDPair, therefore coincides with the classical PD-envelope.*

Proof. Note that $B''\simeq A/I$ is static by Lemma 2.3.37. It follows from Lemma 2.3.41 that $B\otimes_{\mathbb{Z}}^{\mathbb{L}}\mathbb{Q}\simeq A$ is static. Let $(f_1,\dots,f_r)$ be a Koszul-regular sequence which generates I. Fix a prime $p\in\mathbb{N}_{>0}$. Let A_0 denote $A/^{\mathbb{L}}p$. We follow the argument in [Bha12a, Lem 3.41]:

$$\begin{aligned}\varphi_{A_0}^*((A/I)/^{\mathbb{L}}p)&\simeq\varphi_{A_0}^*(A_0/^{\mathbb{L}}(f_1))\otimes_{A_0}^{\mathbb{L}}\cdots\otimes_{A_0}^{\mathbb{L}}\varphi_{A_0}^*(A_0/^{\mathbb{L}}f_r)\\&\simeq(A_0/^{\mathbb{L}}f_1^p)\otimes_{A_0}^{\mathbb{L}}\cdots\otimes_{A_0}^{\mathbb{L}}(A_0/^{\mathbb{L}}f_r^p)\\&\simeq A_0/^{\mathbb{L}}(f_1^p,\dots,f_r^p)\\&\simeq(A/^{\mathbb{L}}(f_1^p,\dots,f_r^p))/^{\mathbb{L}}p\end{aligned}$$

Note that since $(f_1, \ldots, f_r)$ is Koszul-regular, so is $(f_1^p, \ldots, f_r^p)$, which implies that $\pi_i(\varphi_{A_0}^*((A/I)/^{\mathbb{L}}p)) \cong 0$ for $i \neq 0, 1$. It follows from Corollary 2.3.66 and the base change property (Lemma 2.3.46) that $\pi_i(B/^{\mathbb{L}}p) \cong 0$ for $i \neq 0, 1$. The result then follows from Lemma 2.3.69. □

2.3.5. Illusie's question Given a ring A and an ideal $I \subseteq A$ generated by a Koszul-regular sequence, let (B, J, γ) denote the PD-envelope of (A, I). It is known that the canonical comparison map $\Gamma_{A/I}^*(I/I^2) \to J^{[*]}/J^{[*+1]}$ is an isomorphism, cf. [Ber74, I. Prop 3.4.4], where $J^{[*]}$ are divided powers of J in B. In [Ill72, VIII. Ques 2.2.4.2], Illusie asked whether this holds for quasiregular pairs (A, I). The answer is affirmative, and the goal of this section is to furnish a proof by our theory of animated PD-pairs.

Our strategy is similar to Subsection 2.3.4: both the animated PD-envelope and the PD-envelope of a pair (A, I) admit a canonical filtration, and there is a natural comparison between the two. Although for general quasiregular pairs (A, I) we do not know whether the animated PD-envelope coincides with the PD-envelope, the comparison map induces equivalences on graded pieces. The associated graded of the animated PD-envelope admits a natural structure of divided power algebra, and an element tracing proves that the equivalence coincides with the comparison map in Illusie's question.

We start with the *PD-filtration* on animated PD-pairs. We refer to Subsection 2.2.4 for concepts and notations about filtrations. Recall that given a PD-pair $(A, I, \gamma) \in \mathrm{PDPair}$ and a natural number $n \in \mathbb{N}$, the classical divided power ideal $I^{[n]} \subseteq A$ is the ideal generated by elements $\gamma_{i_1}(x_1) \cdots \gamma_{i_k}(x_k)$ where $x_1, \ldots, x_k \in I$ and $(i_1, \ldots, i_k) \in \mathbb{N}^k$ with $i_1 + \cdots + i_k \geq n$. For example, for $(\Gamma_{\mathbb{Z}}(x) \twoheadrightarrow \mathbb{Z}) \in \mathrm{PDPair}$ with kernel I, the kernel $I^{[n]} \subseteq \Gamma_{\mathbb{Z}}(x)$ is generated by $\{\gamma_i(x) \mid i \geq n\}$ (which is different from the ideal $(\gamma_n(x))$). The *classical PD-filtration* on A is given by $A \supseteq I \supseteq I^{[2]} \supseteq \cdots$ endowing A with the structure of filtered ring. A filtered ring is naturally a (nonnegatively) filtered $\mathbb{E}_\infty$-ring, and we get a functor $\mathrm{PDFil} : \mathrm{PDPair} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$.

DEFINITION 2.3.73. *The* PD-filtration functor $\mathbb{L}\,\mathrm{PDFil} : \mathrm{AniPDPair} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *is defined to be the left derived functor (Proposition B.0.10) of the composite functor* $\mathcal{E}^0 \hookrightarrow \mathrm{PDPair} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$. *For an animated PD-pair* $(A \twoheadrightarrow A'', \gamma) \in \mathrm{AniPDPair}$*, we will call the image the* $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebra* A *with* PD-filtration.

Remark 2.3.74. By the same argument, the PD-filtration in fact gives rise to a structure of filtered derived ring (Remark 2.2.42), which we will not use in this article.

Similar to Corollary 2.3.54, by Proposition B.0.10, sifted-colimit-preserving property of the (derived) divided power functor $\Gamma^* : \mathrm{Ani}(\mathrm{Mod}) \to \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(D(\mathbb{Z})))$ and the concrete analysis of $(A, I, \gamma) \in \mathcal{E}^0$, we get

LEMMA 2.3.75. *For every* $(A \twoheadrightarrow A'', \gamma) \in \mathrm{AniPDPair}$*, there exists an equivalence*

$$\Gamma_{A''}^*(\mathrm{gr}^1(\mathbb{L}\,\mathrm{PDFil}(A \twoheadrightarrow A'', \gamma))) \to \mathrm{gr}^*(\mathbb{L}\,\mathrm{PDFil}(A \twoheadrightarrow A'', \gamma))$$

of graded $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras which is functorial in* $(A \twoheadrightarrow A'', \gamma) \in \mathrm{AniPDPair}$.

Furthermore, we can compare the adic filtration on an animated pair and the PD-filtration on the animated PD-filtration. We first compare them on $\mathcal{D}^0$, then extend the comparison to AniPair by Proposition B.0.10, obtaining

LEMMA 2.3.76. *For every* $(A \twoheadrightarrow A'') \in \mathrm{AniPair}$*, let* $(B \twoheadrightarrow B'', \gamma) \in \mathrm{AniPDPair}$ *denote its animated PD-envelope. Then there is a canonical comparison map*

$$\mathrm{gr}^*(\mathbb{L}\,\mathrm{AdFil}(A \twoheadrightarrow A'')) \to \mathrm{gr}^*(\mathbb{L}\,\mathrm{PDFil}(B \twoheadrightarrow B'', \gamma))$$

of graded $\mathbb{E}_\infty$*-*$\mathbb{Z}$*-algebras which is functorial in* $(A \twoheadrightarrow A'') \in \mathrm{AniPair}$. *Furthermore, this map induces equivalences in* $D(\mathbb{Z})$ *when* $* = 0, 1$.

Analogous to Subsection 2.3.4, by universal property of left Kan extensions, there exists a essentially unique comparison map c_{PDFil} from the composite functor $\mathrm{AniPair} \to \mathrm{AniPDPair} \xrightarrow{\mathbb{L}\mathrm{PDFil}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ to the composite functor $\mathrm{AniPair} \to \mathrm{Pair} \to \mathrm{PDPair} \xrightarrow{\mathrm{PDFil}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$, where $\mathrm{AniPair} \to \mathrm{AniPDPair}$ is the animated PD-envelope functor and $\mathrm{AniPair} \to \mathrm{Pair}$ is the localization in Remark 2.3.18. The main result of this subsection is the following:

PROPOSITION 2.3.77. *The comparison map* c_{PDFil} *constructed above becomes an equivalence after composition* $\mathrm{QReg} \hookrightarrow \mathrm{AniPair} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \xrightarrow{\mathrm{gr}^*} \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(\mathbb{Z}))$.

Remark 2.3.78. As seen in Question 3, we do not know whether the comparison is an equivalence when we replace $\mathrm{gr}^*: \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \to \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(\mathbb{Z}))$ by $\mathrm{Fil}^0: \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \to \mathrm{CAlg}_{\mathbb{Z}}$, though it is true under assumptions of Corollary 2.3.68, which is the only obstruction for the comparison map to be a filtered equivalence.

We start to prove this. Unfortunately, we are unable to establish a strong result like Lemma 2.3.55 essentially due to the complication discussed in Warning 2.3.31. Our trick is to show that after replacing gr^* by $\mathrm{gr}^{[0,n)}$, both functors satisfy a common universal property.

As in Subsection 2.3.4, we can forget the $\mathbb{E}_\infty$-algebra structure then replace gr^* by $\mathrm{gr}^{[0,n)}$: $\mathrm{DF}^{\geq 0}(\mathbb{Z}) \to D(\mathbb{Z}), F \mapsto \mathrm{cofib}(\mathrm{Fil}^n F \to \mathrm{Fil}^0 F)$, i.e., it is equivalent to show that the natural comparison $c^{[0,n)}_{\mathrm{PDFil}}$ from the composite functor

$$\mathrm{AniPair} \to \mathrm{AniPDPair} \xrightarrow{\mathbb{L}\mathrm{PDFil}} \mathrm{DF}^{\geq 0}(D(\mathbb{Z})) \xrightarrow{\mathrm{gr}^{[0,n)}} D(\mathbb{Z})$$

to the composite functor

$$\mathrm{AniPair} \to \mathrm{Pair} \hookrightarrow \mathrm{PDPair} \xrightarrow{\mathrm{PDFil}} \mathrm{DF}^{\geq 0}(D(\mathbb{Z})) \xrightarrow{\mathrm{gr}^{[0,n)}} D(\mathbb{Z})$$

is an equivalence after restricting to the full subcategory $\mathrm{QReg} \subseteq \mathrm{AniPair}$. Note that the composite functor $\mathrm{gr}^{[0,n)} \circ \mathrm{PDFil}$ is concretely given by $(A, I, \gamma) \mapsto A / I^{[n]}$, which motivates the following definition:

DEFINITION 2.3.79. *We say that a PD-pair* (A, I, γ) *is* PD-nilpotent of height $n \in \mathbb{N}$ *if* $I^{[n]} = 0$. *We will denote by* $\mathrm{PDPair}^{[n]} \subseteq \mathrm{PDPair}$ *the full subcategory spanned by PD-nilpotent PD-pairs of height* n.

The following lemma could be checked directly, or (as ∞-categories) by invoking [Lur09, Prop 5.2.7.8]:

LEMMA 2.3.80. *The full subcategory* $\mathrm{PDPair}^{[n]} \hookrightarrow \mathrm{PDPair}$ *is reflective of which the localization* $\mathrm{Loc}^{[n]}: \mathrm{PDPair} \to \mathrm{PDPair}^{[n]}$ *is given by killing the higher divided powers:* $(A, I, \gamma) \mapsto (A / I^{[n]}, I A / I^{[n]}, \bar{\gamma})$ *where* $\bar{\gamma}(\bar{x}) = \overline{\gamma(x)}$ *for all* $x \in I$ *and* $\bar{x}, \overline{\gamma(x)}$ *are images of* $x, \gamma(x)$ *in* $A / I^{[n]}$.

Then the composite functor $\mathrm{gr}^{[0,n)} \circ \mathrm{PDFil}: \mathrm{PDPair} \to D(\mathbb{Z}), (A, I, \gamma) \mapsto A / I^{[n]}$ could be rewritten as the composite $\mathrm{PDPair} \to \mathrm{PDPair}^{[n]} \hookrightarrow \mathrm{AniPDPair} \to D(\mathbb{Z})$ where the last functor is the functor $\mathrm{AniPDPair} \to D(\mathbb{Z}), (A \twoheadrightarrow A'', \gamma) \mapsto A$. We now show that the composite functor $\mathrm{gr}^{[0,n)} \circ \mathbb{L}\,\mathrm{PDFil}$: $\mathrm{AniPDPair} \to D(\mathbb{Z})$ could also factor through $\mathrm{AniPDPair} \to D(\mathbb{Z})$. In fact, it is a "derived" version of the previous factorization.

Consider the composite functor $\mathcal{E}^0 \to \mathrm{PDPair}^{[n]} \hookrightarrow \mathrm{AniPDPair}$ where the first functor $\mathcal{E}^0 \to \mathrm{PDPair}^{[n]}$ is the restriction of the localization $\mathrm{Loc}^{[n]}: \mathrm{PDPair} \to \mathrm{PDPair}^{[n]}$ to the full subcategory $\mathcal{E}^0 \subseteq \mathrm{PDPair}$. Let $\mathrm{Red}^{[n]}$ its left derived functor (Proposition B.0.10) $\mathrm{AniPDPair} \to \mathrm{AniPDPair}$. We compose $\mathrm{Red}^{[n]}$ with the functor $\mathrm{AniPDPair} \to D(\mathbb{Z}), (A \twoheadrightarrow A'', \gamma) \mapsto A$ described above, we get a functor $\mathrm{AniPDPair} \to D(\mathbb{Z})$, which is equivalent to the composite functor $\mathrm{gr}^{[0,n)} \circ \mathbb{L}\,\mathrm{PDFil}$ by Proposition B.0.10 since both functors preserves filtered colimits and geometric realizations and they are canonically identified on the full subcategory $\mathcal{E}^0 \subseteq \mathrm{AniPDPair}$.

Then there is an essentially unique comparison map $c_{\mathrm{PDRed}}^{[n]}$ from the composite functor

$$\mathrm{AniPair} \to \mathrm{AniPDPair} \xrightarrow{\mathrm{Red}^{[n]}} \mathrm{AniPDPair}$$

which preserves filtered colimits and geometric realizations, to the composite functor

$$\mathrm{AniPair} \to \mathrm{Pair} \hookrightarrow \mathrm{PDPair} \xrightarrow{\mathrm{Loc}^{[n]}} \mathrm{PDPair}^{[n]} \hookrightarrow \mathrm{AniPDPair}$$

which is equivalent to $c_{\mathrm{PDFil}}^{[0,n)}$ after composing the sifted-colimit-preserving functor $\mathrm{AniPDPair} \to D(\mathbb{Z})$ by checking on the full subcategory $\mathcal{D}^0 \subseteq \mathrm{AniPair}$ and the universal property of the left Kan extension. It remains to show that

Lemma 2.3.81. *The comparison map* $c_{\mathrm{PDRed}}^{[n]}$ *of functors* $\mathrm{AniPair} \rightrightarrows \mathrm{AniPDPair}$ *becomes an equivalence after restricting to the full subcategory* $\mathrm{QReg} \subseteq \mathrm{AniPair}$.

Proof. It follows from Lemmas 2.3.75 and 2.3.76, Corollary 2.3.54, and Proposition 2.3.32 and the fact that the derived divided powers Γ^* of a flat module is flat therefore static, that the essential image of the composite functor

$$\mathrm{QReg} \hookrightarrow \mathrm{AniPair} \to \mathrm{AniPDPair} \xrightarrow{\mathrm{Red}^{[n]}} \mathrm{AniPDPair} \tag{2.3.2}$$

lies in the full subcategory $\mathrm{PDPair} \subseteq \mathrm{AniPDPair}$. We first show that the essential image further lies in the full subcategory $\mathrm{PDPair}^{[n]} \subseteq \mathrm{PDPair}$.

We fix a quasiregular pair $(A, I) \in \mathrm{QReg}$. Let $(C, K, \gamma) \in \mathrm{PDPair}$ denote the image of $(A, I) \in \mathrm{QReg}$ under the composite functor (2.3.2). Since (A, I) could be rewritten as a sifted colimit $\mathrm{colim}_{j \in \mathcal{I}} (B_j, J_j)$ taken in AniPair, where $(B_j, J_j) \in \mathcal{D}^0$. Let $(C_j, K_j, \gamma_j) \in \mathcal{E}^0$ be the PD-envelope of (B_j, J_j). Then $(C, K, \gamma) \simeq \mathrm{colim}_{j \in \mathcal{I}} \big(C_j / K_j^{[n]}, K_j C_j / K_j^{[n]}, \gamma_j\big)$ taken in AniPDPair. For every $x_1,\ldots,x_m \in K$ and $i_1, \ldots, i_m \in \mathbb{N}$ such that $i_1 + \cdots + i_m \geq n$, we need to show that $\gamma_{i_1}(x_1) \cdots \gamma_{i_m}(x_m) = 0$. The elements $x_1,\ldots,x_m$ define a map $\varphi: (\Gamma_{\mathbb{Z}}(X_1,\ldots,X_m) \twoheadrightarrow \mathbb{Z}, \delta) \to (C, K, \gamma)$ in $\mathrm{PDPair} \subseteq \mathrm{AniPDPair}$. Since $(\Gamma_{\mathbb{Z}}(X_1, \ldots, X_m) \twoheadrightarrow \mathbb{Z}, \delta) \in \mathrm{AniPDPair}$ is compact and projective and $\mathcal{I}$ is sifted, the map φ factors as $(\Gamma_{\mathbb{Z}}(X_1, \ldots, X_m) \twoheadrightarrow \mathbb{Z}, \delta) \to \big(C_j / K_j^{[n]}, K_j C_j / K_j^{[n]}, \gamma_j\big) \to \mathrm{colim}_{k \in \mathcal{I}} \big(C_k / K_k^{[n]}, K_k C_k / K_k^{[n]}, \gamma_k\big)$ for some $j \in \mathcal{I}$. Then the element $\gamma_{i_1}(X_1) \cdots \gamma_{i_m}(X_m) \in \Gamma_{\mathbb{Z}}(X_1, \ldots, X_m)$ is killed by the first map, hence $\gamma_{i_1}(x_1) \cdots \gamma_{i_m}(x_m) = 0$.

Note that the composite of left adjoints $\mathrm{Pair} \to \mathrm{PDPair} \xrightarrow{\mathrm{Loc}^{[n]}} \mathrm{PDPair}^{[n]}$ preserves small colimits, $(C, K, \gamma) \in \mathrm{PDPair}^{[n]}$ is isomorphic to the image of $(A, I) \in \mathrm{QReg} \subseteq \mathrm{Pair}$ under this composite functor and the map $(A, I) \to (C, K)$ is the unit map under this isomorphism. The result then follows from the uniqueness of universal objects. ☐

Remark 2.3.82. In fact, there is an ∞-category $\mathrm{AniPDPair}^{[n]}$ of *animated PD-pairs PD-nilpotent of height* n, defined to be the nonabelian derived category of the essential image of $\mathcal{E}^0 \subseteq \mathrm{Pair}$ under the functor $\mathrm{Loc}^{[n]}: \mathrm{PDPair} \to \mathrm{PDPair}^{[n]}$. Then there is a pair of adjoint functors $\mathrm{AniPDPair} \rightleftarrows \mathrm{AniPDPair}^{[n]}$ by Corollary 2.2.2. Furthermore, by mimicking the proof of Lemma 2.3.13, the canonical functor $\mathrm{PDPair}^{[n]} \to \mathrm{AniPDPair}^{[n]}$ is fully faithful. This leads to an alternative proof of Lemma 2.3.81. Although the functor $\mathrm{PDPair}^{[n]} \to \mathrm{PDPair}$ is fully faithful, we conjecture that the functor $\mathrm{AniPDPair}^{[n]} \to \mathrm{AniPDPair}$ is not fully faithful, similar to the fact that the forgetful functor $D(\mathbb{Z}/n\mathbb{Z}) \to D(\mathbb{Z})$ is not fully faithful though $\mathrm{Mod}_{\mathbb{Z}/n\mathbb{Z}} \to \mathrm{Ab}$ is so.

Now we answer Illusie's question:

Proposition 2.3.83. *For every quasiregular pair* $(A, I) \in \mathrm{QReg}$*, let* (B, J, γ) *denote its PD-envelope. Then the canonical map* $\Gamma_{A/I}^*(I/I^2) \to \bigoplus J^{[*]}/J^{[*+1]}$ *of graded rings induced by* $\gamma_*: I \to I$ *is an equivalence.*

Proof. It follows from Corollary 2.3.54, Lemmas 2.3.76 and 2.3.75, and Proposition 2.3.77 that there is a comparison map $\Gamma_{A/I}^*(I/I^2) \to \bigoplus J^{[*]}/J^{[*+1]}$ of graded rings. Then the result follows from element tracing, a modification of the proof of Corollary 2.3.57 by replacing $(\mathbb{Z}[X_1, \ldots, X_n], (X_1, \ldots, X_n))$ with $(\Gamma_{\mathbb{Z}}(X_1, \ldots, X_n) \twoheadrightarrow \mathbb{Z}, \gamma)$. ☐

2.4. Derived crystalline cohomology

In this section, we define and study the *Hodge-filtered derived crystalline cohomology*, a filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra functorially associated to an animated PD-pair $(A \twoheadrightarrow A'', \gamma)$ along with a map $A'' \to R$ of animated rings. To do so, we will introduce an auxiliary construction, the *Hodge-filtered derived de Rham cohomology*, functorially associated to a map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)$ of animated PD-pairs, which will be proved independent of the choice of B, and then we define the Hodge-filtered derived crystalline cohomology for $(A \twoheadrightarrow A'', \gamma)$ along with $A'' \to R$ as the Hodge-filtered derived de Rham cohomology of the map $(A \twoheadrightarrow A'', \gamma) \to (\mathrm{id}_R : R \to R, 0)$. Furthermore, we also define the *cohomology of the affine crystalline site* which could be endowed with Hodge-filtration. The Hodge-filtered derived de Rham cohomology is, roughly speaking, equivalent to the *relative animated PD-envelope* whenever $A'' \to R$ is surjective (Proposition 2.4.64), and the Hodge-filtered derived de Rham cohomology is equivalent to the cohomology of the affine crystalline site with Hodge filtration when $\pi_0(R)$ is a finitely generated $\pi_0(A'')$-algebra (Proposition 2.4.66) or when R is a quasisyntomic A''-algebra (Proposition 2.4.87). Furthermore, the cohomology of the affine crystalline site is equivalent to the classical crystalline cohomology when everything is classically given, at least up to p-completion, due to the fact that our theory is non-completed (Proposition 2.4.90).

Remark 2.4.1. Our theory is characteristic-independent. As a cost, the derived de Rham cohomology does not coincide with algebraic de Rham cohomology even under smoothness condition, although this is true when the base is of characteristic p. In particular, for a map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)$ of animated PD-pairs where A is an animated $\mathbb{Q}$-algebra, the underlying $\mathbb{E}_\infty$-ring of our Hodge-filtered derived de Rham cohomology is constantly A, cf. Lemma 2.4.11. However, in this case, the non-completed crystalline cohomology (Definition 2.4.88) is also A, so the derived de Rham cohomology is as "bad" as the non-completed derived crystalline cohomology. On the other hand, the Hodge-filtration allows us to recover the "correct" cohomology theory in characteristic 0 after taking Hodge completion by [Bha12a, Rem 2.6].

As a corollary, we deduce that the (usual) derived de Rham cohomology $\mathrm{dR}_{\mathbb{Z}/\mathbb{Z}[x]}$ is, as an $\mathbb{E}_\infty$-$\mathbb{Z}[x]$-algebra, equivalent to the PD-polynomial algebra $\Gamma_{\mathbb{Z}}(x)$. Bhatt showed an p-completed version of this [Bha12a, Thm 3.27].

Remark 2.4.2. In fact, our theory stems from the observation that the p-completed derived de Rham cohomology $(\mathrm{dR}_{\mathbb{Z}/\mathbb{Z}[x]})^\wedge_p$ coincides with the p-completed PD-polynomial ring $\Gamma_{\mathbb{Z}}(x)^\wedge_p$, and the rationalization becomes $\mathbb{Q}[x]$.

The virtue of our Hodge-filtered derived crystalline cohomology is that it preserves small colimits. We will show that this implies several properties of derived crystalline cohomology, such as "Künneth formula" and base change property (Corollaries 2.4.29, 2.4.30, and 2.4.31).

Remark 2.4.3. In our future work, we will show that our Hodge-filtered derived de Rham cohomology admits a natural enrichment to derived PD-pairs, Remark 2.3.27, and the Hodge filtration is given by the PD-filtration of the derived PD-pair in question.

2.4.1. Derived de Rham cohomology

In this subsection, we define the derived de Rham cohomology for maps of animated PD-pairs. We need the definition of *modules of PD-differentials*[2.4.1].

Definition 2.4.4. ([Sta21, Tag 07HQ]) *Let $(A, I, \gamma) \to (B, J, \delta)$ be a map of PD-pairs and M an B-module. A* PD A-derivation *into M is a map $\theta : B \to M$ which is additive, $\theta(a) = 0$ for $a \in A$, satisfies the Leibniz rule $\theta(b\, b') = b\, \theta(b') + b'\, \theta(b)$ and that*

$$\theta(\delta_n(x)) = \delta_{n-1}(x)\, \theta(x)$$

2.4.1. It is about differentials preserving PD-structure, rather than a module with a PD-structure.

for all $n \geq 1$ *and* $x \in J$.

In this situation, there exists a universal PD A-derivation

$$\mathrm{d}_{(B,J)/(A,I)}: B \to \Omega^1_{(B,J)/(A,I)}$$

such that for any PD A-derivation $\theta: B \to M$, there exists a unique B-linear map $\xi: \Omega^1_{(B,J)/(A,I)} \to M$ such that $\theta = \xi \circ \mathrm{d}_{(B,J)/(A,I)}$. We also call $\Omega^1_{(B,J)/(A,I)}$ the module of PD-differentials.

Remark 2.4.5. In Definition 2.4.4, the PD-structure on A is irrelevant. However, we will soon see that the derived version of module of PD-differentials does depend on the PD-structure on A.

Definition 2.4.6. ([Sta21, Tag 07HZ]) *Let $(A, I, \gamma) \to (B, J, \delta)$ be a map of PD-pairs such that $\Omega^1_{(B,J)/(A,I)}$ is a flat B-module*[2.4.2]*. The* de Rham complex $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ *is given by $\Omega^i_{(B,J)/(A,I)} = \bigwedge^i_B \Omega^1_{(B,J)/(A,I)}$ and $\mathrm{d}: \Omega^i_{(B,J)/(A,I)} \to \Omega^{i+1}_{(B,J)/(A,I)}$ is the unique A-linear map determined by*

$$\mathrm{d}(f_0\,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_i) = \mathrm{d}f_0 \wedge \cdots \wedge \mathrm{d}f_i$$

We recall that a *commutative differential graded A-algebra* (abbrev. *A-CDGA*) is a commutative algebra object in the symmetric monoidal abelian 1-category $\mathrm{Ch}(\mathrm{Mod}_A)$ of chain complexes[2.4.3] in static A-modules for a ring A. Then any nonpositively graded A-CDGA gives rise to an $\mathbb{E}_\infty$-A-algebra, and in particular, the de Rham complex constructed above gives rise to the *de Rham cohomology* as an $\mathbb{E}_\infty$-A-algebra.

To see this, we need the filtered derived ∞-category $\mathrm{DF}(A)$ along with the Day convolution reviewed in Subsection 2.2.4. Indeed, we can identify the heart $\mathrm{DF}(A)^\heartsuit$ with respect to the Beilinson t-structure (Proposition 2.2.41) with the abelian 1-category $\mathrm{Ch}(\mathrm{Mod}_A)$. Furthermore, the fully faithful embedding $\mathrm{Ch}(\mathrm{Mod}_A) \hookrightarrow \mathrm{DF}(A)$ is lax symmetric monoidal (Lemma 2.2.40). Thus an A-CDGA gives rise to an $\mathbb{E}_\infty$-algebra in $\mathrm{DF}(A)$.

Remark 2.4.7. When restricting to the full subcategory $\mathrm{Ch}_{\gg -\infty}(\mathrm{Mod}_A^\flat) \subseteq \mathrm{Ch}(\mathrm{Mod}_A)$ spanned by bounded below chain complexes of flat A-modules, the fully faithful embedding $\mathrm{Ch}_{\gg -\infty}(\mathrm{Mod}_A^\flat) \hookrightarrow \mathrm{DF}(A)$ is in fact symmetric monoidal. We will refer to this later.

The embedding $\mathrm{Ch}(\mathrm{Mod}_A) \hookrightarrow \mathrm{DF}(A)$ restricts to a lax symmetric monoidal embedding $\mathrm{Ch}_{\leq 0}(\mathrm{Mod}_A) \to \mathrm{DF}^{\geq 0}(A)$. Thus a nonpositively graded A-CDGA gives rise to an $\mathbb{E}_\infty$-algebra in $\mathrm{DF}^{\geq 0}(A)$, which is mapped to an $\mathbb{E}_\infty$-A-algebra by the symmetric monoidal functor $\mathrm{DF}^{\geq 0}(A) \to D(A)$.

Remark 2.4.8. The composite functor $\mathrm{Ch}_{\leq 0}(\mathrm{Mod}_A) \hookrightarrow \mathrm{DF}^{\geq 0}(A) \to D_{\leq 0}(A)$ maps any complex to its underlying module spectrum.

Furthermore, the truncation map $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d}) \to \Omega^0_{(B,J)/(A,I)} = B$ is a map of CDGAs, where B is concentrated in degree 0. Passing to the cohomology, we get a map of $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, called the *de Rham cohomology* of $(A, I) \to (B, J)$.

Now we define the derived de Rham cohomology for PD-pairs. By Corollary 2.2.9, the ∞-category $\mathrm{dRCon} := \mathrm{Fun}(\Delta^1, \mathrm{AniPDPair})$ (abbrev. for de Rham context) admits a set of compact projective generators given by maps of PD-pairs of the form $(\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]) \to (\Gamma_{\mathbb{Z}[X,X']}(Y, Y') \twoheadrightarrow \mathbb{Z}[X, X'])$ where each of X, Y, X', Y' consists of a finite set (including empty) of variables. These objects span a full subcategory $\mathrm{dRCon}^0 \subseteq \mathrm{dRCon}$ stable under finite coproducts. Then it follows from Proposition B.0.12 that there is an equivalence $\mathcal{P}_\Sigma(\mathrm{dRCon}^0) \to \mathrm{dRCon}$ of ∞-categories. The de Rham cohomology, along with the truncation map mentioned above, restricts to a functor $\mathrm{dRCon}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}_{\mathbb{Z}})$ where $\mathrm{CAlg}_{\mathbb{Z}}$ is the ∞-category of $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras.

2.4.2. We assume the flatness only to avoid the appearance of the ordinary tensor product $\otimes$ and the exterior power $\bigwedge$, since for flat modules, these coincide with the derived versions. In fact, we only need the very special case that $((A, I, \gamma) \to (B, J, \delta)) \in \mathrm{dRCon}$ defined before Definition 2.4.9.

2.4.3. We identify cochain complexes (K^*, d) with chain complexes (K_{-*}, d).

DEFINITION 2.4.9. *The* derived de Rham cohomology functor $\mathrm{dR}_{\cdot/\cdot}:\mathrm{dRCon}\to\mathrm{CAlg}_{\mathbb{Z}}$ *along with a canonical map* $\mathrm{dR}_{(B\twoheadrightarrow B'',\delta)/(A\twoheadrightarrow A'',\gamma)}\to B$ *of functors* $\mathrm{dRCon}\rightrightarrows\mathrm{CAlg}_{\mathbb{Z}}$ *is defined to be the left derived functor (Proposition B.0.10) of the functor* $\mathrm{dRCon}^0\to\mathrm{Fun}(\Delta^1,\mathrm{CAlg}_{\mathbb{Z}})$ *described above. Given a map* $(A\twoheadrightarrow A'',\gamma)\to(B\twoheadrightarrow B'',\delta)$ *of animated PD-pairs, its* derived de Rham cohomology*, i.e. the image under the derived de Rham cohomology functor, is denoted by* $\mathrm{dR}_{(B\twoheadrightarrow B'',\delta)/(A\twoheadrightarrow A'',\gamma)}$*, or simply* $\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$ *when there is no ambiguity.*

We first explain that this is a generalization of classical derived de Rham cohomology. We recall that the functor $\mathrm{Ani}(\mathrm{Ring})\to\mathrm{AniPDPair}, A\mapsto(\mathrm{id}_A:A\to A,0)$ is fully faithful (Lemma 2.3.39), thus so is the functor $\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))\to\mathrm{dRCon}$.

LEMMA 2.4.10. *The composite functor* $\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))\to\mathrm{dRCon}\xrightarrow{\mathrm{dR}_{\cdot/\cdot}}\mathrm{CAlg}_{\mathbb{Z}}, (A\to B)\mapsto \mathrm{dR}_{(\mathrm{id}_B:B\to B,0)/(\mathrm{id}_A:A\to A,0)}$ *is equivalent to the classical derived de Rham cohomology functor* $(A\to B)\mapsto\mathrm{dR}_{B/A}$.

Proof. The crucial point is that, $\mathrm{Fun}(\Delta^1,\mathrm{Ani}(\mathrm{Ring}))$ is projectively generated for which $\{(\mathbb{Z}[X]\to\mathbb{Z}[X,Y])\}$ forms a set of compact projective generators, which follows from Corollary 2.2.9 and Lemma 2.3.39. The result then follows from Proposition B.0.10 and the definition of the classical derived de Rham cohomology functor. □

We compute concretely the de Rham complex on dRCon^0. Fix an object $(\Gamma_{\mathbb{Z}[X']}(Y')\twoheadrightarrow\mathbb{Z}[X'])\to(\Gamma_{\mathbb{Z}[X,X']}(Y,Y')\twoheadrightarrow\mathbb{Z}[X,X'])\in\mathrm{dRCon}^0$, to simplify notations, we will write $A:=\Gamma_{\mathbb{Z}[X']}(Y'), A'':=\mathbb{Z}[X']$. Then this object could be rewritten as $(A\twoheadrightarrow A'',\gamma)\to(B:=\Gamma_{A[X]}(Y)\twoheadrightarrow A''[X],\tilde{\gamma})$ where $X=(x_1,\ldots,x_m)$ and $Y=(y_1,\ldots,y_n)$ with the module of PD-differentials $\Omega^1=B\,\mathrm{d}x_1\oplus\cdots\oplus B\,\mathrm{d}x_n\oplus B\,\mathrm{d}y_1\oplus\cdots\oplus B\,\mathrm{d}y_n$ and the universal PD-derivation $B\to\Omega^1$ is determined by $\mathrm{d}(X^\alpha\,\gamma_\beta(Y))=\sum_{i=1}^m\alpha_i\,x_1^{\alpha_1}\cdots x_i^{\alpha_i-1}\cdots x_m^{\alpha_m}\,\gamma_\beta(Y)\,\mathrm{d}x_i+\sum_{j=1}^n X^\alpha\,\gamma_{\beta_1}(y_1)\cdots\gamma_{\beta_j-1}(y_j)\cdots\gamma_{\beta_n}(y_n)\,\mathrm{d}y_j$ (with multi-index product).

As we mentioned earlier, derived de Rham cohomology is considered uninteresting in characteristic 0. Informally, the derived de Rham cohomology $\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$ is functorially equivalent to A after rationalization. More precisely, we will show that

LEMMA 2.4.11. *There is a comparison map* $A\to\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$ *of functors* $\mathrm{dRCon}\rightrightarrows\mathrm{CAlg}_{\mathbb{Z}}$ *which becomes an equivalence after composing with the rationalization* $\cdot\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Q}:\mathrm{CAlg}_{\mathbb{Z}}\to\mathrm{CAlg}_{\mathbb{Q}}$.

Proof. We first construct the comparison map in question. We have the composite of forgetful functors $\mathrm{AniPDPair}\to\mathrm{AniPair}\to\mathrm{Ani}(\mathrm{Ring})\to\mathrm{CAlg}_{\mathbb{Z}}, (A\twoheadrightarrow A'',\gamma)\mapsto A$. Further composing with the evaluation map $\mathrm{dRCon}\to\mathrm{AniPDPair}$ at $[0]\in\Delta^1$, we get a functor $\mathrm{dRCon}\to\mathrm{CAlg}_{\mathbb{Z}}, ((A\twoheadrightarrow A'',\gamma)\to(B\twoheadrightarrow B'',\delta))\mapsto A$. We restrict this functor to dRCon^0, getting a functor $\mathrm{dRCon}^0\to\mathrm{CAlg}_{\mathbb{Z}}$, which coincides with the composite functor $\mathrm{dRCon}^0\to\mathrm{Ring}=\mathrm{CAlg}(\mathrm{Ab})\hookrightarrow\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))\to\mathrm{CAlg}_{\mathbb{Z}}$ given by the "same" formula $((A\twoheadrightarrow A'',\gamma)\to(B\twoheadrightarrow B'',\delta))\mapsto A$. Note that there is a canonical map of functors from $\mathrm{dRCon}^0\to\mathrm{Ring}\to\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))$ to the de Rham complex functor $\mathrm{dRCon}^0\to\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))$, which is given by the A-CDGA structure on the de Rham complex. Now Proposition B.0.10 gives us a comparison map of the left derived functors $\mathrm{dRCon}\rightrightarrows\mathrm{CAlg}_{\mathbb{Z}}$.

It remains to see that this comparison map is an equivalence after rationalization. First, we note that the rationalization $\mathrm{CAlg}_{\mathbb{Z}}\to\mathrm{CAlg}_{\mathbb{Q}}$ preserves small colimits, and in particular, filtered colimits and geometric realizations, it follows from Proposition B.0.10 that both functors are still left derived functors after rationalization, therefore it suffices to check the equivalence on dRCon^0. The Poincaré lemma imply that the comparison map of functors $\mathrm{dRCon}^0\rightrightarrows\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))$ becomes a homotopy equivalence after composing with $\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))\to\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Mod}_{\mathbb{Q}}))$, which implies that it becomes an equivalence after composing with $\mathrm{CAlg}(\mathrm{Ch}_{\leq 0}(\mathrm{Ab}))\to\mathrm{CAlg}_{\mathbb{Q}}$ by Remark 2.4.8. □

Another consequence of this computation is that the de Rham cohomology functor $\mathrm{dRCon}^0\to\mathrm{CAlg}_{\mathbb{Z}}$ preserves finite coproducts, which follows from the fact that the de Rham cohomology functor $\mathrm{dRCon}^0\to\mathrm{CAlg}(\mathrm{Ch}_{\gg-\infty}(\mathrm{FreeAb}))$ preserves finite coproducts, and that the composite functor $\mathrm{Ch}_{\gg-\infty}(\mathrm{FreeAb})\hookrightarrow\mathrm{DF}^{\geq 0}(\mathbb{Z})\to D(\mathbb{Z})$ is symmetric monoidal, cf. Remark 2.4.7. By Proposition B.0.10, we have

Lemma 2.4.12. *The derived de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ *preserves small colimits.*

Now we show that the derived de Rham cohomology associated to the map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)$ does not depend on B, and define the derived crystalline cohomology. To formally define the ∞-category of animated PD-pairs $(A \twoheadrightarrow A'', \gamma)$ along with a map $A'' \to R$ of animated rings, we need the concept of comma categories in Subsection 2.2.3. Consider the comma category $\mathrm{CrysCon} := \mathrm{AniPDPair} \times_{\mathrm{Ani(Ring)}} \mathrm{Fun}(\Delta^1, \mathrm{Ani(Ring)})$ (abbrev. for crystalline context) where the functor $\mathrm{AniPDPair} \to \mathrm{Ani(Ring)}$ is the composite functor $\mathrm{AniPDPair} \to \mathrm{AniPair} \to \mathrm{Ani(Ring)}, (A \twoheadrightarrow A'', \gamma) \mapsto A''$ and the functor $\mathrm{Fun}(\Delta^1, \mathrm{Ani(Ring)}) \to \mathrm{Ani(Ring)}$ is the evaluation $(A'' \to R) \mapsto A''$ at $0 \in \Delta^1$. It follows from Corollary 2.2.22 that CrysCon admits a set of compact projective generators of the form $((\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X], \gamma), \mathbb{Z}[X] \to \mathbb{Z}[X, Z])$ where each of X, Y, Z consists of a finite set of variables, which spans a full subcategory $\mathrm{CrysCon}^0 \subseteq \mathrm{CrysCon}$ stable under finite coproducts.

We note that there is a canonical functor $\mathrm{dRCon} \to \mathrm{CrysCon}$ induced by the evaluation $\mathrm{dRCon} = \mathrm{Fun}(\Delta^1, \mathrm{AniPDPair}) \xrightarrow{\mathrm{ev}_{[0]}} \mathrm{AniPDPair}$ and the functor $\mathrm{dRCon} \to \mathrm{Fun}(\Delta^1, \mathrm{Ani(Ring)})$ which is itself induced by the composite of the forgetful functors $\mathrm{AniPDPair} \to \mathrm{AniPair} \to \mathrm{Ani(Ring)}, (A \twoheadrightarrow A'', \gamma) \mapsto A''$. Concretely, the functor $\mathrm{dRCon} \to \mathrm{CrysCon}$ is given by $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \mapsto ((A \twoheadrightarrow A'', \gamma), A'' \to B'')$. Since both functors preserves small colimits (we have used Proposition 2.3.34), we deduce that

Lemma 2.4.13. *The canonical functor* $\mathrm{dRCon} \to \mathrm{CrysCon}$ *preserves small colimits.*

It follows from Proposition B.0.10 that $\mathrm{dRCon} \to \mathrm{CrysCon}$ is the left derived functor of the composite functor $\mathrm{dRCon}^0 \to \mathrm{CrysCon}^0 \to \mathrm{CrysCon}, ((\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]) \to (\Gamma_{\mathbb{Z}[X,X']}(Y, Y') \twoheadrightarrow \mathbb{Z}[X, X'])) \mapsto ((\Gamma_{\mathbb{Z}[X]}(Y) \twoheadrightarrow \mathbb{Z}[X]), \mathbb{Z}[X] \to \mathbb{Z}[X, X'])$. It then follows from Corollary 2.2.2 that

Lemma 2.4.14. *The canonical functor* $\mathrm{dRCon} \to \mathrm{CrysCon}$ *admits a right adjoint* $\mathrm{CrysCon} \to \mathrm{dRCon}$ *which preserves sifted colimits.*

One can verify (see also Lemma 2.3.39) that

Lemma 2.4.15. *The right adjoint* $\mathrm{CrysCon} \to \mathrm{dRCon}$ *is concretely given by* $((A \twoheadrightarrow A'', \gamma), A'' \to R) \mapsto ((A \twoheadrightarrow A'', \gamma) \to (\mathrm{id}_R : R \to R, 0))$.

In particular, the counit map is an equivalence, therefore the functor $\mathrm{CrysCon} \to \mathrm{dRCon}$ is fully faithful. The unit map between functors $\mathrm{dRCon} \rightrightarrows \mathrm{dRCon}$ is concretely given by $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \to ((A \twoheadrightarrow A'', \gamma) \to (\mathrm{id}_{B''} : B'' \to B'', 0))$. Applying the derived de Rham functor $\mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$, we get the comparison map $\mathrm{dR}_{(B \twoheadrightarrow B'', \delta)/(A \twoheadrightarrow A'', \gamma)} \to \mathrm{dR}_{(\mathrm{id}_{B''}, 0)/(A \twoheadrightarrow A'', \gamma)}$. The independence is formally formulated as follows:

Proposition 2.4.16. *The map* $\mathrm{dR}_{(B \twoheadrightarrow B'', \delta)/(A \twoheadrightarrow A'', \gamma)} \to \mathrm{dR}_{(\mathrm{id}_{B''}, 0)/(A \to A'', \gamma)}$ *of functors* $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}_{\mathbb{Z}}$ *constructed above is an equivalence functorial in* $((A \twoheadrightarrow A'', \gamma) \to (B \to B'', \delta)) \in \mathrm{dRCon}$. *In other words, the derived de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}_{\mathbb{Z}}$ *is* $(\mathrm{dRCon} \to \mathrm{CrysCon})$*-invariant (Definition 2.2.57).*

Proof. Both functors preserves sifted colimits, so by Proposition B.0.10, it suffices to establish the equivalence for the full subcategory $\mathrm{dRCon}^0 \subseteq \mathrm{dRCon}$. For every $(\Gamma_{\mathbb{Z}[X']}(Y') \twoheadrightarrow \mathbb{Z}[X']) \to (\Gamma_{\mathbb{Z}[X,X']}(Y, Y') \twoheadrightarrow \mathbb{Z}[X, X']) \in \mathrm{dRCon}^0$ simply denoted by $((A \twoheadrightarrow A'', \gamma) \to (\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X], \gamma))$, we need to show that the map

$$\mathrm{dR}_{(\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X])/(A \twoheadrightarrow A'')} \to \mathrm{dR}_{(\mathrm{id}_{A''[X]} : A''[X] \to A''[X], 0)/(A \twoheadrightarrow A'')}$$

is an equivalence. Note that the constructed map

$$(\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X], \gamma) \to (\mathrm{id}_{A''[X]} : A''[X] \to A''[X], 0)$$

in $\mathrm{AniPDPair}_{/(A \twoheadrightarrow A'', \gamma)}$ factors as

$$(\Gamma_{A[X]}(Y) \twoheadrightarrow A''[X], \gamma) \xrightarrow{\alpha} (A[X] \twoheadrightarrow A''[X], \gamma) \xrightarrow{\beta} (\mathrm{id}_{A''[X]} : A''[X] \to A''[X], 0)$$

Thus it suffices to show that both maps α and β induces equivalences after passing to the functor $\mathrm{dR}_{\cdot/(A\twoheadrightarrow A'',\gamma)}:\mathrm{AniPDPair}_{/(A\twoheadrightarrow A'',\gamma)}\to\mathrm{CAlg}_{\mathbb{Z}}$. Note that $(A[X]\twoheadrightarrow A''[X],\gamma)\in\mathrm{dRCon}^0$, $\mathrm{dR}_{\alpha/(A\twoheadrightarrow A'',\gamma)}$ could be computed by de Rham complexes, which corresponds a homotopy equivalence of de Rham complexes by the divided power Poincaré's lemma [Sta21, Tag 07LC].

It remains to show that $\mathrm{dR}_{\beta/(A\twoheadrightarrow A'',\gamma)}$ is also an equivalence. For this, we need to resolve $(\mathrm{id}_{A''[X]}:A''[X]\to A''[X],0)$ simplicially under $(A[X]\twoheadrightarrow A''[X],\gamma)$. Recall that $A=\Gamma_{\mathbb{Z}[X']}(Y')$ and $A''=\mathbb{Z}[X']$. The key point is that we can resolve A'' simplicially by divided power polynomial A-algebras, in the same way as resolving $\mathbb{Z}$ simplicially by polynomial $\mathbb{Z}[t]$-algebras, which essentially follows from a bar construction of $\mathbb{N}$, see [Bha12a, Rem 3.31]. For every divided power polynomial A-algebra $\Gamma_A(Z)$, $(\Gamma_{A[X]}(Z)\twoheadrightarrow A''[X],\gamma)$ belongs to dRCon^0, and the map $\mathrm{dR}_{(A[X]\twoheadrightarrow A''[X])/(A\twoheadrightarrow A'')}\to\mathrm{dR}_{(\Gamma_{A[X]}(Z)\twoheadrightarrow A''[X])/(A\twoheadrightarrow A'')}$ (functorial in $\Gamma_A(Z)$) is an equivalence again by the divided power Poincaré's lemma [Sta21, Tag 07LC]. It follows that $\mathrm{dR}_{\beta/(A\twoheadrightarrow A'',\gamma)}$ is indeed an equivalence. □

In view of Proposition 2.2.56, we define the derived crystalline cohomology functor which corresponds to the $(\mathrm{dRCon}\to\mathrm{CrysCon})$-invariant functor $\mathrm{dR}_{\cdot/\cdot}$:

DEFINITION 2.4.17. *The* derived crystalline cohomology functor $\mathrm{CrysCoh}:\mathrm{CrysCon}\to\mathrm{CAlg}_{\mathbb{Z}}$ *is defined to be the composite* $\mathrm{CrysCon}\to\mathrm{dRCon}\xrightarrow{\mathrm{dR}_{\cdot/\cdot}}\mathrm{CAlg}_{\mathbb{Z}}$.

NOTATION 2.4.18. *We will denote the derived crystalline cohomology of* $((A\twoheadrightarrow A'',\gamma),A''\to R)\in\mathrm{CrysCon}$ *by* $\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma)}$ *(or* $\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ *even* $\mathrm{CrysCon}_{R/A}$ *when there is no ambiguity).*

Now we show that

PROPOSITION 2.4.19. *The derived crystalline cohomology functor* $\mathrm{CrysCon}\to\mathrm{CAlg}_{\mathbb{Z}}$ *preserves small colimits.*

Proof. The functor $\mathrm{CrysCon}\to\mathrm{dRCon}$ preserves sifted colimits and $\mathrm{dR}_{\cdot/\cdot}:\mathrm{dRCon}\to\mathrm{CAlg}_{\mathbb{Z}}$ preserves small colimits, it follows that the derived crystalline cohomology functor CrysCoh preserves sifted colimits. By Proposition B.0.10, it remains to show that $\mathrm{CrysCoh}|_{\mathrm{CrysCon}^0}$ preserves finite coproducts. The point is that every $(\Gamma_{\mathbb{Z}[X']}(Y')\twoheadrightarrow\mathbb{Z}[X'],\mathbb{Z}[X']\to\mathbb{Z}[X,X'])\in\mathrm{CrysCon}^0$ lifts to $(\Gamma_{\mathbb{Z}[X']}(Y')\twoheadrightarrow\mathbb{Z}[X'])\to(\Gamma_{\mathbb{Z}[X,X']}(Y,Y')\twoheadrightarrow\mathbb{Z}[X,X'])\in\mathrm{dRCon}^0$, the functor $\mathrm{dRCon}^0\to\mathrm{CrysCon}^0$ preserves finite coproducts, and the functor $\mathrm{dR}_{\cdot/\cdot}$ preserves finite coproducts. □

Now we apply the discussions in Subsection 2.2.3 to deduce some formal properties. First, by Remark 2.2.24, we have

COROLLARY 2.4.20. *The derived crystalline cohomology is compatible with base change. More precisely, let* $((A\twoheadrightarrow A'',\gamma_A),A''\to R)\in\mathrm{CrysCon}$ *and let* $(A\twoheadrightarrow A'',\gamma_A)\to(B\twoheadrightarrow B'',\gamma_B)$ *be a map of animated PD-pairs. Then the canonical map*

$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma_A)}\otimes^{\mathbb{L}}_A B\longrightarrow\mathrm{CrysCoh}_{(R\otimes^{\mathbb{L}}_{A''}B'')/(B\twoheadrightarrow B'',\gamma_B)}$$

is an equivalence.

Next, by Remark 2.2.28, we have

COROLLARY 2.4.21. *The derived crystalline cohomology is symmetric monoidal. More precisely, let* $(A\twoheadrightarrow A'',\gamma_A)\in\mathrm{AniPDPair}$ *and let* $A\to R$, $A\to S$ *be two maps of animated rings. Then the canonical map*

$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}\otimes^{\mathbb{L}}_A\mathrm{CrysCoh}_{S/(A\twoheadrightarrow A'')}\longrightarrow\mathrm{CrysCoh}_{(R\otimes^{\mathbb{L}}_{A''}S)/(A\twoheadrightarrow A'')}$$

is an equivalence.

Finally, by Remark 2.2.31, we have

Corollary 2.4.22. *The derived crystalline cohomology is transitive. More precisely, let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs, and let* $B'' \to R$ *be a map of animated rings. Then the canonical map*

$$\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{CrysCoh}_{B''/(A \twoheadrightarrow A'')}} B \longrightarrow \mathrm{CrysCoh}_{R/(B \twoheadrightarrow B'')}$$

is an equivalence, where the map $\mathrm{CrysCoh}_{B''/(A \twoheadrightarrow A'')} \to B$ *is* $\mathrm{CrysCoh}_{B''/(A \twoheadrightarrow A'')} \to \mathrm{CrysCoh}_{B''/(B \twoheadrightarrow B'')} \simeq B$.

Remark 2.4.23. In particular, if we take $(A \twoheadrightarrow A'', \gamma_A) = (\mathbb{Z}, 0, 0)$ in Corollary 2.4.22, we see that, fix an animated PD-pair $(B \twoheadrightarrow B'', \gamma_B)$, any derived crystalline cohomology $\mathrm{CrysCoh}_{R/(B \twoheadrightarrow B'')}$ is completely determined by the derived de Rham cohomology $\mathrm{dR}_{R/\mathbb{Z}}$. However, without the theory of derived crystalline cohomology, we do not know how to construct the map $\mathrm{dR}_{B''/\mathbb{Z}} \to B$ in terms of the PD-structure on $B \twoheadrightarrow B''$.

2.4.2. Filtrations

In this subsection, we will define the *Hodge filtration* on the derived de Rham cohomology and show that most of our previous discussions are compatible with the Hodge filtration. Furthermore, in characteristic p, we will define the *conjugate filtration*, which is of technical importance to control the cohomology. We start with the definition of the Hodge filtration.

Definition 2.4.24. (cf. [BO78, §6.13]) *Let* $(A, I, \gamma) \to (B, J, \delta)$ *be a map of PD-pairs such that* $\Omega^1_{(B,J)/(A,I)}$ *is a flat B-module. The* Hodge filtration Fil^*_H *on the de Rham complex* $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ *is given by the differential graded ideals* $\mathrm{Fil}^m_H \Omega^*_{(B,J)/(A,I)} := J^{[m-*]} \Omega^*_{(B,J)/(A,I)} \subseteq \Omega^*_{(B,J)/(A,I)}$.

As CDGAs give rise to $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, (nonnegatively) filtered CDGAs give rise to (nonnegatively) filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras. Moreover, the truncation map $(\Omega^*_{(B,J)/(A,I)}, \mathrm{d}) \to B$ is a map of filtered CDGAs, which gives rise to a map of filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras. Thus we get a functor $\mathrm{dRCon}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})))$.

Definition 2.4.25. *The* Hodge-filtered derived de Rham cohomology functor $\mathrm{Fil}^*_H \mathrm{dR}_{./.} : \mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *together with a canonical map* $\mathrm{Fil}^*_H \mathrm{dR}_{(B \twoheadrightarrow B'')/(A \twoheadrightarrow A'')} \to \mathrm{Fil}^*_{\mathrm{PD}} B$ *is defined to be the left derived functor (Proposition B.0.10) of the functor* $\mathrm{dRCon}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})))$ *above, where* $\mathrm{Fil}^*_{\mathrm{PD}} B$ *is the image of* $(B \twoheadrightarrow B'', \gamma_B) \in \mathrm{AniPDPair}$ *under the PD-filtration functor* $\mathbb{L}\,\mathrm{PDFil} : \mathrm{AniPDPair} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *(Definition 2.3.73).*

Most of properties in Subsection 2.4.1 hold with a similar proof:

Lemma 2.4.26. *The composite* $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring})) \to \mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$, $(A \to B) \mapsto \mathrm{Fil}^*_H \mathrm{dR}_{(\mathrm{id}_B : B \to B, 0)/(\mathrm{id}_A : A \to A, 0)}$ *is equivalent to the classical Hodge-filtered derived de Rham cohomology functor* $(A \to B) \mapsto \mathrm{Fil}^*_H \mathrm{dR}_{B/A}$.

Lemma 2.4.27. *The map in Lemma 2.4.11 admits a natural enrichment, that is to say, a map* $\mathrm{Fil}^*_{\mathrm{PD}} A \to \mathrm{Fil}^*_H \mathrm{dR}_{(B \twoheadrightarrow B'')/(A \twoheadrightarrow A'')}$ *of functors* $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$*. As in Lemma 2.4.11, this map becomes an equivalence after rationalization* $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z})) \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Q}))$.

Lemma 2.4.28. *The Hodge-filtered derived de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *preserves small colimits.*

Similar to Corollaries 2.4.20, 2.4.21, and 2.4.22, we have

Corollary 2.4.29. *The Hodge-filtered derived crystalline cohomology is compatible with base change. More precisely, let* $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$ *and let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs. Then the canonical map*

$$\mathrm{Fil}_H \mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \otimes^{\mathbb{L}}_{\mathrm{Fil}_{\mathrm{PD}} A} \mathrm{Fil}_{\mathrm{PD}} B \longrightarrow \mathrm{Fil}_H \mathrm{CrysCoh}_{(R \otimes^{\mathbb{L}}_{A''} B'')/(B \twoheadrightarrow B'', \gamma_B)}$$

is an equivalence.

COROLLARY 2.4.30. *The derived crystalline cohomology is symmetric monoidal. More precisely, let* $(A \twoheadrightarrow A'', \gamma_A) \in \mathrm{AniPDPair}$ *and let* $A \to R$, $A \to S$ *be two maps of animated rings. Then the canonical map*

$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}} \mathrm{Fil}_H\,\mathrm{CrysCoh}_{S/(A\twoheadrightarrow A'')} \to \mathrm{Fil}_H\,\mathrm{CrysCoh}_{(R\otimes^{\mathbb{L}}_{A''}S)/(A\twoheadrightarrow A'')}$$

is an equivalence, where the tensor product on the left is relative to $\mathrm{Fil}_{\mathrm{PD}}\,A$.

COROLLARY 2.4.31. *The derived crystalline cohomology is transitive. More precisely, let* $(A \twoheadrightarrow A'', \gamma_A) \to (B \twoheadrightarrow B'', \gamma_B)$ *be a map of animated PD-pairs, and let* $B'' \to R$ *be a map of animated rings. Then the canonical map*

$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')} \otimes^{\mathbb{L}}_{\mathrm{Fil}_H\,\mathrm{CrysCoh}_{B''/(A\twoheadrightarrow A'')}} \mathrm{Fil}_{\mathrm{PD}}\,B \longrightarrow \mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(B\twoheadrightarrow B'')}$$

is an equivalence, where the map $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{B''/(A\twoheadrightarrow A'')} \to B$ *is equivalent to the map* $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{B''/(A\twoheadrightarrow A'')} \to \mathrm{Fil}_H\,\mathrm{CrysCoh}_{B''/(B\twoheadrightarrow B'')} \simeq \mathrm{Fil}_{\mathrm{PD}}\,B$.

And this allows us to define the Hodge-filtration on the derived crystalline cohomology, due to the following proposition, which follows from the proof of Proposition 2.4.16 by replacing the Poincaré lemma by the filtered Poincaré lemma, cf. [BO78, Thm 6.13]:

PROPOSITION 2.4.32. *The map* $\mathrm{Fil}_H^*\,\mathrm{dR}_{(B\twoheadrightarrow B'',\delta)/(A\twoheadrightarrow A'',\gamma)} \to \mathrm{Fil}_H^*\,\mathrm{dR}_{(\mathrm{id}_{B''},0)/(A\to A'',\gamma)}$ *of functors* $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *induced by the counit map associated to* $((A\twoheadrightarrow A'',\gamma)\to(B\twoheadrightarrow B'',\delta)) \in \mathrm{dRCon}$ *is an equivalence. In other words, the Hodge-filtered de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *is* $(\mathrm{dRCon}\to\mathrm{CrysCon})$*-invariant (Definition 2.2.57).*

DEFINITION 2.4.33. *The* Hodge-filtered derived crystalline cohomology functor $\mathrm{Fil}_H^*\,\mathrm{CrysCoh} : \mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *is defined to be the composite* $\mathrm{CrysCon} \to \mathrm{dRCon} \xrightarrow{\mathrm{Fil}_H^*\,\mathrm{dR}_{\cdot/\cdot}} \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$.

PROPOSITION 2.4.34. *The Hodge-filtered derived crystalline cohomology functor* $\mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ *preserves small colimits.*

Now we come to the characteristic $p>0$ case. We start with an analysis of the Frobenius map on an animated PD-$\mathbb{F}_p$-pair. Let $(A,I,\gamma) \in \mathrm{AniPDPair}^0_{\mathbb{F}_p}$ be an animated PD-$\mathbb{F}_p$-pair of the form $\Gamma_{\mathbb{F}_p[X]}(Y) \twoheadrightarrow \mathbb{F}_p[X]$. We also have similar definitions for $\mathrm{dRCon}_{\mathbb{F}_p}, \mathrm{dRCon}^0_{\mathbb{F}_p}$ and $\mathrm{CrysCon}_{\mathbb{F}_p}$, $\mathrm{CrysCon}^0_{\mathbb{F}_p}$, and a parallel theory for $\mathbb{F}_p$-stuff. We first point out that, by Corollary 2.4.20 along with the proof of Lemma 2.2.37 (to compare with Lemmas 2.3.45 and 2.3.46), we have

LEMMA 2.4.35. *The derived crystalline cohomology* $\mathrm{CrysCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(D(\mathbb{F}_p))$ *fits into the commutative diagram*

$$\begin{tikzcd} \mathrm{CrysCon}_{\mathbb{F}_p} \arrow[r] \arrow[d] & \mathrm{CrysCon} \arrow[d] \\ \mathrm{CAlg}(D(\mathbb{F}_p)) \arrow[r] & \mathrm{CAlg}(D(\mathbb{Z})) \end{tikzcd}$$

of ∞-categories, where the horizontal arrows are forgetful functors. The same for the derived de Rham cohomology. Furthermore, this diagram is left-adjointable (roughly speaking, if we replace the horizontal arrows by their left adjoints, it is still a commutative diagram of ∞-categories).

Then the Frobenius map $\varphi_A : A \to A$ factors uniquely through the quotient map $A \twoheadrightarrow A/I$, which gives rise to a map $A/I \to A$. It then follows from Proposition B.0.10 that

LEMMA 2.4.36. *For any animated PD-$\mathbb{F}_p$-pair* $(A\twoheadrightarrow A'', \gamma) \in \mathrm{AniPDPair}_{\mathbb{F}_p}$*, the Frobenius map* $\varphi_A : A \to A$ *factors functorially through the map* $A \twoheadrightarrow A''$*, which gives rise to the a map* $A'' \to A$*, denoted by* $\varphi_{(A\twoheadrightarrow A'',\gamma)}$ *or* $\varphi_{A\twoheadrightarrow A''}$ *when there is no ambiguity (when* $(A\twoheadrightarrow A'',\gamma)$ *comes from a PD-$\mathbb{F}_p$-pair* (A,I,γ)*, it will also be denoted by* $\varphi_{(A,I,\gamma)}$ *or* $\varphi_{(A,I)}$*).*

Now we point out that in the char p-case, the de Rham complex is "Frobenius-linear" (compare with Definition 2.3.59): given an object $(A, I, \gamma) \to (B, J, \delta)$ in $\mathrm{dRCon}^0_{\mathbb{F}_p}$, each graded piece $\Omega^i_{(B,J,\delta)/(A,I,\gamma)}$ admits a natural B-module structure therefore also a $\varphi^*_{(A,I)}(B/J)$-module structure induced by the map $\varphi^*_{(A,I)}(B/J) := (B/J) \otimes^{\mathbb{L}}_{A/I,\varphi_{(A,I)}} A \to B$, the linearization of $\varphi_{(B,J)}\colon B/J \to B$. Furthermore, the differential d is $\varphi^*_{(A,I)}(B/J)$-linear, which makes the de Rham complex $(\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$ a $\varphi^*_{(A,I)}(B/J)$-CDGA. In other words, there is a map $\varphi^*_{(A,I)}(B/J) \to (\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$ of $\mathbb{F}_p$-CDGAs, where $\varphi^*_{(A,I)}(B/J)$ is concentrated in degree 0.

The derived de Rham cohomology $\mathrm{dR}_{(B,J,\delta)/(A,I,\gamma)}$ is computed by the de Rham complex $(\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$. The Whitehead tower $(\tau_{\geq n} \mathrm{dR}_{(B,J,\delta)/(A,I,\gamma)})_{n\in(\mathbb{Z},\geq)}$ defines a nonpositive[2.4.4] exhaustive filtration, thus the map $\varphi^*_{(A,I)}(B/J) \to (\Omega^*_{(B,J,\delta)/(A,I,\gamma)}, \mathrm{d})$ is a map of filtered $\mathbb{F}_p$-CDGAs (where $\varphi^*_{(A,I)}(B/J)$ is trivially filtered), which gives rise to a map of filtered $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras. Combined with the map above, we get a functor $\mathrm{dRCon}^0_{\mathbb{F}_p} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$.

Definition 2.4.37. *The* conjugate-filtered derived de Rham cohomology functor $\mathrm{Fil}^*_{\mathrm{conj}}\mathrm{dR}_{\cdot/\cdot}\colon \mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *along with the* structure map $\varphi^*_{(A\twoheadrightarrow A'')}(B'') \to \mathrm{Fil}^*_{\mathrm{conj}}\mathrm{dR}_{(B\twoheadrightarrow B'',\delta)/(A\twoheadrightarrow A'',\gamma)}$ *is defined to be the left derived functor (Proposition B.0.10) of the functor* $\mathrm{dRCon}^0_{\mathbb{F}_p} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$ *above.*

It follows either from Proposition B.0.10 and Lemma 2.2.43 or the fact that $\mathrm{AniPair} \simeq \mathcal{P}_\Sigma(\mathcal{D}^0) \subseteq \mathcal{P}(\mathcal{D}^0)$ is stable under filtered colimits (Proposition B.0.7) that

Lemma 2.4.38. *The conjugate filtration on the derived de Rham cohomology is exhaustive.*

We now prove the corresponding results of Subsection 2.4.1 for the conjugate filtration.

Lemma 2.4.39. *The conjugate-filtered derived de Rham cohomology functor* $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *preserves small colimits (note that so does the functor* $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}_{\mathbb{F}_p}, ((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \mapsto \varphi^*_{(A\twoheadrightarrow A'')}(B'')$*).*

Proof. First, we note that, for any connective $\mathbb{E}_\infty$-ring A, the Whitehead-tower functor $D(A) \to \mathrm{DF}(A), M \mapsto (\tau_{\geq n} M)_{n\in(\mathbb{Z},\geq)}$ is canonically lax symmetric monoidal (recall that $\mathrm{DF}(A)$ is endowed with the Day convolution). We give an informal description: given $M, N \in D(A)$, for all $m, n \in \mathbb{Z}$, the canonical map $\tau_{\geq m} M \to M$ and $\tau_{\geq n} N \to N$ gives rise to a map $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N) \to M \otimes^{\mathbb{L}}_A N$. Since $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N)$ is $(m+n)$-connective, this gives rise to a map $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N) \to \tau_{\geq m+n}(M \otimes^{\mathbb{L}}_A N)$. Assembling these maps, we get the lax symmetric monoidal structure. Next, when A is given by a field, in particular, $A = \mathbb{F}_p$, the structure above is in fact symmetric monoidal, since $(\tau_{\geq m} M) \otimes^{\mathbb{L}}_A (\tau_{\geq n} N) \to \tau_{\geq m+n}(M \otimes^{\mathbb{L}}_A N)$ is an equivalence for all $m, n \in \mathbb{Z}$.

Now recall that in a symmetric monoidal ∞-category, finite coproducts of commutative algebra objects are given by tensor products. It follows from Lemma 2.4.12 that the conjugate-filtered derived de Rham cohomology functor $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ is the left derived functor of a finite-coproduct-preserving functor, and then the result follows from Proposition B.0.10. ☐

Note that by the divided power Poincaré's lemma [Sta21, Tag 07LC], the conjugate filtration on the divided power polynomial algebra is trivial. The proof of Proposition 2.4.16 leads to

Proposition 2.4.40. *The natural transformation* $\mathrm{Fil}^*_{\mathrm{conj}}\mathrm{dR}_{(B\twoheadrightarrow B'',\delta)/(A\twoheadrightarrow A'',\gamma)} \to \mathrm{Fil}^*_{\mathrm{conj}}\mathrm{dR}_{(\mathrm{id}_{B''},0)/(A\to A'',\gamma)}$ *of functors* $\mathrm{dRCon} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{Z}))$ *induced by the counit map associated to* $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \in \mathrm{dRCon}$ *is an equivalence. In other words, the conjugate-filtered de Rham cohomology functor* $\mathrm{dRCon} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{Z}))$ *is* $(\mathrm{dRCon} \to \mathrm{CrysCon})$*-invariant (Definition 2.2.57) (note that so is the functor* $\mathrm{dRCon}_{\mathbb{F}_p} \to \mathrm{CAlg}_{\mathbb{F}_p}$, $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'', \delta)) \mapsto \varphi^*_{(A\twoheadrightarrow A'')}(B'')$*).*

Definition 2.4.41. *The* conjugate-filtered derived crystalline cohomology functor $\mathrm{Fil}^*_{\mathrm{conj}}\mathrm{CrysCoh}\colon \mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *along with the* structure map $\varphi^*_{(A\twoheadrightarrow A'',\gamma)}(R) \to \mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma)}$ *is defined to be the composite* $\mathrm{CrysCon} \to \mathrm{dRCon} \to \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$*, where the later functor is the conjugate-filtered derived de Rham cohomology functor combined with the structure map.*

2.4.4. In the literature, the conjugate filtration is increasing. We make it decreasing by negating the sign.

By Lemma 2.4.38, we have

LEMMA 2.4.42. *The conjugate filtration on the derived crystalline cohomology is exhaustive.*

Similar to Proposition 2.4.19, we have

PROPOSITION 2.4.43. *The conjugate-filtered derived crystalline cohomology functor* $\mathrm{CrysCon} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *preserves small colimits.*

Now we analyze the associated graded pieces of the conjugate filtration. Let $(A, I, \gamma) \to (B, J, \delta)$ be an element in dRCon^0. We recall the inverse[2.4.5] Cartier map $C^{-1}: \varphi^*_{(A,I)}(\Omega^\star_{(B/J)/(A/I)}) \to H^\star(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ of graded $\varphi^*_{(A,I)}(B/J)$-algebras (where $\star$ is the grading), then we deduce that this is in fact an isomorphism. Our presentation is adapted from the proof of [Kat70, Thm 7.2].

- $\star = \mathbf{0}$. This is the composite map $\varphi^*_{(A,I)}(B/J) \to B \to H^0(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$, i.e., the $\varphi^*_{(A,I)}(B/J)$-algebra structure on $H^0(\Omega_{(B,J)/(A,I)}, \mathrm{d})$.

- $\star = \mathbf{1}$. Consider the map $B \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ of sets given by $f \mapsto [f^{p-1} \mathrm{d} f]$. We first check that this map is additive: in $\Omega^1_{\mathbb{Z}[u,v]/\mathbb{Z}}$, we have

$$\begin{aligned}(u+v)^{p-1}\mathrm{d}(u+v) - u^{p-1}\mathrm{d}u - v^{p-1}\mathrm{d}v &= \frac{1}{p}(\mathrm{d}((u+v)^p) - \mathrm{d}(u^p) - \mathrm{d}(v^p))\\ &= \frac{1}{p}\mathrm{d}\left(\sum_{j=1}^{p-1}\binom{p}{j}u^j v^{p-1-j}\right)\\ &= \mathrm{d}\left(\sum_{j=1}^{p-1}\frac{1}{p}\binom{p}{j}u^j v^{p-1-j}\right)\end{aligned}$$

  We deduce the additivity by the map $\mathbb{Z}[u,v] \to B, u \mapsto f, v \mapsto g$.

  Now we note that the map $f \mapsto [f^{p-1} \mathrm{d} f]$ satisfies Leibniz rule (recall that $H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ is a $\varphi^*_{(A,I)}(B/J)$-module, therefore a B/J-module). Indeed, $[(f g)^{p-1} \mathrm{d}(f g)] = f^p [g^{p-1} \mathrm{d} g] + g^p [f^{p-1} \mathrm{d} f]$.

  Thus we get a derivation $B/J \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$, which gives rise to a B/J-linear map $\Omega^1_{(B/J)/(A/I)} \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ and after linearization, we get $\varphi^*_{(A,I)} \Omega^1_{(B/J)/(A,I)} \to H^1(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$.

- $\star > \mathbf{1}$. Taking the exterior power of the map for $\star = 1$.

Now we show the Cartier isomorphism:

LEMMA 2.4.44. *Let* $(A, I, \gamma) \to (B, J, \delta)$ *be an element in* $\mathrm{CrysCon}^0_{\mathbb{F}_p}$. *Then the inverse Cartier map* $C^{-1}: \varphi^*_{(A,I)}(\Omega^\star_{(B/J)/(A/I)}) \to H^\star(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$ *is an isomorphism of graded* $\varphi^*_{(A,I)}(B/J)$-*algebras.*

Proof. Recall that (B, J, δ) is of the form $(\Gamma_{A[X]}(Y) \twoheadrightarrow (A/I)[X], \delta)$. It is then direct to check that the inverse Cartier map C^{-1} factors as $\varphi^*_{(A,I)}(\Omega^\star_{(B/J)/(A/I)}) \to H^\star(\Omega^*_{(A[X], IA[X])/(A,I)}, \mathrm{d}) \to H^\star(\Omega^*_{(B,J)/(A,I)}, \mathrm{d})$, where the first map is the inverse Cartier map associated to $(A \twoheadrightarrow A/I, \gamma) \to (A[X], I A[X], \gamma)$, and the second map is an isomorphism by the divided power Poincaré's lemma [Sta21, Tag 07LC].

Thus we can assume that $(B, J, \delta) = (A[X], I A[X], \gamma)$. In this case, the inverse Cartier map is base-changed from that for $(A, 0, 0) \to (A[X], 0, 0)$ along $(A, 0, 0) \to (A, I, \gamma)$, thus we can assume that $I = 0$, which is [Kat70, Thm 7.2]. □

It then follows from Proposition B.0.10 that

2.4.5. A priori, the "inverse" Cartier map C^{-1} is not defined to be the inverse of a map, but just defined to be a map.

Proposition 2.4.45. *There exists a natural isomorphism*[2.4.6]

$$C^{-1}\colon \varphi^*_{(A\twoheadrightarrow A'')}(\textstyle\bigwedge^\star_{B''}\mathbb{L}_{B''/A''})[-\star]\to \mathrm{gr}^{-\star}_{\mathrm{conj}}\,\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$$

in $\mathrm{CAlg}(\mathrm{Gr}^{\leq 0}(D(\varphi^*_{A\twoheadrightarrow A''}(B''))))$*, called the* derived Cartier isomorphism *(cf. [Bha12a, Prop 3.5]), which is functorial*[2.4.7] *in* $((A\twoheadrightarrow A'',\gamma)\to(B\twoheadrightarrow B'',\delta))\in\mathrm{dRCon}_{\mathbb{F}_p}$*.*

Note that both functors are $(\mathrm{dRCon}_{\mathbb{F}_p}\to\mathrm{CrysCon}_{\mathbb{F}_p})$-invariant (Definition 2.2.57), it follows from Proposition 2.2.56 that

Proposition 2.4.46. *There exists a natural isomorphism*

$$C^{-1}\colon \varphi^*_{(A\twoheadrightarrow A'')}(\textstyle\bigwedge^\star_R\mathbb{L}_{R/A''})[-\star]\to \mathrm{gr}^{-\star}_{\mathrm{conj}}\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$$

in $\mathrm{CAlg}(\mathrm{Gr}^{\leq 0}(D(\varphi^*_{A\twoheadrightarrow A''}(R))))$*, called the* derived Cartier isomorphism*, which is functorial in* $((A\twoheadrightarrow A'',\gamma),A''\to R)\in\mathrm{CrysCon}_{\mathbb{F}_p}$*.*

2.4.3. Relative animated PD-envelope

As in the classical case [Sta21, Tag 07H9], there is a relative version of animated PD-envelope, which is needed to study the derived crystalline cohomology, which is defined by the adjunction of undercategories:

Lemma 2.4.47. (dual to [Lur09, Prop 5.2.5.1]) *Let* $\mathcal{C}\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{D}$ *be an adjoint pair of* ∞*-categories. Assume that the* ∞*-category* $\mathcal{D}$ *admits pushouts and let* $D\in\mathcal{D}$ *be an object. Then*

1. *The induced functor* $g\colon\mathcal{D}_{D/}\to\mathcal{C}_{GD/}$ *admits a left adjoint* f*.*
2. *The functor* f *is equivalent to the composition*

$$\mathcal{C}_{GD/}\xrightarrow{f'}\mathcal{D}_{FGD/}\xrightarrow{f''}\mathcal{D}_{D/}$$

where f' *is induced by* F *and* f'' *is induced by the pushout along the counit map* $FGD\to D$*.*

We note that this construction is functorial in $D\in\mathcal{D}$*.*

Notation 2.4.48. *We denote the comma category* $\mathrm{AniPDPair}\times_{\mathrm{AniPair}}\mathrm{Fun}(\Delta^1,\mathrm{AniPair})$ *by* PDEnvCon*, an object of which is denoted by* $(A\twoheadrightarrow A'',\gamma)\to(B\twoheadrightarrow B'')$*, instead of the cumbersome notation* $((A\twoheadrightarrow A'',\gamma),(A\twoheadrightarrow A'')\to(B\twoheadrightarrow B''))$*.*

Definition 2.4.49. *Let* $(A\twoheadrightarrow A'',\gamma)\in\mathrm{AniPDPair}$ *be an animated PD-pair. The* (relative) animated PD-envelope *of an animated pair in* $\mathrm{AniPair}_{(A\twoheadrightarrow A'')/}$ *is the image under the functor* $\mathrm{AniPair}_{(A\twoheadrightarrow A'')/}\to\mathrm{AniPDPair}_{(A\twoheadrightarrow A'',\gamma)/}$ *induced by the animated PD-envelope functor* $\mathrm{AniPair}\to\mathrm{AniPDPair}$ *by Lemma 2.4.47.*

Concretely, let $B\twoheadrightarrow B''$ *be an object in* $\mathrm{AniPair}_{(A\twoheadrightarrow A'')/}$ *and let* $(C\twoheadrightarrow A'',\gamma_C)$ *and* $(D\twoheadrightarrow B'',\gamma_D)$ *denote the animated PD-envelopes of* $A\twoheadrightarrow A''$ *and* $B\twoheadrightarrow B''$ *respectively (we have tacitly used Lemma 2.3.37). Then the relative animated PD-envelope of* $B\twoheadrightarrow B''$ *is given by* $(A\twoheadrightarrow A'',\gamma)\amalg_{(C\twoheadrightarrow A'',\gamma_C)}(D\twoheadrightarrow B'',\gamma_D)$ *where the map* $(C\twoheadrightarrow A'',\gamma_C)\to(A\twoheadrightarrow A'',\gamma)$ *is the counit map associated to* $(A\twoheadrightarrow A'',\gamma)\in\mathrm{AniPDPair}$ *and the map* $(C\twoheadrightarrow A'',\gamma_C)\to(D\twoheadrightarrow B'',\gamma_D)$ *is the image of* $(A\twoheadrightarrow A'')\to(B\twoheadrightarrow B'')$ *under the animated PD-envelope functor.*

This defines the (relative) animated PD-envelope functor $\mathrm{RelPDEnv}\colon\mathrm{PDEnvCon}\to\mathrm{Fun}(\Delta^1,\mathrm{AniPDPair})$.

Example 2.4.50. Let $(A\twoheadrightarrow A'',\gamma)\in\mathrm{AniPDPair}$ be an animated PD-pair. Then the animated PD-envelope of $A\twoheadrightarrow A''$ relative to $(A\twoheadrightarrow A'',\gamma)$ is given by $(A\twoheadrightarrow A'',\gamma)$. This follows from the fact that $A\twoheadrightarrow A''$ relative to $(A\twoheadrightarrow A'',\gamma)$ is the base change of $\mathrm{id}_{\mathbb{Z}}\colon\mathbb{Z}\to\mathbb{Z}$ relative to $(\mathrm{id}_{\mathbb{Z}}\colon\mathbb{Z}\to\mathbb{Z},0)$ along the map $(\mathrm{id}_{\mathbb{Z}}\colon\mathbb{Z}\to\mathbb{Z},0)\to(A\twoheadrightarrow A'',\gamma)$ of animated PD-pairs. Compare with Lemma 2.4.56.

2.4.6. To avoid the ambiguity of symbols, we suppress the asterisk on Fil* to avoid confusion with the pullback symbol φ^*.

2.4.7. Here we use the same convention as in Remark 2.3.61.

Example 2.4.51. Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{AniPDPair}$ be an animated PD-pair. Then the animated PD-envelope of $\mathrm{id}_{A''}: A'' \twoheadrightarrow A''$ relative to $(A \twoheadrightarrow A'', \gamma)$ is given by $(\mathrm{id}_{A''}: A'' \to A'', 0)$. This follows from checking the universal property of the unit map at $\mathrm{id}_{A''}: A'' \twoheadrightarrow A''$ of the adjunction $\mathrm{AniPair}_{(A \twoheadrightarrow A'')/} \rightleftarrows \mathrm{AniPDPair}_{(A \twoheadrightarrow A'', \gamma)/}$.

It follows immediately from Lemma 2.3.41 that

LEMMA 2.4.52. *Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{AniPDPair}$ be an animated PD-pair, $(B \twoheadrightarrow B'') \in \mathrm{AniPair}_{(A \twoheadrightarrow A'')/}$ an animated pair under $A \twoheadrightarrow A''$. Let $(C \twoheadrightarrow B'', \delta)$ denote its relative animated PD-envelope. Then the unit map $(B \twoheadrightarrow B'') \to (C \twoheadrightarrow B'')$ becomes an equivalence after rationalization.*

Recall that given a PD-pair (A, I, γ) and a map $(A, I) \to (B, J)$ of pairs with $A \to B$ being flat, the PD-structure γ extends to B, i.e, there exists a unique PD-structure $\bar{\gamma}$ on (B, IB) such that the map $(A, I) \to (B, J)$ of pairs gives rise to a map $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ of PD-pairs. Then the PD-envelope of (B, J) with respect to (A, I, γ) is the same as that with respect to the PD-pair $(B, IB, \bar{\gamma})$, which corresponds to the crystalline cohomology of B/J with respect to $(B, IB, \bar{\gamma})$. We now show an animated analogue (without flatness).

Let $\mathrm{CrysCon}_{\mathrm{surj}}$ denote the full subcategory $\mathrm{AniPDPair} \times_{\mathrm{Ani(Ring)}} \mathrm{Fun}(\Delta^1, \mathrm{Ani(Ring)})_{\geq 0} \subseteq \mathrm{CrysCon}$ spanned by objects $((A \twoheadrightarrow A'', \gamma), A'' \to R)$ such that $A'' \to R$ is also surjective. There is a canonical functor $\mathrm{PDEnvCon} \to \mathrm{CrysCon}_{\mathrm{surj}}$ given as follows: for every object $((A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'')) \in \mathrm{PDEnvCon}$, we get the commutative diagram

$$\begin{array}{ccc} A & \twoheadrightarrow & A'' \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & B'' \end{array}$$

in Ani(Ring), which gives rise to two surjective maps $B \otimes_A^{\mathbb{L}} A'' \twoheadrightarrow B''$ and $B \twoheadrightarrow B \otimes_A^{\mathbb{L}} A''$. Furthermore, the later admits a PD-structure: it is the underlying animated pair of the pushout $(\mathrm{id}_B : B \to B, 0) \amalg_{(\mathrm{id}_A: A \to A, 0)} (A \twoheadrightarrow A'', \gamma)$ in AniPDPair. We denote by $(B \twoheadrightarrow B \otimes_A^{\mathbb{L}} A'', \delta)$ this pushout. Then we get an object $((B \twoheadrightarrow B \otimes_A^{\mathbb{L}} A'', \delta), B \otimes_A^{\mathbb{L}} A'' \twoheadrightarrow B'')$ in PDEnvCon.

One verifies that

LEMMA 2.4.53. *The functor* $\mathrm{PDEnvCon} \to \mathrm{CrysCon}_{\mathrm{surj}}$ *constructed above admits a fully faithful right adjoint* $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{PDEnvCon}$ *given by* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \mapsto ((A \twoheadrightarrow A'', \gamma) \to (A \twoheadrightarrow R))$.

Thus $\mathrm{CrysCon}_{\mathrm{surj}}$ could be seen as a reflective subcategory (Definition 2.2.44) of PDEnvCon. Now we claim that

LEMMA 2.4.54. *The relative animated PD-envelope functor* $\mathrm{PDEnvCon} \to \mathrm{Fun}(\Delta^1, \mathrm{AniPDPair})$ *is* $(\mathrm{PDEnvCon} \to \mathrm{CrysCon}_{\mathrm{surj}})$*-invariant (Definition 2.2.57).*

Proof. For every object $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow B'')$ in PDEnvCon, we have a map $(A \twoheadrightarrow A'', \gamma) \to (B \twoheadrightarrow A'' \otimes_A^{\mathbb{L}} B, \delta)$ of animated PD-pairs. By the concrete description of the relative animated PD-envelope functor, it suffices to show that this map along with the counit maps forms a pushout diagram of animated PD-pairs. As discussed above, $(B \twoheadrightarrow A'' \otimes_A^{\mathbb{L}} B, \delta)$ is the pushout $(\mathrm{id}_B : B \to B, 0) \amalg_{(\mathrm{id}_A: A \to A, 0)} (A \twoheadrightarrow A'', \gamma)$. The counit maps for $(\mathrm{id}_A, 0)$ and $(\mathrm{id}_B, 0)$ are identities (Lemma 2.3.39). The result then follows from Proposition 2.3.34, which implies that counit maps are compatible with small colimits. □

Consequently, in order to study the relative animated PD-envelope functor, it suffices to study the composite $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{PDEnvCon} \to \mathrm{Fun}(\Delta^1, \mathrm{AniPDPair})$. By abuse of terminology, we will simply denote this functor as RelPDEnv as well and call the image (or after evaluation at $1 \in \Delta^1$) *the animated PD-envelope* of an object $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathrm{surj}}$. We remark that the functor $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{PDEnvCon}$ preserves small colimits by Proposition 2.3.34, therefore so does the composite functor.

We note that $\mathrm{CrysCon}_{\mathrm{surj}}$ is projectively generated: let $\mathrm{CrysCon}^0_{\mathrm{surj}} \subseteq \mathrm{CrysCon}_{\mathrm{surj}}$ be the full subcategory spanned by objects $((\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y, Z], \gamma), \mathbb{Z}[Y, Z] \twoheadrightarrow \mathbb{Z}[Z])$ for all finite sets X, Y, Z.

Lemma 2.4.55. *The full subcategory* $\mathrm{CrysCon}_{\mathrm{surj}}^0 \subseteq \mathrm{CrysCon}_{\mathrm{surj}}$ *constitutes a set of compact projective generators for* $\mathrm{CrysCon}_{\mathrm{surj}}$.

Proof. We only sketch the proof, which is similar to that of Theorem 2.3.23. The key observation is that the composite of forgetful functors $\mathrm{CrysCon}_{\mathrm{surj}}^0 \to \mathrm{Fun}(\Delta^2, \mathrm{Ani}(\mathrm{Ring}))_{\mathrm{surj}} \to \mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})_{\mathrm{surj}}, ((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \mapsto (A \twoheadrightarrow A'' \twoheadrightarrow R)$, which preserves filtered colimits and geometric realizations by Proposition 2.3.34, admits a left adjoint, where $\mathrm{Fun}(\Delta^2, \mathcal{C})_{\mathrm{surj}} \subseteq \mathrm{Fun}(\Delta^2, \mathcal{C})$ is the full subcategory spanned by $(X \to Y \to Z) \in \mathrm{Fun}(\Delta^2, \mathcal{C})$ such that $X \to Y$ and $Y \to Z$ are surjective, for $\mathcal{C} = D(\mathbb{Z})_{\geq 0}$ and $\mathcal{C} = \mathrm{Ani}(\mathrm{Ring})$.

The ∞-category $\mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})_{\mathrm{surj}}$ admits a set $\{\mathbb{Z}X \oplus \mathbb{Z}Y \oplus \mathbb{Z}Z \twoheadrightarrow \mathbb{Z}Y \oplus \mathbb{Z}Z \twoheadrightarrow \mathbb{Z}Z \mid X, Y, Z \in \mathrm{Fin}\}$ of compact projective generators which spans the full subcategory $\mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})_{\mathrm{surj}}^0$, which follows from the fact that the left adjoint to the left derived functor $\mathcal{P}_\Sigma(\mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})_{\mathrm{surj}}^0) \to \mathrm{Fun}(\Delta^2, D(\mathbb{Z})_{\geq 0})_{\mathrm{surj}}^0$ is conservative (cf. the proof of [Lur17, Prop 25.2.1.2]).

The result then follows from Proposition B.0.15. ☐

By Proposition B.0.10, the functor $\mathrm{CrysCon}_{\mathrm{surj}} \to \mathrm{Fun}(\Delta^1, \mathrm{AniPDPair})$ is the left derived functor of the restricted functor $\mathrm{CrysCon}_{\mathrm{surj}}^0 \to \mathrm{Fun}(\Delta^1, \mathrm{AniPDPair})$, which is concretely given as follows:

Lemma 2.4.56. *The relative animated PD-envelope of an object* $((\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y, Z], \gamma), \mathbb{Z}[Y, Z] \twoheadrightarrow \mathbb{Z}[Z]) \in \mathrm{CrysCon}_{\mathrm{surj}}^0$ *is functorially given by* $(\Gamma_{\mathbb{Z}[Z]}(X, Y) \twoheadrightarrow \mathbb{Z}[Z], \tilde{\gamma}) \in \mathrm{AniPDPair}_{(\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y,Z], \gamma)/}$, *i.e. coincides with the classical relative PD-envelope.*

Proof. First, by the adjointness, there exists a functorial comparison map from the relative animated PD-envelope to $(\Gamma_{\mathbb{Z}[Z]}(X, Y) \twoheadrightarrow \mathbb{Z}[Z], \tilde{\gamma})$. It suffices to show that this is an equivalence.

In this case, $((\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y, Z], \gamma), \Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Z]) \in \mathrm{PDEnvCon}$ is the base change of $((\mathrm{id}_{\mathbb{Z}[Y,Z]}, 0), \mathbb{Z}[Y, Z] \twoheadrightarrow \mathbb{Z}[Z]) \in \mathrm{PDEnvCon}$ along $(\mathrm{id}_{\mathbb{Z}[Y,Z]}, 0) \to (\Gamma_{\mathbb{Z}[Y,Z]}(X) \twoheadrightarrow \mathbb{Z}[Y, Z], \gamma)$. The result then follows from the base-change property of the relative adjunction, along with the simple fact that the (absolute) animated PD-envelope of $\mathbb{Z}[Y, Z] \twoheadrightarrow \mathbb{Z}[Z]$ is $(\Gamma_{\mathbb{Z}[Z]}(Y) \twoheadrightarrow \mathbb{Z}[Z], \gamma)$. ☐

As a generalization of Definition 2.3.59, we now introduce the *conjugate filtration* on the relative animated PD-envelope in char p. Let $(A \twoheadrightarrow A'', \gamma) \in \mathrm{PDPair}_{\mathbb{F}_p}$ be a PD-pair and $I \subseteq A''$ an ideal. We recall that there is a canonical "Frobenius" map $\varphi_{A \twoheadrightarrow A''}: A'' \to A$ by Lemma 2.4.36. We suppose that $\varphi_{A \twoheadrightarrow A''}$ is flat[2.4.8]. Let (B, J, δ) denote the classical PD-envelope of $(A \twoheadrightarrow A''/I)$ relative to $(A \twoheadrightarrow A'', \gamma)$. We note that $B/J \cong A''/I$. As in the absolute case, due to the PD-structure (B, J, δ), there is a canonical $\varphi_{A \twoheadrightarrow A''}^*(A''/I)$-algebra structure on B, and we consider the nonpositive filtration on B given by $\mathrm{Fil}^{-n}(B)$ for $n \in \mathbb{N}$ being the $\varphi_{A \twoheadrightarrow A''}^*(A''/I)$-submodule of B generated by $\{\gamma_{i_1 p}(f_1) \cdots \gamma_{i_m p}(f_m) \mid i_1 + \cdots + i_m \leq n$ and $f_1, \ldots, f_m \in I\}$. We have the following relative version of Lemma 2.3.58, for which the proof of [Bha12a, Lem 3.42] adapts:

Lemma 2.4.57. *Let* $(A \twoheadrightarrow A'', \gamma)$ *be a PD-*$\mathbb{F}_p$*-pair such that* $\varphi_{A \twoheadrightarrow A''}$ *is flat, and let* $I \subseteq A''$ *be an ideal such that* I/I^2 *is a flat* A''/I*-module. The relative PD-envelope* (B, J, δ) *and the filtration* $\mathrm{Fil}^* B$ *are constructed above.*

Then there is a comparison map $\varphi_{A \twoheadrightarrow A''}^*(\Gamma_{A''/I}^i(I/I^2)) \to \mathrm{gr}^{-i} B$ *of* $\varphi_{A \twoheadrightarrow A''}^*(A''/I)$*-modules induced by the maps* $(\gamma_{ip})_{i \in \mathbb{N}}$ *(as in Lemma 2.3.58) which is functorial in* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow A''/I)$ *in a subcategory of* $\mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$*. Furthermore, if* $I \subseteq A''$ *is generated by a Koszul-regular sequence*[2.4.9]*, then the comparison map above is an isomorphism.*

Definition 2.4.58. *The* conjugate filtration functor (on the animated PD-envelope) $\mathbb{L}\,\mathrm{ConjFil}: \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *together with the* structure map *of functors* $\mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p))$ *from* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \mapsto \varphi_{A \twoheadrightarrow A''}^*(R) = R \otimes_{A'', \varphi_{A \twoheadrightarrow A''}}^{\mathbb{L}} A$ *to* $\mathbb{L}\,\mathrm{ConjFil}$ *is defined to be the left derived functor (Proposition B.0.10) of* $\mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}^0 \ni ((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow A''/I) \mapsto (\varphi_{A \twoheadrightarrow A''}^*(A''/I) \to \mathrm{Fil}^* B) \in \mathrm{Fun}(\Delta^1, \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\mathbb{F}_p)))$ *constructed above.*

2.4.8. This is satisfied when $(A \twoheadrightarrow A'', \gamma) \in \mathcal{E}^0$, which is the only case that we need to develop the theory. For more examples, see Remark 2.4.62.

2.4.9. We only need the simple case that $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow A''/I) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{conj}}$, which "simplifies" the proof in the sense that a "brute-force" computation suffices.

As in the absolute case (including Remark 2.3.61), it follows from Lemma 2.4.57 that

COROLLARY 2.4.59. *For every* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$, *there exists an equivalence*

$$\varphi^*_{A\twoheadrightarrow A''}(\Gamma^i_R(\mathrm{gr}^1(\mathbb{L}\,\mathrm{AdFil}(A'' \twoheadrightarrow R)))) \to \mathrm{gr}^{-i}(\mathbb{L}\,\mathrm{ConjFil}((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R))$$

in $D(\varphi^*_{A\twoheadrightarrow A''}(R))_{\geq 0}$ *for all* $i \in \mathbb{N}$ *which is functorial in* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$.

As in the absolute case, we have

COROLLARY 2.4.60. *For every* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$ *such that* $A'' \twoheadrightarrow R$ *is a quasiregular animated pair, let* $(B \twoheadrightarrow R, \delta)$ *denote the relative animated PD-envelope. Then* B *is a flat* $\varphi^*_{A\twoheadrightarrow A''}(R)$*-module.*

and similar to Proposition 2.3.72, we have

PROPOSITION 2.4.61. *Let* $(A \twoheadrightarrow A'', \gamma) \in \mathrm{PDPair}$ *be a PD-pair and* $I \subseteq A''$ *an ideal generated by a Koszul-regular sequence. Let* $(B \twoheadrightarrow B'', \delta)$ *denote the relative animated PD-envelope of* $((A \twoheadrightarrow A'', \gamma), A'' \to A''/I) \in \mathrm{CrysCon}$. *Then* $(B \twoheadrightarrow B'', \delta)$ *is a PD-pair, therefore coincides with the classical relative PD-envelope.*

Remark 2.4.62. More precisely, in Corollary 2.3.66, the map $\varphi^*_{A\twoheadrightarrow A''}(R) \to B$ is induced by the Frobenius map $\varphi_{B\twoheadrightarrow R}: R \to B$ (which could be seen by left deriving the special case that $((A \twoheadrightarrow A'', \gamma), A'' \to R) \in \mathrm{CrysCon}^0_{\mathbb{F}_p,\mathrm{surj}}$). In particular, if the Frobenius map $\varphi_{A\twoheadrightarrow A''}: A'' \to A$ is flat, then so is the Frobenius map $\varphi_{B\twoheadrightarrow R}: R \to B$.

For example, when R is a quasiregular semiperfect $\mathbb{F}_p$-algebra [BMS19, Def 8.8], we set $(A \twoheadrightarrow A'', \gamma) = (\mathrm{id}_{R^\flat}: R^\flat \to R^\flat, 0)$ and the map $A'' \to R$ to be the canonical map, by definition, $R^\flat$ is a perfect $\mathbb{F}_p$-algebra therefore $\varphi_{R^\flat}$ is flat. Then the animated PD-envelope $B \twoheadrightarrow R$ of $A'' \to R$ satisfies the condition that the Frobenius map $\varphi_{B\to R}: R \to B$ is flat and hence B is static. It follows that $(R \twoheadrightarrow B, \delta)$ is a PD-pair (Proposition 2.3.32).

Note that the associated graded pieces of derived crystalline cohomology and relative animated PD-envelope of a "surjective" crystalline context $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$, with respect to conjugate filtrations, are equivalent by Corollaries 2.4.59 and 2.3.54 and Proposition 2.4.46. In fact, we have

LEMMA 2.4.63. *There is a canonical equivalence*

$$\mathrm{Fil}_{\mathrm{conj}}\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma)} \to \mathbb{L}\,\mathrm{ConjFil}((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R)$$

in $\mathrm{CAlg}(\mathrm{DF}^{\leq 0}(\varphi^*_{A\twoheadrightarrow A''}(R)))$ *which is functorial*[2.4.10] *in* $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R) \in \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$.

Proof. We first point out how to produce the comparison map of underlying $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras, i.e., ignoring the $\varphi^*_{A\to A''}(R)$-algebra structures and conjugate filtrations. This is logically not necessary but it benefits our understanding. Given $((A \twoheadrightarrow A'', \gamma), A'' \twoheadrightarrow R)$, let $(B \twoheadrightarrow R, \delta)$ denote its relative animated PD-envelope. It follows from Proposition 2.4.16 that the crystalline cohomology $\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma)}$ is naturally equivalent to the derived de Rham cohomology $\mathrm{dR}_{(B\twoheadrightarrow R,\delta)/(A\twoheadrightarrow A'',\gamma)}$, and by definition, it is equipped with a map $\mathrm{dR}_{(B\twoheadrightarrow R,\delta)/(A\twoheadrightarrow A'',\gamma)} \to B$ of $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras, which gives rise to the underlying comparison map that we want.

By Lemma 2.4.55 and Proposition B.0.10, it suffices to construct the equivalence restricted to the full subcategory $\mathrm{CrysCon}^0_{\mathbb{F}_p,\mathrm{surj}} \subseteq \mathrm{CrysCon}_{\mathbb{F}_p,\mathrm{surj}}$, i.e., to establish the equivalence for all $((\Gamma_{\mathbb{F}_p[Y,Z]}(X) \twoheadrightarrow \mathbb{F}_p[Y, Z], \gamma_0), \mathbb{F}_p[Y, Z] \twoheadrightarrow \mathbb{F}_p[Z]) \in \mathrm{CrysCon}^0_{\mathbb{F}_p,\mathrm{surj}}$. This is essentially [Bha12a, Lem 3.29 & Thm 3.27]. We will briefly sketch the argument. The preceding paragraph has already established a comparison map of underlying $\mathbb{E}_\infty$-$\mathbb{F}_p$-algebras. The key point is that both sides are static: the relative animated PD-envelope is static by definition, and the derived crystalline cohomology is static by Cartier isomorphism (Proposition 2.4.46) and the fact that static modules are closed under extension and filtered colimits, see Corollary 2.3.68 for a similar argument. Then the result follows from explicit simplicial resolution. □

2.4.10. Here we apply the same convention as in Remark 2.3.61.

We now deduce the integral version of the comparison above. We recall that $\mathbb{L}\,\mathrm{PDFil}: \mathrm{AniPDPair} \to \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$ is the PD-filtration functor (Definition 2.3.73), and Fil_H is the Hodge-filtration.

Proposition 2.4.64. *There is a canonical equivalence*

$$\mathrm{Fil}_H\,\mathrm{CrysCoh} \to \mathbb{L}\,\mathrm{PDFil} \circ \mathrm{RelPDEnv}$$

of functors $\mathrm{CrysCon}_{\mathrm{surj}} \rightrightarrows \mathrm{CAlg}(\mathrm{DF}^{\geq 0}(\mathbb{Z}))$.

Proof. The comparison map is established in the same way as in the proof of Lemma 2.4.63. It suffices to show that this is an equivalence.

We first show that this becomes an equivalence after passing to underlying $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, i.e. ignoring the Hodge filtration. By conservativity of the forgetful functor $\mathrm{CAlg}_{\mathbb{Z}} \to D(\mathbb{Z})$, it suffices to show the equivalence for underlying $\mathbb{Z}$-module spectra, which follows from Lemmas 2.4.63, 2.4.52, and 2.4.11.

To establish the equivalence of filtered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, it remains to show that the comparison map induces equivalences after passing to associated graded pieces, and by Lemma 2.2.1, it suffices to prove the result restricted to the full subcategory $\mathrm{CrysCon}^0_{\mathrm{surj}} \subseteq \mathrm{CrysCon}_{\mathrm{surj}}$, which is essentially due to [Ill72, Cor VIII.2.2.8], see [Bha12a, Rem 3.33]. □

2.4.4. Affine crystalline site

We now turn to the site-theoretic aspects of the derived crystalline cohomology by showing that the derived crystalline cohomology is equivalent to the cohomology of the *affine crystalline site* under a mild smoothness condition. We warn the reader again that our theory is non-completed. Fix a crystalline context $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$.

Definition 2.4.65. *The* affine crystalline site $\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))$ *is defined to be the opposite ∞-category of animated PD-pairs $(B \twoheadrightarrow B'', \gamma_B)$ under $(A \twoheadrightarrow A'', \gamma_A)$ along with an equivalence $R \xrightarrow{\simeq} B''$ of A-algebras, depicted by the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & & B \arrow[d] \\ A'' \arrow[r] & R \arrow[r, "\simeq"] & B'' \end{tikzcd}$$

which we will simply denoted by $\left(R \xrightarrow{\simeq} B'' \twoheadleftarrow B\right) \in \mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))$. More formally, it is the homotopy fiber of the functor $\mathrm{AniPDPair}_{(A \twoheadrightarrow A'')/} \to \mathrm{Ani}(\mathrm{Ring})_{A''/}, (B \twoheadrightarrow B'', \gamma_B) \mapsto B''$ at the object $R \in \mathrm{Ani}(\mathrm{Ring})_{A''/}$. The endowed Grothendieck topology is indiscrete.

The structure presheaf $\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A))^{\mathrm{op}} \to \mathrm{CAlg}_A$*, denoted by $\mathcal{O}_{\mathrm{CRIS}(R/(A \twoheadrightarrow A'', \gamma_A))}$ (or simply $\mathcal{O}$ when there is no ambiguity), is induced by the evaluation* $\mathrm{AniPDPair}_{(A \twoheadrightarrow A'', \gamma_A)/} \to \mathrm{Ani}(\mathrm{Ring})_{A/} \to \mathrm{CAlg}_{A/}$*. Concretely, it is given by $\left(R \xrightarrow{\simeq} B'' \twoheadleftarrow B\right) \mapsto B$.*

Although the affine crystalline site is not small, the cohomology of the structure presheaf exists in CAlg_A by Čech-Alexander calculation (which we will reproduce in Proposition 2.4.70). We will simply call it the *cohomology of the crystalline site* and denote by $R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A)), \mathcal{O})$. Furthermore, the structure sheaf admits the PD-filtration (Definition 2.3.73), which gives rise to a filtration on the cohomology of the crystalline site, called the *Hodge-filtration* and denoted by Fil_H. We now have a comparison between the derived crystalline cohomology and the cohomology of the crystalline site, which becomes an equivalence after Hodge-completion:

Proposition 2.4.66. *There is a natural comparison map*

$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A \twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H\,R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A'', \gamma_A)), \mathcal{O})$$

in the ∞-category $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$. After passing to the associated graded pieces, i.e. composition with the functor $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A)) \to \mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(A))$, the comparison map above becomes an equivalence. Moreover, when $\pi_0(R)$ is a finitely generated $\pi_0(A'')$-algebra, then the comparison map is an equivalence.

We need some preparation about cosimplicial objects in ∞-categories.

DEFINITION 2.4.67. ([LUR09, DEF 6.1.2.2]) *Let $\mathcal{C}$ be an ∞-category. A* cosimplicial object *of $\mathcal{C}$ is a functor $X^\bullet:\mathbf{\Delta}\to\mathcal{C}$. The value of this functor at $[\nu]\in\mathbf{\Delta}$ is denoted by X^ν. A* map of cosimplicial objects *$X^\bullet\to Y^\bullet$ is simply a map of functors.*

We note that there are two inclusions $\{0\}\hookrightarrow[1]\hookleftarrow\{1\}$ viewed as two maps $[0]\rightrightarrows[1]$, and a constant map $[1]\to[0]$, which induce three functors i_0, i_1: $\mathbf{\Delta}\simeq\mathbf{\Delta}_{/[0]}\rightrightarrows\mathbf{\Delta}_{/[1]}$ and $\delta:\mathbf{\Delta}_{/[1]}\to\mathbf{\Delta}_{/[0]}\simeq\mathbf{\Delta}$. For any ∞-category $\mathcal{C}$, let i_0^* (resp. i_1^*) denote the induced functor $\mathrm{Fun}(\mathbf{\Delta}_{/[1]},\mathcal{C})\to\mathrm{Fun}(\mathbf{\Delta},\mathcal{C})$, and let δ^* denote the induced functor $\mathrm{Fun}(\mathbf{\Delta},\mathcal{C})\to\mathrm{Fun}(\mathbf{\Delta}_{/[1]},\mathcal{C})$.

DEFINITION 2.4.68. ([LUR17, DEF 7.2.1.6]) *Let $\mathcal{C}$ be an ∞-category and let f and g be two maps $X^\bullet\rightrightarrows Y^\bullet$ of cosimplicial objects. A* simplicial homotopy from f to g *is a map $h:\delta^*(X^\bullet)\to\delta^*(Y^\bullet)$ of functors $\mathbf{\Delta}_{/[1]}\rightrightarrows\mathcal{C}$ such that the map $i_0^*(h):X^\bullet\to Y^\bullet$ (resp. $i_1^*(h):X^\bullet\to Y^\bullet$), being a map of cosimplicial objects, is equivalent to f (resp. g). When $X^\bullet=Y^\bullet$, we say that the simplicial homotopy $h:\delta^*(X^\bullet)\to\delta^*(X^\bullet)$ is* constant *if it is equivalent to $\mathrm{id}_{\delta^*(X^\bullet)}$.*

LEMMA 2.4.69. *Let $\mathcal{C}$ be an ∞-category and $X^\bullet,Y^\bullet$ two cosimplicial objects of which the totalization exist in $\mathcal{C}$. Let f and g be two maps $X^\bullet\rightrightarrows Y^\bullet$ of cosimplicial objects such that there exists a simplicial homotopy from f to g. Then the maps f,g induces equivalent*[2.4.11] *maps $\lim_{\mathbf{\Delta}}X^\bullet\rightrightarrows\lim_{\mathbf{\Delta}}Y^\bullet$ of totalizations.*

Proof. (DENIS NARDIN) For every cosimplicial object $X^\bullet$ in $\mathcal{C}$, there are two observations:

1. The canonical map $\lim_{\mathbf{\Delta}_{/[1]}}\delta^*(X^\bullet)\to\lim_{\mathbf{\Delta}}X^\bullet$ is an equivalence (this involves the existence of the limit as the source). Indeed, it suffices to show that the map $\delta:\mathbf{\Delta}_{/[1]}\to\mathbf{\Delta}$ is coinitial. By Joyal's version of Quillen's Theorem A [Lur09, Thm 4.1.3.1], it suffices to show that, for every $[n]\in\mathbf{\Delta}$, the category $\mathbf{\Delta}_{/[1]}\times_{\mathbf{\Delta}}\mathbf{\Delta}_{/[n]}$ is weakly contractible. Its geometric realization is $\Delta^1\times\Delta^n$, which is known to be weakly contractible.
2. The two maps $\lim_{\mathbf{\Delta}}X=\lim_{\mathbf{\Delta}}i_\nu^*\delta^*(X^\bullet)\to\lim_{\mathbf{\Delta}_{/[1]}}\delta^*(X^\bullet)$ for $\nu=0,1$ are equivalences, and these two maps are equivalent. Indeed, both are inverses of the equivalence $\lim_{\mathbf{\Delta}_{/[1]}}\delta^*(X^\bullet)\to\lim_{\mathbf{\Delta}}X^\bullet$ above.

Note that the map $\lim_{\mathbf{\Delta}}f$ (resp. $\lim_{\mathbf{\Delta}}g$) could be identified with the composite

$$\lim_{\mathbf{\Delta}}X=\lim_{\mathbf{\Delta}}i_\nu^*\delta^*(X^\bullet)\longrightarrow\lim_{\mathbf{\Delta}_{/[1]}}\delta^*(X^\bullet)\xrightarrow{\lim_{\mathbf{\Delta}_{/[1]}}(h)}\lim_{\mathbf{\Delta}_{/[1]}}\delta^*(Y^\bullet)\longrightarrow\lim_{\mathbf{\Delta}}Y^\bullet$$

for $\nu=0$ (resp. $\nu=1$). The result then follows. □

Proof of Proposition 2.4.66. There is a map from the constant presheaf $\underline{\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma_A)}}$ on the affine crystalline site to the structure presheaf $\mathcal{O}$ given by the canonical map in Definition 2.4.25, which induces the comparison map in question.

Now we show that this map becomes an equivalence after passing to the associated graded pieces. We first note that, when the map $A''\to R$ is surjective, i.e. $((A\twoheadrightarrow A'',\gamma_A),A''\to R)\in\mathrm{CrysCon}_{\mathrm{surj}}$, the result follows directly from Proposition 2.4.64. Our strategy is to reduce the general case to this special case via Čech-Alexander computation.

We pick a polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P\twoheadrightarrow R$ of A-algebras. Let $P^\bullet\to R$ denote the Čech conerve of the object $P\to R$ in the ∞-category $(\mathrm{Ani}(\mathrm{Ring})_{A/})_{/R}$. Concretely, it is given by $P^\nu:=P^{\otimes_A^{\mathbb{L}}(\nu+1)}$, and the map $P^\nu\to R$ is simply given by the composite map $P^\nu\to P\to R$ which is surjective. In other words, we get a cosimplicial object $(P^\bullet\twoheadrightarrow R)\in\mathrm{Fun}(\Delta,\mathrm{AniPair}_{(A\twoheadrightarrow A'')/})$. Let $(D^\bullet\twoheadrightarrow R,\gamma_{D^\bullet})\in\mathrm{Fun}(\Delta,\mathrm{AniPDPair}_{(A\twoheadrightarrow A'',\gamma_A)/})$ denote the cosimplicial relative animated PD-envelope, i.e. applying the functor $\mathrm{AniPair}_{(A\twoheadrightarrow A'')/}\to\mathrm{AniPDPair}_{(A\twoheadrightarrow A'',\gamma_A)/}$ (Definition 2.4.49) pointwise. This effectively gives rise to a cosimplicial object $\Delta\to\mathrm{Cris}(R/(A\twoheadrightarrow A'',\gamma_A))^{\mathrm{op}}$. Composing with the Hodge-filtered presheaf $\mathrm{Fil}_H\,\mathcal{O}:\mathrm{Cris}(R/(A\twoheadrightarrow A'',\gamma_A))^{\mathrm{op}}\to\mathrm{CAlg}(\mathrm{DF}^{\geq0}(A))$, we get a cosimplicial filtered $\mathbb{E}_\infty$-A-algebra $\Delta\to\mathrm{CAlg}(\mathrm{DF}^{\geq0}(A))$, the limit of which computes the cohomology $\mathrm{Fil}_H\,R\Gamma(\mathrm{Cris}(R/(A\twoheadrightarrow A'',\gamma_A)),\mathcal{O})$. In plain terms, this cosimplicial filtered $\mathbb{E}_\infty$-A-algebra is just the PD-filtration of the cosimplicial animated PD-pair $(D^\bullet\twoheadrightarrow R,\gamma_{D^\bullet})$.

2.4.11. Or called "homotopic". We avoid the terminology "homotopic" to avoid confusion with the simplicial homotopy.

For this cosimplicial object, the comparison map constructed above is concretely given by

$$\mathrm{Fil}_H\,\mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(A\twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_{\mathrm{PD}}\, D^\bullet \tag{2.4.1}$$

Now Proposition 2.4.64 and Lemma 2.4.53 gives us an equivalence

$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})} \to \mathrm{Fil}_{\mathrm{PD}}\, D^\bullet$$

which is effectively given by

$$\mathrm{Fil}_H\,\mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})} \to \mathrm{Fil}_{\mathrm{PD}}\, D^\bullet$$

by chasing the proof. In other words, (2.4.1) could be rewritten as the natural map

$$\mathrm{Fil}_H\,\mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(A\twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H\,\mathrm{dR}_{(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})}$$

or equivalently, the natural map

$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})}$$

It remains to show that this cosimplicial map gives rise to an equivalence after taking the limit, i.e., the totalization, and passing to associated graded pieces. We isolate the remaining part into Lemma 2.4.77. ☐

Before proving Lemma 2.4.77, we isolate an important observation in the previous proof into a proposition:

Proposition 2.4.70. *For every crystalline context* $((A \twoheadrightarrow A'', \gamma_A), A'' \to R) \in \mathrm{CrysCon}$*, the followings are equivalent:*

1. *The comparison map in Proposition 2.4.66 is an equivalence.*
2. *There exists a polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$ of A-algebras, and letting $P^\bullet \to R$ denote the Čech conerve of $P \to R$ in the ∞-category* $(\mathrm{Ani}(\mathrm{Ring})_{A/})_{/R}$ *as in the proof of Proposition 2.4.66, then the natural maps*
$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'', \gamma_A)} \to \mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})} \tag{2.4.2}$$
form a limit diagram in $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$.
3. *For all polynomial A-algebras P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$ of A-algebras, and letting $P^\bullet \to R$ denote the Čech conerve of $P \to R$ in the ∞-category* $(\mathrm{Ani}(\mathrm{Ring})_{A/})_{/R}$*, then the natural maps (2.4.2) form a limit diagram in* $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))$.
4. *(After proving Lemma 2.4.77) There exists a (or equivalently, for every) polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$ of A-algebras, and letting $P^\bullet \to R$ denote the Čech conerve of $P \to R$ in the ∞-category* $(\mathrm{Ani}(\mathrm{Ring})_{A/})_{/R}$ *as in the proof of Proposition 2.4.66, then the natural maps*
$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'', \gamma_A)} \to \mathrm{CrysCoh}_{R/(P^\bullet \twoheadrightarrow P^\bullet \otimes^{\mathbb{L}}_A A'', \gamma_{P^\bullet})}$$
form a limit diagram in $\mathrm{CAlg}(D(A))$.

In order to deal with associated graded pieces of the Hodge filtration, we need a variant of the *Katz-Oda filtration* in [GL20, Cons 3.12]. We need an auxiliary construction:

Definition 2.4.71. *The* cotangent complex functor $\mathbb{L}_{\cdot/\cdot}\colon \mathrm{dRCon} \to \mathrm{Ani}(\mathrm{Mod})$ *is defined to be the left derived functor (Proposition B.0.10) of the functor* $\mathrm{dRCon}^0 \to \mathrm{Ani}(\mathrm{Mod}), ((A, I, \gamma_A) \to (B, J, \gamma_B)) \mapsto (B, \Omega^1_{(B,J)/(A,I)})$.

The proof of Lemma 2.4.10 leads to

Lemma 2.4.72. *The composite functor* $\mathrm{Fun}(\Delta^1, \mathrm{Ani}(\mathrm{Ring})) \to \mathrm{dRCon} \to \mathrm{Ani}(\mathrm{Mod})$ *is equivalent to the classical cotangent complex functor.*

We now introduce the *"stupid" filtration* Fil_B on the Hodge-filtered derived de Rham cohomology $\mathrm{Fil}_H \mathrm{dR}_{\cdot/\cdot}$. For each $((A,I,\gamma_A)\to(B,J,\gamma_B))\in\mathrm{dRCon}^0$, consider the filtration $(\Omega^{\geq n}_{(B,J,\gamma_B)/(A,I,\gamma_A)}, \mathrm{d})_{n\in(\mathbb{N},\geq)}$ of the Hodge-filtered CDGA, which gives rise to a bifiltered $\mathbb{E}_\infty$-$\mathbb{Z}$-algebra. By Proposition B.0.10, we get a functor $\mathrm{CAlg}(\mathrm{Fun}((\mathbb{N},\geq)\times(\mathbb{N},\geq),D(\mathbb{Z})))$, $((A\twoheadrightarrow A'',\gamma_A)\to(B\twoheadrightarrow B'',\gamma_B))\mapsto\mathrm{Fil}_B\,\mathrm{Fil}_H\,\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$.

Warning 2.4.73. Unlike the Hodge filtration, the "stupid" filtration does not descend to CrysCon, that is to say, it depends on the choice of B in question.

We now analyze the associated graded pieces with respect to the "stupid" filtration:

LEMMA 2.4.74. *Let $((A\twoheadrightarrow A'',\gamma_A)\to(B\twoheadrightarrow B'',\gamma_B))\in\mathrm{dRCon}$ be a de Rham context. Then associated graded pieces $\mathrm{gr}^i_B\,\mathrm{Fil}_H\,\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$ could be functorially identified with $\mathrm{ins}^i(\bigwedge^i_B \mathbb{L}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}[-i])\otimes^{\mathbb{L}}_B\mathrm{Fil}_{\mathrm{PD}}\,B$ as a $\mathrm{Fil}_{\mathrm{PD}}\,B$-module in $\mathrm{DF}^{\geq 0}(B)$ (where $\cdot\otimes^{\mathbb{L}}_B\mathrm{Fil}_{\mathrm{PD}}\,B$ is the base change from $\mathrm{DF}^{\geq 0}(B)$ to the ∞-category of $\mathrm{Fil}_{\mathrm{PD}}\,B$-modules). Furthermore, $\mathrm{Fil}^i_B\,\mathrm{gr}^j_H\,\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}\simeq 0$ when $i>j$.*

Proof. By Proposition B.0.10, it suffices to check the equivalences on dRCon^0, which follows from definitions. □

We are now ready to introduce the Katz-Oda filtration:

DEFINITION 2.4.75. (CF. [GL20, CONS 3.12]) *Let $(A\twoheadrightarrow A'',\gamma_A)\to(B\twoheadrightarrow B'',\gamma_B)$ be a map of animated PD-pairs and $B''\to R$ a map of animated rings. The* Katz-Oda filtration *on the Hodge-filtered derived crystalline cohomology $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ rewritten as $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}\otimes^{\mathbb{L}}_{\mathrm{Fil}_H\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}}\mathrm{Fil}_H\,\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$ is induced by the "stupid" filtration on $\mathrm{Fil}_H\,\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}$.*

We now have

LEMMA 2.4.76. (CF. [GL20, LEM 3.13]) *Let $(A\twoheadrightarrow A'',\gamma_A)\to(B\twoheadrightarrow B'',\gamma_B)$ be a map of animated PD-pairs and $B''\to R$ a map of animated rings. Then*

1. *The associated graded pieces $\mathrm{gr}^i_{\mathrm{KO}}\,\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ are functorially equivalent to*
$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(B\twoheadrightarrow B'')}\otimes^{\mathbb{L}}_{\mathrm{Fil}_{\mathrm{PD}}B}(\mathrm{ins}^i(\textstyle\bigwedge^i_B\mathbb{L}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}[-i])\otimes^{\mathbb{L}}_B\mathrm{Fil}_{\mathrm{PD}}\,B)$$
as $\mathrm{Fil}_{\mathrm{PD}}\,B$-modules in $\mathrm{DF}^{\geq 0}(\mathbb{Z})$ for all $i\in\mathbb{N}$, where the functor ins^i is defined in Subsection 2.2.4.
2. *The induced Katz-Oda filtration on $\mathrm{gr}^*_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ is complete. In fact, for $i>j$, we have $\mathrm{Fil}^i_{\mathrm{KO}}\,\mathrm{gr}^j_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}\simeq 0$.*

Proof. We have seen (Corollary 2.4.31) that the canonical map
$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}\otimes^{\mathbb{L}}_{\mathrm{Fil}_H\mathrm{dR}_{(B\twoheadrightarrow B'')/(A\twoheadrightarrow A'')}}\mathrm{Fil}_{\mathrm{PD}}\,B\to\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(B\twoheadrightarrow B'')}$$
is an equivalence. Then both follow from Lemma 2.4.74. □

The convergence of Katz-Oda filtration on associated graded pieces is the key to Lemma 2.4.77.

LEMMA 2.4.77. *In Proposition 2.4.70, the maps (2.4.2) form a limit diagram after passing to the associated graded pieces, i.e. after composition with the functor $\mathrm{CAlg}(\mathrm{DF}^{\geq 0}(A))\to\mathrm{CAlg}(\mathrm{Gr}^{\geq 0}(A))$. Furthermore, if the $\pi_0(A'')$-algebra $\pi_0(R)$ is of finite type, then the maps (2.4.2) form a limit diagram.*

Proof. Note that the map (2.4.2) is the canonical map $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}\to\mathrm{gr}^{0,(\nu)}_{\mathrm{KO}}\,\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$, the Katz-Oda filtration with respect to the cosimplicial system $((A\twoheadrightarrow A'',\gamma_A)\to(P^\nu\to P^\nu\otimes^{\mathbb{L}}_A A'',\gamma_{P^\nu}),P^\nu\to R)$. By the completeness in Lemma 2.4.76, for the equivalence on associated graded pieces, it suffices to show that, for every $i\in\mathbb{N}_{>0}$, the totalization $\lim_{\nu\in\mathbf{\Delta}}\mathrm{gr}^{i,(\nu)}_{\mathrm{KO}}\,\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ is contractible.

The key observation is that $\mathrm{ins}^i(\bigwedge^i \mathbb{L}_{(P^\bullet\to P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}[-i])$ is homotopy equivalent to 0 as a cosimplicial $B^\bullet$-module spectrum [Bha12b, Lem 2.6] when $i>0$ (this is of course false when $i=0$). It follows that the cosimplicial object $\mathrm{gr}^{i,(\bullet)}_{\mathrm{KO}}\,\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ is homotopy equivalent to 0 as $\mathrm{Fil}_{\mathrm{PD}}\,B$-modules by [Sta21, Tag 07KQ] and Lemma 2.4.76.

Finally, if $\pi_0(R)$ is a finitely generated $\pi_0(A'')$-algebra, we could pick a polynomial A-algebra P of finitely many variables along with a surjection $P\twoheadrightarrow R$. In this case, the Katz-Oda filtration is finite and the above argument works. □

Warning 2.4.78. One should be careful about homotopy equivalences. In an earlier draft of this article, we came up with the following "proof": the Hodge-filtered derived de Rham cohomology $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ could be rewritten as

$$\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}\otimes^{\mathbb{L}}_{\mathrm{Fil}_H\mathrm{dR}_{(P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}}\mathrm{Fil}_H\,\mathrm{dR}_{(P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}$$

and since the map $A\to P^\bullet$ is a homotopy equivalence as A-algebras, the map $\mathrm{Fil}_H\,\mathrm{dR}_{(P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}\to\mathrm{Fil}_{\mathrm{PD}}\,P^\bullet$ is also a homotopy equivalence "therefore" the constant cosimplicial algebra $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}$ is homotopy equivalent to $\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'')}\otimes^{\mathbb{L}}_{\mathrm{Fil}_H\mathrm{dR}_{(P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A'')/(A\twoheadrightarrow A'')}}\mathrm{Fil}_{\mathrm{PD}}\,P^\bullet\simeq\mathrm{Fil}_H\,\mathrm{CrysCoh}_{R/(P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A'')}$ therefore the conditions in Proposition 2.4.70.

This argument is incorrect: when playing with homotopy equivalences, one cannot replace the base cosimplicial algebra by a homotopy equivalent algebra without justification. In fact, the last homotopy equivalence obtained above is also incorrect: if it *were* the case, we consider the special case that $(A\twoheadrightarrow A'',\gamma_A)$ is given by $(\mathrm{id}_A: A\to A, 0)$, and $\mathrm{CrysCoh}_{R/P^\bullet}$ is just the animated PD-envelope of $P^\bullet\twoheadrightarrow R$ (see the proof of Proposition 2.4.66). We inspect the homotopy equivalence of cosimplicial objects that we assumed:

$$\mathrm{dR}_{R/A}\simeq^{\mathrm{HoEq}}\mathrm{dR}_{R/P^\bullet}$$

when A is a static $\mathbb{F}_p$-algebra and R is a smooth A-algebra such that $\mathrm{dR}_{R/A}$ is not static, the map $P^\bullet\twoheadrightarrow R$ is Koszul regular and the derived de Rham cohomology $\mathrm{dR}_{R/P^\bullet}$ is simply the PD-envelope, therefore static. Applying π_i to the homotopy equivalence, where $i\neq 0$ is so chosen that $\pi_i(\mathrm{dR}_{R/A})\neq 0$, we get a contradiction.

In view of this warning, our proof of Lemma 2.4.77 tells us that the associated graded pieces with respect to the Katz-Oda filtration are homotopy equivalent, but the homotopy equivalences could not be glued, even after forgetting all the richer structures to the underlying ∞-category $D(\mathbb{Z})$.

When the $\pi_0(A)$-algebra $\pi_0(R)$ is not of finite type, we can still prove that the comparison map is an equivalence with mild smoothness of $A''\to R$ (Proposition 2.4.87). We start with another sufficient condition in characteristic p which is essentially a variant of [LL20, Prop 2.17] by Proposition 2.4.70.

Lemma 2.4.79. *Let $((A\twoheadrightarrow A'',\gamma_A), A''\to R)\in\mathrm{CrysCon}_{\mathbb{F}_p}$. Suppose that*

1. *The cotangent complex $\mathbb{L}_{R/A''}\in D_{\geq 0}(R)$ has Tor-amplitude in $[0,1]$.*
2. *The derived Frobenius twist $\varphi^*_{A\twoheadrightarrow A''}(R)$ (see Lemma 2.4.36) is bounded above, i.e. $\pi_i(\varphi^*_{A\twoheadrightarrow A''}(R))\cong 0$ for $i\gg 0$.*

Then the comparison map in Proposition 2.4.66 is an equivalence.

Proof. Our proof is also adapted from [LL20, Prop 2.17]. By Proposition 2.4.70 and Lemma 2.4.77, it suffices to that the natural maps

$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma_A)}\longrightarrow\mathrm{CrysCoh}_{R/(P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A'',\gamma_{P^\bullet})}\tag{2.4.3}$$

form a limit diagram in $\mathrm{CAlg}(D(A))$. We endow both sides with conjugate filtration (Definition 2.4.41), and show that this forms in fact a limit diagram in $\mathrm{CAlg}(\mathrm{DF}^{\leq 0}(A))$.

We show that, after passing to associated graded pieces with respect to the conjugate filtration, the maps (2.4.3) form a limit diagram, which implies that the natural maps (2.4.3) form limit diagrams after passing to finite level of quotients, and then we control the convergence to deduce the result. To show the result for associated graded pieces, by Proposition 2.4.46, it suffices to show that the maps

$$\varphi^*_{A\twoheadrightarrow A''}(\textstyle\bigwedge^\star_R \mathbb{L}_{R/A''})[-\star] \longrightarrow \varphi^*_{P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A''}(\textstyle\bigwedge^\star_R \mathbb{L}_{R/(P^\bullet\otimes^{\mathbb{L}}_A A'')})[-\star] \tag{2.4.4}$$

form a limit diagram in $\mathrm{Gr}^{\geq 0}(D(A))$.

Let $R_1 := \varphi^*_{A\twoheadrightarrow A''}(R)$. Note that the Frobenius map $\varphi_{P^\bullet\twoheadrightarrow P^\bullet\otimes^{\mathbb{L}}_A A''}$ factors as $P^\bullet \otimes^{\mathbb{L}}_A A'' \to \varphi^*_A(P^\bullet) \to P^\bullet$ where the second map is the Frobenius map of $P^\bullet$ relative to A. Then the maps (2.4.4) could be rewritten as the maps

$$\textstyle\bigwedge^\star_{R_1} \mathbb{L}_{R_1/A}[-\star] \longrightarrow (\bigwedge^\star_{R_1} \mathbb{L}_{R_1/\varphi^*_A(P^\bullet)})[-\star] \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet$$

or equivalently, the maps

$$\mathrm{gr}^\star_H \mathrm{dR}_{R_1/A} \longrightarrow \mathrm{gr}^\star_H \mathrm{dR}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet$$

by an inverse application of Lemma 2.4.74 (recall that for derived de Rham cohomology of animated rings, the "stupid" filtration coincides with the Hodge filtration). We again consider the Katz-Oda filtration associated to the cosimplicial system $A \to \varphi^*_A(P^\bullet) \to R_1$ (Lemma 2.4.76) and by completeness, we could pass to associated graded pieces for $i = 0$:

$$\mathrm{gr}^\star_H \mathrm{dR}_{R_1/\varphi^*_A(P^\bullet)} \longrightarrow \mathrm{gr}^\star_H \mathrm{dR}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet \tag{2.4.5}$$

and $i \in \mathbb{N}_{>0}$:

$$\mathrm{gr}^\star_H \mathrm{dR}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} \big(\textstyle\bigwedge^i_{\varphi^*_A(P^\bullet)} \mathbb{L}_{\varphi^*_A(P^\bullet)/A}[-i]\big) \longrightarrow 0$$

As in Lemma 2.4.77, the later maps constitute a homotopy equivalence by [Bha12b, Lem 2.6] and [Sta21, Tag 07KQ], therefore constitutes a limit diagram by Lemma 2.4.69. On the other hand, by Lemma 2.4.82, the maps (2.4.5) constitute a limit diagram.

Now we control the convergence. Again by Lemma 2.4.74, we rewrite the maps (2.4.5) as the maps

$$\textstyle\bigwedge^\star_{R_1} \mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-\star] \longrightarrow (\bigwedge^\star_{R_1} \mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-\star]) \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet$$

Now consider the transitivity sequence

$$\mathbb{L}_{\varphi^*_A(P^\bullet)/A} \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} R_1 \longrightarrow \mathbb{L}_{R_1/A} \longrightarrow \mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}$$

For every static R_1-module M, we get the fiber sequence

$$\mathbb{L}_{\varphi^*_A(P^\bullet)/A} \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} M \longrightarrow \mathbb{L}_{R_1/A} \otimes^{\mathbb{L}}_{R_1} M \longrightarrow \mathbb{L}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{R_1} M$$

Since $\mathbb{L}_{R/A''} \in D_{\geq 0}(R)$ has Tor-amplitude in $[0,1]$, so does $\mathbb{L}_{R_1/A} \in D_{\geq 0}(R_1)$, therefore $\pi_j(\mathbb{L}_{R_1/A} \otimes^{\mathbb{L}}_{R_1} M) \cong 0$ for $j \neq 0, 1$. Note that $\mathbb{L}_{\varphi^*_A(P^\bullet)/A}$ is a flat $\varphi^*_A(P^\bullet)$-module. It follows that $\pi_j(\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{R_1} M) \cong 0$ for $j \neq 0, 1$. Furthermore, since $\varphi^*_A(P^\bullet) \to R_1$ is surjective, $\pi_0(\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)} \otimes^{\mathbb{L}}_{R_1} M) \cong 0$. It follows that $\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-1]$ is a flat R_1-module, and so is $\bigwedge^\star_{R_1} \mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-\star] \simeq \Gamma^\star_{R_1}(\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-1])$. By assumption, R_1 is bounded above, therefore so is $\bigwedge^\star_{R_1} \mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-\star]$.

It remains to show that the associated graded pieces are uniformly bounded above, which implies that (2.4.3) form a limit diagram, by Lemma 2.4.84 and that the conjugate filtration is exhaustive (Lemma 2.4.42). Suppose that the homotopy groups of R_1 are concentrated in the range $[a,b]$, then the associated graded pieces of the target could be rewritten as $\Gamma^\star_{R_1}(\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-1]) \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet$, where $\Gamma^\star_{R_1}(\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-1])$ is a flat R_1-module therefore the homotopy groups of it is also concentrated in the range $[a,b]$. Since the relative Frobenius $\varphi^*_A(P^\bullet) \to P^\bullet$ is flat, we get $\pi_j(\Gamma^\star_{R_1}(\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-1]) \otimes^{\mathbb{L}}_{\varphi^*_A(P^\bullet)} P^\bullet) \cong \pi_j(\Gamma^\star_{R_1}(\mathbb{L}_{R_1/\varphi^*_A(P^\bullet)}[-1])) \otimes_{\pi_0(\varphi^*_A(P^\bullet))} \pi_0(P^\bullet) \cong 0$ for $j \notin [a,b]$. $\square$

We need the following lemmas:

Lemma 2.4.80. *Let $\mathcal{C}$ be an ∞-category which admits finite coproducts. Let $\varnothing$ denote the initial object of $\mathcal{C}$, and let X, Y be two objects of $\mathcal{C}$. Then for any two maps $g_0, g_1 \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the induced maps $X^\bullet \rightrightarrows Y^\bullet$ of Čech conerves $X^\bullet$ of X (i.e. of $\varnothing \to X$) and $Y^\bullet$ of Y (i.e. of $\varnothing \to Y$) are homotopic. More precisely, there exists a simplicial homotopy from $g_0^\bullet$ to $g_1^\bullet$ which is functorial in g_0 and g_1. In particular, if $X = Y$ and $g_0 = g_1$, then the simplicial homotopy is constant.*

Proof. We start with the special case that $\mathcal{C}$ is a 1-category. We define the simplicial homotopy $h: \delta^*(X^\bullet) \to \delta^*(Y^\bullet)$ as follows: for every $(\alpha_n: [n] \to [1]) \in \mathbf{\Delta}_{/[1]}$, we note that $(X^\bullet)(\alpha_n) = X^n = X \amalg \cdots \amalg X$ and $(Y^\bullet)(\alpha_n) = Y^n = Y \amalg \cdots \amalg Y$, and we set $h_{\alpha_n} = \coprod_{i=0}^n g_{\alpha_n(i)}: X^n \to Y^n$. By construction, $i_0^*(h)^\nu = h_{[\nu] \xrightarrow{0} [1]} = \coprod_{i=0}^n g_0 = g_0^\nu$ and $i_1^*(h)^\nu = g_1^\nu$.

We need to check that this is a map of functors. For every map $\psi: (\alpha_n: [n] \to [1]) \to (\alpha_m: [m] \to [1])$ in $\mathbf{\Delta}_{/[1]}$, we need to check that the diagram

$$\begin{tikzcd} X^n \arrow[r, "h_{\alpha_n}"] \arrow[d, "\psi_*"] & Y^n \arrow[d, "\psi_*"] \\ X^m \arrow[r, "h_{\alpha_m}"] & Y^m \end{tikzcd}$$

commutes, where the vertical maps $\psi_*: X^n \to X^m$ and $\psi_*: Y^n \to Y^m$ are induced by ψ. For this end, let $j_i: X \to X^n$ be the i-th canonical map for $0 \leq i \leq n$.

Then the composite $h_{\alpha_n} \circ j_i: X \to X^n \to Y^n$ could be rewritten as the composite $j_i \circ g_{a_n(i)}: X \to Y \to Y^n$, and the composite $\psi_* \circ h_{\alpha_n} \circ j_i: X \to Y^m$ is equivalent to the composite $j_{\psi(i)} \circ g_{\alpha_n(i)}: X \to Y \to Y^m$. Similarly, the composite $\psi_* \circ j_i: X \to X^n \to X^m$ is equivalent to the $\psi(i)$-th canonical map $j_{\psi(i)}: X \to X^m$, and the composite $h_{\alpha_m} \circ \psi_* \circ j_i$ could be identified with the composite $j_{\psi(i)} \circ g_{\alpha_m(\psi(i))}: X \to Y \to Y^m$.

Since $\alpha_m(\psi(i)) = \alpha_n(i)$, it follows that $\psi_* \circ h_{\alpha_n} \circ j_i = h_{\alpha_m} \circ \psi_* \circ j_i$ for every $0 \leq i \leq n$. It follows that $\psi_* \circ h_{\alpha_n} = h_{\alpha_m} \circ \psi_*$. The other claims for the 1-category $\mathcal{C}$ follow directly from the construction.

Now we claim that the result for ∞-categories follows from that for 1-categories. The point is that there exists a universal 1-category[2.4.12] $\mathcal{C}_0$ along with two objects $X_0, Y_0 \in \mathcal{C}$ and two maps $X_0 \rightrightarrows Y_0$, which admits all finite products, such that for every ∞-category $\mathcal{C}$ as in the assumption of this lemma, there exists an essentially unique functor $\mathcal{C}_0 \to \mathcal{C}$ which preserves finite coproducts: let K be the diagram $\bullet \rightrightarrows \bullet$, and then take the presheaf ∞-category $\mathcal{P}(K) = \mathrm{Fun}(K^{\mathrm{op}}, \mathcal{S})$. Then we can take $\mathcal{C}_0$ to be the full subcategory of $\mathcal{P}(K)$ spanned by finite coproducts of the two vertices of K. □

Corollary 2.4.81. *Let $\mathcal{C}$ be an ∞-category with finite coproducts, and two objects X, Y in $\mathcal{C}$. Let $i: X \to Y$ be a map which admits a left inverse $r: Y \to X$. Then there is a "strong deformation retract", i.e. a simplicial homotopy from $\mathrm{id}_{Y^\bullet}$ to $i^\bullet \circ r^\bullet$, which restricts to a constant simplicial homotopy of $X^\bullet$ along $i^\bullet: X^\bullet \to Y^\bullet$, where $X^\bullet$ (resp. $Y^\bullet$) is the Čech conerve of X (resp. Y), and $i^\bullet: X^\bullet \to Y^\bullet$ and $r^\bullet: Y^\bullet \to X^\bullet$ are induced simplicial maps.*

Proof. We apply Lemma 2.4.80 to $\mathrm{id}_Y, i \circ r \in \mathrm{Hom}_{\mathcal{C}}(Y, Y)$, getting the desired simplicial homotopy. To see the later statement, it suffices to inspect the commutative diagram

$$\begin{tikzcd} Y \arrow[r, shift left, "\mathrm{id}_Y"] \arrow[r, shift right, "i \circ r"'] & Y \\ X \arrow[u] \arrow[r, shift left, "\mathrm{id}_X"] \arrow[r, shift right, "\mathrm{id}_X"'] & X \arrow[u] \end{tikzcd}$$

and invoke the functoriality. □

Lemma 2.4.82. *Let $A \in \mathrm{CAlg}^{\mathrm{cn}}$ be a connective $\mathbb{E}_\infty$-ring and let $B \to C$ be a faithfully flat map of connective $\mathbb{E}_\infty$-A-algebras. Let $B^\bullet$ (resp. $C^\bullet$) denote the Čech conerve of the map $A \to B$ (resp. $A \to C$). Then for any cosimplicial $B^\bullet$-module $N^\bullet$, the natural cosimplicial map*

$$N^\bullet \longrightarrow N^\bullet \otimes_{B^\bullet}^{\mathbb{L}} C^\bullet$$

2.4.12. This is informed to us by Denis Nardin.

induces an equivalence after totalization $\lim_{\bullet\in\mathbf{\Delta}}$ *in* $D(A)$*, where the cosimplicial map* $B^\bullet\to C^\bullet$ *is induced by* $B\to C$.

Proof. Let $D^{\bullet,\bullet}$ denote the cosimplicial Čech conerve of $B^\bullet\to C^\bullet$ (each $D^{\nu,\bullet}$ is the Čech conerve of $B^\nu\to C^\nu$), which is a bicosimplicial object in CAlg_A. We note that there is a unique cosimplicial map $B^\bullet\to D^{\bullet,\mu}$ for all $[\mu]\in\mathbf{\Delta}$. Consider the bicosimplicial object $M^{\bullet,\bullet}$:

$$\begin{array}{rcl}\mathbf{\Delta}^2 & \longrightarrow & D(A)\\ ([\nu],[\mu]) & \longmapsto & N^\nu\otimes^{\mathbb{L}}_{B^\nu}D^{\nu,\mu}\end{array}$$

and its limit $I:=\lim_{([\nu],[\mu])}M^{\nu,\mu}$. The map which we need to show to be an equivalence factors as $\lim_{[\nu]}N^\nu\to\lim_{([\nu],[\mu])}M^{\nu,\mu}\to\lim_{[\nu]}M^{\nu,0}\simeq\lim_{[\nu]}N^\nu\otimes^{\mathbb{L}}_{B^\nu}C^\nu$. It suffices to show that both maps are equivalences.

For the first map, in fact, for every $[\nu]\in\mathbf{\Delta}$, the map $N^\nu\to\lim_{[\mu]\in\mathbf{\Delta}}M^{\nu,\mu}$ is an equivalence by faithfully flat descent.

For the second map, since $\mathbf{\Delta}^{\mathrm{op}}_{\mathrm{inj}}\to\mathbf{\Delta}^{\mathrm{op}}$ is cofinal [Lur09, Lem 6.5.3.7] where $\mathbf{\Delta}_{\mathrm{inj}}\subseteq\mathbf{\Delta}$ is the (non-full) subcategory with strictly increasing maps $[m]\to[n]$, we can replace $\lim_{\mathbf{\Delta}}(\cdot)$ by $\lim_{\mathbf{\Delta}_{\mathrm{inj}}}(\cdot)$. By [Lur09, Cor 4.4.4.10], it suffices to show that, for every injective map $[\mu_1]\to[\mu_2]$ in $\mathbf{\Delta}$, the induced map $\lim_{[\nu]}M^{\nu,\mu_1}\to\lim_{[\nu]}M^{\nu,\mu_2}$ is an equivalence. Every injective map $[\mu_1]\to[\mu_2]$ admits a retract in $\mathbf{\Delta}$, therefore by Corollary 2.4.81, the induced map $D^{\bullet,\mu_1}\to D^{\bullet,\mu_2}$ is a homotopy equivalence of cosimplicial $\mathbb{E}_\infty$-$B^\bullet$-algebras, therefore $M^{\bullet,\mu_1}\to M^{\bullet,\mu_2}$ is a homotopy equivalence of cosimplicial A-modules by [Sta21, Tag 07KQ]. The result then follows from Lemma 2.4.69. □

Remark 2.4.83. When A is a static $\mathbb{F}_p$-algebra, B is a polynomial A-algebra and $C=B\otimes^{\mathbb{L}}_{A,\varphi_A}A$ is the Frobenius twist of B, we recover [BS19, Lem 5.4].

LEMMA 2.4.84. *Let* $(M_i^\bullet)_{i\in(\mathbb{Z}_{\le 0},\ge)}\in\mathrm{Fun}(\mathbf{\Delta}\times(\mathbb{Z}_{\le 0},\ge),\mathrm{Sp})$ *be a cosimplicial filtered spectra. Suppose that it is uniformly bounded above, i.e. there exists* $N\in\mathbb{N}$ *such that for every* $i\in\mathbb{Z}_{\le 0}$ *and* $\nu\in\mathbf{\Delta}$*, we have* $\pi_j(M_i^\nu)\cong 0$ *for all* $j>N$*. Let* $M^\bullet:=\mathrm{colim}_{i\to-\infty}M_i^\bullet$*. Then the canonical map* $\mathrm{colim}_{i\to-\infty}\lim_{\nu\in\mathbf{\Delta}}M_i^\nu\to\lim_{\nu\in\mathbf{\Delta}}M^\nu$ *is an equivalence.*

Proof. We could rewrite $\lim_{\nu\in\mathbf{\Delta}}$ as $\lim_{n\to\infty}\lim_{\nu\in\mathbf{\Delta}_{\le[n]}}$. Furthermore, the functor $\mathbf{\Delta}^{\mathrm{inj}}_{\le[n]}\to\mathbf{\Delta}_{\le[n]}$ is right cofinal, therefore we can replace $\lim_{\nu\in\mathbf{\Delta}_{\le[n]}}$ by $\lim_{\nu\in\mathbf{\Delta}^{\mathrm{inj}}_{\le[n]}}$ which is a finite limit, therefore commutes with $\mathrm{colim}_{i\to-\infty}$. For any cosimplicial spectrum $X^\bullet$, there is a canonical map $\lim_{\nu\in\mathbf{\Delta}}X^\nu\to\lim_{\nu\in\mathbf{\Delta}_{\le[n]}}X^\nu$. If $X^\bullet$ is assumed to be uniformly bounded above, then the coconnectivity of $\mathrm{fib}(\lim_{\nu\in\mathbf{\Delta}}X^\nu\to\lim_{\nu\in\mathbf{\Delta}_{\le[n]}}X^\nu)$ tends to $-\infty$ as $n\to\infty$ by [Lur17, Cor 1.2.4.18]. The result then follows. □

For the integral version, we need to introduce the following concept of smoothness:

DEFINITION 2.4.85. (CF. [BMS19, DEF 4.9]) *We say that a map* $R\to S$ *of animated rings is* quasisyntomic *if it is flat and the cotangent complex* $\mathbb{L}_{S/R}$ *has* Tor*-amplitude in* $[0,1]$.

Example 2.4.86. Any smooth map, or more generally, any syntomic map of static rings is quasisyntomic.

We now phrase the integral comparison:

PROPOSITION 2.4.87. *Let* $((A\twoheadrightarrow A'',\gamma_A),A''\to R)\in\mathrm{CrysCon}$ *such that* A *is bounded above (that is,* $\pi_n(A)\cong 0$ *for* $n\gg 0$*) and the map* $A''\to R$ *is quasisyntomic. Then the comparison map in Proposition 2.4.66 is an equivalence.*

Proof. We again appeal to Proposition 2.4.70. It suffices to show that the map

$$\mathrm{CrysCoh}_{R/(A\twoheadrightarrow A'',\gamma_A)}\longrightarrow\lim_{\nu\in\mathbf{\Delta}}\mathrm{CrysCoh}_{R/(P^\nu\twoheadrightarrow P^\nu\otimes^{\mathbb{L}}_A A'',\gamma_{P^\nu})}$$

is an equivalence of $\mathbb{Z}$-module spectra (since the forgetful functor is conservative), which could be checked by base change along $\mathbb{Z} \to \mathbb{Z}/p$ for all prime numbers $p \in \mathbb{N}_{>0}$ and along $\mathbb{Z} \to \mathbb{Q}$. The latter follows from Lemma 2.4.11 and that the map $A \to P$ is faithfully flat therefore the canonical map $A \to \lim_{\nu \in \mathbf{\Delta}} P^\nu$ is an equivalence (in fact, this is induced by a homotopy equivalence of cosimplicial objects, but we do not need this). For every prime number p, by base change property (Lemma 2.4.35) and Lemma 2.4.79, where the flatness of $A'' \to R$ implies the flatness of $A/^{\mathbb{L}}p \to \varphi^*_{A/^{\mathbb{L}}p \twoheadrightarrow A''/^{\mathbb{L}}p}(R/^{\mathbb{L}}p)$, therefore the Frobenius twist in question is bounded above. □

Finally, we want to compare the cohomology of the affine crystalline site and the classical crystalline cohomology. We first describe a non-complete variant of the classical affine crystalline site, which we will name after *static affine crystalline site*.

Definition 2.4.88. *Let $(A, I, \gamma_A) \in \mathrm{PDPair}$ be a PD-pair and let $A/I \to R$ be a map of rings. Note that $((A \twoheadrightarrow A/I, \gamma_A), A/I \to R) \in \mathrm{CrysCon}$ is a crystalline context. The* static affine crystalline site $\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A))$ *is the full subcategory of* $\mathrm{Cris}(R/(A \twoheadrightarrow A/I, \gamma_A))$ *spanned by those $(B \twoheadrightarrow B/J, \gamma_B)$ along with a map $R \to B/J$, i.e., the animated PD-pair in question is given by a PD-pair, equipped with the indiscrete topology.*

We note that the structure presheaf $\mathcal{O}$ on $\mathrm{Cris}(R/(A \twoheadrightarrow A/I, \gamma_A))$ restricts to a presheaf $\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A))$, still called the *structure presheaf*, which is canonically equipped with PD-filtration, of which the cohomology is called the *cohomology of the static crystalline site* (resp. *Hodge-filtered cohomology of the static crystalline site*), denoted by $R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$ (resp. $\mathrm{Fil}_H R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$. By definition, there is a comparison map $\mathrm{Fil}_H R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A/I, \gamma_A)), \mathcal{O}) \to \mathrm{Fil}_H R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$ of filtered $\mathbb{E}_\infty$-A-algebras.

Warning 2.4.89. Here the PD-filtration is that for animated PD-envelope, although we are considering PD-pairs. However, when $I = 0$, thanks to Proposition 2.3.77, we can consider the classical PD-envelope instead.

Now the cohomology of the affine crystalline site coincides with the classical version:

Proposition 2.4.90. *Let $(A, I, \gamma_A) \in \mathrm{PDPair}$ be a PD-pair and $A/I \to R$ a quasisyntomic map of rings (R is static by flatness). Then the comparison map*

$$\mathrm{Fil}_H R\Gamma(\mathrm{Cris}(R/(A \twoheadrightarrow A/I, \gamma_A)), \mathcal{O}) \to \mathrm{Fil}_H R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$$

of filtered $\mathbb{E}_\infty$-A-algebras constructed above is an equivalence.

Proof. We adapt the Čech-Alexander computation in Proposition 2.4.66. We pick a polynomial A-algebra P (of possibly infinitely many variables) along with a surjection $P \twoheadrightarrow R$. Let $P^\bullet \to R$ denote the Čech conerve of the object $P \to R$ in $(\mathrm{Alg}_A)_{/R}$. Concretely, $P^\nu = P^{\otimes_A(\nu+1)}$. Note that since $A \to P$ is flat, the classical tensor product coincides with the derived tensor product, therefore the cosimplicial pair $P^\bullet \to R$ coincides with the cosimplicial animated pair in the proof of Proposition 2.4.66, and then $\mathrm{Fil}_H R\Gamma(\mathrm{Cris}^{\mathrm{st}}(R/(A, I, \gamma_A)), \mathcal{O})$ is computed by the classical PD-envelope of $P^\bullet \twoheadrightarrow R$ with respect to (A, I, γ_A), equipped with the PD-filtration.

Let $(D^\bullet \twoheadrightarrow R, \gamma_{D^\bullet})$ denote the cosimplicial animated PD-envelope of $(P^\bullet \twoheadrightarrow R)$ relative to (A, I, γ_A). It suffices to show that $(D^\nu \twoheadrightarrow R, \gamma_{D^\bullet})$ is given by a PD-pair for all $\nu \in \mathbf{\Delta}$, or equivalently, the underlying animated ring D^ν is static, by virtue of Proposition 2.3.32 and Lemma 2.3.37, which follows from Lemma 2.4.91 below. □

Lemma 2.4.91. *Let $(A \twoheadrightarrow A'', \gamma_A)$ be an animated PD-pair, $A'' \to R$ a quasisyntomic map of animated rings, P a polynomial A-algebra (of possibly infinitely many variables) and $P \twoheadrightarrow R$ a surjection of A-algebras. Let $(D \twoheadrightarrow R, \gamma_D)$ denote the animated PD-envelope of $P \twoheadrightarrow R$ relative to $(A \twoheadrightarrow A'', \gamma_A)$. Then D is a flat A-module.*

Proof. This is a "quasi" variant of "flatness of PD-envelope" [BS19, Lem 2.42]. By Lemma 2.3.71, it suffices to show that $D \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$ is a flat $A \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Q}$-module, and for every prime $p \in \mathbb{N}$, $D/^{\mathbb{L}}p$ is a flat $A/^{\mathbb{L}}p$-module.

By Lemma 2.4.52, the map $P\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Q}\to D\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Q}$ is an equivalence. Since $A\to P$ is flat, so is the map $A\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Q}\to D\otimes^{\mathbb{L}}_{\mathbb{Z}}\mathbb{Q}$.

For every prime $p\in\mathbb{N}$, by base change property (a relative version of Lemma 2.3.46, with a similar proof), $D_0:=D/^{\mathbb{L}}p$ is the animated PD-envelope of $P/^{\mathbb{L}}p\twoheadrightarrow R/^{\mathbb{L}}p$ relative to the animated PD-pair $(A\twoheadrightarrow A'',\gamma_A)$. To simplify notations, we let $P_0:=P/^{\mathbb{L}}p, R_0:=R/^{\mathbb{L}}p, A_0:=A/^{\mathbb{L}}p$, $A_0'':=A''/^{\mathbb{L}}p$. Since $A\to R$ is quasisyntomic, so is $A_0\to R_0$. Consider the transitivity sequence

$$\mathbb{L}_{P_0/A_0}\otimes^{\mathbb{L}}_{P_0}R_0\longrightarrow\mathbb{L}_{R_0/A_0}\longrightarrow\mathbb{L}_{R_0/P_0}$$

For every static R_0-module M, we get a fiber sequence

$$\mathbb{L}_{P_0/A_0}\otimes^{\mathbb{L}}_{P_0}M\longrightarrow\mathbb{L}_{R_0/A_0}\otimes^{\mathbb{L}}_{R_0}M\longrightarrow\mathbb{L}_{R_0/P_0}\otimes^{\mathbb{L}}_{R_0}M$$

Since P_0 is a polynomial A_0-algebra, $\mathbb{L}_{P_0/A_0}$ is a flat P_0-module. The map $A_0\to R_0$ is quasisyntomic, therefore $\pi_*(\mathbb{L}_{R_0/A_0}\otimes^{\mathbb{L}}_{R_0}M)\cong 0$ for $*\neq 0,1$. It follows that $\pi_*(\mathbb{L}_{R_0/P_0}\otimes^{\mathbb{L}}_{R_0}M)\cong 0$ for $*\neq 0,1$. Furthermore, since $P_0\to R_0$ is surjective, $\pi_0(\mathbb{L}_{R_0/P_0}\otimes^{\mathbb{L}}_{R_0}M)\cong 0$. It follows that $P_0\twoheadrightarrow R_0$ is a quasiregular animated pair. By Corollary 2.4.60, D_0 is a flat $\varphi^*_{P_0\twoheadrightarrow P_0\otimes^{\mathbb{L}}_{A_0}A_0''}(R_0)$-module where $\varphi_{P_0\twoheadrightarrow P_0\otimes^{\mathbb{L}}_{A_0}A_0''}:P_0\otimes^{\mathbb{L}}_{A_0}A_0''\to P_0$ is the Frobenius map (Lemma 2.4.36). It remains to see that the composite map $A_0\to P_0\to\varphi^*_{P_0\twoheadrightarrow P_0\otimes^{\mathbb{L}}_{A_0}A_0''}(R_0)=R_0\otimes^{\mathbb{L}}_{P_0\otimes^{\mathbb{L}}_{A_0}A_0''}P_0$ is flat, where the second map is the "map into the second factor".

We note that the Frobenius $\varphi_{P_0\twoheadrightarrow P_0\otimes^{\mathbb{L}}_{A_0}A_0''}$ factors as $P_0\otimes^{\mathbb{L}}_{A_0}A_0''\to\varphi^*_{A_0}(P_0)\to P_0$ where the second map is the Frobenius of P_0 relative to A_0. Let R_1 denote $R_0\otimes^{\mathbb{L}}_{A_0'',\varphi_{A_0\twoheadrightarrow A_0''}}A_0$. Since $A_0''\to R_0$ is flat, so is $A_0\to R_1$, and we have

$$\varphi^*_{P_0\twoheadrightarrow P_0\otimes^{\mathbb{L}}_{A_0}A_0''}(R_0)\simeq R_1\otimes^{\mathbb{L}}_{\varphi^*_{A_0}(P_0)}P_0$$

as a pushout of A_0-algebras. The relative Frobenius $\varphi^*_{A_0}(P_0)\to P_0$ is flat, therefore the map $R_1\to R_1\otimes^{\mathbb{L}}_{\varphi^*_{A_0}(P_0)}P_0$. The result then follows since flatness is stable under composition. □

Remark 2.4.92. If we examine the proof of Lemma 2.4.91 closely, we see that, instead of being a polynomial, what we really need to impose on the map $A\to P$ is that the map is *quasismooth* (i.e. it is flat and $\mathbb{L}_{P/A}$ is a flat P-module), and for every prime $p\in\mathbb{N}$, the Frobenius of $P/^{\mathbb{L}}p$ relative to $A/^{\mathbb{L}}p$ is flat.

2.5. Animated prismatic structures

We fix a prime $p\in\mathbb{N}$. In this section, we will develop the theory of animated δ-rings, that of animated δ-pairs, a non-complete theory of prisms and prove a variant of the Hodge–Tate comparison, from which we deduce a result about "flat covers of the final object". Almost every ring that we will discuss is a $\mathbb{Z}_{(p)}$-algebra, we will simply denote $\mathrm{AniPair}_{\mathbb{Z}_{(p)}}$ by AniPair and $\mathrm{AniPDPair}_{\mathbb{Z}_{(p)}}$ by AniPDPair.

2.5.1. Animated δ-rings and δ-pairs

In this section, we will define *animated δ-rings* and *animated δ-pairs* and discuss the interaction between the δ-structure and the PD-structure. Recall that

Definition 2.5.1. ([BS19, Def 2.1]) *A* δ-ring *is a pair (R,δ) where R is a $\mathbb{Z}_{(p)}$-algebra and $\delta:R\to R$ is an endomorphism of the underlying set R such that*

1. *$\delta(x+y)=\delta(x)+\delta(y)-P(x,y)$ for all $x,y\in R$ where $P(X,Y)\in\mathbb{Z}[X,Y]$ is the polynomial*

$$\frac{(X+Y)^p-X^p-Y^p}{p}:=\sum_{j=1}^{p-1}\frac{1}{p}\binom{p}{j}X^{p-j}Y^j$$

2. *$\delta(xy)=x^p\,\delta(y)+y^p\,\delta(x)+p\,\delta(x)\,\delta(y)$.*
3. *$\delta(1)=0$.*

A map $f:(R,\delta)\to(S,\delta)$ of δ-rings *is a map* $f:R\to S$ *of rings such that* $f\circ\delta=\delta\circ g$ *as maps of sets. These form the* 1-category of δ-rings*, denoted by* Ring_δ*.*

Remark 2.5.2. ([BS19, Rem 2.2]) Given a δ-ring (R,δ), we write $\varphi:R\to R$ for the map $x\mapsto x^p+p\,\delta(x)$. Then φ is a ring endomorphism of R which lifts the Frobenius map $R/p\to R/p$, i.e. $\varphi(x)\equiv x^p\ (\mathrm{mod}\,p)$ for every $x\in R$.

The 1-category Ring_δ admits an initial object $\mathbb{Z}_{(p)}$ [BS19, Ex 2.6], and more generally, all small colimits and small limits, and the forgetful functor $\mathrm{Ring}_\delta\to\mathrm{Alg}_{\mathbb{Z}_{(p)}}$ preserves them [BS19, Rem 2.7]. The forgetful functor $\mathrm{Ring}_\delta\to\mathrm{Set}$ admits a left adjoint $\mathrm{Set}\to\mathrm{Ring}_\delta$, which sends a set S to the free δ-ring generated by S, denoted by $\mathbb{Z}_{(p)}\{S\}$. Indeed, when $S=\{x\}$ is a singleton, it is given by the free δ-ring $\mathbb{Z}_{(p)}\{x\}$ of which the underlying $\mathbb{Z}_{(p)}$-algebra is isomorphic to the polynomial $\mathbb{Z}_{(p)}$-algebra $\mathbb{Z}_{(p)}[x,\delta(x),\delta^2(x),\ldots]$ [BS19, Lem 2.11], and the general case follows by taking the coproduct of S-copies of $\mathbb{Z}_{(p)}\{x\}$. It then follows from Corollary 2.2.3 that

Lemma 2.5.3. *The* 1*-category* Ring_δ *is* 1*-projectively generated, therefore presentable.*

By the adjoint functor theorem, the forgetful functor $\mathrm{Ring}_\delta\to\mathrm{Alg}_{\mathbb{Z}_{(p)}}$ admits a left adjoint. A further application of Corollary 2.2.3 leads to

Lemma 2.5.4. *There is a pair* $\mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})\rightleftarrows\mathrm{Ani}(\mathrm{Ring}_\delta)$ *of adjoint functors, being the animation of the pair* $\mathrm{Alg}_{\mathbb{Z}_{(p)}}\rightleftarrows\mathrm{Ring}_\delta$ *of adjoint functors. We will call the functor* $\mathrm{Ani}(\mathrm{Ring}_\delta)\to\mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})$ *the* free animated δ-ring functor[2.5.1]*. The functor* $\mathrm{Ani}(\mathrm{Ring}_\delta)\to\mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})$*, called the* forgetful functor*, is conservative and preserves small colimits (and as a right adjoint, it preserves small limits as well).*

Definition 2.5.5. *The* ∞-category of animated δ-rings *is defined to be the animation* $\mathrm{Ani}(\mathrm{Ring}_\delta)$*, of which an object is called an* animated δ-ring*, formally denoted by* (R,δ) *where* R *is the image of* (R,δ) *under the forgetful functor* $\mathrm{Ani}(\mathrm{Ring}_\delta)\to\mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_p})$*, or simply by* R *when the* δ*-structure is unambiguously obvious.*

Concretely, a set of compact projective generators for $\mathrm{Ani}(\mathrm{Ring}_\delta)$ is given by free δ-rings generated by a finite set, which spans a full subcategory $\mathrm{Ring}_\delta^0\subseteq\mathrm{Ring}_\delta$. Recall that $\mathrm{Ring}_\delta\hookrightarrow\mathrm{Ani}(\mathrm{Ring}_\delta)$ is a full subcategory (Remark B.0.35). Now we characterize this full subcategory in terms of the underlying animated ring:

Lemma 2.5.6. *Let* $(R,\delta)\in\mathrm{Ani}(\mathrm{Ring}_\delta)$ *be an animated* δ*-ring. Then the followings are equivalent:*

1. *The animated* δ*-ring* $(R,\delta)\in\mathrm{Ani}(\mathrm{Ring}_\delta)$ *is* n*-truncated.*
2. *The underlying animated ring* $R\in\mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})$ *is* n*-truncated.*
3. *For every* $m\in\mathbb{N}_{>n}$*, the homotopy group* $\pi_m(R)$ *vanishes.*

Proof. The equivalence of parts 2 and 3 is [Lur18b, Prop 25.1.3.3]. On the other hand, part 1 is equivalent to say that, for every free δ-ring F generated by a finite set, the mapping anima $\mathrm{Map}_{\mathrm{Ani}(\mathrm{Ring}_\delta)}(F,R)$ is n-truncated by [Lur09, Rem 5.5.8.26]. Since any such F is a finite coproduct of $\mathbb{Z}_{(p)}\{x\}$, it is equivalent to $\mathrm{Map}_{\mathrm{Ani}(\mathrm{Ring}_\delta)}(\mathbb{Z}_{(p)}\{x\},R)$ being n-truncated, which is equivalent to part 3 since

$$\mathrm{Map}_{\mathrm{Ani}(\mathrm{Ring}_\delta)}(\mathbb{Z}_{(p)}\{x\},R)\simeq\mathrm{Map}_{\mathcal{S}}(\{x\},R)\simeq R$$ □

Now we define the Frobenius map on animated δ-rings. We note that the identity functor $\mathrm{id}:\mathrm{Ani}(\mathrm{Ring}_\delta)\to\mathrm{Ani}(\mathrm{Ring}_\delta)$ is the animation of the identity functor $\mathrm{id}:\mathrm{Ring}_\delta\to\mathrm{Ring}_\delta$.

Definition 2.5.7. *The* Frobenius endomorphism *is the endomorphism of the identity functor* $\mathrm{id}:\mathrm{Ani}(\mathrm{Ring}_\delta)\to\mathrm{Ani}(\mathrm{Ring}_\delta)$ *defined to be the animation of the Frobenius endomorphism (described in Remark 2.5.2) of the identity functor* $\mathrm{id}:\mathrm{Ring}_\delta\to\mathrm{Ring}_\delta$*.*

2.5.1. The non-animated version was called the "δ-envelope" in [GLQ20, Def 1.1].

Recall that a *δ-pair* is the datum (A, I) of a δ-ring A along with an ideal $I \subseteq A$ [BS19, Def 3.2]. Similar to animated pairs, we have an "animated version" of δ-pairs:

DEFINITION 2.5.8. *The* ∞-category of animated δ-pairs $\mathrm{AniPair}_\delta$ *is defined to be the fiber product* $\mathrm{Ani}(\mathrm{Ring}_\delta) \times_{\mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})} \mathrm{AniPair}$ *where the functor* $\mathrm{Ani}(\mathrm{Ring}_\delta) \to \mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})$ *is the forgetful functor and the functor* $\mathrm{AniPair} \to \mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})$ *is the evaluation* $(A \twoheadrightarrow A'') \mapsto A$*. An* animated δ-pair *is an object in* $\mathrm{AniPair}_\delta$ *which we will denote by* $((A, \delta), A \twoheadrightarrow A'')$*, or simply by* $A \twoheadrightarrow A''$ *when there is no ambiguity.*

It follows from Lemma 2.5.4 and [Lur09, Lem 5.4.5.5] which characterizes colimits in the fiber products, that

LEMMA 2.5.9. *The* ∞*-category* $\mathrm{AniPair}_\delta$ *is cocomplete, and the forgetful functor* $\mathrm{AniPair}_\delta \to \mathrm{AniPair}$ *is conservative and preserves small colimits.*

Explicitly, an *animated δ-pair* is given by an animated δ-ring (A, δ) along with a surjection $A \twoheadrightarrow A''$ of animated $\mathbb{Z}_{(p)}$-algebras. Since $\mathrm{Pair} \subseteq \mathrm{AniPair}$ is a full subcategory (Proposition 2.3.17) and so is $\mathrm{Ring}_\delta \subseteq \mathrm{Ani}(\mathrm{Ring}_\delta)$ (Remark B.0.35), the 1-category of δ-pairs is a full subcategory of the ∞-category of animated δ-pairs. Similar to the ∞-category of animated pairs, we have

LEMMA 2.5.10. *The forgetful functor* $\mathrm{AniPair}_\delta \to \mathrm{AniPair}$ *admits a left adjoint, and the* ∞*-category* $\mathrm{AniPair}_\delta$ *is projectively generated.*

Proof. The left adjoint $\mathrm{AniPair} \to \mathrm{AniPair}_\delta$ concretely given by $(A \twoheadrightarrow A'') \mapsto ((A^\delta, \delta), (A^\delta \twoheadrightarrow A'' \otimes^{\mathbb{L}}_A A^\delta))$ where A^δ is the image of $A \in \mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}})$ under the free animated δ-ring functor $\mathrm{Ani}(\mathrm{Alg}_{\mathbb{Z}_{(p)}}) \to \mathrm{Ani}(\mathrm{Ring}_\delta)$. Now the result follows from Corollary 2.2.3 and Lemmas 2.5.3 and 2.5.4. $\square$

Concretely, a set of compact projective generators for $\mathrm{AniPair}_\delta$ is given by the set $\{(\mathbb{Z}_{(p)}\{X, Y\}, (Y)) \mid X, Y \in \mathrm{Fin}\}$ of δ-pairs, which spans a full subcategory $\mathrm{AniPair}_\delta^0 \subseteq \mathrm{AniPair}_\delta$. Now we turn to the PD-structure. Recall that

LEMMA 2.5.11. ([BS19, LEM 2.11]) *The Frobenius endomorphism* $\varphi_{\mathbb{Z}_{(p)}\{x\}}: \mathbb{Z}_{(p)}\{x\} \to \mathbb{Z}_{(p)}\{x\}$ *on the free* δ*-ring* $\mathbb{Z}_{(p)}\{x\}$*, which is in fact induced by* $x \mapsto \varphi(x) = x^p + p\,\delta(x)$*, is faithfully flat. The same holds for free* δ*-rings generated by arbitrary sets (not-necessarily finite).*

We remark that, thanks to Lemma 2.3.71, it is not necessary to pass to the polynomial ring of finitely many variables to invoke the fiberwise criterion of flatness.

We now relate δ-structure to divided powers. Note that, for any p-torsion free $\mathbb{Z}_{(p)}$-algebra A, any element $y \in A$ and any $n \in \mathbb{N}$, we have

$$\frac{y^n}{n!} \in \frac{y^n}{p^{v_p(n!)}} \mathrm{GL}_1(\mathbb{Z}_{(p)})$$

In particular, $y^p/p!$ (resp. $y^{p^2}/(p^2)!$) differs multiplicatively from y^p/p (resp. y^{p^2}/p^{p+1}) by a unit. When A is a p-torsion free δ-ring, we have $\varphi(y) = y^p + p\,\delta(y)$ and $y^p/p! \in A[p^{-1}]$ belongs to A if and only if $\varphi(y)$ is divisible by p.

Now we define the animated δ-ring $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$ to be the pushout of the diagram

$$\begin{tikzcd}
\mathbb{Z}_{(p)}\{y\} \arrow[r, "y \mapsto pz"] \arrow[d, "y \mapsto \varphi(x)"'] & \mathbb{Z}_{(p)}\{z\} \\
\mathbb{Z}_{(p)}\{x\} &
\end{tikzcd}$$

in the ∞-category $\mathrm{Ani}(\mathrm{Ring}_\delta)$. Since the Frobenius map $\varphi: \mathbb{Z}_{(p)}\{y\} \to \mathbb{Z}_{(p)}\{x\}$ is faithfully flat, so is the map $\mathbb{Z}_{(p)}\{z\} \to \mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$. It follows that $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$ is static and p-torsion-free by Remark 2.3.63, therefore it is a δ-ring by Lemma 2.5.6 (this is essentially [BS19, Lem 2.36]). We need another characterization of the underlying ring of $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$:

LEMMA 2.5.12. ([BS19, LEM 2.36]) *There is a natural isomorphism*

$$D_{\mathbb{Z}_{(p)}\{x\}}(x) \longrightarrow \mathbb{Z}_{(p)}\{x, \varphi(x)/p\}$$

of p-torsion-free $\mathbb{Z}_{(p)}$-algebras.

This map transfers the surjective map $D_{\mathbb{Z}_{(p)}\{x\}}(x) \twoheadrightarrow \mathbb{Z}_{(p)}\{x\}/(x)$ to a surjective map $\mathbb{Z}_{(p)}\{x, \varphi(x)/p\} \twoheadrightarrow \mathbb{Z}_{(p)}\{x\}/(x)$, the existence of which does not seem to be *a priori* clear (which is implicitly involved in [BS19, Lem 2.35]).

Note that since $x \in D_{\mathbb{Z}_{(p)}\{x\}}(x)$ is a non-zero-divisor, the map from the animated PD-envelope of $(\mathbb{Z}_{(p)}\{x\}, (x))$ to the classical PD-envelope is an equivalence, by base change of $(\mathbb{Z}_{(p)}[x], (x))$ along the flat map $\mathbb{Z}_{(p)}[x] \to \mathbb{Z}_{(p)}\{x\} \simeq \mathbb{Z}_{(p)}[x, \delta(x), \delta^2(x), \ldots]$, or alternatively by Proposition 2.3.72. We could replace x by a finite number of variables, which leads to

COROLLARY 2.5.13. *There exists a canonical δ-pair structure on the animated PD-envelope of every δ-pair $(\mathbb{Z}_{(p)}\{X, Y\}, (Y)) \in \mathrm{AniPair}_\delta^0$. More formally, there exists a canonical functor $\mathrm{AniPair}_\delta^0 \to \mathrm{AniPair}_\delta$ which fits into a commutative diagram*

$$\begin{tikzcd}
\mathrm{AniPair}_\delta^0 \arrow[rr, dashed] \arrow[d] & & \mathrm{AniPair}_\delta \arrow[d] \\
\mathrm{AniPair} \arrow[r, "\mathrm{AniPDEnv}"] & \mathrm{AniPDPair} \arrow[r] & \mathrm{AniPair}
\end{tikzcd}$$

of ∞-categories.

Proof. The functoriality of $\mathrm{AniPair}_\delta^0 \to \mathrm{AniPair}_\delta$ needs explanation: a map $(\mathbb{Z}_{(p)}\{X, Y\}, (Y)) \to (\mathbb{Z}_{(p)}\{X', Y'\}, (Y'))$ of δ-pairs induces a map $(\mathbb{Q}\{X, Y\}, (Y)) \to (\mathbb{Q}\{X', Y'\}, (Y'))$ of pairs after inverting p which is "Frobenius"-equivariant, where $\mathbb{Q}\{X, Y\} := \mathbb{Z}_{(p)}\{X, Y\}[p^{-1}]$. A careful v_p-analysis implies that this map restricts to a map $\mathbb{Z}_{(p)}\{X, Y, \varphi(Y)/p\} \to \mathbb{Z}_{(p)}\{X', Y', \varphi(Y')/p\}$ of $\mathbb{Z}_{(p)}$-subalgebras, which gives rise to the functoriality. □

It follows from Propositions B.0.10 and 2.3.34, Lemma 2.5.9, and Corollary 2.5.13 that

COROLLARY 2.5.14. *There exists a canonical animated δ-pair structure on the animated PD-envelope of every animated δ-pair. More formally, there exists a canonical functor $\mathrm{AniPair}_\delta \to \mathrm{AniPair}_\delta$ which fits into a commutative diagram*

$$\begin{tikzcd}
\mathrm{AniPair}_\delta \arrow[rr, dashed] \arrow[d] & & \mathrm{AniPair}_\delta \arrow[d] \\
\mathrm{AniPair} \arrow[r, "\mathrm{AniPDEnv}"] & \mathrm{AniPDPair} \arrow[r] & \mathrm{AniPair}
\end{tikzcd}$$

of ∞-categories. Moreover, the functor $\mathrm{AniPair}_\delta \to \mathrm{AniPair}_\delta$ preserves small colimits.

We give an analysis of the conjugate filtration on the PD-envelope of $(\mathbb{F}_p\{x\}, (x))$ where $\mathbb{F}_p\{x\} := \mathbb{Z}_{(p)}\{x\}/^{\mathbb{L}}p$, which is the base change of the PD-envelope of $(\mathbb{Z}_{(p)}\{x\}, (x))$ along $\mathbb{Z}_{(p)} \to \mathbb{F}_p$. Recall that

1. The (animate) PD-envelope $D_{\mathbb{F}_p[x]}(x)$ is a free $\mathbb{F}_p[x]/(x^p)$-module generated by the set $\{\gamma_{kp}(x) \mid k \in \mathbb{N}\}$ of divided powers of x.
2. For $i \in \mathbb{N}_{\geq 0}$, the $(-i)$-th piece of the conjugate filtration of $D_{\mathbb{F}_p[x]}(x)$ is generated by $\{\gamma_{kp}(x) \mid k \leq i\}$ as an $\mathbb{F}_p[x]/(x^p)$-submodule.

By the base change property (Lemma 2.3.46), we have

1. The (animate) PD-envelope $D_{\mathbb{F}_p\{x\}}(x)$ is a free $\mathbb{F}_p\{x\}/(x^p)$-module generated by the set $\{\gamma_{kp}(x) \mid k \in \mathbb{N}\}$.

2. For $i\in\mathbb{N}_{\geq 0}$, the $(-i)$-th piece of the conjugate filtration of $D_{\mathbb{F}_p\{x\}}(x)$ is generated by $\{\gamma_{kp}(x)\,|\,k\leq i\}$ as an $\mathbb{F}_p\{x\}/(x^p)$-submodule.

We follow the argument of [BS19, Lem 2.35]: for every $y\in\mathbb{Z}_{(p)}\{x\}$ with $y^p/p\in\mathbb{Z}_{(p)}\{x\}$, we have

$$\begin{aligned}\delta\left(\frac{y^p}{p}\right) &= \frac{1}{p}\left(\frac{\varphi(y)^p}{p}-\left(\frac{y^p}{p}\right)^p\right)\\ &= \frac{(y^p+p\,\delta(y))^p}{p^2}-\frac{y^{p^2}}{p^{p+1}}\\ &= \frac{1}{p^2}\left(y^{p^2}+p^2\,y^{p(p-1)}\,\delta(y)+\sum_{k=0}^{p-2}\binom{p}{k}\,y^{kp}\,(p\,\delta(y))^{p-k}\right)-\frac{y^{p^2}}{p^{p+1}}\\ &= \frac{p^{p-1}-1}{p^{p+1}}\,y^{p^2}+y^{p(p-1)}\,\delta(y)+\sum_{k=0}^{p-2}p^{p-2-k}\binom{p}{k}\,y^{kp}\,\delta(y)^{p-k} \end{aligned}\tag{2.5.1}$$

Letting $z=x^p/p$, it follows from $p^{p-1}-1\in\mathrm{GL}_1(\mathbb{Z}_{(p)})$ that

1. The set $\{z^{a_0}\,\delta(z)^{a_1}\,(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r}\,|\,r\in\mathbb{N},0\leq a_0,a_1,\ldots,a_r<p\}$ forms a basis of the free $\mathbb{F}_p\{x\}/(x^p)$-module $\mathbb{Z}_{(p)}\{x,\varphi(x)/p\}/^{\mathbb{L}}p\simeq D_{\mathbb{F}_p\{x\}}(x)$.
2. For every $i\in\mathbb{N}$, the $(-i)$-th piece of the conjugate filtration of $D_{\mathbb{F}_p\{x\}}(x)$ is generated by $\{z^{a_0}\,\delta(z)^{a_1}\,(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r}\,|\,0\leq a_0,a_1,\ldots,a_r<p,a_0+a_1\,p+a_2\,p^2+\cdots+a_r\,p^r\leq i\}$.

Remark 2.5.15. In a bit more imprecise terms, $\delta^k(z)$ differs from $\gamma_{p^k}(x)$ up to a unit, modulo "lower terms".

This generalizes to multivariable case with the same argument:

LEMMA 2.5.16. *Let $(A,I):=(\mathbb{Z}_{(p)}\{X,Y\},(Y))\in\mathrm{AniPair}_\delta^0$ be a δ-pair and let (B,J,γ) be the (animated) PD-envelope of $(\mathbb{F}_p\{X,Y\},(Y))$. Let $Y=\{y_1,y_2,\ldots\}$ and $z_j:=\varphi(y_j)/p$. Then*

1. *The $\varphi_A^*(A/I)$-module B is freely generated by the subset $\{\prod_{j,k}(\delta^k(z_j))^{a_{j,k}}\,|\,0\leq a_{j,k}<p\}\subseteq B$.*
2. *For every $i\in\mathbb{N}$, the $(-i)$-th piece of the conjugate filtration of B is generated by $\{\prod_{j,k}(\delta^k(z_j))^{a_{j,k}}\,|\,0\leq a_{j,k}<p,\sum_{j,k}a_{j,k}\,p^k\leq i\}$ as a $\varphi_A^*(A/I)$-submodule.*

2.5.2. Oriented prisms

In this subsection, we will study animated δ-rings viewed as "non-complete oriented prisms". Recall that a *orientable prism* is a δ-pair (A,I) such that the ideal $I\subseteq A$ is principal, the δ-ring A is I-torsion free, derived (p,I)-complete, and $p\in I+\varphi(I)\,A$ [BS19, Def 3.2]. For technical reasons, we will study the "non-complete" analogues where the completeness and the torsion-freeness are dropped.

We fix a δ-ring A along with a chosen non-zero-divisor $d\in A$. In practice, we are only interested in the special case that $A=\mathbb{Z}_{(p)}\{d\}$ and some variants like $A=\mathbb{Z}_{(p)}\{d,\delta(d)^{-1}\}$. We denote by $\mathrm{Ring}_{\delta,A}$ the 1-category $(\mathrm{Ring}_\delta)_{A/}$ of δ-A-algebras. It follows from Lemma 2.2.10 that

LEMMA 2.5.17. *The 1-category $\mathrm{Ring}_{\delta,A}$ is 1-projectively generated, therefore presentable. A set of compact 1-projective generators is given by $\{A\{X\}:=A\otimes^{\mathbb{L}}_{\mathbb{Z}_{(p)}}\mathbb{Z}_{(p)}\{X\}\,|\,X\in\mathrm{Fin}\}$, which spans a full subcategory of $\mathrm{Ring}_{\delta,A}$ denoted by $\mathrm{Ring}^0_{\delta,A}$.*

DEFINITION 2.5.18. *Let B be an animated δ-ring. The* ∞-category of animated δ-B-algebras *is defined to be the undercategory $\mathrm{Ani}(\mathrm{Ring}_\delta)_{B/}$. When B is static, it is equivalent to the animation $\mathrm{Ani}(\mathrm{Ring}_{\delta,B})$ by Corollary 2.2.14.*

By Lemma 2.4.47, we get an adjunction $\mathrm{Alg}_A\rightleftarrows\mathrm{Ring}_{\delta,A}$, where the forgetful functor $\mathrm{Ring}_{\delta,A}\to\mathrm{Alg}_A$ preserves all small colimits (and as a right adjoint, it preserves small limits as well). It follows from Corollary 2.2.3 that

Lemma 2.5.19. *There is a pair* $\mathrm{Ani}(\mathrm{Alg}_A) \rightleftarrows \mathrm{Ani}(\mathrm{Ring}_{\delta,A})$ *of adjoint functors, being the animation of the pair* $\mathrm{Alg}_A \rightleftarrows \mathrm{Ring}_{\delta,A}$ *of adjoint functors. We will call the functor* $\mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \to \mathrm{Ani}(\mathrm{Alg}_A)$ *the* free animated δ-A-algebra functor*. The functor* $\mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \to \mathrm{Ani}(\mathrm{Alg}_A)$*, called the* forgetful functor*, is conservative and preserves small colimits (and as a right adjoint, it preserves small limits as well).*

Definition 2.5.20. *The* ∞-category of animated δ-A-pairs $\mathrm{AniPair}_{\delta,A}$ *is defined to be the undercategory* $(\mathrm{AniPair}_\delta)_{(\mathrm{id}_A: A \to A)/}$*, which is equivalent to the fiber product* $\mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \times_{\mathrm{Ani}(\mathrm{Alg}_A)} \mathrm{AniPair}_A$ *by [Lur09, Lem 5.4.5.4].*

The set $\{(A\{X,Y\},(Y)) \mid X,Y \in \mathrm{Fin}\}$ form a set of compact projective generators for $\mathrm{AniPair}_{\delta,A}$ by by Lemma 2.2.10, which spans a full subcategory $\mathrm{AniPair}_{\delta,A}^0 \subseteq \mathrm{AniPair}_{\delta,A}$. It follows from Lemmas 2.4.47 and 2.5.10 that

Lemma 2.5.21. *The forgetful functor* $\mathrm{AniPair}_{\delta,A} \to \mathrm{AniPair}_A$ *admits a left adjoint.*

There is a canonical functor $\mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \to \mathrm{AniPair}_{\delta,A}$ given by $B \mapsto (B \twoheadrightarrow B/^{\mathbb{L}} d)$. We observe that

Lemma 2.5.22. *The functor* $\mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \to \mathrm{AniPair}_{\delta,A}, B \mapsto (B \twoheadrightarrow B/^{\mathbb{L}} d)$ *admits a left adjoint* $\mathrm{AniPair}_{\delta,A} \to \mathrm{Ani}(\mathrm{Ring}_{\delta,A})$*, given by the left derived functor (Proposition B.0.10) of* $\mathrm{AniPair}_{\delta,A}^0 \to \mathrm{Ani}(\mathrm{Ring}_{\delta,A}), (A\{X,Y\},(Y)) \mapsto A\{X,Y/d\}$ *where* $A\{X,Y/d\}$ *is an abbreviation for the free* δ*-*A*-algebra* $A\{x_1,x_2,\ldots,y_1/d,y_2/d,\ldots\}$[2.5.2].

Proof. Let G denote the functor $\mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \to \mathrm{AniPair}_{\delta,A}, B \mapsto (B \twoheadrightarrow B/^{\mathbb{L}} d)$. Then we have a functor $F: \mathrm{AniPair}_\delta \to \mathrm{Fun}(\mathrm{Ani}(\mathrm{Ring}_\delta),\mathcal{S})^{\mathrm{op}}$ which preserves small colimits and sends $(B \twoheadrightarrow B'') \in \mathrm{AniPair}_{\delta,A}^0$ to the functor $\mathrm{Map}_{\mathrm{AniPair}_{\delta,A}}(B \twoheadrightarrow B'', G(\cdot))$. By Proposition B.0.10, it is the left derived functor of its restriction to the full subcategory $\mathrm{AniPair}_{\delta,A}^0 \subseteq \mathrm{AniPair}_{\delta,A}$.

We now show that, for every $(A\{X,Y\},(Y)) \in \mathrm{AniPair}_{\delta,A}^0$, the functor $F((A\{X,Y\},(Y)))$ is equivalent to the functor $\mathrm{Map}_{\mathrm{Ani}(\mathrm{Ring}_{\delta,A})}(A\{X,Y/d\},\cdot)$. In other words, the essential image of $F|_{\mathrm{AniPair}_{\delta,A}^0}$ lies in the full subcategory $\mathrm{Ring}_{\delta,A}^0 \hookrightarrow \mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \hookrightarrow \mathrm{Fun}(\mathrm{Ani}(\mathrm{Ring}_{\delta,A}),\mathcal{S})^{\mathrm{op}}$. By adjunctions $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0}) \rightleftarrows \mathrm{AniPair}_A \rightleftarrows \mathrm{AniPair}_{\delta,A}$ (Definition 2.3.15 and Lemma 2.5.21), we have

$$\begin{aligned}
F(A\{X,Y\},(Y))(B) &\simeq \mathrm{Map}_{\mathrm{AniPair}_A}(A[X,Y] \twoheadrightarrow A[X], B \twoheadrightarrow B/^{\mathbb{L}} d) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, D(\mathbb{Z})_{\geq 0})}\Big(X\,\mathbb{Z} \oplus Y\,\mathbb{Z} \leftarrow Y\,\mathbb{Z}, B \xleftarrow{d} B\Big) \\
&\simeq B^{\mathrm{Card}(Y)} \times B^{\mathrm{Card}(X)} \\
&\simeq \mathrm{Map}_{\mathrm{Ani}(\mathrm{Ring}_{\delta,A})}(A\{X,Y/d\}, B)
\end{aligned}$$

which are functorial in $B \in \mathrm{Ani}(\mathrm{Ring}_{\delta,A})$ (note that naively speaking, the "values" of Y/d correspond to the "preimages" of Y under the map $B \xleftarrow{d} B$, therefore the formal notation Y/d).

Since the Yoneda embedding $\mathrm{Ani}(\mathrm{Ring}_{\delta,A}) \hookrightarrow \mathrm{Fun}(\mathrm{Ani}(\mathrm{Ring}_{\delta,A}),\mathcal{S})^{\mathrm{op}}$ is stable under small colimits, it follows that the essential image of F lies in $\mathrm{Ani}(\mathrm{Ring}_{\delta,A})$, which proves that G admits a left adjoint $L: \mathrm{AniPair}_{\delta,A} \to \mathrm{Ani}(\mathrm{Ring}_{\delta,A})$, and that $L(A\{X,Y/d\}) \simeq A\{X,Y/d\}$.

We still need to show that $L|_{\mathrm{AniPair}_{\delta,A}^0}$ coincides with the functor defined in the obvious way. We have shown this objectwise, and since $L(\mathrm{AniPair}_{\delta,A}^0)$ lies in the full subcategory $\mathrm{Ring}_{\delta,A}^0 \hookrightarrow \mathrm{Ani}(\mathrm{Ring}_{\delta,A})$ which is a 1-category, we only need to show that the image of morphisms coincide with the "obvious" choice, i.e. without higher categorical complication. This can be checked putting different d-torsion-free δ-A-algebras $B \in \mathrm{Ring}_{\delta,A}$ into the functorial isomorphism

$$\mathrm{Hom}_{\mathrm{Pair}_{\delta,A}}((A\{X,Y\},(Y)),(B,(d))) \cong \mathrm{Hom}_{\mathrm{Ring}_{\delta,A}}(A\{X,Y/d\},B)$$

given by the adjunction. □

Now we introduce a variant of Definition 2.5.20:

2.5.2. These generators y_i/d are in fact formal variables z_i. This notation indicates that the counit map $(A\{X,Y\},(Y)) \to (A\{X,Y/d\} \twoheadrightarrow A\{X,Y/d\}/^{\mathbb{L}} d)$ is induced by $x_i \mapsto x_i$ and $y_i \mapsto z_i\, d$.

DEFINITION 2.5.23. *The* ∞-category of animated δ-(A, d)-pairs $\mathrm{AniPair}_{\delta,(A,d)}$ *is defined to be the undercategory* $(\mathrm{AniPair}_{\delta,A})_{(A\twoheadrightarrow A/^{\mathbb{L}}d)/}$*, which is equivalent to the fiber product* $\mathrm{Ani}(\mathrm{Ring}_{\delta,A})\times_{\mathrm{Ani}(\mathrm{Alg}_A)}\mathrm{AniPair}_{(A\twoheadrightarrow A/^{\mathbb{L}}d)/}$ *by [Lur09, Lem 5.4.5.4].*

By Lemma 2.2.10, we have

LEMMA 2.5.24. *The ∞-category $\mathrm{AniPair}_{\delta,(A,d)}$ is projectively generated. A set of compact projective generators is given by $\{(A\{X,Y\},(d,Y))\,|\,X,Y\in\mathrm{Fin}\}$, which spans a full subcategory of $\mathrm{AniPair}_{\delta,(A,d)}$ denoted by $\mathrm{AniPair}^0_{\delta,(A,d)}$.*

Note that A is initial in $\mathrm{AniPair}_{\delta,A}$. It follows from Lemmas 2.4.47 and 2.5.22 that

COROLLARY 2.5.25. *The functor $\mathrm{Ani}(\mathrm{Ring}_{\delta,A})\to\mathrm{AniPair}_{\delta,(A,d)}, B\mapsto(B\twoheadrightarrow B/^{\mathbb{L}}d)$ admits a left adjoint $\mathrm{AniPair}_{\delta,(A,d)}\to\mathrm{Ani}(\mathrm{Ring}_{\delta,A})$, which will be denoted by* PrismEnv[2.5.3]*, given by the left derived functor (Proposition B.0.10) of $\mathrm{AniPair}^0_{\delta,(A,d)}\to\mathrm{Ani}(\mathrm{Ring}_{\delta,A}),(A\{X,Y\},(d,Y))\mapsto A\{X,Y/d\}$.*

Furthermore, for every $B\in\mathrm{Ani}(\mathrm{Ring}_{\delta,A})$, by unrolling the definitions, the counit map $\mathrm{PrismEnv}(B\twoheadrightarrow B/^{\mathbb{L}}d)\to B$ is an equivalence, therefore

LEMMA 2.5.26. *The functor $\mathrm{Ani}(\mathrm{Ring}_{\delta,A})\to\mathrm{AniPair}_{\delta,(A,d)}$ is in fact fully faithful, the image of which is a reflective subcategory (Definition 2.2.44).*

The following concept is not strictly necessary, but it would help us to understand when we need to "divide by d":

DEFINITION 2.5.27. *Let A be a δ-ring and $d\in A$ a non-zero-divisor. Let $M\in D(A/^{\mathbb{L}}d)$ be a $A/^{\mathbb{L}}d$-module spectrum. For every $n\in\mathbb{Z}$, the n-th* Breuil–Kisin twist *of M with respect to (A,d), denoted by $M\{n\}$, is defined to be $M\otimes^{\mathbb{L}}_{A/^{\mathbb{L}}d}(dA/d^2A)^{\otimes^{\mathbb{L}}_{A/^{\mathbb{L}}d}n}$.*

Note that when $d\in A$ is a non-zero-divisor, the $A/^{\mathbb{L}}d$-module $d^nA/d^{n+1}A$ is a free of rank 1, therefore equivalent to $A/^{\mathbb{L}}d$. The Breuil–Kisin twists are strictly necessary when we want to generalize to non-orientable prisms. In our case, we understand $M\{1\}$ "formally multiplied by d" and $M\{-1\}$ "formally divided by d", just as the formal notations y_i/d in Lemma 2.5.22.

Finally, we introduce a variant of the concept of *distinguished elements* [BS19, Def 2.19]:

DEFINITION 2.5.28. *Let A be a δ-ring. We say that an element $d\in A$ is* weakly distinguished *if the ideal $(d,\delta(d))$ is the unital ideal A, or equivalently, $\delta(d)$ is invertible in A/d.*

Remark 2.5.29. Let A be a δ-ring and $d\in\mathrm{Rad}(A)$ an element in the Jacobson radical. Then d is weakly distinguished if and only if it is distinguished.

The following lemma is a motivation for the introduction of weakly distinguished elements:

LEMMA 2.5.30. (CF. [BS19, LEM 2.23]) *Let A be a δ-ring, $I=(d)\subseteq A$ a principal ideal. Then for any invertible element $u\in\mathrm{GL}_1(A)$, the principal ideals $\delta(d)(A/I)$ and $\delta(ud)(A/I)$ are the same. In particular, when I is generated by a non-zero-divisor, the principal ideal $\delta(d)(A/I)$ does not depend on the choice of the generator $d\in I$.*

Proof. We have $\delta(ud)=\varphi(u)\,\delta(d)+\delta(u)\,d^p\equiv\varphi(u)\,\delta(d)\pmod{ud}$. Since u is invertible, so is $\varphi(u)$, and the result follows. □

COROLLARY 2.5.31. *Let A be a δ-ring, $I\subseteq A$ a principal ideal generated by a non-zero-divisor. Then the followings are equivalent:*

1. *There exists a weakly distinguished generator d of I.*
2. *Every generator d of I is weakly distinguished.*

2.5.3. This is understood as a "non-complete" prismatic envelope when p lies in the Jacobson radical $\mathrm{Rad}(A)$ and $d\in A$ is weakly distinguished (Definition 2.5.28).

Remark 2.5.32. (BHATT) We have a variant of Corollary 2.5.31 which does not involve non-zero-divisors, by replacing a principal ideal I by the equivalence classes of maps $A \to A$ of A-modules, and the proof of Lemma 2.5.30 implies that the concept of "weakly distinguished" is invariant under this equivalence. More generally, we can consider the equivalence classes of an invertible A-module I along with a map $I \to A$, and define the concept of such a map $I \to A$ being weakly distinguished when $p \in \mathrm{Rad}(A)$. This generalizes to animated δ-rings.

Recall that an animated ring A is *p-local* if the element $p \in \pi_0(A)$ lies in the Jacobson radical[2.5.4] $\mathrm{Rad}(\pi_0(A))$.

LEMMA 2.5.33. *Let A be a p-local δ-ring and $d \in A$ a weakly distinguished element. Then for every $n \in \mathbb{N}$, $\varphi^n(\delta(d))$ is invertible in A/d.*

Proof. By induction, it suffices to show that, for every $u \in A$ of which the image in A/d is invertible, then so is the image of $\varphi(u)$ in A/d. It follows from the identity $\varphi(u) = u^p + p\,\delta(u)$, since the image of u^p in A/d invertible, and $p \in \mathrm{Rad}(A/d)$. □

2.5.3. Conjugate filtration

In this subsection, we will introduce the *conjugate filtration* on "non-complete prismatic envelopes", which plays a similar role as the conjugate filtrations on animated PD-envelopes and derived crystalline cohomology. Let A be a p-local δ-ring, $d \in A$ a weakly distinguished non-zero-divisor. To simplify the presentation, we mostly concentrate on the "single variable" case: $\mathrm{PrismEnv}(A\{y\}, (d, y)) /^{\mathbb{L}} d \simeq A\{y/d\} /^{\mathbb{L}} d$ as an $A\{y\}/^{\mathbb{L}}(d, y)$-algebra (or module).

First, note that the identity

$$\begin{aligned} \delta(u^p) &= (\varphi(u^p) - u^{p^2})/p \\ &= ((u^p + p\,\delta(u))^p - u^{p^2})/p \\ &= \sum_{k=1}^{p} \binom{p}{k} u^{p(p-k)}\, p^{k-1}\, \delta(u)^k \end{aligned} \tag{2.5.2}$$

holds in the free δ-ring $\mathbb{Z}_{(p)}\{u\}$, therefore it is an identity in any δ-ring.

We now compute $\delta^n(y)$ in terms of $\delta^n(z)$ where $y = z\,d$ in the free δ-A-algebra $A\{y\}$:

$$\begin{aligned} \delta(y) &= \delta(z\,d) \\ &= \delta(z)\,\varphi(d) + z^p\,\delta(d) \\ \delta^2(y) &= \delta(\delta(z)\,\varphi(d) + z^p\,\delta(d)) \\ &= \delta(\delta(z)\,\varphi(d)) + \delta(z^p\,\delta(d)) - \underbrace{\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} (\delta(z)\,\varphi(d))^{p-k}\,(z^p\,\delta(d))^k}_{=:R_2} \\ &= \delta^2(z)\,\varphi^2(d) + \delta(z)^p\,\delta(\varphi(d)) + \delta(z^p)\,\delta(\varphi(d)) + z^{p^2}\,\delta^2(d) - R_2 \\ &= \delta^2(z)\,\varphi^2(d) + \varphi(\delta(d))\,(1 + p^{p-1})\,\delta(z)^p + \sum_{k=1}^{p-1} \cdots + z^{p^2}\,\delta^2(d) - R_2 \end{aligned}$$

where we used the fact that $\varphi \circ \delta = \delta \circ \varphi$ and (2.5.2) (which leads to the summand $\sum_{k=1}^{p-1} \cdots$), and in general, we have

LEMMA 2.5.34. *Let $A\{z\}$ be the free δ-A-algebra and $y := z\,d$. For every $n \in \mathbb{N}$, there exists a unique polynomial $P_n \in A[X_0, \ldots, X_{n-1}]$ with $\deg_{X_{n-1}} P_n \leq p$ such that*

$$\delta^n(y) = \delta^n(z)\,\varphi^n(d) + P_n(z, \delta(z), \ldots, \delta^{n-1}(z))$$

Moreover, there exists a unique $Q_n \in A[X_0, \ldots, X_{n-1}]$ with $\deg_{X_{n-1}} Q_n < p$ such that $P_n = a_n\,\varphi^{n-1}(\delta(d))\,X_{n-1}^p + Q_n$ where a_n are partial sums $\sum_{k=0}^{n-1} p^{k(p-1)}$ of the geometric progression $(p^{k(p-1)})_{k \in \mathbb{N}}$. Note that $a_n \in \mathrm{GL}_1(\mathbb{Z}_{(p)})$ for $n > 0$. On the other hand, if we endow X_i with degree p^i, then P_n is homogeneous of degree p^n.

2.5.4. The *Jacobson radical* $\mathrm{Rad}(A)$ of a ring A is defined to be the subset (and *a fortiori*, the ideal) of elements $x \in A$ such that for every $a \in A$, the element $1 + a\,x$ is invertible in A.

Proof. The uniqueness follows from the freeness. We prove the existence inductively on $n\in\mathbb{N}$. When $n=0$, this is obvious. Now let $n\in\mathbb{N}_{>0}$, and assume that this is true for every $m<n$, Now we have

$$\begin{aligned}\delta^n(y) &= \delta(\delta^{n-1}(y))\\ &= \delta(\delta^{n-1}(z)\,\varphi^{n-1}(d)+P_{n-1}(z,\delta(z),\ldots,\delta^{n-2}(z)))\\ &= \delta(\delta^{n-1}(z)\,\varphi^{n-1}(d))+\delta(P_{n-1}(z,\delta(z),\ldots,\delta^{n-2}(z)))-R_n\end{aligned}$$

where $\delta(\delta^{n-1}(z)\,\varphi^{n-1}(d))=\delta^n(z)\,\varphi^n(d)+(\delta^{n-1}(z))^p\,\varphi^{n-1}(\delta(d))$ and

$$R_n:=\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}(\delta^{n-1}(z)\,\varphi^{n-1}(d))^{p-k}\,(P_{n-1}(z,\delta(z),\ldots,\delta^{n-2}(z)))^k$$

Note that the "degree" of $\delta^{n-1}(z)$ in R_n is strictly less than p. Let $b_{n-1}=a_{n-1}\,\varphi^{n-2}(\delta(d))$, we have

$$\begin{aligned}\delta(P_{n-1}(z,\delta(z),\ldots)) &= \delta(b_{n-1}\,(\delta^{n-2}(z))^p+Q_{n-1}(z,\delta(z),\ldots))\\ &= \delta(b_{n-1}\,(\delta^{n-2}(z))^p)+\delta(Q_{n-1}(z,\delta(z),\ldots))-\underbrace{\sum_{k=1}^{p-1}\cdots}_{=:R'_n}\\ &= \varphi(b_{n-1})\,\delta((\delta^{n-2}(z))^p)+\delta(Q_{n-1}(\cdots))+\delta(b_{n-1})\,(\delta^{n-2}(z))^{p^2}\\ &\quad -R'_n\end{aligned}$$

and only $\varphi(b_{n-1})\,\delta((\delta^{n-2}(z))^p)$ has contribution on $\delta^{n-1}(z)^p$, and

$$\delta((\delta^{n-2}(z))^p)=\sum_{k=1}^{p}\binom{p}{k}(\delta^{n-2}(z))^{p(p-k)}\,p^{k-1}\,(\delta^{n-1}(z))^k$$

has contribution on $\delta^{n-1}(z)^p$ only at $k=p$, i.e. $p^{p-1}\delta^{n-1}(z)^p$. Note that $\varphi(b_{n-1})=a_{n-1}\,\varphi^{n-1}(\delta(d))$, the result then follows. □

We now rewrite $A\{y\}\to A\{z\}, y\mapsto z\,d$ as the sequential composite (i.e. the $A\{y\}$-algebra $A\{z\}$ is equivalent to the sequential colimit of)

$$A\{y\}\longrightarrow A\{y\}\otimes^{\mathbb{L}}_{B_0}C_0\longrightarrow A\{y\}\otimes^{\mathbb{L}}_{B_1}C_1\longrightarrow\cdots \tag{2.5.3}$$

where $A_n:=A[z,\ldots,\delta^{n-1}(z)]$, $B_n:=A_n[\delta^n(y)]$ and $C_n:=A_n[\delta^n(z)]$ are polynomial algebras, and the map $B_n\to C_n$ is given by the evaluation $\delta^n(y)\mapsto\delta^n(z)\,\varphi^n(d)+P_n(z,\delta(z),\ldots,\delta^{n-1}(z))$ by Lemma 2.5.34. Thus $B_n\to C_n$ could be written as the composite (where we replace $\delta^n(y)$ by u and $\delta^n(z)$ by v)

$$B_n=A_n[u]\to A_n[u,v]/(u-\varphi^n(d)\,v-P_n(z,\delta(z),\ldots,\delta^{n-1}(z)))\cong A_n[v]=C_n \tag{2.5.4}$$

In other words, $B_n\to C_n$ is essentially formally adjoining[2.5.5] $(\delta^n(y)-P_n(z,\delta(z),\ldots,\delta^{n-1}(z)))/\varphi^n(d)$ to B_n as an (animated) A-algebra, and the $A\{y\}$-algebra $A\{z\}$ is obtained by formally adjoining $(\delta^n(y)-P_n(z,\delta(z),\ldots,\delta^{n-1}(z)))/\varphi^n(d)$ iteratively from $A\{y\}$.

The *conjugate filtration* on $A\{y/d\}/^{\mathbb{L}}d$ is given by $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{y/d\}/^{\mathbb{L}}d)$ being the $A\{y\}/^{\mathbb{L}}(y,d)$-submodule of $A\{y/d\}/^{\mathbb{L}}d$ spanned by $\{(y/d)^{a_0}\,\delta(y/d)^{a_1}\,(\delta^2(y/d))^{a_2}\cdots(\delta^r(y/d))^{a_r}\mid r\in\mathbb{N}, 0\le a_0,a_1,\ldots,a_r<p\}$. Passing to the multivariable version, we get:

LEMMA 2.5.35. *Let A be a p-local δ-ring and $d\in A$ a weakly distinguished non-zero-divisor. Then there exists a canonical functor* $\mathrm{Fil}^*_{\mathrm{conj}}(\mathrm{PrismEnv}(\cdot)/^{\mathbb{L}}d):\mathrm{AniPair}^0_{\delta,(A,d)}\to\mathrm{CAlg}(\mathrm{DF}^{\le 0}(A/^{\mathbb{L}}d))$ *which preserves finite coproducts, along with a functorial map* $\mathrm{Fil}^*_{\mathrm{conj}}(\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d)\to\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d$*, understood as the* conjugate filtration *on* $\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d$*, such that*

1. *The conjugate filtration is exhaustive, that is to say, the induced map* $\mathrm{Fil}^{-\infty}_{\mathrm{conj}}(\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d)\to\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d$ *is an equivalence in $D(A)$.*

2.5.5. Note that this is true although $\varphi^n(d)$ is not necessarily a non-zero-divisor.

2. *The filtration* $\mathrm{Fil}^*_{\mathrm{conj}}(\mathrm{PrismEnv}(A\{y\} \twoheadrightarrow A\{y\}/^{\mathbb{L}}(y,d))/^{\mathbb{L}}d)$ *coincides with the filtration* $\mathrm{Fil}^*_{\mathrm{conj}}(A\{y/d\}/^{\mathbb{L}}d)$ *constructed above.*
3. *The maps* $\mathrm{Fil}^{-i}_{\mathrm{conj}}(\mathrm{PrismEnv}(A\{x\} \twoheadrightarrow A\{x\}/^{\mathbb{L}}d)/^{\mathbb{L}}d) \to \mathrm{PrismEnv}(A\{x\} \twoheadrightarrow A\{x\}/^{\mathbb{L}}d)/^{\mathbb{L}}d \simeq A\{x\}/^{\mathbb{L}}d$ *are equivalences for all* $i \in \mathbb{N}$*, that is to say, the conjugate filtration on* $(A/^{\mathbb{L}}d)\{x\}$ *is "constant*[2.5.6]*".*

Proof. The conjugate filtration on each object $\mathrm{PrismEnv}(B,J)$ for $(B,J) \in \mathrm{AniPair}^0_{\delta,(A,d)}$ is completely determined by these properties and that the functor preserves finite coproducts, since every (B,J) could be written as a coproduct of $A\{x\} \twoheadrightarrow A\{x\}/^{\mathbb{L}}d$ and $A\{y\} \twoheadrightarrow A\{y\}/^{\mathbb{L}}(y,d)$. Concretely, $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X,Y/d\}/^{\mathbb{L}}d)$ are generated, as an $A\{X,Y\}/(d,Y)$-submodule, by "standard monomials" $\prod_{(r,y)\in E} \delta^r(y/d)$ "of total degree $\leq i$" where $E \subseteq \mathbb{N} \times Y$ is a finite subset and the element $\delta^r(y/d)$ is of degree p^r for $r \in \mathbb{N}$ and $y \in Y$. One verifies that this indeed gives rise to a functor.

Alternatively, if we further assume that d is weakly transversal (Definition 2.5.39), then we can invoke Lemma 2.5.42 to reduce significantly the computations. □

Definition 2.5.36. *Let A be a p-local δ-ring and $d \in A$ a weakly distinguished non-zero-divisor. Then the* conjugate filtration *on* $\mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ *for* $(B \twoheadrightarrow B'') \in \mathrm{AniPair}_{\delta,(A,d)}$ *is given by the left derived functor (Proposition B.0.10)* $\mathrm{AniPair}_{\delta,(A,d)} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(A/^{\mathbb{L}}d))$ *of the functor* $\mathrm{AniPair}^0_{\delta,(A,d)} \to \mathrm{CAlg}(\mathrm{DF}^{\leq 0}(A/^{\mathbb{L}}d))$ *in Lemma 2.5.35.*

It follows from Lemma 2.2.43 that

Lemma 2.5.37. *Let A be a p-local δ-ring and $d \in A$ a weakly distinguished non-zero-divisor. Then the* conjugate filtration *on* $\mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ *for every* $(B \twoheadrightarrow B'') \in \mathrm{AniPair}_{\delta,(A,d)}$ *is exhaustive, i.e.* $\mathrm{Fil}^{-\infty}\,\mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d \to \mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ *is an equivalence.*

We now analyze the "denominators" $\varphi^n(d)$ when A is p-local and d is weakly distinguished:

Lemma 2.5.38. (cf. [AL19b, Lem 3.5]) *Let A be a p-local δ-ring and $d \in A$ a weakly distinguished element. Then for every $n \in \mathbb{N}_{>0}$, there exists a unit $u \in \mathrm{GL}_1(A/d)$ such that $\varphi^n(d) \equiv p\,u \pmod{d}$.*

Proof. We will construct inductively on $n \in \mathbb{N}_{>0}$ a sequence $(u_n)_n \in A^{\mathbb{N}_{>0}}$ such that for every $n \in \mathbb{N}_{>0}$, the image of u_n in A/d is invertible, and $\varphi^n(d) - d^{p^n} = p\,u_n$. We take $u_1 = \delta(d)$, and suppose that u_m are already constructed for $1 \leq m < n$, then

$$\begin{aligned}
\varphi^n(d) &= \varphi^{n-1}(\varphi(d))\\
&= \varphi^{n-1}(d^p + p\,\delta(d))\\
&= (\varphi^{n-1}(d))^p + p\,\varphi(\delta(d))\\
&= (d^{p^{n-1}} + p\,u_{n-1})^p + p\,\varphi(\delta(d))\\
&= d^{p^n} + p\left(\varphi(\delta(d)) + \sum_{k=1}^{p} \binom{p}{k} d^{p^{n-1}(p-k)}\, p^{k-1}\, u_{n-1}^k\right)
\end{aligned}$$

We pick $u_n = \delta(d) + \sum_{k=1}^{p} \binom{p}{k} d^{p^{n-1}(p-k)}\, p^{k-1}\, u_{n-1}^k$. Note that the second summand $\sum_{k=1}^{p} \cdots$ is canonically divisible by p (separating the cases $k=0$ and $k>1$), thus $u_n \equiv \delta(d) \pmod{p}$ of which the image in $A/(p,d)$ is invertible. The result then follows from the fact that $p \in \mathrm{Rad}(A/d)$. □

We introduce the following temporary terminology:

Definition 2.5.39. *Let A be a δ-ring. We say that an element $d \in A$ is* weakly transversal *if it is weakly distinguished and the sequence (d,p) is regular in A, that is to say, d is a non-zero-divisor and A/d is p-torsion-free.*

2.5.6. More precisely, it is constant after restriction to $\mathbb{Z}_{\leq 0}$, but this restriction is expected as the conjugate filtration is non-positive.

Recall that for a ring A, the *Zariski localization* of A along an ideal $I \subseteq A$ is defined to be the localization of A at the multiplicative set $1+I$. The image of I in $(1+I)^{-1}A$ lies in the Jacobson radical.

Example 2.5.40. The element d in the p-local δ-ring $\mathbb{Z}_{(p)}\{d, \delta(d)^{-1}\}_{(p)}$ is weakly transversal. In fact, this special case suffices for our applications.

Now we assume that $d \in A$ is weakly transversal. In the "single variable" case $A\{y\}/^{\mathbb{L}}(d, y) \to A\{y/d\}/^{\mathbb{L}}d$, by Lemmas 2.5.38 and 2.5.34, the sequence $(z, \delta(z), \delta^2(z), \dots)$ forms a system similar to that of divided p^r-powers $(\gamma_{p^r})_{r\in\mathbb{N}}$ up to a multiplication of a unit after modulo d:

$$\begin{aligned} p\,\delta(z) &\equiv -a_1\,\delta(d)\,z^p \,(\operatorname{mod} B)\\ p\,\delta^2(z) &\equiv -a_2\,\varphi(\delta(d))\,\delta(z)^p \,(\operatorname{mod} B[\delta(z)])\\ p\,\delta^3(z) &\equiv -a_3\,\varphi^2(\delta(d))\,\delta^2(z)^p \,(\operatorname{mod} B[\delta(z), \delta^2(z)])\end{aligned}$$

where $B := A\{y\}/^{\mathbb{L}}(d, y)$ and $a_n\,\varphi^{n-1}(\delta(d)) \in \mathrm{GL}_1(A/d)$ (cf. Remark 2.5.15). We now translate this observation to an analysis of the conjugate filtration, which seems hard to attack directly. We look at the maps $B_0/^{\mathbb{L}}(d, y) \to C_0/^{\mathbb{L}}d$ and $B_n/^{\mathbb{L}}d \to C_n/^{\mathbb{L}}d$ for $n \in \mathbb{N}_{>0}$ induced by the map (2.5.4). We first note that the map $B_0/^{\mathbb{L}}(d, y) \to C_0/^{\mathbb{L}}d$ is the polynomial algebra in single variable z.

If we further (derived) modulo p, we see that $B_n/^{\mathbb{L}}(d, p) \to C_n/^{\mathbb{L}}(d, p)$ for $n \in \mathbb{N}_{>0}$ is killing a polynomial $\delta^n(y) - P_n(z, \delta(z), \ldots, \delta^{n-1}(z))$ monic in $\delta^{n-1}(z)$ of degree p, and then adjoining a formal variable $\delta^n(z)$. In view of (2.5.3), we see that the map $A\{y\}/^{\mathbb{L}}(d, p, y) \xrightarrow{y\mapsto zd} A\{z\}/^{\mathbb{L}}(d, p)$ is the composition of consecutively adjoining a root of a monic polynomial of degree p, and consequently, as a $A\{y\}/^{\mathbb{L}}(d, p, y)$-module, $A\{z\}/^{\mathbb{L}}(d, p)$ is freely generated by $\{z^{a_0}\delta(z)^{a_1}(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r} \mid r \in \mathbb{N}, 0 \le a_0, a_1, \dots, a_r < p\}$.

On the other hand, if we invert p, we see that, for every $n \in \mathbb{N}_{>0}$, the maps $(B_n/^{\mathbb{L}}d)[p^{-1}] \to (C_n/^{\mathbb{L}}d)[p^{-1}]$ are equivalences, therefore $(A\{y\}/^{\mathbb{L}}(d, y))[p^{-1}] \to (A\{z\}/^{\mathbb{L}}d)[p^{-1}]$ is the polynomial algebra in one variable z.

The mod p conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{z\}/^{\mathbb{L}}(d, p))/^{\mathbb{L}}p$ is then freely generated by $\{z^{a_0}\delta(z)^{a_1}(\delta^2(z))^{a_2}\cdots(\delta^r(z))^{a_r} \mid r \in \mathbb{N}, 0 \le a_0, a_1, \dots, a_r < p, a_0 + p\,a_1 + \cdots + p^r a_r \le i\}$. On the other hand, the rationalized conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{z\}/^{\mathbb{L}}(d, p))[p^{-1}]$ is given by the $(A\{y\}/^{\mathbb{L}}(d, y))[p^{-1}]$-polynomials in z of degree $\le i$. This follows from the following lemma, which can be established by induction on n:

LEMMA 2.5.41. *In the rationalized free δ-ring $\mathbb{Z}_{(p)}\{x\}[p^{-1}] \cong \mathbb{Q}[x, \varphi(x), \varphi^2(x), \ldots]$, for every $n \in \mathbb{N}$, the image of $\delta^n(x) \in \mathbb{Z}_{(p)}\{x\}$ in $\mathbb{Q}[x, \varphi(x), \varphi^2(x), \ldots]$ is given by a polynomial $D_n(x, \varphi(x), \ldots, \varphi^n(x))$ such that $\deg_x D_n = p^n$ with leading term $(-p^{-1})^{1+p+\cdots+p^n} x^{p^n}$ for all $n \in \mathbb{N}$.*

We summarize the "multi-variable" version as follows:

LEMMA 2.5.42. *Let A be a p-local δ-ring and $d \in A$ a weakly transversal element. Let $(A\{X, Y\}, (d, Y)) \in \mathrm{AniPair}^0_{\delta,(A,d)}$. Then*

1. *The generator $\left\{\prod_{(r,y)\in E}\delta^r(y/d)\right\}_E$ for $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X, Y/d\}/^{\mathbb{L}}d)$ as an $A\{X, Y\}/^{\mathbb{L}}(d, Y)$-submodule, "of total degree $\le i$" where $E \subseteq \mathbb{N} \times Y$ is a finite subset and the element $\delta^r(y/d)$ is of degree p^r, becomes an basis after (derived) modulo p. This also holds for $i = +\infty$.*

2. *The $(-i)$-th piece of the rationalized conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X, Y/d\}/^{\mathbb{L}}d)[p^{-1}] \subseteq (A\{X, Y/d\}/^{\mathbb{L}}d)[p^{-1}]$ is given by the $A\{X, Y\}/^{\mathbb{L}}(d, Y)$-polynomials in variables Y/d of total degree $\le i$. This also holds for $i = +\infty$.*

Furthermore, an element $x \in A\{X, Y/d\}/^{\mathbb{L}}d$ belongs to the $(-i)$-th piece of the conjugate filtration $\mathrm{Fil}^{-i}_{\mathrm{conj}}(A\{X, Y/d\}/^{\mathbb{L}}d)$ if and only if so does it after (derived) modulo p and after rationalization.

Remark 2.5.43. In some vague terms, in Lemma 2.5.42, the derived modulo p is about "controlling the denominators", and the rationalization is about "controlling the degree".

Recall that for every $(B,J)\in\mathrm{AniPair}^0_{\delta,(A,d)}$, there exists a canonical map $B/^{\mathbb{L}}d\simeq B\otimes^{\mathbb{L}}_A(A/^{\mathbb{L}}d)\to B/J$ which is in fact surjective. Then we have the following "multivariable" version:

LEMMA 2.5.44. *Let A be a p-local δ-ring and $d\in A$ a weakly transversal element. For every $(A\{X,Y\},(d,Y))=:(B,J)\in\mathrm{AniPair}^0_{\delta,(A,d)}$, let $K:=\ker(B/^{\mathbb{L}}d\to B/J)$. Note that K/K^2 is naturally a B/J-module. Then there exists a comparison map*

$$\Gamma^*_{B/J}((K/K^2)\{-1\})\longrightarrow \mathrm{gr}^{-*}_{\mathrm{conj}}(\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d)$$

of graded B/J-algebras induced by $[\gamma_n(z_i)]\mapsto\prod_{j=0}^r\left(\frac{\delta^j(z_i)}{-a_j\,\varphi^j(\delta(d))}\right)^{n_j}$ where $Y=\{y_1,\dots\}$, $z_i=y_i/p$ and $n=\sum_{j=0}^r n_j p^j$ is the p-adic expansion of n. The comparison map is functorial in $(B,J)\in\mathrm{AniPair}^0_{\delta,(A,d)}$.

Proof. The comparison map is induced by $\left[\gamma_n\left(\frac{y}{d}\right)\right]\mapsto\prod_{j=0}^r\left(\frac{\delta^j(y/d)}{-a_j\,\varphi^j(\delta(d))}\right)^{n_j}$ for every y in the ideal (d,Y). To see that this is well-defined, the most nontrivial part is to show that this vanishes when $y\in(d,Y^2)$. By the multiplicity of the conjugate filtration, we can assume that $n_r=1$ and $n_j=0$ for $j\neq r$, and it suffices to analyze $\delta^r(y/d)$ when $y\in(d,Y^2)$, which can be reduced to the special case that $y=y_1\,y_2$ where $y_1,y_2\in Y$.

By Lemma 2.5.41, the element $\delta^r(y_1\,y_2/d)\in A[p^{-1}][X,Y/d,\varphi(Y),\varphi^2(Y/d),\dots]$ is a polynomial in $y_1\,y_2/d=y_1\,z_2,\varphi(y_1\,y_2),\dots,\varphi^r(y_1\,y_2)$. The crucial point is that $y_1\,y_2/d=(y_1/d)\,(y_2/d)\,d=0$ in $A\{X,Y/d\}/^{\mathbb{L}}d$, therefore after rationalization, $\delta^r(y_1\,y_2/d)$ lies in $\mathrm{Fil}^0_{\mathrm{conj}}(\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d)[p^{-1}]$.

By Lemma 2.5.34, $\delta^r(y_1\,z_2)=\delta^r(z_2)\,\varphi^r(y_1)+P_r(z_2,\dots,\delta^{r-1}(z_2))$ where P_r is an $A\{y_1\}$-polynomial. Note that $\varphi^r(y_1)=\varphi^r(z_1\,d)=\varphi^r(z_1)\,\varphi^r(d)\equiv 0\ (\mathrm{mod}\,(d,p))$ by Lemma 2.5.38. Since P_r is homogeneous of degree p^r when $\deg(\delta^j(z_2))=p^j$, it follows that for every monomial $\prod_j T_j^{n_j}$ of P_r, there exists a j such that $n_j\geq p$, but then $\delta^j(z_2)^{n_j}$ is a linear combination of basis elements in Lemma 2.5.42 which shows that $\prod_j(\delta^j(z_2))^{n_j}\in\mathrm{Fil}^{-(p^r-1)}_{\mathrm{conj}}(\mathrm{PrismEnv}(B,J)/^{\mathbb{L}}d)/^{\mathbb{L}}p$. The result then follows from the last part of Lemma 2.5.42. □

It again follows from Lemma 2.5.42, via derived modulo p and rationalization, that

LEMMA 2.5.45. *Let A be a p-local δ-ring and $d\in A$ a weakly transversal element. For every $(B,J)\in\mathrm{AniPair}^0_{\delta,(A,d)}$, the comparison map in Lemma 2.5.44 is an equivalence.*

After such a long march, let us harvest the Hodge–Tate comparison, which is a prismatic analogue of Corollary 2.3.60. Note that for every $(B\twoheadrightarrow B'')\in\mathrm{AniPair}_{\delta,(A,d)}$, note that the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B\\ \downarrow & & \downarrow\\ A/^{\mathbb{L}}d & \longrightarrow & B'' \end{array}$$

induces a natural map $B/^{\mathbb{L}}d\simeq B\otimes^{\mathbb{L}}_A(A/^{\mathbb{L}}d)\to B''$ which is surjective, that is to say, $B/^{\mathbb{L}}d\twoheadrightarrow B''$ is an animated pair. It then follows from Lemma 2.5.45 and Proposition B.0.10 that

THEOREM 2.5.46. (HODGE–TATE) *Let A be a p-local δ-ring and $d\in A$ a weakly transversal element. Then for every animated δ-(A,d)-pair $(B\twoheadrightarrow B'')\in\mathrm{AniPair}_{\delta,(A,d)}$, there exists a canonical equivalence*

$$\Gamma^i_{B''}(\mathrm{gr}^1(\mathbb{L}\,\mathrm{AdFil}(B/^{\mathbb{L}}d\twoheadrightarrow B''))\{-1\})\longrightarrow\mathrm{gr}^{-i}_{\mathrm{conj}}(\mathrm{PrismEnv}(B\twoheadrightarrow B'')/^{\mathbb{L}}d)$$

which is functorial in $(B\twoheadrightarrow B'')\in\mathrm{AniPair}_{\delta,(A,d)}$, where $\mathbb{L}\,\mathrm{AdFil}$ is the adic filtration functor defined before Corollary 2.3.54.

Let R be an $\mathbb{E}_1$-ring. Recall that a right R-module M is *faithfully flat* if it is flat (Definition 2.3.62) and $\pi_0(M)$ is a faithfully flat right $\pi_0(R)$-module. A map $R\to S$ of $\mathbb{E}_\infty$-rings is *faithfully flat* if S is faithfully flat as an R-module. There is a useful characterization of faithfully flat algebras:

LEMMA 2.5.47. ([LUR04, LEMMA 5.5]) *Let $f: R \to S$ be a map of static (commutative) rings. Then f is faithfully flat if and only if f is flat, injective and that* $\mathrm{coker}(f)$ *taken in the category of R-modules is flat.*

LEMMA 2.5.48. *Let $f: R \to S$ a map of $\mathbb{E}_\infty$-rings. If f is faithfully flat, then* $\mathrm{cofib}(f)$ *taken in the ∞-category of R-module spectra is flat. The converse is true if R is supposed to be connective.*

Proof. Assume first that f is faithfully flat. Let $M := \mathrm{coker}(\pi_0(R) \to \pi_0(S))$. By Lemma 2.5.47, the map $\pi_0(R) \to \pi_0(S)$ is injective and the $\pi_0(R)$-module M is flat, Then for every $n \in \mathbb{Z}$, we have the exact sequence

$$\mathrm{Tor}_1^{\pi_0(R)}(\pi_n(R), M) \to \pi_n(R) \to \pi_n(R) \otimes_{\pi_0(R)} \pi_0(S) \to \pi_n(R) \otimes_{\pi_0(R)} M \to 0$$

which implies that the map $\pi_n(R) \to \pi_n(R) \otimes_{\pi_0(R)} \pi_0(S)$ is injective. Since f is flat, the canonical map $\pi_n(R) \otimes_{\pi_0(R)} \pi_0(S) \to \pi_n(S)$ is an isomorphism, therefore the map $\pi_n(R) \to \pi_n(S)$ is injective. Then the long exact sequence associated to the fiber sequence $R \to S \to \mathrm{cofib}(f)$ splits into short exact sequences

$$0 \longrightarrow \pi_n(R) \longrightarrow \pi_n(S) \longrightarrow \pi_n(\mathrm{cofib}(f)) \longrightarrow 0$$

which implies that the canonical map $M \to \pi_0(\mathrm{cofib}(f))$ is an isomorphism. Furthermore, we have a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \pi_n(R) \arrow[r] \arrow[d, "\sim"] & \pi_n(R) \otimes_{\pi_0(R)} \pi_0(S) \arrow[r] \arrow[d, "\sim"] & \pi_n(R) \otimes_{\pi_0(R)} M \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \pi_n(R) \arrow[r] & \pi_n(S) \arrow[r] & \pi_n(\mathrm{cofib}(f)) \arrow[r] & 0 \end{tikzcd}$$

By the short five lemma, the map $\pi_n(R) \otimes_{\pi_0(R)} M \to \pi_n(\mathrm{cofib}(f))$ is an isomorphism, therefore $\mathrm{cofib}(f)$ is flat.

Now we assume that R is connective and that $\mathrm{cofib}(f)$ is flat. By definition, $\mathrm{cofib}(f)$ is connective and so is S by the fiber sequence $R \to S \to \mathrm{cofib}(f)$. For every static R-module M, we have the fiber sequence

$$M \longrightarrow M \otimes_R^{\mathbb{L}} S \longrightarrow M \otimes_R^{\mathbb{L}} \mathrm{cofib}(f)$$

By flatness of $\mathrm{cofib}(f)$ and [Lur17, Prop 7.2.2.13], $M \otimes_R^{\mathbb{L}} \mathrm{cofib}(f)$ is static, therefore so is $M \otimes_R^{\mathbb{L}} S$. It then follows from [Lur17, Thm 7.2.2.15] that S is a flat R-module. It remains to show that the map $\pi_0(R) \to \pi_0(S)$ is faithfully flat. By Lemma 2.5.47, it suffices to show that $\pi_0(R) \to \pi_0(S)$ is injective and $\mathrm{coker}(\pi_0(R) \to \pi_0(S))$ is flat. The first follows from the connectivity of $\mathrm{cofib}(f)$, and the later follows from the isomorphism $\mathrm{coker}(\pi_0(R) \to \pi_0(S)) \cong \pi_0(\mathrm{cofib}(f))$ and the flatness of $\mathrm{cofib}(f)$. $\square$

Now we have a prismatic analogue of Corollary 2.3.66, with a similar argument:

PROPOSITION 2.5.49. *Let A be a p-local δ-ring and $d \in A$ a weakly transversal element. Let $(B \twoheadrightarrow B'') \in \mathrm{AniPair}_{\delta,(A,d)}$ be an animated δ-(A,d)-pair such that the canonical animated pair $B/^{\mathbb{L}}d \twoheadrightarrow B''$ is quasiregular. Then the unit map $B'' \to \mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ is faithfully flat.*

Proof. By Theorem 2.5.46 and the quasiregularity of $B/^{\mathbb{L}}d \twoheadrightarrow B''$, for every $i \in \mathbb{N}$, the B''-module $\mathrm{gr}_{\mathrm{conj}}^{-i}(\mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d)$ is flat. By Lemma 2.3.64, for every $i \in \mathbb{N}_{>0}$, $\mathrm{cofib}(B'' \to \mathrm{Fil}_{\mathrm{conj}}^{-i}(\mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d))$ is flat. Since the conjugate filtration is exhaustive (Lemma 2.5.35) and the collection of flat modules is stable under filtered colimits [Lur17, Lem 7.2.2.14(1)], we get $\mathrm{cofib}(B'' \to \mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d)$ is a flat B''-module. Then the result follows from Lemma 2.5.48. $\square$

Remark 2.5.50. In Proposition 2.5.49, if we further assume that B'' is static, then so is $\mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$. This does not imply that $\mathrm{PrismEnv}(B \twoheadrightarrow B'')$ is static. However, it implies that, after taking d-completion, $\mathrm{PrismEnv}(B \twoheadrightarrow B'')$ becomes static which should be understood as a "static d-completed envelope".

Remark 2.5.51. There is a p-completed analogue of Proposition 2.5.49: suppose that the animated pair $B/^{\mathbb{L}}d \twoheadrightarrow B''$ is p-completely[2.5.7] quasiregular, that is to say, the shifted cotangent complex $\mathbb{L}_{B''/(B/^{\mathbb{L}}d)}[-1]$ is a p-completely flat B''-module, then the same proof shows that the unit map $B'' \to \mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ is p-completely faithfully flat (i.e., it becomes faithfully flat after derived modulo p).

In particular, if if (B, d) is a bounded oriented prism [BS19, Def 3.2] and that B'' is static and has bounded p-power torsion, then the p-completion of $\mathrm{PrismEnv}(B \twoheadrightarrow B'')/^{\mathbb{L}}d$ is static. Moreover, by [BS19, Lem 3.7(2,3)], the (p, d)-completion of $C := \mathrm{PrismEnv}(B \twoheadrightarrow B'')$ is static and thus it follows from a formal argument that $(C^{\wedge}_{(p,d)}, d)$ is the prismatic envelope of the δ-pair $B \twoheadrightarrow B''$ as long as it is d-torsion free. In other words, we generalize [BS19, Prop 3.13] by weakening regularity to quasiregularity.

As we will explain in Section 2.6, we prefer to deal with completeness (and complete flatness) in the general framework of condensed (or more precisely, solid) mathematics.

We record a simple corollary which furnishes a quite general class of "flat covers of the final object" in the *affine prismatic site* (similar to Definition 2.4.65) which will be studied in a future work. For this, we need the following definition:

Definition 2.5.52. *Let A be a δ-ring, $d \in A$ an element and B an animated δ-A-algebra. The* ∞-category of δ-(B, d)-pairs*, denoted by* $\mathrm{AniPair}_{\delta,(B,d)}$*, is defined to be the undercategory* $(\mathrm{AniPair}_{\delta,(A,d)})_{(B \twoheadrightarrow B/^{\mathbb{L}}d)/}$.

Let A be a p-local δ-ring and $d \in A$ a weakly distinguished non-zero-divisor. Let B be an animated δ-A-algebra, and R an animated $B/^{\mathbb{L}}d$-algebra. Similar to Definition 2.4.65, we can consider the category of animated δ-B-algebras C along with a map[2.5.8] $R \to C/^{\mathbb{L}}d$ of animated $B/^{\mathbb{L}}d$-algebras, which we will denoted by $R \to C/^{\mathbb{L}}d \twoheadleftarrow C$, depicted by the commutative diagram

$$\begin{tikzcd} B \arrow[rr] \arrow[d, two heads] & & C \arrow[d, two heads] \\ B/^{\mathbb{L}}d \arrow[r] & R \arrow[r] & C/^{\mathbb{L}}d \end{tikzcd}$$

More formally, this is the fiber product $\mathrm{Ani}(\mathrm{Ring}_{\delta})_{B/} \times_{\mathrm{Ani}(\mathrm{Ring})_{(B/^{\mathbb{L}}d)/}} \mathrm{Ani}(\mathrm{Ring})_{R/}$ of ∞-categories, the opposite category of which will be denoted by $\Delta(R/(B,d))$[2.5.9]. In a future work, we will show that the ∞-category $\Delta(R/(B,d))$ admits a Grothendieck topology given by flat covers.

Now let P be an animated δ-B-algebra along with a surjection $P \twoheadrightarrow R$ of animated B-algebras such that the cotangent complex $\mathbb{L}_{P/B}/^{\mathbb{L}}d$ is a flat $P/^{\mathbb{L}}d$-module.

Remark 2.5.53. We note that such P exists in abundance. For example, this happens when R is a smooth $B/^{\mathbb{L}}d$-algebra which admits a smooth B-lift P with a δ-structure compatible with that on B, or P is a polynomial B-algebra $B[x_i]$ (of possibly infinitely many variables) with $\delta(x_i) = 0$ along with a surjection $P \twoheadrightarrow R$ of animated B-algebras.

Then the animated pair $P \twoheadrightarrow R$ admits a canonical animated δ-(B, d)-pair structure, and thus the animated δ-ring $\mathrm{PrismEnv}(P \twoheadrightarrow R)$ gives rise to an object of $\Delta(R/(B,d))$ (by abuse of notation, we will still denote by $\mathrm{PrismEnv}(P \twoheadrightarrow R)$ the object of $\Delta(R/(B,d))$).

Remark 2.5.54. By Lemma 2.5.26, when $P \twoheadrightarrow R$ is "already" a non-completed prism in the sense that the induced map $P/^{\mathbb{L}}d \to R$ is an equivalence, the non-completed prismatic envelope $\mathrm{PrismEnv}(P \twoheadrightarrow R)$ is equivalent to P itself.

2.5.7. "p-complete" concepts are usually applied to p-complete objects. However, this is not necessary because we can always derived p-complete a non-complete object.

2.5.8. Unlike the crystalline case, here we do not assume that the map $R \to C/^{\mathbb{L}}d$ is an equivalence.

2.5.9. In [BS19], they used the notation $(R/A)_{\Delta}$. However, this notation is usually devoted to topoi (such as X_{et} and X_{cris}). We therefore adopt the traditional notation for sites.

For any object $(R \to C/^{\mathbb{L}} d \twoheadleftarrow C) \in \Delta(R/(B,d))$, by unrolling the definitions, the product of $(R \to C/^{\mathbb{L}} d \twoheadleftarrow C)$ and $\mathrm{PrismEnv}(P \twoheadrightarrow R)$ in $\Delta(R/(B,d))$ is given by $\mathrm{PrismEnv}(P \otimes_B^{\mathbb{L}} C \twoheadrightarrow R)$. We have therefore a map $C/^{\mathbb{L}} d \to \mathrm{PrismEnv}(P \otimes_B^{\mathbb{L}} C \twoheadrightarrow C/^{\mathbb{L}} d)/^{\mathbb{L}} d$ of animated R-algebras. The following proposition is essentially equivalent to the "flat cover of the final object", cf. [Cha20, Prop 1.1.2][2.5.10].

PROPOSITION 2.5.55. *Let A be a p-local δ-ring and $d \in A$ a weakly transversal element. Let B be an animated δ-A-algebra and $P \twoheadrightarrow R$ an animated δ-(B,d)-pair such that the cotangent complex $\mathbb{L}_{P/B}/^{\mathbb{L}} d$ is a flat $P/^{\mathbb{L}} d$-module. Then for every $(R \to C/^{\mathbb{L}} d \twoheadleftarrow C) \in \Delta(R/(B,d))$, the map $C/^{\mathbb{L}} d \to \mathrm{PrismEnv}(P \otimes_B^{\mathbb{L}} C \twoheadrightarrow C/^{\mathbb{L}} d)/^{\mathbb{L}} d$ is faithfully flat.*

Proof. By Proposition 2.5.49, it suffices to show that the map $(P/^{\mathbb{L}} d) \otimes_{B/^{\mathbb{L}} d}^{\mathbb{L}} (C/^{\mathbb{L}} d) \to C/^{\mathbb{L}} d$ is quasiregular. To simplify the notations, let $P'' := P/^{\mathbb{L}} d$, $B'' := B/^{\mathbb{L}} d$ and $C'' := C/^{\mathbb{L}} d$. We have the transitivity sequence

$$\mathbb{L}_{(P'' \otimes_{B''}^{\mathbb{L}} C'')/C''} \otimes_{P'' \otimes_{B''}^{\mathbb{L}} C''}^{\mathbb{L}} C'' \to \mathbb{L}_{C''/C''} \simeq 0 \to \mathbb{L}_{C''/(P'' \otimes_{B''}^{\mathbb{L}} C'')}$$

associated to the maps $C'' \to P'' \otimes_{B''}^{\mathbb{L}} C'' \to C''$ whose composite is $\mathrm{id}_{C''}$. Note that $\mathbb{L}_{(P'' \otimes_{B''}^{\mathbb{L}} C'')/C''} \simeq \mathbb{L}_{P''/B''} \otimes_{B''}^{\mathbb{L}} C''$ is a flat $P'' \otimes_{B''}^{\mathbb{L}} C''$-module. It follows that $\mathbb{L}_{C''/(P'' \otimes_{B''}^{\mathbb{L}} C'')}[-1]$ is a flat C''-module. □

We first learned the possibility of such kind of result from [MT, Prop 3.4] (which is closely related to [Cha20, Prop 1.1.2]). Later we came up with an argument which is essentially equivalent to the proof of Proposition 2.5.55, but the foundation was lacking then, therefore the current article could be understood as paving the way to this proof. Now we want to point out that, with minor modifications, this proof would imply [MT, Prop 3.4] and the relevant technical lemmas in the recent works by Y. Tian and by A. Ogus [Ogu21] announced in Illusie conference. Furthermore, when the proper foundation is laid, the same proof would lead to a flat cover of the final object in the *absolute prismatic site*, and in particular, it would recover [AL19a, Lem 5.2.8]. We now show this implication.

As in Remark 2.5.51, we assume that (B,d) is a bounded oriented prism, R is derived p-complete and the map $B/^{\mathbb{L}} d \to R$ is a p-completely quasisyntomic (i.e. the map $B/^{\mathbb{L}} d \to R$ is p-completely flat and the cotangent complex $\mathbb{L}_{R/(B/^{\mathbb{L}} d)}$ has p-complete Tor-amplitude in $[0,1]$ as an R-module spectrum). Then by [BMS19, Lem 4.7], R is static and has bounded p-power torsion. Let P be a derived (p,d)-complete animated δ-B-algebra which is (p,d)-completely quasismooth (i.e. the map $B \to P$ is (p,d)-completely flat and the cotangent complex $\mathbb{L}_{P/B}$ is a (p,d)-completely flat B-module). Then by [BS19, Lem 3.7(2,3)], P is static and for every $n \in \mathbb{N}$, the multiplication map $d^n : P \to P$ is injective and P/d^n has bounded p-power torsion.

Now suppose that we are given a surjection $P \twoheadrightarrow R$ of B-algebras. Then by Remark 2.5.51, the derived (p,d)-completion of $\mathrm{PrismEnv}(P \twoheadrightarrow R)$ is static and the prism defined by this (p,d)-completed algebra is the prismatic envelope in the sense of [BS19, Prop 3.13], where the d-torsion-freeness follows from the complete flatness of $B \to P$ and [BS19, Lem 3.7(2)]. Moreover, since both $B/^{\mathbb{L}} d \to R$ and $R \to \mathrm{PrismEnv}(P \twoheadrightarrow R)/^{\mathbb{L}} d$ is p-completely flat, the map $B \to \mathrm{PrismEnv}(P \twoheadrightarrow R)$ is (p,d)-completely flat (this in fact generalizes the flatness in [BS19, Prop 3.13]). The proof of Proposition 2.5.55 shows that

PROPOSITION 2.5.56. *Let (B,d) be a bounded oriented prism, R a derived p-complete and p-completely quasisyntomic B/d-algebra. Let P be a derived (p,d)-complete animated δ-B-algebra which is (p,d)-completely quasismooth over B, equipped with a surjection $P \twoheadrightarrow R$ of B-algebras. Then the (p,d)-completion of* $\mathrm{PrismEnv}(P \twoheadrightarrow R)$ *is static which gives rise to a prism (C,d) in the prismatic site*[2.5.11] *defined in [BS19, Def 4.1] of R relative to the base prism (B,d). Furthermore, (C,d) is a flat cover of the final object in this site.*

2.5.10. This characterization was already implicit in the Faltings's proof of "independence of the choice of the framing".

2.5.11. It is the non-animated but (p,d)-completed version of our $\Delta(R/(B,d))$.

This implies virtually all the similar technical cover results for relative prismatic site mentioned above, cf. Remark 2.5.54.

Remark 2.5.57. For the absolute prismatic site, the proof also works in the special case of [AL19a, Lem 5.2.8], but we are not aware of a statement as general as Proposition 2.5.56.

2.6. Sketch of an analytic theory

As we mentioned before, our theory of crystalline cohomology is non-completed. That is to say, given a smooth $\mathbb{F}_p$-algebra R, the derived crystalline cohomology of R with respect to the PD-pair $(\mathbb{Z}_p, (p))$ does not coincide with the classical crystalline cohomology of R. One could recover the classical version by taking the derived p-completion. Instead, we take a more systematic approach: we will introduce the concept of *analytic PD-pairs* $(\mathcal{A} \twoheadrightarrow \mathcal{A}'', \gamma, \mathcal{M})$ which serves as rudiments of *analytic crystalline cohomology* in a future work. In particular, roughly speaking, we will put a topology on the PD-pair $(\mathbb{Z}_p, (p))$ along with an analytic structure, such that the analytic crystalline cohomology of a smooth $\mathbb{F}_p$-algebra is essentially the classical crystalline cohomology equipped with the p-adic topology.

Here are some very succinct recollection of condensed mathematics [Sch19b]: given two topological spaces X, Y, we will denote by $C(X, Y)$ the set of continuous maps $X \to Y$. Recall that CHaus is the category of compact Hausdorff spaces and ExtrDisc $\subseteq$ CHaus is the full subcategory spanned by *extremally disconnected sets*, that is, the projective objects in the category CHaus. Concretely, a compact Hausdorff space X is extremally disconnected if for every surjective map $Y \to Z$ of compact Hausdorff spaces, the induced map $C(X, Y) \to C(X, Z)$ is also surjective [Sta21, Tag 08YN]. Every extremally disconnected set is profinite [Sta21, Tag 08YI]. Given a presentable ∞-category $\mathcal{C}$, the *∞-category* $\mathrm{Cond}(\mathcal{C})$ *of condensed objects in* $\mathcal{C}$ is defined to be the full subcategory of the functor category $\mathrm{Fun}(\mathrm{ExtrDisc}^{\mathrm{op}}, \mathcal{C})$ spanned by those functors $F: \mathrm{ExtrDisc}^{\mathrm{op}} \to \mathcal{C}$ which preserve finite limit along with a set-theoretic technical condition that there is a strong limit cardinal κ such that the functor F is left Kan extended from the full subcategory $\mathrm{ExtrDisc}_\kappa^{\mathrm{op}} \subseteq \mathrm{ExtrDisc}^{\mathrm{op}}$ spanned by κ-small extremally disconnected sets, i.e., the cardinality of the underlying set is less than κ.

Analytic (PD-)pairs

In this subsection, we will first introduce the concept of condensed (PD-)pairs, and then we indicate how to put an analytic structure.

Definition 2.6.1. *A* condensed pair *(resp.* condensed PD-pair*)*[2.6.1] *is a condensed object in the ∞-category* AniPair *(resp.* AniPDPair*). The ∞-category of condensed pairs (resp. condensed PD-pairs) is denoted by* Cond(AniPair) *(resp.* Cond(AniPDPair)*).*

Example 2.6.2. There is a canonical way to view an animated pair (resp. animated PD-pair) as a condensed animated pair (resp. condensed animated PD-pair). More precisely, there is a canonical functor AniPair $\to$ Cond(AniPair) (resp. AniPDPair $\to$ Cond(AniPDPair)). We explain the functor AniPair $\to$ Cond(AniPair) in more details: given a pair $(A, I) \in \mathrm{AniPair}^0$ (which was denoted by $\mathcal{D}^0$ in Subsection 2.3.2), note that for every extremally disconnected set S, $(C(S, A), C(S, I))$ is naturally a ring-ideal pair, which defines a functor $\mathrm{AniPair}^0 \to \mathrm{Cond(Pair)} \subseteq \mathrm{Cond(AniPair)}$. Then the functor AniPair $\to$ Cond(AniPair) is defined to be the left derived functor (Proposition B.0.10) of the functor $\mathrm{AniPair}^0 \to \mathrm{Cond(AniPair)}$ above.

Example 2.6.3. Consider the condensed ring $\mathbb{Z}_p$ given by the functor $\mathrm{ExtrDisc} \ni S \mapsto C(S, \mathbb{Z}_p) \in$ Ring along with the surjective map $\mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p$ of condensed rings, or more precisely, profinite rings, where the surjectivity follows from S being projective in CHaus. This gives rise to a condensed pair.

Moreover, for every $S \in \mathrm{ExtrDisc}$, the ring $C(S, \mathbb{Z}_p)$ is p-torsion free and there is a canonical PD-structure on the pair $C(S, \mathbb{Z}_p) \twoheadrightarrow C(S, \mathbb{F}_p)$, which gives rise to a condensed object in PDPair $\subseteq$ AniPDPair, thus we get a condensed PD-pair $(\mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p, \gamma)$.

2.6.1. We ignore the adjective "animated" to make the terminology shorter.

Example 2.6.4. ([SCH19A, PROP 6.8 & THM 6.9]) Let $0<r<1$. Consider the condensed ring $\mathbb{Z}((T))_{>r}$ given by $S\mapsto\bigcup_{\substack{r<\tilde{r}<1\\ c\geq 0}}\{\sum_{n\gg-\infty}a_nT^n\mid a_n\in C(S,\mathbb{Z})\text{ and }\sum_{n\gg-\infty}|a_n|\,\tilde{r}^n\leq c\}\subseteq\mathbb{Z}((T))(S)$. Let $0<r'<r$, then the map $\theta_{r'}:\mathbb{Z}((T))_{>r}\to\mathbb{R}$ induced by $T\mapsto r'$ is surjective on $\{*\}\in\mathrm{ExtrDisc}$ with the kernel generated by a non-zero-divisor. Furthermore, it is surjective as a map of condensed sets by [Sch19a, Prop 7.2]. Thus we get a condensed pair $\mathbb{Z}((T))_{>r}\twoheadrightarrow\mathbb{R}$.

Concretely, it follows from Theorem 2.3.23 that a condensed pair is given by a *surjective* map $A\twoheadrightarrow A''$ of condensed animated rings, where the surjectivity means that the induced map $\pi_0(A)\to\pi_0(A'')$ of static condensed rings is surjective. As in Notation 2.3.25, we will denote a condensed PD-pair by $(A\twoheadrightarrow A'',\gamma)$. We need the following result:

LEMMA 2.6.5. *The animated PD-envelope functor* $\mathrm{AniPair}\to\mathrm{AniPDPair}$ *preserves finite products.*

Proof. This follows from Lemmas 2.3.46 and 2.3.41 and the conjugate filtration. □

Note that, given a small category $\mathcal{C}$ which admits finite products, we have an adjunction $\mathrm{Fun}(\mathcal{C},\mathrm{AniPair})\rightleftarrows\mathrm{Fun}(\mathcal{C},\mathrm{AniPDPair})$ induced by the adjunction $\mathrm{AniPair}\rightleftarrows\mathrm{AniPDPair}$. This is a direct corollary of the alternative definition of adjoint functors in [Lur20, Tag 02EP][2.6.2]. Restricting to full subcategories spanned by functors which preserves finite products, we get an adjunction $\mathrm{Fun}^{\pi}(\mathcal{C},\mathrm{AniPair})\rightleftarrows\mathrm{Fun}^{\pi}(\mathcal{C},\mathrm{AniPDPair})$. Setting $\mathcal{C}=\mathrm{ExtrDisc}_{\kappa}^{\mathrm{op}}$ and taking the colimit over all uncountable strong limit cardinals κ, we get

COROLLARY 2.6.6. *There is a canonical pair* $\mathrm{Cond}(\mathrm{AniPair})\rightleftarrows\mathrm{Cond}(\mathrm{AniPDPair})$ *of adjoint functors. We will call the functor* $\mathrm{Cond}(\mathrm{AniPair})\to\mathrm{Cond}(\mathrm{AniPDPair})$ *the* condensed PD-envelope functor.

DEFINITION 2.6.7. *An* analytic pair *(resp.* analytic PD-pair*) is given by the datum of a normalized [Sch19a, Def 12.9] analytic ring* $(\mathcal{A},\mathcal{M})$ *along with a condensed pair* $\mathcal{A}\twoheadrightarrow\mathcal{A}''$ *(resp. a condensed PD-pair* $(\mathcal{A}\to\mathcal{A}'',\gamma)$*), which will be denoted by* $(\mathcal{A},\mathcal{M})\twoheadrightarrow\mathcal{A}''$ *(resp.* $((\mathcal{A},\mathcal{M})\twoheadrightarrow\mathcal{A}'',\gamma)$*. The* ∞*-category* AnPair *of analytic pairs (resp.* AnPDPair *of analytic PD-pairs) is defined to be the fiber product* $\mathrm{AnRing}\times_{\mathrm{Cond}(\mathrm{Ani}(\mathrm{Ring}))}\mathrm{Cond}(\mathrm{AniPair})$ *(resp.* $\mathrm{AnRing}\times_{\mathrm{Cond}(\mathrm{Ani}(\mathrm{Ring}))}\mathrm{Cond}(\mathrm{AniPDPair})$*).*

Remark 2.6.8. The ∞-categories in Definition 2.6.7 are *a priori* simplicial sets. They are effectively ∞-categories since $\mathrm{Cond}(\mathrm{AniPDPair})\to\mathrm{Cond}(\mathrm{Ani}(\mathrm{Ring}))$ and $\mathrm{Cond}(\mathrm{AniPair})\to\mathrm{Cond}(\mathrm{Ani}(\mathrm{Ring}))$ are categorical fibrations.

Remark 2.6.9. We remark that, given an analytic pair $(\mathcal{A},\mathcal{M})\twoheadrightarrow\mathcal{A}''$, there is a canonical normalized analytic structure $\mathcal{M}''$ on $\mathcal{A}''$ given by $S\mapsto\mathcal{M}(S)\otimes_{\mathcal{A}}^{\mathbb{L}}\mathcal{A}''$ (see [Sch19a, Prop 12.8] for the associative analogue), which gives rise to a map $(\mathcal{A},\mathcal{M})\to(\mathcal{A}'',\mathcal{M}'')$ of analytic rings, therefore our notation $(\mathcal{A},\mathcal{M})\twoheadrightarrow\mathcal{A}''$ is an abuse of notation of $(\mathcal{A},\mathcal{M})\twoheadrightarrow(\mathcal{A}'',\mathcal{M}'')$.

Example 2.6.10. Any condensed pair $A\twoheadrightarrow A''$ (resp. condensed PD-pair $(A\twoheadrightarrow A'',\gamma)$) gives rise to an analytic pair (resp. analytic PD-pair) by taking the trivial analytic structure $\mathcal{M}[S]=A[S]$ for $S\in\mathrm{ExtrDisc}$.

Example 2.6.11. Consider the analytic ring $\mathbb{Z}_{p,\blacksquare}$ given by the condensed ring $\mathbb{Z}_p$ along with the functor $S\mapsto\mathbb{Z}_{p,\blacksquare}[S]:=\mathbb{Z}_p[S]^{\blacksquare}$ (see [Sch19a, Prop 7.9], essentially because $\mathbb{Z}_p$ is a compact idempotent in the derived category $D(\mathbb{Z}_{\blacksquare})$ of solid abelian groups). By Example 2.6.3, we get an analytic PD-pair $(\mathbb{Z}_{p,\blacksquare}\twoheadrightarrow\mathbb{F}_p,\gamma)$.

Example 2.6.12. ([SCH19A, THM 6.9]) Let $0<r'<1$ and $0<p<1$. Set $r=(r')^p$. Then combining with Example 2.6.4, we get an analytic pair $(\mathbb{Z}((T))_{>r},\mathcal{M})\twoheadrightarrow\mathbb{R}$ of analytic rings induced by $T\mapsto r'$, where the induced analytic structure on $\mathbb{R}$ is precisely given by $\mathcal{M}_{<p}$. It would be interesting if we could apply techniques in arithmetic geometry to this analytic pair to study the homotopy theory of real manifolds.

2.6.2. We are informed by Denis NARDIN in private conversation.

Now we prove the existence of *analytic PD-envelope*.

Proposition 2.6.13. *The forgetful functor* AnPDPair $\to$ AnPair *admits a left adjoint, called the* analytic PD-envelope functor.

Proof. Let $(\mathcal{A},\mathcal{M})\twoheadrightarrow\mathcal{A}''$ be an analytic pair. Let $(\mathcal{B}\twoheadrightarrow\mathcal{A}'',\gamma)$ be the condensed PD-envelope of $\mathcal{A}\twoheadrightarrow\mathcal{A}''$. Then similar to Remark 2.6.9, the unit map $\mathcal{A}\to\mathcal{B}$ gives rise to an analytic structure $\mathcal{N}$ on $\mathcal{B}$, given by $\mathrm{ExtrDisc}\ni S\mapsto\mathcal{N}[S]:=\mathcal{M}[S]\otimes^{\mathbb{L}}_{\mathcal{A}}\mathcal{B}$. One then checks the initiality of $((\mathcal{B},\mathcal{N})\twoheadrightarrow\mathcal{A}''$, $\gamma)$. $\square$

Appendix B

Animations and projectively generated categories

In this appendix, we recollect basic category-theoretic facts about animations [CS19] and projectively generated categories needed in the text.

B.0.1. Projectively generated ∞-categories We recall that $\mathcal{S}$ is the ∞-category of animae (see Section 2.1). In very few places, we will encounter large animae of which the ∞-category will be denoted by $\hat{\mathcal{S}}$. One way to distinguish small objects and large objects is to fix a Grothendieck universe.

Definition B.0.1. ([Lur09, Rem 5.5.8.20]) *Let $\mathcal{C}$ be a cocomplete ∞-category and $X\in\mathcal{C}$ an object. We say that X is* compact and projective*, or that X is a* compact projective object*, if the functor $\mathcal{C}\to\hat{\mathcal{S}}, Y\mapsto\mathrm{Map}_{\mathcal{C}}(X,Y)$ corepresented by X commutes with filtered colimits and geometric realizations.*

Remark B.0.2. Here we need $\hat{\mathcal{S}}$ in lieu of $\mathcal{S}$ because the ∞-category $\mathcal{C}$ is not necessarily locally small. In practice, the ∞-categories that we encounter, e.g. projectively generated ∞-categories, are *a fortiori* locally small, but not necessarily a priori locally small.

Definition B.0.3. ([Lur09, Def 5.5.8.23]) *Let $\mathcal{C}$ be a cocomplete ∞-category and $S\subseteq\mathcal{C}$ a (small) collection of objects of $\mathcal{C}$. We say that S is a* set of compact projective generators *for $\mathcal{C}$ if the following conditions are satisfied:*

1. *Each element of S is a compact projective object of $\mathcal{C}$.*
2. *The full subcategory of $\mathcal{C}$ spanned by finite coproducts of elements of S is essentially small.*
3. *The set S generates $\mathcal{C}$ under small colimits.*

We say that an ∞-category $\mathcal{C}$ is projectively generated *if it is cocomplete and there exists a set S of compact projective generators for $\mathcal{C}$.*

Remark B.0.4. Let $\mathcal{C}$ be a cocomplete ∞-category and $\mathcal{C}_0\subseteq\mathcal{C}$ an essentially small full subcategory. Then we will abuse the terminology by saying that $\mathcal{C}_0$ is a set of compact projective generators for $\mathcal{C}$ if a skeleton of $\mathcal{C}_0$ is a set of compact generators for $\mathcal{C}$.

A closely related concept is that of *sifted colimits* which is based on the following definition:

Definition B.0.5. ([Lur09, Def 5.5.8.1]) *A simplicial set K is called* sifted *if it satisfies the following conditions:*

1. *The simplicial set K is nonempty.*
2. *The diagonal map $K\to K\times K$ is cofinal.*

Projectively generated ∞-categories are essentially determined by a set of compact projective generators. More precisely, we have the following:

Notation B.0.6. ([Lur09, Def 5.5.8.8]) *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. We let $\mathcal{P}_\Sigma(\mathcal{C})$ denote the full subcategory of $\mathcal{P}(\mathcal{C}):=\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$ spanned by those functors $\mathcal{C}^{\mathrm{op}}\to\mathcal{S}$ which preserves finite products.*

Proposition B.0.7. ([Lur09, Prop 5.5.8.10]) *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. Then*

1. *The ∞-category $\mathcal{P}_\Sigma(\mathcal{C})$ is an accessible localization of $\mathcal{P}(\mathcal{C})$, therefore presentable.*

2. *The Yoneda embedding $j:\mathcal{C}\to\mathcal{P}(\mathcal{C})$ factors through $\mathcal{P}_\Sigma(\mathcal{C})$. Moreover, the induced functor $\mathcal{C}\to\mathcal{P}_\Sigma(\mathcal{C})$ preserves finite coproducts.*

3. *Let $\mathcal{D}$ be a presentable ∞-category and let $\mathcal{P}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors. Then G factors through $\mathcal{P}_\Sigma(\mathcal{C})$ if and only if $f=F\circ j:\mathcal{C}\to\mathcal{D}$ preserves finite coproducts.*

4. *The full subcategory $\mathcal{P}_\Sigma(\mathcal{C})\subseteq\mathcal{P}(\mathcal{C})$ is stable under sifted colimits.*

We recall that, for a small ∞-category $\mathcal{C}$, $\mathrm{Ind}(\mathcal{C})\subseteq\mathcal{P}(\mathcal{C})$ is the full subcategory generated under filtered colimits by the essential image of the Yoneda embedding $\mathcal{C}\to\mathcal{P}(\mathcal{C})$, [Lur09, Prop 5.3.5.3 & Cor 5.3.5.4]. It follows from [Lur09, Prop 5.3.5.11] that

LEMMA B.0.8. *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. Then the fully faithful embedding $\mathcal{C}\hookrightarrow\mathcal{P}_\Sigma(\mathcal{C})$ extends uniquely to a functor $\mathrm{Ind}(\mathcal{C})\to\mathcal{P}_\Sigma(\mathcal{C})$ which preserves filtered colimit. This functor $\mathrm{Ind}(\mathcal{C})\to\mathcal{P}_\Sigma(\mathcal{C})$ is fully faithful.*

LEMMA B.0.9. *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts. Then the ∞-category $\mathcal{P}_\Sigma(\mathcal{C})$ is projectively generated for which $\mathcal{C}\subseteq\mathcal{P}_\Sigma(\mathcal{C})$ is a set of projective generators. In fact, for any $X\in\mathcal{P}_\Sigma(\mathcal{C})$, there exists a simplicial object $U_\bullet:\mathbf{\Delta}^{\mathrm{op}}\to\mathrm{Ind}(\mathcal{C})$ whose colimit is X.*

Proof. First, since $\mathcal{P}_\Sigma(\mathcal{C})\subseteq\mathcal{P}(\mathcal{C})$ is a accessible localization, $\mathcal{P}_\Sigma(\mathcal{C})$ is presentable [Lur09, Rem 5.5.1.6] therefore cocomplete. Since $\mathcal{P}_\Sigma(\mathcal{C})\subseteq\mathcal{P}(\mathcal{C})$ is stable under sifted colimits (Proposition B.0.7), the objects of $\mathcal{C}$ are compact and projective. The last statement then follows from [Lur09, Lem 5.5.8.14]. □

PROPOSITION B.0.10. ([LUR09, PROP 5.5.8.15]) *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts and let $\mathcal{D}$ be an ∞-category which admits filtered colimits and geometric realizations. Let $\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}),\mathcal{D})$ denote the full subcategory spanned by those functors $\mathcal{P}_\Sigma(\mathcal{C})\to\mathcal{D}$ which preserve filtered colimits and geometric realizations. Then*

1. *Composition with the Yoneda embedding $j:\mathcal{C}\to\mathcal{P}_\Sigma(\mathcal{C})$ induces an equivalence $\theta:\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}),\mathcal{D})\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$ of categories. The inverse θ^{-1} is given by the left Kan extension along j. In this case, we will call $\theta^{-1}(f)$ the* left derived functor *of $f\in\mathrm{Fun}(\mathcal{C},\mathcal{D})$.*

2. *Any functor $g\in\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}),\mathcal{D})$ preserves sifted colimits.*

3. *Assume that $\mathcal{D}$ admits finite coproducts. A functor $g\in\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}),\mathcal{D})$ preserves small colimits if and only if $g\circ j$ preserves finite coproducts.*

PROPOSITION B.0.11. ([LUR09, PROP 5.5.8.22]) *Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts, $\mathcal{D}$ an ∞-category which admits filtered colimits and geometric realizations, and $F:\mathcal{P}_\Sigma(\mathcal{C})\to\mathcal{D}$ a left derived functor of $f=F\circ j:\mathcal{C}\to\mathcal{D}$, where $j:\mathcal{C}\to\mathcal{P}_\Sigma(\mathcal{D})$ denotes the Yoneda embedding. Consider the following conditions:*

1. *The functor f is fully faithful.*

2. *The essential image of f consists of compact projective objects of $\mathcal{D}$.*

3. *The ∞-category $\mathcal{D}$ is generated by the essential image of f under filtered colimits and geometric realizations.*

If 1 and 2 are satisfied, then F is fully faithful. Moreover, F is an equivalence if and only if 1, 2 and 3 are satisfied.

PROPOSITION B.0.12. ([LUR09, PROP 5.5.8.25]) *Let $\mathcal{C}$ be a projectively generated ∞-category with a set S of compact projective generators for $\mathcal{C}$. Then*

1. *Let $\mathcal{C}^0\subseteq\mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects in S. Then $\mathcal{C}^0$ is essentially small, and the left derived functor $F:\mathcal{P}_\Sigma(\mathcal{C}^0)\to\mathcal{C}$ is an equivalence of ∞-categories. In particular, $\mathcal{C}$ is a compactly generated presentable ∞-category.*

2. *Let $C\in\mathcal{C}$ be an object. The following conditions are equivalent:*

   a. *The object $C\in\mathcal{C}$ is compact and projective.*

b. *The functor $\mathcal{C} \to \mathcal{S}$ corepresented by C preserves sifted colimits.*

c. *There exists an object $C' \in \mathcal{C}^0$ such that C is a retract of C'.*

Proof. We explain more details of the first point than [Lur09, Prop 5.5.8.25]. It follows from definitions that $\mathcal{C}^0$ is essentially small. Then it follows from Proposition B.0.11 that the left derived functor $F: \mathcal{P}_\Sigma(\mathcal{C}^0) \to \mathcal{C}$ is fully faithful. Since $\mathcal{C}^0 \subseteq \mathcal{C}$ is stable under finite coproducts taken in $\mathcal{C}$, the embedding $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ preserves finite coproducts. It follows from Proposition B.0.10 that F preserves small colimits, thus the essential image of F is stable under small colimits. By assumption, S generates $\mathcal{C}$ under small colimits, therefore F is essentially surjective. □

COROLLARY B.0.13. *Let $\mathcal{C}$ be a projectively generated ∞-category and let $\mathcal{D}$ be an ∞-category which admits filtered colimits and geometric realizations. If a functor $\mathcal{C} \to \mathcal{D}$ preserves filtered colimits and geometric realizations, then it also preserve sifted colimits.*

Warning B.0.14. Filtered colimits and geometric realizations are sifted colimits, therefore if a functor preserves sifted colimits, then it also preserves filtered colimits and geometric realizations. However, the converse is in general false.

The following proposition is extremely useful to detect projectively generated ∞-categories:

PROPOSITION B.0.15. ([LUR17, COR 4.7.3.18]) *Given a pair $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ of adjoint functors between ∞-categories. Assume that*

1. *The ∞-category $\mathcal{D}$ admits filtered colimits and geometric realizations, and the functor G preserves filtered colimits and geometric realizations.*
2. *The ∞-category $\mathcal{C}$ is projectively generated.*
3. *The functor G is conservative.*

Then

1. *The ∞-category $\mathcal{D}$ is projectively generated.*
2. *An object $D \in \mathcal{D}$ is compact and projective if and only if there exists a compact projective object $C \in \mathcal{C}$ such that D is a retract of $F(C)$.*
3. *The functor G preserves all sifted colimits.*

B.0.2. Projectively generated n-categories

In this subsection, we will briefly describe the n-categorical analogue of Subsection B.0.1. We say that an anima X is n-truncated for $n \in \mathbb{N}_{\geq 0}$ if the homotopy groups $\pi_i(X, x) = 0$ for every point $x \in X$ and every $i \in \mathbb{N}_{>n}$, and (-1)-truncated if X is either empty or contractible, and (-2)-truncated if $X = \varnothing$. An ∞-category $\mathcal{C}$ is an n-category [Lur09, Prop 2.3.4.18] if for every pair $(X, Y) \in \mathcal{C} \times \mathcal{C}$ of objects, the mapping anima $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is $(n-1)$-truncated. We will denote by $\mathcal{S}_{\leq n}$ the ∞-category of n-truncated animae, and by $\hat{\mathcal{S}}_{\leq n}$ the ∞-category of large n-truncated animae.

Remark B.0.16. 1-categories are just categories in the classical category theory. If we define ∞-categories as quasicategories as in [Lur09], this identification is given by the nerve construction. Since in our texts, categories often mean ∞-categories, we usually add "1-" to avoid possible ambiguities.

In fact, for the text, we only need results for $n = 1$ (and $n = \infty$ in some sense), but the generalization to general $n \in \mathbb{N}_{>0}$ is quite cost-free.

PROPOSITION B.0.17. ([LUR09, COR 2.3.4.8]) *Let $\mathcal{C}$ be an n-category and K a simplicial set. Then $\mathrm{Fun}(K, \mathcal{C})$ is an n-category.*

DEFINITION B.0.18. *Let $\mathcal{C}$ be a cocomplete n-category and $X \in \mathcal{C}$ an object. We say that X is* compact and n-projective*, or that X is a* compact n-projective object*, if the functor $\mathcal{C} \to \hat{\mathcal{S}}_{\leq n-1}$, $Y \mapsto \mathrm{Map}_{\mathcal{C}}(X, Y)$ corepresented by X commutes with filtered colimits and geometric realizations.*

Remark B.0.19. In fact, an object $X \in \mathcal{C}$ is called n-projective if and only if the functor $\mathcal{C} \to \hat{\mathcal{S}}_{\leq n-1}, Y \mapsto \mathrm{Map}_{\mathcal{C}}(X, Y)$ corepresented by X commutes with geometric realizations. In particular, when $\mathcal{C}$ is an abelian 1-category, an object $X \in \mathcal{C}$ is 1-projective if and only if it is a "projective object" of the abelian 1-category $\mathcal{C}$.

Remark B.0.20. Let $\mathcal{C}$ be a cocomplete n-category and $X \in \mathcal{C}$ a compact n-projective object. In general, X is not a compact projective object of $\mathcal{C}$ as an ∞-category. In fact, the inclusion $\hat{\mathcal{S}}_{\leq n-1} \to \hat{\mathcal{S}}$ does *not* commute with geometric realizations. That is to say, for general simplicial objects $Y_\bullet: \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, the geometric realization $|\mathrm{Map}_{\mathcal{C}}(X, Y_\bullet)|_{\bullet \in \mathbf{\Delta}^{\mathrm{op}}}$ is not in general $(n-1)$-truncated.

Remark B.0.21. There is another way to characterize geometric realizations in an n-category $\mathcal{C}$. In fact, the fully faithful embedding $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ is "n-cofinal", therefore the geometric realization of a simplicial object $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ exists if and only if colimit of the composite functor $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ exists, and the two colimits are equivalent. Furthermore, for any diagram $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \to \mathcal{C}$, the left Kan extension along $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]} \hookrightarrow \mathbf{\Delta}^{\mathrm{op}}$ always exists. Thus for a cocomplete n-category $\mathcal{C}$, an object $X \in \mathcal{C}$ is n-projective if and only if the functor $\mathrm{Map}_{\mathcal{C}}(X, \cdot)$ corepresented by X preserves $\mathbf{\Delta}^{\mathrm{op}}_{\leq [n]}$-indexed colimits. See [Nar16] and the proof of [Lur17, Lem 1.3.3.10].

DEFINITION B.0.22. *Let $\mathcal{C}$ be a cocomplete n-category and $S \subseteq \mathcal{C}$ a (small) collection of objects of $\mathcal{C}$. We say that S is a* set of compact n-projective generators *for $\mathcal{C}$ if the following conditions are satisfied:*

1. *Each element of S is a compact n-projective object of $\mathcal{C}$.*
2. *The full subcategory of $\mathcal{C}$ spanned by finite coproducts of elements of S is essentially small.*
3. *The set S generates $\mathcal{C}$ under small colimits.*

We say that an n-category $\mathcal{C}$ is n-projectively generated *if it is cocomplete and there exists a set S of compact n-projective generators for $\mathcal{C}$.*

Similar to ∞-categories, we have

NOTATION B.0.23. *Let $\mathcal{C}$ be a small n-category which admits finite coproducts. We let $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ denote the full subcategory of $\mathcal{P}_n(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{\leq n-1})$ spanned by those functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}_{\leq n-1}$ which preserves finite products.*

PROPOSITION B.0.24. *Let $\mathcal{C}$ be a small n-category which admits finite coproducts. Then*

1. *The ∞-category $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ is an accessible localization of $\mathcal{P}_n(\mathcal{C})$, therefore presentable.*
2. *The Yoneda embedding $j: \mathcal{C} \to \mathcal{P}_n(\mathcal{C})$ factors through $\mathcal{P}_{\Sigma,n}(\mathcal{C})$. Moreover, the induced functor $\mathcal{C} \to \mathcal{P}_{\Sigma,n}(\mathcal{C})$ preserves finite coproducts.*
3. *Let $\mathcal{D}$ be a presentable n-category and let $\mathcal{P}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors. Then G factors through $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ if and only if $f = F \circ j: \mathcal{C} \to \mathcal{D}$ preserves finite coproducts.*
4. *The full subcategory $\mathcal{P}_{\Sigma,n}(\mathcal{C}) \subseteq \mathcal{P}_n(\mathcal{C})$ is stable under sifted colimits*[B.0.1].

LEMMA B.0.25. *Let $\mathcal{C}$ be a small n-category which admits finite coproducts. Then the fully faithful embedding $\mathcal{C} \hookrightarrow \mathcal{P}_{\Sigma,n}(\mathcal{C})$ extends uniquely to a functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{P}_{\Sigma,n}(\mathcal{C})$ which preserves filtered colimit. This functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{P}_{\Sigma,n}(\mathcal{C})$ is fully faithful.*

LEMMA B.0.26. *Let $\mathcal{C}$ be a small n-category which admits finite coproducts. Then the n-category $\mathcal{P}_{\Sigma,n}(\mathcal{C})$ is n-projectively generated for which $\mathcal{C} \subseteq \mathcal{P}_{\Sigma,n}(\mathcal{C})$ is a set of n-projective generators. In fact, for any $X \in \mathcal{P}_{\Sigma,n}(\mathcal{C})$, there exists a simplicial object $U_\bullet: \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Ind}(\mathcal{C})$ (or equivalently, a diagram $\mathbf{\Delta}^{\mathrm{op}}_{\leq n} \to \mathrm{Ind}(\mathcal{C})$ by Remark B.0.21) whose colimit is X.*

B.0.1. We do not introduce n-sifted diagrams, so *a priori* it is a sifted diagram defined in [Lur09, Def 5.5.8.1]. However, here one can replace sifted diagrams by n-sifted diagram. See Remark B.0.21.

Proposition B.0.27. *Let $\mathcal{C}$ be a small n-category which admits finite coproducts and let $\mathcal{D}$ be an n-category which admits filtered colimits and geometric realizations. Let $\mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}),\mathcal{D})$ denote the full subcategory spanned by those functors $\mathcal{P}_{\Sigma,n}(\mathcal{C})\to\mathcal{D}$ which preserve filtered colimits and geometric realizations. Then*

1. *Composition with the Yoneda embedding $j:\mathcal{C}\to\mathcal{P}_{\Sigma,n}(\mathcal{C})$ induces an equivalence $\theta:\mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}),\mathcal{D})\to\mathrm{Fun}(\mathcal{C},\mathcal{D})$ of categories. The inverse θ^{-1} is given by the left Kan extension along j. In this case, we will call $\theta^{-1}(f)$ the* left derived functor *of $f\in\mathrm{Fun}(\mathcal{C},\mathcal{D})$.*
2. *Any functor $g\in\mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}),\mathcal{D})$ preserves sifted colimits.*
3. *Assume that $\mathcal{D}$ admits finite coproducts. A functor $g\in\mathrm{Fun}_\Sigma(\mathcal{P}_{\Sigma,n}(\mathcal{C}),\mathcal{D})$ preserves small colimits if and only if $g\circ j$ preserves finite coproducts.*

Proposition B.0.28. *Let $\mathcal{C}$ be a small n-category which admits finite coproducts, $\mathcal{D}$ an n-category which admits filtered colimits and geometric realizations, and $F:\mathcal{P}_{\Sigma,n}(\mathcal{C})\to\mathcal{D}$ a left derived functor of $f=F\circ j:\mathcal{C}\to\mathcal{D}$, where $j:\mathcal{C}\to\mathcal{P}_{\Sigma,n}(\mathcal{D})$ denotes the Yoneda embedding. Consider the following conditions:*

1. *The functor f is fully faithful.*
2. *The essential image of f consists of compact n-projective objects of $\mathcal{D}$.*
3. *The n-category $\mathcal{D}$ is generated by the essential image of f under filtered colimits and geometric realizations.*

If 1 and 2 are satisfied, then F is fully faithful. Moreover, F is an equivalence if and only if 1, 2 and 3 are satisfied.

Proposition B.0.29. *Let $\mathcal{C}$ be a n-projectively generated n-category with a set S of compact n-projective generators for $\mathcal{C}$. Then*

1. *Let $\mathcal{C}^0\subseteq\mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects in S. Then $\mathcal{C}^0$ is essentially small, and the left derived functor $F:\mathcal{P}_{\Sigma,n}(\mathcal{C}^0)\to\mathcal{C}$ is an equivalence of n-categories. In particular, $\mathcal{C}$ is a compactly generated presentable n-category.*
2. *Let $C\in\mathcal{C}$ be an object. The following conditions are equivalent:*
   a. *The object $C\in\mathcal{C}$ is compact and n-projective.*
   b. *The functor $\mathcal{C}\to\mathcal{S}_{\leq n-1}$ corepresented by C preserves sifted colimits.*
   c. *There exists an object $C'\in\mathcal{C}^0$ such that C is a retract of C'.*

Corollary B.0.30. *Let $\mathcal{C}$ be a projectively generated n-category and let $\mathcal{D}$ be an n-category which admits filtered colimits and geometric realizations. If a functor $\mathcal{C}\to\mathcal{D}$ preserves filtered colimits and geometric realizations, then it also preserve sifted colimits.*

As in the case of ∞-categories, the following proposition is essential:

Proposition B.0.31. ([Lur17, Cor 4.7.3.18]) *Given a pair $\mathcal{C}\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{D}$ of adjoint functors between n-categories. Assume that*

1. *The n-category $\mathcal{D}$ admits filtered colimits and geometric realizations, and the functor G preserves filtered colimits and geometric realizations.*
2. *The n-category $\mathcal{C}$ is n-projectively generated.*
3. *The functor G is conservative.*

Then

1. *The n-category $\mathcal{D}$ is n-projectively generated.*
2. *An object $D\in\mathcal{D}$ is compact and n-projective if and only if there exists a compact n-projective object $C\in\mathcal{C}$ such that D is a retract of $F(C)$.*
3. *The functor G preserves all sifted colimits.*

B.0.3. Animation of n-projectively generated n-categories In this subsection, we describe a procedure, called *animation*, introduced in [CS19, §5.1], to produce a projectively generated ∞-category from an n-projectively generated n-category. Roughly speaking, this projectively generated ∞-category is determined by a set of compact n-projective generators for the n-category in question.

DEFINITION B.0.32. *Let $\mathcal{C}$ be an n-projectively generated n-category. We choose a set $S \subseteq \mathcal{C}$ of compact n-projective generators for $\mathcal{C}$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects in S. Then the* animation *of $\mathcal{C}$, denoted by* $\mathrm{Ani}(\mathcal{C})$, *is defined to be the projectively generated ∞-category $\mathcal{P}_\Sigma(\mathcal{C}^0)$.*

Remark B.0.33. The definition of the animation does not depend on the choice of the set of compact n-projective generators. The key is that if S' is another compact n-projective generators, then it follows from Proposition B.0.29 that every object $X' \in S'$ is a retract of an object $X \in \mathcal{C}^0$ in Definition B.0.32. The same applies to the discussions below.

Example B.0.34. Let Ab be the abelian category of abelian groups. Then $\mathrm{Ani}(\mathrm{Ab})$ coincides with the (connective) derived category $D_{\geq 0}(\mathrm{Ab})$.

Remark B.0.35. In the context of Definition B.0.32, we have $\mathcal{C} \simeq \mathcal{P}_{\Sigma,n}(\mathcal{C}^0)$ by Proposition B.0.29 and $\mathrm{Ani}(\mathcal{C}) \simeq \mathcal{P}_\Sigma(\mathcal{C}^0)$. It follows that the n-category $\mathcal{C}$ could be identified with n-truncated objects in $\mathrm{Ani}(\mathcal{C})$. In particular, there exists a left adjoint $\tau_{\leq n-1} : \mathrm{Ani}(\mathcal{C}) \to \mathcal{C}$ to the fully faithful embedding $\mathcal{C} \hookrightarrow \mathrm{Ani}(\mathcal{C})$, cf. [Lur09, Rem 5.5.8.26].

We now discuss the animation of functors.

DEFINITION B.0.36. ([CS19, §5.1.4]) *Let $\mathcal{C}, \mathcal{D}$ be two n-projectively generated n-categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then the* animation *of the functor F, denoted by* $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$, *is defined as follows:*

We choose a set $S \subseteq \mathcal{C}$ of compact n-projective generators for $\mathcal{C}$. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be the full subcategory spanned by finite coproducts of the objects in S. Then the functor $F : \mathcal{C} \to \mathcal{D}$ gives rise to the composite $\mathcal{C}^0 \to \mathcal{C} \to \mathcal{D} \to \mathrm{Ani}(\mathcal{D})$. We define $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$ *to be the left derived functor (in Proposition B.0.10) of $\mathcal{C}^0 \to \mathrm{Ani}(\mathcal{D})$.*

Example B.0.37. Let $F : \mathrm{Ab} \to \mathrm{Ab}$ be an additive functor. Then the animation $\mathrm{Ani}(F) : \mathrm{Ani}(\mathrm{Ab}) \to \mathrm{Ani}(\mathrm{Ab})$ coincides with the left derived functor $\mathbb{L}F : D_{\geq 0}(\mathrm{Ab}) \to D_{\geq 0}(\mathrm{Ab})$ in homological algebra.

It follows from Propositions B.0.27, B.0.29, B.0.10, and B.0.12 that

COROLLARY B.0.38. *In Definition B.0.36, if F preserves sifted colimits (cf. Corollary B.0.30), then so does* $\mathrm{Ani}(F)$. *Furthermore, if F preserves small colimits, then so does* $\mathrm{Ani}(F)$.

In homological algebra, there is a natural comparison map $H_0 \circ \mathbb{L}F \to F \circ H_0$, which becomes an equivalence when F is assumed to be right exact. Now we study the animated analogue. In the context of Definition B.0.36, the composite functor $\mathrm{Ani}(\mathcal{C}) \xrightarrow{\tau_{\leq n-1}} \mathcal{C} \xrightarrow{F} \mathcal{D} \xhookrightarrow{j_{\mathcal{D}}} \mathrm{Ani}(\mathcal{D})$ is an extension of the composite functor $\mathcal{C} \xrightarrow{F} \mathcal{D} \hookrightarrow \mathrm{Ani}(\mathcal{D})$. Since $\mathrm{Ani}(F) : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{D})$ is the left Kan extension, there exists an essentially unique map $\mathrm{Ani}(F) \to j_{\mathcal{D}} \circ F \circ \tau_{\leq n-1}$ of functors $\mathrm{Ani}(\mathcal{C}) \rightrightarrows \mathrm{Ani}(\mathcal{D})$. By adjunction, we get a canonical map $\tau_{\leq n-1} \circ \mathrm{Ani}(F) \to F \circ \tau_{\leq n-1}$ of functors $\mathrm{Ani}(\mathcal{C}) \rightrightarrows \mathcal{D}$.

LEMMA B.0.39. ([CS19, §5.1.4]) *In Definition B.0.36, suppose that the functor $F : \mathcal{C} \to \mathcal{D}$ (between n-categories) preserves sifted colimits. Then the map $\tau_{\leq n-1} \circ \mathrm{Ani}(F) \to F \circ \tau_{\leq n-1}$ of functors constructed above is an equivalence of functors.*

Proof. First, note that the map $\tau_{\leq n-1} \circ \mathrm{Ani}(F) \to F \circ \tau_{\leq n-1}$ of functors $\mathrm{Ani}(\mathcal{C}) \rightrightarrows \mathcal{D}$ is an equivalence of functors after composing with the inclusion $\mathcal{C}^0 \hookrightarrow \mathrm{Ani}(\mathcal{C})$. We claim that both functors $\tau_{\leq n-1} \circ \mathrm{Ani}(F)$ and $F \circ \tau_{\leq n-1}$ preserve sifted colimits, thus belonging to $\mathrm{Fun}_\Sigma(\mathrm{Ani}(\mathcal{C}), \mathcal{D})$ which becomes an equivalence after mapped along $\mathrm{Fun}_\Sigma(\mathrm{Ani}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and hence by Proposition B.0.10, the constructed map of functors is an equivalence.

In fact, since $\tau_{\leq n-1}$ is a left adjoint, therefore commutes with small colimits, which implies that $\tau_{\leq n-1} \circ \mathrm{Ani}(F)$ commutes with sifted colimits. On the other hand, $F : \mathcal{C} \to \mathcal{D}$ is a functor which preserves sifted colimits, therefore also preserves sifted colimits since $\mathcal{C}, \mathcal{D}$ are n-categories. Thus $F \circ \tau_{\leq n-1}$ also preserves sifted colimits. □

In homological algebra, leftly deriving functors is not compatible with compositions, therefore neither is animation of functors in general. However, recall that with some acyclicity conditions [Sta21, Tag 015M], there is a compatibility of leftly deriving functors and compositions. Here is such a condition in the world of animations:

Proposition B.0.40. ([CS19, Prop 5.1.5]) *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three n-projectively generated n-categories and $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{E}$ two functors preserving sifted colimits (cf. Corollary B.0.30). Then*

1. *There is a natural transformation from the composite $\mathrm{Ani}(G) \circ \mathrm{Ani}(F)$ to $\mathrm{Ani}(G \circ F)$ (In fact, for this, we only need that G preserves sifted colimits).*
2. *Let $\mathcal{C}^0 \subseteq \mathcal{C}$ and $\mathcal{D}^0 \subseteq \mathcal{D}$ be full subcategories determined by a choice of set of compact n-projective generators as in Definition B.0.32. If either $F(\mathcal{C}^0) \subseteq \mathrm{Ind}(\mathcal{D}^0)$ in $\mathcal{D}$ or $(\mathrm{Ani}(G))(F(\mathcal{C}^0)) \subseteq \mathcal{E}$ in $\mathrm{Ani}(\mathcal{E})$, then the natural transformation $\mathrm{Ani}(G) \circ \mathrm{Ani}(F) \to \mathrm{Ani}(G \circ F)$ is an equivalence.*

Appendix C

Stable symmetric monoidal ∞-categories

In this section, we recall some basic facts about stable symmetric monoidal ∞-categories that we need in our text. Roughly speaking, a symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ is an ∞-category with an operation $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is commutative and associative "up to coherent homotopies". The precise definition is the following:

Definition C.0.1. ([Lur17, Def 2.0.0.7 & Rem 2.1.2.20]) *A symmetric monoidal ∞-category is a coCartesian fibration of simplicial sets $p: \mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ with the following property:*

For each $n \in \mathbb{N}_{\geq 0}$, the maps $(\rho^i : \langle n \rangle \to \langle 1 \rangle)_{1 \leq i \leq n}$ induce functors $\rho^i_! : \mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ which determines an equivalence $\mathcal{C}^{\otimes}_{\langle n \rangle} \xrightarrow{\simeq} (\mathcal{C}^{\otimes}_{\langle 1 \rangle})^n$, where ρ^i is the map given by

$$\rho^i(j) := \begin{cases} 1, & i = j \\ *, & \textit{otherwise} \end{cases}$$

In particular, when $n = 0$, we see that $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$ is a singleton.

In this case, we will refer to the fiber $\mathcal{C} := \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ as the underlying ∞-category of $\mathcal{C}^{\otimes}$. The active morphism $\beta : \langle 2 \rangle \to \langle 1 \rangle, 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1$ induces the tensor bifunctor $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We will abuse terminology by referring to $\mathcal{C}$ or $(\mathcal{C}, \otimes)$ as a symmetric monoidal ∞-category. The unique map $\langle 0 \rangle \to \langle 1 \rangle$ induces a functor $\mathcal{C}^{\otimes}_{\langle 0 \rangle} \to \mathcal{C}^{\otimes}_{\langle 1 \rangle} = \mathcal{C}$ which determines a distinguished object of $\mathcal{C}$, which we will denote by $\mathbf{1}_{\mathcal{C}}$.

Most symmetric monoidal ∞-categories that we consider are presentable and the tensor product behaves well with colimits:

Definition C.0.2. ([Nik16, §2]) *A symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ is called* presentable symmetric monoidal *if the underlying ∞-category $\mathcal{C}$ is presentable and the tensor bifunctor $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable.*

In most cases in which we are interested, the underlying ∞-category $\mathcal{C}$ is stable, which satisfies some compatibility conditions formulated as follows:

Definition C.0.3. ([Nik16, Def 4.1]) *A symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ is called* stable symmetric monoidal *if the underlying ∞-category $\mathcal{C}$ is stable and the tensor bifunctor $\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite colimits separately in each variable.*

Corollary C.0.4. *A presentable symmetric monoidal ∞-category $(\mathcal{C}, \otimes)$ is stable symmetric monoidal if and only if the underlying ∞-category $\mathcal{C}$ is stable.*

We usually need to impose a t-structure which is compatible with the symmetric monoidal structure:

Definition C.0.5. *Let $(\mathcal{C}, \otimes)$ be a stable symmetric monoidal ∞-category. A* symmetric monoidal t-structure *on the underlying stable ∞-category $\mathcal{C}$ is a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\geq 0} \in \mathcal{C}$ is closed under tensor product: $X \otimes Y \in \mathcal{C}_{\geq 0}$ for all $X, Y \in \mathcal{C}_{\geq 0}$. In this case, $(\mathcal{C}_{\geq 0}, \otimes)$ is a symmetric monoidal ∞-category and the inclusion $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ is symmetric monoidal.*

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