A note on representation stability of FB-modules

Alberto Arabia*

May 16, 2018

Abstract. An FB-module is, after Thomas Church and Benson Farb ([1]), a countable family $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$ of finite dimensional linear representations W_m of the symmetric groups $\mathcal{S}_m$, over the field of rational numbers $\mathbb{Q}$, hereafter *an* $\mathcal{S}_m$*-module*. The study of the asymptotic behavior of an FB-module is the main motivation of their work. Among the different types of stability they consider, two play a central rôle : the *representation stability* and the *character polynomiality*, which we now recall.

• An FB-module $\mathcal{W}$ is said to be *(eventually) representation stable* (RS), if there exists $N \in \mathbb{N}$, such that

$$W_m \sim \bigoplus\nolimits_\lambda {V_{\lambda[m]}}^{\boldsymbol{n}_\lambda}\,, \quad \text{for all } m \geq N,$$

where $\lambda := (\lambda_1, \ldots, \lambda_\ell)$ is a partition verifying $|\lambda| + \lambda_1 \leq N$, where $V_{\lambda[m]}$ is the simple $\mathcal{S}_m$-module associated with the partition $\lambda[m] := (m - |\lambda|, \lambda_1, \ldots, \lambda_\ell)$, and where $\{\boldsymbol{n}_\lambda\}$ is a family of natural numbers independent of m. The smallest such N is *the rank of representation stability of* $\mathcal{W}$, it will be denoted by '$\mathrm{rank}_{\mathrm{RS}}(\mathcal{W})$'.

• An FB-module $\mathcal{W}$ is said to be *(eventually) of polynomial character* (PC), if there exist $N \in \mathbb{N}$ and a polynomial $P_{\mathcal{W}} \in \mathbb{Q}[X_1, \ldots, X_N]$, such that

$$\chi_{W_m}(g) = P_{\mathcal{W}}(X_1(g), \ldots, X_N(g))\,, \quad \text{for all } m \geq N \text{ and } g \in \mathcal{S}_m\,,$$

where χ_{W_m} is the character of the $\mathcal{S}_m$-module W_m, and $X_i(g)$ is the number of i-cycles in the decomposition of g as product of disjoint cycles in $\mathcal{S}_m$. The polynomial $P_{\mathcal{W}}$, which is unique, is *the polynomial character of* $\mathcal{W}$. The smallest such N is *the rank of polynomiality of* $\mathcal{W}$, it will be denoted by '$\mathrm{rank}_{\mathrm{PC}}(\mathcal{W})$'.

The purpose of these notes is to present à self-contained proof of the fact that these two properties are equivalent. While the implication (RS) $\Rightarrow$ (PC) is a simple consequence of the Frobenius character formula, the converse does not seem to be documented and motivates the present work. More precisely, we prove :

Theorem (4.1.1). *Let* $\mathcal{W}$ *be an* FB*-module.*

a) *If* $\mathcal{W}$ *is* (RS) *for* $m \geq N$, *then* $\mathcal{W}$ *is* (PC) *for* $m \geq N$.
b) *If* $\mathcal{W}$ *is* (PC) *for* $m \geq N$ *with polynomial character* $P_{\mathcal{W}}$, *then* $\mathcal{W}$ *is* (RS) *for* $m \geq \max\{N, 2\deg_{\boldsymbol{w}}(P_{\mathcal{W}})\}$, *where* $\deg_{\boldsymbol{w}}(P_{\mathcal{W}})$ *is the degree of the* $P_{\mathcal{W}}$ *under the convention that* $\deg_{\boldsymbol{w}}(X_i) := i$.

We will exhibit FB-modules $\mathcal{W}$ such that $\mathrm{rank}_{\mathrm{PC}}(\mathcal{W}) = 0$, and $\mathrm{rank}_{\mathrm{RS}}(\mathcal{W}) = N$, for N arbitrarily big. In particular, there are no universal upper bounds for the numbers $\mathrm{rank}_{\mathrm{RS}}(\mathcal{W})/\,\mathrm{rank}_{\mathrm{PC}}(\mathcal{W})$ or $\mathrm{rank}_{\mathrm{RS}}(\mathcal{W}) - \mathrm{rank}_{\mathrm{PC}}(\mathcal{W})$ (*cf.* proposition 2.3.4-(a)).

* Université Paris Diderot-Paris 7, IMJ-PRG, CNRS, Bâtiment Sophie Germain, bureau 608, Case 7012, 75205. Paris Cedex 13, France. Contact : `alberto.arabia@imj-prg.fr`.

Contents

Warning. The next two sections introduce notations and some well-known properies of FI-modules, which advanced readers can skip.

1. Preliminaries

1.1. General notations

- Given a group G, we denote by G-mod the category of G-modules, *i.e.* of finite dimensional linear representations of G over the field of rational numbers $\mathbb{Q}$.
- The symmetric group $\mathcal{S}_m$ is the group of bijections of the interval of natural numbers $[\![1, m]\!]$. For $n \le m$, the inclusion $\mathcal{S}_n \subseteq \mathcal{S}_m$ identifies a permutation g of $[\![1, n]\!]$ with its extension to $[\![1, m]\!]$ that fixes all $i > n$.
- $\mathbb{Q}_m$ and $\epsilon(\mathbb{Q})_m$ denote respectively the *trivial* and the *alternating or signature* representation of the group $\mathcal{S}_m$.
- $\mathcal{S}_a \boxtimes \mathcal{S}_b$ is the stabilizer of the partition $[\![1, a+b]\!] = [\![1, a]\!] \sqcup [\![a+1, a+b]\!]$.
- If W_a is an $\mathcal{S}_a$-module and W_b is an $\mathcal{S}_b$-module, we denote by $W_a \boxtimes W_b$ the tensor product $W_a \otimes_{\mathbb{Q}} W_b$ endowed with the $\mathcal{S}_a \boxtimes \mathcal{S}_b$-module structure defined by the componentwise action $(g_a, g_b)(w_a \otimes x_b) := g_a(w_a) \otimes g_b(w_b)$.
- A *partition* of $m \in \mathbb{N}\backslash\{0\}$ is any decreasing sequence of natural numbers $\lambda := (\lambda_1 \ge \ldots \ge \lambda_\ell > 0)$ such that $m = \sum_i \lambda_i$. It is also denoted as the m-tuple $(1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, m^{\boldsymbol{n}_m})$ where $\boldsymbol{n}_k := \#\{i \mid \lambda_i = k\}$, so that $m = \sum_i i\, \boldsymbol{n}_i$. The notation $\lambda \vdash m$ says that λ is a partition of m, and $|\lambda|$ is used for the number partitioned by λ. The partition λ is *empty* if $|\lambda| = 0$.
- $\mathbb{Q}_{\text{cl}}(\mathcal{S}_m)$ denotes the $\mathbb{Q}$-algebra of *rational class functions* of $\mathcal{S}_m$. These are the functions $f : \mathcal{S}_m \to \mathbb{Q}$ which are constant along the conjugacy classes of $\mathcal{S}_m$, *i.e.* such hat $f(gxg^{-1}) = f(x)$, $\forall g, x \in \mathcal{S}_m$. The scalar product

  $$\langle\!\langle\, \_ \mid \_ \,\rangle\!\rangle_{\mathcal{S}_m} : \mathbb{Q}_{\text{cl}}(\mathcal{S}_m) \times \mathbb{Q}_{\text{cl}}(\mathcal{S}_m) \to \mathbb{Q}$$

  is defined by

  $$\langle\!\langle\, f_1 \mid f_2 \,\rangle\!\rangle_{\mathcal{S}_m} := \frac{1}{|\mathcal{S}_m|} \sum_{g \in \mathcal{S}_m} f_1(g)\, f_2(g^{-1})\,.$$

- If W_m is an $\mathcal{S}_m$-module, $\chi_{W_m} : \mathcal{S}_m \to \mathbb{Q}$ denotes its character. The *Schur's orthogonality relations* state that if V_1 and V_2 are *simple* $\mathcal{S}_m$-modules, then $\langle\!\langle\, \chi_{V_1} \mid \chi_{V_2} \,\rangle\!\rangle_{\mathcal{S}_m}$ is equal to 1 if V_1 is isomorphic to V_2, and to 0 otherwise.

From now, and up to the end of this preliminary section, we will be recalling concepts and terminology coming from Church's and Farb's works (*cf.* [1]).

1.2. The categories of FB and FI modules

- **FB** denotes the category of Finite sets and Bijections. An FB-*module* is, by definition, a *covariant* functor from the category **FB** to the category $\mathbf{Vec}_f(k)$ of finite dimensional $\mathbb{Q}$-vector spaces and $\mathbb{Q}$-linear maps:

$$\mathcal{W} : \mathbf{FB} \rightsquigarrow \mathbf{Vec}_f(k)\,.$$

  To give an FB-module $\mathcal{W}$ is then equivalent to give the countable collection $\{W_m := \mathcal{W}([\![1,m]\!])\}_{m\in\mathbb{N}}$, where W_m is an $\mathcal{S}_m$-module. A morphism of FB-modules $f : \mathcal{W} \to \mathcal{Z}$ corresponds then to a family $\{f_m : W_m \to Z_m\}_{m\in\mathbb{N}}$ of morphisms of $\mathcal{S}_m$-modules. We thus have a canonical identification:

$$\mathrm{Mor}_{\mathrm{FB}}(\mathcal{W},\mathcal{Z}) = \prod\nolimits_{m\in\mathbb{N}} \mathrm{Hom}_{\mathcal{S}_m}(W_m, Z_m)$$

  The category of FB-modules will be denoted by **FB**-mod. It is a semi-simple abelian category.

- **FI** denotes the category of Finite sets and Injections. An FI-*module* is a *covariant functor* from **FI** to the category $\mathbf{Vec}_f(k)$ of finite dimensional $\mathbb{Q}$-vector spaces and $\mathbb{Q}$-linear maps:

$$\mathcal{W} : \mathbf{FI} \rightsquigarrow \mathbf{Vec}_f(k)\,.$$

  To give an FI-module is thus equivalent to give

  FI-1) an FB-module $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$;

  FI-2) for all $m \in \mathbb{N}$, an *interior* map $\phi(\mathcal{W})_m : W_m \to W_{m+1}$ (in short ϕ_m), which is a $\mathbb{Q}$-linear map such that, for all $g \in \mathcal{S}_m \subseteq \mathcal{S}_{m+1}$, one has

$$\phi_m(g \cdot w) = g \cdot \phi_m(w)\,.$$

  FI-3) for all $n \geq m$, the image of $\phi_{n,m} := \phi_{n-1} \circ \cdots \circ \phi_m$ must satisfy:

$$\phi_{n,m}(W_m) \subseteq (W_n)^{\mathbf{1}_m \boxtimes \mathcal{S}_{n-m}}\,.$$

  Under this equivalence, a morphism of FI-modules $f : \mathcal{W} \to \mathcal{Z}$ is simply a morphism of FB-modules which is compatible with the interior maps ϕ_m, *i.e.* such that the diagrams

$$\begin{array}{ccc} W_m & \xrightarrow{\phi(\mathcal{W})_m} & W_{m+1} \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle f_{m+1}} \\ Z_m & \xrightarrow{\phi(\mathcal{Z})_m} & Z_{m+1} \end{array}$$

  are commutative.

  The category of FI-modules will be denoted by **FI**-mod. It is an abelian category, which is *not* semi-simple.

1.2.1. Comments

a) The category FB-mod is equivalent to the full subcategory of FI-mod of FI-modules whose interior maps are null.

b) A more interesting subcategory of **FI**-mod is that of the FI-modules whose interior maps are *injective* and *(eventually) exhaustive*, *i.e.* such that the image $\phi_m(W_m)$ generates W_{m+1} as $\mathcal{S}_{m+1}$-module (for large m). Among these, there are the FI-modules $\mathcal{V}_\lambda$'s, which bind up all the simple $\mathcal{S}_m$-modules $V_{\lambda[m]}$ (*cf.* 1.6.1). We will see that an FB-module $\mathcal{W}$ which is (PC), is asymptotically isomorphic to a finite direct sum of $\mathcal{V}_\lambda$'s, and, as such, it admits a structure of FI-module with interior maps injective and eventually exhaustive (see 4.1.2). From this perspective, the (PC) property is a *numerical* condition on FB-modules revealing the existence of a nontrivial structure of FI-module on $\mathcal{W}$: that of *representation stable* FI-modules (*cf.* 1.7.1).

1.2.2. The *stupid* truncations

The functor $(\_)_{\geq \ell} : \mathbf{FI}\text{-mod} \rightsquigarrow \mathbf{FI}\text{-mod}$ that "*truncates*" an FI-module $\{W_m\}_{m\in\mathbb{N}}$ by replacing by $\mathbf{0}$ its terms W_m for $m < \ell$, is an additive exact functor. There is a natural inclusion $(\_)_{\geq \ell} \rightarrowtail \mathrm{id}_{\mathbf{FI}}$ whose cokernel is the truncation $(\_)_{<\ell}$ which replaces the terms W_m for $m \geq \ell$ by $\mathbf{0}$. We thus have short exact sequences

$$\mathbf{0} \to (\mathcal{W})_{\geq \ell} \rightarrowtail \mathcal{W} \twoheadrightarrow (\mathcal{W})_{<\ell} \to \mathbf{0}\,,$$

which are natural with respect to $\mathcal{W}$. The full subcategory $\mathbf{FI}\text{-mod}_{\geq \ell}$ of FI-modules $\mathcal{W}$ such that the inclusion $\mathcal{W}_{\geq \ell} \rightarrowtail \mathcal{W}$ is an isomorphism is an abelian subcategory, and the same for the full subcategory $\mathbf{FI}\text{-mod}_{<\ell}$ of FI-modules $\mathcal{W}$ such that the quotient $\mathcal{W} \twoheadrightarrow \mathcal{W}_{<\ell}$ is an isomorphism. One has,

$$\mathrm{Ext}^i_{\mathrm{FI}}\big(\mathbf{FI}\text{-mod}_{\geq \ell}, \mathbf{FI}\text{-mod}_{\leq \ell}\big) = 0\,, \quad \forall i > 0\,.$$

The intersection $\mathbf{FI}\text{-mod}_{\geq \ell} \cap \mathbf{FI}\text{-mod}_{\leq \ell}$ is the (semi-simple) category $\mathcal{S}_\ell$-mod.

1.3. Projective FI-modules

An obvious way to construct FI-modules of the type described in 1.2.1-(b) is to start off with a given representation W_n of some $\mathcal{S}_n$ and define, for all $m \in \mathbb{N}$:

$$\mathcal{P}(W_n)_m := \begin{cases} \mathbf{0}\,, & \text{if } m < n, \\ \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m}\big(W_n \boxtimes k_{m-n}\big)\,, & \text{otherwise.} \end{cases}$$

For each $m \geq n$, the composition of the following natural maps ι_m and κ_{m+1}:

$$\begin{array}{ccc} W_n \boxtimes k_{m-n} & \xhookrightarrow{\ \iota_m\ } & W_n \boxtimes k_{m+1-n} \\ & \overset{\psi_m}{\searrow} & \big\downarrow {\scriptstyle \kappa_{m+1}} \\ & & \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m+1-n}}^{\mathcal{S}_{m+1}}\big(W_n \boxtimes k_{m+1-n}\big) \end{array}$$

gives the map ψ_m whose image is invariant under $\big(\mathbf{1}_n \boxtimes \mathcal{S}_{m+1-n}\big)$, something that implies that the induced maps

$$\phi_m := \mathrm{ind}(\psi_m) : \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m}\big(W_n \boxtimes k_{m-n}\big) \to \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m+1-n}}^{\mathcal{S}_{m+1}}\big(W_n \boxtimes k_{m+1-n}\big)$$

satisfy the requirements which make of the family

$$\mathcal{P}(W) := \{\phi_m : \mathcal{P}(W_n)_m \to \mathcal{P}(W_n)_{m+1}\}_{m\in\mathbb{N}}\,,$$

an FI-module.

1.3.1. Proposition. *Let W_n be a representation of $\mathcal{S}_n$.*

a) *The interior maps of $\mathcal{P}(W_n)$ are injective $\forall m \in \mathbb{N}$, and exhaustive $\forall m \geq n$.*
b) *There is a natural identification of functors*

$$\mathrm{Mor}_{\mathbf{FI}}(\mathcal{P}(W_n), \_) = \mathrm{Hom}_{\mathcal{S}_n}(W_n, (\_)_n)\,.$$

c) *$\mathcal{P}(W_n)$ is a projective* FI*-module, and it is a simple projective* FI*-modules if and only if W_n is a simple $\mathcal{S}_n$-module. All simple projective* FI*-modules are of this form.*
d) *The category* **FI**-mod *has enough projective objects.*

Proof. Left to the reader. □

1.4. Young diagrams and Pieri's rule

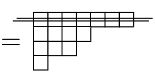
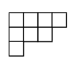
We recall the re-parametrization of irreducible representations of the symmetric groups introduced by Church and Farb.

1.4.1. The *socle* and the *weight* of a partition

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ be a non-empty partition of $m := \sum_i \lambda_i$.

- The *socle* of λ is the sub-partition $\underline{\lambda} := (\lambda_2, \ldots, \lambda_\ell)$.
- The *weight* of λ is the number $\boldsymbol{w}(\lambda) := |\underline{\lambda}| = \lambda_2 + \cdots + \lambda_\ell$.

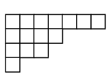
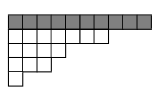
Notice that the map $\lambda \mapsto \underline{\lambda}$ is injective from the set of partitions of $m \in \mathbb{N}$, so that it amounts the same giving $(\lambda \vdash m)$ or giving the pair $(m, \underline{\lambda})$. In terms of Young diagrams, in order to get the socle $\underline{\lambda}$ of λ, one simply erases the first row of the Young diagram corresponding to λ. The picture is thus:

$\lambda :=$  $\mapsto$ $\underline{\lambda} :=$  .

Conversely, given $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and any $m \geq |\lambda| + \lambda_1$, Church and Farb introduce the notation

$$\lambda[m] := (m - |\lambda|, \lambda_1, \ldots, \lambda_\ell)\,,$$

which, in terms of Young diagrams, corresponds to simply add a first row with as many boxes as is necessary to raise the total number of boxes from $|\lambda|$ to m.

$\lambda :=$  $\mapsto$ $\lambda[m] :=$  .

Notice the following obvious facts:

$$\lambda[m] \vdash m\,, \quad \underline{\lambda[m]} = \lambda\,, \quad \lambda = \underline{\lambda}[\,|\lambda|\,]\,.$$

1.4.2. Irreducible representations of the symmetric groups

The irreducible representations of $\mathcal{S}_n$ are parametrized by the partitions $\nu \vdash n$. The simple $\mathcal{S}_n$-module associated with ν is denoted by V_ν. The following proposition recalls a very fundamental and basic fact about the representations of the symmetric groups (see [2], chap. 4, thm. 4.3, pp. 46–).

1.4.3. Proposition. *The irreducible representations of the symmetric groups over a field k of characteristic zero are defined over the field of rational numbers. In particular, if W_m is a $\mathbb{Q}[\mathcal{S}_m]$-module of finite dimension, the character*

$$\chi_{W_m} : \mathcal{S}_m \to k\,, \quad g \mapsto \mathrm{tr}(g : W_m \to W_m)\,,$$

a priori with values in k, takes its values in the ring of integers $\mathbb{Z} \subseteq k$.

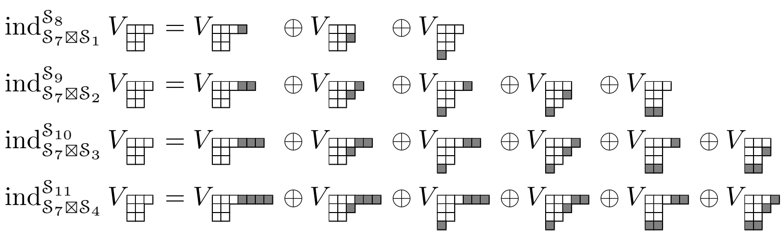
Hint. Because W_m is defined over the rationals, the traces are rationals and are sums of roots of unity, hence algebraic integers, hence integersz. □

1.4.4. Pieri's rule

Pieri's rule ([1]) gives the irreducible factors of the terms $\mathcal{P}(V_\nu)_m$, for any given partition $\nu \vdash n \le m$. The rule says that in the decomposition

$$\mathcal{P}(V_\nu)_m := \mathrm{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_m} \big(V_\nu \boxtimes k_{m-n}\big) = \bigoplus_{\mu \vdash m} V_\mu{}^{\boldsymbol{n}_\nu(\mu)}\,, \qquad (1)$$

the nonzero multiplicities $\boldsymbol{n}_\nu(\mu)$ are all equal to 1, and the corresponding partitions μ are those obtained from ν by adding $m-n$ boxes in *different* columns. For example, if $\nu := (3,2,2)$ and $m \in \{8,9,10,11\}$, we have



and, a key observation, due to Church and Farb, is that for $m \ge |\nu| + \nu_1$ (=10), the set of socles of the Young diagrams appearing in the decomposition (1) becomes constant. The following lemma then follows in an obvious way.

1.4.5. Lemma. *Let $\nu \vdash n > 0$.*

a) *For all $\mu \vdash m \ge n$, such that $\mathbf{n}_\nu(\mu) \ne 0$, the weight of μ verifies*

$$\boldsymbol{w}(\nu) \le \boldsymbol{w}(\mu) \le |\nu|\,,$$

and one has $\begin{cases} \boldsymbol{w}(\mu) = \boldsymbol{w}(\nu) & \iff \mu = \underline{\nu}[m] \\ \boldsymbol{w}(\mu) = |\nu| & \iff (m \ge |\nu| + \nu_1)\ \&\ \big(\mu = \nu[m]\big)\,. \end{cases}$

[1] For a thorough introduction to these rules, read the paragraph §4.3, p. 54–62, on Fulton-Harris' book [2], and also the Littlewood-Richardson rules in its appendix A, p. 451.

b) *Let* $m_0 = |\nu| + \nu_1$ *and let* $\mathcal{P}_\nu$ *be a set of partitions* μ *such that*

$$\operatorname{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m_0-n}}^{\mathcal{S}_{m_0}} \big(V_\nu \boxtimes k_{m_0-n})\big) = \bigoplus\nolimits_{\mu \in \mathcal{P}_\nu} V_{\mu[m_0]} \,.$$

Then, for all $m \geq m_0$,

$$\operatorname{ind}_{\mathcal{S}_n \boxtimes \mathcal{S}_{m-n}}^{\mathcal{S}_{m}} \big(V_\nu \boxtimes k_{m-n})\big) = \bigoplus\nolimits_{\mu \in \mathcal{P}_\nu} V_{\mu[m]} \,.$$

Proof. Left to the reader. □

1.5. The *weight* of a representation and of an FI-module

We extend the definition of *weight*, from partitions (1.4.1) to representations and, more generally, to FB-modules.

- The *weight of an* $\mathcal{S}_m$*-module* W_m is the upper bound of the weights of the partitions associated with its irreducible factors, *i.e.* if

$$W_m \sim \bigoplus\nolimits_{\mu \vdash m} {V_\tau}^{\boldsymbol{n}_\mu}$$

  then

$$\boldsymbol{w}\,(W_m) := \sup \big\{ \boldsymbol{w}\,(\mu) \bigm| \boldsymbol{n}_\mu \neq 0 \big\} \,.$$

  For example, as a consequence of 1.4.5-(a), we have

$$\begin{cases} \boldsymbol{w}\,(\nu) \leq \boldsymbol{w}\,(\mathcal{P}(V_\nu))_m \leq |\nu| \,, & \forall m \geq |\nu| \,, \text{ and} \\ \qquad\qquad \boldsymbol{w}\,(\mathcal{P}(V_\nu))_m = |\nu| \,, & \forall m \geq |\nu| + \nu_1 \,. \end{cases} \qquad (*)$$

- The *weight of an* FB*-module* $\mathcal{W} := (W_m)$ is the upper-bound of the weights of its terms, *i.e.*

$$\boldsymbol{w}\,(\mathcal{W}) := \sup \big\{ \boldsymbol{w}\,(W_m) \big\}_{m \in \mathbb{N}} \,.$$

- The *weight at infinity of an* FB*-module* $\mathcal{W} := (W_m)$ is

$$\boldsymbol{w}_\infty(\mathcal{W}) := \lim_{N \mapsto +\infty} \boldsymbol{w}\,(\mathcal{W}_{\geq N}) \,.$$

  –It is easy to see that $\boldsymbol{w}_\infty(\mathcal{W}) = \boldsymbol{w}\,(\mathcal{W}_{\geq N})$, for some $N \gg 0$.
  –We have $\boldsymbol{w}\,(\mathcal{P}(V_\lambda)) = \boldsymbol{w}_\infty(\mathcal{P}(V_\lambda)) = |\lambda| \,.$

1.5.1. The *weight* truncations

Let $p \in \mathbb{N}$. Given an $\mathcal{S}_m$-module W_m, denote by $(W_m)_{\boldsymbol{w}>p}$ the sum of the irreducible factors of W_m of weight $> p$.

- By Pieri's rule 1.4.5-(a), if $\mathcal{W} := \{\phi_m : W_m \to W_{m+1}\}_{m \in \mathbb{N}}$ is an FI-module, one has $\phi_m\big((W_m)_{\boldsymbol{w}>p}\big) \subseteq \big((W_{m+1})_{\boldsymbol{w}>p}\big)$, in which case, the family

$$\mathcal{W}_{\boldsymbol{w}>p} := \big\{ \phi_m(W_m)_{\boldsymbol{w}>p} \to (W_m)_{\boldsymbol{w}>p} \big\}_{m \in \mathbb{N}}$$

  is a sub-FI-module of $\mathcal{W}$.

- Let $\mathcal{W}_{\boldsymbol{w}\le p} := \mathcal{W}/\mathcal{W}_{\boldsymbol{w}>p}$. The short exact sequence

$$\mathbf{0} \to \mathcal{W}_{\boldsymbol{w}>p} \to \mathcal{W} \to \mathcal{W}_{\boldsymbol{w}\le p} \to \mathbf{0}$$

is natural with respect to $\mathcal{W}$.

1.5.2. Remark. The following are easy consequences of the definitions.

a) The full subcategory $\mathbf{FI}\text{-mod}_{\boldsymbol{w}>p}$ of FI-modules $\mathcal{W}$ such that the inclusion $\mathcal{W}_{\boldsymbol{w}>p} \rightarrowtail \mathcal{W}$ is an isomorphism is an abelian subcategory.

b) The full subcategory $\mathbf{FI}\text{-mod}_{\boldsymbol{w}\le p}$ of FI-modules $\mathcal{W}$ such that the quotient $\mathcal{W} \twoheadrightarrow \mathcal{W}_{\boldsymbol{w}\le p}$ is an isomorphism is an abelian subcategory.

c) $\mathrm{Ext}^i_{\mathrm{FI}}\big(\mathbf{FI}\text{-mod}_{\boldsymbol{w}>p}, \mathbf{FI}\text{-mod}_{\boldsymbol{w}\le p}\big) = 0$, for all $i \in \mathbb{N}$.

1.6. The FI-module $\mathcal{V}_\lambda$

Given a partition λ, consider the projective FI-module $\mathcal{P}(V_\lambda)$ introduced in 1.3. The lemma 1.4.5-(a) says that for all $m \ge |\lambda|$ the smallest weight of the irreducible factors of the terms $\mathcal{P}(V_\lambda)_m$ is exactly $\boldsymbol{w}(\lambda)$, which is the weight of a unique factor: the simple $\mathcal{S}_m$-module $V_{\underline{\lambda}[m]}$ with multiplicity 1. The following proposition results from this simple observation and the weight filtration 1.5.1.

1.6.1. Proposition. *Let λ be a nonempty partition.*

a) *The terms of the quotient* FI*-module*

$$\mathcal{V}_\lambda := \mathcal{P}(V_\lambda)_{\boldsymbol{w}\le\boldsymbol{w}(\lambda)} = \big\{\phi_m : \mathcal{V}_{\lambda,m} \to \mathcal{V}_{\lambda,m+1}\big\}_{m\in\mathbb{N}}$$

are

$$\mathcal{V}_{\lambda,m} = \begin{cases} \mathbf{0} \ , & \text{for all } m < |\lambda|, \\ V_{\underline{\lambda}[m]} \ , & \text{otherwise.} \end{cases}$$

The interior maps ϕ_m are injective, and are exhaustive for $m \ge |\lambda|$.

b) *If $\mathcal{V}'_\lambda := \{\phi'_m : \mathcal{V}_{\lambda,m} \to \mathcal{V}_{\lambda,m+1}\}_{m\in\mathbb{N}}$ is an* FI*-module such that ϕ'_m is injective and is exhaustive for all $m \ge N$, then $(\mathcal{V}'_\lambda)_{\ge N}$ and $(\mathcal{V}_\lambda)_{\ge N}$* (cf. *1.2.2) are isomorphic* FI*-modules for $m \ge N$.*

Proof. Left to the reader. □

1.7. Representation stability

1.7.1. Representation stable FI-modules

An FI-module $\mathcal{W} = \{\phi_m : W_m \to W_{m+1}\}_{m\in\mathbb{N}}$ is said to be *representation stable for $m \ge N$*, in short $(\mathrm{RS})_{m\ge N}$, if the following conditions are satisfied.

RS-1) The interior maps ϕ_m are injective for $m \ge 0$, and exhaustive for $m \ge N$.

RS-2) For all $m \ge N$, we have

$$W_m \sim \bigoplus_{|\lambda|\le N} V_{\underline{\lambda}[m]}{}^{\boldsymbol{n}_\lambda},$$

where the $\boldsymbol{n}_\lambda$ are independent of $m \ge N$.

We denote by $\mathrm{rank}_{\mathrm{RS}}(\mathcal{W})$, the smallest such N, and we call it the *rank of representation stability of $\mathcal{W}$.*

1.7.2. Examples

a) $\mathcal{P}(V_\lambda)$ is $(\mathrm{RS})_{m\geq|\lambda|+\lambda_1}$, (lemma 1.4.5-(b)).
b) $\mathcal{P}(W_n)$ $(\mathrm{RS})_{m\geq 2n}$, (consequence of lemma 1.4.5-(b)).
c) $\mathcal{V}_\lambda$ is $(\mathrm{RS})_{m\geq|\lambda|}$, (by definition).

1.7.3. Representation stable FB-modules

An FB-module $\mathcal{W} = \{W_m\}_{m\in\mathbb{N}}$ is said to be *representation stable for* $m \geq N$, in short $(\mathrm{RS})_{m\geq N}$, if the previous condition (RS-2) is satisfied.

In that case, we say that $\mathcal{W}$ and $\bigoplus_{\lambda\in P} \mathcal{V}_\lambda^{\boldsymbol{n}_\lambda}$ are *asymptotically isomorphic as* FB-*modules*, and we will write

$$\boxed{\mathcal{W}_{\geq N} \sim \bigoplus\nolimits_{|\lambda|\leq N} (\mathcal{V}_\lambda)^{\boldsymbol{n}_\lambda}_{\geq N}} \tag{2}$$

2. Character polynomiality of FI-modules

2.1. Character polynomiality

We denote by $\mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m)$ the $\mathbb{Q}$-algebra of *class functions* defined on $\mathcal{S}_m$ with values in $\mathbb{Q}$, *i.e.* functions $f : \mathcal{S}_m \to \mathbb{Q}$ which are constant on each conjugacy classe of $\mathcal{S}_m$. We denote by $\mathbb{Q}[\overline{X}]$ the ring of polynomials with coefficients in $\mathbb{Q}$, and in countably many variables $X_1, X_2, \dots$, endowed with the grading '$\deg_{\boldsymbol{w}}$' that stipulates that :

$$\deg_{\boldsymbol{w}}(X_i) := i\,.$$

2.1.1.The *weight* of a polynomial. In the sequel, in order avoid confusions with the usual degree $\deg(P)$ of a polynomial $P\in A[\overline{X}]$, the one which stipulates that $\deg(X_i) = 1$, we will call $\deg_{\boldsymbol{w}}(P)$ the *weight* of P.

2.1.2. Proposition. *Denote by $X_{m,i} : \mathcal{S}_m \to \mathbb{N}$ the class function which assigns to $g \in \mathcal{S}_m$, the number $X_{m,i}(g)$ of i-cycles in the decomposition of g as product of disjoint cycles in $\mathcal{S}_m$.*

a) *The map*

$$\begin{aligned} \rho_m : \mathbb{Q}[\overline{X}] &\longrightarrow \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m) \\ X_i &\longmapsto (g \mapsto X_{m,i}(g)) \end{aligned} \tag{3}$$

is an homomorphism of $\mathbb{Q}$-algebras whose kernel contains the polynomials

$$\big(X_1 + 2X_2 + \cdots + mX_m - m\big) \quad \textit{and} \quad \big(X_i(X_i-1)\cdots\big(X_i - \lfloor m/i\rfloor\big)\big)\,. \tag{4}$$

The restriction

$$\rho_m : \mathbb{Q}[X_1,\dots,X_{m-1}] \twoheadrightarrow \mathbb{Q}_{\mathrm{cl}}(\mathcal{S}_m)$$

is surjective. In particular, the characters of $\mathcal{S}_m$ are represented by polynomials with rational coefficients and in the variables $X_1,\dots,X_{m-1}$.

b) *For $n \leq m$ and $g\in\mathcal{S}_n$, we have*

(i) $\rho_m(X_1)(\iota g) = \rho_n(X_1)(g) + (m-n)\,,$
(ii) $\rho_m(X_i)(\iota g) = \rho_n(X_i)(g)\,,\ \forall i > 1\,.$

$$\begin{array}{ccc} \mathcal{S}_n & \xrightarrow{\rho_n} & \\ {\scriptstyle\iota}\downarrow & & A \\ \mathcal{S}_m & \xrightarrow{\rho_m} & \end{array}$$

Hint. (a) The fact that the polynomials (4) belong to $\ker(\rho_m)$ is clear. Next, to see that ρ_m is surjective, it suffices to show that the characteristic function of a conjugacy class of $\mathfrak{S}_m$ can be realized as a polynomial in $X_1, \ldots, X_m$.

For $k \in [\![1, m]\!]$, let $R_k(Z) := Z(Z-1)\cdots\widehat{(Z-k)}\cdots(Z-m)$ and consider

$$D_k(Z) := R_k(Z)/R_k(k) \in \mathbb{Q}[Z]\,.$$

This polynomial has the property that

$$\rho_m(D_k(X_i)(g) = \begin{cases} 1\,, & \text{if } X_i(g) = k\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

So that, if $\sum_i i\,\boldsymbol{n}_i = m$, we get

$$\rho_m(D_{\boldsymbol{n}_1}(X_1)D_{\boldsymbol{n}_2}(X_2)\cdots D_{\boldsymbol{n}_m}(X_m))(g) = \begin{cases} 1\,, & \text{if } g \text{ is of type } (1^{\boldsymbol{n}_1}, \ldots, m^{\boldsymbol{n}_m}) \\ 0\,, & \text{otherwise.} \end{cases}$$

(b) is clear. □

2.1.3. Convention. *In order to alleviate notations, we will simply write $X_i(g)$ for $X_{m,i}(g)$, and this, despite the possible ambiguity of '$X_1(g)$' (cf.* 2.1.2-(b-i)).

2.1.4. Definition. An FB-module $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$ is said to be of *polynomial character for $m \geq N$*, in short $(\mathrm{PC})_{m\geq N}$, if there exists a polynomial $P_{\mathcal{W}} \in \mathbb{Q}[\overline{X}]$ such that

$$\chi_{W_m} = \rho_m(P_{\mathcal{W}})\,, \quad \forall m \geq N\,.$$

We denote by $\mathrm{rank}_{\mathrm{PC}}(\mathcal{W})$, the smallest such N, and we call it the *rank of character polynomiality of $\mathcal{W}$.*

2.1.5. Proposition and definition. *The polynomial $P_{\mathcal{W}}$ that asymptotically represents the characters of the terms W_m of an* FB*-module $\mathcal{W} := \{W_m\}_{m\in\mathbb{N}}$ is unique. It is called the* polynomial character of the FB-module $\mathcal{W}$.

Proof. Indeed, if $P'_{\mathcal{W}}$ were another polynomial representing χ_{W_m} for $m \ggg 0$, the difference $Q := P_{\mathcal{W}} - P'_{\mathcal{W}}$, that we may assume to belong to $\mathbb{Q}[X_1, \ldots, X_N]$, would be a polynomial representing the zero class function for all $m \ggg N$.

If Q is not the null polynomial, we can write it as a polynomial in X_N with coefficients Q_i in $\mathbb{Q}[X_1, \ldots, X_{N-1}]$:

$$Q = Q_0 + Q_1 X_N + Q_2 X_N^2 + \cdots + Q_r X_N^r\,, \text{ and } Q_r \neq 0. \tag{$*$}$$

Now, for *any* family of numbers $\bar{a} := \{a_1, \ldots, a_{N-1} \subseteq \mathbb{N}\}$, and for any $i \in \mathbb{N}$, it is easy to find $m_i \ggg N$ and $g_i \in \mathfrak{S}_{m_i}$ such that $X_1(g_i) = a_1, \ldots, X_{N-1}(g_i) = a_{N-1}$ and $X_N(g_i) > i$. In that case $Q(\bar{a}, X_N)$ has infinitely many roots and is, therefore, the null polynomial in X_N. In particular, $Q_r(\bar{a}) = 0$ for all choices of $\bar{a}$, which is only possible if Q_r is the null polynomial in $\mathbb{Q}[X_1, \ldots, X_{N-1}]$, contrary to its definition $(*)$. The polynomial Q must therefore be the null polynomial. □

2.2. Frobenius character formula

Given a partition $\lambda := (\lambda_1, \ldots, \lambda_\ell) \vdash m$, Frobenius gave a celebrated formula to compute the character χ_{V_λ} of the simple $\mathcal{S}_m$-module V_λ. The important point for us about this formula is that it gives an expression of χ_{V_λ} as a polynomial *only depending on the socle* $\underline{\lambda}$. As a consequence, the same polynomial expresses the characters of all the terms in the FI-module $\mathcal{V}_{\lambda,m}$, for $m \geq |\lambda|$, something that self-explains the character polynomiality of the FI-module $\mathcal{V}_\lambda = \{V_{\lambda,m}\}_m$.

2.2.1. Frobenius polynomial for χ_{V_λ}

Following Macdonald in his book [3] (ex. I.7.14, p. 122), let $y := \{y_1, \ldots, y_\ell\}$ be a set of ℓ abstract variables, where $\ell := \ell(\lambda)$. The *discriminant of* y is the antisymmetric homogeneous polynomial

$$\Delta(y) := \prod\nolimits_{i<j} (y_i - y_j)$$

and, for $d \in \mathbb{N}$, the *d-power sum of* y is the symmetric homogeneous polynomial

$$P_d(y) := y_1^d + \cdots + y_\ell^d\,.$$

The value $\chi_{V_\lambda}(g)$ for $g \in \mathcal{S}_m$, is, after Frobenius, the coefficient of the monomial

$$y_1^{\lambda_1+(\ell-1)} y_2^{\lambda_2+(\ell-2)} y_3^{\lambda_3+(\ell-3)} \cdots y_\ell^{\lambda_\ell}\,,$$

in the development of the product

$$\Delta(y) \left(\prod\nolimits_{d\geq 1} P_d(y)^{X_d(g)} \right). \tag{5}$$

This coefficient, denoted by $\boldsymbol{X}_\lambda$, is a polynomial in $\mathbb{Q}[\overline{X}]$, we call it *the Frobenius polynomial for* χ_{V_λ}.

2.2.2. Proposition. *The Frobenius polynomial $\boldsymbol{X}_\lambda$ for χ_{V_λ} only depends on the socle $\underline{\lambda}$ of λ, it belongs to the ring $\mathbb{Q}[X_1, \ldots, X_{\lambda_2+\ell-2}]$ and its weight is:*

$$\deg_{\boldsymbol{w}}(\boldsymbol{X}_\lambda) = \boldsymbol{w}(\lambda)\,.$$

The characters of $\mathcal{S}_m$ can thus be represented by polynomials in $\mathbb{Q}[X_1, \ldots, X_{m-1}]$ of weights $\leq m-1$.

Proof. Because the polynomial (5) is homogeneous, we can make $y_1 = 1$ without loosing information. In that case, $\chi_{V_\lambda}(g)$ is the coefficient in the monomial

$$y_2^{\lambda_2+(\ell-2)} y_3^{\lambda_3+(\ell-3)} \cdots y_\ell^{\lambda_\ell}\,,$$

after the development of the product

$$\Delta(\tilde{y}) \left(\prod\nolimits_{j>1} (1 - y_j) \right) \left(\prod\nolimits_{d\geq 1} (1 + P_d(\tilde{y}))^{X_d(g)} \right) \tag{‡‡}$$

where $\tilde{y} := \{y_2, \ldots, y_\ell\}$. But, in this product the first factor $\Delta(\tilde{y})$ is already homogeneous of total degree $(\ell-2) + (\ell-3) + \cdots$, so that we have to seek, in the development of the remaining factors, terms whose total degree is bounded by $|\underline{\lambda}| = \lambda_2 + \cdots + \lambda_\ell$. But then, since we have

$$\big(1 + P_d\big)^{X_d} = 1 + \binom{X_d}{1} P_d + \binom{X_d}{2} P_d^2 + \binom{X_d}{3} P_d^3 + \cdots$$

and because $\deg_{\mathrm{tot}}(P_d^a) = ad$, we conclude that

- If $d > \lambda_2 + \ell - 2$, the factor $\big(1 + P_d\big)^{X_d}$ only contributes to χ_{V_λ} with its term 1^{X_d}, so that it can be neglected. The product symbol $\prod_{d\geq 1}$ in (‡‡) can therefore be replaced by $\prod_{d=1}^{\lambda_2+\ell-2}$.
- The coefficient $\binom{X_d}{j}$ is a polynomial of degree j in X_d and appears attached to monomials in $\tilde{y}$ of total degree jd. We can thus conclude that after development, the expression of χ_{V_λ} is a polynomial in $X_1, \ldots, X_{\lambda_2+\ell-2}$ of *weight* $|\underline{\lambda}| = \boldsymbol{w}(\lambda)$. □

The following corollary of proposition 2.2.2 is now immediate from the definition of representation stable FB-modules 1.7.3.

2.2.3. Corollary *An* FB*-module which is* $(\mathrm{RS})_{\geq N}$*, is also* $(\mathrm{PC})_{\geq N}$.

Proof. Left to the reader. □

2.3. Basic examples of character polynomiality of FI-modules

2.3.1. The ℓ-cycles of $[\![1, m]\!]$

Given $m, \ell \in \mathbb{N}$, we denote by Γ_ℓ^m the set of ℓ-tuples $(i_1, \ldots, i_\ell)$ of pairwise distinct elements of $[\![1, m]\!]$ modulo cyclic permutation, *i.e.* such that

$$(i_1, \ldots, i_\ell) = (i_2, \ldots, i_\ell, i_1) = (i_3, \ldots, i_\ell, i_1, i_2) = \cdots$$

The symmetric group $\mathcal{S}_m$ acts on Γ_ℓ^m by

$$g \cdot (i_1, \ldots, i_\ell) = (g(i_1), g(i_2), \ldots, g(i_\ell))\,. \tag{6}$$

The elements of Γ_ℓ^m are called *the* ℓ*-cycles of* $[\![1, m]\!]$.

2.3.2. Comments

a) Given $g \in \mathcal{S}_m$, the set $[\![1, m]\!]$ is decomposed in $\langle g\rangle$-orbits, each of which can be endowed with a cyclic order defined by g. For example, if $x \in [\![1, m]\!]$, we may consider the ordering $(x \to g(x) \to g^2(x) \to \cdots \to x)$, which gives the well-known *decomposition of* $g \in \mathcal{S}_m$ *as product of disjoint cycles.*

b) For $m \geq \ell$, there is a difference between the cases $\ell = 1$ and $\ell > 1$.

   i) For $\ell = 1$, we have $\Gamma_1^m = [\![1, m]\!]$ endowed with the standard action of $\mathcal{S}_m$.

   ii) For any $\ell > 0$, define the *support* of an ℓ-cycle $\gamma := (i_1, \ldots, i_\ell)$ to be the set of its coordinates $\{\!\{\gamma\}\!\} := \{i_1, \ldots, i_\ell\} \subseteq [\![1, m]\!]$. Then let:

$$\tilde{\gamma} \in \mathcal{S}_m := \begin{cases} \tilde{\gamma}(i_j) := i_{j+1\,(\mathrm{mod}\,\ell)}\,, & \text{for } i_j \in \{\!\{\gamma\}\!\}, \\ \tilde{\gamma}(x) := x\,, & \text{for } x \notin \{\!\{\gamma\}\!\}. \end{cases}$$

If $\ell=1$, the map $(\tilde{\_}):\mathit{\Gamma}_1^m\to\mathcal{S}_m$, is the constant map $\gamma\mapsto\mathbf{1}_m$, whereas, if $\ell>1$, the map $(\tilde{\_}):\mathit{\Gamma}_\ell^m\subseteq\mathcal{S}_m$ is *injective*, and the action (6) of $\mathcal{S}_m$ on $\mathit{\Gamma}_\ell^m$ appears to be also induced by the conjugation action of $\mathcal{S}^m$ on itself, *i.e.*:

$$\widetilde{g\cdot\gamma}=g\,\tilde{\gamma}\,g^{-1}\,.$$

We will identify γ and $\tilde{\gamma}$ if no confusion is likely to arise. In this sense, for $g\in\mathcal{S}_m$, the set of fixed points $(\mathit{\Gamma}_\ell^m)^g:=\{\gamma\in\mathit{\Gamma}_\ell^m \mid g\cdot\gamma=\gamma\}$ is:

$$(\mathit{\Gamma}_\ell^m)^g=\{\ell\text{-cycles }\gamma\in\mathcal{S}_m \mid g\gamma=\gamma g\}\,. \tag{7}$$

iii) For $m,\ell>0$, we have

$$|\mathit{\Gamma}_\ell^m|=\frac{m(m-1)\cdots(m-(\ell-1))}{\ell}\,.$$

2.3.3. The FI-modules $\mathbb{E}_\nu$

For $m,\ell>0$, let $\mathbb{E}_\ell^m$ be the $\mathbb{Q}$-vector space spanned by the set $\mathit{\Gamma}_\ell^m$ of ℓ-cycles of $[\![1,m]\!]$, *i.e.*

$$\mathbb{E}_\ell^m:=\bigoplus\nolimits_{\gamma\in\mathit{\Gamma}_\ell^m}\mathbb{Q}\cdot\gamma\,.$$

Endow it with the linear action of $\mathcal{S}_m$ induced by its action on the basis $\mathit{\Gamma}_\ell^m$.

Notice that, according to this definition, $\mathbb{E}_\ell^m=\mathbf{0}$, for all $m<\ell$. On the other hand, for all $m\geq n$, the set $\mathit{\Gamma}_\ell^n$ is a subset of $\mathit{\Gamma}_\ell^m$ invariant under the action of $\mathbf{1}_n\boxtimes\mathcal{S}_{m-n}$, so that the natural inclusions $\mathit{\Gamma}_\ell^m\subseteq\mathit{\Gamma}_\ell^{m+1}$ induce the interior maps (clearly injective) of an FI-module

$$\boxed{\mathbb{E}_\ell:=\{\phi(\mathbb{E}_\ell)_m:\mathbb{E}_\ell^m\rightarrowtail\mathbb{E}_\ell^{m+1}\}}$$

In this, since the natural maps between orbit spaces: $\mathcal{S}_m\backslash\mathit{\Gamma}_\ell^m\to\mathcal{S}_{m+1}\backslash\mathit{\Gamma}_\ell^{m+1}$ are bijective for $m\geq\ell$, the interior maps $\phi(\mathbb{E}_\ell)_m$ are exhaustive for $m\geq\ell$.

More generally, if $\nu:=(1^{\boldsymbol{n}_1},2^{\boldsymbol{n}_2},\ldots,N^{\boldsymbol{n}_N})=(\nu_1\geq\nu_2\geq\cdots\geq\nu_\ell)$ is a nonempty partition, let

$$\begin{aligned}\mathbb{E}_\nu^m&:=(\mathbb{E}_1^m)^{\otimes\boldsymbol{n}_1}\otimes(\mathbb{E}_2^m)^{\otimes\boldsymbol{n}_2}\otimes\cdots\otimes(\mathbb{E}_N^m)^{\otimes\boldsymbol{n}_N}\,,\\&:=\mathbb{E}_{\nu_1}\otimes\mathbb{E}_{\nu_2}\otimes\cdots\otimes\mathbb{E}_{\nu_\ell}\,,\end{aligned}$$

and define $\phi(\mathbb{E}_\nu)_m:\mathbb{E}_\nu^m\to\mathbb{E}_\nu^{m+1}$ as the tensor product of interior maps

$$\begin{aligned}\phi(\mathbb{E}_\nu)_m&:=\phi(\mathbb{E}_1)^{\otimes\boldsymbol{n}_1}\otimes\cdots\otimes\phi(\mathbb{E}_N)^{\otimes\boldsymbol{n}_N}\,,\\&:=\phi(\mathbb{E}_{\nu_1})\otimes\cdots\otimes\phi(\mathbb{E}_{\nu_\ell})\,.\end{aligned}$$

The family

$$\boxed{\mathbb{E}_\nu:=\{\phi(\mathbb{E}_\nu)_m:\mathbb{E}_\nu^m\rightarrowtail\mathbb{E}_\nu^{m+1}\}_m}$$

is then an FI-module (with interior maps clearly injective).

2.3.4. Proposition

a) *For $m, \ell \in \mathbb{N}$, the character $\chi_{\mathbb{E}_\ell^m}$ is expressed by the following polynomial of $\mathbb{Q}[X_1, \ldots, X_\ell]$ of weight ℓ and independent of $m \in \mathbb{N}$:*

$$\boldsymbol{E}_\ell = X_\ell + \sum\nolimits_{ed=\ell\,,\, e\neq 1} \phi(d)\, \frac{d^{e-1}}{e}\, X_d(X_d-1)\cdots\big(X_d-(e{-}1)\big)\,, \qquad (*_\ell)$$

where ϕ is the Euler's totient function.

The FI*-module $\mathbb{E}_\ell := \{\mathbb{E}_\ell^m \rightarrowtail \mathbb{E}_\ell^{m+1}\}_m$ has the following ranks*

$$\mathrm{rank}_{\mathrm{PC}}(\mathbb{E}_\ell) = 0 \quad \textit{and} \quad \mathrm{rank}_{\mathrm{RS}}(\mathbb{E}_\ell) = 2\ell\,.$$

And the same, if considered as an FB*-module.*

b) *Given a sequence $\overline{Z} := (Z_1, \ldots, Z_N)$ of polynomials of $\mathbb{Q}[\overline{X}]$, and given $d \in \mathbb{N}$, we denote by $\mathbb{Q}^{\leq d}[Z_1, \ldots, Z_N]$ the subspace of polynomials of weight $\leq d$ relative to $\overline{Z}$,* i.e. *the subspace spanned by the elements $Z_1^{a_1} \cdots Z_N^{a_N}$ where $\sum_i i\, a_i \leq d$. Then, for all $N \in \mathbb{N}$, the natural inclusion:*

$$\mathbb{Q}^{\leq d}[\boldsymbol{E}_1, \ldots, \boldsymbol{E}_N] \subseteq \mathbb{Q}^{\leq d}[X_1, \ldots, X_N]$$

is an equality.

c) *For every nonempty partition $\nu := (1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, N^{\boldsymbol{n}_N})$, we have*

i) $\boldsymbol{w}(\mathbb{E}_\nu^m) \leq \min\{m, |\nu|\}$.

ii) *The* FI*-module $\mathbb{E}_\nu := \{\mathbb{E}_\nu^m \rightarrowtail \mathbb{E}_\nu^{m+1}\}_m$ is* $(\mathrm{PC})_{m\geq 0}$ *and* $(\mathrm{RS})_{m\geq 2|\nu|}$.

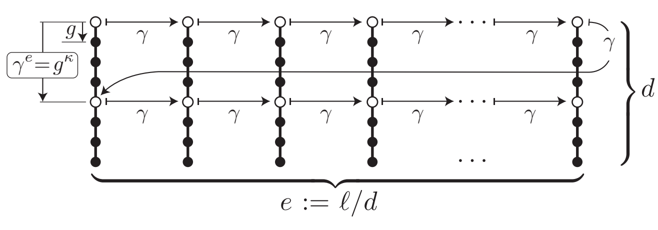
Proof. (a) Since the linear action of $g \in \mathcal{S}_m$ is induced by its action on the basis $\varGamma_\ell^m \subseteq \mathbb{E}_\ell^m$, the trace $\chi_{\mathbb{E}_\ell^m}(g)$ is the cardinality of the set $(\varGamma_\ell^m)^g$ of fixed ℓ-cycles.

When $m < \ell$, the set $\varGamma_\ell^m$ is empty and $\mathbb{E}_\ell^m = 0$. Also, since in $\mathcal{S}_m$ there is no permutation g such that $\rho_m(X_d)(g) \geq \ell/d$, we necessary have that $\rho_m(\boldsymbol{E}_\ell) = 0$. This states (a) when $m < \ell$. Suppose now that $m \geq \ell$.

Following 2.3.2-(b), we have two cases to consider:

– $\boldsymbol{\ell = 1}$. Then $\varGamma_1^m = [\![1, m]\!]$, $\chi_{\mathbb{E}_1^m}(g) = |[\![1, m]\!]^g| = X_1(g)$, and (a) is obvious.

– $\boldsymbol{\ell > 1}$. The set $(\varGamma_\ell^m)^g$ identifies, after 2.3.2-(b-ii)-(7), with the set of ℓ-cycles $\gamma \in \mathcal{S}_m$ such that $g\gamma = \gamma g$. As a consequence, $g\{\!\{\gamma\}\!\} = \{\!\{\gamma\}\!\}$ and the set $\{\!\{\gamma\}\!\}$ appears endowed with two actions commuting to each other. But then, since the action of γ on $\{\!\{\gamma\}\!\}$ is transitive, the $\langle g\rangle$-orbits in it are equipotent. Denote by d their common cardinality, and set $e := \ell/d$. Then, $g^\kappa = \gamma^e$, (or, equivalently, $g = \gamma^{\kappa e}$) for some $\kappa \in [\![1, d]\!]$, relatively prime to d.



Now, the map

$$\xi_{\kappa,e} : \varGamma_\ell^m \to \mathcal{S}_m\,, \quad \gamma \mapsto \gamma^{\kappa e}\,,$$

is clearly a fibration over its image, which is the set of products of disjoint d-cycles. Hence, if π is any such product, we have

$$\left|\xi_{\kappa,e}^{-1}(\pi)\right| = \frac{\dfrac{m(m-1)\cdots(m-\ell+1)}{\ell}}{\dfrac{m(m-1)\cdots(m-\ell+1)}{d^e e!}} = d^{e-1}(e-1)!\,.$$

And, since the number of products π made up of disjoint d-cycles of g is given by the combinatorial number $\binom{X_d(g)}{e}$, we finally get the equality

$$\begin{aligned}\left|(\mathit{\Gamma}_\ell^m)^g\right| &= \sum\nolimits_{de=\ell} \phi(d)\, d^{e-1}(e-1)!\,\binom{X_d(g)}{e}\,,\\ &= \sum\nolimits_{de=\ell} \phi(d)\,\frac{d^{e-1}}{e}\, X_d(g)\big(X_d(g)-1\big)\cdots\big(X_d(g)-(e{-}1)\big)\,.\end{aligned}$$

In both cases, $\ell = 1$ and $\ell > 1$, the expression of $|(\mathit{\Gamma}_\ell^m)^g|$ is independent of $m \geq \ell$, which implies that the FI-module $\mathbb{E}_\ell$ is $(\mathrm{PC})_{m\geq 0}$.

To finish, we observe that since the action of $\mathcal{S}_m$ on $\mathit{\Gamma}_\ell^m$ is transitive, we have an isomorphism of $\mathcal{S}_m$-space :

$$\mathit{\Gamma}_\ell^m = \mathcal{S}_m\cdot(1,\dots,\ell) \simeq \mathcal{S}_m \times_{\langle\gamma\rangle\boxtimes\mathcal{S}_{m-\ell}} \{\mathrm{pt}\}\,,$$

as $\mathrm{Fix}_{\mathcal{S}_m}(\gamma) = \langle\gamma\rangle\boxtimes\mathcal{S}_{m-\ell}$. Therefore,

$$\mathbb{E}_\ell^m = \mathrm{ind}_{\mathcal{S}_\ell\boxtimes\mathcal{S}_{m-\ell}}^{\mathcal{S}_m}\Big(\big(\mathbb{Q}[\mathcal{S}_\ell]\otimes_{\langle(1,\dots,\ell)\rangle}\mathbb{Q}\big)\boxtimes k_{m-\ell}\Big)\,.$$

and $\mathbb{E}_\ell$ is the projective FI-module $\mathcal{P}(\mathbb{Q}[\mathcal{S}_\ell/\langle(1,\dots,\ell)\rangle])$ for which we have already shown that it is $(\mathrm{RS})_{m\geq 2\ell}$ (*cf.* 1.7.2-(b)).

We can be a little more precise if we observe that the simple $\mathcal{S}_\ell$-module $\mathbb{Q}_\ell$ appears with multiplicity 1 in $\mathbb{Q}[\mathcal{S}_\ell]\otimes_{\langle(1,\dots,\ell)\rangle}\mathbb{Q}$, which is the case because :

$$\mathrm{Hom}_{\mathcal{S}_\ell}(\mathbb{Q}[\mathcal{S}_\ell]\otimes_{\langle(1,\dots,\ell)\rangle}\mathbb{Q},\mathbb{Q}_\ell) = \mathrm{Hom}_{\langle(1,\dots,\ell)\rangle}(\mathbb{Q},\mathbb{Q}) = \mathbb{Q}\,.$$

We deduce that $\mathbb{E}_\ell$ contains as factor the sub-FI-module $\mathcal{P}(\mathbb{Q}_\ell) = \mathcal{P}(V_{0[\ell]})$ whose rank of representation stability is exactly 2ℓ. Therefore, $\mathrm{rank}_{\mathrm{RS}}(\mathbb{E}_\ell) = 2\ell$.

(b) In (a), the equation $(*_\ell)$ shows that $\boldsymbol{E}_\ell$ is equal to X_ℓ, modulo a polynomial of weight ℓ in the variables X_d for $d|\ell$ and $d\neq\ell$. (In particular, $\boldsymbol{E}_1 = X_1$.) Now, given $N\geq 1$, the system of equations $(*_\ell)$, for $\ell\leq N$, can be inverted by recursively introducing some polynomials $Q_\ell^m(Z_1,\dots,Z_\ell)\in\mathbb{Q}[\overline{Z}]$ of weight ℓ, such that

$$X_\ell = Q_\ell^m(\boldsymbol{E}_1,\dots,\boldsymbol{E}_\ell)\,,\quad \forall\ell\leq N\,.$$

The equality,

$$\mathbb{Q}[X_1,\dots,X_N] = \mathbb{Q}[\boldsymbol{E}_1,\dots,\boldsymbol{E}_N]\,,$$

but also,

$$\mathbb{Q}^{\leq d}[X_1,\dots,X_N] = \mathbb{Q}^{\leq d}[\boldsymbol{E}_1,\dots,\boldsymbol{E}_N]\,,\quad \forall d\in\mathbb{N}\,,$$

then follows straightforward.

(c) Since the action of $\mathcal{S}_m$ on $\mathbb{E}_\nu^m := \mathbb{E}_{\nu_1}^m \otimes \cdots \otimes \mathbb{E}_{\nu_\ell}^m$ is induced by its component-wise action on the basis $\Gamma_\nu^m := \Gamma_{\nu_1}^m \times \Gamma_{\nu_2}^m \times \cdots \times \Gamma_{\nu_\ell}^m$, the structure of $\mathbb{Q}[\mathcal{S}_m]$-module of $\mathbb{E}_\nu^m$ follows from the structure of $\mathcal{S}_m$-space of Γ_ν^m. Given $\bar{\gamma} := (\gamma_1, \ldots, \gamma_\ell) \in \Gamma_\nu^m$, we have

$$\mathcal{S}_m \cdot \bar{\gamma} = \mathcal{S}_m / \operatorname{Stab}_{\mathcal{S}_m}(\bar{\gamma}) = \mathcal{S}_m \times_{\operatorname{Stab}_{\mathcal{S}_m}(\{\!\{\bar{\gamma}\}\!\})} \left(\mathcal{S}_{\{\!\{\bar{\gamma}\}\!\}} / \operatorname{Stab}_{\mathcal{S}_{\{\!\{\bar{\gamma}\}\!\}}}(\bar{\gamma})\right) \tag{‡}$$

where we set $\{\!\{\bar{\gamma}\}\!\} := \{\!\{\gamma_1\}\!\} \cup \cdots \cup \{\!\{\gamma_\ell\}\!\}$.

As a consequence the $\mathbb{Q}[\mathcal{S}_m]$-module $\mathbb{E}_\nu^m$ is a direct sum of induced modules of the form:

$$\operatorname{ind}_{\mathcal{S}_{\{\!\{\bar{\gamma}\}\!\}} \boxtimes \mathcal{S}_{\complement\{\!\{\bar{\gamma}\}\!\}}}^{\mathcal{S}_m} W_n \boxtimes k_{m-n}\,,$$

where $n := |\{\!\{\bar{\gamma}\}\!\}| \leq \min\{m, |\nu|\}$ and $W_n := \mathcal{S}_{\{\!\{\bar{\gamma}\}\!\}} / \operatorname{Stab}_{\mathcal{S}_{\{\!\{\bar{\gamma}\}\!\}}}(\bar{\gamma})$.

When $m \geq |\nu|$, the corollary 1.4.5-(a) to Pieri's rule, gives the inequality

$$\boldsymbol{w}(\nu) \leq \boldsymbol{w}(\mathbb{E}_\nu^m) \leq |\nu|\,,$$

and (c-i) follows.

(c-ii) Since $\chi_{\mathbb{E}_\nu^m} = \chi_{\mathbb{E}_1^m}^{\boldsymbol{n}_1} \cdots \chi_{\mathbb{E}_N^m}^{\boldsymbol{n}_N}$ and since $\mathbb{E}_\ell$ is $(\mathrm{PC})_{m\geq 0}$ after (a), we immediately get the fact that $\mathbb{E}_\nu$ is $(\mathrm{PC})_{m\geq 0}$.

To finish, we notice that, on the one hand, the number of terms of the form (‡) is exactly de number of $\mathcal{S}_m$-orbits in Γ_ν^m, and, on the other hand, that the increasing sequence $\mathcal{S}_m \backslash \Gamma_\nu^m \subseteq \mathcal{S}_{m+1} \backslash \Gamma_\nu^{m+1}$ stabilizes for $m \geq |\nu|$. Therefore, the truncated FI-module $(\mathbb{E}_\nu)_{\geq|\nu|}$ is isomorphic to the sum of truncated projective FI-modules $\mathcal{P}(W_n)_{\geq|\nu|}$, all being $(\mathrm{RS})_{m\geq 2|\nu|}$, after 1.7.2-(b).

We can be more precise if we observe that for $m \geq |\nu|$ the module

$$R_m := \operatorname{ind}_{\mathcal{S}_{\nu_1} \boxtimes \mathcal{S}_{\nu_2} \boxtimes \cdots \boxtimes \mathcal{S}_{\nu_\ell} \boxtimes \mathcal{S}_{m-|\nu|}}^{\mathcal{S}_m} \mathbb{Q}$$

is a factor of $\mathbb{E}_\nu^m$. Now, the ideas used in 1.3, show that the FB-module $\{R_m\}_m$ canonically determines a projective FI-module $\mathcal{R} := \{R_m \to R_{m+1}\}_m$, and thanks to an inductive argument on the number ℓ of terms in ν one can prove, as in (a), that $\operatorname{rank}_{\mathrm{RS}}(\mathcal{R}) = 2|\nu|$ as an FI or FB module. Details are left to the reader. □

3. Polynomial weight vs. Representation weight

3.1. On the minimal polynomial weight of a character

The Frobenius polynomial $\boldsymbol{X}_\lambda \in \mathbb{Q}[\overline{X}]$ is of weight $\boldsymbol{w}(\lambda)$, and there are infinitely many polynomials $P_\lambda \in \mathbb{Q}[X_1, \ldots, X_{m-1}]$ such that $\rho_m(P_\lambda) = \rho_m(\boldsymbol{X}_\lambda) = \chi_{V_\lambda}$. In this section, we push further the study of the relationship between weights of polynomials and weights of representations. We will see that $\boldsymbol{w}(W_m)$ is the lowest possible weight for a polynomial $P_{W_m} \in \mathbb{Q}[X_1, \ldots, X_{m-1}]$ verifying $\rho_m(P_{W_m}) = \chi_{W_m}$. This is for example the case of $\boldsymbol{X}_{W_m} := \sum_\lambda \boldsymbol{n}_\lambda \boldsymbol{X}_\lambda$, where V_λ is an irreducible factor of W_m of multiplicity $\boldsymbol{n}_\lambda$. Moreover, this last polynomial is the only satisfying these conditions if and only if $\boldsymbol{w}(W_m) \leq m/2$.

3.1.1. Theorem

a) *For $d, m \in \mathbb{N}$, denote by $\rho_{m|d}$ the restriction of $\rho_m : \mathbb{Q}[\overline{X}] \to \mathbb{Q}_{\rm cl}(\mathcal{S}_m)$ to the subspace $\mathbb{Q}^{\leq d}[\overline{X}]$ of polynomials of weight $\leq d$. The image of the map*

$$\rho_{m|d} : \big(\mathbb{Q}^{\leq d}[\overline{X}]\big) \to \mathbb{Q}_{\rm cl}(\mathcal{S}_m)$$

is the subspace

$$\mathbb{Q}_{\rm cl}^{\leq d}(\mathcal{S}_m) := \big\langle\, \chi_{V_\lambda} \,\big|\, \big(|\lambda| = m\big) \,\&\, \big(\, \boldsymbol{w}(\lambda) \leq d\big)\big\rangle_{\mathbb{Q}}\,, \tag{8}$$

and $\ker(\rho_{m|d}) = 0$ *if and only if* $d \leq m/2$.

b) *Let $P \in \mathbb{Q}[\overline{X}]$.*
   i) *The scalar product $\langle\!\langle\, 1 \,|\, P \,\rangle\!\rangle_{\mathcal{S}_m}$ is constant for $m \geq \deg_{\boldsymbol{w}}(P)$.*
   ii) *The scalar product $\langle\!\langle\, \chi_{V_\lambda} \,|\, P \,\rangle\!\rangle_{\mathcal{S}_m}$ vanishes for $\lambda \vdash m$ s.t. $\boldsymbol{w}(\lambda) > \deg_{\boldsymbol{w}}(P)$.*
   iii) *Let W_m be a representation of $\mathcal{S}_m$. If $\rho_m(P) = \chi_{W_m}$, then:*

$$\deg_{\boldsymbol{w}}(P) \geq \boldsymbol{w}(W_m)\,.$$

c) *Let $W_m = \bigoplus_{\lambda \vdash m} V_\lambda{}^{\boldsymbol{n}_\lambda}$ be a representation of $\mathcal{S}_m$. The polynomial*

$$\boxed{\boldsymbol{X}_{W_m} := \sum\nolimits_{\lambda \vdash m} \boldsymbol{n}_\lambda \boldsymbol{X}_\lambda \in \mathbb{Q}[X_1, \ldots, X_{m-1}]}$$

where $\boldsymbol{X}_\lambda$ is the Frobenius polynomial for χ_{V_λ} (2.2.2), always verifies

$$\rho_m(\boldsymbol{X}_{W_m}) = \chi_{W_m} \quad \text{and} \quad \deg_{\boldsymbol{w}}(\boldsymbol{X}_{W_m}) = \boldsymbol{w}(W_m)\,,$$

and it is the only polynomial verifying simultaneously these two equalities if and only if $\boldsymbol{w}(W_m) \leq m/2$.

Proof. (a) In the equality (8), the inclusion '$\supseteq$' is 2.2.2. For the converse, it suffices to restrict oneself to the case of a monomial $\rho_m(X_1^{\boldsymbol{n}_1} \cdots X_d^{\boldsymbol{n}_d})$ for $\sum_i i\,\boldsymbol{n}_i \leq d$, or, what amounts to the same thing, thanks to 2.3.4-(b), to show that

$$(\chi_{\mathbb{E}_1^m})^{\boldsymbol{n}_1} \cdots (\chi_{\mathbb{E}_d^m})^{\boldsymbol{n}_d} \in \big\langle \chi_{V_\lambda} \,\big|\, \big(|\lambda| = m\big) \,\&\, \big(\, \boldsymbol{w}(\lambda) \leq d\,\big)\big\rangle_{\mathbb{Q}}\,.$$

Here, one recognizes, at the left, the character of $\mathbb{E}_\nu^m$, for $\nu := (1^{\boldsymbol{n}_1}, \ldots, d^{\boldsymbol{n}_d})$. Now, since we already know after 2.3.4-(c), that $\boldsymbol{w}(\mathbb{E}_\nu^m) \leq |\nu|$, we conclude that $\boldsymbol{w}(\mathbb{E}_\nu^m) \leq d$. The irreducible factors of $\mathbb{E}_\nu^m$ are therefore of the form V_λ with $\boldsymbol{w}(\lambda) \leq d$, which settles the inclusion '$\subseteq$'.

For the last claim about $\ker(\rho_{m|d})$, notice that the space $\mathbb{Q}^{\leq d}[X_1, \ldots, X_d]$ admits as basis the set of monomials

$$M(d) := \big\{X_1^{\boldsymbol{n}_1} X_2^{\boldsymbol{n}_2} \ldots X_d^{\boldsymbol{n}_d} \,\big|\, \textstyle\sum_{i=1}^d i\,\boldsymbol{n}_i \leq d\,\big\}\,,$$

while the space $\mathbb{Q}_{\rm cl}^{\leq d}(\mathcal{S}_m)$ has a basis indexed by the following set of partitions:

$$L(d) := \big\{\underline{\lambda} := (1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, \lambda_2^{\boldsymbol{n}_{\lambda_2}}) \,\big|\, \big(\textstyle\sum_{i=1}^{\lambda_2} i\,\boldsymbol{n}_i \leq d\big) \,\&\, \big(\textstyle\sum_{i=1}^{\lambda_2} i\,\boldsymbol{n}_i + \lambda_2 \leq m\big)\big\}\,,$$

because $\{\chi_{V_\lambda} \mid \lambda \vdash m\}$ is linearly independent after Schur orthogonality.

Since $\lambda_2 \leq d$, the map $\xi : L(d) \to M(d)$, which associates $(1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, \lambda_2^{\boldsymbol{n}_{\lambda_2}})$ with the monomial $X_1^{\boldsymbol{n}_1} X_2^{\boldsymbol{n}_2} \ldots X_{\lambda_2}^{\boldsymbol{n}_{\lambda_2}}$ is well defined and injective. The condition for ξ to be bijective is that λ_2 be able to take the value d, in which case $\sum_{i=1,\ldots,\lambda_2} i\,\boldsymbol{n}_i = d$, and that condition appears to be just that $2d \leq m$.

(b-i) Again, thanks to 2.3.4-(b), it suffices to prove that

$$\langle\!\langle\, 1 \,\big|\, \chi_{\mathbb{E}_1^m}(g)^{\boldsymbol{n}_1}\chi_{\mathbb{E}_2^m}(g)^{\boldsymbol{n}_2}\cdots\chi_{\mathbb{E}_r^m}(g)^{\boldsymbol{n}_r}\,\rangle\!\rangle_{\mathcal{S}_m}\,, \tag{$\diamond$}$$

is constant for all $m \geq \sum_{i=1}^r i\,\boldsymbol{n}_i$. But ($\diamond$) is nothing but the dimension of the subspace of invariant tensors in the $\mathcal{S}_m$-module

$$\mathbb{E}_\nu^m := (\mathbb{E}_1^m)^{\otimes\boldsymbol{n}_1} \otimes (\mathbb{E}_2^m)^{\otimes\boldsymbol{n}_2} \otimes\cdots\otimes (\mathbb{E}_r^m)^{\otimes\boldsymbol{n}_r}\,,$$

where $\nu := (1^{\boldsymbol{n}_1}, 2^{\boldsymbol{n}_2}, \ldots, r^{\boldsymbol{n}_r})$.

Let $\nu = (\nu_1 \geq \nu_2 \geq \cdots \geq \nu_\ell)$. The canonical basis $\mathcal{B}_\nu^m$ of $\mathbb{E}_\nu^m$ is the set of tensors

$$\gamma_1 \otimes \gamma_2 \otimes \cdots \otimes \gamma_\ell\,,$$

where γ_i is a cycle of length ν_i of $\mathcal{S}_m$. The basis $\mathcal{B}_\nu^m$ is clearly in bijection with the set $\mathcal{T}_\nu^m$ of Young *tableaux* τ of shape ν, with the i'th row filled with elements of $[\![1, m]\!]$ in a way they represent a cycle of length ν_i of $\mathcal{S}_m$. The action of $\mathcal{S}_m$ on $\mathcal{B}_\nu^m$ induces in $\mathcal{T}_\nu^m$ the natural action of $\mathcal{S}_m$ on Young tableaux.

Because of these identifications, the dimension of $(\mathbb{E}_\nu^m)^{\mathcal{S}_m}$ is the cardinality of the orbit space $\mathcal{T}_\nu^m/\mathcal{S}_m$, and the stability we are seeking to prove, is equivalent to the fact that the natural map

$$\mathcal{T}_\nu^m/\mathcal{S}_m \to \mathcal{T}_\nu^{m+1}/\mathcal{S}_{m+1}\,,\ [\tau\,(\mathrm{mod}\,\mathcal{S}_m)] \mapsto [\tau\,(\mathrm{mod}\,\mathcal{S}_{m+1})]\,,$$

is a bijection. But the necessary and sufficient condition for this is precisely that the total number of boxes $|\nu|$ be smaller or equal to m, since, in that case, a single permutation $g \in \mathcal{S}_m$ will allow to renumber all the boxes simultaneously with numbers in the interval $[\![1, |\nu|]\!]$. Hence ($\diamond$) is constant for $m \geq |\nu| = \sum_i i\,\boldsymbol{n}_i$.

(b-ii) After (a), $\rho_m(P)$ belongs to the linear span of the characters χ_{V_ν} of $\mathcal{S}_m$, with $\boldsymbol{w}(\nu) \leq \deg_{\boldsymbol{w}}(P)$, and, tanks to Schur orthogonality, these characters are orthogonal to any χ_{V_λ} with $\boldsymbol{w}(\lambda) > \deg_{\boldsymbol{w}}(P)$, hence the claim.

(b-iii) Indeed, if we had $\deg_{\boldsymbol{w}}(P) < \boldsymbol{w}(W_m)$ then, for any irreducible factor V_λ of W_m of weight $\boldsymbol{w}(V_\lambda) = \boldsymbol{w}(W_m)$, we would get, after (b-ii):

$$0 = \langle\!\langle\, \chi_{V_\lambda} \,|\, P \,\rangle\!\rangle_{\mathcal{S}_m} = \langle\!\langle\, \chi_{V_\lambda} \,|\, W_m \,\rangle\!\rangle_{\mathcal{S}_m} \neq 0\,,$$

which is a contradiction. Hence $\deg_{\boldsymbol{w}}(P) \geq \boldsymbol{w}(W_m)$.

(c) Immediate consequence of (b-iii) and the study of $\ker(\rho_{m|d})$ in (a). □

3.1.2. Comments

- The proposition 2.3.4-(a) showed that, for fixed $m \geq N$, the character of the tensor product $\mathbb{E}_\nu^m := (\mathbb{E}_1^m)^{\otimes\boldsymbol{n}_1} \otimes (\mathbb{E}_2^m)^{\otimes\boldsymbol{n}_2} \otimes\cdots\otimes (\mathbb{E}_N^m)^{\otimes\boldsymbol{n}_N}$, view as an $\mathcal{S}_m$-module, is given by the polynomial: $\boldsymbol{E}_\nu := \boldsymbol{E}_1^{\boldsymbol{n}_1}\boldsymbol{E}_2^{\boldsymbol{n}_2}\cdots\boldsymbol{E}_N^{\boldsymbol{n}_N}$ of $\mathbb{Q}[X_1,\ldots,X_N]$, whose weight $\deg_{\boldsymbol{w}}(\boldsymbol{E}_\nu) = \sum_i i\,\boldsymbol{n}_i$ can be arbitrarily big. On the other hand, theorem 3.1.1-(c) showed that the same character is given the polynomial $\boldsymbol{X}_{\mathbb{E}_\nu^m}$ of $\mathbb{Q}[X_1,\ldots,X_{m-1}]$, of weight $\boldsymbol{w}(\mathbb{E}_\nu^m) < m$.

  The reason of this disagreement is that, while $\boldsymbol{E}_\nu$ expresses a priori all the characters in the family $\{\chi_{\mathbb{E}_\nu^m}\}_{m\geq N}$ *simultaneously*, the Frobenius polynomial $\boldsymbol{X}_{\mathbb{E}_\nu^m}$ expresses a priori only one of them: $\chi_{\mathbb{E}_\nu^m}$.

Heuristically speaking, the elements in $\ker(\rho_m)$ allow to reduce the weight of $\boldsymbol{E}_\nu$ to raise the require bounding by m. In the next section, this distinction will be meaningful.

- It is also worth noting that while the explicit writing of $\boldsymbol{E}_\nu$ is very simple, thanks to formula 2.3.4-(a), the writing of $\mathbf{X}_{\mathbb{E}_\nu^m}$ can be quite involved insofar it is based on the knowledge of the set of multiplicities $\boldsymbol{n}_\lambda$ of the irreducible factors V_λ in the decomposition $\mathbb{E}_\nu^m = \bigoplus_{\lambda \vdash m} V_\lambda^{\boldsymbol{n}_\lambda}$, and also in the explicit description of each $\mathbf{X}_\lambda$, for $\boldsymbol{n}_\lambda \neq 0$.

The following is a useful corollary to theorem 3.1.1-(c)

3.1.3. Corollary. *Let $\mathcal{W}$ be an* FB*-module which is* $(\mathrm{PC})_{m\geq N}$ *and has polynomial character $P_{\mathcal{W}}$.*

a) $P_{\mathcal{W}} = \mathbf{X}_{W_m}$, *for all* $m \geq \max\{2\deg_{\boldsymbol{w}}(P_{\mathcal{W}}), N\}$.

b) $\boldsymbol{w}(W_m) = \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$, *for all* $m \geq \max\{2\deg_{\boldsymbol{w}}(P_{\mathcal{W}}), N\}$.

c) $\boldsymbol{w}(\mathcal{W}_{\geq N}) = \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$.

Proof. (a,b) Immediate after 3.1.1-(c).

(c) For all $m \geq N$, we have $\rho_m(P_{\mathcal{W}}) = \chi_{W_m}$ so that $\boldsymbol{w}(W_m) \leq \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$, again after 3.1.1-(c). Therefore, $\sup_{m\geq N}\{\boldsymbol{w}(W_m)\} = \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$, after (b). □

4. (PC) versus (RS)

4.1. The equivalence

We show the equivalence, for an FB-module $\mathcal{W}$, between the properties of being (RS) and of being (PC). In it, two more elements will deserve special attention. First, the ranks of validity of the properties, and, second, the weight of the polynomial $P_{\mathcal{W}}$. The complete statement is the following.

4.1.1. Theorem. *Let $\mathcal{W}$ be an* FB*-module.*

a) *If $\mathcal{W}$ is* (RS) *for $m \geq N$, then $\mathcal{W}$ is* (PC) *for $m \geq N$.*

b) *If $\mathcal{W}$ is* (PC) *for $m \geq N$, and has polynomial character $P_{\mathcal{W}}$, then $\mathcal{W}$ is* (RS) *for* $m \geq \max\{2\deg_{\boldsymbol{w}}(P_{\mathcal{W}}), N\}$.

Proof. (a) This is corollary 2.2.3. (b) Set $d := \deg_{\boldsymbol{w}}(P_{\mathcal{W}})$. After 3.1.3-(a), we know that $P_{\mathcal{W}} = \mathbf{X}_{W_m}$ for all $m \geq 2d$, so that, if we decompose W_{2d} in its simple $\mathbb{Q}[\mathbb{S}_{2d}]$-submodules:

$$W_{2d} = \bigoplus_{|\lambda|\leq d} (V_{\lambda[2d]})^{\boldsymbol{n}_\lambda},$$

and if we denote $\lambda' := (\lambda_1, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$ for $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, then the FI-module

$$\mathcal{W}' := \bigoplus_{|\lambda|\leq d} (\mathcal{V}_{\lambda'})^{\boldsymbol{n}_\lambda},$$

which is clearly $(\mathrm{RS})_{m\geq 2d}$, will be such that, by construction:

$$P_{\mathcal{W}'} = P_{\mathcal{W}}.$$

As a consequence, there exists an isomorphism of FB-modules:

$$\mathcal{W}_{\geq \max\{2d,N\}} \sim \mathcal{W}'_{\geq \max\{2d,N\}}\,, \qquad (\diamond)$$

and $\mathcal{W}$ is (RS) for $m \geq \max\{2d, N\}$, as stated. □

4.1.2. Remark. The proof of the last theorem, shows in ($\diamond$) that an FB-module which is (PC) is asymptotically isomorphic, as FB-module, to an (RS)-module.

5. Addendum on the weight of a tensor product

5.1. Reinterpretation of the weight of a representation

The theorem 3.1.1-(c) gives an alternative definition of the weight of a representation W_m of $\mathcal{S}_m$ as the lowest possible weight of the polynomials expressing the character χ_{W_m}. The following theorem then easily follows from corollary 3.1.3.

5.1.1. Proposition

a) *If $W_{1,m}$ and $W_{2,m}$ are representations of $\mathcal{S}_m$, then*

$$\boldsymbol{w}\,(W_{1,m} \otimes W_{2,m}) \leq \boldsymbol{w}\,(W_{1,m}) + \boldsymbol{w}\,(W_{2,m})\,.$$

b) *If $\mathcal{W}_1$ and $\mathcal{W}_2$ are* FB*-modules which are* $(\mathrm{PC})_{m\geq N}$*, then*

$$\boldsymbol{w}\,(W_{1,m} \otimes W_{2,m}) = \boldsymbol{w}\,(W_{1,m}) + \boldsymbol{w}\,(W_{2,m})\,,$$

for all $m \geq 2(\boldsymbol{w}\,(W_{1,m}) + \boldsymbol{w}\,(W_{2,m}))$. In particular,

$$\boldsymbol{w}\,((\mathcal{W}_1 \otimes \mathcal{W}_2)_{\geq N}) = \boldsymbol{w}\,((\mathcal{W}_1)_{\geq N}) + \boldsymbol{w}\,((\mathcal{W}_2)_{\geq N})$$

c)
$$\boldsymbol{w}\,(V_{\lambda[m]} \otimes V_{\mu[m]}) = |\lambda| + |\mu|, \quad \forall m \geq 2(|\lambda| + |\mu|).$$

Proof. (a) As $\chi_{W_1\otimes W_2} = \chi_{W_1}\chi_{W_2}$, we get

$$\begin{aligned}\boldsymbol{w}\,(W_1 \otimes W_2) &\leq_{(1)} \deg_{\boldsymbol{w}}(\mathbf{X}_{W_1}\mathbf{X}_{W_2}) \\ &= \deg_{\boldsymbol{w}}(\mathbf{X}_{W_1}) + \deg_{\boldsymbol{w}}(\mathbf{X}_{W_2}) \\ &=_{(2)} \deg_{\boldsymbol{w}}(W_1) + \deg_{\boldsymbol{w}}(W_2)\,,\end{aligned}$$

inequality '$\leq_{(1)}$' after 3.1.1-(b-iii), and equality '$=_{(2)}$' after 3.1.1-(c).

(b) Let $d_i := \deg_{\boldsymbol{w}}(P_{\mathcal{W}_i})$. We know after corollary 3.1.3-(a), that $P_{\mathcal{W}_i} = \mathbf{X}_{W_{i,m}}$, for $m \geq 2\max\{d_1, d_2\}$. Hence, $\mathbf{X}_{W_{1,m}}\mathbf{X}_{W_{2,m}} = \mathbf{X}_{W_{1,m}\otimes W_{2,m}}$, for $m \geq 2(d_1 + d_2)$, after the same corollary. Therefore,

$$\boldsymbol{w}\,(W_{1,m} \otimes W_{2,m}) = \boldsymbol{w}\,(W_{1,m}) + \boldsymbol{w}\,(W_{2,m})\,,$$

for $m \geq 2(d_1 + d_2)$, again after 3.1.1-(c).

(c) Particular case of (b). □

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