On Equivariant Poincaré Duality, Gysin Morphisms and Euler Classes

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January 2017

Abstract. Given a connected compact Lie group $\boldsymbol{G}$, we set up the formalism of the $\boldsymbol{G}$-equivariant Poincaré duality for oriented $\boldsymbol{G}$-manifolds, following the work of J.-L. Brylinski leading to the spectral sequence

$$\mathbf{Ext}_{H_{\boldsymbol{G}}}(H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}),H_{\boldsymbol{G}}) \Rightarrow H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}].$$

The equivariant Gysin functors

$$\begin{aligned} (\_)_! &: \boldsymbol{G}\text{-}\boldsymbol{Man}_\pi \rightsquigarrow \mathcal{D}^+(\mathrm{DGM}(H_{\boldsymbol{G}})), \quad \boldsymbol{M} \rightsquigarrow \Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}], \quad f \rightsquigarrow f_! \\ \text{resp. } (\_)_* &: \boldsymbol{G}\text{-}\boldsymbol{Man} \rightsquigarrow \mathcal{D}^+(\mathrm{DGM}(H_{\boldsymbol{G}})), \quad \boldsymbol{M} \rightsquigarrow \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}], \quad f \rightsquigarrow f_* \end{aligned}$$

are then the covariant functors from the category of oriented $\boldsymbol{G}$-manifolds and proper (resp. unrestricted) maps, to the derived category of the category of differential graded modules over $H_{\boldsymbol{G}}$, defined as the composition of the Cartan complex of equivariant differential forms functor $\Omega_{\boldsymbol{G},\mathrm{c}}(\_)$ (resp. $\Omega_{\boldsymbol{G}}(\_)$) with the duality functor $\mathbb{R}\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}((\_),H_{\boldsymbol{G}})$ and the equivariant Poincaré adjunction $\mathbb{D}_{\boldsymbol{G}}(\boldsymbol{M}) : \Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \mathbb{R}\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}\big(\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}),H_{\boldsymbol{G}}\big)$ (resp. $\mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{M}) : \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \mathbb{R}\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M}),H_{\boldsymbol{G}}\big)$). Equivariant Euler classes are next introduced for any closed embedding $i : \boldsymbol{N} \subseteq \boldsymbol{M}$ as $\mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{N},\boldsymbol{M}\big) := i^*i_!(1)$ where $i^*i_! : H_{\boldsymbol{G}}(\boldsymbol{N}) \to H_{\boldsymbol{G}}(\boldsymbol{N})$ is the push-pull operator. We end recalling localization and fixed point theorems.

About this work. These notes were originally intended as an appendix to a book on the foundations of equivariant cohomology. The idea of introducing Gysin morphisms through an equivariant Poincaré duality formalism *à la Grothendieck-Verdier* has many theoretical advantages and is somewhat uncommon in the equivariant setting, warranting publication of these notes.

Contents

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1. Nonequivariant Background

1.1. Category of Cochain Complexes

1.1.1. Fields in Use. Unless otherwise specified, $\mathbb{K}$ denotes either the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$.

1.1.2. Vector Spaces Pairings. Whenever $\mathbb{K}$ is understood the expression "*vector space*" means vector space over $\mathbb{K}$. If V is a vector space, we denote by $V^\vee := \mathrm{Hom}_{\mathbb{K}}(V,\mathbb{K})$ its *dual*.

Given a bilinear map $\beta : V \times W \to \mathbb{K}$, also called *a pairing*, consider the two linear maps $\gamma_\beta : V \to W^\vee$ and $\rho_\beta : W \to V^\vee$, defined by $\gamma_\beta(v)(w) = \rho_\beta(w)(v) := \beta(v,w)$ and respectively called the *left and right adjoint maps associated with* β. One says that β *is a nondegenerate pairing* whenever the adjoint maps are injective, and one says that β *is a perfect pairing* whenever they are bijective. For example, the canonical pairing $V^\vee \times V \to \mathbb{K}$, $(\lambda,v) \mapsto \lambda(v)$, is always nondegenerate and it is perfect if and only if V is finite dimensional.

1.1.3. Graded Vector Spaces. A *graded space* is a family $\boldsymbol{V} := \{V^m\}_{m\in\mathbb{Z}}$ of vector spaces. A *graded homomorphism* $\boldsymbol{\alpha} : \boldsymbol{V} \to \boldsymbol{W}$ *of degree* $d = \deg(\boldsymbol{\alpha})$ is a family of linear maps $\{\alpha_m : V^m \to W^{m+d}\}_{m\in\mathbb{Z}}$, composition of such is defined degree by degree, i.e. $\boldsymbol{\beta} \circ \boldsymbol{\alpha} = \{\beta_{m+d} \circ \alpha_m\}_{m\in\mathbb{Z}}$. One has $\deg(\boldsymbol{\alpha} \circ \boldsymbol{\beta}) = \deg\boldsymbol{\alpha} + \deg\boldsymbol{\beta}$.

We denote by $\mathrm{Homgr}^d_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W})$ the space of graded homomorphisms of degree d and by $\mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W})$ the graded space of all graded homomorphisms, i.e. the family

$$\mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W}) = \big\{\mathrm{Homgr}^d_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W})\big\}_{d\in\mathbb{Z}}.$$

When $d = 0$, we may write $\mathrm{Homgr}_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W})$ for $\mathrm{Homgr}^0_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W})$.

1.1.4 The *category* $\mathrm{GV}(\mathbb{K})$ *of graded vector spaces* is the category whose objects are graded spaces and whose *morphisms* are the graded homomorphisms of degree 0. We denote equivalently $\mathbf{Mor}_{\mathrm{GV}(\mathbb{K})}(\boldsymbol{V},\boldsymbol{W}) := \mathrm{Homgr}_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W})$ the set of morphisms from $\boldsymbol{V}$ to $\boldsymbol{W}$.

1.1.5. Differential Graded Vector Space. A *differential graded vector space* $(\boldsymbol{V},\boldsymbol{d})$, *a complex* in short, is a graded vector space $\boldsymbol{V}$ together a *differential* or *coboundary* $\boldsymbol{d} : \mathrm{Endgr}^1(\boldsymbol{V})$ such that $\boldsymbol{d}^2 = 0$. A *morphism of complexes* $\boldsymbol{\alpha} : (\boldsymbol{V},\boldsymbol{d}) \to (\boldsymbol{V}',\boldsymbol{d}')$ is a morphism $\boldsymbol{\alpha} \in \mathrm{Homgr}_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{V}')$ commuting with differentials, i.e. $\boldsymbol{\alpha} \circ \boldsymbol{d} = \boldsymbol{d}' \circ \boldsymbol{\alpha}$. The complexes and their morphisms constitute the *category* $\mathrm{DGM}(\mathbb{K})$ *of differential graded vector spaces*.

1.1.6 A morphism of complexes $\boldsymbol{\alpha} : (\boldsymbol{V},\boldsymbol{d}) \to (\boldsymbol{V}',\boldsymbol{d}')$ induces a morphism between the graded spaces of cohomologies $H(\boldsymbol{\alpha}) : H(\boldsymbol{V},\boldsymbol{d}) \to H(\boldsymbol{V}',\boldsymbol{d}')$. The morphism $\boldsymbol{\alpha}$ is a *quasi-isomorphism, quasi-injection, quasi-surjection*, whenever $H(\boldsymbol{\alpha})$ is respectively, an isomorphism, injection, surjection.

1.1.7 Let $m \in \mathbb{Z}$. If L is a vector space, we denote by $L[m]$ the graded space defined by $L[m]^{-m} = L$ and $L[m]^n = 0$ if $n + m \neq 0$. If $\alpha : V \to W$ is a linear map, we denote by $\alpha[m] : V[m] \to W[m]$ the morphism of graded spaces equal to α in degree $-m$ and 0 otherwise. The correspondence $L \rightsquigarrow (L[m],\mathbf{0})$ and $\alpha \rightsquigarrow \alpha[m]$ is a functor

$$[m] : \mathrm{Vec}(\mathbb{K}) \to \mathrm{DGM}(\mathbb{K}).$$

More generally, If $\boldsymbol{V}$ is a graded space we denote by $\boldsymbol{V}[m]$ the graded space $(\boldsymbol{V}[m])^i = \boldsymbol{V}^{m+i}$, and if $\boldsymbol{\alpha} : \boldsymbol{V} \to \boldsymbol{W}$ is a graded homomorphism we denote by $\boldsymbol{\alpha}[m] : \boldsymbol{V}[m] \to \boldsymbol{W}[m]$ the graded homomorphism $\boldsymbol{\alpha}[m]_i = \alpha_{m+i}$. Next, if $(\boldsymbol{V},\boldsymbol{d})$ is a complex, $(\boldsymbol{V},\boldsymbol{d})[m]$ is the complex $\big(\boldsymbol{V}[m],(-1)^m\boldsymbol{d}[m]\big)$. The correspondence $(\boldsymbol{V},\boldsymbol{d}) \rightsquigarrow (\boldsymbol{V},\boldsymbol{d})[m]$, $\boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\alpha}[m]$ is the *m-th shift functor*

$$[m] : \mathrm{DGM}(\mathbb{K}) \rightsquigarrow \mathrm{DGM}(\mathbb{K}).$$

1.1.8 Given two complexes $(\boldsymbol{V},\boldsymbol{d})$ and $(\boldsymbol{V}',\boldsymbol{d}')$, we recall the definition of the complexes

$$\big(\mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{V}'),\boldsymbol{D}\big) \quad \text{and} \quad \big((\boldsymbol{V} \otimes_{\mathbb{K}} \boldsymbol{V}')^\bullet,\boldsymbol{\Delta}\big).$$

As graded vector spaces they are

$$m \in \mathbb{Z} \mapsto \begin{cases} \mathbf{Hom}^m_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{V}') = \mathrm{Homgr}^m_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{V}') \\ \quad (\boldsymbol{V} \otimes_{\mathbb{K}} \boldsymbol{V}')^m = \prod_{b+a=m} V^a \otimes_{\mathbb{K}} V'^b \end{cases}$$

and their differentials are

$$\begin{cases} \qquad D_m(f) = \boldsymbol{d}' \circ f - (-1)^m f \circ \boldsymbol{d} \\ \Delta_m(v \otimes v') = \boldsymbol{d}(v) \otimes v' + (-1)^{|v|} v \otimes \boldsymbol{d}'(v') \end{cases}$$

where $v \otimes v' \in V^{|v|} \otimes V'^{|v'|}$. ([1])

1.1.9. Exercise. Verify that the following complexes coincide as graded vector spaces but not as complexes even though they are naturally isomorphic.

$$\mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{V}[n],\boldsymbol{W}[m]) \simeq \mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{W})[m-n]$$
$$\boldsymbol{V}[n] \otimes \boldsymbol{W}[m] \simeq (\boldsymbol{V} \otimes \boldsymbol{W})[m+n]$$

1.1.10 Given a morphism of complexes $\boldsymbol{\varphi} : (\boldsymbol{V},\boldsymbol{d}) \to (\boldsymbol{W},\boldsymbol{d})$ the map

$$\mathbf{Hom}^m_{\mathbb{K}}(\boldsymbol{W},\boldsymbol{V}') \to \mathbf{Hom}^m_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{V}'), \quad \boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} \circ \boldsymbol{\varphi},$$

is well defined for all $m \in \mathbb{Z}$ and commutes with differentials so that one has a morphism of complexes

$$\mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{\alpha},\boldsymbol{V}') : \big(\mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{W},\boldsymbol{V}'),\boldsymbol{D}\big) \to \big(\mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{V}'),\boldsymbol{D}\big).$$

The correspondence $(\boldsymbol{V},\boldsymbol{d}) \rightsquigarrow \mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{V},\boldsymbol{V}')$, $\boldsymbol{\alpha} \rightsquigarrow \mathbf{Hom}^\bullet_{\mathbb{K}}(\boldsymbol{\alpha},\boldsymbol{V}')$ is then a *contravariant* functor

$$\mathbf{Hom}^\bullet_{\mathbb{K}}(\_,\boldsymbol{V}') : \mathrm{DGM}(\mathbb{K}) \rightsquigarrow \mathrm{DGM}(\mathbb{K}).$$

1.1.11. The Dual Complex. The functor $\mathbf{Hom}^\bullet_{\mathbb{K}}(\_,\mathbb{K}[0])$ is the *duality functor*, simply denoted by $(\_)^\vee := \mathbf{Hom}^\bullet_{\mathbb{K}}(\_,\mathbb{K}[0])$

$$(\_)^\vee : \mathrm{DGM}(\mathbb{K}) \rightsquigarrow \mathrm{DGM}(\mathbb{K}).$$

The complex $(\boldsymbol{V},\boldsymbol{d})^\vee$ is called *the dual complex associated with* $(\boldsymbol{V},\boldsymbol{d})$. One has

$$(\boldsymbol{V}^\vee)^m = \mathrm{Hom}_{\mathbb{K}}(V^{-m},\mathbb{K}), \quad D_m = (-1)^{m+1\,t} d_{-(m+1)}$$

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[1] It is worth noting that these formulae are inspired by the super Lie bracket equalities

$$[\![d,f]\!] = df - (-1)^{|d||f|} fd \quad \text{and} \quad [\![d,ab]\!] = [\![d,a]\!] + (-1)^{|a||d|} a[\![d,b]\!]$$

where $[\![d,[\![d,\_]\!]]\!] = 0$ is an immediate consequence of $|d| = 1$ and $d^2 = 0$.

1.1.12. Remark. One must take care that the natural embedding of vector spaces $V \subseteq V^{\vee\vee}$ gives only an inclusion of complexes $(\boldsymbol{V}, -\boldsymbol{d}) \subseteq (\boldsymbol{V}, \boldsymbol{d})^{\vee\vee}$ where the sign of the differential has changed ! The canonical isomorphism

$$\boldsymbol{\epsilon} : (\boldsymbol{V}, \boldsymbol{d}) \to (\boldsymbol{V}, -\boldsymbol{d}), \quad \epsilon_m = (-1)^m \mathrm{id}_{V^m} \qquad (\boldsymbol{\epsilon})$$

is then necessary to get a canonical embedding $(\boldsymbol{V}, \boldsymbol{d}) \hookrightarrow (\boldsymbol{V}, \boldsymbol{d})^{\vee\vee}$.

The next statement will be used without mention, it is left as an exercise.

1.1.13. Proposition

a) *A morphism of complexes* $\boldsymbol{\alpha} : (\boldsymbol{V}, \boldsymbol{d}) \to (\boldsymbol{V}', \boldsymbol{d}')$ *is a quasi-isomorphism if and only if* $\boldsymbol{\alpha}^\vee$ *is so.*

b) *There exists a canonical isomorphism between the cohomology of the dual and the dual of the cohomology, i.e.*

$$\boldsymbol{h}\big((\boldsymbol{V}, \boldsymbol{d})^\vee\big) \xrightarrow[\simeq]{} \big(\boldsymbol{h}(\boldsymbol{V}, \boldsymbol{d})\big)^\vee .$$

where $\boldsymbol{h}$ *denotes the graded vector space of the cohomologies of a complex.*

1.2. Some Categories of Manifolds

1.2.1. Manifolds. The names *manifold* and *map* (when applied to manifolds) will be shortcuts for *real differentiable manifold* and *smooth map*. Manifolds are equidimensionnal, i.e. all their connected components have the same dimension, unless otherwise specified. The notation "$d_{\boldsymbol{M}}$", unless otherwise indicated, will always denote the dimension on $\boldsymbol{M}$.

1.2.2 $\boldsymbol{Man}$ (resp. $\boldsymbol{Man}^{\mathrm{or}}$) denotes the *category of (equidimensional) manifolds (resp. oriented) and smooth maps*. Over $\boldsymbol{Man}$ one has the *(real) de Rham complex* contravariant functor

$$\Omega(\_) : \boldsymbol{Man} \rightsquigarrow \mathrm{DGM}(\mathbb{R})$$

and the *de Rham cohomology* contravariant functor

$$H(\_) : \boldsymbol{Man} \rightsquigarrow \mathrm{Mod}^{\mathbb{N}}(\mathbb{R}).$$

1.2.3 $\boldsymbol{Man}_\pi$ (resp. $\boldsymbol{Man}_\pi^{\mathrm{or}}$) denotes the subcategory of $\boldsymbol{Man}$ (resp. $\boldsymbol{Man}^{\mathrm{or}}$) with the same objects but with only **proper** maps. Over $\boldsymbol{Man}_\pi$ one has, in addition to the previous functors, the *(real) de Rham complex with* ***compact*** *supports* contravariant functor:

$$\Omega_{\mathrm{c}}(\_) : \boldsymbol{Man}_\pi \rightsquigarrow \mathrm{DGM}(\mathbb{R})$$

and the *de Rham cohomology with compact support* contravariant functor

$$H_{\mathrm{c}}(\_) : \boldsymbol{Man}_\pi \rightsquigarrow \mathrm{DGM}(\mathbb{R}).$$

The inclusion $\Omega_{\mathrm{c}}(\_) \subseteq \Omega(\_)$ induces a morphism of functors $H_{\mathrm{c}}(\_) \to H(\_)$.

1.2.4. $\boldsymbol{G}$-Manifolds. Let $\boldsymbol{G}$ denote a Lie group. A manifold endowed with a smooth action of $\boldsymbol{G}$ is called *a* $\boldsymbol{G}$*-manifold*. A map $f : \boldsymbol{M} \to \boldsymbol{N}$ between $\boldsymbol{G}$-manifolds is called *a* $\boldsymbol{G}$*-equivariant* if it commutes with the action of $\boldsymbol{G}$. The class of $\boldsymbol{G}$-manifolds and $\boldsymbol{G}$-equivariant maps constitutes the category $\boldsymbol{G}$-$\boldsymbol{Man}$. The categories $\boldsymbol{G}$-$\boldsymbol{Man}^{\mathrm{or}}$, $\boldsymbol{G}$-$\boldsymbol{Man}_\pi$, $\boldsymbol{G}$-$\boldsymbol{Man}_\pi^{\mathrm{or}}$ are the analogues of those already introduced in this section.

1.3. Poincaré Pairing

The reference for this section is [BT] (**I** §5 p. 44). Let $\boldsymbol{M}$ be an oriented manifold. The composition of the bilinear map $\Omega^{d_M-i}(\boldsymbol{M})\otimes\Omega^i_{\mathrm{c}}(\boldsymbol{M})\to\Omega^{d_M}_{\mathrm{c}}(\boldsymbol{M})$, $\alpha\otimes\beta\mapsto\alpha\wedge\beta$, with integration $\int_{\boldsymbol{M}}:\Omega^{d_M}_{\mathrm{c}}(\boldsymbol{M})\to\mathbb{R}$, gives rise to a nondegenerate pairing

$$\begin{aligned} I\!\!P(\boldsymbol{M}):\Omega^{d_M-i}(\boldsymbol{M})\otimes\Omega^i_{\mathrm{c}}(\boldsymbol{M}) &\longrightarrow \mathbb{R}\\ \alpha\otimes\beta \quad &\longmapsto \int_{\boldsymbol{M}}\alpha\wedge\beta \end{aligned} \tag{$I\!\!P$}$$

inducing the *Poincaré pairing in cohomology*

$$\begin{aligned} \mathscr{P}(\boldsymbol{M}):H^{d_M-i}(\boldsymbol{M})\otimes H^i_{\mathrm{c}}(\boldsymbol{M}) &\longrightarrow \mathbb{R}\\ [\alpha]\otimes[\beta] \quad &\longmapsto \int_{\boldsymbol{M}}[\alpha]\cup[\beta] \end{aligned} \tag{$\mathscr{P}$}$$

The left adjoint map associated with $I\!\!P$ is

$$\begin{aligned} I\!\!D(\boldsymbol{M}):\Omega^{d_M-i}(\boldsymbol{M}) &\longrightarrow \Omega^i_{\mathrm{c}}(\boldsymbol{M})^\vee\\ \alpha \quad &\longmapsto I\!\!D(\alpha):=\Big(\beta\mapsto\int_{\boldsymbol{M}}\alpha\wedge\beta\Big) \end{aligned} \tag{$I\!\!D$}$$

and one has

$$\begin{aligned} I\!\!D(\boldsymbol{M})\big((-1)^{d_M}\boldsymbol{d}\alpha\big)(\beta) &= \int_{\boldsymbol{M}}(-1)^{d_M}\boldsymbol{d}\alpha\wedge\beta\\ &= \int_{\boldsymbol{M}}(-1)^{d_M+|\alpha|+1}\alpha\wedge\boldsymbol{d}\beta = (-1)^{|\beta|}I\!\!D(\boldsymbol{M})(\alpha)(\boldsymbol{d}\beta), \end{aligned}$$

Hence, following the conventions introduced in 1.1.7 and 1.1.8, $I\!\!D(\boldsymbol{M})$ is a morphism of complexes from $\Omega(\boldsymbol{M})[d_{\boldsymbol{M}}]$ to $\Omega_{\mathrm{c}}(\boldsymbol{M})^\vee$.

1.3.1. Exercise. Show that ($I\!\!P$) is a nondegenerate pairing.

1.3.2. Theorem Poincaré duality theorem. *Let $\boldsymbol{M}$ be an oriented manifold.*

a) *The morphism of complexes, called* the Poincaré morphism,

$$I\!\!D(\boldsymbol{M}):\Omega(\boldsymbol{M})[d_{\boldsymbol{M}}]\hookrightarrow\Omega_{\mathrm{c}}(\boldsymbol{M})^\vee \tag{$*$}$$

is a quasi-isomorphism, i.e. the morphism of graded spaces it induces in cohomology

$$\mathscr{D}(\boldsymbol{M}):H(\boldsymbol{M})[d_{\boldsymbol{M}}]\xrightarrow[\simeq]{}H_{\mathrm{c}}(\boldsymbol{M})^\vee, \tag{$**$}$$

is an isomorphism.

b) *The Poincaré pairing in cohomology*

$$\mathscr{P}(\boldsymbol{M}):H(\boldsymbol{M})\otimes H_c(\boldsymbol{M})\longrightarrow\mathbb{R} \tag{$***$}$$

is always nondegenerate. *It is* perfect *(see 1.1.2) if and only if $H(\boldsymbol{M})$ is finite dimensional.*

Proof. The (a) part (cf. [BT] p. 44–, for details) states the bijectivity of the left adjoint map associated with $\mathscr{P}$. Then, for each fixed i, one obtains by duality the bijectivity of $\mathscr{D}_i^\vee:(H^{d_M-i}_{\mathrm{c}}(\boldsymbol{M}))^{\vee\vee}\to H^i(\boldsymbol{M})^\vee$ and the composition of this map with the canonical embedding $\epsilon_i:H^{d_M-i}_{\mathrm{c}}\hookrightarrow(H^{d_M-i}_{\mathrm{c}})^{\vee\vee}$ is the right adjoint map $\rho_{\mathscr{P}}:H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}]\to H(\boldsymbol{M})^\vee$ (see 1.1.2). The "finite dimensional" condition then ensures the bijectivity of ϵ_i, hence of $\rho_{\mathscr{P}}$. □

1.3.3. Exercise. Let $\boldsymbol{M}$ and $\boldsymbol{N}$ be oriented manifolds. We denote by

$$\mathbb{D}(\_) : \Omega(\_)[d\_] \to \Omega(\_)_{\rm c}^{\vee}, \quad \mathbb{D}(\alpha)(\beta) = \int \alpha \wedge \beta$$

the left adjoint map of the Poincaré pairing, and by $\mathbb{D}'(\_) : \Omega_{\rm c}(\_)[d\_] \to \Omega(\_)^{\vee}$, $\mathbb{D}'(\beta)(\alpha) = \int \alpha \wedge \beta$ the right adjoint map. A pair (L,R) of morphisms of complexes $L : \Omega(\boldsymbol{N}) \to \Omega(\boldsymbol{M})$ and $R : \Omega_{\rm c}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \Omega_{\rm c}(\boldsymbol{N})[d_{\boldsymbol{N}}]$ is a *(Poincaré) adjoint pair* whenever

$$\int_{\boldsymbol{M}} L(\alpha) \wedge \beta = \int_{\boldsymbol{N}} \alpha \wedge R(\beta)$$

for all $\alpha \in \Omega(\boldsymbol{N})$ and $\beta \in \Omega_{\rm c}(\boldsymbol{M})$. Show that

a) If (L,R_1) and (L,R_2) are adjoint pairs, then $R_1 = R_2$. One says that $R = R_1$ is the *(Poincaré) right adjoint of* L.

b) If (L_1,R) and (L_2,R) are adjoint pairs, then $L_1 = L_2$. One says that $L = L_1$ is the *(Poincaré) left adjoint of* R.

c) If (L,R) is an adjoint pair, then

$$\mathbb{D} \circ L = R^{\vee} \circ \mathbb{D}, \qquad \mathbb{D}' \circ R = L^{\vee} \circ \mathbb{D}',$$

i.e. the following diagrams are commutative

$$\begin{array}{ccc} \Omega(\boldsymbol{M}) & \xrightarrow[(\simeq)]{\mathbb{D}(\boldsymbol{M})} & \Omega_{\rm c}(\boldsymbol{M})^{\vee} \\ {\scriptstyle L}\uparrow & & \uparrow{\scriptstyle R^{\vee}} \\ \Omega(\boldsymbol{N}) & \xrightarrow[(\simeq)]{\mathbb{D}(\boldsymbol{N})} & \Omega_{\rm c}(\boldsymbol{N})^{\vee} \end{array} \qquad \begin{array}{ccc} \Omega_{\rm c}(\boldsymbol{M}) & \xrightarrow{\mathbb{D}'(\boldsymbol{M})} & \Omega(\boldsymbol{M})^{\vee} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle L^{\vee}} \\ \Omega_{\rm c}(\boldsymbol{N}) & \xrightarrow{\mathbb{D}'(\boldsymbol{N})} & \Omega(\boldsymbol{N})^{\vee} \end{array}$$

d) Do the exercise in the cohomological framework, i.e. replace Poincaré pairing ($\mathbb{P}$) by ($\mathscr{P}$), $\mathbb{D}$ by $\mathscr{D} : H[d] \to H_{\rm c}^{\vee}$, $\mathbb{D}'$ by $\mathscr{D}' : H_{\rm c}[d] \to H^{\vee}$, and define the notion of *(Poincaré) adjoint pair in cohomology*.

Show that if (L,R) is an adjoint pair of morphisms of complexes, then $(H(L),H_{\rm c}(R))$ is an adjoint pair in cohomology so that one has

$$\mathscr{D} \circ H(L) = H_{\rm c}(R)^{\vee} \circ \mathscr{D}, \qquad \mathscr{D}' \circ H_{\rm c}(R) = H(L)^{\vee} \circ \mathscr{D}'.$$

In particular, $H(L)$ *identifies with the dual of* $H_{\rm c}(R)$ *via Poincaré duality.*

1.3.4. Remark. We shall see that, given $f : \boldsymbol{M} \to \boldsymbol{N}$, the pullback morphism $f^* : \Omega(\boldsymbol{N}) \to \Omega(\boldsymbol{M})$ may or may not admit a right Poincaré adjoint at the complexes level, but that it will always do so at the cohomology level, this right adjoint is the Gysin morphism $f_* : H_{\rm c}(\boldsymbol{M}) \to H_{\rm c}(\boldsymbol{N})$, so that $(H(f^*),f_*)$ is a Poincaré adjoint pair in cohomology. On the other hand, when f is a proper map, the pullback $f^* : \Omega_{\rm c}(\boldsymbol{N}) \to \Omega_{\rm c}(\boldsymbol{M})$ is also well defined and one may look for a left Poincaré adjoint to f^*, i.e. some morphism $L : \Omega(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \Omega(\boldsymbol{N})[d_{\boldsymbol{N}}]$

$$\int_{\boldsymbol{N}} L(\alpha) \wedge \beta = \int_{\boldsymbol{M}} \alpha \wedge f^*(\beta).$$

Again, this will sometimes be possible at the complex level and will always be possible at the cohomology level leading to the notion of *the Gysin morphism for proper maps* $f_! : H(\boldsymbol{M}) \to H(\boldsymbol{N})$, so that $(f_!,H_{\rm c}(f^*))$ is a Poincaré adjoint pair in cohomology.

1.4. Manifolds and maps of Finite de Rham Type

1.4.1. Definitions. A manifold $\boldsymbol{M}$ is said to be *of finite (de Rham) type* whenever its de Rham cohomology ring $H(\boldsymbol{M})$ is finite dimensional. A map between manifolds $f : \boldsymbol{M} \to \boldsymbol{N}$ si said to be *of finite (de Rham) type* if $\boldsymbol{N}$ is the union of a countable ascending chain $\mathscr{U} := \{U_0 \subseteq U_1 \subseteq \cdots\}$ of open subspaces of finite type such that each subspace $f^{-1}(U_m) \subseteq \boldsymbol{M}$ is of finite type.

1.4.2. Remarks

a) By Poincaré duality (1.3.2), $\boldsymbol{M}$ is of finite type if and only if its de Rham cohomology with compact support $H_c(\boldsymbol{M})$ is finite dimensional.

b) A compact manifold is of finite type ([BT] **5.3** pp. 42-43). An oriented manifold is of finite type if and only if its Poincaré pairing in cohomology is perfect (1.3.2-(b)), which will be used in our discussion of the Gysin morphism.

c) Since any manifold is the union of a countable ascending chain $\{\uparrow U_m\}$ of open submanifolds of finite type (cf. 1.4.4), any locally trivial fibration $f : \boldsymbol{M} \to \boldsymbol{N}$ is of finite type (exercise).

1.4.3. Ascending Chain Property. Although general manifolds need not be of finite type, they are always the inductive limit of such. More precisely, any manifold $\boldsymbol{M}$ is the union of an ascending chain $\{U_0 \subseteq U_1 \subseteq \cdots\}$ of open subsets of finite type of $\boldsymbol{M}$. This weaker finiteness property, sufficient for our needs, is generally proved by a riemannian geometry argument ([2]). When a manifold is endowed with the action of a Lie group $\boldsymbol{G}$, we will require in addition that each U_n be $\boldsymbol{G}$-stable.

1.4.4. Proposition. *Let $\boldsymbol{G}$ be a compact Lie Group. A $\boldsymbol{G}$-manifold $\boldsymbol{M}$ is the union of a countable ascending chain $\mathscr{U} := \{U_0 \subseteq U_1 \subseteq \cdots\}$ of $\boldsymbol{G}$-stable open subsets of $\boldsymbol{M}$ of finite type.*

The next sections recall some facts needed in the proof of this proposition.

1.4.5. Existence of Proper Functions. Recall that a map between manifolds $f : \boldsymbol{M} \to \boldsymbol{N}$ is said to be *proper* whenever $f^{-1}(\boldsymbol{F})$ is compact for any compact subset $\boldsymbol{F}$ of $\boldsymbol{N}$. The aim of this section is to show that on a $\boldsymbol{G}$-manifold there are always proper $\boldsymbol{G}$-invariant functions.

Since the existence of proper functions over compact manifolds is clear, let $\boldsymbol{M}$ be a noncompact manifold. Fix a countable, locally finite covering $\mathscr{U} := \{U_n\}_{n\in\mathbb{N}}$ of $\boldsymbol{M}$, where each U_n is a *relatively compact* open subset of $\boldsymbol{M}$, and note that the noncompactness of $\boldsymbol{M}$ implies that the family is necessarily infinite. Next, fix a smooth partition of unity $\{\phi_n\}_{n\in\mathbb{N}}$ subordinate to $\mathscr{U}$. This means in particular that for each $n \in \mathbb{N}$, the equality $\phi_n(x) = 0$ holds whenever $x \notin U_n$. Then one has for every $N \in \mathbb{N}$

$$1 = \sum\nolimits_{n>N} \phi_n(x), \quad \forall x \notin U_0 \cup \cdots \cup U_N. \qquad (\diamond)$$

[2] In these notes, a *good cover of $\boldsymbol{M}$* (also known as *Leray cover*) is a finite open covering $\mathscr{U} = \{U_i \mid i = 1,\ldots,r\}$ of $\boldsymbol{M}$ such that all intersections $U_{i_1} \cap \ldots \cap U_{i_k}$ are either vacuous or acyclic ([BT], p. 5). The existence of good covers is established in *loc.cit.* §5, p. 42.

Now, for every $x \in \boldsymbol{M}$, the infinite sum $\Phi(x) := \sum_{n\in\mathbb{N}} n\cdot\phi_n(x)$ is actually finite and smooth with respect to $x \in \boldsymbol{M}$, as it is a locally finite sum of smooth functions.

1.4.6. Lemma. *The positive function $\Phi : \boldsymbol{M} \to \mathbb{R}$ is unbounded and proper.*

Proof. By property ($\diamond$) one has

$$\Phi(x) \geqslant \sum_{n>N} n\cdot\phi_n(x) > N\Big(\sum_{n>N} \phi_n(x)\Big) = N, \quad \forall x \notin U_0 \cup \cdots \cup U_N, \quad (\diamond\diamond)$$

which clearly implies that Φ is an unbounded function on $\boldsymbol{M}$. Now, to see that Φ is proper, remark that if $\boldsymbol{F} \subseteq \mathbb{R}$ is compact, then $\boldsymbol{F} \subseteq [-N,N]$ for some $N \in \mathbb{N}$ and $\Phi^{-1}(\boldsymbol{F}) \subseteq U_0 \cup \cdots \cup U_N$ by ($\diamond\diamond$). But the closure $\overline{U_0 \cup \cdots \cup U_N}$ is a compact subset of $\boldsymbol{M}$ because each $\overline{U}_i$ is assumed compact. As a closed subset of a compact set, $\Phi^{-1}(\boldsymbol{F})$ is compact. □

As a corollary of the preceding lemma one has:

1.4.7. Proposition. *Manifolds $\boldsymbol{M}$ endowed with a smooth action of a compact Lie group $\boldsymbol{G}$ admit proper $\boldsymbol{G}$-invariant positive functions $\Phi : \boldsymbol{M} \to \mathbb{R}$.*

Proof. If $\boldsymbol{M}$ is compact, any positive *constant* map Φ will do. If $\boldsymbol{M}$ is not compact, let $\phi : \boldsymbol{M} \to \mathbb{R}$ denote any unbounded proper positive function (see 1.4.6), and set:

$$\Phi(x) := \int_{\boldsymbol{G}} \phi(g\cdot x)\, dg,$$

where dg is a $\boldsymbol{G}$-invariant form of top degree on $\boldsymbol{G}$, such that $1 = \int_{\boldsymbol{G}} dg$. The correspondence $x \mapsto \Phi(x)$ is clearly a well-defined nonnegative unbounded $\boldsymbol{G}$-invariant function of $\boldsymbol{M}$ into $\mathbb{R}$. Now, for each $N \in \mathbb{N}$, the set

$$\boldsymbol{M}_N := \boldsymbol{G}\cdot\phi^{-1}([-N,N])$$

is compact and $\boldsymbol{G}$-stable, and if $y \notin \boldsymbol{M}_N$, one has $\phi(g\cdot y) > N$ for all $g \in \boldsymbol{G}$, so that

$$\Phi(y) = \int_{\boldsymbol{G}} \phi(g\cdot y)\, dg > N, \qquad (\diamond\diamond\diamond)$$

and properness of Φ follows by the same argument as in lemma 1.4.6: If $\boldsymbol{F}$ is a compact subset of $\mathbb{R}$, then $\boldsymbol{F} \subseteq [-N,N]$ for some $N \in \mathbb{N}$, and $\Phi^{-1}(\boldsymbol{F}) \subseteq \boldsymbol{M}_N$ by ($\diamond\diamond\diamond$). Then $\Phi^{-1}(\boldsymbol{F})$ is compact since it is closed in the compact set $\boldsymbol{M}_N$. □

1.5. Manifolds With Boundary

The following is well known.

1.5.1. Proposition. *The interior of a compact manifold with boundary is of finite type.*

Proof. Let $\boldsymbol{B}$ be a compact manifold with boundary and let $\boldsymbol{M}$ be its interior. Gluing $\boldsymbol{B}$ with itself along its boundary $\partial\boldsymbol{B}$, one gets the "double" $\boldsymbol{B} \sqcup_{\partial\boldsymbol{B}} \boldsymbol{B}$, which is a compact manifold without boundary. Then, from the long exact sequence of de Rham cohomology with compact support (see 1.11.1-(a))

$$\cdots \longrightarrow H^i_{\mathrm{c}}(\boldsymbol{M}) \times H^i_{\mathrm{c}}(\boldsymbol{M}) \longrightarrow H^i(\boldsymbol{B} \sqcup_{\partial\boldsymbol{B}} \boldsymbol{B}) \longrightarrow H^0 i(\partial\boldsymbol{B}) \longrightarrow \cdots,$$

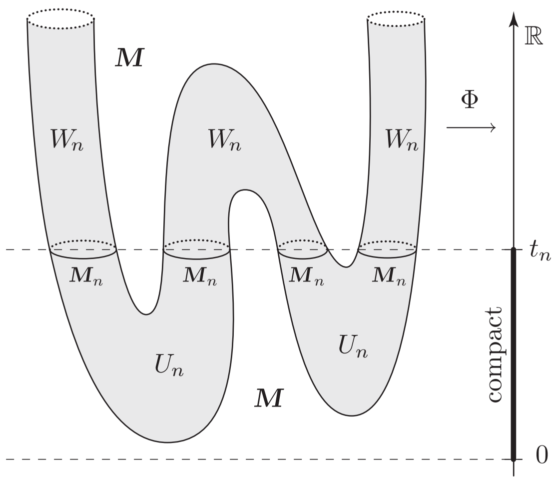
where $H^*(\boldsymbol{B} \sqcup_{\partial \boldsymbol{B}} \boldsymbol{B})$ and $H^*(\partial \boldsymbol{B})$ are finite-dimensional, the finiteness of $H_c^*(\boldsymbol{M})$ follows immediately. The finiteness of $H^*(\boldsymbol{M})$ results from Poincaré duality $H^*(\boldsymbol{M}) \cong H_c^*(\boldsymbol{M})^\vee$. □

1.6. Proof of Proposition 1.4.4

Recall that the connected components of a manifold $\boldsymbol{M}$ are always open and closed submanifolds of $\boldsymbol{M}$. In particular, if $\boldsymbol{M} = \coprod_{\mathfrak{a} \in \mathfrak{A}} \boldsymbol{C}_{\mathfrak{a}}$ denotes the decomposition of $\boldsymbol{M}$ in connected components, the indexing set $\mathfrak{A}$ is finite or countable, and the restriction of a proper function $\Phi : \boldsymbol{M} \to \mathbb{R}$ to each $\boldsymbol{C}_i$ remains proper.

If all the connected components of $\boldsymbol{M}$ are compact, we may index them by natural numbers $\boldsymbol{C}_0, \boldsymbol{C}_1, \ldots$ and define $U_n := \boldsymbol{C}_0 \cup \boldsymbol{C}_1 \cup \cdots \cup \boldsymbol{C}_n$. Each U_n is then open in $\boldsymbol{M}$ and is also a compact manifold, hence it is of finite type. The ascending chain $\{U_0 \subseteq U_1 \subseteq \cdots\}$ satisfies the conditions of the proposition.

If $\boldsymbol{M}$ contains a noncompact connected component $\boldsymbol{C}$, fix any proper positive $\boldsymbol{G}$-invariant function $\Phi : \boldsymbol{M} \to \mathbb{R}$, which is possible due to 1.4.7, and note that $\Phi(\boldsymbol{C})$ is necessarily unbounded, since otherwise $\boldsymbol{C} \subseteq \Phi^{-1}([0,T])$ for some $T \in \mathbb{R}$, and $\boldsymbol{C}$ would be compact as Φ is proper over $\boldsymbol{C}$. Moreover, there exists $N \in \mathbb{N}$ such that $\Phi(\boldsymbol{M}) \supseteq \Phi(\boldsymbol{C}) \supseteq (]N,+\infty[)$, since $\Phi(\boldsymbol{C})$ is unbounded *and* connected. Now, by Sard's theorem, the interior of the set of *critical* values of $\Phi : \boldsymbol{M} \to \mathbb{R}$ is empty so that there exists an unbounded increasing sequence of positive real numbers $\{N < t_0 < \cdots < t_n < \cdots\}_{n \in \mathbb{N}}$ which are *regular* values of Φ. Each subset $\boldsymbol{M}_n := \Phi^{-1}(t_n)$ is then a submanifold of codimension 1 in $\boldsymbol{M}$ and, moreover, it is compact and $\boldsymbol{G}$-stable since Φ is proper and $\boldsymbol{G}$-invariant. Similarly, the sets $U_n := \Phi^{-1}(]-\infty,t_n[)$ and $W_n := \Phi^{-1}(]t_n,+\infty[)$, clearly nonempty, are $\boldsymbol{G}$-stable open subsets of $\boldsymbol{M}$. One then easily checks that $\overline{U}_n = U_n \sqcup \boldsymbol{M}_n$ and $\overline{W}_n = \boldsymbol{M}_n \sqcup W_n$ are in fact $\boldsymbol{G}$-manifolds with boundary $\boldsymbol{M}_n$ embedded in $\boldsymbol{M}$. Furthermore, $\overline{U}_n$ is compact as one has $\overline{U}_n := \Phi^{-1}(]-\infty,t_n]) = \Phi^{-1}([0,t_n])$ since Φ is positive.



We can then apply proposition 1.5.1 and state that U_n is a $\boldsymbol{G}$-stable open subset of finite type of $\boldsymbol{M}$. The ascending chain $\{U_0 \subseteq U_1 \subseteq \cdots\}$ satisfies the conditions of the proposition. □

1.7. The Gysin Functor

In this section we dualize $\mathbb{D}(\boldsymbol{M}) : \Omega(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \Omega_{\mathrm{c}}(\boldsymbol{M})^{\vee}$, getting an injection $\mathscr{D}'(\boldsymbol{M}) : H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \hookrightarrow H(\boldsymbol{M})^{\vee}$ whose image will be shown to be functorial on the category $\boldsymbol{Man}^{\mathrm{or}}$ of oriented manifolds.

1.7.1. The Right Adjoint Map. In 1.3 we introduced the left adjoint map associated with Poincaré pairing, i.e. the quasi-isomorphism

$$\mathbb{D}(\boldsymbol{M}) : \Omega(\boldsymbol{M})[d_{\boldsymbol{M}}] \xrightarrow[(\simeq)]{} \Omega_{\mathrm{c}}(\boldsymbol{M})^{\vee}.$$

By duality, this map yields $\mathbb{D}(\boldsymbol{M})^{\vee} : \Omega_{\mathrm{c}}(\boldsymbol{M})^{\vee\vee} \to \Omega(\boldsymbol{M})[d_{\boldsymbol{M}}]^{\vee}$ which is also a quasi-isomorphism and, composed with the embedding $\Omega_{\mathrm{c}}(\boldsymbol{M}) \subseteq \Omega_{\mathrm{c}}(\boldsymbol{M})^{\vee\vee}$, gives rise to the injection and quasi-injection (1.3.1, 1.1.6)

$$\begin{array}{ccccc} \big(\Omega_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}],\boldsymbol{d}\big) & \xhookrightarrow{\subseteq} & \big(\Omega_{\mathrm{c}}(\boldsymbol{M})^{\vee\vee}[d_{\boldsymbol{M}}],-\boldsymbol{d}\big) & \xrightarrow[(\simeq)]{\mathbb{D}^{\vee}} & \big(\Omega(\boldsymbol{M})^{\vee},-\boldsymbol{d}\big) \\ & & \underbrace{\qquad\qquad\qquad}_{\mathbb{D}'(\boldsymbol{M})} & & \end{array}$$

(See 1.1.11 for the sign of differentials.) One has (cf. 1.3.3)

$$\mathbb{D}'(\boldsymbol{M})(\beta) = \Big(\alpha \mapsto \int_{\boldsymbol{M}} \alpha \wedge \beta\Big),$$

which clearly it is the *right adjoint map associated with the Poincaré pairing* $\mathbb{P}$.

The following proposition paraphrases the statement 1.3.2-(b).

1.7.2. Proposition. *Let* $\boldsymbol{M}$ *be an oriented manifold.*

a) *The morphism of complexes*

$$\mathbb{D}'(\boldsymbol{M}) : \big(\Omega_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}],\boldsymbol{d}\big) \hookrightarrow \big(\Omega(\boldsymbol{M})^{\vee},-\boldsymbol{d}\big)$$

is always an injection and a quasi-injection. We will denote by

$$\mathscr{D}'(\boldsymbol{M}) : H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \hookrightarrow H(\boldsymbol{M})^{\vee}$$

the induced injection in cohomology.

b) *The morphism* $\mathscr{D}'(\boldsymbol{M})$ *is an isomorphism if and only if* $\boldsymbol{M}$ *is of finite type.*

1.7.3. The Gysin Morphism. The last statement shows that in the oriented finite type case, compact support cohomology canonically coincides with the dual of arbitrary support cohomology so that if $\boldsymbol{M}$ and $\boldsymbol{N}$ are such, the diagram

$$\begin{array}{ccc} H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] & \xhookrightarrow[\simeq]{\mathscr{D}'(\boldsymbol{M})} & H(\boldsymbol{M})^{\vee} \\ {\scriptstyle f_*}\downarrow & \oplus & \downarrow{\scriptstyle H(f^*)^{\vee}} \\ H_{\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}] & \xhookrightarrow[\simeq]{\mathscr{D}'(\boldsymbol{N})} & H(\boldsymbol{N})^{\vee} \end{array} \qquad (\mathscr{D}')$$

can be closed in one and only one way with the morphism of graded spaces $f_* : H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}]$. It is then clear that the correspondence defined over the category of oriented finite type manifolds, that assigns $\boldsymbol{M} \rightsquigarrow \boldsymbol{M}_* := H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}]$ and $f \rightsquigarrow f_*$, is a covariant functor.

When the manifold $\boldsymbol{N}$ in $(\mathscr{D}')$ is oriented but not of finite type, $\mathscr{D}'(\boldsymbol{N})$ is still an injection but it is no longer surjective so that it is not obvious that the diagram can be closed. Statement (b) in the following theorem gives a positive answer to this question showing that the image of $\mathscr{D}'(\_)$ is "stable under pullbacks". Hence, it will always be possible to induce $f_* : \boldsymbol{M}_* \to \boldsymbol{N}_*$, *the Gysin morphism associated with* f. After that, the correspondence $\boldsymbol{M} \rightsquigarrow \boldsymbol{M}_* := H_c(\boldsymbol{M})[d_{\boldsymbol{M}}]$ and $f \rightsquigarrow f_*$ will appear to be a covariant functor defined over the ***whole*** category $\boldsymbol{Man}^{\mathrm{or}}$, *the Gysin functor*.

1.7.4. Theorem and definitions

a) *Let* $\boldsymbol{M}$ *be oriented and endow its open subsets with induced orientations. For any inclusion of open subsets* $i : V \subseteq W$*, denote by* $i_* : \Omega_c(V) \to \Omega_c(W)$ *the map that assigns to* $\beta \in \Omega_c(V)$ *its extension by zero to* W*, called* the pushforward of β. *Then, the following diagrams*

$$\begin{array}{ccccccc} \Omega_c(V)[d_{\boldsymbol{M}}] & \xhookrightarrow{I\!\!D'(V)} & \Omega(V)^\vee & \qquad & H_c(V)[d_{\boldsymbol{M}}] & \xhookrightarrow{\mathscr{D}'(V)} & H(V)^\vee \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle (i^*)^\vee} & & {\scriptstyle H_c(i_*)}\downarrow & & \downarrow{\scriptstyle H(i^*)^\vee} \\ \Omega_c(W)[d_{\boldsymbol{M}}] & \xhookrightarrow{I\!\!D'(W)} & \Omega(W)^\vee & & H_c(W)[d_{\boldsymbol{M}}] & \xhookrightarrow{\mathscr{D}'(W)} & H(W)^\vee \end{array}$$

are commutative, i.e. (i^*,i_*) *is a Poincaré adjoint pair (1.3.3).*

b) *For any map* $f : \boldsymbol{M} \to \boldsymbol{N}$ *between oriented manifolds, one has the diagram*

$$\begin{array}{ccc} H_c(\boldsymbol{M})[d_{\boldsymbol{M}}] & \xhookrightarrow{\mathscr{D}'(\boldsymbol{M})} & H(\boldsymbol{M})^\vee \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle H(f^*)^\vee} \\ H_c(\boldsymbol{N})[d_{\boldsymbol{N}}] & \xhookrightarrow{\mathscr{D}'(\boldsymbol{N})} & H(\boldsymbol{N})^\vee \end{array} \qquad (\mathscr{D}')$$

where $H(f^*)^\vee\big(\mathrm{Im}(\mathscr{D}'(\boldsymbol{M}))\big) \subseteq \mathrm{Im}(\mathscr{D}'(\boldsymbol{N}))$*, so that there exists one and only one morphism of graded spaces*

$$f_* : H_c(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow H_c(\boldsymbol{N})[d_{\boldsymbol{N}}] \qquad (\diamond)$$

called the Gysin morphism associated with f*, making* $(\mathscr{D}')$ *commutative, i.e.* $(H(f^*),f_*)$ *is a Poincaré adjoint pair in cohomology, which means that, for any* $[\alpha] \in H(\boldsymbol{N})$ *and* $[\beta] \in H_c(\boldsymbol{M})$*, the equation in* X*,*

$$\int_{\boldsymbol{M}} f^*[\alpha] \cup [\beta] = \int_{\boldsymbol{N}} [\alpha] \cup X, \qquad (\diamond\diamond)$$

admits one and only one solution in $H_c(\boldsymbol{N})$*, namely* $X = f_*[\beta]$.

Furthermore, f_* *in* $(\diamond)$ *is a morphism of* $H(\boldsymbol{N})$*-modules, i.e. the equality, called the* projection formula,

$$f_*\big(f^*[\alpha] \cup [\beta]\big) = [\alpha] \cup f_*([\beta]) \qquad (\diamond\diamond\diamond)$$

holds for all $[\alpha] \in H(\boldsymbol{N})$ *and* $[\beta] \in H_c(\boldsymbol{M})$.

c) *The correspondence*

$$(\_)_* : \boldsymbol{Man}^{\mathrm{or}} \rightsquigarrow \mathrm{GV}(\mathbb{R}) \qquad \text{with} \qquad \begin{cases} \boldsymbol{M} \rightsquigarrow \boldsymbol{M}_* := H_c(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ f \rightsquigarrow f_* \end{cases}$$

is a covariant functor. It will be called the Gysin functor.

d) *If* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *are oriented of finite type, then* $f^* : H(\boldsymbol{N}) \to H(\boldsymbol{M})$ *is an isomorphism if and only if the Gysin morphism* $f_* : H_c(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_c(\boldsymbol{N})[d_{\boldsymbol{N}}]$ *is also an isomorphism.*

Proof. (a) The commutativity results from the equality

$$\int_V \alpha|_V \wedge \beta = \int_W \alpha \wedge i_*\beta$$

for $\alpha \in \Omega(W)$ and $\beta \in \Omega_c(V)$, which is evident since the support of $\alpha \wedge i_*\beta$ is contained in V.

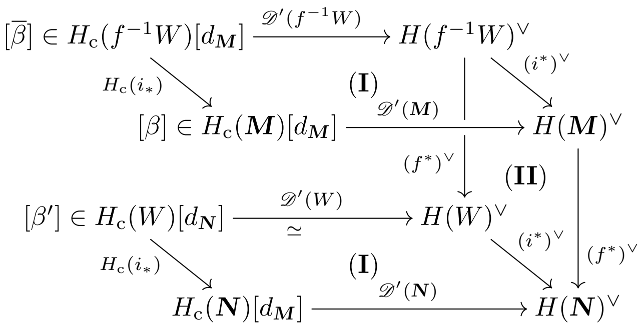
(b) One must verify that, given $[\beta] \in H_c(\boldsymbol{M})$, the linear form

$$[\alpha] \in H(\boldsymbol{N}) \mapsto \int_{\boldsymbol{M}} f^*[\alpha] \cup [\beta]$$

is of the form

$$[\alpha] \in H(\boldsymbol{N}) \mapsto \int_{\boldsymbol{N}} [\alpha] \cup [\beta']$$

for some $[\beta'] \in H_c(\boldsymbol{N})$. Now, thanks to proposition 1.4.4, there exists an open subset $W \in \boldsymbol{N}$ of finite type such that $f^{-1}W$ contains the support of β, denoted $\overline{\beta} := \beta_{|f^{-1}W}$. One then has the following commutative diagram:



where subdiagrams (**I**) are commutative after (c) and the commutativity of (**II**) is just functoriality of pullbacks.

Following the arrows, we see that

$$\begin{aligned}(f^*)^\vee \circ \mathscr{D}'(\boldsymbol{M})([\beta]) &= (i^*)^\vee \circ (f^*)^\vee \circ \mathscr{D}'(f^{-1}W)([\overline{\beta}]) \\ &= (i^*)^\vee \circ \mathscr{D}'(W)([\beta']) \\ &= \mathscr{D}'(\boldsymbol{N}) \circ H_c(i_*)([\beta'])\end{aligned}$$

where $[\beta'] \in H_c(W)[d_{\boldsymbol{N}}]$ verifies

$$\mathscr{D}'(W)([\beta']) = (f^*)^\vee \circ \mathscr{D}'(f^{-1}W)([\overline{\beta}])$$

which is possible since $\mathscr{D}'(W)$ is **surjective** as W is of finite type !

The statement about the equation $(\diamond\diamond)$ is clear and implies formally the projection formula since

$$\begin{aligned}\int_{\boldsymbol{N}} [\omega] \cup f_*\big(f^*[\alpha] \cup [\beta]\big) &= \int_{\boldsymbol{M}} f^*[\omega] \cup f^*[\alpha] \cup [\beta] \\ &= \int_{\boldsymbol{M}} f^*([\omega] \cup [\alpha]) \cup [\beta] = \int_{\boldsymbol{N}} [\omega] \cup [\alpha] \cup f_*[\beta].\end{aligned}$$

Finally, (c) is trivial since $\mathscr{D}'$ is bijective over its image, and (d) is clear. □

1.7.5. Remark. It is important to note that the main ingredients in the proof are (i) the Poincaré pairings, (ii) Poincaré duality, (iii) the ascending chain property (1.4.3). In later sections of these notes we will show that all these ingredients exist also in the equivariant setting so that the last theorem and its proof will extend *verbatim* to $\boldsymbol{G}$-manifolds and $\boldsymbol{G}$-equivariant cohomology.

1.7.6. Exercise. Let $f:\boldsymbol{M}\to\boldsymbol{N}$ be a map between oriented manifolds. Show that the dual of the Gysin morphism $f_*:H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}]\to H_{\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}]$ coincides, via Poincaré duality, with the *pullback morphism* $f^*:H(\boldsymbol{N})\to H(\boldsymbol{M})$.

1.7.7. The Image of $\mathscr{D}'(\boldsymbol{M})$. The next proposition will be used when extending the Gysin functor to the equivariant context. It gives a description of the image of $\mathscr{D}'(\boldsymbol{M})$ in terms of ascending chains of open finite type subsets of $\boldsymbol{M}$, which was the main reason why we proved that such coverings always exist (see 1.4.4).

1.7.8. Proposition. *Let $\mathscr{U}$ be a filtrant open covering* ([3]) *of a manifold $\boldsymbol{M}$.*

a) *Let $i:V\subseteq W$ denote an inclusion of open subsets of $\boldsymbol{M}$.*

The map $i_*:\Omega_{\mathrm{c}}(V)\subseteq\Omega_{\mathrm{c}}(W)$, that assigns to $\beta\in\Omega_{\mathrm{c}}(V)$ the differential form $i_*(\beta)\in\Omega_{\mathrm{c}}(W)$ equal to β over V and 0 otherwise, is a well-defined morphism of complexes inducing in cohomology the morphism of graded spaces $H_{\mathrm{c}}(i_*):H_{\mathrm{c}}(V)\to H_{\mathrm{c}}(W)$. One has also the morphism of complexes $i^*:\Omega(W)\to\Omega(V)$ that restricts a differential form of W to V, and the corresponding morphism of graded spaces $H(i^*):H(W)\to H(V)$.

These constructions, applied to the elements of $\mathscr{U}$, give rise to the inductive systems $\{\Omega_{\mathrm{c}}(U)\}_{U\in\mathscr{U}}$ and $\{H_{\mathrm{c}}(U)\}_{U\in\mathscr{U}}$, and to the projective systems $\{\Omega(U)\}_{U\in\mathscr{U}}$ and $\{H(U)\}_{U\in\mathscr{U}}$, whence the canonical maps

$$\nu:\varinjlim_{U\in\mathscr{U}}\Omega_{\mathrm{c}}(U)\to\Omega_{\mathrm{c}}(\boldsymbol{M})\quad\text{and}\quad H(\nu):\varinjlim_{U\in\mathscr{U}}H_{\mathrm{c}}(U)\to H_{\mathrm{c}}(\boldsymbol{M}),$$

$$\mu:\Omega(\boldsymbol{M})\to\varprojlim_{U\in\mathscr{U}}\Omega(U)\quad\text{and}\quad H(\mu):H(\boldsymbol{M})\to\varprojlim_{U\in\mathscr{U}}H(U).$$

All these maps are bijective.

b) *Suppose $\boldsymbol{M}$ is oriented, then the map*

$$\boxed{\begin{array}{rcl}\mathbb{D}'(\mathscr{U}):\big(\Omega_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}],\mathbf{d}\big)&\longrightarrow&\varinjlim_{U\in\mathscr{U}}\big(\Omega(U)^\vee,-\mathbf{d}\big)\\ \beta&\mapsto&\Big(\alpha\longmapsto\displaystyle\int_{\boldsymbol{M}}\alpha\wedge\beta\Big)\end{array}}$$

is a well-defined morphism of complexes inducing in cohomology the map

$$\mathscr{D}'(\mathscr{U}):H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}]\to\varinjlim_{U\in\mathscr{U}}H(U)^\vee$$

c) *Suppose further that each $U\in\mathscr{U}$ is of finite type. Then $\mathbb{D}'(\mathscr{U})$ is a quasi-isomorphism, and one has*

$$\mathrm{Im}(\mathscr{D}'(\boldsymbol{M}))=\varinjlim_{U\in\mathscr{U}}H(U)^\vee\subseteq H(\boldsymbol{M})^\vee\qquad(\diamond)$$

Moreover, the adjoint $\mathscr{D}'(\mathscr{U})^\vee$ canonically identifies with $\mathscr{D}(\boldsymbol{M})$; more precisely, the following diagram is commutative:

$$\begin{array}{ccc}\varprojlim_{U\in\mathscr{U}}H(U)[d_{\boldsymbol{M}}]=(\varinjlim_{U\in\mathscr{U}}H(U)^\vee)^\vee[d_{\boldsymbol{M}}]&\xrightarrow{\mathscr{D}'(\mathscr{U})^\vee}&H_{\mathrm{c}}(\boldsymbol{M})^\vee\\ \uparrow\simeq&&\|\\ H(\boldsymbol{M})[d_{\boldsymbol{M}}]&\xrightarrow{\quad\mathscr{D}(\boldsymbol{M})\quad}&H_{\mathrm{c}}(\boldsymbol{M})^\vee\end{array}$$

---

[3]We recall that $\mathscr{U}=\{U_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$ is said *filtrant* whenever for all $U_1,U_2\in\mathscr{U}$ there exists $U_3\in\mathscr{U}$ such that $(U_1\cup U_2)\subseteq U_3$.

Proof. (a) The map $\nu : \varinjlim_{U\in\mathscr{U}} \Omega_c^*(U) \to \Omega_c^*(\boldsymbol{M})$ is injective since it's the limit of a filtrant inductive system of injective maps. The image of ν is the union of $\Omega_c^*(U)$ for the same reason. Now, if $\omega \in \Omega_c^*(\boldsymbol{M})$, its support, being compact, is contained in some $U \in \mathscr{U}$ so that ω is the pushforward of $\omega_{|U} \in \Omega_c^*(U)$. This justifies the equality $\Omega_c^*(\boldsymbol{M}) = \bigcup_{U\in\mathscr{U}} \Omega_c^*(U)$ and proves that ν is surjective. Standard arguments on the homology of filtrant inductive systems of complexes prove that $H(\nu)$ is bijective.

The map $\mu : \Omega(\boldsymbol{M}) \to \varprojlim_{U\in\mathscr{U}} \Omega(U)$ is injective, since a differential form is null if and only if it is locally null. To see it is also surjective, let $\{\alpha_U \in \Omega(U)\}_{U\in\mathscr{U}}$ be a given projective system of differential forms, and note that for any $x \in \boldsymbol{M}$, the element $\tilde{\alpha}(x) := \alpha_U(x)$ is well defined since if $x \in U_1 \in \mathscr{U}$ and $x \in U_2 \in \mathscr{U}$, one chooses $U_3 \in \mathscr{U}$ s.t. $U_1 \cup U_2 \subseteq U_3$, in which case $\alpha_{U_1}(x) = \alpha_{U_3}(x) = \alpha_{U_2}(x)$. Likewise, one verifies the differentiability of $\tilde{\alpha}$. It is clear that $\tilde{\alpha}_{|U} = \alpha_U$, which ends the proof that μ is surjective.

It remains only to justify why $H(\mu)$ is bijective. This is immediate when $\boldsymbol{M}$ is orientable, since $H(\mu)$ is then just the Poincaré dual of $H_c(\nu)$ which has already been shown to be bijective. Otherwise, when $\boldsymbol{M}$ is not orientable, we lift $\mathscr{U}$ to the orientation manifold $\tilde{\boldsymbol{M}}$ associated with $\boldsymbol{M}$ through the canonical $\mathbb{Z}/2\mathbb{Z}$-covering $p : \tilde{\boldsymbol{M}} \twoheadrightarrow \boldsymbol{M}$, setting therefore $\tilde{\mathscr{U}} := \{\tilde{U} := p^{-1}(U) \mid U \in \mathscr{U}\}$. As $\tilde{\boldsymbol{M}}$ is orientable, the map $H(\tilde{\boldsymbol{M}}) \to \varprojlim_{U\in\mathscr{U}} H(\tilde{U})$ is now bijective, and because this map is also compatible with the reversing-orientation action of $\mathbb{Z}/2\mathbb{Z}$, it induces a bijection between invariants subspaces $H(\tilde{\boldsymbol{M}})^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow[\simeq]{} \varprojlim_{U\in\mathscr{U}} H(\tilde{U})^{\mathbb{Z}/2\mathbb{Z}}$, and one concludes since $H(U) = H(\tilde{U})^{\mathbb{Z}/2\mathbb{Z}}$.

(b) Endow each $U \in \mathscr{U}$ with the induced orientation. Taking the inductive limit of the maps $\mathbb{D}'(U) : H_c(U)[d_{\boldsymbol{M}}] \to H(U)^\vee$ and applying (a) one sees immediately that $\mathbb{D}(\mathscr{U}) = \varinjlim_{U\in\mathscr{U}} \mathbb{D}'(U)$.

(c) By 1.7.2 the maps $\mathbb{D}'(U) : H_c(U)[d_{\boldsymbol{M}}] \to H(U)^\vee$ are quasi-isomorphisms for each $U \in \mathscr{U}$, hence $\mathbb{D}(\mathscr{U}) = \varinjlim_{U\in\mathscr{U}} \mathbb{D}'(U)$ is also a quasi-isomorphism since $\mathscr{U}$ is filtrant. The rest of the statement is then clear by duality. □

1.8. The Gysin Functor for Proper Maps

In this section, the Gysin morphism for compact supports $f_* : H_c(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_c(\boldsymbol{N})[d_{\boldsymbol{N}}]$ will be extended to arbitrary supports $f_! : H(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H(\boldsymbol{N})[d_{\boldsymbol{N}}]$ when $f : \boldsymbol{M} \to \boldsymbol{N}$ is a **proper** map. As we will see this case is much simpler than the general one as it results immediately from Poincaré duality.

When $f : \boldsymbol{M} \to \boldsymbol{N}$ is proper, the pullback $f^* : \Omega(\boldsymbol{N}) \to \Omega(\boldsymbol{M})$ respects compact supports and induces a morphism of complexes $f^* : \Omega_c(\boldsymbol{N}) \to \Omega_c(\boldsymbol{M})$, giving rise to the *covariant* functor from $\boldsymbol{Man}_\pi$ to $\mathrm{Vec}(\mathbb{K})$

$$\boldsymbol{M} \rightsquigarrow H_c(\boldsymbol{M})^\vee, \qquad f \rightsquigarrow H_c(f^*)^\vee.$$

When $\boldsymbol{M}$ is oriented, $\mathbb{D}'(\boldsymbol{M})$ may be extended from $\Omega_c(\boldsymbol{M})$ to $\Omega(\boldsymbol{M})$ by setting (see 1.7.1)

$$\mathbb{D}'(\boldsymbol{M})(\alpha) = \Big(\beta \mapsto \int_{\boldsymbol{M}} \beta \wedge \alpha\Big), \quad \forall \alpha \in \Omega(\boldsymbol{M}), \quad \forall \beta \in \Omega_c(\boldsymbol{M}),$$

so that the diagram

$$\begin{array}{ccc} \Omega(\boldsymbol{M}) & \xrightarrow[(\simeq)]{I\!\!D'(\boldsymbol{M})} & \Omega_{\mathrm{c}}(\boldsymbol{M})^{\vee} \\ {\scriptstyle\subseteq}\uparrow & & \uparrow\!\!\uparrow \\ \Omega_{\mathrm{c}}(\boldsymbol{M}) & \xrightarrow{I\!\!D'(\boldsymbol{M})} & \Omega(\boldsymbol{M})^{\vee} \end{array}$$

is commutative, and, moreover, with its first line a *quasi-isomorphism* as it is simply the Poincaré duality map $I\!\!D(\boldsymbol{M})$ up to ± 1.

1.8.1. Definition. If $f:\boldsymbol{M}\to\boldsymbol{N}$ is a proper map between oriented manifolds, the *Gysin morphism associated with* f is the map $f_!: H(\boldsymbol{M})[d_{\boldsymbol{M}}]\to H(\boldsymbol{N}[d_{\boldsymbol{N}}])$ making commutative the diagram

$$\begin{array}{ccc} H(\boldsymbol{M})[d_{\boldsymbol{M}}] & \xrightarrow[\simeq]{\mathscr{D}'(\boldsymbol{M})} & H_{\mathrm{c}}(\boldsymbol{M})^{\vee} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle H_{\mathrm{c}}(f^*)^{\vee}} \\ H(\boldsymbol{N})[d_{\boldsymbol{N}}] & \xrightarrow[\simeq]{\mathscr{D}'(\boldsymbol{N})} & H_{\mathrm{c}}(\boldsymbol{N})^{\vee} \end{array}$$

The next theorem, analog to 1.7.4 and almost immediate, is left as an exercise.

1.8.2. Theorem and definitions

a) *For $\beta\in H_{\mathrm{c}}(\boldsymbol{N})$ and $\alpha\in H(\boldsymbol{M})$, the equation in X,*

$$\int_{\boldsymbol{M}} f^*[\beta]\cup[\alpha] = \int_{\boldsymbol{N}}[\beta]\cup X, \qquad (\star\star)$$

admits one and only one solution in $H(\boldsymbol{N})$, namely $X=f_![\alpha]$.

Furthermore, $f_!$ is a morphism of $H_{\mathrm{c}}(\boldsymbol{N})$-modules, i.e. the equality, called the projection formula for proper maps,

$$f_!\big(f^*[\beta]\cup[\alpha]\big) = [\beta]\cup f_![\alpha] \qquad (\star\star\star)$$

holds for all $[\beta]\in H_{\mathrm{c}}(\boldsymbol{N})$ and $[\alpha]\in H(\boldsymbol{M})$.

b) *The following correspondence is a covariant functor:*

$$f_! : \boldsymbol{Man}^{\mathrm{or}}_{\pi} \rightsquigarrow \mathrm{GV}(\mathbb{R}) \qquad \textit{with} \qquad \begin{cases} \boldsymbol{M}\rightsquigarrow \boldsymbol{M}_! := H(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ f \rightsquigarrow f_! \end{cases}$$

We will refer to it as the Gysin functor for proper maps.

c) *The pullback $f^*: H_{\mathrm{c}}(\boldsymbol{N})\to H_{\mathrm{c}}(\boldsymbol{M})$ is an isomorphism if and only if the Gysin morphism $f_!: H(\boldsymbol{M})[d_{\boldsymbol{M}}]\to H(\boldsymbol{N})[d_{\boldsymbol{N}}]$ is also an isomorphism.*

d) *The natural map $\phi(\_): H_{\mathrm{c}}(\_)[d_\_]\to H(\_)[d_\_]$ (1.2.3) is a homomorphism of Gysin functors $(\_)_*\to(\_)_!$ over the category $\boldsymbol{Man}^{\mathrm{or}}_{\pi}$, i.e. the diagrams*

$$\begin{array}{ccc} H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] & \xrightarrow{\phi(\boldsymbol{M})} & H(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_!} \\ H_{\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}] & \xrightarrow{\phi(\boldsymbol{N})} & H(\boldsymbol{N})[d_{\boldsymbol{N}}] \end{array}$$

are natural and commutative.

1.9. Principal Examples of Gysin Morphisms

1.9.1. Universal Property of the Gysin Morphism. This property is the statement (b) in theorem 1.7.4, which says that if $f : \boldsymbol{M} \to \boldsymbol{N}$ is a map between oriented manifolds, then for each $[\beta] \in H_{\mathrm{c}}(\boldsymbol{M})$, the element $f_*([\beta]) \in H_{\mathrm{c}}(\boldsymbol{N})$ is determined by the equality, for all $[\alpha] \in H(\boldsymbol{N})$,

$$\boxed{\int_{\boldsymbol{M}} f^*[\alpha] \cup [\beta] = \int_{\boldsymbol{N}} [\alpha] \cup f_*[\beta]} \tag{$\diamond\diamond$}$$

The pair (f_*, f^*) is a Poincaré *adjoint pair* in cohomology (1.3.3).

1.9.2. Constant Map. Let $\boldsymbol{M}$ be oriented and denote by $c_{\boldsymbol{M}} : \boldsymbol{M} \to \{\bullet\}$ the constant map. One applies ($\diamond\diamond$) taking $\alpha = 1$:

$$c_{\boldsymbol{M}*}([\beta]) = \int_{\{\bullet\}} 1 \cup c_{\boldsymbol{M}*}[\beta] = \int_{\boldsymbol{M}} \beta.$$

so that the Gysin morphism $c_{\boldsymbol{M}*} : H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\mathrm{c}}(\{\bullet\}) = \mathbb{R}$ is the integration map, Poincaré dual of the graded algebra homomorphism $c^*_{\boldsymbol{M}} : \mathbb{R} \to H(\boldsymbol{M})$.

1.9.3. Exercise. Show that $c^*_{\boldsymbol{M}} : \Omega(\{\bullet\}) \to \Omega(\boldsymbol{M})$ admits a right Poincaré adjoint at the complex level, i.e. $c_{\boldsymbol{M}*} : \Omega_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \Omega(\{\bullet\})$.

1.9.4. Open Embedding. Let $\boldsymbol{M}$ be oriented. Given an open embedding $i : U \subseteq \boldsymbol{M}$, endow U with the induced orientation. For any $\beta \in \Omega_{\mathrm{c}}(U)$ one has the tautological equality:

$$\int_U \alpha|_U \wedge \beta = \int_{\boldsymbol{M}} \alpha \wedge i_*\beta \tag{$*$}$$

where $i_*\beta \in \Omega_{\mathrm{c}}(\boldsymbol{M})$ denotes the extension by zero of β. The Gysin morphism $i_* : H_{\mathrm{c}}(U)[d_U] \to H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}]$ is therefore the pushforward $i_* = H_{\mathrm{c}}(i_*)[d_{\boldsymbol{M}}]$ (see 1.7.8-(a)). Note also that the equality ($*$) shows that the pair (i^*, i_*) is a Poincaré adjoint pair (1.3.3).

1.9.5. Locally Trivial Fibration. Let $\pi : \boldsymbol{E} \to \boldsymbol{B}$ be a locally trivial fibration with base space $\boldsymbol{B}$ (connected for simplicity) and total space $\boldsymbol{E}$ both assumed oriented, with fiber $\boldsymbol{F}$ of dimension $d_{\boldsymbol{F}}$ endowed with the induced orientation. Under these assumptions one has the *operation of integration along* $\boldsymbol{F}$ (see [BT] **I**§6 pp. 61-63) which is a morphism of complexes

$$\int_{\boldsymbol{F}} : \Omega_{\mathrm{c}}(\boldsymbol{E})[d_{\boldsymbol{F}}] \to \Omega_{\mathrm{c}}(\boldsymbol{B})$$

satisfying

$$\int_{\boldsymbol{E}} \pi^*\alpha \wedge \beta = \int_{\boldsymbol{B}} \Big(\alpha \wedge \int_{\boldsymbol{F}} \beta\Big), \tag{$*$}$$

so that after the adjunction property ($\diamond\diamond$), one has $\pi_* = \int_{\boldsymbol{F}}[d_{\boldsymbol{B}}]$ and the Gysin morphism is the shift of integration along fibers. Note again that ($*$) shows that the pair $(\pi^*, \int_{\boldsymbol{F}}[d_{\boldsymbol{B}}])$ is a Poincaré adjoint pair.

1.9.6. Proposition. *Let* $(\pi, \boldsymbol{V}, \boldsymbol{B})$ *and* $(\pi, \boldsymbol{V}', \boldsymbol{B}')$ *be two oriented locally trivial fibrations. Let* $g : \boldsymbol{B}' \to \boldsymbol{B}$ *be a* ***proper*** *map and assume the following diagram cartesian:*

$$\begin{array}{ccc} \boldsymbol{V}' & \xrightarrow{g} & \boldsymbol{V} \\ {\scriptstyle\pi}\downarrow & \Box & \downarrow{\scriptstyle\pi} \\ \boldsymbol{B}' & \xrightarrow{g} & \boldsymbol{B} \end{array}$$

i.e. $\boldsymbol{V}' = \big\{(b', v) \in \boldsymbol{B}' \times \boldsymbol{V} \;\big|\; g(b') = \pi(v)\big\}$. *Then*

$$\begin{cases} g^* \circ \pi_* = \pi_* \circ g^* : H_c(\boldsymbol{V}) \to H_c(\boldsymbol{B}') \\ \pi^* \circ g_! = g_! \circ \pi^* : H(\boldsymbol{B}') \to H(\boldsymbol{V}) \end{cases}$$

Hint. By adjointness, the first equality is equivalent to the second. The first equality follows from the equality for differential forms $g^*\big(\int_{\boldsymbol{F}} \omega\big) = \int_{\boldsymbol{F}} g^*(\omega)$ for all $\omega \in \Omega_c(\boldsymbol{V})$, that may be verified locally in $\boldsymbol{B}'$ (*loc.cit.*). ⊡

1.9.7. Zero Section of a Vector Bundle. Let $(\pi, \boldsymbol{V}, \boldsymbol{B})$ be a vector bundle and assume $\boldsymbol{B}$ and $\boldsymbol{V}$ oriented. The *zero section map* $\sigma : \boldsymbol{B} \to \boldsymbol{V}$ is a closed embedding, hence proper, so that we have the Gysin morphism for proper maps $\sigma_! : H(\boldsymbol{B}) \to H(\boldsymbol{V})$. By the adjunction property $(\star\star)$ (see 1.8.2-(a)), one has for all $\beta \in H_c(\boldsymbol{V})$ and $\alpha \in H(\boldsymbol{B})$

$$\begin{aligned} \int_{\boldsymbol{V}} [\beta] \cup \sigma_!([\alpha]) &= \int_{\boldsymbol{B}} \sigma^*[\beta] \cup [\alpha] = \int_{\boldsymbol{B}} \sigma^*[\beta] \cup \sigma^*(\pi^*[\alpha]) \\ &= \int_{\boldsymbol{B}} \sigma^*([\beta] \cup \pi^*[\alpha]) \cup 1 = \int_{\boldsymbol{V}} [\beta] \cup \pi^*[\alpha] \cup \sigma_!(1) \end{aligned} \tag{$\diamond$}$$

where $\Phi := \sigma_!(1)$ is *the Thom class of the pair* $(\boldsymbol{B}, \boldsymbol{V})$. The Gysin morphism associated with the zero section of a fiber bundle

$$\sigma_! : H(\boldsymbol{B})[d_{\boldsymbol{B}}] \to H(\boldsymbol{V})[d_{\boldsymbol{V}}] \tag{!}$$

is then the multiplication by the Thom class

$$\sigma_!([\alpha]) = \pi^*[\alpha] \cup \Phi. \tag{!!}$$

Finally, note that $\sigma_!$ is generally not an isomorphism, since it identifies, via Poincaré duality, with the dual of the proper pullback $\sigma^* : H_c(\boldsymbol{V}) \to H_c(\boldsymbol{B})$ (see 1.8.1) which is generally not an isomorphism ([4]).

It can be seen ([BT] §I.6 p. 64) that if $\alpha \in H_c(\boldsymbol{B})$, then $\pi^*[\alpha] \cup \Phi$ naturally belongs to $H_c(\boldsymbol{V})$ so that the Gysin morphism

$$\sigma_* : H_c(\boldsymbol{B})[d_{\boldsymbol{B}}] \to H_c(\boldsymbol{V})[d_{\boldsymbol{V}}] \tag{$\star$}$$

is given by the same equality (!!),

$$\sigma_*([\beta]) = \pi^*[\beta] \wedge \Phi. \tag{$\star\star$}$$

On the other hand, the Poincaré lemma for vector bundles asserts that the pullback $\pi^* : H(\boldsymbol{B}) \to H(\boldsymbol{V})$ is an isomorphism and this implies, via Poincaré duality (see 1.7.6), that $\pi_* : H_c(\boldsymbol{V})[d_{\boldsymbol{V}}] \to H_c(\boldsymbol{B})[d_{\boldsymbol{B}}]$ is also an isomorphism. Now, by functoriality, one has $\pi_* \circ \sigma_* = \mathrm{id}$, so that σ_* is also an isomorphism. This isomorphism is *the Thom isomorphism*.

[4] For example, if $\boldsymbol{B}$ is compact, $H_c(\boldsymbol{B}) = H(\boldsymbol{B}) = H(\boldsymbol{V})$ and σ^* would give a graded isomorphism $H_c(\boldsymbol{V}) \simeq H(\boldsymbol{V})$, and by Poincaré duality $H^0(\boldsymbol{V}) \simeq H^{d_{\boldsymbol{V}}}(\boldsymbol{V})$, which impossible if $\boldsymbol{V}$ is a vector bundle of positive dimension over $\boldsymbol{B}$.

1.9.8. Proposition. *Let $(\pi, \boldsymbol{V}, \boldsymbol{B})$ and $(\pi, \boldsymbol{V}', \boldsymbol{B}')$ be two oriented vector bundles and assume the cartesian diagram in 1.9.6 with $g : \boldsymbol{B}' \to \boldsymbol{B}$* **proper**. *Denote by $\sigma : \boldsymbol{B} \to \boldsymbol{V}$ and $\sigma : \boldsymbol{B}' \to \boldsymbol{V}'$ the zero section maps. The diagram*

$$\begin{array}{ccc} \boldsymbol{B}' & \xrightarrow{g} & \boldsymbol{B} \\ {\scriptstyle\sigma}\downarrow & \square & \downarrow{\scriptstyle\sigma} \\ \boldsymbol{V}' & \xrightarrow{g} & \boldsymbol{V} \end{array}$$

is cartesian and the equalities $\begin{cases} g^* \circ \sigma_* = \sigma_* \circ g^* : H_c(\boldsymbol{B}) \to H_c(\boldsymbol{V}') \\ \sigma^* \circ g_! = g_! \circ \sigma^* : H(\boldsymbol{V}') \to H(\boldsymbol{B}) \end{cases}$ *hold.*

Hint. It is a corollary of 1.9.6 since σ_* is the inverse of π_*. ⊡

1.10. Constructions of Gysin Morphisms

In this last preliminary section we summarize the steps in the construction of the Gysin morphisms.

1.10.1. The Proper Case. Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be a **proper** map of oriented manifolds. To $\alpha \in \Omega(\boldsymbol{M})$ we assign the linear form on $\Omega_c(\boldsymbol{N})$ defined by $I\!D'(f)(\alpha) : \beta \mapsto \int_{\boldsymbol{M}} f^*\beta \wedge \alpha$. In this way we obtain diagram

$$\begin{array}{ccc} \Omega(\boldsymbol{M})[d_{\boldsymbol{M}}] & \overset{f_!}{\dashrightarrow} & \Omega(\boldsymbol{N})[d_{\boldsymbol{N}}] \\ & \searrow^{I\!D'(f)} \quad \oplus & \downarrow{\scriptstyle I\!D'(\boldsymbol{N})\ \text{(quasi-iso)}} \\ & & \Omega_c(\boldsymbol{N})^\vee \end{array}$$

which may be closed in cohomology, since $I\!D'(\boldsymbol{N})$ is a quasi-isomorphism. Note that the closing arrow $f_!$, the Gysin morphism for proper maps, in general exists *only* at the cohomology level.

1.10.2. The General Case. Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be a map of oriented manifolds. To $\beta \in \Omega_c(\boldsymbol{M})$ we assign the linear form on $\Omega(\boldsymbol{N})$ defined by $I\!D'(f)(\beta) : \alpha \mapsto \int_{\boldsymbol{M}} f^*\alpha \wedge \beta$. In this way we obtain the diagram

$$\begin{array}{ccc} \Omega_c(\boldsymbol{M})[d_{\boldsymbol{M}}] & \overset{f_*}{\dashrightarrow} & \Omega_c(\boldsymbol{N})[d_{\boldsymbol{N}}] \\ & \searrow^{I\!D'(f)} \quad \oplus & \downarrow{\scriptstyle I\!D'(\boldsymbol{N})\ \left(\text{quasi-iso if } \boldsymbol{N} \text{ is of finite type}\right)} \\ & & \Omega(\boldsymbol{N})^\vee \end{array}$$

which may be closed in cohomology (as in the proper case), when **$\boldsymbol{N}$ is of finite type**, since then $I\!D'(\boldsymbol{N})$ is a quasi-isomorphism (1.7.2-(b)).

When $\boldsymbol{N}$ is not of finite type, one fixes any filtrant covering $\mathscr{U}$ of $\boldsymbol{N}$ made up of open finite type subsets of $\boldsymbol{N}$ (see 1.4.4), and replaces $I\!D'(\boldsymbol{N})$ with $I\!D'(\mathscr{U})$. In this way, we get (see 1.7.8-(b,c)), the following diagram:

$$\begin{array}{ccccc} \Omega_c(\boldsymbol{M})[d_{\boldsymbol{M}}] & \overset{f_*}{\dashrightarrow} & \Omega_c(\boldsymbol{N})[d_{\boldsymbol{N}}] & \overset{=}{\cdots\cdots} & \Omega_c(\boldsymbol{N})[d_{\boldsymbol{N}}] \\ & \searrow^{I\!D'(f,\mathscr{U})} \quad \oplus & \downarrow{\scriptstyle I\!D'(\mathscr{U})\ \text{(quasi-iso)}} & & \vdots{\scriptstyle I\!D'(\boldsymbol{N})} \\ & & \varinjlim_{U \in \mathscr{U}} \Omega(U)^\vee & \underset{\subseteq}{\dashrightarrow} & \Omega(\boldsymbol{N})^\vee \end{array}$$

where $I\!D'(f,\mathscr{U})$ is defined as follows. For $\beta \in \Omega_c(\boldsymbol{M})$ denote by $|\beta|$ its support and by $\mathscr{U}_\beta \subseteq \mathscr{U}$ the system consisting of $U \in \mathscr{U}$ s.t. $|\beta| \subseteq f^{-1}U$. One has a natural map $\varinjlim_{\mathscr{U}_\beta} \Omega(U)^\vee \to \varinjlim_{\mathscr{U}} \Omega(U)^\vee$ (which is in fact is bijective). Now, for every $U \in \mathscr{U}_\beta$ the linear map $\left(\int_{\boldsymbol{M}} f^*(\_) \wedge \beta\right) : \Omega(U) \to \mathbb{R}$, is well defined and is compatible with restriction, so that it defines an element of $\varinjlim_{\mathscr{U}_\beta} \Omega(U)^\vee$, and then of $\varinjlim_{\mathscr{U}} \Omega(U)^\vee$. This element is $I\!D'(f,\mathscr{U})(\beta)$ by definition.

The closing arrow f_*, *the Gysin morphism associated with a general map* f, is then defined in cohomology as the composition $\mathscr{D}'(\mathscr{U})^{-1} \circ H(I\!D(f,\mathscr{U}))$.

1.10.3. Remark. In all cases, the Gysin morphism appears as the composition of a morphism of complexes with the "inverse" of a quasi-isomorphism, which obviously is possible in cohomology but also in the *derived category of complexes* since this is its main property, i.e. a morphism in derived category is an isomorphism if and only if it induces an isomorphism in cohomology. Gysin morphisms are well defined morphisms of the derived category of complexes of vector spaces.

1.11. Exercises

1.11.1. Gysin Long Exact Sequence. Let $i : \boldsymbol{F} \subseteq \boldsymbol{M}$ be a closed embedding of oriented manifolds. Assume $\boldsymbol{F}$ compact, for simplicity. Put $\boldsymbol{U} := \boldsymbol{M}\backslash\boldsymbol{F}$ and $j : \boldsymbol{U} \subseteq \boldsymbol{M}$ the canonical injection.

a) i) Let $\mathscr{F}$ denote the set of open neighborhood of $\boldsymbol{F}$. Restriction morphisms $R^{\boldsymbol{W}}_{\boldsymbol{V}} : \Omega(\boldsymbol{W}) \to \Omega(\boldsymbol{V})$ for all $\boldsymbol{W} \supseteq \boldsymbol{V} \supseteq \boldsymbol{F}$, give rise to a filtrant inductive system $\{R^{\boldsymbol{W}}_{\boldsymbol{V}} \mid \boldsymbol{W} \supseteq \boldsymbol{V} \text{ in } \mathscr{F}\}$ and a canonical morphism of complexes $R^{\boldsymbol{M}}_{\mathscr{F}} : \Omega(\boldsymbol{M}) \to \varinjlim_{\boldsymbol{V}\in\mathscr{F}} \Omega(\boldsymbol{V})$. Show that the short sequence

$$\boldsymbol{0} \to \Omega_c(\boldsymbol{U}) \xrightarrow{j_*} \Omega_c(\boldsymbol{M}) \xrightarrow{R^{\boldsymbol{M}}_{\mathscr{F}}} \varinjlim_{\mathscr{F}} \Omega(\boldsymbol{V}) \to \boldsymbol{0}$$

where j_* is the pushforward morphism, is exact.

ii) Restrictions $R^{\boldsymbol{V}}_{\boldsymbol{F}} : \Omega(V) \to \Omega(\boldsymbol{F})$ for $\boldsymbol{V} \supseteq \boldsymbol{F}$, define a morphism of the inductive system $\{R^{\boldsymbol{W}}_{\boldsymbol{V}} \mid \boldsymbol{W} \supseteq \boldsymbol{V} \text{ in } \mathscr{F}\}$ into $\Omega(\boldsymbol{F})$. Denote by $R^{\mathscr{F}}_{\boldsymbol{F}} := \varinjlim_{\mathscr{F}} R^{\boldsymbol{V}}_{\boldsymbol{F}}$. Show that

$$R^{\mathscr{F}}_{\boldsymbol{F}} : \varinjlim_{\mathscr{F}} \Omega(\boldsymbol{V}) \to \Omega(\boldsymbol{F})$$

is a quasi-isomorphism.

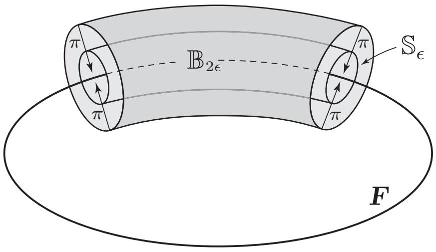
iii) Derive the existence of *the long exact sequence of compact support cohomology*

$$\cdots \to H^k_c(\boldsymbol{U}) \xrightarrow{i_*} H^k_c(\boldsymbol{M}) \xrightarrow{i^*} H^k(\boldsymbol{F}) \xrightarrow{c_k} H^{k+1}_c(\boldsymbol{U}) \to \cdots \qquad (\diamond)$$

iv) Endow $\boldsymbol{M}$ with a Riemannian metric $d : \boldsymbol{M} \times \boldsymbol{M} \to \mathbb{R}$. For each $\epsilon \in \mathbb{R}$, denote

$$\begin{cases} I\!B_\epsilon(\boldsymbol{F}) := \{m \in \boldsymbol{M} \mid d(m,\boldsymbol{F}) < \epsilon\} \\ \mathbb{S}_\epsilon(\boldsymbol{F}) := \{m \in \boldsymbol{M} \mid d(m,\boldsymbol{F}) = \epsilon\} \end{cases}$$

If ϵ is small enough, $I\!B_{2\epsilon}(\boldsymbol{F})$ is a fiber bundle with fiber $\mathbb{R}^{d_M - d_F}$ over $\boldsymbol{F}$ via the geodesic projection $\pi : I\!B_{2\epsilon} \to \boldsymbol{F}$. By restriction, $\pi : \mathbb{S}_\epsilon \to \boldsymbol{F}$ is a fiber bundle with compact fiber $\mathbb{S}^{d_{\boldsymbol{M}}-d_{\boldsymbol{F}}-1}$. Denote by $\ell : \mathbb{S}_\epsilon \hookrightarrow \mathbb{B}_{2\epsilon}\backslash \boldsymbol{F}$ the canonical injection. We have the following maps



$$\begin{array}{ccccc} \mathbb{S}_\epsilon & \xrightarrow{\ \ell\ } & \mathbb{B}_{2\epsilon}\backslash \boldsymbol{F} & \xrightarrow[\subseteq]{j_\epsilon} & \boldsymbol{U} \\ {\scriptstyle \pi}\downarrow & & & & \downarrow {\scriptstyle j}\ \subseteq \\ \boldsymbol{F} & & \xhookrightarrow[\subseteq]{\quad i \quad} & & \boldsymbol{M} \end{array}$$

Show that the connecting morphism $c : H(\boldsymbol{F}) \to H_c(\boldsymbol{U})[1]$ is given by the composition of the following morphisms

$$\underbrace{H(\boldsymbol{F}) \xrightarrow{\pi^*} H_c(\mathbb{S}_\epsilon) \xrightarrow{\ell_*[-d_{\mathbb{S}_\epsilon}]} H_c(\mathbb{B}_{2\epsilon}\backslash\boldsymbol{F})[1] \xrightarrow{j_{\epsilon,*}} H_c(\boldsymbol{U})[1]}_{c}$$

where $\ell_* : H_c(\mathbb{S}_\epsilon)[d_{\mathbb{S}_\epsilon}] \to H_c(\mathbb{B}_{2\epsilon}\backslash\boldsymbol{F})[d_{\mathbb{B}_{2\epsilon}}]$ denotes the Gysin morphism associated with ℓ.

b) i) Dualizing and shifting the long exact sequence of compact support ($\diamond$), justify the exactness of the *Gysin long exact sequence*

$$\xrightarrow{\delta[-1]} H(\boldsymbol{F})[d_{\boldsymbol{F}} - d_{\boldsymbol{M}}] \xrightarrow{i_*[-d_{\boldsymbol{M}}]} H(\boldsymbol{M}) \xrightarrow{j^*} H(\boldsymbol{U}) \xrightarrow{\delta} \qquad (\diamond\diamond)$$

where $i : \boldsymbol{F} \to \boldsymbol{N}$ and $j : U \to \boldsymbol{N}$ are the canonical injections and δ is adjoint to the shift of the connecting morphism c in ($\diamond$).

ii) Show that the connecting morphism $\delta : H(\boldsymbol{U}) \to H(\boldsymbol{F})[-(d_{\boldsymbol{M}} - d_{\boldsymbol{F}} - 1)]$ is simply the restriction to $\mathbb{S}_\epsilon$ followed by integration along fibers of π

$$\delta(\alpha) = \int_{\mathbb{S}^{d_{\boldsymbol{M}}-d_{\boldsymbol{F}}-1}} \alpha|_{\mathbb{S}_\epsilon}\,.$$

1.11.2. Lefschetz Fixed Point Theorem. Let $\boldsymbol{M}$ be a oriented manifold. Denote by $\delta : \boldsymbol{M} \to \boldsymbol{M} \times \boldsymbol{M}$ the diagonal embedding $x \mapsto (x,x)$ and let $\Delta := \mathrm{Im}(\delta)$. Given $f : \boldsymbol{M} \to \boldsymbol{M}$, denote $\mathrm{Gr}(f) : \boldsymbol{M} \to \boldsymbol{M} \times \boldsymbol{M}$ *the graph map* $x \mapsto (f(x),x)$. The *Lefschetz class of* f is by definition

$$L(f) := \mathrm{Gr}(f)^*(\delta_!(1)) \in H^{d_M}(\boldsymbol{M}),$$

and its *Lefschetz number* is $\Lambda_f := \int_{\boldsymbol{M}} L(f)$.

a) Explain de following the equalities

$$\Lambda_f := \int_{\boldsymbol{M}} \mathrm{Gr}(f)^*(\delta_!(1)) = \int_{\boldsymbol{M}\times\boldsymbol{M}} \delta_!(1) \cup \mathrm{Gr}(f)_!(1) = (-1)^{d_M} \int_{\boldsymbol{M}} \delta^*(\mathrm{Gr}(f)_!(1)). \quad (\diamond)$$

b) Assuming that f has no fixed points, show that the Gysin morphism

$$\mathrm{Gr}(f)_! : H(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H(\boldsymbol{M} \times \boldsymbol{M})[2d_{\boldsymbol{M}}]$$

factorizes through $H_c(\boldsymbol{M} \times \boldsymbol{M}\backslash\Delta)$ so that $\Lambda_f = 0$.

c) From now on suppose that $\boldsymbol{M}$ is orientable. Let $\mathscr{B} := \{e_i\}_{i\in \boldsymbol{I}}$ be a graded basis of $H(\boldsymbol{M})$ and let $\mathscr{B}' := \{e'_i\}_{i\in\boldsymbol{I}}$ denote the Poincaré dual basis of $\mathscr{B}$, i.e. such that $e_i \cup e'_j = \delta_{i,j}[\zeta]$, where $[\zeta]$ denotes the fundamental class of $\boldsymbol{M}$. Using the projection formula for $\delta : \boldsymbol{M} \to \boldsymbol{M} \times \boldsymbol{M}$ show that

$$\delta_*(1) = \sum\nolimits_{i \in \boldsymbol{I}} (-1)^{\deg(e_i)}\, e_i \otimes e'_i,$$

Prove the equality: $\displaystyle\int_{\boldsymbol{M}} \delta_!(1)|_\Delta = \sum_{k \in \mathbb{N}} (-1)^k \dim\big(H^k(\boldsymbol{M})\big)$.

d) Combining ($\diamond$) with the last result, show the *Lefschetz fixed point formula*

$$\Lambda_f = \sum\nolimits_{k \in \mathbb{N}} (-1)^k \operatorname{Tr}\big(f^* : H^k(\boldsymbol{M}) \to H^k(\boldsymbol{M})\big)$$

In particular, if this alternating sum doesn't vanish, f has fixed points !

1.12. Conclusion. We have reached the end of the preliminaries on Poincaré duality and Gysin morphism in the nonequivariant setting. As shown, the key ingredient is Poincaré duality so that, in order to extend the constructions to $\boldsymbol{G}$-manifolds, we propose ourselves to follow the same approach. It will therefore be necessary first to introduce Poincaré pairings and Poincaré duality in the $\boldsymbol{G}$-equivariant framework. We devote section 4 entirely to this subject. In section 5, the $\boldsymbol{G}$-equivariant Gysin morphisms associated with equivariant maps will then be defined following the same procedures described in 1.10.

2. Equivariant Background

2.1. Category of Cochain $\mathfrak{g}$-Complexes

2.1.1. Fields in Use. Unless otherwise stated, Lie groups and Lie algebras, vector spaces, complexes of vector spaces, linear maps, tensor products and related stuff, will be defined over the field of real numbers $\mathbb{R}$.

2.1.2. $\mathfrak{g}$-modules. Let $\mathfrak{g}$ be a real *Lie algebra*. A *representation of* $\mathfrak{g}$, also called *a* $\mathfrak{g}$*-module*, will be a real vector space V together with a Lie algebra homomorphism $\rho_V : \mathfrak{g} \to \operatorname{End}_{\mathbb{R}}(V)$. For simplicity, the notation "$Y \cdot v$" will frequently replace "$\rho_V(v)$" when the representation is understood.

The *trivial representation of* $\mathfrak{g}$ *on a vector space* V, is the one where $\rho_V = 0$.

Given $\mathfrak{g}$-modules V and W, a $\mathfrak{g}$*-module morphism from* V *to* W is a linear map $\lambda : V \to W$ s.t. $\lambda \circ \rho_V(Y) = \rho_W(Y) \circ \lambda$ for all $Y \in \mathfrak{g}$. We denote by $\operatorname{Hom}_{\mathfrak{g}}(V,W)$ the subspace of $\operatorname{Hom}_{\mathbb{R}}(V,W)$ of such maps.

A $\mathfrak{g}$-module V is said to be:

- *simple or irreducible*, if it is nonzero and has no nontrivial submodules;
- *semisimple*, if it is a direct sum of irreducible $\mathfrak{g}$-modules;
- *reducible* if it is a direct sum of two nonzero $\mathfrak{g}$-modules;
- *completely reducible* if it is a direct sum of irreducible modules;

The $\mathfrak{g}$-modules and their morphisms constitute a category, the *category of* $\mathfrak{g}$*-modules* denoted by $\operatorname{Mod}(\mathfrak{g})$.

2.1.3. Exercise. Let V be a $\mathfrak{g}$-module. Show the equivalence of:

a) V is completely reducible.
b) V is a sum of irreducible modules.
c) If W is a submodule of V then $V = V' \oplus W$ for some submodule V'.

2.1.4. Exercise. Given a $\mathfrak{g}$-module V, denote by $V^{\mathfrak{g}}$ the subspace of $\mathfrak{g}$-*invariant vectors of* V, i.e. of $v \in V$, such that $Y \cdot v = 0$ for all $Y \in \mathfrak{g}$.

a) Show that for all $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V,W)$, $\varphi(V^{\mathfrak{g}}) \subseteq W^{\mathfrak{g}}$. Derive the fact that the correspondence $V \rightsquigarrow V^{\mathfrak{g}}$, $\varphi \rightsquigarrow \varphi|_{V^{\mathfrak{g}}}$ is fonctorial from $\mathrm{Mod}(\mathfrak{g})$ intito $\mathrm{Vec}(\mathbb{R})$.

b) Endow $\mathbb{R}$ with the trivial action of $\mathfrak{g}$. Show that the map

$$\mathrm{Hom}_{\mathfrak{g}}(\mathbb{R},V) \to V^{\mathfrak{g}}, \quad \varphi \mapsto \varphi(1),$$

is a natural isomorphism of functors $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{R},\_) \to (\_)^{\mathfrak{g}}$. In particular, $(\_)^{\mathfrak{g}}$ is left exact but not necessarily exact.

2.1.5. Differential Graded $\mathfrak{g}$-Complexes. A *differential graded* $\mathfrak{g}$*-complex*, a $\mathfrak{g}$*-complex* in short, is a quadruple $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$ where:

- $(\boldsymbol{C},\boldsymbol{d})$ is a complex in $\mathrm{DGM}(\mathbb{R})$ (cf. 1.1.5);
- $\boldsymbol{\theta} : \mathfrak{g} \to \mathbf{End}_{\mathrm{GV}(\mathbb{R})}(\boldsymbol{C})$ is a Lie algebra morphism, the $\mathfrak{g}$*-derivation* ([5]);
- $\boldsymbol{c} : \mathfrak{g} \to \mathbf{Mor}_{\mathrm{GV}(\mathbb{R})}(\boldsymbol{C},\boldsymbol{C}[-1])$ is a linear map, the $\mathfrak{g}$*-contraction*;

such that, for all $X,Y \in \mathfrak{g}$

$$\begin{cases} \text{i) } \boldsymbol{c}(X)\circ\boldsymbol{c}(Y) + \boldsymbol{c}(Y)\circ\boldsymbol{c}(X) = 0 \\ \text{ii) } \boldsymbol{d}\circ\boldsymbol{c}(X) + \boldsymbol{c}(X)\circ\boldsymbol{d} = \boldsymbol{\theta}(X) \\ \text{iii) } \boldsymbol{\theta}(Y)\circ\boldsymbol{c}(X) - \boldsymbol{c}(X)\circ\boldsymbol{\theta}(Y) = \boldsymbol{c}([Y,X]) \end{cases} \tag{$\diamond$}$$

2.1.6. Remark. From $(\diamond)$-(ii), one immediately obtains $\boldsymbol{d}\circ\boldsymbol{\theta}(\_) = \boldsymbol{\theta}(\_)\circ\boldsymbol{d}$ which implies that $\boldsymbol{\theta}$ naturally induces an action of $\mathfrak{g}$ on the cohomology of $(\boldsymbol{C},\boldsymbol{d})$. However, that same condition shows that $\boldsymbol{c}(X)$ is a homotopy for $\boldsymbol{\theta}(X)$, so that this induced action is in fact trivial.

2.1.7. Morphisms of $\mathfrak{g}$-Complexes. A *morphism of graded* $\mathfrak{g}$*-complexes*, or *morphism of* $\mathfrak{g}$*-complexes* in short, $\boldsymbol{\alpha} : (\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c}) \to (\boldsymbol{D},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$, is a morphism of complexes $\boldsymbol{\alpha} : (\boldsymbol{C},\boldsymbol{d}) \to (\boldsymbol{D},\boldsymbol{d})$ commuting with derivations and contractions, i.e. such that $\boldsymbol{\alpha}\circ\boldsymbol{\theta} = \boldsymbol{\theta}\circ\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}\circ\boldsymbol{c} = \boldsymbol{c}\circ\boldsymbol{\alpha}$.

2.1.8. Category of $\mathfrak{g}$-Complexes. The $\mathfrak{g}$-complexes $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$ and their morphisms constitute the *category of* $\mathfrak{g}$*-complexes* denoted by $\mathrm{DGM}(\mathfrak{g},\mathbb{R})$.

In the sequel, a $\mathfrak{g}$-complex $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$ may be denoted by $(\boldsymbol{C},\boldsymbol{d})$ and even simply $\boldsymbol{C}$, whenever the remaining data are understood.

2.1.9. Split $\mathfrak{g}$-Complexes. Given an inclusion of $\mathfrak{g}$-modules $N \subseteq M$, we will use the notation "$N|M$" to express that the natural map

$$\mathrm{Hom}_{\mathfrak{g}}(V,M) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(V,M/N) \tag{$\ddagger$}$$

is **surjective** for all **finite** dimensional $\mathfrak{g}$-module V.

[5] Recall that given two $\mathbb{Z}$-graded vector spaces $\boldsymbol{C}$ and $\boldsymbol{D}$, we denote by $\mathbf{Mor}_{\mathrm{GV}(\mathbb{R})}(\boldsymbol{C},\boldsymbol{D})$ the group of graded homomorphisms of degree zero from $\boldsymbol{C}$ into $\boldsymbol{D}$. The terminology *derivation* comes from the fact that in the main case where $(\boldsymbol{C},\boldsymbol{d})$ is the de Rham complex of a $\boldsymbol{G}$-manifold, the group $\boldsymbol{G}$ acts on $(\boldsymbol{C},\boldsymbol{d})$ by differential graded algebra automorphisms, so that the infinitesimal action of its Lie algebra $\mathfrak{g} := \mathrm{Lie}(\boldsymbol{G})$ will be by differential graded algebra derivations.

Exercise. Show that the condition $N|M$ is equivalent to the fact that for every $\mathfrak{g}$-submodule $M' \subseteq M$ such that $N \subseteq M'$ is of finite codimension, there exists a $\mathfrak{g}$-submodule $H \subseteq M'$ such that $M' = H \oplus N$.

Definition. For a $\mathfrak{g}$-complex $(\boldsymbol{C},\boldsymbol{d})$, let $B^i := \operatorname{im}(d_{i-1})$ and $Z^i := \ker(d_i)$ respectively be *the* $\mathfrak{g}$*-submodules of* i*-coboundaries and* i*-cocycles of* $(\boldsymbol{C},\boldsymbol{d})$. The $\mathfrak{g}$-complex $(\boldsymbol{C},\boldsymbol{d})$ will be called $\mathfrak{g}$*-split* whenever one has

$$B^i|Z^i|C^i, \quad \text{for all } i \in \mathbb{Z}.$$

2.1.10. Lemma. *Keep the above notations and prove the following,*

a) *If* $N|M$, *the natural map* $\dfrac{M^{\mathfrak{g}}}{N^{\mathfrak{g}}} \longrightarrow \left(\dfrac{M}{N}\right)^{\mathfrak{g}}$ *is an isomorphism.* $(\diamond)$
b) *The condition* $B^i|Z^i$ *is equivalent to the fact that* $(Z^i)^{\mathfrak{g}} \to (Z^i/B^i)$ *is surjective, and it is also equivalent to the existence of a* $\mathfrak{g}$*-submodule* H^i *of* Z^i *such that* $Z^i = B^i \oplus H^i$, *in which case* H^i *is a trivial* $\mathfrak{g}$*-module isomorphic to* Z^i/B^i.
c) *A* $\mathfrak{g}$*-complex* $(\boldsymbol{C},\boldsymbol{d})$ *such that each* C^i *is completely reducible, is* $\mathfrak{g}$*-split.*

Proof

a) After 2.1.4, the functor $(\_)^{\mathfrak{g}}$ is isomorphic to $\operatorname{Hom}_{\mathfrak{g}}(\mathbb{R};\_)$ and the sequence $\mathbf{0} \to N^{\mathfrak{g}} \to M^{\mathfrak{g}} \to (M/N)^{\mathfrak{g}}$ is left exact. The split condition ensures it is also right exact.
b) Recall that $\mathcal{H}^i := Z^i/B^i$ is a trivial $\mathfrak{g}$-module (see 2.1.6). Following (a), the split condition immediately gives the surjection $(Z^i)^{\mathfrak{g}} \twoheadrightarrow (\mathcal{H}^i)^{\mathfrak{g}} = \mathcal{H}^i$. Conversely, one clearly has $\operatorname{Hom}_{\mathfrak{g}}(\mathcal{H}^i,\_) = \operatorname{Hom}_{\mathbb{R}}(\mathcal{H}^i,(\_)^{\mathfrak{g}})$ and, thereafter, the commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathfrak{g}}(\mathcal{H}^i,Z^i) & \longrightarrow & \operatorname{Hom}_{\mathfrak{g}}(\mathcal{H}^i,\mathcal{H}^i) \\ \| & & \| \\ \operatorname{Hom}_{\mathbb{R}}(\mathcal{H}^i,(Z^i)^{\mathfrak{g}}) & \twoheadrightarrow & \operatorname{Hom}_{\mathbb{R}}(\mathcal{H}^i,\mathcal{H}^i) \end{array}$$

where the surjectivity of the second line implies the surjectivity of the first one. In particular, there exists $\sigma \in \operatorname{Hom}_{\mathfrak{g}}(\mathcal{H}^i,Z^i)$ such that $\pi \circ \sigma = \mathrm{id}$ where $\pi : Z^i \twoheadrightarrow \mathcal{H}^i$ denotes the canonical projection. Setting $H^i := \operatorname{Im}(\sigma)$ completes de proof.
c) Clear from exercise 2.1.3. □

2.1.11. Proposition. *Let* $(\boldsymbol{C},\boldsymbol{d})$ *be a* $\mathfrak{g}$*-split* $\mathfrak{g}$*-complex.*

a) *The inclusion* $\boldsymbol{C}^{\mathfrak{g}} \subseteq \boldsymbol{C}$ *is a quasi-isomorphism*
b) *If* V *is a finite dimensional* ***semi-simple*** $\mathfrak{g}$*-module, the inclusions*

$$\begin{array}{ccccc} V^{\mathfrak{g}} \otimes \boldsymbol{C} & \supseteq & V^{\mathfrak{g}} \otimes \boldsymbol{C}^{\mathfrak{g}} & \subseteq & (V \otimes \boldsymbol{C})^{\mathfrak{g}} \\ \mathbf{Hom}^{\bullet}_{\mathbb{R}}(V^{\mathfrak{g}},\boldsymbol{C}) & \supseteq & \mathbf{Hom}^{\bullet}_{\mathbb{R}}(V^{\mathfrak{g}},\boldsymbol{C}^{\mathfrak{g}}) & \subseteq & \mathbf{Hom}^{\bullet}_{\mathfrak{g}}(V,\boldsymbol{C}) \end{array}$$

are quasi-isomorphisms.

Proof

a) Immediate from (2.1.10-(a)).
b) Let us first show that if W is a simple $\mathfrak{g}$-module different from $\mathbb{R}$, the complexes $(W \otimes \boldsymbol{C})^{\mathfrak{g}}$ and $\mathbf{Hom}^{\bullet}_{\mathfrak{g}}(W,\boldsymbol{C})$ are acyclic.

It suffices to treat only the $\mathbf{Hom}^\bullet$ case, since one has

$$\mathbf{Hom}^\bullet_{\mathfrak{g}}(W,\boldsymbol{C}) = \mathbf{Hom}^\bullet_{\mathbb{R}}(W,\boldsymbol{C})^{\mathfrak{g}} = (W^\vee \otimes \boldsymbol{C})^{\mathfrak{g}}.$$

An i-cocycle of $\mathbf{Hom}^\bullet_{\mathfrak{g}}(W,\boldsymbol{C})$ is a $\mathfrak{g}$-module morphism $\lambda : W \to C^i$ such that $\boldsymbol{d}\circ\lambda = 0$, i.e. such that $\mathrm{im}(\lambda) \subseteq Z^i$. But the composition of λ with the surjection $Z^i \twoheadrightarrow Z^i/B^i$ is null since $\mathfrak{g}$ acts trivially on cohomology, so that in fact $\mathrm{im}(\lambda) \subseteq B^i$. Now, thanks to the fact that $Z^i|C^i$, we can lift $\lambda : W \to B^i$ to $\mu : W \to C^{i-1}$ and we have thus proved that $\lambda = \boldsymbol{d}\circ\mu$, i.e. that λ is a coboundary.

If V is a semisimple $\mathfrak{g}$-module, one decomposes V as $V^{\mathfrak{g}} \oplus W$, where W is a direct sum of simple $\mathfrak{g}$-modules different from $\mathbb{R}$. Then

$$\mathbf{Hom}^\bullet_{\mathfrak{g}}(V,\boldsymbol{C}) = \mathbf{Hom}^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},\boldsymbol{C}) \oplus \mathbf{Hom}^\bullet_{\mathfrak{g}}(W,\boldsymbol{C})$$

is quasi-isomorphic to $\mathbf{Hom}^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},\boldsymbol{C})$ after the previous paragraph. But

$$\mathbf{Hom}^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},\boldsymbol{C}) = \mathbf{Hom}^\bullet_{\mathfrak{g}}(V^{\mathfrak{g}},\boldsymbol{C}^{\mathfrak{g}}) = \mathbf{Hom}^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},\boldsymbol{C}^{\mathfrak{g}}),$$

so that $\mathbf{Hom}^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},\boldsymbol{C}^{\mathfrak{g}}) \subseteq \mathbf{Hom}^\bullet_{\mathfrak{g}}(V,\boldsymbol{C})$ is clearly a quasi-isomorphism.

Finally, that the inclusion $\mathbf{Hom}^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},\boldsymbol{C}^{\mathfrak{g}}) \subseteq \mathbf{Hom}^\bullet_{\mathbb{R}}(V^{\mathfrak{g}},\boldsymbol{C})$ is a quasi-isomorphism results from (a) since $V^{\mathfrak{g}} \simeq \mathbb{R}^r$ and the inclusion being considered becomes simply $\prod_{1\leqslant i\leqslant r}\boldsymbol{C}^{\mathfrak{g}} \subseteq \prod_{1\leqslant i\leqslant r}\boldsymbol{C}$. □

2.2. Equivariant Cohomology of $\mathfrak{g}$-Complexes

2.2.1. The symmetric Algebra of $\mathfrak{g}^\vee$. Let $\boldsymbol{S}(\mathfrak{g})$ be the ring of polynomial maps from $\mathfrak{g}$ to $\mathbb{R}$, graded by twice the polynomial degree and denote by $S^d(\mathfrak{g})$ the subspace of elements of degree d, in particular $\boldsymbol{S}^2(\mathfrak{g}) = \mathfrak{g}^\vee$ and $\boldsymbol{S}^m(\mathfrak{g}) = 0$ for every odd integer m.

Let $\boldsymbol{\theta} : \mathfrak{g} \to \mathrm{Der}_{\mathbb{R}}(\boldsymbol{S}(\mathfrak{g}))$ denote the coadjoint representation.

Fix for later use a vector space basis $\{e_i\}$ of $\mathfrak{g}$, of dual basis $\{e^i\}$.

2.2.2. Cartan Complexes. Given a $\mathfrak{g}$-complex $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$, we are interested in the polynomial maps $\omega : \mathfrak{g} \ni Y \mapsto \omega(Y) \in \boldsymbol{C}$, i.e. the elements $\omega \in \boldsymbol{S}(\mathfrak{g}) \otimes \boldsymbol{C}$. The Lie algebra $\mathfrak{g}$ acts on each $\boldsymbol{S}^a(\mathfrak{g}) \otimes C^b$ by the formula

$$\boldsymbol{\theta}(Y)(P\otimes\mu) := \boldsymbol{\theta}(Y)(P)\otimes\mu + P\otimes\boldsymbol{\theta}(Y)(\mu), \qquad \forall Y \in \mathfrak{g}.$$

A polynomial map $Y \mapsto \omega(Y)$ is then $\mathfrak{g}$-invariant if and only if it satisfies the equality

$$\theta(X)(\omega(Y)) + \omega([X,Y]) = 0,$$

for all $X,Y \in \mathfrak{g}$. Put

$$\boxed{\boldsymbol{C}_{\mathfrak{g}} := (\boldsymbol{S}(\mathfrak{g})\otimes\boldsymbol{C})^{\mathfrak{g}} = \bigoplus\nolimits_{k\in\mathbb{Z}} \boldsymbol{C}^k_{\mathfrak{g}}} \tag{$\boldsymbol{C}_{\mathfrak{g}}$}$$

where $\boldsymbol{C}^k_{\mathfrak{g}} := \sum_{a+b=k}\big(\boldsymbol{S}^a(\mathfrak{g})\otimes C^b\big)^{\mathfrak{g}}$. The $\boldsymbol{S}(\mathfrak{g})$-linear map $\boldsymbol{d}_{\mathfrak{g}} : \boldsymbol{C}_{\mathfrak{g}} \to \boldsymbol{C}_{\mathfrak{g}}$,

$$\boxed{\boldsymbol{d}_{\mathfrak{g}}(1\otimes\omega) = 1\otimes\boldsymbol{d}\omega + \sum\nolimits_i e^i\otimes\boldsymbol{c}(e_i)\omega} \tag{$\boldsymbol{d}_{\mathfrak{g}}$}$$

is a morphism of graded spaces of degree $+1$. It verifies $\boldsymbol{d}^2_{\mathfrak{g}} = \sum_i e^i\otimes\boldsymbol{\theta}(e_i)$, so that, over $\boldsymbol{C}_{\mathfrak{g}}$, one has

$$\boldsymbol{d}^2_{\mathfrak{g}} = \sum_i e^i\boldsymbol{\theta}(e_i)\otimes\mathrm{id}.$$

But $\Xi := \sum_i e^i \boldsymbol{\theta}(e_i)$ is the null operator on $\boldsymbol{S}(\mathfrak{g})$. Indeed, since it acts as a derivation on $\boldsymbol{S}(\mathfrak{g})$, it suffices to show that it vanishes on any $\lambda \in \mathfrak{g}^\vee$, i.e. that $\Xi(\lambda)(e_j) = 0$ for all j, which comes from the straightforward computation

$$\Xi(\lambda)(e_j) = \Big(\sum_i e^i \theta(e_i)(\lambda)\Big)(e_j) = \sum_i e^i(e_j)\lambda([e_i,e_j]) = \lambda([e_j,e_j]) = 0.$$

Hence, $\boldsymbol{d}_\mathfrak{g}^2 = 0$ in $\boldsymbol{C}_\mathfrak{g}$. This $\boldsymbol{d}_\mathfrak{g} \in \mathrm{Endgr}^1_{\boldsymbol{S}(\mathfrak{g})^\mathfrak{g}}(\boldsymbol{C}_\mathfrak{g})$ is the *Cartan differential*.

2.2.3. Definition. The pair $\big(\boldsymbol{C}_\mathfrak{g}, \boldsymbol{d}_\mathfrak{g}\big)$ is a complex. It is *the Cartan (equivariant) complex associated with the* $\mathfrak{g}$*-complex* $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$, and the cohomology of $\big(\boldsymbol{C}_\mathfrak{g}, \boldsymbol{d}_\mathfrak{g}\big)$ is its $\mathfrak{g}$*-equivariant cohomology*, denoted in the sequel by

$$\boxed{H_\mathfrak{g}(\boldsymbol{C}) := \boldsymbol{h}\big(\boldsymbol{C}_\mathfrak{g}, \boldsymbol{d}_\mathfrak{g}\big)}$$

2.2.4. *Important Remark.* The graded space $\boldsymbol{C}_\mathfrak{g}$ is an $\boldsymbol{S}(\mathfrak{g})^\mathfrak{g}$-graded module (4.1.3), the differential $\boldsymbol{d}_\mathfrak{g}$ is $\boldsymbol{S}(\mathfrak{g})^\mathfrak{g}$-linear, and the cohomology $H_\mathfrak{g}(\boldsymbol{C})$ is an $\boldsymbol{S}(\mathfrak{g})^\mathfrak{g}$-graded module.

2.2.5 Any morphism of $\mathfrak{g}$-complexes $\boldsymbol{\alpha} : (\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c}) \to (\boldsymbol{D},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$ induces a canonical $\boldsymbol{S}(\mathfrak{g})$-linear morphism of complexes $\alpha_\mathfrak{g} : \boldsymbol{C}_\mathfrak{g} \to \boldsymbol{D}_\mathfrak{g}$ by the formula $\boldsymbol{\alpha}_\mathfrak{g} = \mathrm{id} \otimes \boldsymbol{\alpha}$.

2.2.6. Theorem. *With the above notations one has,*

a) *The correspondence* $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c}) \rightsquigarrow (\boldsymbol{C}_\mathfrak{g},\boldsymbol{d})$ *and* $\boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\alpha}_\mathfrak{g}$ *is a covariant functor from* $\mathrm{DGM}(\mathfrak{g},\mathbb{R})$ *into* $\mathrm{DGM}(\mathbb{R})$*.*

b) *For every* $\mathfrak{g}$*-complex* $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$*, there exists a spectral sequence converging to* $H_\mathfrak{g}(\boldsymbol{C})$ *with*

$$\big(\mathbb{E}_0^{p,q} = \big(\boldsymbol{S}^p(\mathfrak{g}) \otimes C^q\big)^\mathfrak{g},\ d_0 = 1 \otimes \boldsymbol{d}\big) \Rightarrow H_\mathfrak{g}^{p+q}(\boldsymbol{C}).$$

c) *Let* $\boldsymbol{G}$ *be a compact Lie group,* $\mathfrak{g} := \mathrm{Lie}(\boldsymbol{G})$ *and* $\boldsymbol{C}$ *end* $\boldsymbol{D}$ *two* $\mathfrak{g}$***-split*** $\mathfrak{g}$*-complexes (2.1.9).*

   i) *The* $(\mathbb{E}_2,d_2)$ *spectral sequence term in (b) is given by*

$$\Big(\mathbb{E}_2^{p,q} = \boldsymbol{S}^p(\mathfrak{g})^\mathfrak{g} \otimes H^q(\boldsymbol{C}),\ d_2 = \textstyle\sum_i e^i \otimes \boldsymbol{c}(e_i)\Big) \Rightarrow H_\mathfrak{g}^{p+q}(\boldsymbol{C}).$$

   ii) *If* $H^m(\boldsymbol{C}) = 0$ *for all odd (or for all even)* m*, then*

$$H_\mathfrak{g}(\boldsymbol{C}) = \boldsymbol{S}(\mathfrak{g})^\mathfrak{g} \otimes \boldsymbol{h}(\boldsymbol{C}).$$

   iii) *If* $\boldsymbol{\alpha} : \boldsymbol{C} \to \boldsymbol{D}$ *is a quasi-isomorphism of* $\mathfrak{g}$*-complexes,* $\boldsymbol{\alpha}_\mathfrak{g} : \boldsymbol{C}_\mathfrak{g} \to \boldsymbol{D}_\mathfrak{g}$ *is a quasi-isomorphism.*

d) *Let* $\boldsymbol{G}$ *be a* **commutative** *compact Lie group and* $\mathfrak{g} := \mathrm{Lie}(\boldsymbol{G})$*,*

   i) *For every* $\mathfrak{g}$*-complex* $(\boldsymbol{C},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$*, the subcomplex* $(\boldsymbol{C}^\mathfrak{g},\boldsymbol{d})$ *is stable under* $\boldsymbol{\theta}$ *and* $\boldsymbol{c}$*, i.e.* $(\boldsymbol{C}^\mathfrak{g},\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$ *is a well defined* $\mathfrak{g}$*-complex.*

   ii) *If* $j : \boldsymbol{C}^\mathfrak{g} \hookrightarrow \boldsymbol{C}$ *denotes the inclusion map,* $j_\mathfrak{g}$ *is a quasi-isomorphism.*

   iii) *The* $(\mathbb{E}_2,d_2)$ *spectral sequence term in (b) is given by*

$$\Big(\mathbb{E}_2^{p,q} = \boldsymbol{S}^p(\mathfrak{g}) \otimes H^q(\boldsymbol{C}^\mathfrak{g}),\ d_2 = \textstyle\sum_i e^i \otimes \boldsymbol{c}(e_i)\Big) \Rightarrow H_\mathfrak{g}^{p+q}(\boldsymbol{C})$$

iv) *If* $H^m((\boldsymbol{C})^{\mathfrak{g}}) = 0$ *for all odd (or for all even)* m, *then*

$$H_{\mathfrak{g}}(\boldsymbol{C}) = \boldsymbol{S}(\mathfrak{g}) \otimes \boldsymbol{h}(\boldsymbol{C}^{\mathfrak{g}}).$$

v) *If* $\boldsymbol{\alpha} : \boldsymbol{C}^{\mathfrak{g}} \to \boldsymbol{D}^{\mathfrak{g}}$ *is a quasi-isomorphism,* $\boldsymbol{\alpha}_{\mathfrak{g}}$ *is a quasi-isomorphism.*

Proof

a) Clear.

b) For $m \in \mathbb{Z}$, let $K_m = \big(\boldsymbol{S}^{\geqslant m}(\mathfrak{g}) \otimes \boldsymbol{C}\big)^{\mathfrak{g}}$. Each K_m is clearly a sub-complex of $(\boldsymbol{C}_{\mathfrak{g}}, \boldsymbol{d}_{\mathfrak{g}})$ and $\big(\boldsymbol{C}_{\mathfrak{g}} = K_0 \supseteq K_1 \supseteq \cdots\big)$ is a *regular* decreasing filtration of $(\boldsymbol{C}_{\mathfrak{g}}, \boldsymbol{d}_{\mathfrak{g}})$ (see [Go] §4 pp. 76-) giving rise to the stated spectral sequence.

c) i) The assumption that $\boldsymbol{G}$ is compact ensures that each (finite dimensional) $\mathfrak{g}$-module $\boldsymbol{S}^p(\mathfrak{g})$ is semisimple. Proposition 2.1.11-(b) may be used, and $\big((\boldsymbol{S}^p(\mathfrak{g}) \otimes \boldsymbol{C})^{\mathfrak{g}}, 1 \otimes \boldsymbol{d}\big)$ is quasi-isomorphic to $(\boldsymbol{S}^p(\mathfrak{g})^{\mathfrak{g}} \otimes \boldsymbol{C}, 1 \otimes \boldsymbol{d})$. Consequently $(\mathbb{E}_0, d_0)$ in (b) is quasi-isomorphic to $\big(\boldsymbol{S}(\mathfrak{g})^{\mathfrak{g}} \otimes \boldsymbol{C}, 1 \otimes \boldsymbol{d}\big)$ and $\mathbb{E}_1^{p,q} = \boldsymbol{S}^p(\mathfrak{g})^{\mathfrak{g}} \otimes H^q(\boldsymbol{C})$. But the differential $d_1 : \mathbb{E}_1^{p,q} \to \mathbb{E}_1^{p+1,q}$ is null since the $\boldsymbol{S}(\mathfrak{g})$ vanishes in odd degrees, therefore $\mathbb{E}_1 = \mathbb{E}_2$, which completes the proof of the claim.

ii) Since the differential d_r is of total degree 1 and that $\mathbb{E}_r^{p,q} = 0$ if p or q is odd for all $r \geqslant 2$, one has $d_r = 0$ for $r \geqslant 2$, and $\mathbb{E}_2 = \mathbb{E}_\infty$.

iii) Follows immediately from (c-i).

d) i) We must check that $\boldsymbol{\theta}(Y)\boldsymbol{c}(X)\boldsymbol{C}^{\mathfrak{g}} = 0$ for all $X, Y \in \mathfrak{g}$, but, on $\boldsymbol{C}^{\mathfrak{g}}$ one has $\boldsymbol{\theta}(Y)\boldsymbol{c}(X) = \boldsymbol{\theta}(Y)\boldsymbol{c}(X) + \boldsymbol{c}(X)\boldsymbol{\theta}(Y) = \boldsymbol{c}([Y,X]) = \boldsymbol{c}(0)$ since $\mathfrak{g}$ is abelian and from property (iii) of $\mathfrak{g}$-complexes (see 2.1.5-($\diamond$)).

ii,iii,iv,v) Left to the reader. □

2.2.7. Split $\boldsymbol{G}$-Complexes. It's worth noting that the proof of 2.2.6-(c) makes use of the split condition (2.1.9) *only* for the finite dimensional sub-$\mathfrak{g}$-modules $V \in \boldsymbol{S}(\mathfrak{g})$, whose $\mathfrak{g}$-module structure is obtained by differentiating its natural $\boldsymbol{G}$-module structure.

The split condition 2.1.9 can easily be adapted to the context of $\boldsymbol{G}$-modules. For any inclusion of $\boldsymbol{G}$-modules $N \subseteq M$ one writes "$N|M$" whenever the natural map

$$\mathrm{Hom}_{\boldsymbol{G}}(V, M) \longrightarrow \mathrm{Hom}_{\boldsymbol{G}}(V, M/N) \tag{‡}$$

is **surjective** for all **finite dimensional** $\boldsymbol{G}$-module V.

2.2.8. Definition. A complex of $\boldsymbol{G}$-modules $(\boldsymbol{C}, \boldsymbol{d})$ is said to be $\boldsymbol{G}$-*split* whenever $B^i|Z^i|C^i$, for all $i \in \mathbb{Z}$.

The proof of the following proposition is the same as 2.1.11.

2.2.9. Proposition. *Let* $(\boldsymbol{C}, \boldsymbol{d})$ *be a* $\boldsymbol{G}$*-split complex of* $\boldsymbol{G}$*-modules such that the natural action of* $\boldsymbol{G}$ *in cohomology is trivial. Then,*

a) *The inclusion* $\boldsymbol{C}^{\boldsymbol{G}} \subseteq \boldsymbol{C}$ *is quasi-isomorphism.*

b) *If* V *is a* ***semisimple*** *finite dimensional* $\boldsymbol{G}$*-module, the inclusions*

$$\begin{array}{ccccc} V^{\boldsymbol{G}} \otimes \boldsymbol{C} & \supseteq & V^{\boldsymbol{G}} \otimes \boldsymbol{C}^{\boldsymbol{G}} & \subseteq & (V \otimes \boldsymbol{C})^{\boldsymbol{G}} \\ \mathbf{Hom}^{\bullet}_{\mathbb{R}}(V^{\boldsymbol{G}}, \boldsymbol{C}) & \supseteq & \mathbf{Hom}^{\bullet}_{\mathbb{R}}(V^{\boldsymbol{G}}, \boldsymbol{C}^{\boldsymbol{G}}) & \subseteq & \mathbf{Hom}^{\bullet}_{\boldsymbol{G}}(V, \boldsymbol{C}) \end{array}$$

are quasi-isomorphisms.

3. Equivariant Cohomology of G-Manifolds

3.1. Equivariant Differential Forms

3.1.1. Fields in Use. Unless otherwise stated, manifolds, Lie groups and Lie algebras, vector spaces, complexes of vector spaces, linear maps, tensor products and related stuff, will be defined over the field of real numbers $\mathbb{R}$.

3.1.2. G-Derivations and Contractions. Let $\boldsymbol{G}$ be a **connected** Lie group. Denote by $\mathfrak{g} := \operatorname{Lie}(\boldsymbol{G}) = T_e\boldsymbol{G}$ the Lie algebra of $\boldsymbol{G}$ endowed with the adjoint action. As in 2.2.1, let $\boldsymbol{S}(\mathfrak{g})$ be the ring of polynomial maps from $\mathfrak{g}$ to $\mathbb{R}$, graded by twice the polynomial degree.

Let $\boldsymbol{M}$ be a $\boldsymbol{G}$-manifold. Each $Y \in \mathfrak{g}$ defines a vector field on $\boldsymbol{M}$ by setting

$$\vec{Y}(m) := \frac{d}{dt}\Big(t \mapsto \exp(tY)\cdot m\Big)_{t=0}$$

Let $\vec{Y}\cdot\omega$ denote the *contraction* of the differential form $\omega \in \Omega(\boldsymbol{M})$ by the vector field $\vec{Y}$. The map $\boldsymbol{c}(Y) : \Omega(\boldsymbol{M}) \to \Omega(\boldsymbol{M})$, $\omega \mapsto \vec{Y}\cdot\omega$, is then an *antiderivation* of degree -1 and the map $\boldsymbol{c} : \mathfrak{g} \to \mathbf{Mor}_{\mathrm{GV}(\mathbb{K})}(\Omega(\boldsymbol{M}),\Omega(\boldsymbol{M})[-1])$ verifies the condition (i) for $\mathfrak{g}$-complexes (see 2.1.5-($\diamond$)).

The Lie derivative with respect to the vector field $\vec{Y}$, gives a Lie algebra representation $\boldsymbol{\theta} : \mathfrak{g} \to \mathbf{End}_{\mathrm{GV}(\mathbb{K})}(\Omega(\boldsymbol{M}))$ by *algebra derivations*.

Both of the operators $\boldsymbol{\theta}(Y)$ and $\boldsymbol{c}(Y)$, resp. *the $\boldsymbol{G}$-derivations and the $\boldsymbol{G}$-contractions* stabilizes the subcomplex of compact support differential forms, and $(\Omega(\boldsymbol{M}),\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$ and $(\Omega_{\mathrm{c}}(\boldsymbol{M}),\boldsymbol{d},\boldsymbol{\theta}\,\boldsymbol{c})$ become $\mathfrak{g}$-complexes in the sense of 2.1.5.

3.1.3. Definition. Let $\boldsymbol{G}$ be a compact connected Lie group. The *complex of $\boldsymbol{G}$-equivariant differential forms, resp. with compact support, of $\boldsymbol{M}$*, are the following Cartan complexes (2.2.3)

$$\begin{aligned}\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M}),\boldsymbol{d}_{\boldsymbol{G}}\big) &:= \big(\Omega(\boldsymbol{M})_{\mathfrak{g}},\boldsymbol{d}_{\mathfrak{g}}\big) = \Big(\big(\boldsymbol{S}(\mathfrak{g})\otimes\Omega(\boldsymbol{M})\big)^{\boldsymbol{G}},\boldsymbol{d}_{\mathfrak{g}}\Big)\\ \textit{resp. } \big(\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}),\boldsymbol{d}_{\boldsymbol{G}}\big) &:= \big(\Omega_{\mathrm{c}}(\boldsymbol{M})_{\mathfrak{g}},\boldsymbol{d}_{\mathfrak{g}}\big) = \Big(\big(\boldsymbol{S}(\mathfrak{g})\otimes\Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\boldsymbol{G}},\boldsymbol{d}_{\mathfrak{g}}\Big).\end{aligned}$$

Their cohomology, denoted by $H_{\boldsymbol{G}}(\boldsymbol{M})$, resp. $H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$, are the *$\boldsymbol{G}$-equivariant cohomology, resp. with compact support, of $\boldsymbol{M}$.*

In the case where $\boldsymbol{M} = \{\bullet\}$, we have $H_{\boldsymbol{G}}(\{\bullet\}) = \boldsymbol{S}(\mathfrak{g})^{\boldsymbol{G}} = \boldsymbol{S}(\mathfrak{g})^{\mathfrak{g}}$. The notation "$H_{\boldsymbol{G}}$" stands for "$H_{\boldsymbol{G}}(\{\bullet\})$".

The Cartan complexes $\Omega_{\boldsymbol{G}}(\boldsymbol{M})$,$\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ and the equivariant cohomology spaces $H_{\boldsymbol{G}}(\boldsymbol{M})$ and $H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ are $H_{\boldsymbol{G}}$-graded modules (cf. 4.1.3).

3.1.4. Proposition. *Let $\boldsymbol{G}$ be a compact connected Lie group.*

a) *The complexes of $\boldsymbol{G}$-modules $(\Omega(\boldsymbol{M}),\boldsymbol{d})$ and $(\Omega_{\mathrm{c}}(\boldsymbol{M}),\boldsymbol{d})$ are $\boldsymbol{G}$-split (2.2.7). In particular, if $\boldsymbol{C}$ denotes $(\Omega(\boldsymbol{M}),\boldsymbol{d})$ or $(\Omega_{\mathrm{c}}(\boldsymbol{M}),\boldsymbol{d})$, the inclusions*

$$\boldsymbol{S}(\mathfrak{g})^{\boldsymbol{G}}\otimes\boldsymbol{C} \supseteq \boldsymbol{S}(\mathfrak{g})^{\boldsymbol{G}}\otimes\boldsymbol{C}^{\boldsymbol{G}} \subseteq (\boldsymbol{S}(\mathfrak{g})\otimes\boldsymbol{C})^{\boldsymbol{G}}$$

are quasi-isomorphisms.

b) *The correspondence $\boldsymbol{M} \rightsquigarrow (\Omega(\boldsymbol{M}),\boldsymbol{d},\boldsymbol{\theta},\boldsymbol{c})$, $f \rightsquigarrow f^*$ is a contravariant functor from the category of $\boldsymbol{G}$-manifolds into the category of $\boldsymbol{G}$-split $\mathfrak{g}$-complexes.*

c) *The correspondence* $\boldsymbol{M} \rightsquigarrow (\Omega_c(\boldsymbol{M}), \boldsymbol{d}, \boldsymbol{\theta}, \boldsymbol{c})$, $f \rightsquigarrow f^*$ *is a contravariant functor from the category of* $\boldsymbol{G}$*-manifolds and* ***proper*** *maps to the category of* $\boldsymbol{G}$*-split* $\mathfrak{g}$*-complexes.*

d) *There exists a* ***functor*** *on the category* $\boldsymbol{G}$*-**Man*** *of* $\boldsymbol{G}$*-manifolds and* $\boldsymbol{G}$*-equivariant maps that assigns to every* $\boldsymbol{G}$*-manifold* $\boldsymbol{M}$ *a spectral sequence that converges to its equivariant cohomology*

$$\mathbb{E}_2^{p,q} = \boldsymbol{S}^p(\mathfrak{g})^{\mathfrak{g}} \otimes H^q(\boldsymbol{M}) \Rightarrow H^{p+q}_{\boldsymbol{G}}(\boldsymbol{M}).$$

e) *There exists a* ***functor*** *on the category* $\boldsymbol{G}$*-**Man***$_\pi$ *of* $\boldsymbol{G}$*-manifolds and* $\boldsymbol{G}$*-equivariant* ***proper*** *maps that assigns to every* $\boldsymbol{G}$*-manifold* $\boldsymbol{M}$ *a canonical spectral sequence that converges to its equivariant cohomology with compact support*

$$\mathbb{E}_2^{p,q} = \boldsymbol{S}^p(\mathfrak{g})^{\mathfrak{g}} \otimes H^q_c(\boldsymbol{M}) \Rightarrow H^{p+q}_{\boldsymbol{G},c}(\boldsymbol{M}).$$

Proof

a) For $i \in \mathbb{N}$, the *pushforward action* of $\boldsymbol{G}$ on $\Omega^i(\boldsymbol{M})$ is defined as $g_*(\omega) := (g^{-1})^*(\omega)$ for all $g \in \boldsymbol{G}$ and $\omega \in \Omega^i$, so that $(g_1 g_2)_* = g_{1*} \circ g_{2*}$.

If V be is a (smooth) finite dimensional representation of $\boldsymbol{G}$ over $\mathbb{C}$, we make the group $\boldsymbol{G}$ act on $\mathrm{Hom}(V,\Omega^i(\boldsymbol{M}))$ by the formula

$$(g \cdot \lambda)(v) = g_*\big(\lambda(g^{-1}v)\big), \quad \forall \lambda \in \mathrm{Hom}(V,\Omega^i(\boldsymbol{M})),$$

so that λ is a $\boldsymbol{G}$-module morphism if and only if $g \cdot \lambda = \lambda$. We claim that there exists a "symmetrization" operator

$$\Sigma : \mathrm{Hom}(V,\Omega^i(\boldsymbol{M})) \to \mathrm{Hom}(V,\Omega^i(\boldsymbol{M}))^{\boldsymbol{G}}$$

such that $\Sigma^2 = \Sigma$ and $\Sigma(\lambda) = \lambda$ if and only if λ is a $\boldsymbol{G}$-module morphism.

Indeed, let λ be a linear map from V to $\Omega^i(\boldsymbol{M})$. For every i-tuple of vector fields $\{\chi_1,\ldots,\chi_i\}$ over $\boldsymbol{M}$ and each $v \in V$, the real function

$$\boldsymbol{M} \ni x \mapsto \Big(\int_{\boldsymbol{G}} g_*\big(\lambda(g^{-1}v)\big)(x)\big(\chi_1(x),\ldots,\chi_i(x)\big)\, dg\Big) \in \mathbb{R}$$

where dg is a $\boldsymbol{G}$-invariant form of top degree on $\boldsymbol{G}$, such that $1 = \int_{\boldsymbol{G}} dg$, is a smooth function **because $\boldsymbol{V}$ is finite dimensional**, and it depends linearly on $v \in V$, and multilinearly and antisymmetrically on the χ_*. We therefore have an i-differential form which we denote by

$$\Sigma(\lambda)(v) := \int_{\boldsymbol{G}} g_*\big(\lambda(g^{-1}v)\big)\, dg, \tag{$*$}$$

and whose fundamental properties are

- $\Sigma(\boldsymbol{d} \circ \lambda) = \boldsymbol{d} \circ \Sigma(\lambda)$;
- $\Sigma(\lambda) : V \to \Omega^i(\boldsymbol{M})$ is a $\boldsymbol{G}$-module morphism;
- $\Sigma(\lambda) = \lambda$ if λ is already a $\boldsymbol{G}$-module morphism.

We can now resume the proof that $Z^i(\boldsymbol{M}) | \Omega^i(\boldsymbol{M})$. Given a $\boldsymbol{G}$-module morphism $\mu \in \mathrm{Hom}_{\boldsymbol{G}}(V, B^{i+1}(\boldsymbol{M}))$, there always exists a linear map $\lambda : V \to \Omega^i(\boldsymbol{M})$ lifting μ, i.e. such that $\mu = \boldsymbol{d} \circ \lambda$, but then one applies the symmetrization operator Σ and one gets $\mu = \Sigma(\mu) = \Sigma(\boldsymbol{d} \circ \lambda) = \boldsymbol{d} \circ \Sigma(\lambda)$, which shows that the $\boldsymbol{G}$-module morphism $\Sigma(\lambda)$ lifts μ.

For $Z^i_{\rm c}(\boldsymbol{M})|\Omega^i_{\rm c}(\boldsymbol{M})$, note that, since V is finite dimensional, the supports of the elements in $\lambda(V)$ are contained in one and the same compact subset $\boldsymbol{C}\subseteq\boldsymbol{M}$, but then the supports of the $g_*(\lambda(g^{-1}v))$ in $(*)$ are contained in $\boldsymbol{G}{\cdot}\boldsymbol{C}$ which is obviously compact. Therefore, given $\lambda: V\to\Omega_{\rm c}(\boldsymbol{M})$, one gets a linear map $\Sigma(V): V\to\Omega_{\rm c}(\boldsymbol{M})$ which is a $\boldsymbol{G}$-module morphism, and the preceding arguments apply to the compactly supported case.

To prove that $B^i(\boldsymbol{M})|Z^i(\boldsymbol{M})$, il suffices, from 2.1.10-(b), to show that every cocycle is cohomologous to a $\boldsymbol{G}$-invariant cocycle. But before doing so, let us recall a general homotopy argument. Given a smooth map $\varphi:\mathbb{R}{\times}\boldsymbol{M}\to\boldsymbol{N}$, if $\omega\in\Omega^i(\boldsymbol{N})$ the pullback $\varphi^*\omega$ belongs to $\Omega^i(\mathbb{R}{\times}\boldsymbol{M})$, i.e. is a section of the exterior algebra bundle of the cotangent bundle $T^*(\mathbb{R}{\times}\boldsymbol{M})$ of $\mathbb{R}{\times}\boldsymbol{M}$. Now, the canonical decomposition $T^*(\mathbb{R}{\times}\boldsymbol{M})$ as the direct sum of cotangent bundles $T^*(\mathbb{R})\oplus T^*(\boldsymbol{M})$, gives rise to a canonical decomposition of the i-th exterior power of the cotangent bundle

$$\bigwedge^i T^*(\mathbb{R}{\times}\boldsymbol{M})=\bigwedge^i(T^*\boldsymbol{M})\oplus\Big(T^*(\mathbb{R})\otimes\bigwedge^{i-1}(T^*\boldsymbol{M})\Big).$$

Consequently, the pullback $\varphi^*(\omega)$ canonically decomposes as

$$\varphi^*(\omega)(t,x)=\alpha(t,x)+dt\wedge\beta(t,x),$$

where α (resp. β) is a section of the vector bundle $\bigwedge^i T^*(\boldsymbol{M})$ (resp. $\bigwedge^{i-1}T^*(\boldsymbol{M})$) over the base space $\mathbb{R}{\times}\boldsymbol{M}$.

When ω is in addition a cocycle, so is $\varphi^*(\omega)$ and, in view of the previous decomposition, this amounts to the following two conditions

$$\boldsymbol{d}\alpha(t,x)=0,\qquad \frac{\partial}{\partial t}\alpha(t,x)=\boldsymbol{d}\beta(t,x),$$

where $\boldsymbol{d}$ is the coboundary in $\Omega(\boldsymbol{M})$ (t is then assumed constant). In particular, if $\varphi_t:\boldsymbol{M}\to\boldsymbol{N}$ denotes the map $x\mapsto\varphi(t,x)$, we get

$$\begin{aligned}\varphi_t^*(\omega)-\varphi_0^*(\omega)&=\alpha(t)-\alpha(0)\\&=\int_0^t\frac{\partial}{\partial t}\alpha(t)dt=\int_0^t\boldsymbol{d}\beta(t)dt=\boldsymbol{d}\Big(\int_0^t\beta(t)dt\Big),\end{aligned}\tag{$**$}$$

and the cocycles $\varphi_t^*(\omega)$ are all cohomologous to $\varphi_0^*(\omega)$.

At this point it is worth noting that this process also gives a canonical element $\varpi(x)=\int_0^1\beta(t,x)\,dt\in\Omega^{i-1}(\boldsymbol{M})$, depending on ω and such that $\varphi_1^*(\omega)-\varphi_0^*(\omega)=\boldsymbol{d}\varpi$.

Under the hypothesis of our proposition, a first consequence of these notes, is that if $\omega\in Z^i(\boldsymbol{M})$ then $g^*\omega$ is cohomologous to ω for all $g\in\boldsymbol{G}$. Indeed, since $\boldsymbol{G}$ is connected, there is a smooth path $\gamma:\mathbb{R}\to\boldsymbol{G}$ such that $\gamma(0)=e$ and $\gamma(1)=g$, and then taking $\varphi:\mathbb{R}{\times}\boldsymbol{M}\to\boldsymbol{M}$, $(t,x)\mapsto\gamma(t){\cdot}x$, one concludes that $g^*\omega=\gamma_1^*(\omega)\sim\gamma_0^*(\omega)=\omega$.

More generally, given any diffeomorphism $\phi:\mathbb{R}^{d_G}\to\boldsymbol{G}$ onto an open subset $U\subseteq\boldsymbol{G}$, one defines a smooth multiplicative action of $\mathbb{R}$ over U by setting $t\star g:=\phi(t\cdot\phi^{-1}(g))$ for all $t\in\mathbb{R}$ and $g\in U$, and considers, for each

$g \in U$, the map $\varphi_g : \mathbb{R}\times\boldsymbol{M} \to \boldsymbol{M}$, $\varphi_g(t,x) = (t \star g)x$. After that, if ω is a cocycle of $\Omega^i(\boldsymbol{M})$ we will have

$$g^*\omega - g_0^*\omega = \boldsymbol{d}\Big(\int_0^1 \beta(t,g)\,dt\Big), \qquad (***)$$

with $g_0 := \phi(0)$ and where $\beta(t,g)$ denotes a family of elements of $\Omega^{i-1}(\boldsymbol{M})$ depending smoothly on $(t,g) \in \mathbb{R}\times U$, i.e. for any $(i-1)$-tuple $(\chi_1,\dots,\chi_{i-1})$ of vector fields over $\boldsymbol{M}$, the following map is smooth:

$$\mathbb{R}\times U\times\boldsymbol{M} \ni (t,g,x) \mapsto \beta(t,g,x)(\chi_1(x),\dots,\chi_{i-1}(x)) \in \mathbb{R}.$$

We now come to a key point. If in addition, one has a compactly supported function $\rho : U \to \mathbb{R}$, then, for any top degree form dg on $\boldsymbol{G}$, one has

$$\begin{aligned}\int_{\boldsymbol{G}}\rho(g)\,g^*\omega\,dg &= \int_{\boldsymbol{G}}\rho(g)\big(g^*\omega - g_0^*\omega\big)\,dg + \Big(\int_{\boldsymbol{G}}\rho(g)\,dg\Big)g_0^*\omega\\ &= \boldsymbol{d}\Big(\int_{\boldsymbol{G}}\int_0^1 \rho(g)\beta(t,g)\,dg\Big) + \Big(\int_{\boldsymbol{G}}\rho(g)\,dg\Big)g_0^*\omega\end{aligned}$$

where $\int_{\boldsymbol{G}}\int_0^1\rho(g)\beta(t,g)\,dg$ is a **smooth** differential form over $\boldsymbol{M}$. But, as we already show that $g_0^*\omega \sim \omega$, since $\boldsymbol{G}$ is connected, we may conclude that

$$\int_{\boldsymbol{G}}\rho(g)\,g^*\omega\,dg \sim \Big(\int_{\boldsymbol{G}}\rho(g)\,dg\Big)\omega,$$

something that is satisfied by any compactly supported function $\rho : \boldsymbol{G} \to \mathbb{R}$ whose support is contained in any open subset of $\boldsymbol{M}$ diffeomorphic to $\mathbb{R}^d_{\boldsymbol{G}}$.

If we now make use of the fact that $\boldsymbol{G}$ is compact (which we haven't done so far), we can choose the form dg to be $\boldsymbol{G}$-invariant such that $\int_{\boldsymbol{G}} dg = 1$, and we can fix a smooth partition of unity $\{\rho_i\}$ subordinate to a finite good cover (cf. ([2])) of $\boldsymbol{G}$. Then

$$\begin{aligned}\Sigma(\omega) := \int_{\boldsymbol{G}} g^*\omega\,dg &= \int_{\boldsymbol{G}}\sum_i \rho_i(g)\,g^*\omega\,dg = \sum_i\int_{\boldsymbol{G}}\rho_i(g)\,g^*\omega\,dg\\ &\sim \Big(\sum_i\int_{\boldsymbol{G}}\rho_i(g)\,dg\Big)\omega = \Big(\int_{\boldsymbol{G}}\sum_i\rho_i(g)\,dg\Big)\omega = \omega\end{aligned}$$

where, obviously, $\Sigma(\omega)$ is a ***G*-invariant** cocycle, which completes the poof that $B^i(\boldsymbol{M})|Z^i(\boldsymbol{M})$ as $\boldsymbol{G}$-modules.

If we denote by $|\_|$ the support of a differential form, we see in what precedes that for $t \in [0,1]$ and $g \in \boldsymbol{G}$ one has

$$\begin{cases} |\beta(t)| \subseteq \gamma([0,1])\cdot|\omega| & \text{in } (**)\\ |\rho(g)\beta(t,g)| \subseteq ([0,1]\star|\rho|)|\omega| & \text{in } (***)\end{cases}$$

so that if $|\omega|$ is compact, the previous arguments show that $\Sigma(\omega) - \omega$ is in fact the differential of a compactly supported differential form, i.e. we have also proved that $B^i_{\mathrm{c}}(\boldsymbol{M})|Z^i_{\mathrm{c}}(\boldsymbol{M})$.

b,c,d) Follow by (a) and 2.2.6 by interchanging $\mathfrak{g}$ and $\boldsymbol{G}$, by 2.2.7 and 2.2.9. □

3.1.5. Exercise and remarks. Show that the conclusion in 3.1.4-(a) does not change if we weaken the connectedness hypothesis of $\boldsymbol{G}$ to simply require the action of $\boldsymbol{G}$ on $\boldsymbol{C}$ to be homotopically trivial. Show that this arrives in particular when, $\boldsymbol{G}$ being connected, one is interested in $H_{\boldsymbol{H}}(\boldsymbol{M})$ where $\boldsymbol{H}$ is a closed subgroup of $\boldsymbol{G}$, connected or not. In that case, if $\boldsymbol{H}_\circ$ denotes the connected component of $1 \in \boldsymbol{H}$, one has $H_{\boldsymbol{H}}(\boldsymbol{M}) = H_{\boldsymbol{H}_\circ}(\boldsymbol{M})^W$ and $H_{\boldsymbol{H},\mathrm{c}}(\boldsymbol{M}) = H_{\boldsymbol{H}_\circ,\mathrm{c}}(\boldsymbol{M})^W$, where $W = \boldsymbol{H}/\boldsymbol{H}_\circ$.

3.2. The Borel Construction

3.2.1. The Classifying Space. Let $\boldsymbol{G}$ be a compact connected Lie group and $\mathbb{E}_{\boldsymbol{G}}$ a *universal fiber bundle for* $\boldsymbol{G}$. Recall that this topological space is the limit of an inductive system in the category of (right) $\boldsymbol{G}$-manifolds $\{\mathbb{E}_{\boldsymbol{G}}(n) \to \mathbb{E}_{\boldsymbol{G}}(n+1)\}_{n\in\mathbb{N}}$, where $\mathbb{E}_{\boldsymbol{G}}(n)$ is compact, connected, oriented, n-acyclic and, moreover, the action of $\boldsymbol{G}$ on $\mathbb{E}_{\boldsymbol{G}}(n)$ is free. A *classifying space of* $\boldsymbol{G}$ is then the quotient manifold $\mathbb{B}_{\boldsymbol{G}} = \mathbb{E}_{\boldsymbol{G}}/\boldsymbol{G}$, limit of the inductive system in the category of manifolds $\{\mathbb{B}_{\boldsymbol{G}}(n) \to \mathbb{B}_{\boldsymbol{G}}(n+1)\}$ where each $\mathbb{B}_{\boldsymbol{G}}(n) := \mathbb{E}_{\boldsymbol{G}}(n)/\boldsymbol{G}$ is compact, simply connected since $\boldsymbol{G}$ is connected, and oriented.

3.2.2 Given a $\boldsymbol{G}$-manifold $\boldsymbol{M}$, the quotient $\boldsymbol{M}/\boldsymbol{G}$ may lack good differentiability properties since the action of $\boldsymbol{G}$ is not, in general, a free action. A key idea to deal with this issue, dating to the 1950s, is to replace the $\boldsymbol{G}$-manifold $\boldsymbol{M}$ by the product $\mathbb{E}_{\boldsymbol{G}} \times \boldsymbol{M}$ endowed with the *diagonal action* of $\boldsymbol{G}$, $g\cdot(e,x) := (eg^{-1},gx)$. Now, because $\mathbb{E}_{\boldsymbol{G}}$ is "contractible", the topological space $\mathbb{E}_{\boldsymbol{G}} \times \boldsymbol{M}$ has the same homotopy type as $\boldsymbol{M}$ and moreover has the advantage that $\boldsymbol{G}$ acts freely on it. The quotient space is denoted, following Armand Borel ([6]):

$$\boxed{\boldsymbol{M}_{\boldsymbol{G}} := (\mathbb{E}_{\boldsymbol{G}} \times \boldsymbol{M})/\boldsymbol{G}}$$

The natural fibration of fiber $\boldsymbol{M}$:

$$\begin{array}{ccc} \boldsymbol{M}_{\boldsymbol{G}} := \mathbb{E}_{\boldsymbol{G}} \times_{\boldsymbol{G}} \boldsymbol{M} & \xrightarrow{\pi_{\boldsymbol{M}}} & \mathbb{E}_{\boldsymbol{G}}/\boldsymbol{G} =: \mathbb{B}_{\boldsymbol{G}} \\ [e,x] & \longmapsto & [e] \end{array}$$

establishes an important link between the three spaces $\boldsymbol{M}, \boldsymbol{M}_{\boldsymbol{G}}, \mathbb{B}_{\boldsymbol{G}}$. Finally, if $f : \boldsymbol{M} \to \boldsymbol{N}$ is a $\boldsymbol{G}$-equivariant map, the induced map $f_{\boldsymbol{G}} : \boldsymbol{M}_{\boldsymbol{G}} \to \boldsymbol{N}_{\boldsymbol{G}}$, $[e,m] \mapsto [e,f(m)]$, is well defined and the diagram

$$\begin{array}{ccc} \boldsymbol{M}_{\boldsymbol{G}} & \xrightarrow{f_G} & \boldsymbol{N}_{\boldsymbol{G}} \\ {\scriptstyle\pi_{\boldsymbol{M}}}\downarrow & & \downarrow{\scriptstyle\pi_{\boldsymbol{N}}} \\ \mathbb{B}_{\boldsymbol{G}} & = & \mathbb{B}_{\boldsymbol{G}} \end{array}$$

is clearly commutative.

3.2.3. Definition. The functor $\boldsymbol{M} \rightsquigarrow \boldsymbol{M}_{\boldsymbol{G}}$, $f \rightsquigarrow f_{\boldsymbol{G}}$, from the category of $\boldsymbol{G}$-manifolds to the category of fiber spaces over the classifying space $\mathbb{B}_{\boldsymbol{G}}$, is called *the Borel construction*

3.2.4 Although the topological space $\boldsymbol{M}_{\boldsymbol{G}}$ is not a manifold, it is the limit of an inductive system of such. Indeed, for each $n \in \mathbb{N}$, since the compact group $\boldsymbol{G}$ acts freely on the product manifold $\mathbb{E}_{\boldsymbol{G}}(n) \times \boldsymbol{M}$, the topological quotient

[6] Confer §3 of chapter IV in [Bo$_2$], especially the remark §3.9, reproduced at the end of these notes, where Borel cites previous works of Conner and of Shapiro using this construction in some special cases.

$\boldsymbol{M}_{\boldsymbol{G}}(n) = \mathbb{E}_{\boldsymbol{G}}(n) \times_{\boldsymbol{G}} \boldsymbol{M}$ has a natural manifold structure, canonically oriented whenever $\boldsymbol{M}$ is so. One gets an inductive system in the category of manifolds $\{\mu_n : \boldsymbol{M}_{\boldsymbol{G}}(n) \to \boldsymbol{M}_{\boldsymbol{G}}(n+1)\}_{n\in\mathbb{N}}$ with $\boldsymbol{M}_{\boldsymbol{G}} = \varinjlim \boldsymbol{M}_{\boldsymbol{G}}(n)$, and even an inductive system in the category of fibrations with fiber $\boldsymbol{M}$ and **compact** base spaces

$$\begin{array}{ccccccc} \dashrightarrow & \boldsymbol{M}_{\boldsymbol{G}}(n) & \xrightarrow{\mu_n} & \boldsymbol{M}_{\boldsymbol{G}}(n+1) & \dashrightarrow \cdots & & \boldsymbol{M}_{\boldsymbol{G}} \\ & {\scriptstyle \pi_{\boldsymbol{M},n}}\downarrow & & {\scriptstyle \pi_{\boldsymbol{M},n+1}}\downarrow & & & {\scriptstyle \pi_{\boldsymbol{M}}}\downarrow \\ \dashrightarrow & \mathbb{B}_{\boldsymbol{G}}(n) & \xrightarrow{\beta_n} & \mathbb{B}_{\boldsymbol{G}}(n+1) & \dashrightarrow \cdots & & \mathbb{B}_{\boldsymbol{G}} \end{array}$$

giving rise to the projective system of de Rham complexes Rham

$$\big\{\Omega^*(\boldsymbol{M}_{\boldsymbol{G}}(n+1)) \xrightarrow{\mu_n^*} \Omega^*(\boldsymbol{M}_{\boldsymbol{G}}(n))\big\}_{n\in\mathbb{N}}$$

and the projective system of the Rham cohomology

$$\big\{H^d(\boldsymbol{M}_{\boldsymbol{G}}(n+1)) \xrightarrow{H^d(\mu_n^*)} H^d(\boldsymbol{M}_{\boldsymbol{G}}(n))\big\}_{n\in\mathbb{N}}$$

for each $d \in \mathbb{N}$, which has the remarkable property that, for a given d, the system is stationary, i.e. $H^d(\mu_n^*)$ is bijective for sufficiently large n.

The same remarks hold for the compact support case since the maps μ_n are proper. One then has the projective system of de Rham complexes

$$\big\{\Omega^*_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n+1)) \xrightarrow{\mu_n^*} \Omega^*_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n))\big\}_{n\in\mathbb{N}}$$

and; for each $d \in \mathbb{N}$, the stationary projective systems of the Rham cohomology

$$\big\{H^d_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n+1)) \xrightarrow{H^d(\mu_n^*)} H^d_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n))\big\}_{n\in\mathbb{N}}.$$

3.2.5. Remark. One can show that in both cases $H^d(\mu_n^*)$ is bijective for all $n > d+1$. The projective limit of $\{H^d(\boldsymbol{M}_{\boldsymbol{G}}(n))\}_{n\in\mathbb{N}}$ identifies then canonically with the d-th **singular** cohomology $H^d(\boldsymbol{M}_{\boldsymbol{G}};\mathbb{R})$, and the projective limit of $\{H^d_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n))\}_{n\in\mathbb{N}}$ with the d-th **singular** cohomology of **vertical compact support** $H^d_{\mathrm{c.v.}}(\boldsymbol{M}_{\boldsymbol{G}};\mathbb{R})$. Using methods of [Ca$_1$, Ca$_2$] one obtains canonical isomorphisms

$$H_{\boldsymbol{G}}(\boldsymbol{M}) \simeq H(\boldsymbol{M}_{\boldsymbol{G}};\mathbb{R}) \quad \text{and} \quad H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) \simeq H_{\mathrm{c.v.}}(\boldsymbol{M}_{\boldsymbol{G}};\mathbb{R}).$$

3.2.6. Serre Spectral Sequences. The fibrations $\pi_{\boldsymbol{M},n}$ in 3.2.4 are Serre fibrations and as such, give rise to a projective system of spectral sequences

$$\begin{cases} \mathbb{E}_2^{p,q}(\boldsymbol{M}_{\boldsymbol{G}}(n)) := H^p(\mathbb{B}_{\boldsymbol{G}}(n)) \otimes H^q(\boldsymbol{M}) \Rightarrow H^{p+q}(\boldsymbol{M}_{\boldsymbol{G}}(n)) \\ \mathbb{E}_{\mathrm{c},2}^{p,q}(\boldsymbol{M}_{\boldsymbol{G}}(n)) := H^p(\mathbb{B}_{\boldsymbol{G}}(n)) \otimes H^q_{\mathrm{c}}(\boldsymbol{M}) \Rightarrow H^{p+q}_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n)) \end{cases}$$

whose limits are the *(Serre) spectral sequence associated with* $\pi_{\boldsymbol{M}}$.

$$\begin{cases} \mathbb{E}_2^{p,q}(\boldsymbol{M}_{\boldsymbol{G}}) := H^p(\mathbb{B}_{\boldsymbol{G}}) \otimes H^q(\boldsymbol{M}) \\ \mathbb{E}_{\mathrm{c},2}^{p,q}(\boldsymbol{M}_{\boldsymbol{G}}) := H^p(\mathbb{B}_{\boldsymbol{G}}) \otimes H^q_{\mathrm{c}}(\boldsymbol{M}) \end{cases} \qquad (\mathbb{E}(\boldsymbol{M}_{\boldsymbol{G}}))$$

3.2.7. Proposition. *The Serre spectral sequences* $(\mathbb{E}(\boldsymbol{M}_{\boldsymbol{G}}))$ *associated with the fibration* $\pi_{\boldsymbol{M}} : \boldsymbol{M}_{\boldsymbol{G}} \to \mathbb{B}_{\boldsymbol{G}}$ *canonically identifies with the spectral sequences already met in 3.1.4-(d).*

Proof. Implicit in [Ca$_1$, Ca$_2$]. □

3.2.8. Exercise. Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be a $\boldsymbol{G}$-equivariant map between oriented $\boldsymbol{G}$-manifolds. For each $n \in \mathbb{N}$, as in 3.2.4, denote by $f_{\boldsymbol{G}}(n) : \boldsymbol{M}_{\boldsymbol{G}}(n) \to \boldsymbol{N}_{\boldsymbol{G}}(n)$ the corresponding induced map over $\mathbb{B}_{\boldsymbol{G}}(n)$.

a) Show that the following diagrams are cartesian with $\mu(n)$ and $\nu(n)$ proper.

$$\begin{array}{ccc} \boldsymbol{M}_{\boldsymbol{G}}(n) & \xrightarrow{\mu(n)} & \boldsymbol{M}_{\boldsymbol{G}}(n+1) \\ {\scriptstyle f_{\boldsymbol{G}}(n)}\downarrow & & \downarrow{\scriptstyle f_{\boldsymbol{G}}(n+1)} \\ \boldsymbol{N}_{\boldsymbol{G}}(n) & \xrightarrow{\nu(n)} & \boldsymbol{N}_{\boldsymbol{G}}(n+1) \end{array}$$

b) Prove the following equalities

$$\begin{cases} \nu(n)^* \circ f_{\boldsymbol{G}}(n+1)_* = f_{\boldsymbol{G}}(n)_* \circ \mu(n)^* \\ f_{\boldsymbol{G}}(n+1)^* \circ \nu(n)_! = \mu(n)_! \circ f_{\boldsymbol{G}}(n)^* \end{cases}$$

c) When $f : \boldsymbol{M} \to \boldsymbol{N}$ is moreover a closed embedding, one defines the *equivariant cohomology with support in* $\boldsymbol{M}$ by

$$H_{\boldsymbol{G},\boldsymbol{M}}(\boldsymbol{N}) := H_{\boldsymbol{M}_{\boldsymbol{G}}}(\boldsymbol{N}_{\boldsymbol{G}}).$$

Show that there exists a convergent spectral sequence $(E_{\boldsymbol{M}\subseteq\boldsymbol{N},r}, d_r)$

$$E^{p,q}_{\boldsymbol{M}\subseteq\boldsymbol{N},2} := H^p(\mathbb{B}_{\boldsymbol{G}}) \otimes H^q_{\boldsymbol{M}}(\boldsymbol{N}) \Rightarrow H^{p+q}_{\boldsymbol{G},\boldsymbol{M}}(\boldsymbol{N}).$$

4. Equivariant Poincaré Duality

4.1. Differential Graded Modules over a Graded Algebra

4.1.1. Graded Algebras. A *graded algebra* is a graded vector space $\boldsymbol{A} \in \mathrm{GV}(\mathbb{R})$ with a family of bilinear maps $\cdot : A^a \times A^b \to A^{a+b}$ such that the triple $(\boldsymbol{A}, 0, +, \cdot)$ is an $\mathbb{R}$-algebra.

4.1.2. Examples

a) For a graded vector space $\boldsymbol{N} \in \mathrm{GV}(\mathbb{R})$, the space of graded endomorphisms $(\mathbf{End}^{\bullet}_{\mathbb{R}}(\boldsymbol{N}), 0, +, \mathrm{id}, \circ)$ (1.1.3) is a noncommutative graded algebra.

b) $H_{\boldsymbol{G}} = \boldsymbol{S}(\mathfrak{g})^{\mathfrak{g}}$ is a positively and evenly graded commutative algebra.

c) $\Omega(\boldsymbol{M})$ and $\Omega_{\boldsymbol{G}}(\boldsymbol{M})$ are positively graded anticommutative algebras.

d) $\Omega_{\mathrm{c}}(\boldsymbol{M})$ and $\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ are positively graded anticommutative algebras, with no unit element whenever $\boldsymbol{M}$ is not compact.

4.1.3. Graded Modules. An $H_{\boldsymbol{G}}$*-graded module*, $H_{\boldsymbol{G}}$*-gm* in short, is a graded space $\boldsymbol{V} \in \mathrm{GV}(\mathbb{R})$ together with a homomorphism $H_{\boldsymbol{G}} \to \mathrm{Endgr}^0_{\mathbb{R}}(\boldsymbol{V})$ of graded algebras of degree 0. Given two $H_{\boldsymbol{G}}$-gm's $\boldsymbol{V}$ and $\boldsymbol{W}$, a *graded homomorphism of* $H_{\boldsymbol{G}}$*-gm's of degree* d *from* $\boldsymbol{V}$ *to* $\boldsymbol{W}$ is a graded homomorphism of graded spaces $\boldsymbol{\alpha} : \boldsymbol{V} \to \boldsymbol{W}$ of degree d (1.1.3), which is compatible with the action of $H_{\boldsymbol{G}}$, i.e. $\boldsymbol{\alpha}(P \cdot \boldsymbol{v}) = P \cdot \boldsymbol{\alpha}(\boldsymbol{v})$ for all $P \in H_{\boldsymbol{G}}$ and $\boldsymbol{v} \in \boldsymbol{V}$. We denote by $\mathrm{Homgr}^d_{H_{\boldsymbol{G}}}(\boldsymbol{V}, \boldsymbol{W})$ the space of such homomorphisms and by

$$\mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}(\boldsymbol{V}, \boldsymbol{W}) = \big\{\mathrm{Homgr}^d_{H_{\boldsymbol{G}}}(\boldsymbol{V}, \boldsymbol{W})\big\}_{d\in\mathbb{Z}}$$

the graded space of *graded homomorphisms of* $H_{\boldsymbol{G}}$*-gm's*.

When $d = 0$, we may write $\mathrm{Homgr}_{H_{\boldsymbol{G}}}(\boldsymbol{V}, \boldsymbol{W})$ instead of $\mathrm{Homgr}^0_{H_{\boldsymbol{G}}}(\boldsymbol{V}, \boldsymbol{W})$.

4.1.4. Example. Examples 4.1.2-(c,d) are examples of $H_{\boldsymbol{G}}$-graded modules.

4.1.5 The *category* $\mathrm{GM}(H_{\boldsymbol{G}})$ *of* $H_{\boldsymbol{G}}$*-graded modules* is the category whose objects are the $H_{\boldsymbol{G}}$-gm and whose *morphisms* are the graded homomorphisms of degree 0. We will equivalently write $\mathbf{Mor}_{\mathrm{GM}(H_{\boldsymbol{G}})}(\boldsymbol{V},\boldsymbol{W})$ and $\mathrm{Homgr}_{H_{\boldsymbol{G}}}(\boldsymbol{V},\boldsymbol{W})$ the set of morphisms from $\boldsymbol{V}$ to $\boldsymbol{W}$.

4.1.6 A direct sum $\bigoplus_{\mathfrak{a}\in\mathfrak{A}} H_{\boldsymbol{G}}[m_{\mathfrak{a}}]$, with $m_{\mathfrak{a}}\in\mathbb{Z}$, is called *a free* $H_{\boldsymbol{G}}$*-graded module*.

4.1.7. Proposition

a) *An object* $\boldsymbol{V}\in\mathrm{GM}(H_{\boldsymbol{G}})$ *is* projective (resp. injective) *if and only if the functor* $\mathbf{Hom}^\bullet(\boldsymbol{V},\_):\mathrm{GM}(H_{\boldsymbol{G}})\rightsquigarrow\mathrm{GM}(H_{\boldsymbol{G}})$ (resp. $\mathbf{Hom}^\bullet(\_,\boldsymbol{V})$) *is exact.*

b) *The category* $\mathrm{GM}(H_{\boldsymbol{G}})$ *is an abelian category with enough injective and projective objects. The cohomological dimension of* $\mathrm{GM}(H_{\boldsymbol{G}})$ *is finite and equals the rank of* $\boldsymbol{G}$.

Proof. (a) is an immediate consequence of the direct decomposition of functors

$$\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}(\_,\_)=\bigoplus\nolimits_{m\in\mathbb{Z}}\mathrm{Homgr}_{H_{\boldsymbol{G}}}(\_,\_[m])=\bigoplus\nolimits_{m\in\mathbb{Z}}\mathrm{Homgr}_{H_{\boldsymbol{G}}}(\_[-m],\_)\,.$$

(b) – For the injectivity properties, let $\{v_{\mathfrak{a}}\}_{\mathfrak{a}\in\mathfrak{A}}$ be a family of *homogeneous* generators for $\boldsymbol{V}\in\mathrm{GM}(H_{\boldsymbol{G}})$ and consider, for each $\mathfrak{a}\in\mathfrak{A}$, the map $\gamma_{\mathfrak{a}}:H_{\boldsymbol{G}}[-d_{\mathfrak{a}}]\to\boldsymbol{V}$, $x\mapsto xv_{\mathfrak{a}}$ which is clearly a morphism in $\mathrm{GM}(H_{\boldsymbol{G}})$. The sum

$$\sum\nolimits_{\mathfrak{a}\in\mathfrak{A}}\gamma_{\mathfrak{a}}:\bigoplus\nolimits_{\mathfrak{a}\in\mathfrak{A}}H_{\boldsymbol{G}}[-d_{\mathfrak{a}}]\twoheadrightarrow\boldsymbol{V}\qquad(\diamond)$$

represents $\boldsymbol{V}$ as the quotient in $\mathrm{GM}(H_{\boldsymbol{G}})$ of a free, and thus projective, $H_{\boldsymbol{G}}$-gm.

– For the injectivity properties we reproduce the proof of theorem 1.2.2 in [Go] §1.4 in the context of graded rings.

The correspondence $\boldsymbol{V}\rightsquigarrow\widehat{\boldsymbol{V}}:=\mathbf{Hom}^\bullet_{\mathbb{Z}}(\boldsymbol{V},(\mathbb{Q}/\mathbb{Z})[0])\qquad(\diamond\diamond)$

is an additive contravariant functor from the category of *left* (resp. *right*) $H_{\boldsymbol{G}}$-gm to the category of *right* (resp. *left*) $H_{\boldsymbol{G}}$-gm ([7]), and is exact, by (a), since

$$\mathrm{Homgr}_{\mathbb{Z}}(\_,(\mathbb{Q}/\mathbb{Z})[0])=\mathrm{Hom}_{\mathbb{Z}}((\_)^0,\mathbb{Q}/\mathbb{Z})$$

and since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.

Lemma 1. The map $\nu(\boldsymbol{V}):\boldsymbol{V}\to\widehat{\widehat{\boldsymbol{V}}}$, $v\mapsto(\gamma\mapsto\gamma(v))$ *is an* injective *morphism.*

Proof of lemma 1. Because $\nu(\boldsymbol{V})$ is clearly a morphism of graded modules, it is injective if and only if it doesn't kill any homogeneous nonzero element. If $0\neq v\in\boldsymbol{V}^d$, the subgroup $\mathbb{Z}{\cdot}v\subseteq\boldsymbol{V}^d$ is isomorphic to some $\mathbb{Z}/n\mathbb{Z}$ for $n\neq\pm1$, and there exists a nonzero homomorphism $\gamma'':\mathbb{Z}{\cdot}v\to\mathbb{Q}/\mathbb{Z}$ (exercise), restriction of some $\gamma':\boldsymbol{V}^d\to\mathbb{Q}/\mathbb{Z}$ (thanks to the injectivity of $\mathbb{Q}/\mathbb{Z}$). Extend this γ' to the whole of $\boldsymbol{V}$, assigning zero on the homogeneous factors $\boldsymbol{V}^e$ when $e\neq d$. This last extension, denoted by $\gamma:\boldsymbol{V}\to\mathbb{Q}/\mathbb{Z}$, is a graded morphism of degree $-d$ and verifies $\nu(\boldsymbol{V})(v)(\gamma)=\gamma(v)\neq0$ by construction, so that $\nu(\boldsymbol{V})(v)\neq0$, which completes the proof of lemma 1. ⊟

[7] If $\boldsymbol{N}$ is a *right* $H_{\boldsymbol{G}}$-gm, the structure of *left* $H_{\boldsymbol{G}}$-module of $\mathbf{Hom}^\bullet_{\mathbb{Z}}(\boldsymbol{N},(\mathbb{Q}/\mathbb{Z})[0])$ is given by $(x{\cdot}\gamma)(y):=\gamma(yx)$ for all $x\in H_{\boldsymbol{G}}$ and $y\in\boldsymbol{N}$. If $\boldsymbol{N}$ is a *left* $H_{\boldsymbol{G}}$-gm, the structure of *right* $H_{\boldsymbol{G}}$-module of $\mathbf{Hom}^\bullet_{\mathbb{Z}}(\boldsymbol{N},(\mathbb{Q}/\mathbb{Z})[0])$ is given by $(\gamma{\cdot}x)(y):=\gamma(xy)$ for all $x\in H_{\boldsymbol{G}}$ and $y\in\boldsymbol{N}$.

Lemma 2. For any free right $H_{\boldsymbol{G}}$*-gm* $\boldsymbol{F}$*, the* left $H_{\boldsymbol{G}}$*-gm* $\widehat{\boldsymbol{F}}$ *is injective.*

Proof of lemma 2. We recall (cf. [Bk] Chap. II, §4, Prop. 1) that for any left $H_{\boldsymbol{G}}$-dgm $\boldsymbol{N}$, the maps

$$\begin{array}{ccc} \mathbf{Hom}^\bullet_{H_G}(\boldsymbol{N},\mathbf{Hom}^\bullet_{\mathbb{Z}}(H_{\boldsymbol{G}},(\mathbb{Q}/\mathbb{Z})[0])) & \rightleftarrows & \mathbf{Hom}^\bullet_{\mathbb{Z}}(\boldsymbol{N},(\mathbb{Q}/\mathbb{Z})[0]) \\ \gamma & \longmapsto & \big(v\mapsto \gamma(v)(1)\big) \\ \big(v\mapsto (x\mapsto \xi(xv))\big) & \longleftarrow & \xi \end{array}$$

are isomorphisms of graded vector spaces each inverse to the other. It follows that $\mathbf{Hom}^\bullet_{\mathbb{Z}}(H_{\boldsymbol{G}},(\mathbb{Q}/\mathbb{Z})[0]))$ is an injective left $H_{\boldsymbol{G}}$-gm if and only if the functor $\mathbf{Hom}^\bullet_{\mathbb{Z}}(\_,(\mathbb{Q}/\mathbb{Z})[0])$ is exact, but this is equivalent, by (a), to the exactness of the functor $\mathrm{Homgr}_{\mathbb{Z}}(\_,(\mathbb{Q}/\mathbb{Z})[0]) = \mathrm{Hom}_{\mathbb{Z}}((\_)^0,\mathbb{Q}/\mathbb{Z})$, which is clear since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. ⊟

Now, if $\boldsymbol{V}$ is a left $H_{\boldsymbol{G}}$-gm, fix some epimorphism of right $H_{\boldsymbol{G}}$-gm $\pi : \boldsymbol{F} \twoheadrightarrow \widehat{\boldsymbol{V}}$ where $\boldsymbol{F}$ is free as in (⋄). The morphism $\widehat{\pi} : \widehat{\widehat{\boldsymbol{V}}} \to \widehat{\boldsymbol{F}}$ is injective and composed with $\nu(\boldsymbol{V}) : \boldsymbol{V} \to \widehat{\widehat{\boldsymbol{V}}}$, injective by lemma 1, we get an injective morphism $\boldsymbol{V} \hookrightarrow \widehat{\boldsymbol{F}}$ of left $H_{\boldsymbol{G}}$-gm, where $\widehat{\boldsymbol{F}}$ is an injective left $H_{\boldsymbol{G}}$-gm by lemma 2. This completes the proof of the existence of enough injective objects in $\mathrm{GM}(H_{\boldsymbol{G}})$.

The statement about $\dim_{\mathrm{ch}}(\mathrm{GM}(H_{\boldsymbol{T}}))$ results from the fact (Chevalley's theorem) that $H_{\boldsymbol{G}}$ is a polynomial algebra in $\mathrm{rk}(\boldsymbol{G})$ variables. One may then refer to Hilbert's Syzygy Theorem (cf. [J] p. 385, and Ex. 2, p. 387). □

4.1.8. Exercise. Let $\boldsymbol{A}$ be a graded $\mathbb{R}$-algebra which is an integral domain.

a) Show that $S^{-1}\boldsymbol{A}$, where S denotes the multiplicative system of homogeneous nonzero elements of $\boldsymbol{A}$, is an injective object of $\mathrm{GM}(\boldsymbol{A})$. Also, prove that the canonical inclusion $\boldsymbol{A} \hookrightarrow S^{-1}\boldsymbol{A}$ is an injective envelope for $\boldsymbol{A}$.
b) Show that when $\mathrm{rk}(\boldsymbol{G}) > 0$, the degrees of a non trivial injective object of $\mathrm{GM}(H_{\boldsymbol{G}})$ cannot be bounded below ([8]).
c) Show that if $\boldsymbol{V} \in \mathrm{GM}(H_{\boldsymbol{G}})$ is positively graded, it admits projective resolutions in $\mathrm{GM}(H_{\boldsymbol{G}})$ all of whose terms are positively graded.

The next two sections are straightforward generalizations of sections 1.1.5 and 1.1.8 from graded vector spaces to $H_{\boldsymbol{G}}$-graded modules.

4.1.9. Differential Graded Modules. An $H_{\boldsymbol{G}}$*-differential graded module*, $H_{\boldsymbol{G}}$*-dgm* in short, is a pair $(\boldsymbol{V},\boldsymbol{d})$ with $\boldsymbol{V} \in \mathrm{GM}(H_{\boldsymbol{G}})$ and $\boldsymbol{d} \in \mathrm{Endgr}^1_{H_G}(\boldsymbol{V})$, called *differential*, is such that $(\boldsymbol{V},\boldsymbol{d})$ is a complex, i.e. $\boldsymbol{d}^2 = 0$. A *morphism of* $H_{\boldsymbol{G}}$*-dgm* $\boldsymbol{\alpha} : (\boldsymbol{V},\boldsymbol{d}) \to (\boldsymbol{V}',\boldsymbol{d}')$ is a morphism of $H_{\boldsymbol{G}}$-gm's which is also a morphism of complexes, i.e. $\boldsymbol{d}' \circ \boldsymbol{\alpha} = \boldsymbol{\alpha} \circ \boldsymbol{d}$. The $H_{\boldsymbol{G}}$-dgm's and their morphisms constitute the *category* $\mathrm{DGM}(H_{\boldsymbol{G}})$ *of* $H_{\boldsymbol{G}}$*-differential graded modules*. The category $\mathrm{DGM}(H_{\boldsymbol{G}})$ is an abelian category.

4.1.10. The $\mathbf{Hom}^\bullet(\_,\_)$ and $(\_\otimes\_)^\bullet$ Bi-functors. Given two $H_{\boldsymbol{G}}$-dgm's $(\boldsymbol{V},\boldsymbol{d})$ and $(\boldsymbol{V}',\boldsymbol{d}')$, we recall the definition of the $H_{\boldsymbol{G}}$-dgm's

$$\big(\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},\boldsymbol{V}'),\boldsymbol{D}_\bullet\big) \quad \text{and} \quad \big((\boldsymbol{V}\otimes_{H_G}\boldsymbol{V}')^\bullet,\boldsymbol{\Delta}_\bullet\big).$$

[8] A graded space $\boldsymbol{V}$ is said to be *bounded below (resp. above)*, if there exists $N \in \mathbb{Z}$ such that $\boldsymbol{V}^i = 0$ for all $i < N$ (resp. $i > N$). The graded algebra $H_{\boldsymbol{G}}$ is bounded below by 0.

As $H_{\boldsymbol{G}}$-graded modules they are defined by

$$m \mapsto \begin{cases} \mathbf{Hom}^m_{H_{\boldsymbol{G}}}(\boldsymbol{V},\boldsymbol{V}') := \mathrm{Homgr}^m_{H_{\boldsymbol{G}}}(\boldsymbol{V},\boldsymbol{V}') \\ (\boldsymbol{V} \otimes_{H_{\boldsymbol{G}}} \boldsymbol{V}')^m := \pi(\boldsymbol{V} \otimes_{\mathbb{R}} \boldsymbol{V}')^m \end{cases}$$

where $\pi : \boldsymbol{V} \otimes_{\mathbb{R}} \boldsymbol{V}' \twoheadrightarrow \boldsymbol{V} \otimes_{H_{\boldsymbol{G}}} \boldsymbol{V}'$, $v \otimes v' \mapsto [v \otimes v']$, is the canonical (graded) surjection (see remark 4.1.12). The differentials $\boldsymbol{D}_\bullet$ and $\boldsymbol{\Delta}_\bullet$ are:

$$\begin{cases} \boldsymbol{D}_m(f) = \boldsymbol{d}' \circ f - (-1)^m f \circ \boldsymbol{d} \\ \boldsymbol{\Delta}_m([v \otimes v']) = [\boldsymbol{d}(v) \otimes v'] + (-1)^{|v|}[v \otimes \boldsymbol{d}'(v')] \end{cases}$$

where $v \otimes v' \in V^{|v|} \otimes V'^{|v'|}$ and $|v| + |v'| = m$.

The fact that $\boldsymbol{D}$ and $\boldsymbol{\Delta}$ are $H_{\boldsymbol{G}}$-linear results from the fact that $H_{\boldsymbol{G}}$ is graded only by **even** degrees (!).

4.1.11 These constructions are natural w.r.t. each side entry which means that one has in fact defined two bi-functors

$$\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}((\_),(\_)) : \mathrm{DGM}(H_{\boldsymbol{G}}) \times \mathrm{DGM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{DGM}(H_{\boldsymbol{G}})$$
$$((\_) \otimes_{H_{\boldsymbol{G}}} (\_))^\bullet : \mathrm{DGM}(H_{\boldsymbol{G}}) \times \mathrm{DGM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{DGM}(H_{\boldsymbol{G}})$$

which are bi-additive and have the usual variances and exactnesses. For example, the first one is contravariant and left exact on the left entry, and covariant and left exact on the right entry, while the second one is bi-covariant and right exact.

4.1.12. Remark. Some care must be taken with the tensor product since it hides some subtleties. A good way to understand it is to note that $\boldsymbol{V} \otimes_{H_{\boldsymbol{G}}} \boldsymbol{V}'$ is the quotient of the graded space $\boldsymbol{V} \otimes_{\mathbb{R}} \boldsymbol{V}'$ by the subspace $\boldsymbol{W}$ spanned by the tensors $Pv \otimes v' - v \otimes Pv'$ with $P \in H_{\boldsymbol{G}}$ and $(v,v') \in \boldsymbol{V} \times \boldsymbol{V}'$ both *homogeneous*. One then shows that $\boldsymbol{W}$ is a graded subcomplex of $(\boldsymbol{V} \otimes_{\mathbb{R}} \boldsymbol{V}',\boldsymbol{\Delta})$, so that the canonical surjection $\pi : (\boldsymbol{V} \otimes_{\mathbb{R}} \boldsymbol{V}',\boldsymbol{\Delta}) \twoheadrightarrow (\boldsymbol{V} \otimes_{\mathbb{R}} \boldsymbol{V}',\boldsymbol{\Delta})/\boldsymbol{W}$ is an epimorphism of graded complexes, therefore inducing over $\boldsymbol{V} \otimes_{H_{\boldsymbol{G}}} \boldsymbol{V}'$ a structure of $H_{\boldsymbol{G}}$-dgm. Again, a key point is that $H_{\boldsymbol{G}}$ is graded only by **even** degrees.

4.1.13. The Dual Complex. In section 1.1.11 we introduced the duality functor $\mathbf{Hom}^\bullet_{\mathbb{K}}(\_,\mathbb{K}) : \mathrm{DGM}(\mathbb{K}) \rightsquigarrow \mathrm{DGM}(\mathbb{K})$ and noted that it was an exact functor (1.1.13). In the framework of $H_{\boldsymbol{G}}$-dgm's, the corresponding functor is the *$H_{\boldsymbol{G}}$-duality functor*

$$\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}(\_,H_{\boldsymbol{G}}) : \mathrm{DGM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{DGM}(H_{\boldsymbol{G}})$$

which is generally **not** exact, **nor does** it respect quasi-isomorphisms.

4.1.14. The Forgetful Functor. If we disregard differentials, $H_{\boldsymbol{G}}$-dgm's simply appear as $H_{\boldsymbol{G}}$-gm's, and likewise for morphisms. Forgetting the complex structure gives the *forgetful functor* $o : \mathrm{DGM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{GM}(H_{\boldsymbol{G}})$ which is exact and will often be implicit in some of our considerations.

4.2. Deriving Functors

4.2.1. Deriving Functors Defined on the Category GM($H_{\boldsymbol{G}}$). We have already shown (4.1.7) that the abelian category $\mathrm{GM}(H_{\boldsymbol{G}})$ has enough projective and injective objects ([9]). We will now recall the definition of the *left and right*

[9] See Grothendieck [Gr], chapter **I**, Thm. 1.10, p. 135.

derived functors associated with an additive functor $\boldsymbol{F} : \mathbf{Ab}' \to \mathbf{Ab}$ between abelian categories where $\mathbf{Ab}'$ has enough projective and injective objects.

The *left and right derived functors*, respectively $\mathbb{L}_\star \boldsymbol{F} : \mathbf{Ab}' \rightsquigarrow \mathcal{K}_\star(\mathbf{Ab})$ and $\mathbb{R}^\star \boldsymbol{F} : \mathbf{Ab}' \rightsquigarrow \mathcal{K}^\star(\mathbf{Ab})$ ([10]) applied to an object $\boldsymbol{V} \in \mathbf{Ab}'$ are defined by the following steps. First, choose an injective and a projective resolution of $\boldsymbol{V}$,

$$\mathbf{0} \longrightarrow \boldsymbol{V} \xrightarrow{\ \epsilon\ } \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \mathcal{I}^2 \xrightarrow{d_2} \cdots$$

$$\cdots \xrightarrow{d_{-2}} \mathcal{P}^{-2} \xrightarrow{d_{-1}} \mathcal{P}^{-1} \xrightarrow{d_0} \mathcal{P}^0 \xrightarrow{\ \epsilon\ } \boldsymbol{V} \longrightarrow \mathbf{0}.$$

Next, let $\mathcal{I}^\star \boldsymbol{V}$ stand for the truncated complex $\big(0 \to \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \cdots\big)$, and $\mathcal{P}^\star \boldsymbol{V}$ for $\big(\cdots \xrightarrow{d_{-1}} \mathcal{P}^{-1} \xrightarrow{d_0} \mathcal{P}^0 \to \mathbf{0}\big)$, and set

$$\begin{cases} \mathbb{L}^\star \boldsymbol{F}(\boldsymbol{V}) := \boldsymbol{F}(\mathcal{P}^\star \boldsymbol{V}) \\ \mathbb{R}^\star \boldsymbol{F}(\boldsymbol{V}) := \boldsymbol{F}(\mathcal{I}^\star \boldsymbol{V}) \end{cases} \tag{$*$}$$

One proves that the complexes $(*)$ are homotopically independent of the chosen resolutions so that each canonically defines an object of $\mathcal{K}^*(\mathbf{Ab})$.

As the targets of the derived functors $\mathbb{R}^\star \boldsymbol{F}$ and $\mathbb{L}^\star \boldsymbol{F}$ are complexes, one is interested in their cohomologies. Their classical notations are

$$\begin{cases} (\mathbb{R}^i \boldsymbol{F})(\boldsymbol{V}) := H^i(\mathbb{R}^\star(\_)) \\ (\mathbb{L}^i \boldsymbol{F})(\_) := H^i(\mathbb{L}^\star \boldsymbol{F}(\_)). \end{cases}$$

It is easily seen from the above definitions that the *augmentation morphisms* of complexes $\epsilon : \boldsymbol{V}[0] \to \mathcal{I}^\star$ and $\epsilon : \mathcal{P}_\star \to \boldsymbol{V}[0]$, give rise to natural morphisms of complexes $\boldsymbol{F}(\boldsymbol{V}[0]) \to (\mathbb{R}^\star \boldsymbol{F})(\boldsymbol{V})$ and $(\mathbb{L}^\star \boldsymbol{F})(\boldsymbol{V}) \to \boldsymbol{F}(\boldsymbol{V}[0])$, inducing canonical morphisms

$$\boldsymbol{F}(\boldsymbol{V}) \to (\mathbb{R}^0 \boldsymbol{F})(\boldsymbol{V}) \quad \text{and} \quad (\mathbb{L}^0 \boldsymbol{F})(\boldsymbol{V}) \to \boldsymbol{V}.$$

These are isomorphisms whenever $\boldsymbol{F}$ is respectively left and right exact.

4.2.2. Simple Complex Associated with a Double Complex. The category $\mathcal{C}^\natural(\mathbf{Ab})$ of (cochain) complexes of an abelian category $\mathbf{Ab}$ is again an abelian category so that we can look at the category $\mathcal{C}^{\star,\natural}(\mathbf{Ab}) := \mathcal{C}^\star(\mathcal{C}^\natural(\mathbf{Ab}))$ of (cochain) complexes of $\mathcal{C}^\natural(\mathbf{Ab})$ also called *double (cochain) complexes of* $\mathbf{Ab}$. A double complex $\mathbf{N}^{\star,\natural} := (\mathbf{N}^{\star,\natural}, \delta_{\star,\natural}, d_{\star,\natural}) \in \mathcal{C}^{\star,\natural}(\mathbf{Ab})$ is generally represented as a two dimensional ladder all of whose subdiagrams are commutative.

$$\begin{array}{ccccccccc}
 & & \uparrow{\scriptstyle d_{i-1,j+1}} & & \uparrow{\scriptstyle d_{i,j+1}} & & \uparrow{\scriptstyle d_{i+1,j+1}} & & \\
\xrightarrow{\delta_{i-2,j+1}} & & \mathbf{N}^{i-1,j+1} & \xrightarrow{\delta_{i-1,j+1}} & \mathbf{N}^{i,j+1} & \xrightarrow{\delta_{i,j+1}} & \mathbf{N}^{i+1,j+1} & \xrightarrow{\delta_{i+1,j+1}} & \\
 & & \uparrow{\scriptstyle d_{i-1,j}} & & \uparrow{\scriptstyle d_{i,j}} & & \uparrow{\scriptstyle d_{i+1,j}} & & \\
\xrightarrow{\delta_{i-2,j}} & & \mathbf{N}^{i-1,j} & \xrightarrow{\delta_{i-1,j}} & \mathbf{N}^{i,j} & \xrightarrow{\delta_{i,j}} & \mathbf{N}^{i+1,j} & \xrightarrow{\delta_{i+1,j}} & \\
 & & \uparrow{\scriptstyle d_{i-1,j-1}} & & \uparrow{\scriptstyle d_{i,j-1}} & & \uparrow{\scriptstyle d_{i+1,j-1}} & & \\
\xrightarrow{\delta_{i-2,j-1}} & & \mathbf{N}^{i-1,j-1} & \xrightarrow{\delta_{i-1,j-1}} & \mathbf{N}^{i,j-1} & \xrightarrow{\delta_{i,j-1}} & \mathbf{N}^{i+1,j-1} & \xrightarrow{\delta_{i+1,j-1}} & \\
 & & \uparrow{\scriptstyle d_{j-2}} & & \uparrow{\scriptstyle d_{j-2}} & & \uparrow{\scriptstyle d_{j-2}} & &
\end{array}$$

[10] $\mathcal{K}^\star(\mathbf{Ab})$ (resp. $\mathcal{K}_\star(\mathbf{Ab})$) is the category of cochain (resp. chain) complexes of $\mathbf{Ab}$ whose morphisms are the morphisms of complexes modulo homotopy.

The *simple (or total) complex* associated with$\boldsymbol{N}^{\star,\natural}$ is the complex $(\mathbf{Tot}^\circ(\boldsymbol{N}^{\star,\natural}), D_\circ)$, where, for all $m \in \mathbb{Z}$,

$$\begin{cases} \mathbf{Tot}^m(\boldsymbol{N}^{\star,\natural}) := \bigoplus_{m=a+b} \boldsymbol{N}^{i,j} \\ \quad D_m(\boldsymbol{n}_{i,j}) := \boldsymbol{d}_{i,j}(\boldsymbol{n}_{i,j}) + (-1)^j \delta_{i,j}(\boldsymbol{n}_{i,j}) \end{cases} \qquad \begin{array}{ccc} \boldsymbol{N}^{i,j+1} & & \\ \boldsymbol{d}_{i,j} \uparrow & & \\ \boldsymbol{N}^{i,j} & \xrightarrow{(-1)^j \delta_{i,j}} & \boldsymbol{N}^{i+1,j} \end{array}$$

In this way, one obtains an additive exact functor

$$\mathbf{Tot}^\circ := \mathcal{C}^{\star,\natural}(\mathbf{Ab}) \rightsquigarrow \mathcal{C}^\circ(\mathbf{Ab}).$$

4.2.3. Spectral Sequences Associated with Double Complexes. The double complex $\boldsymbol{N}^{\star,\natural}$ is said to be *of the first quadrant* if $\{(i,j) \mid \boldsymbol{N}^{i,j} \neq \mathbf{0}\} \subseteq \mathbb{N} \times \mathbb{N}$. As explained in [Go] (§4.8, p. 86), one assigns to this kind of double complex, two *regular* decreasing filtrations of $(\mathbf{Tot}^\circ(\boldsymbol{N}^{\star,\natural}), D_\circ)$. The first is relative to the *line* $\natural$*-filtration* $\mathbf{Tot}^\circ(\boldsymbol{N}^{\star,\natural})_\ell := \mathbf{Tot}^\circ(\boldsymbol{N}^{\star,\natural\geq\ell})$, and the second to the *column* $\star$*-filtration* $\mathbf{Tot}^\circ(\boldsymbol{N}^{\star,\natural})_c := \mathbf{Tot}^\circ(\boldsymbol{N}^{\star\geq c,\natural})$. Each filtration gives rise to a spectral sequence converging to the cohomology of $(\mathbf{Tot}^\circ \boldsymbol{N}^{\star,\natural}, D_\circ)$, respectively

$$\begin{cases} {}'\mathbb{E}_2^{p,q} := H_\natural^p H_\star^q(\boldsymbol{N}^{\star,\natural}) \Rightarrow H_\circ^{p+q}(\mathbf{Tot}^\circ \boldsymbol{N}^{\star,\natural}, D_\circ), \\ {}''\mathbb{E}_2^{p,q} := H_\star^p H_\natural^q(\boldsymbol{N}^{\star,\natural}) \Rightarrow H_\circ^{p+q}(\mathbf{Tot}^\circ \boldsymbol{N}^{\star,\natural}, D_\circ), \end{cases}$$

where $H_\star$ (resp. $H_\natural$) is the cohomology w.r.t. $\delta_\star$ (resp. $d_\natural$).

4.2.4. The $\mathbb{R}^\star\mathbf{Hom}^\bullet_{H_G}(\_,\_)$ and $(\_)\otimes^{\mathbb{L}^\star}_{H_G}(\_)$ Bi-functors. Given two $H_{\boldsymbol{G}}$-graded modules $\boldsymbol{V}, \boldsymbol{W} \in \mathrm{GM}(H_{\boldsymbol{G}})$, we may consider the four functors

$$\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V}, \_), \quad \mathbf{Hom}^\bullet_{H_G}(\_, \boldsymbol{W}), \quad \boldsymbol{V} \otimes_{H_G} (\_), \quad (\_) \otimes_{H_G} \boldsymbol{W},$$

where the first two are left exact and the other two are right exact.

Now, given projective resolutions $\mathcal{P}^\natural(\boldsymbol{V}) \to \boldsymbol{V}$, $\mathcal{P}^\natural(\boldsymbol{W}) \to \boldsymbol{W}$ and an injective resolution $\boldsymbol{W} \to \mathcal{I}^\star(\boldsymbol{W})$ in $\mathrm{GM}(H_{\boldsymbol{G}})$, we have natural morphisms of double complexes

$$\begin{array}{ccccc} \mathbf{Hom}(\mathcal{P}^\natural(\boldsymbol{V}), \boldsymbol{W}[0]^\star) & \longrightarrow & \mathbf{Hom}(\mathcal{P}^\natural(\boldsymbol{V}), \mathcal{I}^\star\boldsymbol{W}) & \longleftarrow & \mathbf{Hom}(\boldsymbol{V}[0]^\natural, \mathcal{I}^\star\boldsymbol{W}) \\ \mathcal{P}^\natural(\boldsymbol{V}) \otimes \boldsymbol{W}[0]^\star & \longrightarrow & \mathcal{P}^\natural(\boldsymbol{V}) \otimes \mathcal{P}^\star(\boldsymbol{W}) & \longleftarrow & \boldsymbol{V}[0]^\natural \otimes \mathcal{P}^\star(\boldsymbol{W}) \end{array}$$

([11]) where, to avoid confusion, we omit the indication of the $\bullet$-grading, giving rise to canonical morphisms of complexes on $H_{\boldsymbol{G}}$-gm

$$\begin{array}{ccccc} \mathbf{Hom}(\mathcal{P}^\natural(\boldsymbol{V}), \boldsymbol{W}) & \longrightarrow & \mathbf{Tot}\,\mathbf{Hom}(\mathcal{P}^\natural(\boldsymbol{V}), \mathcal{I}^\star\boldsymbol{W}) & \longleftarrow & \mathbf{Hom}(\boldsymbol{V}, \mathcal{I}^\star\boldsymbol{W}) \\ \mathcal{P}^\natural(\boldsymbol{V}) \otimes \boldsymbol{W} & \longrightarrow & \mathbf{Tot}\left(\mathcal{P}^\natural(\boldsymbol{V}) \otimes \mathcal{P}^\star(\boldsymbol{W})\right) & \longleftarrow & \boldsymbol{V} \otimes \mathcal{P}^\star(\boldsymbol{W}) \end{array} \tag{‡}$$

The following proposition is classical (*loc. cit.*).

4.2.5. Proposition. *The morphisms* (‡) *are quasi-isomorphisms.* ([12])

Sketch of the proof. For the first line of (‡), one notes that the morphisms of complexes are compatible with line and column filtrations of double complexes of the first quadrant. In the case of

$$\mathbf{Hom}(\mathcal{P}^\natural(\boldsymbol{V}), \boldsymbol{W}) \to \mathbf{Tot}\,\mathbf{Hom}(\mathcal{P}^\natural(\boldsymbol{V}), \mathcal{I}^\star\boldsymbol{W}),$$

[11] By $\boldsymbol{W}[0]^\bullet$ we denote the complex satisfying $\boldsymbol{W}[0]^0 = \boldsymbol{W}$ and $\boldsymbol{W}[0]^i = \mathbf{0}$ for $i \neq 0$.
[12] In fact they are homotopic equivalences, but we won't need to be so precise.

since for each $i \in \mathbb{Z}$ the map $\mathbf{Hom}(\mathcal{P}^i(\boldsymbol{V}), \boldsymbol{W}) \to \mathbf{Tot}\mathbf{Hom}(\mathcal{P}^i(\boldsymbol{V}),\mathcal{I}^\star \boldsymbol{W})$ is a quasi-isomorphism, the induced map on the $''\mathbb{E}$ terms of the associated spectral sequences (4.2.3) is an isomorphism and we conclude. The case of

$$\mathbf{Tot}\mathbf{Hom}(\mathcal{P}^\natural(\boldsymbol{V}),\mathcal{I}^\star \boldsymbol{W}) \longleftarrow \mathbf{Hom}(\boldsymbol{V},\mathcal{I}^\star \boldsymbol{W})$$

is almost the same except that now we must consider the line filtration and use the $'\mathbb{E}$ spectral sequence.

The second line in (‡) is dealt with in the same way after observing that 4.2.3 also applies (dually) to double complexes of the *third* quadrant. ⊡

As a consequence of 4.2.5, in each line of (‡) the complexes represent the *same objet* in the derived category $\mathcal{D}^\star(\mathrm{GM}(H_{\boldsymbol{G}}))$. They are classically denoted by $\mathbb{R}^\star\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V}, \boldsymbol{W})$ and $\boldsymbol{V} \otimes^{\mathbb{L}^\star}_{H_G} \boldsymbol{W}$. The constructions are natural w.r.t. each entry so that we get two bi-functors

$$\begin{aligned} \mathbb{R}^\star\mathbf{Hom}^\bullet_{H_G}((\_),(\_)) &: \mathrm{GM}(H_{\boldsymbol{G}}) \times \mathrm{GM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathcal{D}^\star(\mathrm{GM}(H_{\boldsymbol{G}})) \\ ((\_) \otimes^{\mathbb{L}^\star}_{H_G} (\_))^\bullet &: \mathrm{GM}(H_{\boldsymbol{G}}) \times \mathrm{GM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathcal{D}^\star(\mathrm{GM}(H_{\boldsymbol{G}})) \end{aligned} \qquad (\diamond)$$

which are bi-additive and have the usual variances and exactnesses. They clearly extend the bi-functors in 4.1.11 from $\mathrm{GM}(H_{\boldsymbol{G}})$ to $\mathcal{D}^\star(\mathrm{GM}(H_{\boldsymbol{G}}))$.

4.2.6. The Ext$^\bullet$ and Tor$^\bullet$ Bi-functors. Given $\boldsymbol{V}, \boldsymbol{W} \in \mathrm{GM}(H_{\boldsymbol{G}})$, one defines for $i \in \mathbb{Z}$

$$\begin{cases} \mathbf{Ext}^{i,\bullet}_{H_G}(\boldsymbol{V}, \boldsymbol{W}) := H^i_\star\big(\mathbb{R}^\star\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V}, \boldsymbol{W})\big) \\ \mathbf{Tor}^\bullet_{H_G,i}(\boldsymbol{V}, \boldsymbol{W}) := H^i_\star\big(\boldsymbol{V} \otimes^{\mathbb{L}^\star}_{H_G} \boldsymbol{W}\big) \end{cases}$$

Where $H^i_\star$ is the i'th cohomology functor on $\mathcal{D}^\star(\mathrm{GL}(H_{\boldsymbol{G}}))$.

4.2.7. Defining $\mathbb{R}^\star\mathbf{Hom}^\bullet(\_,H_{\boldsymbol{G}})$ on DGM$(H_{\boldsymbol{G}})$. We proceed as in 4.2.4 except that we will consider only injective resolutions of $H_{\boldsymbol{G}}$ in $\mathrm{GM}(H_{\boldsymbol{G}})$. ([13])

Let $\boldsymbol{V} := (\boldsymbol{V},\boldsymbol{d}) \in \mathrm{DGM}(H_{\boldsymbol{G}})$. For any $\boldsymbol{N} \in \mathrm{GM}(H_{\boldsymbol{G}})$, we already endowed the $H^\bullet_{\boldsymbol{G}}$-graded module $\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},\boldsymbol{N})$ with a canonical structure of $H^\bullet_{\boldsymbol{G}}$-differential graded module $(\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},\boldsymbol{N}),\boldsymbol{D}_\bullet)$ (cf. 4.1.11) in such a way that we obtain a left exact functor

$$\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},\_) : \mathrm{GM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{DGM}(H_{\boldsymbol{G}}).$$

It follows that the target of the functor $\mathbb{R}^\star\mathbf{Hom}^\bullet_{H_G}(\_,H_{\boldsymbol{G}}) := \mathbf{Hom}^\bullet_{H_G}(\_,\mathcal{I}^\star\boldsymbol{N})$ is the category $\mathcal{C}^\star(\mathrm{DGM}(H_{\boldsymbol{G}}))$. The functor transforms homotopies to identities, and respects quasi-isomorphisms, it therefore induces a contravariant functor

$$\mathbb{R}^\star\mathbf{Hom}^\bullet_{H_G}(\_,H_{\boldsymbol{G}}) : \mathrm{DGM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathcal{D}^\star(\mathrm{DGM}(H_{\boldsymbol{G}})), \qquad (\diamond)$$

[13] The good notion of projective resolution for an $H_{\boldsymbol{G}}$-dgm $(\boldsymbol{V},\boldsymbol{d})$ is the one of *simultaneous* projective resolutions. These are resolutions $\cdots \to \mathcal{P}^2 \to \mathcal{P}^1 \to \mathcal{P}^0 \to \boldsymbol{V} \to 0$ $(*)$ in $\mathrm{DGM}(H_{\boldsymbol{T}})$ where $\mathcal{P}^i$ is a projective $H_{\boldsymbol{G}}$-gm's, and such that the *derived sequence* $\cdots \to \boldsymbol{h}\mathcal{P}^2 \to \boldsymbol{h}\mathcal{P}^1 \to \boldsymbol{h}\mathcal{P}^0 \to \boldsymbol{h}\boldsymbol{V} \to 0$ $(**)$ is a projective resolution in $\mathrm{GM}(H_{\boldsymbol{G}})$. When the graded space $\boldsymbol{V}$ is bounded below (cf. ([8])), the complex $(\boldsymbol{V},\boldsymbol{d})$ always admits *simultaneous projective resolutions* and the derived functor $\mathbb{R}^\star\mathbf{Hom}^\bullet((\boldsymbol{V},\boldsymbol{d}),H_{\boldsymbol{G}})$ may as well be defined as $\mathbf{Hom}^\bullet(\mathcal{P}^\natural,H_{\boldsymbol{G}})$, as in the case of $H_{\boldsymbol{G}}$-gm's, but we won't use this point of view in these notes.

which we will call *the derived duality functor*. This functor, composed with the i'th cohomology functor $H^i_\star : \mathcal{D}^\star(\mathrm{DGM}(H_{\boldsymbol{G}})) \rightsquigarrow \mathrm{DGM}(H_{\boldsymbol{G}})$ gives the *i'th extension functor*

$$\mathbf{Ext}^{i,\bullet}_{H_G}(\_,H_{\boldsymbol{G}}) := H^i_\star\big(I\!\!R^\star\mathbf{Hom}^\bullet_{H_G}(\_,H_{\boldsymbol{G}})\big) : \mathrm{DGM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{DGM}(H_{\boldsymbol{G}}).$$

The family (indexed by $\star \in \mathbb{Z}$) of all these functors

$$\mathbf{Ext}^{\star,\bullet}_{H_G}(\_,H_{\boldsymbol{G}}) := H^\star\big(I\!\!R^\star\mathbf{Hom}^\bullet_{H_G}(\_,H_{\boldsymbol{G}})\big) : \mathrm{DGM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{DGM}^{\star,\bullet}(H^\bullet_{\boldsymbol{G}}) \quad (\diamond)$$

where $\mathrm{DGM}^{\star,\bullet}(H^\bullet_{\boldsymbol{G}})$ is the category of $\star$-graded $H^\bullet_{\boldsymbol{G}}$-dgm (the action of $H^\bullet_{\boldsymbol{G}}$ does not affect the $\star$-degree), constitutes a ∂-functor in $\mathcal{K}^\star(\mathrm{DGM}(H_{\boldsymbol{G}}))$.

4.2.8. Spectral Sequences Associated with $I\!\!R^\star\mathbf{Hom}^\bullet_{H_G}(\_,H_{\boldsymbol{G}})$. In the last paragraph we defined the derived duality functor ($\diamond$) for any $H_{\boldsymbol{G}}$-dgm $(\boldsymbol{V},\boldsymbol{d})$ as the complex of $H_{\boldsymbol{G}}$-dgm's

$$I\!\!R^\star\mathbf{Hom}^\bullet_{H_G}((\boldsymbol{V},\boldsymbol{d}),H_{\boldsymbol{G}}) := \mathbf{Hom}^\bullet_{H_G}((\boldsymbol{V},\boldsymbol{d}),(\mathcal{I}^\star H_{\boldsymbol{G}},\delta_\star)), \qquad (\dagger)$$

that will be represented as a double complex with lines indexed by '$\bullet$' and columns by ''$\star$'. The differentials $\boldsymbol{d}$ and $\delta_\star$ commute, $\boldsymbol{d}$ increases de $\bullet$-degree and leaves unchanged the $\star$-degree, while $\delta_\star$ does the opposite.

4.2.9. Proposition *Let* $(\boldsymbol{V},\boldsymbol{d}) \in \mathrm{DGM}(H_{\boldsymbol{G}})$.

a) *There exist convergent spectral sequences*

$$\begin{cases} {}'I\!\!E^{p,q} := H^p_\bullet(\mathbf{Ext}^{q,\bullet}_{H_G}(\boldsymbol{V},H_{\boldsymbol{G}})) \Rightarrow H^{p+q}_\circ\big(\mathbf{Tot}^\circ I\!\!R^\star\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},H_{\boldsymbol{G}}),D_\circ\big) \\ {}''I\!\!E^{p,q} := \mathbf{Ext}^{p,q}_{H_G}(\boldsymbol{h}\boldsymbol{V},H_{\boldsymbol{G}})) \quad\; \Rightarrow H^{p+q}_\circ\big(\mathbf{Tot}^\circ I\!\!R^\star\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},H_{\boldsymbol{G}}),D_\circ\big) \end{cases}$$

b) *If $\boldsymbol{V}$ is projective as $H_{\boldsymbol{G}}$-gm* ([14]), *then the following morphism of complexes induced by the augmentation $\epsilon : H_{\boldsymbol{G}} \to \mathcal{I}^\star$ is a quasi-isomorphism:*

$$\mathbf{Hom}^\bullet_{H_G}((\boldsymbol{V},\boldsymbol{d}),H_G) \xrightarrow{(\epsilon)} \mathbf{Tot}^\circ I\!\!R^\star\mathbf{Hom}^\bullet_{H_G}((\boldsymbol{V},\boldsymbol{d}),H_{\boldsymbol{G}}).$$

c) *If $\boldsymbol{h}\boldsymbol{V}$ is projective as $H_{\boldsymbol{G}}$-gm, then the following natural morphisms of $H_{\boldsymbol{G}}$-gm's are isomorphisms:*

$$\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{h}\boldsymbol{V},H_G) \xrightarrow{(\epsilon)} \mathbf{Tot}^\circ\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{h}\boldsymbol{V},\mathcal{I}^\star) \longrightarrow \boldsymbol{h}\big(\mathbf{Tot}^\circ\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},\mathcal{I}^\star)\big)$$

Proof. (a) By 4.1.7 we can fix an injective resolution $H_{\boldsymbol{G}} \to (\mathcal{I}_\star H_{\boldsymbol{G}},\delta_\star)$ of $H_{\boldsymbol{G}}$-gm of *finite length*, whereby the line $\bullet$-filtration and the column $\star$-filtration are both regular for the total order '$\bullet + \star$'. We have

$$\begin{aligned} ({}'I\!\!E^{p,\star}_0,d_0) &= (\mathrm{Homgr}^p(\boldsymbol{V},\mathcal{I}^* H_{\boldsymbol{G}}),\delta_\star) \\ ({}''I\!\!E^{p,\bullet}_0,d_0) &= \mathbf{Hom}^\bullet((\boldsymbol{V},\boldsymbol{d}),\mathcal{I}^p H_{\boldsymbol{G}}) \end{aligned}$$

and the proposition follows. (b,c) are straightforward consequences of (a). □

4.2.10. Proposition. *Let* $(\boldsymbol{V},\boldsymbol{d})$ *be an* $H_{\boldsymbol{G}}$*-dgm.*

a) *For any $\boldsymbol{N} \in \mathrm{GM}(H_{\boldsymbol{G}})$, there exists a natural morphism of $H_{\boldsymbol{G}}$-modules*

$$\xi(\boldsymbol{V},\boldsymbol{N}) : \boldsymbol{h}(\mathbf{Hom}^\bullet_{H_G}((\boldsymbol{V},\boldsymbol{d}),\boldsymbol{N})) \to \mathbf{Hom}^\bullet_{H_G}(\boldsymbol{h}\boldsymbol{V},\boldsymbol{N})$$

b) *If $\boldsymbol{V}$ and $\boldsymbol{h}\boldsymbol{V}$ are projective (free) $H_{\boldsymbol{G}}$-gm, then $\xi(\boldsymbol{V},H_{\boldsymbol{G}})$ is an isomorphism.*

[14] A projective $H_{\boldsymbol{G}}$-gm is always free, cf. in [J] the corollary of theorem 6.21, p. 386.

c) *Let* $(\boldsymbol{V},\boldsymbol{d})$ *and* $(\boldsymbol{V}',\boldsymbol{d}')$ *be* $H_{\boldsymbol{G}}$*-dgm's where* $\boldsymbol{V}$ *and* $\boldsymbol{V}'$ *are projective (free)* $H_{\boldsymbol{G}}$*-gm's. If* $\boldsymbol{\alpha} : (\boldsymbol{V},\boldsymbol{d}) \to (\boldsymbol{V}',\boldsymbol{d}')$ *is a quasi-isomorphism of* $H_{\boldsymbol{G}}$*-dgm's, the following induced morphism of* $H_{\boldsymbol{G}}$*-dgm's is a quasi-isomorphism:*

$$\mathbf{Hom}^\bullet_{\boldsymbol{H}}(\boldsymbol{\alpha},H_{\boldsymbol{G}}) : \mathbf{Hom}^\bullet_{\boldsymbol{H}}((\boldsymbol{V}',\boldsymbol{d}'),H_{\boldsymbol{G}}) \to \mathbf{Hom}^\bullet_{\boldsymbol{H}}((\boldsymbol{V},\boldsymbol{d}),H_{\boldsymbol{G}})\,.$$

Proof. In order to minimise notations we shall write '$\mathbf{Hom}^\bullet$' for '$\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}$'.

Let $\boldsymbol{Z} \subseteq \boldsymbol{V}$, resp. $\boldsymbol{B} \subseteq \boldsymbol{V}$, denote the $H_{\boldsymbol{G}}$-graded submodules of cocycles, resp. coboundaries, of $(\boldsymbol{V},\boldsymbol{d})$, and let $\boldsymbol{N}$ be any $H_{\boldsymbol{G}}$-graded module.

(a) Applying the functor $\mathbf{Hom}^\bullet(\_,\boldsymbol{N})$ to the short exact sequence:

$$\mathbf{0} \to \boldsymbol{Z} \longrightarrow \boldsymbol{V} \xrightarrow{\boldsymbol{d}} \boldsymbol{B}[1] \to \mathbf{0}\,, \tag{$\dagger$}$$

one gets the left exact sequence of $H_{\boldsymbol{G}}$-complexes

$$\mathbf{0} \to \mathbf{Hom}^\bullet(\boldsymbol{B},\boldsymbol{N})[-1] \xrightarrow{\alpha} \mathbf{Hom}^\bullet(\boldsymbol{V},\boldsymbol{N}) \xrightarrow{\beta} \mathbf{Hom}^\bullet(\boldsymbol{Z},\boldsymbol{N})\,,$$

and the short exact sequence of $H_{\boldsymbol{G}}$-complexes

$$\mathbf{0} \to \mathbf{Hom}^\bullet(\boldsymbol{B},\boldsymbol{N})[-1] \xrightarrow{\alpha} \mathbf{Hom}^\bullet(\boldsymbol{V},\boldsymbol{N}) \xrightarrow{\beta} \boldsymbol{Q}^\bullet(\boldsymbol{Z},\boldsymbol{N}) \to \mathbf{0}\,, \tag{$*$}$$

where $\boldsymbol{Q}^\bullet(\boldsymbol{Z},\boldsymbol{N})$ denotes the image of β. Note that the left and right complexes in $(*)$ have null differentials so that they coincide with their cohomology.

The cohomology sequence associated with $(*)$ is the long exact sequence

$$\xrightarrow{c_{i-1}} \mathbf{Hom}^i(\boldsymbol{B},\boldsymbol{N}) \xrightarrow{a_i} \boldsymbol{h}^i\mathbf{Hom}^\bullet(\boldsymbol{V},\boldsymbol{N}) \xrightarrow{b_i} \boldsymbol{Q}^i(\boldsymbol{Z},\boldsymbol{N}) \xrightarrow{c_i} \mathbf{Hom}^i(\boldsymbol{B},\boldsymbol{N}) \xrightarrow{a_{i+1}}\,,$$

where one easily verifies that c_i is the restriction to $\boldsymbol{Q}^i(\boldsymbol{V},\boldsymbol{N})$ of the natural map $\mathbf{Hom}^\bullet(\boldsymbol{Z},\boldsymbol{N}) \to \mathbf{Hom}^\bullet(\boldsymbol{B},\boldsymbol{N})$ induced by the inclusion $\boldsymbol{B} \subseteq \boldsymbol{Z}$. In this way we obtain the exact triangle of $H_{\boldsymbol{G}}$-graded modules

$$\boldsymbol{h}\mathbf{Hom}^\bullet(\boldsymbol{V},\boldsymbol{N}) \xrightarrow{b} \boldsymbol{Q}^\bullet(\boldsymbol{Z},\boldsymbol{N}) \xrightarrow{c} \mathbf{Hom}^\bullet(\boldsymbol{B},\boldsymbol{N}) \xrightarrow{a[+1]}\,. \tag{$**$}$$

On the other hand, if we apply $\mathbf{Hom}^\bullet(\_,\boldsymbol{N})$ to the short exact sequence

$$\mathbf{0} \to \boldsymbol{B} \subseteq \boldsymbol{Z} \to \boldsymbol{h}\boldsymbol{V} \to \mathbf{0}\,, \tag{$\dagger\dagger$}$$

we obtain the left exact sequence

$$\mathbf{0} \to \mathbf{Hom}^\bullet(\boldsymbol{h}\boldsymbol{V},\boldsymbol{N}) \xrightarrow{b'} \mathbf{Hom}^\bullet(\boldsymbol{Z},\boldsymbol{N}) \xrightarrow{c'} \mathbf{Hom}^\bullet(\boldsymbol{B},\boldsymbol{N})\,,$$

which, joined to $(**)$, gives rise to the following commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccc} & & \boldsymbol{h}\mathbf{Hom}^\bullet(\boldsymbol{V},\boldsymbol{N}) & \xrightarrow{b} & \boldsymbol{Q}^\bullet(\boldsymbol{V},\boldsymbol{N}) & \xrightarrow{c} & \mathbf{Hom}^\bullet(\boldsymbol{B},\boldsymbol{N}) \\ & & \downarrow{\scriptstyle \xi(\boldsymbol{V},\boldsymbol{N})} & & \downarrow{\scriptstyle \subseteq} & \bigoplus & \downarrow{\scriptstyle =} \\ \mathbf{0} & \longrightarrow & \mathbf{Hom}^\bullet(\boldsymbol{h}\boldsymbol{V},\boldsymbol{N}) & \xrightarrow{b'} & \mathbf{Hom}^\bullet(\boldsymbol{Z},\boldsymbol{N}) & \xrightarrow{c'} & \mathbf{Hom}^\bullet(\boldsymbol{B},\boldsymbol{N}) \end{array} \tag{$\mathcal{D}$}$$

establishing the existence of $\xi(\boldsymbol{V},\boldsymbol{N})$.

(b) If $\boldsymbol{h}\boldsymbol{V}$ is projective, the connecting morphism c' is surjective and

$$I\!R^i\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}(\boldsymbol{Z},\boldsymbol{N}) = I\!R^i\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}(\boldsymbol{B},\boldsymbol{N})\,, \quad \forall i \geq 1\,. \tag{$\diamond$}$$

It follows that $\xi(\boldsymbol{V},\boldsymbol{N})$ is bijective if and only if $\boldsymbol{Q}^\bullet(\boldsymbol{V},\boldsymbol{N}) = \mathbf{Hom}^\bullet(\boldsymbol{Z},\boldsymbol{N})$, which is equivalent, as $\boldsymbol{V}$ is projective, to $I\!R^1\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}(\boldsymbol{B}[1],\boldsymbol{N}) = \mathbf{0}$, and to

$$I\!R^1\mathbf{Hom}^\bullet_{H_{\boldsymbol{G}}}(\boldsymbol{Z}[1],\boldsymbol{N}) = \mathbf{0}\,, \tag{$\diamond\diamond$}$$

thanks to $(\diamond)$. Let us prove this equality.

For each $\ell \in \mathbb{Z}$, the projectivity of $\boldsymbol{V}[\ell]$ and the exactness of (†), implies that

$$I\!R^i\,\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{Z}[\ell],\boldsymbol{N}) \simeq I\!R^{i+1}\,\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{B}[\ell{+}1],\boldsymbol{N}), \quad \forall i \geq 1,$$

and, again by (⋄),

$$I\!R^i\,\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{Z}[\ell],\boldsymbol{N}) \simeq I\!R^{i+1}\,\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{Z}[\ell{+}1],\boldsymbol{N}), \quad \forall i \geq 1,$$

so that we have, for all $m \geq 1$

$$I\!R^1\,\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{Z}[1],\boldsymbol{N}) \simeq I\!R^{1+m}\,\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{Z}[1{+}m],\boldsymbol{N}),$$

and (⋄⋄) follows from tha fact that $\dim_{\mathrm{ch}}(\mathrm{GM}(H_{\boldsymbol{G}})) < +\infty$ (4.1.7-(b)).

(c) Consider the exact triangle in $\mathrm{DGM}(H_{\boldsymbol{G}})$

$$(\boldsymbol{V},\boldsymbol{d}) \xrightarrow{\boldsymbol{\alpha}} (\boldsymbol{V}',\boldsymbol{d}') \xrightarrow{p_1} (\hat{c}(\boldsymbol{\alpha}),\Delta) \xrightarrow[{[+1]}]{p_2}$$

where $(\hat{c}(\boldsymbol{\alpha}),\Delta)$ denotes the *cone* of $\boldsymbol{\alpha}$, i.e. the $H_{\boldsymbol{G}}$-gm $\hat{c}(\boldsymbol{\alpha}) := \boldsymbol{V}' \oplus \boldsymbol{V}[1]$ with differential $\Delta(v',w) := (\boldsymbol{d}'v' + \boldsymbol{\alpha}\omega, -\boldsymbol{d}\omega)$. Note that $\boldsymbol{h}(\hat{c}(\boldsymbol{\alpha})) = 0$ since by the universal property of the cone construction, $\hat{c}(\boldsymbol{\alpha})$ is acyclic if and only if $\boldsymbol{\alpha}$ is a quasi-isomorphism. Now, since additive functors respect exactness of triangles and cones, the morphism $\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{\alpha},H_{\boldsymbol{G}})$ is a quasi-isomorphism if and only if the complex $\hat{c}(\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{\alpha},H_{\boldsymbol{G}})) = \mathbf{Hom}^\bullet_{H_G}(\hat{c}(\boldsymbol{\alpha}),H_{\boldsymbol{G}})$ is acyclic. This is indeed the case following (b) because, $\hat{c}(\boldsymbol{\alpha})$ and $\boldsymbol{h}(\hat{c}(\boldsymbol{\alpha}))$ being both projective $H_{\boldsymbol{G}}$-gm, we have $\boldsymbol{h}(\mathbf{Hom}^\bullet_{H_G}(\hat{c}(\boldsymbol{\alpha}),H_{\boldsymbol{G}})) = \mathbf{Hom}^\bullet_{H_G}(\boldsymbol{h}(\hat{c}(\boldsymbol{\alpha})),H_{\boldsymbol{G}}) = 0$. □

4.2.11. Remarks. If one disregards the morphism $\xi(\boldsymbol{V},H_{\boldsymbol{G}})$ in 4.2.10-(a), then the fact that $\boldsymbol{h}(\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{V},H_{\boldsymbol{G}}))$ and $\mathbf{Hom}^\bullet_{H_G}(\boldsymbol{h}\boldsymbol{V},H_{\boldsymbol{G}})$ are isomorphic $H_{\boldsymbol{G}}$-gm's when $\boldsymbol{V}$ and $\boldsymbol{h}\boldsymbol{V}$ are projectives is an immediate result of 4.2.9-(b,c).

The statement 4.2.10-(c) is a straightforward consequence of 4.2.9-(b). Indeed, it claims that the restriction of the functor $\mathbf{Hom}^\bullet_{H_G}(\_,H_{\boldsymbol{G}})$ to the full subcategory of $H_{\boldsymbol{G}}$-dgm's $(\boldsymbol{V},\boldsymbol{d})$ with $\boldsymbol{V}$ projective (free) as $H_{\boldsymbol{G}}$-gm is a derived functor, so that, as such, it preserves quasi-isomorphisms.

4.2.12. Exercise. Prove that $\boldsymbol{V}$ and $\boldsymbol{h}\boldsymbol{V}$ are projective (free) $H_{\boldsymbol{G}}$-gm if and only if $\boldsymbol{Z}$ and $\boldsymbol{B}$ are. *Hint. Follow the ideas in the proof of 4.2.10-(b).*

4.3. Equivariant Integration

Let $\boldsymbol{G}$ be a **compact connected** Lie group and $\boldsymbol{M}$ an oriented $\boldsymbol{G}$-manifold of dimension $d_{\boldsymbol{M}}$.

Extend the $\mathbb{R}$-linear integration map $\int_{\boldsymbol{M}} : \Omega_{\mathrm{c}}(\boldsymbol{M}) \to \mathbb{R}$ by $\boldsymbol{S}(\mathfrak{g})$-linearity to the map

$$\begin{aligned} \int_{\boldsymbol{M}} : \boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M}) &\longrightarrow \boldsymbol{S}(\mathfrak{g}) \\ P \otimes \omega &\longmapsto P \textstyle\int_{\boldsymbol{M}} \omega \end{aligned} \qquad (\diamond)$$

As $\boldsymbol{G}$ acts on $\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})$ by $g\cdot(P\otimes\omega) := g\cdot P \otimes g\cdot\omega$, the above integration map is $\boldsymbol{G}$-equivariant since one has $\int_{\boldsymbol{M}} g\cdot\omega = \int_{\boldsymbol{M}}\omega$, as a consequence of the connectedness of $\boldsymbol{G}$ (see proof 3.1.4-(a)). Therefore, the restriction of (⋄) to the subspace of $\boldsymbol{G}$-equivariant differential forms with compact support

$$\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) := \big(\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\boldsymbol{G}} = \big(\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\mathfrak{g}}$$

takes values in $H_{\boldsymbol{G}} := \boldsymbol{S}(\mathfrak{g})^{\boldsymbol{G}}$ (3.1.3). We denote this restriction by

$$\int_{\boldsymbol{M}} : \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) \to H_{\boldsymbol{G}} \qquad (\diamond\diamond)$$

and call it *the equivariant integration*, which is clearly a morphism of $H_{\boldsymbol{G}}$-graded modules of degree $-d_{\boldsymbol{M}}$.

Now, the graded algebra $\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ has already been endowed with the differential $\boldsymbol{d}_{\boldsymbol{G}}(P\otimes\omega) = P\otimes\boldsymbol{d}\omega + \sum_i Pe^i\otimes\boldsymbol{c}(e_i)\omega$ (see 3.1.3, 2.2.2-($\boldsymbol{d}_{\mathfrak{g}}$)), and a homogeneous equivariant form $\zeta\in\Omega^d_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ of total degree d decomposes in a unique way as a sum

$$\zeta = \sum_{0\le i\le d/2}\Big(\sum_{Q\in\mathscr{B}(i)} Q\otimes\omega_Q\Big)$$

where $\mathscr{B}(i)$ denotes a vector space basis of $S^i(\mathfrak{g})$ and $\omega_Q\in\Omega^{d-2\deg Q}(\boldsymbol{M})$. As a consequence, one easily checks that if ζ is an equivariant coboundary, the terms $Q\otimes\omega_Q$ in the above decomposition such that $\omega_Q\in\Omega_{\mathrm{c}}(\boldsymbol{M})$ is of top degree $d_{\boldsymbol{M}}$ are already coboundaries, i.e. $\omega_Q\in B^{d_M}_{\mathrm{c}}(\boldsymbol{M})$, and consequently $\int_{\boldsymbol{M}}\zeta = 0$. We have thereby proved the following lemma.

4.3.1. Lemma. $\displaystyle\int_{\boldsymbol{M}} \boldsymbol{d}_{\boldsymbol{G}}(\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})) = 0.$

Therefore, the equivariant integration $(\diamond\diamond)$ induces a morphism of $H_{\boldsymbol{G}}$-graded modules of degree $[-d_{\boldsymbol{M}}]$ in cohomology:

$$\int_{\boldsymbol{M}} : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) \to H_{\boldsymbol{G}} \qquad (\diamond\diamond\diamond)$$

also called *equivariant integration*.

4.3.2. Equivariant Integration vs. Integration Along Fibers. As we already pointed out in 3.2.7, $\boldsymbol{G}$-equivariant cohomology is canonically isomorphic to the projective limit of the de Rham cohomologies of the fibered spaces $\pi_n : \boldsymbol{M}_{\boldsymbol{G}}(n) = \mathbb{E}_{\boldsymbol{G}}(n)\times_{\boldsymbol{G}}\boldsymbol{M} \twoheadrightarrow \mathbb{B}_{\boldsymbol{G}}(n)$ (3.2.4). Moreover, for each fixed $d\in\mathbb{N}$ the projective system $\{H^d(\boldsymbol{M}_{\boldsymbol{G}}(n))\}_n$ is stationary and converges to $H^d(\boldsymbol{M}_{\boldsymbol{G}})$. Now, each $\pi_{\boldsymbol{M},n} : \boldsymbol{M}_{\boldsymbol{G}}(n) \to \mathbb{B}_{\boldsymbol{G}}(n)$ is a fibration with oriented base space, whose fiber is the oriented manifold $\boldsymbol{M}$. The operation of integration along $\boldsymbol{M}$ is then well defined $\int_{\boldsymbol{M}} : H_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n))[d_{\boldsymbol{M}}] \to H_{\mathrm{c}}(\mathbb{B}_{\boldsymbol{G}}(n))$ (see 1.9.5) and induces a limit map

$$\pi_{\boldsymbol{M},*} : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H(\mathbb{B}_{\boldsymbol{G}}) = H_{\boldsymbol{G}}(\{\bullet\})$$

Proposition. *The map $\pi_{\boldsymbol{M},*}$ coincides with the equivariant integration.*

Proof. Left to the reader. □

4.4. Equivariant Poincaré Pairing

Equivariant integration is what we need to mimic the nonequivariant Poincaré pairing (1.3) in the equivariant framework.

4.4.1 The composition of the $H_{\boldsymbol{G}}$-bilinear map $\Omega_{\boldsymbol{G}}(\boldsymbol{M})\otimes\Omega_{\boldsymbol{G},c}(\boldsymbol{M}) \to \Omega_{\boldsymbol{G},c}(\boldsymbol{M})$, $\alpha\otimes\beta \mapsto \alpha\wedge\beta$, with equivariant integration $\int_{\boldsymbol{M}} : \Omega_{\boldsymbol{G},c}(\boldsymbol{M}) \to H_{\boldsymbol{G}}$, gives rise to a nondegenerate pairing

$$\begin{aligned} \mathbb{P}_{\boldsymbol{G}}(\boldsymbol{M}) : \Omega_{\boldsymbol{G}}(\boldsymbol{M}) \otimes \Omega_{\boldsymbol{G},c}(\boldsymbol{M}) &\longrightarrow H_{\boldsymbol{G}} \\ \alpha\otimes\beta \quad &\longmapsto \int_{\boldsymbol{M}} \alpha\wedge\beta \end{aligned} \tag{$\mathbb{P}_{\boldsymbol{G}}$}$$

inducing the *Poincaré pairing in equivariant cohomology*

$$\begin{aligned} \mathscr{P}_{\boldsymbol{G}}(\boldsymbol{M}) : H_{\boldsymbol{G}}(\boldsymbol{M}) \otimes H_{\boldsymbol{G},c}(\boldsymbol{M}) &\longrightarrow H_{\boldsymbol{G}} \\ [\alpha]\otimes[\beta] \quad &\longmapsto \int_{\boldsymbol{M}} [\alpha]\cup[\beta] \end{aligned} \tag{$\mathscr{P}_{\boldsymbol{G}}$}$$

4.4.2 The left adjoint map associated with $\mathbb{P}_{\boldsymbol{G}}$ (see 1.3) is now the map

$$\begin{aligned} \mathbb{D}_{\boldsymbol{G}}(\boldsymbol{M}) : \Omega_{\boldsymbol{G}}(\boldsymbol{M}) &\longrightarrow \mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}\big(\Omega_{\boldsymbol{G},c}(\boldsymbol{M}),H_{\boldsymbol{G}}\big) \\ \alpha \quad &\longmapsto \mathbb{D}_{\boldsymbol{G}}(\boldsymbol{M})(\alpha) := \Big(\beta\mapsto \int_{\boldsymbol{M}} \alpha\wedge\beta\Big) \end{aligned} \tag{$\mathbb{D}_{\boldsymbol{G}}$}$$

and one has, following lemma 4.3.1, for α homogeneous

$$\begin{aligned} \mathbb{D}_{\boldsymbol{G}}\big((-1)^{d_M}\mathbf{d}_{\boldsymbol{G}}\alpha\big)(\beta) &= \int_{\boldsymbol{M}} (-1)^{d_M}\mathbf{d}_{\boldsymbol{G}}\alpha\wedge\beta \\ &= \int_{\boldsymbol{M}} (-1)^{d_M+|\alpha|+1}\alpha\wedge\mathbf{d}_{\boldsymbol{G}}\beta = (-1)^{|\beta|}\mathbb{D}_{\boldsymbol{G}}(\alpha)(\mathbf{d}_{\boldsymbol{G}}\beta), \end{aligned}$$

Hence, following the conventions in 1.1.7 and 4.1.11, $\mathbb{D}_{\boldsymbol{G}}(\boldsymbol{M})$ is a morphism of $H_{\boldsymbol{G}}$-graded complexes from $\Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}]$ to $\mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}(\Omega_{\boldsymbol{G},c}(\boldsymbol{M}),H_{\boldsymbol{G}})$.

4.4.3 The right adjoint map associated with $\mathbb{P}_{\boldsymbol{G}}$ (see 1.7.1) is the map

$$\begin{aligned} \mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{M}) : (\Omega_{\boldsymbol{G},c}(\boldsymbol{M}),\mathbf{d}_{\boldsymbol{G}}) &\longrightarrow \big(\mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M}),H_{\boldsymbol{G}}\big),-\mathbf{D}\big) \\ \beta \quad &\longmapsto \mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{M})(\beta) := \Big(\alpha\mapsto \int_{\boldsymbol{M}} \alpha\wedge\beta\Big) \end{aligned} \tag{$\mathbb{D}'_{\boldsymbol{G}}$}$$

which is also a morphism of $H_{\boldsymbol{G}}$-graded complexes.

4.4.4. Exercise. Verify that ($\mathbb{P}_{\boldsymbol{G}}$) is a nondegenerate pairing and that $\mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{M})$ is a morphism of $H_{\boldsymbol{G}}$-graded complexes.

4.5. $\boldsymbol{G}$-Equivariant Poincaré Duality Theorem

4.5.1. Theorem. *Let $\boldsymbol{G}$ be a compact connected Lie group, and $\boldsymbol{M}$ an oriented $\boldsymbol{G}$-manifold of dimension $d_{\boldsymbol{M}}$. Then,*

a) *The $H_{\boldsymbol{G}}$-graded morphism of complexes*

$$\boxed{\mathbb{D}_{\boldsymbol{G}}(\boldsymbol{M}) : \Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}\big(\Omega_{\boldsymbol{G},c}(\boldsymbol{M}),H_{\boldsymbol{G}}\big)}$$

is a quasi-isomorphism.

b) *The morphism $\mathbb{D}_{\boldsymbol{G}}(\boldsymbol{M})$ induces* the "Poincaré morphism in $\boldsymbol{G}$-equivariant cohomology" *(see 4.2.10-(a))*

$$\boxed{\mathscr{D}_{\boldsymbol{G}}(\boldsymbol{M}) : H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}\big(H_{\boldsymbol{G},c}(\boldsymbol{M}),H_{\boldsymbol{G}}\big)}$$

If $H_{\boldsymbol{G},c}(\boldsymbol{M})$ is a free $H_{\boldsymbol{G}}$-module, $\mathscr{D}_{\boldsymbol{G}}(\boldsymbol{M})$ is an isomorphism.

c) *There are natural spectral sequence converging to* $H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}]$

$$\begin{cases} I\!\!E_2^{p,q}(\boldsymbol{M}) = \big(\mathbf{Ext}^p_{H_G}\big(H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}),H_{\boldsymbol{G}}\big)\big)^q \Rightarrow H_{\boldsymbol{G}}^{p+q}(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ I\!\!F_2^{p,q}(\boldsymbol{M}) = H^p_{\boldsymbol{G}} \otimes_{\mathbb{R}} \mathbf{Hom}^\bullet_{\mathbb{R}}(H^q_{\mathrm{c}}(\boldsymbol{M}),\mathbb{R}) \Rightarrow H_{\boldsymbol{G}}^{p+q}(\boldsymbol{M})[d_{\boldsymbol{M}}] \end{cases}$$

where, in the first one, q denotes the graded vector space degree.

d) *Moreover, if $\boldsymbol{M}$ is of finite type, the $H_{\boldsymbol{G}}$-graded morphism of complexes*

$$\boxed{I\!\!D'_{\boldsymbol{G}}(\boldsymbol{M}) : \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}^\bullet_{H_G}\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M}),H_{\boldsymbol{G}}\big)}$$

is a quasi-isomorphism, and mutatis mutandis for (b) *and* (c).

Proof

a) We recall the filtration of the Cartan complex we already used in the proof of 2.2.6-(b): An equivariant form in $(\Omega_{\boldsymbol{G}}(\boldsymbol{M}),\boldsymbol{d}_{\boldsymbol{G}})$ is said to be *of index m* if it belongs to the subspace

$$\Omega_{\boldsymbol{G}}(\boldsymbol{M})_m := \big(\boldsymbol{S}^{\geqslant m}(\mathfrak{g}) \otimes \Omega(\boldsymbol{M})\big)^{\boldsymbol{G}}.$$

One easily checks that each $\Omega_{\boldsymbol{G}}(\boldsymbol{M})_m$ is stable under the Cartan differential $\boldsymbol{d}_{\boldsymbol{G}}$, that $\Omega_{\boldsymbol{G}}(\boldsymbol{M}) = \Omega_{\boldsymbol{G}}(\boldsymbol{M})_m$ for all $m \leqslant 0$ and that one has a decreasing filtration

$$\Omega_{\boldsymbol{G}}(\boldsymbol{M}) = \Omega_{\boldsymbol{G}}(\boldsymbol{M})_0 \supseteq \Omega_{\boldsymbol{G}}(\boldsymbol{M})_1 \supseteq \Omega_{\boldsymbol{G}}(\boldsymbol{M})_2 \supseteq \cdots \qquad (*)$$

Furthermore, $\Omega^i_{\boldsymbol{G}}(\boldsymbol{M}) \cap \Omega_{\boldsymbol{G}}(\boldsymbol{M})_m = 0$ whenever $m > i$, so that $(*)$ is a *regular filtration* (see [Go] §4 pp. 76-).

In a similar way, $\lambda \in \mathbf{Hom}^\bullet_{H_G}(\Omega_{\boldsymbol{G}}(\boldsymbol{M}),H_{\boldsymbol{G}})$ is said to be *of index m* whenever

$$\lambda\Big(\big(\boldsymbol{S}^a(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\boldsymbol{G}}\Big) \subseteq H^{\geqslant a+m}_{\boldsymbol{G}}, \qquad \forall a \in \mathbb{N},$$

and we denote $\mathbf{Hom}^\bullet_{H_G}(\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}),H_{\boldsymbol{G}})_m$ the subspace of such maps. As before, each of these spaces is is a subcomplex of $(\mathbf{Hom}^\bullet_{H_G}(\Omega_{\boldsymbol{G}}(\boldsymbol{M})),\boldsymbol{D})$ and the decreasing filtration

$$\cdots \supseteq \mathbf{Hom}^\bullet_{H_G}(\Omega_{\boldsymbol{G},\mathrm{c}},H_{\boldsymbol{G}})_m \supseteq \mathbf{Hom}^\bullet_{H_G}(\Omega_{\boldsymbol{G},\mathrm{c}},H_{\boldsymbol{G}})_{m+1} \supseteq \cdots \qquad (**)$$

verifies for each λ homogeneous of degree i

$$a + \dim \boldsymbol{M} + i \geqslant \deg \lambda\Big(\big(\boldsymbol{S}^a(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\boldsymbol{G}}\Big) \geqslant a + i, \qquad \forall a \in \mathbb{N},$$

so that $(**)$ is also regular.

An immediate verification shows that $I\!\!D_{\boldsymbol{G}}(\boldsymbol{M})$ is a morphism of graded filtered modules, i.e.

$$I\!\!D_{\boldsymbol{G}}(\boldsymbol{M})\big(\Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}]_m\big) \subseteq \mathbf{Hom}^\bullet_{H_G}\big(\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}),H_{\boldsymbol{G}}\big)_m, \qquad \forall m \in \mathbb{Z},$$

giving rise, therefore, to a morphism between the associated spectral sequences (see [Go], §4 Thm. 4.3.1, p. 80) whose $I\!\!E_0$ terms are

$$\begin{cases} \big((\boldsymbol{S}(\mathfrak{g}) \otimes \Omega(\boldsymbol{M}))^{\boldsymbol{G}},1 \otimes \boldsymbol{d}\big)[d_{\boldsymbol{M}}] \\ \mathbf{Hom}^\bullet_{H_G}\big(\big(\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\boldsymbol{G}},1 \otimes \boldsymbol{d}\big),H_{\boldsymbol{G}}\big) \end{cases}$$

and which are respectively quasi-isomorphic to

$$\begin{cases} H_{\boldsymbol{G}} \otimes \big(\Omega(\boldsymbol{M}),\boldsymbol{d}\big)[d_{\boldsymbol{M}}] \\ \mathbf{Hom}^{\bullet}_{H_G}\big(H_{\boldsymbol{G}} \otimes \big(\Omega(\boldsymbol{M}),\boldsymbol{d}\big),H_{\boldsymbol{G}}\big) \end{cases}$$

Indeed, the first one is just 3.1.4-(a), and the second one results from the fact that, since $\boldsymbol{G}$ is compact, there is a canonical isomorphism $\boldsymbol{S}(\mathfrak{g}) = H_{\boldsymbol{G}} \otimes_{\mathbb{R}} \mathcal{H}$, where $\mathcal{H}$ denotes the (graded) subspace of $\boldsymbol{G}$-harmonic polynomials of $\boldsymbol{S}(\mathfrak{g})$ (see [Dx], thm. **7.3.5** p. 241, §**8** pp. 277-), so that

$$\big(\big(\boldsymbol{S}(\mathfrak{g}) \otimes_{\mathbb{R}} \Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\boldsymbol{G}},1 \otimes d_{\boldsymbol{M}}\big) = H_{\boldsymbol{G}} \otimes_{\mathbb{R}} \big(\big(\mathcal{H} \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})\big)^{\boldsymbol{G}},1 \otimes d_{\boldsymbol{M}}\big),$$

and the quasi-isomorphisms of 3.1.4-(a)

$$H_{\boldsymbol{G}} \otimes \big(\Omega_c(\boldsymbol{M}),\boldsymbol{d}\big) \supseteq H_{\boldsymbol{G}} \otimes \big(\Omega_c(\boldsymbol{M})^{\boldsymbol{G}},\boldsymbol{d}\big) \subseteq \big((\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c(\boldsymbol{M}))^{\boldsymbol{G}},1 \otimes \boldsymbol{d}\big)$$

are morphisms of complexes of ***free*** $H_{\boldsymbol{G}}$-graded modules. Consequently, the induced morphisms on the corresponding $H_{\boldsymbol{G}}$-dual complexes will still be quasi-isomorphisms (cf. 4.2.10-(c)).

Putting together these observations, the induced morphism on the $\mathbb{E}_1$ terms of the concerned spectral sequences by $\mathbb{D}_{\boldsymbol{G}}(\boldsymbol{M})$, is simply

$$\begin{array}{ccc} H_{\boldsymbol{G}} \otimes H(\boldsymbol{M})[d_{\boldsymbol{M}}] & \xrightarrow{\mathbb{1}\otimes\mathscr{D}(\boldsymbol{M})} & H_{\boldsymbol{G}} \otimes \mathbf{Hom}^{\bullet}_{\mathbb{R}}(H_c(\boldsymbol{M}),\mathbb{R}) \\ & & \| \\ & & \mathrm{Hom}_{H_G}\big((H_{\boldsymbol{G}} \otimes H_c(\boldsymbol{M}),H_{\boldsymbol{G}})\big) \end{array}$$

where one recognizes in $\mathbb{1} \otimes \mathscr{D}(\boldsymbol{M})$ the classical Poincaré duality 1.3.2.

b) This is a straightforward application of proposition 4.2.10 since, as we noted in the previous paragraphs, $\Omega_{\boldsymbol{G}} := \Omega_{\boldsymbol{G}}(\boldsymbol{M})$ is a free $H_{\boldsymbol{G}}$-gm.

c) The first spectral sequence $\mathbb{E}(\boldsymbol{M})$ is just the $''\mathbb{E}$ spectral sequence of 4.2.9 converging to the right hand side of $(\star)$. On the other hand, the spectral sequence, $\mathbb{F}_2^{p,q}(\boldsymbol{M})$, is the one we used in the proof of (a).

d) Left to the reader. □

4.5.2. Torsion-freeness, Freeness and Reflexivity. Proposition 4.5.1-(b,d) shows that the freeness of equivariant cohomology as $H_{\boldsymbol{G}}$-gm is a sufficient condition for equivariant Poincaré duality to hold. The question then arises whether some weaker condition could be equivalent to duality. Apart from freeness, two other properties have been thoroughly study in Allday-Franz-Puppe [AFP].

- Torsion-freeness. An $H_{\boldsymbol{G}}$-gm $\boldsymbol{V}$ is said to be *torsion-free* if, for all $v \in \boldsymbol{V}$,
$$\mathrm{Ann}(v) := \{P \in H_{\boldsymbol{G}} \mid P \cdot v = 0\} = 0.$$
The torsion-freeness of equivariant cohomology is clearly a necessary condition for duality as the modules $\mathbf{Hom}^{\bullet}_{H_G}(\_,H_{\boldsymbol{G}})$ are torsion-free. It is also a sufficient condition for the injectivity of the Poincaré morphism (Prop. 5.9 [AFP], see ex. 6.1.3), but it is not for duality as the explicit examples of Franz-Puppe [FP] (2006) show.
- Reflexivity. An $H_{\boldsymbol{G}}$-gm $\boldsymbol{V}$ is said to be *reflexive* if the natural map
$$\boldsymbol{V} \to \mathbf{Hom}^{\bullet}_{H_G}(\mathbf{Hom}^{\bullet}_{H_G}(\boldsymbol{V},H_{\boldsymbol{G}}),H_{\boldsymbol{G}})$$
is an isomorphism.

  For a finite type manifold $\boldsymbol{M}$, while the reflexivity of $H_{\boldsymbol{G}}(\boldsymbol{M})$ and $H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ are clearly necessary conditions to duality, the converse, which is also true, is more subtle. The equivalence between duality and reflexivity has been established in [AFP] (Prop. 5.10) for $\boldsymbol{G}$ abelian, and in Franz [F_2] (Cor. 5.1) for general $\boldsymbol{G}$. ([15])

The following diagram illustrates the relationship between the different kinds of nontorsions in equivariant cohomology and significant properties of the equivariant Poincaré pairing.

$$\begin{array}{ccccc} \{\texttt{free}\} \subseteq & \{\texttt{reflexive}\} & \subseteq & \{\texttt{torsion-free}\} \\ & \Updownarrow & & \Updownarrow \\ & \left\{\begin{matrix}\text{Perfect}\\ \text{Poincaré pairing}\end{matrix}\right\} & \subseteq & \left\{\begin{matrix}\text{Nondegenerate}\\ \text{Poincaré pairing}\end{matrix}\right\} \end{array}$$

It is worth noting that in [F_1] (2015), Franz gives the first known examples of compact manifolds having reflexive but nonfree equivariant cohomology.

4.6. $\boldsymbol{T}$-Equivariant Poincaré Duality Theorem

When $\boldsymbol{G}$ is a **compact connected torus** $\boldsymbol{T} = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, we have:

$$\begin{cases} H_{\boldsymbol{T}} = \boldsymbol{S}(\mathfrak{t}) \\ \Omega_{\boldsymbol{T}}(\boldsymbol{M}) = \boldsymbol{S}(\mathfrak{t}) \otimes_{\mathbb{R}} \Omega(\boldsymbol{M})^{\boldsymbol{T}} \\ \Omega_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}) = \boldsymbol{S}(\mathfrak{t}) \otimes_{\mathbb{R}} \Omega_{\mathrm{c}}(\boldsymbol{M})^{\boldsymbol{T}} \end{cases}$$

so that

$$\begin{aligned} \mathbf{Hom}_{H_{\boldsymbol{T}}}(\Omega_{\boldsymbol{T},\mathrm{c}}, H_{\boldsymbol{T}}) &= \mathbf{Hom}_{\boldsymbol{S}(\mathfrak{t})}\big(\boldsymbol{S}(\mathfrak{t}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})^{\boldsymbol{T}}, \boldsymbol{S}(\mathfrak{t})\big) \\ &= \mathbf{Hom}_{\mathbb{R}}\big(\Omega_{\mathrm{c}}(\boldsymbol{M})^{\boldsymbol{T}}, \boldsymbol{S}(\mathfrak{t})\big) \\ &= \boldsymbol{S}(\mathfrak{t}) \otimes_{\mathbb{R}} \mathbf{Hom}_{\mathbb{R}}\big(\Omega_{\mathrm{c}}(\boldsymbol{M})^{\boldsymbol{T}}, \mathbb{R}\big) \end{aligned}$$

The left adjoint map $\mathbb{D}_{\boldsymbol{T}}(\boldsymbol{M})$ associated with the $\boldsymbol{T}$-equivariant Poincaré pairing $\mathbb{P}_{\boldsymbol{T}}$ (see 4.4.2) identifies naturally to $\mathbb{1} \otimes \mathbb{D}(\boldsymbol{M})$,

$$\boxed{\begin{array}{ccc} \boldsymbol{S}(\mathfrak{t}) \otimes \Omega(\boldsymbol{M})^{\boldsymbol{T}}[d_{\boldsymbol{M}}] & \xrightarrow[\mathbb{1}\otimes\mathbb{D}(\boldsymbol{M})]{\mathbb{D}_T(\boldsymbol{M})} & \boldsymbol{S}(\mathfrak{t}) \otimes \mathbf{Hom}_{\mathbb{R}}\big(\Omega_{\mathrm{c}}(\boldsymbol{M})^{\boldsymbol{T}}, \mathbb{R}\big) \\ P \otimes \alpha & \longmapsto & P \otimes \Big(\beta \mapsto \displaystyle\int_{\boldsymbol{M}} \alpha \wedge \beta\Big) \end{array}}$$

and the right adjoint map (see 4.4.3) to

$$\boxed{\begin{array}{ccc} \boldsymbol{S}(\mathfrak{t}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M})^{\boldsymbol{T}}[d_{\boldsymbol{M}}] & \xrightarrow[\mathbb{1}\otimes\mathbb{D}'(\boldsymbol{M})]{\mathbb{D}'_T(\boldsymbol{M})} & \boldsymbol{S}(\mathfrak{t}) \otimes \mathbf{Hom}_{\mathbb{R}}\big(\Omega(\boldsymbol{M})^{\boldsymbol{T}}, \mathbb{R}\big) \\ P \otimes \beta & \longmapsto & P \otimes \Big(\alpha \mapsto \displaystyle\int_{\boldsymbol{M}} \alpha \wedge \beta\Big) \end{array}}$$

The following theorem is a particular case of 4.5.1.

4.6.1. Theorem. *Let $\boldsymbol{T}$ be a compact connected torus, and $\boldsymbol{M}$ an oriented $\boldsymbol{T}$-manifold of dimension $d_{\boldsymbol{M}}$.*

a) *The $H_{\boldsymbol{T}}$-graded morphism of complexes*

$$\boxed{\mathbb{D}_{\boldsymbol{T}}(\boldsymbol{M}) : \Omega_{\boldsymbol{T}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}_{H_{\boldsymbol{T}}}\big(\Omega_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}), H_{\boldsymbol{T}}\big)}$$

is a quasi-isomorphism.

[15] The key point is to prove that reflexivity of $H_{\boldsymbol{G}}(\boldsymbol{M})$ implies that $\mathbf{Ext}^i_{H_G}(H_{\boldsymbol{G}}(\boldsymbol{M}), H_{\boldsymbol{G}}) = 0$, for all $i > 0$, in which case the proof 4.2.10-(b) applies and duality follows as in 4.5.1-(b).

b) *The morphism* $\mathbb{D}_{\boldsymbol{T}}(\boldsymbol{M})$ *induces* the "Poincaré morphism in $\boldsymbol{T}$-equivariant cohomology" *(see 4.2.10-(a))*

$$\boxed{\mathscr{D}_{\boldsymbol{T}}(\boldsymbol{M}) : H_{\boldsymbol{T}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}_{H_{\boldsymbol{T}}}\big(H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}), H_{\boldsymbol{T}}\big)}$$

If $H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M})$ *is a free* $H_{\boldsymbol{T}}$*-module,* $\mathscr{D}_{\boldsymbol{T}}(\boldsymbol{M})$ *is an isomorphism.*

c) *There are natural spectral sequences converging to* $H_{\boldsymbol{T}}(\boldsymbol{M})[d_{\boldsymbol{M}}]$

$$\begin{cases} \mathbb{E}_2^{p,q}(\boldsymbol{M}) = \big(\mathbf{Ext}^p_{H_{\boldsymbol{T}}}\big(H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}), H_{\boldsymbol{T}}\big)\big)^q \Rightarrow H^{p+q}_{\boldsymbol{T}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ \mathbb{F}_2^{p,q}(\boldsymbol{M}) = H^p_{\boldsymbol{T}} \otimes_{\mathbb{R}} \mathbf{Hom}_{\mathbb{R}}(H^q_{\mathrm{c}}(\boldsymbol{M}), \mathbb{R}) \Rightarrow H^{p+q}_{\boldsymbol{T}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \end{cases}$$

where, in the first one, q *denotes the graded vector space degree.*

d) *Moreover, if* $\boldsymbol{M}$ *is of finite type, the* $H_{\boldsymbol{G}}$*-graded morphism of complexes*

$$\boxed{\mathbb{D}'_{\boldsymbol{T}}(\boldsymbol{M}) : \Omega_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}_{H_{\boldsymbol{T}}}\big(\Omega_{\boldsymbol{T}}(\boldsymbol{M}), H_{\boldsymbol{T}}\big)}$$

is a quasi-isomorphism, and mutatis mutandis for (b) *and* (c).

Proof (a) Since we have the identification $\mathbb{D}_{\boldsymbol{T}}(\boldsymbol{M}) = \mathbb{1} \otimes \mathbb{D}(\boldsymbol{M})$, we may conclude using 2.2.6-(d). Statements (b,c,d) are particular cases of 4.5.1. □

4.6.2. Remark. Recall that $H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M})$ is a free $H_{\boldsymbol{T}}$-module whenever $\boldsymbol{M}$ has no odd (or no even) degree cohomology with compact support (2.2.6-(d)-(iv)). Obviously, though not very interesting, this is also the case when $\boldsymbol{T}$ acts trivially on $\boldsymbol{M}$, since then $\boldsymbol{c}(Y) = \boldsymbol{\theta}(Y) = 0$, $\forall Y \in \mathfrak{t}$, and $H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}) = H_{\boldsymbol{T}} \otimes_{\mathbb{R}} H_{\mathrm{c}}(\boldsymbol{M})$.

5. Equivariant Gysin Morphism

We now follow the steps in section 1.10 for the construction of the Gysin morphisms in the equivariant framework.

5.1. *G*-Equivariant Gysin Morphism in the General Case

5.1.1. Equivariant Finite de Rham Type Coverings. We have already proved in 1.6 that if $\boldsymbol{G}$ is a compact Lie Group, a $\boldsymbol{G}$-manifold $\boldsymbol{M}$ is the union of a countable ascending chain $\mathscr{U} := \{U_0 \subseteq U_1 \subseteq \cdots\}$ of $\boldsymbol{G}$-stable open subsets of $\boldsymbol{M}$ of finite type.

The following theorem, the equivariant analog of 1.7.8, is a corollary of the $\boldsymbol{G}$-equivariant Poincaré duality theorem 4.5.1. The proof is left to the reader.

5.1.2. Theorem. *Let* $\boldsymbol{G}$ *be a compact connected Lie group, and* $\boldsymbol{M}$ *an oriented* $\boldsymbol{G}$*-manifold of dimension* $d_{\boldsymbol{M}}$*. Then,*

a) *For every filtrant covering* $\mathscr{U}$ *of* $\boldsymbol{M}$ *by* $\boldsymbol{G}$*-stable open subsets, the canonical map* $\varinjlim_{U\in\mathscr{U}} \Omega_{\boldsymbol{G},\mathrm{c}}(U) \to \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ *is bijective, and the map*

$$\mathbb{D}'_{\boldsymbol{G}}(\mathscr{U}) : (\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}), \boldsymbol{d}_{\boldsymbol{G}})[d_{\boldsymbol{M}}] \longrightarrow \varinjlim_{U\in\mathscr{U}} \big(\mathbf{Hom}_{H_{\boldsymbol{G}}}(\Omega_{\boldsymbol{G}}(U), H_{\boldsymbol{G}}), -\boldsymbol{D}\big)$$

analog to 1.7.8-(b) is a well defined morphism of complexes.

b) *Moreover, if the open sets in* $\mathscr{U}$ *are of finite type, the map* $\mathbb{D}'_{\boldsymbol{G}}(\mathscr{U})$ *is a quasi-isomorphism.*

5.1.3 We now closely follow the instructions of section 1.10.2 for the construction of Gysin morphisms.

Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be a $\boldsymbol{G}$-equivariant map between oriented $\boldsymbol{G}$-manifolds. To $\beta \in \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ we assign the linear form on $\Omega_{\boldsymbol{G}}(\boldsymbol{N})$ defined by $I\!D'_{\boldsymbol{G}}(f)(\beta) : \alpha \mapsto \int_{\boldsymbol{M}} f^*\alpha \wedge \beta$. In this way we obtain the diagram

$$\begin{array}{ccc} \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] & \overset{f_*}{\dashrightarrow} & \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}] \\ & \searrow^{I\!D'_{\boldsymbol{G}}(f)} \quad \oplus & \downarrow I\!D'(\boldsymbol{N}) \left(\begin{smallmatrix}\text{quasi-iso if } \boldsymbol{N} \text{ is of} \\ \text{finite type}\end{smallmatrix}\right) \\ & & \mathbf{Hom}_{H_{\boldsymbol{G}}}(\Omega_{\boldsymbol{G}}(\boldsymbol{N}), H_{\boldsymbol{G}}) \end{array}$$

which may be closed in cohomology whenever $\boldsymbol{N}$ is of **finite type**, since $I\!D'_{\boldsymbol{G}}(\boldsymbol{N})$ is then a quasi-isomorphism (4.5.1-(d)).

When $\boldsymbol{N}$ is not of finite type, one fixes any equivariant covering $\mathscr{U}$ of $\boldsymbol{N}$ made up of open finite type subsets (5.1.1), and replaces $I\!D'_{\boldsymbol{G}}(\boldsymbol{N})$ by the morphism $I\!D'_{\boldsymbol{G}}(\mathscr{U})$ of theorem 5.1.2. The diagram

$$\begin{array}{ccc} \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] & \overset{f_*}{\dashrightarrow} & \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}] \\ & \searrow^{I\!D'_{\boldsymbol{G}}(f,\mathscr{U})} \quad \oplus & \downarrow I\!D'_{\boldsymbol{G}}(\mathscr{U}) \text{ (quasi-iso)} \\ & & \varinjlim_{U\in\mathscr{U}} \mathbf{Hom}_{H_{\boldsymbol{G}}}(\Omega_{\boldsymbol{G}}(U), H_{\boldsymbol{G}}) \end{array}$$

where $I\!D'_{\boldsymbol{G}}(f,\mathscr{U})$ is defined as in 1.10.2, may be closed in cohomology since $I\!D'_{\boldsymbol{G}}(\mathscr{U})$ is a quasi-isomorphism. The closing arrow

$$\boxed{f_* : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}]}$$

the equivariant Gysin morphism associated with f, is therefore defined as

$$f_* := \mathscr{D}'_{\boldsymbol{G}}(\mathscr{U})^{-1} \circ \boldsymbol{h}(I\!D(f,\mathscr{U})).$$

5.1.4. Theorem and definitions. *With the above notations,*

a) *The equality*

$$\int_{\boldsymbol{M}} f^*[\alpha] \cup [\beta] = \int_{\boldsymbol{N}} [\alpha] \cup f_*[\beta] \qquad (\diamond\diamond)$$

holds for all $[\alpha] \in H_{\boldsymbol{G}}(\boldsymbol{N})$ *and* $[\beta] \in H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$.

b) *Furthermore,* f_* *is a morphism of* $H_{\boldsymbol{G}}(\boldsymbol{N})$*-modules, i.e. the equality, called the* equivariant projection formula,

$$f_*\big(f^*[\alpha] \cup [\beta]\big) = [\alpha] \cup f_*([\beta]) \qquad (\diamond\diamond\diamond)$$

holds for all $[\alpha] \in H_{\boldsymbol{G}}(\boldsymbol{N})$ *and* $[\beta] \in H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$.

c) *The correspondence*

$$(\_)_* : \boldsymbol{G}\text{-}\boldsymbol{Man}^{\mathrm{or}} \rightsquigarrow \mathrm{GM}(H_{\boldsymbol{G}}) \quad \text{with} \quad \begin{cases} \boldsymbol{M} \rightsquigarrow \boldsymbol{M}_* := H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ f \rightsquigarrow f_* \end{cases}$$

is a covariant functor. We will refer to it as the equivariant Gysin functor.

d) *Suppose that $\boldsymbol{M}$ and $\boldsymbol{N}$ are manifolds of finite type. If the pullback morphism $f^* : H_{\boldsymbol{G}}(\boldsymbol{N}) \to H_{\boldsymbol{G}}(\boldsymbol{M})$ is an isomorphism, then the Gysin morphism $f_* : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}]$ is also.*

Proof. (a) Immediate from de definition of the Gysin morphism.

(b) Unlike the proof of the nonequivariant statement 1.7.4-(b), this claim is no longer a formal consequence of (a) because equivariant cohomology may have torsion elements, something that doesn't affect equivariant integration. Instead, when $\boldsymbol{N}$ is of finite type and since then $\mathbb{D}'(\boldsymbol{N})$ is a quasi-isomorphism, we can check that the following equality holds at the *cochain* level,

$$\mathbb{D}'_{\boldsymbol{G}}(f)(f^*(\alpha) \cup \beta) = \text{“}\mathbb{D}'(\boldsymbol{N})(\alpha \cup f_*(\beta))\text{”} = \mathbb{D}'_{\boldsymbol{G}}(f)(\beta) \circ \mu_{\mathrm{r}}(\alpha), \qquad (\dagger)$$

where the central term is there for purely heuristic reasons and where we denote $\mu_{\mathrm{r}}(\alpha) : \Omega_{\boldsymbol{G}}(\boldsymbol{N}) \to \Omega_{\boldsymbol{G}}(\boldsymbol{N})$ the right multiplication by α, i.e. $\mu_{\mathrm{r}}(\alpha)(\_) = (\_) \cup \alpha$. The identification of the left and right terms in $(\dagger)$ is then a straightforward verification from the definition of $\mathbb{D}'_{\boldsymbol{G}}(f)$. When $\boldsymbol{N}$ is not of finite type, we follow the same arguments with $\mathbb{D}'_{\boldsymbol{G}}(f,\mathscr{U})$ instead of $\mathbb{D}'_{\boldsymbol{G}}(f)$.

(c) is clear. (d) as $f^* : \Omega_{\boldsymbol{G}}(\boldsymbol{N}) \to \Omega_{\boldsymbol{G}}(\boldsymbol{M})$ is a quasi-isomorphism, the induced map $\mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}(\Omega_{\boldsymbol{G}}(\boldsymbol{N}), H_{\boldsymbol{G}}) \to \mathbf{Hom}^{\bullet}_{H_{\boldsymbol{G}}}(\Omega_{\boldsymbol{G}}(\boldsymbol{M}), H_{\boldsymbol{G}})$ is also, following 4.2.10-(c), and one concludes, since $\mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{M})$ and $\mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{N})$ are quasi-isomorphisms. □

5.1.5. Exercise. Prove the following enhancement of the statement 5.1.4-(d). If $\pi : \boldsymbol{V} \to \boldsymbol{B}$ is a vector bundle over an oriented manifold $\boldsymbol{B}$, the map π is of finite type (1.4.1), and $\pi^* : H_{\boldsymbol{G}}(\boldsymbol{B}) \to H_{\boldsymbol{G}}(\boldsymbol{V})$ and $\pi_* : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{V})[d_{\boldsymbol{V}}] \to H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{B})[d_{\boldsymbol{B}}]$ and both isomorphisms (cf. 1.4.2-(c)).

5.2. $\boldsymbol{G}$-Equivariant Gysin Morphism for Proper Maps

Following 1.10.1, let $f : \boldsymbol{M} \to \boldsymbol{N}$ be a **proper** $\boldsymbol{G}$-equivariant map between oriented $\boldsymbol{G}$-manifolds. To $\alpha \in \Omega_{\boldsymbol{G}}(\boldsymbol{M})$ we assign the $H_{\boldsymbol{G}}$-linear form on $\Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})$ defined by $\mathbb{D}'_{\boldsymbol{G}}(f)(\alpha) : \beta \mapsto \int_{\boldsymbol{M}} f^*\beta \wedge \alpha$. In this way we obtain the diagram

$$\begin{array}{ccc} \Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] & \overset{f_!}{\dashrightarrow} & \Omega_{\boldsymbol{G}}(\boldsymbol{N})[d_{\boldsymbol{N}}] \\ & \searrow^{\mathbb{D}'_{\boldsymbol{G}}(f)} \quad \oplus & \downarrow \mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{N}) \text{ (quasi-iso)} \\ & & \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})^{\vee} \end{array}$$

which may be closed in cohomology because $\mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{N})$ is a quasi-isomorphism, as shown in 4.5.1-(a). The closing arrow:

$$\boxed{f_! : H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G}}(\boldsymbol{N})[d_{\boldsymbol{N}}]},$$

the equivariant Gysin morphism associated with a proper map f, is therefore defined as

$$f_! := \mathscr{D}'_{\boldsymbol{G}}(\mathscr{U})^{-1} \circ \boldsymbol{h}(\mathbb{D}'_{\boldsymbol{G}}(f)).$$

5.2.1. Theorem and definitions. *With the above notations,*

a) *The equality*

$$\int_{\boldsymbol{M}} f^*[\beta] \cup [\alpha] = \int_{\boldsymbol{N}} [\beta] \cup f_![\alpha] \tag{$\star\star$}$$

holds for all $[\alpha] \in H_{\boldsymbol{G}}(\boldsymbol{M})$ *and* $[\beta] \in H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})$.

b) *Furthermore,* $f_!$ *is a morphism of* $H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})$*-modules, the equality, called the* equivariant projection formula for proper maps,

$$f_!\big(f^*[\beta] \cup [\alpha]\big) = [\beta] \cup f_![\alpha] \tag{$\star\star\star$}$$

holds for all $[\beta] \in H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})$ *and* $[\alpha] \in H_{\boldsymbol{G}}(\boldsymbol{M})$.

c) *The correspondence*

$$f_! : \boldsymbol{G}\text{-}\boldsymbol{Man}^{\mathrm{or}}_{\pi} \rightsquigarrow \mathrm{GM}(H_{\boldsymbol{G}}) \qquad \textit{with} \qquad \begin{cases} \boldsymbol{M} \rightsquigarrow \boldsymbol{M}_! := H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ f \rightsquigarrow f_! \end{cases}$$

is a covariant functor. We will refer to it as the equivariant Gysin functor for proper maps.

d) *If the pullback morphism* $f^* : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N}) \to H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ *is an isomorphism, the Gysin morphism* $f_! : H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G}}(\boldsymbol{N})[d_{\boldsymbol{N}}]$ *is also an isomorphism.*

e) *The natural map* $\phi(\_) : H_{\boldsymbol{G},\mathrm{c}}(\_)[d_\_] \to H_{\boldsymbol{G}}(\_)[d_\_]$ (1.2.3) *is a homomorphism between the two equivariant Gysin functors* $(\_)_* \to (\_)_!$ *over the category* $\boldsymbol{G}\text{-}\boldsymbol{Man}^{\mathrm{or}}_{\pi}$*, i.e. the diagrams*

$$\begin{array}{ccc} H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) & \xrightarrow{\phi(\boldsymbol{M})} & H_{\boldsymbol{G}}(\boldsymbol{M}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_!} \\ H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N}) & \xrightarrow{\phi(\boldsymbol{N})} & H_{\boldsymbol{G}}(\boldsymbol{N}) \end{array}$$

are naturally commutative.

Proof. Same as the proof of 5.1.4, left to the reader. □

5.3. Comparison Theorems

The next theorem establishes a close connexion between the nonequivariant and the equivariant Gysin morphisms. It is a basic tool for the generalization of known properties of classical Gysin morphisms into the equivariant framework.

5.3.1. Theorem. *Let* $\boldsymbol{G}$ *be a compact connected Lie group and* $f : \boldsymbol{M} \to \boldsymbol{N}$ *a* $\boldsymbol{G}$*-equivariant map between oriented* $\boldsymbol{G}$*-manifolds. There exists a natural morphism of the spectral sequences* $I\!F$ *of theorem* 4.5.1-(c) *converging to the Gysin morphism* $f_* : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}]$,

$$\begin{array}{ccc} I\!F_{\mathrm{c},2}(\boldsymbol{M}) = H_{\boldsymbol{G}} \otimes H_{\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] & \Rightarrow & H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ {\scriptstyle 1\otimes f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ I\!F_{\mathrm{c},2}(\boldsymbol{N}) = H_{\boldsymbol{G}} \otimes H_{\mathrm{c}}(\boldsymbol{N})[d_{\boldsymbol{N}}] & \Rightarrow & H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{N}}] \end{array}$$

and in the proper case to $f_! : H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G}}(\boldsymbol{N})[d_{\boldsymbol{N}}]$,

$$\begin{array}{ccc} I\!\!F_2(\boldsymbol{M}) = H_{\boldsymbol{G}} \otimes H(\boldsymbol{M})[d_{\boldsymbol{M}}] & \Rightarrow & H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ {\scriptstyle 1\otimes f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ I\!\!F_2(\boldsymbol{N}) = H_{\boldsymbol{G}} \otimes H(\boldsymbol{N})[d_{\boldsymbol{N}}] & \Rightarrow & H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{N}}] \end{array}$$

Proof. Clear from the proof of 4.5.1 and the definition of Gysin morphisms. □

5.4. Universal Property of the equivariant Gysin Morphism

Proposition *Let* $f : \boldsymbol{M} \to \boldsymbol{N}$ *be a* $\boldsymbol{G}$*-equivariant map between oriented* $\boldsymbol{G}$*-manifolds.*

a) *A morphism of complexes* $\varphi_* : \Omega_{\boldsymbol{G},c}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \Omega_{\boldsymbol{G},c}(\boldsymbol{N})[d_{\boldsymbol{N}}]$ *induces the equivariant Gysin morphism*

$$f_* : H_{\boldsymbol{G},c}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G},c}(\boldsymbol{N})[d_{\boldsymbol{N}}],$$

if and only if

$$\int_{\boldsymbol{M}} f^*\alpha \wedge \beta = \int_{\boldsymbol{N}} \alpha \wedge \varphi_*\beta, \quad \forall \alpha \in \Omega_{\boldsymbol{G}}(\boldsymbol{N}),\ \forall \beta \in \Omega_{\boldsymbol{G},c}(\boldsymbol{M}).$$

b) *If* f *is a proper, a morphism of complexes* $\varphi_! : \Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \Omega_{\boldsymbol{G}}(\boldsymbol{N})[d_{\boldsymbol{N}}]$ *induces the equivariant Gysin morphism*

$$f_! : H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to H_{\boldsymbol{G}}(\boldsymbol{N})[d_{\boldsymbol{N}}],$$

if and only if

$$\int_{\boldsymbol{M}} f^*\beta \wedge \alpha = \int_{\boldsymbol{N}} \beta \wedge \varphi_!\alpha, \quad \forall \alpha \in \Omega_{\boldsymbol{G}}(\boldsymbol{M}),\ \forall \beta \in \Omega_{\boldsymbol{G},c}(\boldsymbol{N}).$$

5.4.1. Remark. The last proposition is a simple consequence of the definition of the Gysin morphism. But one must beware that, unlike the nonequivariant case (1.9.1), it is generally not true that the equivariant Gysin morphism is characterized by the equality of **cohomology classes**:

$$\int_{\boldsymbol{M}} f^*[\alpha] \cup [\beta] = \int_{\boldsymbol{N}} [\alpha] \cup f_*[\beta], \quad \forall [\alpha] \in H_{\boldsymbol{G}}(\boldsymbol{N}),\ \forall [\beta] \in H_{\boldsymbol{G},c}(\boldsymbol{M}). \qquad (\diamond\diamond)$$

(or $(\star\star)$ for $f_!$ in the proper case). For example, the uniqueness of f_* satisfying the relation $(\diamond\diamond)$, results only from the injectivity of the map:

$$\begin{array}{rcl} I\!\!D_{\boldsymbol{G}}(\boldsymbol{N}) : H_{\boldsymbol{G},c}(\boldsymbol{N}) & \longrightarrow & \mathrm{Hom}_{H_{\boldsymbol{G}}}\big(H_{\boldsymbol{G}}(\boldsymbol{N}), H_{\boldsymbol{G}})\big) \\ {}[\beta] & \longrightarrow & \Big([\alpha] \to \displaystyle\int_{\boldsymbol{N}} [\alpha] \wedge [\beta]\Big) \end{array}$$

a property that is not always satisfied. Indeed, let $\boldsymbol{T}$ be a torus and $\boldsymbol{N}$ a compact oriented $\boldsymbol{T}$-manifold without fixed points. We know from the localization theorem, that $H_{\boldsymbol{T}}(\boldsymbol{N})$ is a torsion $H_{\boldsymbol{T}}$-module and consequently that $\mathbf{Hom}_{H_{\boldsymbol{T}}}(\boldsymbol{H}(\boldsymbol{N}), H_{\boldsymbol{T}}) = 0$, so that $I\!\!D_{\boldsymbol{G}}(\boldsymbol{N})$ is null, although $H_{\boldsymbol{T}}(\boldsymbol{N}) \neq 0$.

Exercise. Let $\boldsymbol{T} = \mathbb{S}^1 \times \mathbb{S}^1$ act on $\boldsymbol{N} = \mathbb{S}^1$ by $(t,u)(v) = uv$.

a) $H_{\boldsymbol{T}} = \mathbb{R}[X,Y]$, $H_{\boldsymbol{T}}(\boldsymbol{N}) = \mathbb{R}[Y]$, $\mathbf{End}_{H_{\boldsymbol{T}}}(H_{\boldsymbol{T}}(\boldsymbol{N})) = \mathbb{R}[Y]$.

b) For any map $f : \boldsymbol{N} \to \boldsymbol{N}$ and any $\lambda \in \mathbf{End}_{H_G}(H_{\boldsymbol{G}}(\boldsymbol{N}))$ one has

$$\int_{\boldsymbol{N}} f^*[\alpha] \cup [\beta] = \int_{\boldsymbol{N}} [\alpha] \cup \lambda[\beta], \quad \forall [\alpha],[\beta] \in H_{\boldsymbol{T}}(\boldsymbol{N}).$$

c) Let $\boldsymbol{N}$ be any oriented $\boldsymbol{G}$-manifold such that $H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N})$ is an $H_{\boldsymbol{G}}$-free module. Show that condition ($\diamond\diamond$) (resp. ($\star\star$) for proper maps) of theorem 5.1.4 (resp. 5.2.1) completely characterizes Gysin morphisms for maps $f : \boldsymbol{M} \to \boldsymbol{N}$.

5.5. Group Restriction

Let $\boldsymbol{G}$ be a compact *connected* Lie group. For any closed subgroup $\boldsymbol{H} \subseteq \boldsymbol{G}$, *connected or not*, and for any $\boldsymbol{G}$-manifold $\boldsymbol{M}$, the canonical projection of Borel constructions $\mathbb{E}\boldsymbol{G} \times_{\boldsymbol{H}} \boldsymbol{M} \twoheadrightarrow \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{M}$ which is a locally trivial fibration with fiber $\boldsymbol{G}/\boldsymbol{H}$, induces by inverse image the *restriction homomorphism* of equivariant cohomology rings

$$\mathrm{Res}^{\boldsymbol{G}}_{\boldsymbol{H}} : H_{\boldsymbol{G}}(\boldsymbol{M}) \to H_{\boldsymbol{H}}(\boldsymbol{M}).$$

At this point, one could react against the possible lack of connectednes of $\boldsymbol{H}$ in so far as this property has been everywhere required in these notes. However, a careful examination shows that connectednes is only needed to ensure that the action of $\boldsymbol{G}$ on $\boldsymbol{M}$ is homotopically trivial, a property that is clearly inherited by any subgroup $\boldsymbol{H}$ of a connected group $\boldsymbol{G}$, whether the subgroup is connected or not (cf. 3.1.5). In that case if $\boldsymbol{H}_\circ$ denotes the connected component of 1 in $\boldsymbol{H}$ and $W_{\boldsymbol{H}} = \boldsymbol{H}/\boldsymbol{H}_\circ$, we have

$$H_{\boldsymbol{H}} = S(\mathfrak{h})^{\boldsymbol{H}} \quad \text{and} \quad H_{\boldsymbol{H}}(\boldsymbol{M}) = H_{\boldsymbol{H}_\circ}(\boldsymbol{M})^{W_{\boldsymbol{H}}}.$$

5.5.1. Theorem. *For any closed subgroup $\boldsymbol{H} \subseteq \boldsymbol{G}$ and any equivariant map $f : \boldsymbol{M} \to \boldsymbol{N}$ between oriented $\boldsymbol{G}$-manifolds, the following diagrams of Gysin morphisms are commutative:*

$$\begin{array}{ccc} H_{\boldsymbol{G}}(\boldsymbol{M}) & \xrightarrow{f_*} & H_{\boldsymbol{G}}(\boldsymbol{N}) \\ {\scriptstyle \mathrm{Res}^G_H}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^G_H} \\ H_{\boldsymbol{H}}(\boldsymbol{M}) & \xrightarrow{f_*} & H_{\boldsymbol{H}}(\boldsymbol{N}) \end{array} \qquad \begin{array}{ccc} H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) & \xrightarrow{f_!} & H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N}) \\ {\scriptstyle \mathrm{Res}^G_H}\downarrow & (f \text{ is proper}) & \downarrow{\scriptstyle \mathrm{Res}^G_H} \\ H_{\boldsymbol{H},\mathrm{c}}(\boldsymbol{M}) & \xrightarrow{f_!} & H_{\boldsymbol{H},\mathrm{c}}(\boldsymbol{N}) \end{array}$$

Proof. For a general map $f : \boldsymbol{M} \to \boldsymbol{N}$ the diagram of induced maps between Borel constructions

$$\begin{array}{ccc} \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{H}} \boldsymbol{M} & \xrightarrow{f} & \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{H}} \boldsymbol{N} \\ {\scriptstyle \pi}\downarrow & \square & \downarrow{\scriptstyle \pi} \\ \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{M} & \xrightarrow{f} & \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{M} \end{array}$$

is *cartesian* and if we endow $\boldsymbol{G}/\boldsymbol{H}$ with an orientation, the integration along the fibers of π enters in the *commutative* diagram of complexes:

$$\begin{array}{ccc} \Omega_{\boldsymbol{H},\mathrm{c}}(\boldsymbol{N}) & \xrightarrow{f^*} & \Omega_{\boldsymbol{H},\mathrm{c}}(\boldsymbol{M}) \\ \int_{G/H}\downarrow & \oplus & \downarrow\int_{G/H} \\ \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N}) & \xrightarrow{f^*} & \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) \end{array}$$

We may then conclude thanks to 5.4-(a) and that $\int_{\boldsymbol{G}/\boldsymbol{H}}$ is adjoint to $\mathrm{Res}^{\boldsymbol{G}}_{\boldsymbol{H}}$.

The case where $f : \boldsymbol{M} \to \boldsymbol{N}$ is proper follows in the same way. □

5.6. Explicit Constructions of Equivariant Gysin Morphisms

Although we gave a universal definition for the equivariant Gysin morphism in the last section, it is worth recalling alternative constructions for some particular maps where there exist explicit morphisms of Cartan complexes inducing the Gysin morphism, just as in the nonequivariant case (1.9).

5.6.1. Constant Map. Let $\boldsymbol{M}$ be an oriented $\boldsymbol{G}$-manifold. The constant map $c_{\boldsymbol{M}} : \boldsymbol{M} \to \{\bullet\}$ is $\boldsymbol{G}$-equivariant, $H_{\boldsymbol{G}}(\{\bullet\}) = H_{\boldsymbol{T}}$ is free and $I\!D_{\boldsymbol{G}}(\{\bullet\})$ is bijective. Therefore, the cohomological adjunction 5.4.1-($\diamond\diamond$) uniquely determines the Gysin morphism and we have, for all $\beta \in \Omega_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M})$:

$$c_{\boldsymbol{M}*}(\beta) = \Big(\int_{\{\bullet\}} 1 \cup c_{\boldsymbol{M}*}[\beta]\Big) = \int_{\boldsymbol{M}} \beta .$$

5.6.2. Equivariant Open Embedding. Let $\boldsymbol{M}$ be an oriented $\boldsymbol{G}$-manifold. If U is a $\boldsymbol{G}$-invariant open set in $\boldsymbol{M}$, denote by $\iota : U \subseteq \boldsymbol{M}$ the injection and endow U with the induced orientation. One has a natural inclusion of Cartan complexes $\iota_{\boldsymbol{G}} : \Omega_{\boldsymbol{G},\mathrm{c}}(U) \to \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$, and the elementary equality

$$\int_{\boldsymbol{U}} \iota_{\boldsymbol{G}}^*(\alpha) \wedge \beta = \int_{\boldsymbol{M}} \alpha \wedge \iota_{\boldsymbol{G},*}(\beta), \quad \forall \alpha \in \Omega_{\boldsymbol{G}}(\boldsymbol{M}),\, \forall \beta \in \Omega_{\boldsymbol{G},\mathrm{c}}(U),$$

shows immediately that the following induced map is the equivariant gysin map:

$$H(\iota_{\boldsymbol{G}*}) : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{U})[d_{\boldsymbol{U}}] \longrightarrow H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}].$$

5.6.3. Equivariant Projection. Given two oriented $\boldsymbol{G}$-manifolds $\boldsymbol{M},\boldsymbol{N}$, denote by $\mathrm{pr} : \boldsymbol{M} \times \boldsymbol{N} \twoheadrightarrow \boldsymbol{N}$, the projection $(x,y) \mapsto y$.

The map

$$\begin{array}{ccc} \Omega_{\mathrm{c}}(\boldsymbol{M}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{N}) & \xrightarrow{\varphi_*} & \Omega_{\mathrm{c}}(\boldsymbol{N}) \\ \nu \otimes \mu & \xmapsto{\varphi_*} & \big(\int_{\boldsymbol{M}} \nu\big)\mu \end{array}$$

is a morphism of $H_{\boldsymbol{G}}$-gm's commuting with $\boldsymbol{G}$-derivations ($\boldsymbol{G}$ is connected), and with $\boldsymbol{G}$-contractions since

$$\begin{aligned} \varphi_*\big(\boldsymbol{c}(X)(\nu \otimes \mu)\big) &= \varphi_*\big(\boldsymbol{c}(X)(\nu) \otimes \mu\big) + (-1)^{\deg \nu} \varphi_*\big(\nu \otimes \boldsymbol{c}(X)(\mu)\big) \\ &= (-1)^{d_M} \varphi_*\big(\nu \otimes \boldsymbol{c}(X)(\mu)\big) = (-1)^{d_M} \boldsymbol{c}(X)\big(\varphi_*(\nu \otimes \mu)\big), \end{aligned}$$

as $\int_{\boldsymbol{M}} \iota(X)\nu = 0$. The morphism φ_* may then be naturally extended to a morphism of Cartan complexes $\varphi_{\boldsymbol{G}*} : \big(\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{N})\big)^{\boldsymbol{G}} \to \big(\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{N})\big)^{\boldsymbol{G}}$ satisfying

$$\int_{\boldsymbol{M}\times\boldsymbol{N}} \mathrm{pr}^*(\alpha) \wedge \beta = \int_{\boldsymbol{N}} \alpha \wedge \varphi_{\boldsymbol{G}*}(\beta), \quad \forall \beta \in \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) \times_{H_{\boldsymbol{G}}} \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N}), \forall \alpha \in \Omega_{\boldsymbol{G}}(\boldsymbol{N}).$$

On the other hand, since the natural map $\Omega_{\mathrm{c}}(\boldsymbol{M}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{N}) \to \Omega_{\mathrm{c}}(\boldsymbol{M} \times \boldsymbol{N})$ is a quasi-isomorphism (Künneth [BT] p. 50), the induced map

$$(\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{N}))^{\boldsymbol{G}} \to (\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_{\mathrm{c}}(\boldsymbol{M} \times \boldsymbol{N}))^{\boldsymbol{G}} = \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M} \times \boldsymbol{N})$$

is also a quasi-isomorphism and one may conclude that

$$\mathrm{pr}_{\boldsymbol{G},*} : H^*_{\boldsymbol{G},c}(\boldsymbol{M} \times \boldsymbol{N})[d_{\boldsymbol{M}}] \longrightarrow H^*_{\boldsymbol{G},c}(\boldsymbol{N})$$

induced by $\varphi_{\boldsymbol{G},*}$ is the equivariant Gysin map associated with pr.

5.6.4. Equivariant Fiber Bundle. Let $(\pi, \boldsymbol{V}, \boldsymbol{B})$ be an oriented $\boldsymbol{G}$-equivariant fiber bundle with fiber $\boldsymbol{F}$. Integration along fibers (see [BT] **I**§6 pp. 61-63) gives a morphism of complexes $\int_{\boldsymbol{F}} : \Omega_c(\boldsymbol{V}) \to \Omega_c(\boldsymbol{B})$ such that if $\psi : \boldsymbol{V} \to \boldsymbol{V}$ is an isomorphism exchanging fibres, then $\int_{\boldsymbol{F}} \circ \psi^* = \psi^* \circ \int_{\boldsymbol{F}}$, consequently $\int_{\boldsymbol{F}}$ is $\boldsymbol{G}$-equivariant. On the other hand, $\int_{\boldsymbol{F}}$ commutes with the contractions $\boldsymbol{c}(X)$. Indeed, since these are local operators, it suffices (modulo unit partitions if necessary) to verify the claim over a trivializing open subset of $\boldsymbol{V}$, i.e. over $\pi^{-1}(U)$ for U s.t. $\pi^{-1}U \sim \boldsymbol{F} \times U$, where we are in the case of a projection already discussed in 5.6.3.

Now, the map $\int_{\boldsymbol{F}} : \boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c(\boldsymbol{V}) \to \boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c(\boldsymbol{B})$, given by $\int_{\boldsymbol{F}} P \otimes \omega := P \otimes \int_{\boldsymbol{F}} \omega$ restricts naturally to $\int_{\boldsymbol{F}} : \Omega_{\boldsymbol{G},c}(\boldsymbol{V})[d_{\boldsymbol{F}}] \to \Omega_{\boldsymbol{G},c}(\boldsymbol{B})$ as a morphism of Cartan complexes satisfying $\int_{\boldsymbol{V}} \pi_* \alpha \wedge \beta = \int_{\boldsymbol{B}} \big(\alpha \wedge \int_{\boldsymbol{F}} \beta\big)$ since it is so in the nonequivariant case 1.9.5-$(*)$.

5.6.5. Zero Section of an Equivariant Vector Bundle

The Equivariant Thom Class. Let $(\pi, \boldsymbol{V}, \boldsymbol{B})$ be a $\boldsymbol{G}$-equivariant oriented vector bundle. In 5.1.5, we pointed out that the Gysin morphism for compact supports $\pi_* : H_c(\boldsymbol{V})[d_{\boldsymbol{F}}] \to H_c(\boldsymbol{B})$ is an **isomorphism**, so that, in particular:

$$H_c^i(\boldsymbol{V}) = 0, \qquad \text{for all } i < d_{\boldsymbol{F}}. \tag{$\diamond$}$$

5.6.6. Proposition and definition. *Assume $\boldsymbol{G}$ is compact and connected.*

a) *There exist homogeneous $\boldsymbol{G}$-equivariant cocycles of total degree $d_{\boldsymbol{F}}$*

$$\Phi_{\boldsymbol{G}} = \Phi^{[d_{\boldsymbol{F}}]} + \Phi^{[d_{\boldsymbol{F}}-2]} + \Phi^{[d_{\boldsymbol{F}}-4]} + \cdots$$

with $\Phi^{[i]} \in \big(\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c^i(\boldsymbol{V})\big)^{\boldsymbol{G}}$ where $\Phi^{[d_{\boldsymbol{F}}]} \in \Omega_c^{d_{\boldsymbol{F}}}(\boldsymbol{V})^{\boldsymbol{G}}$ represents the Thom class of $(\boldsymbol{B}, \boldsymbol{V})$ (see 1.9.7). *Two such cocycles are cohomologous. The map*

$$\begin{array}{ccc} (\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c(\boldsymbol{B}))^{\boldsymbol{G}} & \xrightarrow{\varphi_{\boldsymbol{G},*}} & (\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c(\boldsymbol{V}))^{\boldsymbol{G}}[d_{\boldsymbol{F}}] \\ \nu & \longmapsto & \pi^*\nu \wedge \tilde{\Phi} \end{array}$$

is a morphism of Cartan complexes, and the same with 'Ω' instead of 'Ω_c'.

b) *The zero section $\sigma : \boldsymbol{B} \hookrightarrow \boldsymbol{V}$ of the vector bundle $\pi : \boldsymbol{V} \to \boldsymbol{B}$ is a proper $\boldsymbol{G}$-equivariant map. The* equivariant Gysin morphisms

$$\begin{cases} \sigma_* : H_{\boldsymbol{G},c}(\boldsymbol{B}))[d_{\boldsymbol{B}}] \to H_{\boldsymbol{G},c}(\boldsymbol{V})[d_{\boldsymbol{V}}] \\ \sigma_! : \ H_{\boldsymbol{G}}(\boldsymbol{B}))[d_{\boldsymbol{B}}] \ \to \ H_{\boldsymbol{G}}(\boldsymbol{V})[d_{\boldsymbol{V}}] \end{cases}$$

are both induced by the morphism of complexes $\varphi_{\boldsymbol{G},*}$ of (a).

Proof. (a) Let $n = d_{\boldsymbol{F}}$. Since $\boldsymbol{G}$ is connected and compact, there exists $\Phi^{[n]} \in \Omega_c^n(\boldsymbol{V})^{\boldsymbol{G}}$ representing the Thom class of $\boldsymbol{V}$. We have

$$\boldsymbol{d_G}(\Phi^{[n]}) = \boldsymbol{d}(\Phi^{[n]}) + \boldsymbol{c}(X)\Phi^{[n]} = \boldsymbol{c}(X)\Phi^{[n]},$$

where $\boldsymbol{c}(X)\Phi^{[n]} \in (\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c^{n-1}(\boldsymbol{V}))^{\boldsymbol{G}}$ and $\boldsymbol{d}\big(\boldsymbol{c}(X)\Phi^{[n]}\big) = L(X)\Phi^{[n]} = 0$. But then $\boldsymbol{c}(X)\Phi^{[n]}$ is a coboundary of compact support following $(\diamond)$ and, again thanks to the connectedness of $\boldsymbol{G}$, there exists $\Phi^{[n-2]} \in (\boldsymbol{S}(\mathfrak{g}) \otimes \Omega_c^{n-2}(\boldsymbol{V}))^{\boldsymbol{G}}$ s.t. $\boldsymbol{c}(X)\Phi^{[n]} = \boldsymbol{d}\Phi^{[n-2]}$. The iteration of this procedure, possible because of the vanishing condition $(\diamond)$, leads to the $\boldsymbol{G}$-equivariant cocycle $\Phi_{\boldsymbol{G}}$. The cohomological uniqueness is proved in a similar way. The fact that $\varphi_{\boldsymbol{G},*}$ is compatible with differentials is obvious as $\Phi_{\boldsymbol{G}}$ is a cocycle.

(b) By the universal property of the equivariant Gysin morphisms 5.4, it suffices to verify the equality

$$\int_{\boldsymbol{B}} \sigma^*\alpha \wedge \beta = \int_{\boldsymbol{V}} \alpha \wedge \varphi_{\boldsymbol{G},*}(\beta), \qquad \forall \alpha \in \Omega_{\boldsymbol{G}}(\boldsymbol{V}),\ \forall \beta \in \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{B}).$$

Since $\pi^* : H(\boldsymbol{B}) \to H(\boldsymbol{V})$ is an isomorphism, the same is true in equivariant cohomology following 3.1.4-(d), so that there exists $\alpha' \in \Omega_{\boldsymbol{G}}(\boldsymbol{B})$ s.t. $\alpha \sim \pi^*\alpha'$. We are thus lead to verify that

$$\int_{\boldsymbol{B}} \alpha' \wedge \beta = \int_{\boldsymbol{V}} \pi^*(\alpha' \wedge \beta) \wedge \Phi_{\boldsymbol{G}}, \qquad \forall \alpha' \in \Omega_{\boldsymbol{G}}(\boldsymbol{B}),\ \forall \beta \in \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{B}),$$

and this follows from the universal property of the nonequivariant Thom class (1.9.7) that states that one has: $\int_{\boldsymbol{B}} \omega|_{\boldsymbol{B}} = \int_{\boldsymbol{V}} \omega \wedge \Phi, \quad \forall \omega \in H(\boldsymbol{V})$. □

5.7. Exercises

1) Restate and solve exercise 1.11 in the equivariant framework. In particular:
   - If $i : \boldsymbol{B} \hookrightarrow \boldsymbol{M}$ is a *closed equivariant embedding* of oriented $\boldsymbol{G}$-manifolds, denote by $j : \boldsymbol{U} := \boldsymbol{M}\backslash\boldsymbol{B} \hookrightarrow \boldsymbol{M}$ the complementary open embedding, and justify the existence of the following triangles where the left arrows are Gysin morphisms and the right ones are restriction morphisms.
     i) The equivariant compact support cohomology triangle
     $$H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{U})[d_{\boldsymbol{U}}] \xrightarrow{j_*} H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \xrightarrow{i^*} H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{B})[d_{\boldsymbol{B}}] \xrightarrow[{[+1]}]{} . \qquad (\diamond)$$
     ii) The equivariant Gysin triangle
     $$H_{\boldsymbol{G}}(\boldsymbol{B})[d_{\boldsymbol{B}}] \xrightarrow{i!} H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \xrightarrow{j^*} H_{\boldsymbol{G}}(\boldsymbol{U})[d_{\boldsymbol{U}}] \xrightarrow[{[+1]}]{} \qquad (\diamond\diamond)$$
   - In the equivariant version of the Lefschetz fixed point exercise (1.11.2) you will define the *$\boldsymbol{G}$-equivariant Lefschetz class of f* by
     $$L_{\boldsymbol{G}}(f) := \mathrm{Gr}(f)^*(\delta_!(1)) \in H_{\boldsymbol{G}}^{d_M}(\boldsymbol{M}),$$
     and its *equivariant Lefschetz number* $\Lambda_{\boldsymbol{G},f} := \int_{\boldsymbol{M}} L_{\boldsymbol{G}}(f)$. Prove that
     $$\begin{cases} \mathrm{Res}_1^{\boldsymbol{G}} L_{\boldsymbol{G}}(f) = L(f) \in H^{d_M}(\boldsymbol{M}) \\ \Lambda_{\boldsymbol{G},f} = \Lambda_f \end{cases}$$
     and conclude that the *equivariant Lefschetz number* coincides with the nonequivariant one. In particular, if $H_{\boldsymbol{G}}(\boldsymbol{M})$ is a torsion module (7.3.1), the Euler caracteristic of $\boldsymbol{M}$ is zero.

2) Show that if $f : \boldsymbol{B} \to \boldsymbol{M}$ is $\boldsymbol{G}$-equivariant between oriented $\boldsymbol{G}$-manifolds, the projective limit (see 3.2.5) of nonequivariant Gysin morphisms
   $$\varprojlim_n \big(f(n)_* : H_{\mathrm{c}}(\boldsymbol{B}_{\boldsymbol{G}}(n))[d_{\boldsymbol{B}}] \to H_{\mathrm{c}}(\boldsymbol{M}_{\boldsymbol{G}}(n))[d_{\boldsymbol{M}}]\big)$$
   is well defined and coincides with the equivariant Gysin morphism
   $$f_* : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{B})[d_{\boldsymbol{B}}] \to H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}].$$
   And *mutatis mutandis* for the proper case.

3) i) Show that in 5.6.5, the restriction of the equivariant Thom class to the complement of the zero section, is an equivariant coboundary. (*Hint: remark that* $[\Phi_{\boldsymbol{G}}] = \sigma_!(1)$ *and use* (1)-($\diamond\diamond$)).

ii) (**) Show that the multiplication by $[\Phi_{\boldsymbol{G}}]$ defines a map from $H_{\boldsymbol{G}}(\boldsymbol{B})$ to the equivariant cohomology of $\boldsymbol{V}$ with supports in $\boldsymbol{B}$:

$$(\_) \wedge [\Phi_{\boldsymbol{G}}] : H_{\boldsymbol{G}}(\boldsymbol{B}) \to H_{\boldsymbol{G},\boldsymbol{B}}(\boldsymbol{V}).$$

Show next that this map is an isomorphism. (*Hint: use the spectral sequence of exercise* 3.2.8-(c) *to reduce to the nonequivariant case*).

iii) (**) Extend (ii) to the case of a closed embedding $\boldsymbol{B} \hookrightarrow \boldsymbol{M}$ of oriented manifolds. (*Hint: show that $\boldsymbol{B}$ may be seen as the zero section of a tubular $\boldsymbol{G}$-stable neighborhood $\boldsymbol{B}_\epsilon$ and use (and justify) the fact that the restriction map $H_{\boldsymbol{G},\boldsymbol{B}}(\boldsymbol{M}) = H_{\boldsymbol{G},\boldsymbol{B}}(\boldsymbol{B}_\epsilon)$ is an isomorphism*).

6. The field of fractions of $H_{\boldsymbol{G}}$

6.1. The Localization Functor

Denote by $Q_{\boldsymbol{G}}$ the field of fractions of $H_{\boldsymbol{G}}$. The *localization functor* is the base change functor

$$Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} (\_) : \mathrm{GM}(H_{\boldsymbol{G}}) \rightsquigarrow \mathrm{Vec}(Q_{\boldsymbol{G}})$$

([16]) General result of commutative algebra state for any $H_{\boldsymbol{G}}$-module $\boldsymbol{N}$, the $H_{\boldsymbol{G}}$-module $Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \boldsymbol{N}$ is flat and injective (as in appendix §9). The localization functor is exact and when applied to Cartan complexes, we obtain the *localized Cartan complexes*

$$\big(Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \Omega_{\boldsymbol{G}}(\boldsymbol{M}), \mathrm{id} \otimes \boldsymbol{d_G}\big) \quad \text{and} \quad \big(Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}), \mathrm{id} \otimes \boldsymbol{d_G}\big)$$

whose cohomologies, the *localized equivariant cohomologies*, respectively denoted $Q_{\boldsymbol{G}}(\boldsymbol{M})$ and $Q_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$, satisfy :

$$Q_{\boldsymbol{G}}(\boldsymbol{M}) = Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} H_{\boldsymbol{G}}(\boldsymbol{M}) \quad \text{and} \quad Q_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) = Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}).$$

6.1.1. Localized Equivariant Poincaré Duality. The localized equivariant cohomology is very close to the non equivariant cohomology in that the Poincaré duality pairings are perfect. The following, analog of 4.5.1, simply results from the fact that $Q_{\boldsymbol{G}}$ is a flat and injective $H_{\boldsymbol{G}}$-module (details are left to the reader).

6.1.2. Theorem. *Let $\boldsymbol{G}$ be a compact connected Lie group, and $\boldsymbol{M}$ an oriented $\boldsymbol{G}$-manifold of dimension $d_{\boldsymbol{M}}$. Then,*

a) *The morphism of (nongraded) complexes*

$$I\!\!D_{\boldsymbol{G}}(\boldsymbol{M}) : Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \Omega_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}^{\bullet}_{Q_{\boldsymbol{G}}}\big(Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}), Q_{\boldsymbol{G}}\big)$$

induces an isomorphism

$$\boxed{\mathscr{D}_{\boldsymbol{G}}(\boldsymbol{M}) : Q_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}^{\bullet}_{Q_{\boldsymbol{G}}}\big(Q_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}), Q_{\boldsymbol{G}}\big)}$$

[16] Note that we loose grading in considering this kind of localization. From this point of view, it would have been more clever to tensor by the ring $\boldsymbol{L_G} := S^{-1}H_{\boldsymbol{G}}$, where S denotes the multiplicative system of nonzero homogeneous elements of $H_{\boldsymbol{G}}$. As appendix 9 explains, if $\boldsymbol{N}$ is an $H_{\boldsymbol{G}}$-gm, the $H_{\boldsymbol{G}}$-module $\boldsymbol{L_G} \otimes_{H_{\boldsymbol{G}}} (\_)$ is graded, flat and injective, which is what we really need about localization.

b) *Moreover, if $\boldsymbol{M}$ is of finite type, the morphism of complexes*

$$\mathbb{D}'_{\boldsymbol{G}}(\boldsymbol{M}) : Q_{\boldsymbol{G}} \otimes_{H_G} \Omega_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}^{\bullet}_{Q_G}\big(Q_{\boldsymbol{G}} \otimes_{H_G} \Omega_{\boldsymbol{G}}(\boldsymbol{M}), Q_{\boldsymbol{G}}\big)$$

induces an isomorphism

$$\boxed{\mathscr{D}'_{\boldsymbol{G}}(\boldsymbol{M}) : Q_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \longrightarrow \mathbf{Hom}^{\bullet}_{Q_G}\big(Q_{\boldsymbol{G}}(\boldsymbol{M}), Q_{\boldsymbol{G}}\big)}$$

6.1.3. Exercise. Let $\boldsymbol{M}$ be of finite type. Prove that the torsion-freeness (4.5.2) of $H_{\boldsymbol{G}}(\boldsymbol{M})$ (resp. $H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$) is a necessary and sufficient condition for

$$\mathscr{D}_{\boldsymbol{G}}(\boldsymbol{M}) : H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \to \mathbf{Hom}^{\bullet}_{H_G}\big(H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}), H_{\boldsymbol{G}}\big)$$

(resp. $\mathscr{D}'_{\boldsymbol{G}}(\boldsymbol{M})$) to be injective. Discuss the case where $\boldsymbol{M}$ is not of finite type.

Hint: Let M be $H_{\boldsymbol{G}}$-gm. Show that the canonical map $M \to Q_{\boldsymbol{G}} \otimes_{H_G} M$ is injective if and only if M is torsion-free. Show that if M is also of finite type the natural map $\mathbf{Hom}^{\bullet}_{H_G}\big(M, H_{\boldsymbol{G}}\big) \to \mathbf{Hom}^{\bullet}_{H_G}\big(M, Q_{\boldsymbol{G}}\big)$ induces an isomorphism $Q_{\boldsymbol{G}} \otimes_{H_G} \mathbf{Hom}^{\bullet}_{H_G}\big(M, H_{\boldsymbol{G}}\big) \simeq \mathbf{Hom}^{\bullet}_{Q_G}\big(Q_{\boldsymbol{G}} \otimes_{Q_G} M, Q_{\boldsymbol{G}}\big)$. Apply 6.1.2.

6.1.4. Localized Equivariant Gysin Morphisms. As a consequence of theorem 6.1.2, if $f : \boldsymbol{M} \to \boldsymbol{N}$ is a map between oriented $\boldsymbol{G}$-manifolds, the localized Gysin morphisms

$$\begin{cases} f_* : Q_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M}) \to Q_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N}) \\ f_! : Q_{\boldsymbol{G}}(\boldsymbol{M}) \to Q_{\boldsymbol{G}}(\boldsymbol{N}), \quad \text{if } f \text{ is proper,} \end{cases}$$

are *catacterized*, as in the nonequivariant framework, by the adjoint equalities,

$$\begin{cases} \displaystyle\int_{\boldsymbol{M}} f^*[\beta] \cup [\alpha] = \int_{\boldsymbol{N}} [\beta] \cup f_*[\alpha] \\ \displaystyle\int_{\boldsymbol{M}} f^*[\beta] \cup [\alpha] = \int_{\boldsymbol{N}} [\beta] \cup f_![\alpha], \quad \text{if } f \text{ is proper.} \end{cases}$$

7. Equivariant Euler Classes

The reference for this section is Atiyah-Bott's paper [AB], notably §2 and §3.

7.1. The Nonequivariant Euler Class

Given a pair of oriented manifolds $(\boldsymbol{N}, \boldsymbol{M})$ with $\boldsymbol{N} \subseteq \boldsymbol{M}$, we denote by $\boldsymbol{N}_\epsilon$ a tubular neighborhood of $\boldsymbol{N}$ in $\boldsymbol{M}$. As the inclusion $\boldsymbol{N} \subseteq \boldsymbol{N}_\epsilon$ has the same nature as the inclusion of the zero section of a vector bundle $\sigma : \boldsymbol{B} \subseteq \boldsymbol{V}$ (1.9.7), we may define the *Thom class* $[\Phi(\boldsymbol{N}, \boldsymbol{M})]$ *of the pair* $(\boldsymbol{N}, \boldsymbol{M})$ following the same principle, that is, by means of the Gysin morphism associated with the closed embedding $i : \boldsymbol{N} \subseteq \boldsymbol{M}$. We thus set :

$$\boxed{[\Phi(\boldsymbol{N}, \boldsymbol{M})] := i_!(1) \in H^{d_M - d_N}(\boldsymbol{M})}$$

7.1.1. Definition. The *Euler class* $\mathrm{Eu}\big(\boldsymbol{N},\boldsymbol{M}\big)$ *of the pair* $(\boldsymbol{N},\boldsymbol{M})$ is the restriction of the Thom class to $H(\boldsymbol{N})$ ([17]), i.e. :

$$\mathrm{Eu}\big(\boldsymbol{N},\boldsymbol{M}\big) := i^* i_!(1) = [\Phi(\boldsymbol{N},\boldsymbol{M})]|_{\boldsymbol{N}} \in H^{d_M - d_N}(\boldsymbol{N}). \qquad (\diamond)$$

7.2. $\boldsymbol{G}$-Equivariant Euler Class

The generalization of the concept of Euler class to the equivariant framework is straightforward thanks to the equivariant Gysin morphism formalism: Given a pair of oriented $\boldsymbol{G}$-manifolds $(\boldsymbol{N},\boldsymbol{M})$ with $\boldsymbol{N} \subseteq \boldsymbol{M}$, we denote by $i_{\boldsymbol{G}} : \boldsymbol{N} \subseteq \boldsymbol{M}$ the inclusion map and define the *$\boldsymbol{G}$-equivariant Euler class* $\mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{N},\boldsymbol{M}\big)$ *of the pair* $(\boldsymbol{N},\boldsymbol{M})$ by the same formula $(\diamond)$:

$$\boxed{\mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{N},\boldsymbol{M}\big) := i_{\boldsymbol{G}}^* i_{\boldsymbol{G}!}(1) = [\Phi_{\boldsymbol{G}}(\boldsymbol{N},\boldsymbol{M})]|_{\boldsymbol{N}} \in H_{\boldsymbol{G}}^{d_M - d_N}(\boldsymbol{N})}\,.$$

where $i_{\boldsymbol{G}!} : H_{\boldsymbol{G}}(\boldsymbol{N})[d_{\boldsymbol{N}}] \to H_{\boldsymbol{G}}(\boldsymbol{M})[d_{\boldsymbol{M}}]$ is now the equivariant Gysin morphism.

7.2.1. Exercise. Given oriented $\boldsymbol{G}$-manifolds $\boldsymbol{L} \subseteq \boldsymbol{N} \subseteq \boldsymbol{M}$, prove the folowing formula for nested equivariant Euler classes

$$\boxed{\mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{L},\boldsymbol{M}\big) = \mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{L},\boldsymbol{N}\big) \cup \mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{N},\boldsymbol{M}\big)|_{\boldsymbol{L}}}$$

Hint: Use the projection formula for Gysin morphisms.

7.2.2. $\boldsymbol{G}$-Equivariant Euler Class of Discrete Fixed Point Sets

In the sequel, we denote by $\boldsymbol{M}^{\boldsymbol{G}}$ the subspace of $\boldsymbol{G}$-fixed points of $\boldsymbol{M}$.

When $\boldsymbol{N}$ is a discrete subspace of $\boldsymbol{M}^{\boldsymbol{G}}$, one has

$$\mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{N},\boldsymbol{M}\big) \in H_{\boldsymbol{G}}^{d_M}(\boldsymbol{N}) = \prod\nolimits_{b\in \boldsymbol{N}} S^{d_M}(\mathfrak{g})^{\boldsymbol{G}},$$

and $\mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{N},\boldsymbol{M}\big)$ is simply the family of invariant polynomials

$$\mathrm{Eu}_{\boldsymbol{G}}\big(\boldsymbol{N},\boldsymbol{M}\big) = \big\{\mathrm{Eu}_{\boldsymbol{G}}\big(b,\boldsymbol{M}\big) \in S^{d_M}(\mathfrak{g})^{\boldsymbol{G}}\big\}_{b\in\boldsymbol{N}}.$$

7.2.3. Proposition. *If $\boldsymbol{N}$ is a finite subset of $\boldsymbol{M}^{\boldsymbol{G}}$, one has*

$$\sum\nolimits_{b\in\boldsymbol{N}} \mathrm{Eu}_{\boldsymbol{G}}\big(b,\boldsymbol{M}\big) = \int_{\boldsymbol{M}} \Phi_{\boldsymbol{G}}(\boldsymbol{N},\boldsymbol{M}) \cup \Phi_{\boldsymbol{G}}(\boldsymbol{N},\boldsymbol{M}) \quad \text{and} \quad |\boldsymbol{N}| = \int_{\boldsymbol{M}} \Phi_{\boldsymbol{G}}(\boldsymbol{N},\boldsymbol{M}).$$

Proof. The constant function $\mathbb{1}_{\boldsymbol{N}}$ and, *a fortiori*, the Thom class $\Phi_{\boldsymbol{G}}(\boldsymbol{N},\boldsymbol{M})$, are both of compact supports. The equalities then immediately follow from the adjoint property of the Gysin morphism $i_* : H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{N}) \to H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ which gives:

$$\sum\nolimits_{b\in\boldsymbol{N}} \alpha|_b = \int_{\boldsymbol{M}} i_*(\mathbb{1}_{\boldsymbol{N}}) \cup \alpha, \quad \forall \alpha \in H_{\boldsymbol{G}}(\boldsymbol{M}). \qquad \square$$

7.2.4. $\boldsymbol{G}$- and $\boldsymbol{T}$-Equivariant Euler Classes of a Fixed Point. Let $\boldsymbol{T}$ be the maximal torus of the compact connected Lie group $\boldsymbol{G}$ and denote by $\boldsymbol{T}' := N_{\boldsymbol{G}}(\boldsymbol{T})$ the *normalizer* of $\boldsymbol{T}$ in $\boldsymbol{G}$. We have $\boldsymbol{T} \subseteq \boldsymbol{T}' \subseteq \boldsymbol{G}$ and if we choose $\mathbb{E}\boldsymbol{G}$ as *universal fiber bundle* for any of these groups, we can easily

[17] *Cf.* formula (2.19), p. 5, in *loc.cit.*

compare the corresponding Borel constructions for a given $\boldsymbol{G}$-manifold $\boldsymbol{M}$. In this way we obtain a natural commutative diagram of locally trivial fibrations:

$$\begin{array}{ccccc}
\boldsymbol{M_T} := \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{T}} \boldsymbol{M} & \overset{p}{\twoheadrightarrow} & \boldsymbol{M_{T'}} := \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{T'}} \boldsymbol{M} & \overset{q}{\twoheadrightarrow} & \boldsymbol{M_G} := \mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{B}\boldsymbol{T} & \overset{p}{\twoheadrightarrow} & \mathbb{B}\boldsymbol{T'} & \overset{q}{\twoheadrightarrow} & \mathbb{B}\boldsymbol{G}
\end{array} \tag{$\diamond$}$$

– The *Weyl group of* $(\boldsymbol{G},\boldsymbol{T})$, i.e. the finite group $\boldsymbol{W} := \boldsymbol{N_G}(\boldsymbol{T})/\boldsymbol{T}$, acts on the *right* of $\boldsymbol{M_T}$ and p is the orbit map for this action. In particular, the map $p^* : H_{\boldsymbol{T'}}(\boldsymbol{M}) \to H_{\boldsymbol{T}}(\boldsymbol{M})^{\boldsymbol{W}}$ is an isomorphism.

– The fibers of q are isomorphic to $\boldsymbol{G}/\boldsymbol{T'}$ which is *acyclic in rational cohomology*. Indeed, this space is the orbit space of $\boldsymbol{G}/\boldsymbol{T}$ for the right action of $\boldsymbol{W}$ and we know from an old result of Leray ([18]) that, under this action, $H(\boldsymbol{G}/\boldsymbol{T})$ is the regular representation of $\boldsymbol{W}$. In particular $H(\boldsymbol{G}/\boldsymbol{T'}) = H(\boldsymbol{G}/\boldsymbol{T})^{\boldsymbol{W}} = \mathbb{K}$, which implies that

$$q^* : H_{\boldsymbol{G}}(\boldsymbol{M}) \to H_{\boldsymbol{T'}}(\boldsymbol{M})$$

is an isomorphism.

This is a consequence of the general fact that if $q : \boldsymbol{X} \to \boldsymbol{Y}$ is a locally trivial fibration with acyclic fiber $\boldsymbol{F}$ between manifolds, then $q^* : H(\boldsymbol{Y}) \to H(\boldsymbol{X})$ is an isomorphism. Indeed, if $\mathscr{U} = \{U\}$ is an good cover of $\boldsymbol{Y}$ (cf. ([2])) such that $q : f^{-1}(U) \to U$ is a trivial fibration for all $U \in \mathscr{U}$, then $f^{-1}(U) = U \times \boldsymbol{F}$ and the cover $f^{-1}(\mathscr{U}) := \{f^{-1}(U)\}$ will be also good for $\boldsymbol{X}$. In that case, q^* establishes an *isomorphism* of Čech cohomologies $q^* : \check{H}(\mathscr{U};\boldsymbol{Y}) \to \check{H}(f^{-1}\mathscr{U};\boldsymbol{X})$ which are known to be canonically isomorphic to de Rham cohomologies. By this result, the maps $q^* : H(\mathbb{E}\boldsymbol{G}(m) \times_{\boldsymbol{G}} \boldsymbol{M}) \to H(\mathbb{E}\boldsymbol{G}(m) \times_{\boldsymbol{T'}} \boldsymbol{M})$ are bijective for all finite dimensional approximation $\mathbb{E}\boldsymbol{G}(m)$ of $\mathbb{E}\boldsymbol{G}$, which suffices to our purposes as equivariant cohomology is the projective limit of the cohomologies of these approximations (3.2.5). ([19])

– Summing up, we have the following two canonical isomorphisms

$$\boxed{H_{\boldsymbol{G}}(\boldsymbol{M}) \xrightarrow{q^*} H_{\boldsymbol{T'}}(\boldsymbol{M}) \xrightarrow{p^*} H_{\boldsymbol{T}}(\boldsymbol{M})^{\boldsymbol{W}}} \tag{$\ddagger$}$$

– When $\boldsymbol{M} = \{\bullet\}$, we obtain a commutative diagram of Chern-Weil homomorphisms

$$\begin{array}{ccccc}
\boldsymbol{S}(\mathfrak{g})^{\boldsymbol{G}} & \xrightarrow{\text{Chv}} & \boldsymbol{S}(\mathfrak{t})^{\boldsymbol{W}} & \xrightarrow{\subseteq} & \boldsymbol{S}(\mathfrak{t}) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
H_{\boldsymbol{G}} & \underset{\simeq}{\xrightarrow{q^*}} & H_{\boldsymbol{T'}} & \overset{p^*}{\rightarrowtail} & H_{\boldsymbol{T}}
\end{array} \tag{$\ddagger\ddagger$}$$

where $\text{Chv} : \boldsymbol{S}(\mathfrak{g})^{\boldsymbol{G}} \to \boldsymbol{S}(\mathfrak{t})^{\boldsymbol{W}}$ is the map that associates a symmetric polynomial function on $\mathfrak{g}$ with its restriction to the subspace $\mathfrak{t}$. The diagram already shows that Chv is an isomorphism, a claim known as the *Chevalley isomorphism* following the celebrated, much more general, *Chevalley's restriction theorem*.

[18] The statement appears as the Lemma 27.1 in the thesis of A. Borel, defended at La Sorbonne (with Leray as president) in 1952, ([Bo3], lemme 27.1, p. 193). Borel attributes the result to J. Leray ([L]).

[19] In [AB], p. 4, the interested reader will find partial indications to a different justification, that seems to rely on [Gt].

At this point it is worth noting that for each $b \in \boldsymbol{M}^{\boldsymbol{G}}$ the group $\boldsymbol{G}$ acts naturally on the tangent space $T_b(\boldsymbol{M})$ through a *linear representation.* Now, if we endow $\boldsymbol{M}$ with a $\boldsymbol{G}$-invariant riemannian metric, the exponential map $\exp : T_b(\boldsymbol{M}) \to \boldsymbol{M}$ is a $\boldsymbol{G}$-equivariant diffeomorphism between $T_b(\boldsymbol{M})$ and an open neighborhood of b in $\boldsymbol{M}$, so that the computation of equivariant Euler classes on fixed points may be greatly simplified by linearizing the data. The following proposition deals with the linear case.

7.2.5. Proposition. *Let $\boldsymbol{V}$ be a linear representation of a compact connected Lie group $\boldsymbol{G}$ with maximal torus $\boldsymbol{T}$.*

a) *The equivariant Euler class* $\mathrm{Eu}_{\boldsymbol{T}}\big(0,\boldsymbol{V}\big)$ *belongs to* $\boldsymbol{S}(\mathfrak{t})^{\boldsymbol{W}}$ *and the Chevalley isomorphism* $\mathrm{Chv} : \boldsymbol{S}(\mathfrak{g})^{\boldsymbol{G}} \to \boldsymbol{S}(\mathfrak{t})^{\boldsymbol{W}}$ *exchanges* $\mathrm{Eu}_{\boldsymbol{G}}\big(0,\boldsymbol{V}\big)$ *and* $\mathrm{Eu}_{\boldsymbol{T}}\big(0,\boldsymbol{V}\big)$.

b) *If* $\boldsymbol{V} := \boldsymbol{V}_1 \oplus \boldsymbol{V}_2$ *as* $\boldsymbol{G}$*-module, then* $\mathrm{Eu}_{\boldsymbol{G}}\big(0,\boldsymbol{V}\big) = \mathrm{Eu}_{\boldsymbol{G}}\big(0,\boldsymbol{V}_1\big)\mathrm{Eu}_{\boldsymbol{G}}\big(0,\boldsymbol{V}_2\big)$.

c) *Denote by* $\mathbb{C}(\alpha)$ *the complex vector space* $\mathbb{C}$ *endowed with the representation of* $\boldsymbol{T}$ *corresponding to the (nonzero)* weight $\alpha \in \mathfrak{t}^\vee$, *i.e.* $\exp(tx)(z) = e^{it\alpha(x)}z$, *for all* $t \in \mathbb{R}$ *and* $z \in \mathbb{C}$. *If the decomposition of* $\boldsymbol{V}$ *in irreducible representations of* $\boldsymbol{T}$ *is* $\boldsymbol{V} = \mathbb{R}^{\mu_0} \oplus \bigoplus_\alpha \mathbb{C}(\alpha)^{\mu(\alpha)}$, *then*

$$\mathrm{Eu}_{\boldsymbol{T}}\big(0,\boldsymbol{V}\big) = 0^{\mu_0}\prod\nolimits_\alpha \alpha^{\mu(\alpha)}.$$

d) $\mathrm{Eu}_{\boldsymbol{G}}\big(0,\boldsymbol{V}\big) \neq 0$ *if and only if* $\boldsymbol{V}^{\boldsymbol{T}} = \{0\}$.

Proof. (a) After the natural isomorphism of functors $H_{\boldsymbol{G}}(\_) \simeq H_{\boldsymbol{T}'}(\_)$ of (‡), it suffices to justify the commutativity of the following diagram:

$$\begin{array}{ccccc} H_{\boldsymbol{T}'}(0) & \xrightarrow{i_!} & H_{\boldsymbol{T}'}(\boldsymbol{V}) & \xrightarrow{i^*} & H_{\boldsymbol{T}'}(0) \\ \downarrow{\scriptstyle p^*} & \text{(I)} & \downarrow{\scriptstyle p^*} & \text{(II)} & \downarrow{\scriptstyle p^*} \\ H_{\boldsymbol{T}}(0) & \xrightarrow[i_!]{} & H_{\boldsymbol{T}}(\boldsymbol{V}) & \xrightarrow[i^*]{} & H_{\boldsymbol{T}}(0) \end{array}$$

The commutativity of the subdiagram (II) is obvious. For (I) we check its dual, the diagram

$$\begin{array}{ccc} H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{V}) & \xrightarrow{i^*} & H_{\boldsymbol{T}}(0) \\ \downarrow{\scriptstyle p_!} & (\mathrm{I}^\vee) & \downarrow{\scriptstyle p_!} \\ H_{\boldsymbol{T}',\mathrm{c}}(\boldsymbol{V}) & \xrightarrow[i^*]{} & H_{\boldsymbol{T}'}(0) \end{array}$$

where $p_! = \int_{\boldsymbol{W}}$, which is also clearly commutative.

(b,c,d) left to the reader. *Hint for* (c). Following (b), it suffices to show that $\mathrm{Eu}_{\boldsymbol{T}}\big(0,\mathbb{C}(\alpha)\big) = \alpha$. Taking polar coordinates $(\rho,\theta) \in \mathbb{R}_+ \times [0,2\pi]$ in $\mathbb{C}$, the nonequivariant Thom class $\Phi(0,\mathbb{C})$ is of the form

$$\Phi^{[2]} = \lambda(\rho)\ \rho d\rho \wedge d\theta\,,$$

where $\lambda : \mathbb{R} \to \mathbb{R}$ is a nonnegative differential function with compact support equal to 1 in a neighborhood of 0 and such that $\int_0^\infty\ \lambda(\rho)\ \rho d\rho = 1/2\pi$. As it is clear that $\Phi^{[2]}$ is invariant under the action of the unit circle action, it is

$\boldsymbol{T}$-invariant. We can thus use this differential 2-form to construct an equivariant Thom class following the procedure described in the proof of 5.6.6-(a). We have

$$(d_{\mathrm{t}}\Phi^{[2]})(X) = c(X)\ \lambda(\rho)\ \rho d\rho \wedge d\theta = -2\pi\alpha(X)\ \lambda(\rho)\ \rho d\rho,$$

and $\Phi^{[0]}(X)$ is necessarily equal to

$$\Phi^{[0]}(\rho,\theta)(X) = -2\pi\alpha(X)\Big(\int_0^\rho \lambda(\rho)\ \rho d\rho - \int_0^{+\infty}\lambda(\rho)\ \rho d\rho\Big),$$

since it must be of compact support. In this way we obtain

$$\mathrm{Eu}_{\boldsymbol{T}}\big(0,\mathbb{C}(\alpha)\big) = \Phi_{\boldsymbol{T}}(0,\mathbb{C}(\alpha))_{|0} = \Phi^{[0]}(0)(X) = \alpha(X). \qquad \square$$

7.2.6. Exercise. If $\boldsymbol{G}$ is the *special orthogonal group* SO(3) of the euclidean space $\mathbb{R}^3$, show that $\mathrm{Eu}_{\boldsymbol{G}}\big(0,\mathbb{R}^3\big) = 0$. Conclude that isolated $\boldsymbol{G}$-fixed points may have a null equivariant Euler class when $\boldsymbol{G}$ is nonabelian, contrary to the abelian case.

7.3. Torsions in Equivariant Cohomology Modules

7.3.1. Torsions. The *annihilator of an element* v of an $H_{\boldsymbol{G}}$-gm $\boldsymbol{V}$, is the ideal

$$\mathrm{Ann}(v) := \{P \in H_{\boldsymbol{G}} \mid P \cdot v = 0\}.$$

One says that v is a *torsion element* if $\mathrm{Ann}(v) \neq 0$, otherwise v is a *torsion-free element*. The $H_{\boldsymbol{G}}$-gm $\boldsymbol{V}$ is called a *torsion module* if all its elements are torsion elements, it is called a *torsion-free module* if zero is its the only torsion element, otherwise, it is called a *nontorsion module*.

7.3.2. Exercise. Given an $H_{\boldsymbol{G}}$-gm $\boldsymbol{V}$, let $\tau(\boldsymbol{V})$ be the subset of its torsion elements. Show that

1) $\tau(\boldsymbol{V})$ is a torsion module and the quotient $\varphi(\boldsymbol{V}) := \boldsymbol{V}/\tau(\boldsymbol{V})$ is torsion-free. The natural map: $Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \boldsymbol{V} \to Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \varphi(\boldsymbol{V})$ is an isomorphism.
2) $Q_{\boldsymbol{G}} \otimes_{H_{\boldsymbol{G}}} \boldsymbol{V} = 0$ if and only if $\boldsymbol{V}$ is torsion.
3) $\mathrm{Homgr}_{H_{\boldsymbol{G}}}(\boldsymbol{V},Q_{\boldsymbol{G}}) = 0$ if and only if $\boldsymbol{V}$ is torsion.
4) An inductive limit of torsion modules is a torsion module.
5) A projective limit of torsion modules may be a nontorsion module.

7.3.3. Exercise. The *annihilator of an* $H_{\boldsymbol{G}}$*-module* is the ideal

$$\mathrm{Ann}(\boldsymbol{V}) := \{P \in H_{\boldsymbol{G}} \mid P \cdot \boldsymbol{V} = 0\} = \bigcap\nolimits_{v\in \boldsymbol{V}} \mathrm{Ann}(v).$$

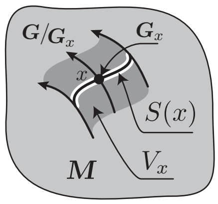
1) Show that if $\mathrm{Ann}(\boldsymbol{V}) \neq 0$, then $\boldsymbol{V}$ is torsion, but the converse may fail. ([20])
2) Show that if $\boldsymbol{V}$ is an $H_{\boldsymbol{T}}$-algebra with unit element, then $\mathrm{Ann}(\boldsymbol{V}) = \mathrm{Ann}(1)$.
3) Let $\{U_1 \subseteq U_2 \subseteq \cdots U_n \cdots\}$ be an increasing family of $\boldsymbol{G}$-stable open subsets of a $\boldsymbol{G}$-manifold $\boldsymbol{M}$ such that $\boldsymbol{M} = \bigcup_n U_n$. Suppose that $H_{\boldsymbol{G}}(U_n)$ and $H_{\boldsymbol{G},\mathrm{c}}(U_n)$ are torsion for all $n \in \mathbb{N}$. Show that $H_{\boldsymbol{G},\mathrm{c}}(\boldsymbol{M})$ is torsion, whereas $H_{\boldsymbol{G}}(\boldsymbol{M})$ may fail to be torsion.

[20] *Hint: For* $P \in H_{\boldsymbol{G}}$, *let* $\boldsymbol{W}(P) := H_{\boldsymbol{G}}/(H_{\boldsymbol{G}} \cdot P)$ *and take* $\boldsymbol{V} := \bigoplus_{P\in H_{\boldsymbol{G}}} \boldsymbol{W}(P)$.

4) In (3) show that $\{\mathrm{Ann}(H_{\boldsymbol{G}}(U_n))\}_n$ is a decreasing sequence of ideals and that
$$\mathrm{Ann}(H_{\boldsymbol{G}}(\boldsymbol{M})) = \bigcap\nolimits_{n\in\mathbb{N}} \mathrm{Ann}(H_{\boldsymbol{G},\mathrm{c}}(U_n)).$$
In particular, if the set $\{\mathrm{Ann}(H_{\boldsymbol{G}}(U_n))\}$ is finite, then $H_{\boldsymbol{G}}(\boldsymbol{M})$ is torsion.

7.3.4. The slice theorem. Given a $\boldsymbol{G}$-manifold $\boldsymbol{M}$, the *slice theorem* ([21]) claims that, through every point $x \in \boldsymbol{M}$ a submanifold $S(x)$ passes, *a slice through* x, such that the map $\boldsymbol{G}/\boldsymbol{G}_x$

$$\boldsymbol{G} \times_{\boldsymbol{G}_x} S(x) \to \boldsymbol{M}, \quad [(g,x)] \mapsto g\cdot x,$$



where $\boldsymbol{G}_x$ denotes the *isotropie group of* x, is a diffeomorphism onto a $\boldsymbol{G}$-stable neighborhood V_x of x. Then
$$H_{\boldsymbol{G}}(V_x) = H(\mathbb{E}\boldsymbol{G} \times_{\boldsymbol{G}} \boldsymbol{G} \times_{\boldsymbol{G}_x} S(x)) = H_{\boldsymbol{G}_x}(S(x)),$$
as well as $H_{\boldsymbol{G},\mathrm{c}}(V_x) = H_{\boldsymbol{G}_x,\mathrm{c}}(V_x)$ (exercise). As a consequence, the $H_{\boldsymbol{G}}$-module structures of $H_{\boldsymbol{G}}(V_x)$ and $H_{\boldsymbol{G},\mathrm{c}}(V_x)$ factorise through the natural ring homomorphism $\rho_x : H_{\boldsymbol{G}} \to H_{\boldsymbol{G}_x}$.

7.3.5. Proposition. *Let* $\boldsymbol{T}$ *be a torus. For every point* x *in a* $\boldsymbol{T}$*-manifold* $\boldsymbol{M}$*, the following equivalences hold.*

a) $\rho_x : H_{\boldsymbol{T}} \to H_{\boldsymbol{T}_x}$ *is injective if and only if* $x \in \boldsymbol{M}^{\boldsymbol{T}}$*. The* $H_{\boldsymbol{T}}$*-modules* $H_{\boldsymbol{G}}(V_x)$ *and* $H_{\boldsymbol{G},\mathrm{c}}(V_x)$ *are torsion if and only if* $x \notin \boldsymbol{M}^{\boldsymbol{T}}$

b) *If* $x \in \boldsymbol{M}^{\boldsymbol{T}}$*, then* $\mathrm{Eu}_{\boldsymbol{T}}\big(x,\boldsymbol{M}\big) \neq 0$ *if and only if* x *is an isolated point of* $\boldsymbol{M}^{\boldsymbol{T}}$*.*

Proof. (a) If $\boldsymbol{T}_x \neq \boldsymbol{T}$, there exist closed subtorus $\boldsymbol{H} \subseteq \boldsymbol{T}$ such that $\boldsymbol{T} = \boldsymbol{H} \times \boldsymbol{T}_x$ and $\dim(\boldsymbol{H}) > 0$, in which case $\ker(\rho_x) = H^+_{\boldsymbol{H}} \otimes H_{\boldsymbol{T}_x} \neq 0$. (b) is 7.2.5-(d). □

7.3.6. Remark. The interesting point of this proposition is that it faithfully translates topological properties of a point in a $\boldsymbol{T}$-manifold into algebraic properties of $H_{\boldsymbol{T}}$-modules, opening the way to the algebraic study of the topology of $\boldsymbol{T}$-spaces. When $\boldsymbol{G}$ is no longer abelian, both claims may fail. For (a), if $\boldsymbol{T}$ is a maximal torus for $\boldsymbol{G}$ and $\boldsymbol{M} = \boldsymbol{G}/\boldsymbol{T}$, the isotropie group of $x = g[\boldsymbol{T}] \in \boldsymbol{M}$ is the maximal torus $\boldsymbol{G}_x = g\boldsymbol{T}g^{-1}$, and ρ_x is the inclusion $H_{\boldsymbol{G}} = (H_{\boldsymbol{T}})^{\boldsymbol{W}} \subseteq H_{\boldsymbol{G}_x}$. Thus, ρ_x is injective although x is not a $\boldsymbol{G}$-fixed point. Exercise 7.2.6 gives a counterexample for (b).

7.3.7. Orbit Type of $\boldsymbol{T}$-Manifolds. The torsions of the $H_{\boldsymbol{T}}$-modules $H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M})$ and $H_{\boldsymbol{T}}(\boldsymbol{M})$ play a central role in the *fixed point theorem.*

When $\boldsymbol{M}^{\boldsymbol{T}} = \emptyset$, the slice theorem and 7.3.5-(a) show that $\boldsymbol{M}$ may be covered by a family of $\boldsymbol{T}$-stable open subspaces V_x where $H_{\boldsymbol{T}}(V_x)$ is killed by the elements of the nontrivial kernel $\rho_x : H_{\boldsymbol{T}} \to H_{\boldsymbol{T}_x}$. But then any finite union of those subspaces will also have torsion equivariant cohomology thanks to Mayer-Vietoris sequences, and, if $\boldsymbol{M}$ is compact, we can already say that $H_{\boldsymbol{T}}(\boldsymbol{M})$ is torsion. When $\boldsymbol{M}$ is not compact we may not be able to conclude the same (cf. 7.3.3-(3)) unless we have some kind of finiteness condition on the kernels of ρ_x. As shown in exercise 7.3.3-(4), such condition may be the finiteness of the

[21] See Hsiang [H] §I.3, p. 11.

set those kernels, or, what amounts to the same, the set of the isotropy groups $\mathcal{O}_{\boldsymbol{T}}(\boldsymbol{M}) := \{\boldsymbol{T}_x \mid x \in \mathbf{X}\}$ which is called the *orbit type of the* $\boldsymbol{T}$*-space* $\boldsymbol{M}$ ([22]).

Definition. A $\boldsymbol{T}$-manifold $\boldsymbol{M}$ is said *of finite orbit type* if $\mathcal{O}_{\boldsymbol{T}}(\boldsymbol{M})$ is finite.

7.3.8. Exercise. Show that a $\boldsymbol{T}$-manifold $\boldsymbol{M}$ is always locally of finite orbit type. In particular, if $\boldsymbol{M}$ is compact, it is of finite orbit type.

Hint. Use the slice theorem. If $x \notin \boldsymbol{M}^{\boldsymbol{T}}$, show that the slice $S(x)$ is a strict submanifold of $\boldsymbol{M}$ stable under $\boldsymbol{G}_x$ and that $\mathcal{O}_{\boldsymbol{G}}(\boldsymbol{G} \cdot S(x)) = \mathcal{O}_{\boldsymbol{G}_x}(S(x))$, then conclude by induction on $\dim(\boldsymbol{M})$. Otherwise, if $x \in \boldsymbol{M}^{\boldsymbol{T}}$, linearize the action as in 7.2.5 and conclude showing that there is a one-to-one correspondence between isotropy groups in the $\boldsymbol{T}$-space $T_x\boldsymbol{M}$ and subsets of the set of nonzero weights of the linear representation of $\boldsymbol{T}$ on $T_x\boldsymbol{M}$.

7.3.9. Proposition. *If $\boldsymbol{M}^{\boldsymbol{T}} = \emptyset$ and $\boldsymbol{M}$ is of finite orbit type, then*

$$H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}) \otimes_{H_{\boldsymbol{T}}} Q_{\boldsymbol{T}} = H_{\boldsymbol{T}}(\boldsymbol{M}) \otimes_{H_{\boldsymbol{T}}} Q_{\boldsymbol{T}} = 0.$$

Proof. – Torsion of $H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M})$. Let $(\mathscr{U},\subseteq)$ be the set of $\boldsymbol{G}$-stable open subspaces $U \subseteq \boldsymbol{M}$, such that $H_{\boldsymbol{T},\mathrm{c}}(U)$ is torsion, partially ordered by set inclusion. The set $\mathscr{U}$ is non empty as it contains every slice neighborhood V_x (7.3.4) and it is an inductive poset by exercise 7.3.3-(3), so that Zorn lemma can be applied. Let U be a maximal element in $\mathscr{U}$. For any $y \in \boldsymbol{M}$, let V_y be a slice neighborhood of y. By the exactness of the Mayer-Vietoris sequence for compact supports:

$$\cdots \to H^0_{\boldsymbol{G},\mathrm{c}}(U\cap V_y) \to H^0_{\boldsymbol{G},\mathrm{c}}(U)\oplus H^0_{\boldsymbol{G},\mathrm{c}}(V_y) \to H^0_{\boldsymbol{G},\mathrm{c}}(U\cup V_y) \to H^0_{\boldsymbol{G},\mathrm{c}}(U\cap V_y)[1] \to,$$

we easily conclude that $H^0_{\boldsymbol{G},\mathrm{c}}(U\cup V_y)$ is torsion. Then $U \supseteq V_y$, by the maximality of U, hence $U = \boldsymbol{M}$.

– *Torsion of $H_{\boldsymbol{T}}(\boldsymbol{M})$.* We cannot use the same argument as in the compact support case because a projective limit of torsion modules is no longer necessarily torsion. The finiteness assumption on the set of orbit types will now be crucial.

Let I be the intersection of all the ideals $\ker(\rho_x : H_{\boldsymbol{T}} \to H_{\boldsymbol{T}_x})$ for $x \in \boldsymbol{M}$. The finiteness of the orbit type of $\boldsymbol{M}$ ensures that $I \neq 0$. Let $(\mathscr{U},\subseteq)$ be the set of $\boldsymbol{G}$-stable open subspaces $U \subseteq \boldsymbol{M}$, such that $I \subseteq \mathrm{Ann}(H_{\boldsymbol{T}}(U))$, partially ordered by set inclusion. The set $\mathscr{U}$ is non empty as it contains every slice neighborhood V_x (7.3.4) and it is an inductive poset by exercise 7.3.3-(4), so that Zorn lemma can be applied. Let U be a maximal element in $\mathscr{U}$. For any $y \in \boldsymbol{M}$, let V_y be a slice neighborhood of y. Thanks to the exactness of the first terms of the Mayer-Vietoris sequence: $0 \to H^0_{\boldsymbol{G}}(U \cup V_y) \to H^0_{\boldsymbol{G}}(U) \oplus H^0_{\boldsymbol{G}}(V_y) \to H^0_{\boldsymbol{G}}(U \cap V_y) \to$, we easily see that $1 \in H^0_{\boldsymbol{G}}(U \cup V_y)$ is killed by I. Then $I \subseteq \mathrm{Ann}(H_{\boldsymbol{G}}(U \cup V_y))$ by 7.3.3-(2) and $U \supseteq V_y$, by the maximality of U, hence $U = \boldsymbol{M}$. □

7.4. Localization Theorems

Given a $\boldsymbol{T}$-manifold $\boldsymbol{M}$ and a *nontrivial closed* subgroup $\{1\} \neq \boldsymbol{H} \subseteq \boldsymbol{T}$, the fixed point set $\boldsymbol{M}^{\boldsymbol{H}} := \{x \in \boldsymbol{M} \mid h \cdot x = x \ \forall h \in \boldsymbol{H}\}$ is a submanifold whose

[22] See [H] chap IV §2, p. 54, for the general definition nottably for non abelian groups.

connected components (not necessarily of equal dimensions) are stable under the action of $\boldsymbol{T}$, and furthermore they are orientable if $\boldsymbol{M}$ is ([23]).

Terminology. An homomorphism of $H_{\boldsymbol{T}}$-modules $\alpha : L \to L'$ will be called an *isomorphism modulo torsion* if its kernel and cokernel are both torsion $H_{\boldsymbol{T}}$-modules, i.e. if the induced homomorphism of $Q_{\boldsymbol{T}}$-modules

$$\alpha \otimes_{H_{\boldsymbol{T}}} \mathrm{id} : L \otimes_{H_{\boldsymbol{T}}} Q_{\boldsymbol{T}} \to L' \otimes_{H_{\boldsymbol{T}}} Q_{\boldsymbol{T}}$$

is an isomorphism.

7.4.1. Proposition. *Let $\boldsymbol{M}$ be an oriented $\boldsymbol{T}$-manifold of finite orbit type. For any $\boldsymbol{H}$ nontrivial closed subgroup of $\boldsymbol{T}$, denote by $\iota_{\boldsymbol{H}} : \boldsymbol{M}^{\boldsymbol{H}} \hookrightarrow \boldsymbol{M}$ the set inclusion. The following morphisms of $H_{\boldsymbol{T}}$-gm ([24]) are isomorphisms modulo torsion.*

$$\textit{Gysin morphisms} \begin{cases} \iota_{\boldsymbol{H}!} : \ H_{\boldsymbol{T}}(\boldsymbol{M}^{\boldsymbol{H}})[d_{\boldsymbol{M}^{\boldsymbol{H}}}] \ \to \ H_{\boldsymbol{T}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \\ \iota_{\boldsymbol{H}*} : H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}^{\boldsymbol{H}})[d_{\boldsymbol{M}^{\boldsymbol{H}}}] \to H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M})[d_{\boldsymbol{M}}] \end{cases}$$

$$\textit{Restriction morphisms} \begin{cases} \iota^*_{\boldsymbol{H}} : H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}) \to H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}^{\boldsymbol{H}}) \\ \iota^*_{\boldsymbol{H}} : \ H_{\boldsymbol{T}}(\boldsymbol{M}) \ \to \ H_{\boldsymbol{T}}(\boldsymbol{M}^{\boldsymbol{H}}) \end{cases}$$

Proof. The kernel and cokernel of the restriction $\iota^*_{\boldsymbol{H}} : H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}) \to H_{\boldsymbol{T},\mathrm{c}}(\boldsymbol{M}^{\boldsymbol{H}})$ lay within $H_{\boldsymbol{T},\mathrm{c}}(U)$, where $U := \boldsymbol{M} \backslash \boldsymbol{M}^{\boldsymbol{H}}$. Now, as the isotropy groups of the points of U are *strict* subgroups of $\boldsymbol{T}$, there are no $\boldsymbol{T}$-fixed points, i.e. $U^{\boldsymbol{T}} = \emptyset$, and we can conclude that $H_{\boldsymbol{T},\mathrm{c}}(U)$ is an $H_{\boldsymbol{T}}$-torsion module by 7.3.9. In particular, any submodule of $H_{\boldsymbol{T},\mathrm{c}}(U)$, viz. the kernel and the cokernel of $\iota^*_{\boldsymbol{H}}$, is a torsion $H_{\boldsymbol{T}}$-module. By duality the same is true for $\iota_{\boldsymbol{H},!} : H_{\boldsymbol{T}}(\boldsymbol{M}^{\boldsymbol{H}}) \to H_{\boldsymbol{T}}(\boldsymbol{M})$.

The other restriction $\iota^*_{\boldsymbol{H}} : H_{\boldsymbol{T}}(\boldsymbol{M}) \to H_{\boldsymbol{T}}(\boldsymbol{M}^{\boldsymbol{H}})$ is a little more tricky as its kernel and cokernel lay within $H_{\boldsymbol{T},U}(\boldsymbol{X})$ which we have not yet proved is an $H_{\boldsymbol{T}}$-torsion module. For that, recall that since one has short exact sequences of local section functors over open subspaces

$$0 \to \varGamma_{U_1 \cap U_1}(\_) \longrightarrow \varGamma_{U_1}(\_) \oplus \varGamma_{U_2}(\_) \longrightarrow \varGamma_{U_1 \cup U_2}(\_) \to 0$$

where $\varGamma_U(\_)$ denotes the kernel of the restriction $\varGamma(\boldsymbol{M}, \_) \to \varGamma(\boldsymbol{M} \backslash U, \_)$, one may follow a Mayer-Vietoris procedure to approach $H_{\boldsymbol{T},U}(\boldsymbol{X})$ by successively adding slice open sets $V_x \subseteq U$ (7.3.4). In this way, to show that $H_{\boldsymbol{T},U}(\boldsymbol{M})$ is a torsion module, it suffices to show that each $H_{\boldsymbol{T},V_x}(\boldsymbol{M})$ is so. Now, this $H_{\boldsymbol{T}}$-module occurs in the exact triangle

$$H_{\boldsymbol{T},V_x}(\boldsymbol{M}) \longrightarrow H(\boldsymbol{M}) \to H_{\boldsymbol{T}}(\boldsymbol{M} \backslash V_x) \to$$

where $\boldsymbol{M} \backslash V_x$ is $\boldsymbol{T}$-equivariantly homotopic to $\boldsymbol{M} \backslash \boldsymbol{T} \cdot x$ since the slice $S(x)$ is a submanifold of $\boldsymbol{M}$, therefore $H_{\boldsymbol{T},V_x}(\boldsymbol{M}) \simeq H_{\boldsymbol{T},\boldsymbol{T}\cdot x}(\boldsymbol{M}) \simeq H_{\boldsymbol{T}}(\boldsymbol{T}\cdot x) = H_{\boldsymbol{T}_x}$, which proves that $H_{\boldsymbol{T},V_x}(\boldsymbol{M})$ is a torsion $H_{\boldsymbol{T}}$-module. □

[23] We recall that the reason for this is that under the action of $\boldsymbol{H}$, the tangent spaces $T_x(\boldsymbol{M})$ for $x \in \boldsymbol{M}^{\boldsymbol{H}}$ split as the direct sum of $T_x(\boldsymbol{M}^{\boldsymbol{H}})$ and a sum of $\boldsymbol{H}$-irreducible two dimension representations $\mathbb{C}(\alpha)$ (cf. 7.2.5-(c)) which are besides canonically oriented by their character. Thus, the orientation of $T_x(\boldsymbol{M}^{\boldsymbol{H}})$ determines that of $T_x(\boldsymbol{M})$ and vice versa.

[24] As the submanifold $\boldsymbol{M}^{\boldsymbol{H}}$ need not be connected nor equidimensionnal the shift indication in a notation as $H_{\boldsymbol{T}}(\boldsymbol{M}^{\boldsymbol{H}})[d_{\boldsymbol{M}^{\boldsymbol{H}}}]$ must be understood component-wise.

7.4.2. Theorem. *Let $\boldsymbol{M}$ be a $\boldsymbol{T}$-oriented manifold of finite orbit type such that $\boldsymbol{M}^{\boldsymbol{T}}$ is a discrete subspace of $\boldsymbol{M}$. Then*

a) *For all $\mu \in H_{\boldsymbol{T},c}(\boldsymbol{M})$ the following* "localization formula" *is satisfied:*

$$\int_{\boldsymbol{M}} \mu = \sum_{x \in \boldsymbol{M}^{\boldsymbol{T}}} \frac{\mu|_x}{\mathrm{Eu}_{\boldsymbol{T}}(x, \boldsymbol{M})}.$$

b) *If $\boldsymbol{M}$ is compact of positive dimension*

$$0 = \sum_{x \in \boldsymbol{M}^{\boldsymbol{T}}} \frac{1}{\mathrm{Eu}_{\boldsymbol{T}}(x, \boldsymbol{M})}.$$

Proof. (a) From 7.4.1, the morphism $i_{\boldsymbol{T},*} : H_{\boldsymbol{T},c}(\boldsymbol{M}^{\boldsymbol{T}}) \to H_{\boldsymbol{T},c}(\boldsymbol{M})$ is an isomorphism modulo torsion, so that it suffices to prove the localization formula for the equivariant Thom classes $\Phi_{\boldsymbol{T}}(x, \boldsymbol{M})$ for all $x \in \boldsymbol{M}^{\boldsymbol{T}}$. But we have already shown that $\int_{\boldsymbol{M}} \Phi_{\boldsymbol{T}}(x, \boldsymbol{M}) = 1$ (7.2.3) and that $\Phi_{\boldsymbol{T}}(x, \boldsymbol{M})|_x = \mathrm{Eu}_{\boldsymbol{T}}(x, \boldsymbol{M})$ by definition. (b) Apply the localization formula to $1 \in H^0_{\boldsymbol{T}}(\boldsymbol{M})$. □

8. Miscellany

Excerpt from [Bo$_2$] (A. Borel, IV-§3 , p. 55, 1960), where, for the first time, a reference to what is nowadays known as *the Borel construction* appears.

> 3.9. REMARK. All our discussion will center around the space X_G, and the remarks 3.6, 3.7 will be basic. Similar arguments have been used by Conner [5] when G is a circle, in rational cohomology. For an algebraic analogue when G is discrete, see [7, Chap. V]. The space X_G and the embedding $F(X; G) \times B_G \subset X_G$ were also mentioned to the author by A. Shapiro. The proof of Smith's theorem 4.3 is also related to that of [2].

The references '[2]', '[5]' and '[7]' correspond to ours [Bo$_1$], [Co] and [Gr].

9. Appendix

We explain the following fact mentioned in footnote ([16]).

Proposition. *Let $\boldsymbol{A} := \boldsymbol{A}^0 \oplus \boldsymbol{A}^1 \oplus \cdots$ be a graded ring. Denote by S the multiplicative system generated by the nonzero graded elements of $\boldsymbol{A}$. The ring $\boldsymbol{L} := S^{-1}\boldsymbol{A}$ is a graded $\boldsymbol{A}$-module such that, for any $\boldsymbol{A}$-graded module $\boldsymbol{N}$, the tensor product $\boldsymbol{L} \otimes_{\boldsymbol{A}} \boldsymbol{N}$ is flat and injective in the category of graded $\boldsymbol{A}$-modules.*

Proof. • $\boldsymbol{L} \otimes \boldsymbol{N}$ **is flat.** For any graded ideal $\boldsymbol{I}$ of $\boldsymbol{A}$, one has the long exact sequence:

$$\mathbf{0} \to \mathbf{Tor}_1^{\boldsymbol{A}}(\boldsymbol{L}, \boldsymbol{A}/\boldsymbol{I}) \longrightarrow \boldsymbol{L} \otimes \boldsymbol{I} \longrightarrow \boldsymbol{L} \longrightarrow \boldsymbol{L} \otimes (\boldsymbol{A}/\boldsymbol{I}) \to \mathbf{0} \qquad (*)$$

where $\boldsymbol{A}/\boldsymbol{I}$ is a torsion graded $\boldsymbol{A}$-module. The annihilators of the elements of $\boldsymbol{A}/\boldsymbol{I}$ are graded ideals, generated, as such, by invertible elements of $\boldsymbol{L}$. Therefore

$$\mathbf{Tor}_1^{\boldsymbol{A}}(\boldsymbol{L}, \boldsymbol{A}/\boldsymbol{I}) = \mathbf{0}, \quad \forall i \in \mathbb{N},$$

and we have from $(*)$ the equality $\boldsymbol{L}\otimes\boldsymbol{I}=\boldsymbol{L}$ from which, we deduce

$$\boldsymbol{L}\otimes\boldsymbol{I}\otimes\boldsymbol{N}=\boldsymbol{L}\otimes\boldsymbol{N}$$

for any $\boldsymbol{A}$-graded module $\boldsymbol{N}$. The *ideal criterion of flatness* applies, and the $\boldsymbol{A}$-graded module $\boldsymbol{L}\otimes\boldsymbol{N}$ is flat.

• **$\boldsymbol{L}\otimes\boldsymbol{N}$ is injective.** Let $\alpha:\boldsymbol{M}_1\subseteq\boldsymbol{M}_2$ be a graded inclusion of graded $\boldsymbol{A}$-modules. We must show that any morphism $\lambda:\boldsymbol{M}_1\to\boldsymbol{L}\otimes\boldsymbol{N}$ of graded $\boldsymbol{A}$-modules can be extended to $\boldsymbol{M}_2$.

$$\begin{array}{ccc} \boldsymbol{M}_1 & \xrightarrow[\subseteq]{\alpha} & \boldsymbol{M}_2 \\ {\scriptstyle\lambda}\downarrow & \swarrow{\scriptstyle\lambda'} & \\ \boldsymbol{L}\otimes\boldsymbol{N} & & \end{array}$$

In the contrary, Zorn's lemma will led us to assume that $\boldsymbol{M}_2\supsetneqq\boldsymbol{M}_1$ and that λ may not be further extended. In particular, $\boldsymbol{A}\cdot m\cap\boldsymbol{M}_1\neq\boldsymbol{0}$ for any homogeneous $m\in\boldsymbol{M}_2\setminus\boldsymbol{M}_1$, hence the quotient $\boldsymbol{M}_2/\boldsymbol{M}_1$ is a torsion module. One then has

$$\boldsymbol{L}\otimes\boldsymbol{M}_1=\boldsymbol{L}\otimes\boldsymbol{M}_2\,,$$

and a contradiction arises as a consequence of the diagram

$$\begin{array}{ccc} \mathrm{Homgr}_{\boldsymbol{A}}(\boldsymbol{M}_2,\boldsymbol{L}\otimes\boldsymbol{N}) & \longrightarrow & \mathrm{Homgr}_{\boldsymbol{A}}(\boldsymbol{M}_1,\boldsymbol{L}\otimes\boldsymbol{N}) \\ {\scriptstyle\cong}\downarrow & & {\scriptstyle\cong}\downarrow \\ \mathrm{Homgr}_{\boldsymbol{L}}(\boldsymbol{L}\otimes\boldsymbol{M}_2,\boldsymbol{L}\otimes\boldsymbol{N}) & \xrightarrow[(=)]{} & \mathrm{Homgr}_{\boldsymbol{L}}(\boldsymbol{L}\otimes\boldsymbol{M}_1,\boldsymbol{L}\otimes\boldsymbol{N}) \end{array}$$

where the horizontal arrows are induced by the inclusion $\boldsymbol{M}_1\subseteq\boldsymbol{M}_2$ and the vertical arrows are the well-known canonical natural isomorphisms. □

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Index