Comparison theorem between the equivariant de Rham cohomology of a G-manifold M and the ordinary cohomology of the homotopic quotient M_G

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1. Description of the problem. Let G be a compact (not necessarily connected) Lie group with Lie algebra $\mathfrak{g}$. We compare, for a given G-manifold M, the ordinary cohomology, with coefficients in the field of real numbers $\mathbb{R}$, of the homotopic quotient

$$M_G := \mathbb{E}G \times_G M\,, \tag{1}$$

with the cohomology of the complex of G-equivariant differential forms

$$\Omega_G(M) := ((S(\mathfrak{g}) \otimes \Omega(M))^G, d_{\mathfrak{g}})\,. \tag{2}$$

We would like to emphasize that as (1) and (2) are functorial on the category G-Man of G-manifolds and equivariant differentiable maps, what we are aiming for is to compare the two contravariant functors

$$G\text{-Man} \ni M \begin{array}{l} \rightsquigarrow H(M_G;\mathbb{R}) \\ \rightsquigarrow H_G(M) := H\big(\Omega_G(M)\big) \end{array} \tag{3}$$

We will do this by constructing a specific isomorphism of graded $\mathbb{R}$-algebras

$$\Phi_M : H(M_G;\mathbb{R}) \simeq H_G(M), \tag{4}$$

which will be functorial for $M \in G$-Man. In the particular case where $M := \{\bullet\}$, we get an isomorphism of graded $\mathbb{R}$-algebras

$$\Phi_{\{\bullet\}} : H(\mathbb{B}G;\mathbb{R}) \simeq S(\mathfrak{g})^G\,. \tag{5}$$

The functors in (3) will therefore have values in the category of $S(\mathfrak{g})^G$-graded algebras. In other words, we will get commutative diagrams of homomorphisms of graded $\mathbb{R}$-algebras

$$G\text{-Man} \ni M \rightsquigarrow \begin{array}{ccc} H(\mathbb{B}G;\mathbb{R}) & \longrightarrow & H(M_G;\mathbb{R}) \\ {\scriptstyle \Phi_{\{\bullet\}}}\big\downarrow\simeq & \bigoplus & {\scriptstyle \Phi_M}\big\downarrow\simeq \\ S(\mathfrak{g})^G & \longrightarrow & H_G(M) \end{array} \tag{6}$$

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2. Constructing Φ_M for free actions. Let G-Man$_f$ denote the full subcategory of G-Man whose objects are the G-manifolds M on which G acts freely. The quotient space M/G then has a unique structure of manifold such that the orbit map $\nu : M \to M/G$ is a locally trivial fibration of manifolds of fiber G.

In that case, the projection onto the second coordinate

$$\xi_M : M_G := \mathbb{E}G \times_G M \twoheadrightarrow M/G, \quad [b,m] \mapsto [m],$$

is a locally trivial fibration with fibers homeomorphic to $\mathbb{E}G$. The map ξ_M is thus a homotopy equivalence, and a natural transformation of functors

$$G\text{-Man}_f \ni (M \xrightarrow{\phi} N) \rightsquigarrow \begin{array}{ccc} M_G & \xrightarrow{\phi_G} & N_G \\ {\scriptstyle \xi_M} \downarrow \wr & & {\scriptstyle \xi_N} \downarrow \wr \\ M/G & \xrightarrow{\overline{\phi}} & N/G \end{array} \tag{7}$$

This is saying that there exists a de Rham model for the algebra $H(M_G;\mathbb{R})$, in this case the de Rham complex $(\Omega(M/G),d)$.

In this situation, the diagram (6) splits naturally in two isomorphisms of $\mathbb{R}$-algebras

$$G\text{-Man} \ni M \rightsquigarrow \left. \begin{array}{c} H(M_G;\mathbb{R}) \\ {\scriptstyle (\xi_M^*)^{-1}} \downarrow \simeq \\ H_{\mathrm{dR}}(M/G) \\ {\scriptstyle \Phi'_M} \downarrow \simeq \\ H_G(M) \end{array} \right\rbrace {\scriptstyle \Phi_M} \tag{8}$$

where Φ'_M is the isomorphism defined by Cartan in his Brussels lectures, and whose construction is functorial on the category G-Man$_f$.

3. Constructing Φ_M for general actions

3.1. The spaces M_G are generally not of finite cohomological dimension, as the example of the classifying space $\mathbb{B}G$ already shows. On the other hand, if the action of G on M is not free, the topological quotient space M/G is no longer necessarily a manifold. So it appears that Cartan's method, which depends on the existence of some kind of differential model for the ordinary cohomology of M_G, becomes useless for comparing $H(M_G;\mathbb{R})$ and $H_G(M)$.

3.2. The universal G-bundle $\mathbb{E}G$, is the topological inductive limit of an increasing sequence (a tower) of G-manifolds

$$\mathbb{E}G(0) \subset \mathbb{E}G(1) \subset \mathbb{E}G(2) \subset \cdots \subset \mathbb{E}G \tag{9}$$

where $\mathbb{E}G(n)$ is compact and n-connected, i.e. such that

$$\Pi_q(\mathbb{E}G(n)) = 0, \quad \forall q \leq n. \tag{10}$$

The family of restriction homomorphisms of algebras

$$\begin{array}{l} H(\mathbb{E}G;\mathbb{R}) \\ \cdots \to H(\mathbb{E}G(n{+}1);\mathbb{R}) \to H(\mathbb{E}G(n);\mathbb{R}) \to H(\mathbb{E}G(n{-}1);\mathbb{R}) \to \cdots \end{array}$$

is then a projective family and induces, as such, an algebra homomorphism:

$$H(\mathbb{E}G;\mathbb{R}) \to \varprojlim H(\mathbb{E}G(n);\mathbb{R}), \tag{11}$$

which will be shown to be an isomorphism (*cf.* 5.1-(a)).

3.3. If X is a G-space, the increasing sequence (9) can be used to realize X_G as the topological inductive limit of the tower of spaces

$$X_G(*) := \big(X_G(0) \subset X_G(1) \subset X_G(2) \subset \cdots \subset X_G(\infty) := X_G\big) \tag{12}$$

where

$$X_G(n) := \mathbb{E}G(n) \times_G X\,.$$

– The correspondence $X \rightsquigarrow X_G(*)$ is functorial relative to $X \in G$-Man.

– Furthermore, if X is a manifold, then the $X_G(n)$'s will also be so, since G is compact and acts freely on $\mathbb{E}G(n) \times X$, which is also a manifold.

The tower (12) induces, as in section 3.2, the projective system of restriction homomorphisms of algebras

$$\begin{array}{l} H(X_G;\mathbb{R}) \\ \cdots \to H(X_G(n{+}1);\mathbb{R}) \to H(X_G(n);\mathbb{R}) \to H(X_G(n{-}1);\mathbb{R}) \to \cdots \end{array}$$

and, therefore, the algebra homomorphism:

$$\boxed{H(X_G;\mathbb{R}) \to \varprojlim H(X_G(n);\mathbb{R})} \tag{13}$$

which will also be shown to be an isomorphism (*cf.* 5.1-(b)).

3.4. Let M be a G-manifold. For each $n \in \mathbb{N}$, the projection of G-manifolds

$$p_n : \mathbb{E}G(n) \times M \to M\,, \quad (x,m) \mapsto m\,, \tag{14}$$

is equivariant and induces an homomorphism of equivariant cohomologies

$$\boxed{p_n^* : H_G(M) \to H_G(\mathbb{E}G(n) \times M) = H_{\mathrm{dR}}(M_G)} \tag{15}$$

where the equality at the right-hand is, once again, justified by Cartan's lectures.

Putting together the family of the projections (14), we get the inductive system

$$\begin{array}{l} \mathbb{E}G(n{-}1) \times M \;\subset\; \mathbb{E}G(n) \times M \;\subset\; \mathbb{E}G(n{-}1) \times M \;\subset\; \cdots \\ \quad \xrightarrow{p_{n+1}} M, \quad \xrightarrow{p_n} M, \quad \xrightarrow{p_{n+1}} M \end{array}$$

which induces, by (15), the projective system of algebra homomorphisms

$$
H_G(M) \xrightarrow{p_{n+1}^*} H_{\mathrm{dR}}(M_G(n{+}1)),\quad H_G(M) \xrightarrow{p_{n}^*} H_{\mathrm{dR}}(M_G(n)),\quad H_G(M) \xrightarrow{p_{n+1}^*} H_{\mathrm{dR}}(M_G(n{-}1))
$$

$$
\cdots \to H_{\mathrm{dR}}(M_G(n{+}1)) \to H_{\mathrm{dR}}(M_G(n)) \to H_{\mathrm{dR}}(M_G(n{-}1)) \to \cdots
$$

and, hence, the algebra homomorphism:

$$
\boxed{H_G(M) \to \varprojlim H_{\mathrm{dR}}(M_G(n))} \tag{16}
$$

which, as before, will be shown to be an isomorphism (*cf.* 5.1-(c)).

3.5. To compare $H(M_G;\mathbb{R})$ and $H_G(M)$, we put together (13) and (16) thanks to the identification $H(M(n);\mathbb{R}) \cong H_{\mathrm{dR}}(M(n))$ given by the de Rham comparison theorem. We thus have three functorial isomorphisms for $M \in G$-Man

$$
H(M_G;\mathbb{R}) \underset{\simeq}{\longrightarrow} \varprojlim H(M_G(n);\mathbb{R}) \cong \varprojlim H_{\mathrm{dR}}(M_G(n)) \underset{\simeq}{\longleftarrow} H_G(M),
$$

the composition of which defines the announced algebra isomorphism

$$
\boxed{\Phi_M : H(M_G;\mathbb{R}) \underset{\simeq}{\longrightarrow} H_G(M)}
$$

functorial relative to $M \in G$-Man.

4. A technical result. The fact that (11), (13) and (16) are isomorphisms, will be consequence of one and the same property that we now establish.

4.1. Theorem. *Let $\varphi : C \to D$ be a morphism of regular filtered graded complexes whose corresponding spectral sequences pages $\mathbb{E}(C)_2$ and $\mathbb{E}(D)_2$ are in the first quadrant, i.e. are such that $\mathbb{E}(C)_2^{p,q} = \mathbb{E}(D)_2^{p,q} = 0$, if $p < 0$ or $q < 0$. Assume that, for some fixed $n \in \mathbb{N}$, the induced homomorphisms*

$$
\mathbb{E}(\varphi)_2^{p,q} : \mathbb{E}(C)_2^{p,q} \to \mathbb{E}(D)_2^{p,q}
$$

are isomorphisms for all $q \leq n$. Then, the induced morphisms in cohomology

$$
H^i(\varphi) : H^i(C) \to H^i(D)
$$

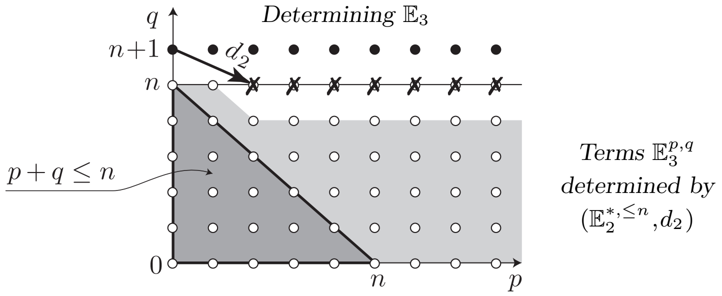
are isomorphisms for all $i \leq n$.

Proof. In the page $(\mathbb{E}_2,d_2)$, one has $d_2(\mathbb{E}_2^{p-2,q+1}) \subset \mathbb{E}_2^{p,q}$, which implies that the determination of $\mathbb{E}_3^{p,q}$ is based on the knowledge of the sub-complex $(\mathbb{E}_2^{*,\leq n},d_2)$ only, except if

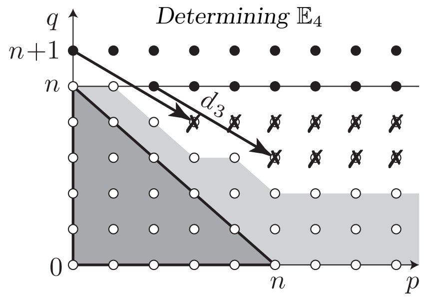
$$
p-2 \geq 0 \quad \text{and} \quad q+1 > n, \tag{$*$}
$$

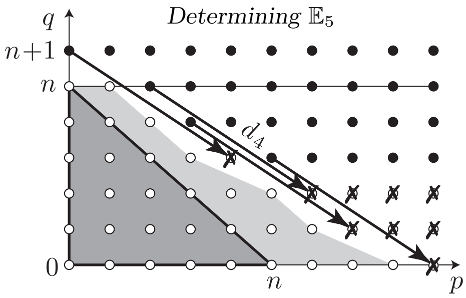
since in that case it may happen that $d_2(\mathbb{E}_2^{p-2,q+1}) \neq 0$, while $\mathbb{E}_2^{p-2,q+1} \not\subset \mathbb{E}_2^{*,\leq n}$. Notice also that if the conditions $(*)$ are satisfied, then $p+q > n+1$.

In the pictures below, the symbol '$\circ$' indicates the terms $\mathbb{E}_{r+1}^{p,q}$ determined only by the sub-complex $(\mathbb{E}_2^{p,\leq n},d_2)$, while a '$\bullet$' indicates those not entirely determined by $(\mathbb{E}_2^{p,\leq n}, d_2)$, and '✗' indicates that $\mathbb{E}_r^{p,q}$ is determined by $(\mathbb{E}_2^{p,\leq n}, d_2)$ but that $\mathbb{E}_{r+1}^{p,q}$ is not so.



The same analysis for the terms $\mathbb{E}_4^{p,q}$ and $\mathbb{E}_5^{p,q}$ gives the pictures

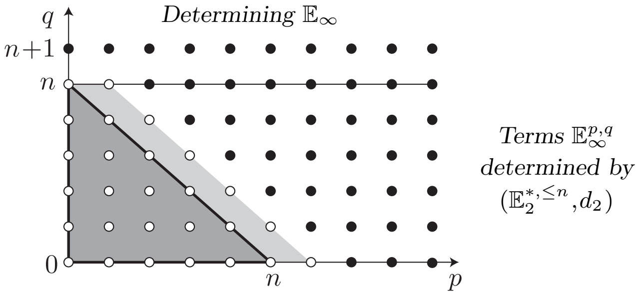




Since the image of

$$d_r : \mathbb{E}_r^{0,n+1} \to \mathbb{E}_r \qquad (\diamond)$$

is contained in the term $\mathbb{E}_r^{r,n+1-r+1}$, which is of total degree $n+2$, we understand that any term in $\mathbb{E}_\infty^{\geq 0, \geq 0}$ of total degree $n+2$ may depend on $\mathbb{E}_2^{0,n+1}$. On the other hand, none of the arrows $(\diamond)$ affect the terms $\mathbb{E}_r^{p,q}$ with $p+q \leq n$ so that, at the end, we can easily justify the following picture showing terms $\mathbb{E}_r^{p,q}$ depending only on the sub-complex $(\mathbb{E}_2^{*,\leq n}, d_2)$ for all $r \geq 2$, hence for $r = \infty$.



It then follows that if $\mathbb{E}(\varphi)_2^{*,\leq n}$ is an isomorphism, the induced homomorphisms

$$\mathbb{E}(\varphi)_r^{p,q} : \mathbb{E}(C)_\infty^{p,q} \to \mathbb{E}(D)_\infty^{p,q},$$

are isomorphisms for all $p+q \leq n$, hence the theorem. □

5. Application of Theorem 4.1. We now prove the statements in 3.5.

5.1. Theorem

a) *Given $n \in \mathbb{N}$, the restriction homomorphism*

$$H^q(\mathbb{E}G;\mathbb{R}) \to H^q(\mathbb{E}G(n);\mathbb{R})$$

is an isomorphism for $q \leq n$. In particular, the algebra homomorphism (11)

$$H(\mathbb{E}G;\mathbb{R}) \to \varprojlim H(\mathbb{E}G(n);\mathbb{R})$$

is an isomorphism.

b) *Given $n \in \mathbb{N}$, and a G-space X, the restriction homomorphism*

$$H^q(X_G;\mathbb{R}) \to H^q(X_G(n);\mathbb{R})$$

is an isomorphism for $q \leq n$. In particular, the algebra homomorphism (13)

$$H(X_G;\mathbb{R}) \to \varprojlim H(X_G(n);\mathbb{R})$$

is an isomorphism.

c) *Given $n \in \mathbb{N}$, and a G-manifold M, the restriction homomorphism* (15)

$$H^q_G(M) \to H^q_{\mathrm{dR}}(M_G(n))$$

is an isomorphism for $q \leq n$. In particular, the algebra homomorphism (16)

$$H_G(M) \to \varprojlim H_{\mathrm{dR}}(M_G(n))$$

is an isomorphism.

Proof. (a) We can assume the group G embedded in the orthogonal group $\mathrm{O}(k)$ for some $k \in \mathbb{N}$. We then set

$$\mathbb{E}G(n) := V(k,k+n+1), \quad (\forall k \geq 1)(\forall n \geq 1), \tag{$\dagger$}$$

where $V(\ell,m)$ denotes the Stiefel manifold of ℓ-tuples $(\vec{v}_1,\ldots,\vec{v}_\ell)$ of orthonormal vectors in $\mathbb{R}^m$, which is easily seen to be compact and $(m-\ell-1)$-connected ([1]).

The manifold $\mathbb{E}G(n)$ is therefore compact and n-connected, and verifies, by Hurewicz theorems,

$$H^q(\mathbb{E}G(n);\mathbb{R}) = 0, \quad \forall q \leq n,$$

One easily sees that $\mathbb{E}G(n) \sim \mathrm{O}(k+n+1)/(\mathbb{1}_k \times \mathrm{O}(n+1))$ and that $\mathrm{O}(k)$ acts freely at the right of $\mathbb{E}G(n)$. In particular, the space $\mathbb{E}G := \bigcup_n \mathbb{E}G(n)$ meets the requirements for a universal principal bundle for $\mathrm{O}(k)$, hence for G.

(b) Since the diagonal action of G on $\mathbb{E}G(n) \times X$ is free, the ideas in section 2 apply. We can therefore replace $X_G(n) \leftrightarrow (\mathbb{E}G(n) \times X)_G$ and consider the natural fibration $\pi : \mathbb{E}G \times_G (\mathbb{E}G(n) \times X) \twoheadrightarrow \mathbb{B}G$ and the corresponding Leray-Serre spectral sequence

$$\mathbb{E}(X_G(n))_2^{p,q} = H^p(\mathbb{B}G;\mathbb{R}) \otimes H^q(\mathbb{E}G(n) \times X) \Rightarrow H^{p+q}(X_G(n))$$

which is contravariant functorial for $(\mathbb{E}G(n) \times X) \in G\text{-}\mathrm{Top}_f$. In particular, we get a morphism of spectral sequences

$$\mathbb{E}(\varphi)_2^{p,q} : \mathbb{E}(X_G(n))_2^{p,q} \to \mathbb{E}(X_G)_2^{p,q}$$

[1] Indeed, for $\ell \leq m+1$, the map $p_\ell : V(\ell,m+1) \to \mathbb{S}^m$, $(\vec{v}_1,\ldots,\vec{v}_\ell) \mapsto \vec{v}_\ell$, is a locally trivial fibration of fiber $V(\ell-1,m)$, and an inductive argument on $\ell \in [\![1m]\!]$, immediately shows that $\Pi_q(V(\ell,m)) = 0$, for all q such that $q + \ell < m$.

which is an isomorphism for all $q \leq n$ since, in that case, by Künneth,

$$\begin{aligned} H^q(I\!\!EG(n){\times}X;\mathbb{R}) &= \bigoplus\nolimits_{a+b=q} H^a(I\!\!EG(n);\mathbb{R}) \otimes H^b(X;\mathbb{R}) \\ &= \bigoplus\nolimits_{a+b=q} H^a(I\!\!EG;\mathbb{R}) \otimes H^b(X;\mathbb{R}) = H^q(I\!\!EG{\times}X;\mathbb{R}). \end{aligned}$$

where $a \leq n$. We can therefore apply theorem 4.1, and (b) follows.

(c) Here, the important fact is that $I\!\!EG(n){\times}M$ is a manifold, in which case we can use Cartan's identification $H_{\mathrm{dR}}(M_G(n)) = H_G(I\!\!EG(n){\times}M)$ and the corresponding spectral sequence for equivariant cohomology (see **6**):

$$\mathbb{E}(I\!\!EG(n){\times}M)_2^{p,q} := H_G^p \otimes H^q(\Omega(I\!\!EG(n){\times}M))$$

where $H_G^{2m} := S^m(\mathfrak{g})^G$ and $H_G^{2m+1} = 0$, for all $m \in \mathbb{N}$.

These constructions are contravariant functorial over G-Man, and for the equivariant projection $p_n : I\!\!EG(n) \times M \to M$, $(x,m) \mapsto m$, give a morphism of spectral sequences for equivariant cohomology

$$\mathbb{E}(p_n^*)_2^{p,q} : \mathbb{E}(M)_2^{p,q} \to \mathbb{E}(I\!\!EG(n){\times}M)_2^{p,q},$$

which is an isomorphism for all $q \leq n$, since the pullbacks

$$p_n^* : H^q(M)) \to H^q(\Omega(I\!\!EG(n){\times}M))$$

are so. We can therefore apply theorem 4.1 and state that the restriction homomorphisms $H_G(M)^q \to H^q_{\mathrm{dR}}(M(n))$ (15) are isomorphisms for all $q \leq n$, which ends the proof of (c). □

6. On the spectral sequence for equivariant cohomology. The complex of equivariant differential forms of a G-manifold M is the complex

$$\Omega_G(M) := ((S(\mathfrak{g}) \otimes \Omega(M))^G, d_{\mathfrak{g}}), \tag{$*$}$$

where, if $\{e_j\}$ is a basis of $\mathfrak{g}$ of dual basis and $\{e^j\}$, the equivariant differential $d_{\mathfrak{g}}$ is given by the expression

$$d_{\mathfrak{g}}(P \otimes \omega) = P \otimes d\omega + \sum\nolimits_j Pe^j \otimes \iota(e_j)(\omega).$$

Denote by $S^a(\mathfrak{g})$ the polynomial functions on $\mathfrak{g}$ of degree a, and define the *principal degree* of $S^a(\mathfrak{g}) \otimes \Omega^b(M)$ by $2a$, and its *total degree* by $2a + b$.

In order to be able to apply theorem 4.1 *without reindexing terms*, we choose to filter the complex $(*)$ by principal degrees in $\mathbb{N}$, and not just in $2\mathbb{N}$.

We therefore set, for all $j \in \mathbb{N}$,

$$\Omega_G(M)_{\geq j} = S^{\geq \frac{j}{2}} \otimes \Omega(M). \tag{$\dagger$}$$

(In particular, $\Omega_G(M)_{\geq j} = \Omega_G(M)_{\geq j+1}$, if j is odd.)

Endowed with the filtration (†), the complex $(\Omega_G(M), d_{\mathfrak{g}})$ is a regular filtered graded complex. We can then apply the general theory of spectral sequences and construct a spectral sequence $(\mathbb{E}(\Omega_G(M))_r, d_r)$, where

$$\mathbb{E}(\Omega_G(M))_2^{p,q} = S^{p/2}(\mathfrak{g})^G \otimes \Omega^q(M),$$

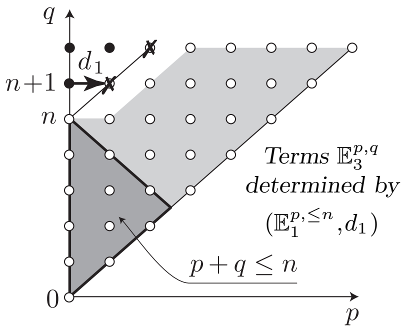
$(S^{p/2}(\mathfrak{g}) = 0$, for p odd) where $d_2 : \mathbb{E}(\Omega_G(M))_2^{p,q} \to \mathbb{E}(\Omega_G(M))_2^{p+2,q-1}$.

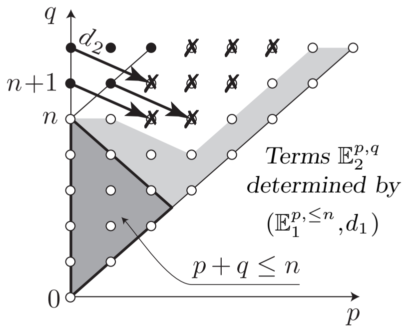
For $r \geq 2$, we then have, as usual, $d_r(\mathbb{E}(\Omega_G(M))_r^{p,q}) \subset \mathbb{E}(\Omega_G(M))_r^{p+r,q-r+1}$, and the spectral sequence converges to the graded $S(\mathfrak{g})^G$-module associated with the filtered module $H_G(M)$, by the decreasing successive images

$$H_G(M)_{\geq k} := \operatorname{im}\big(H(\Omega_G(M)_{\geq j}) \to H_G(M)\big).$$

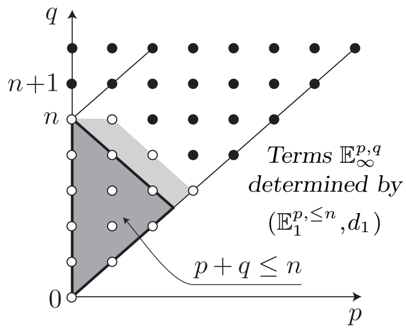
We have
$$\mathbb{E}(\Omega_G(M))_\infty^p = \frac{H_G(M)_{\geq p}}{H_G(M)_{\geq p+1}}.$$

6.1. Comment on a different spectral sequence. In appendix A.9 of ([2]), the complex $\Omega_G(M)$ is put under the form of a bi-complex, which is quite nice, but then one has to prove the analog of theorem 4.1 which gives better bounds for the proposition A.10 in *loc.cit.*. The proof will then be the same, but the pictures must be replaced by the followings





where we see (again) that the terms $\mathbb{E}_r^{p,q}$ with $p+q \leq n$ are determined solely by the sub-complex $(\mathbb{E}_1^{p,\leq n}, d_1)$, as in the proof of 4.1.



These details, which are not difficult to understand, are missing in the proposition A.10. In its proof, it is only observed that under d_r, the superscript q lowers to $q-r+1$, without considering that at the same time the superscript p raises to $p+r$ leaving enough room to preserve the $\mathbb{E}^{p,q}$'s such that $p+q \leq n$.

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[2] Forthcoming book of Loring Tu, *Introductory Lectures on Equivariant Cohomology*, in Annals of Mathematics Studies, Princeton University Press.