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FAMILIES OF BERKOVICH SPACES

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Abstract. — This memoir is devoted to the systematic study of relative properties in the context of Berkovich analytic spaces. We first develop a theory of flatness in this setting. After showing through a counter-example that naive flatness cannot be the right notion because it is not stable under base change, we define flatness by *requiring* invariance under base change, and we study a first important class of flat morphisms, that of quasi-smooth ones.

We then show the existence of local *dévissages* (in the spirit of Raynaud and Gruson's work) for coherent sheaves, which we use, together with a study of the local rings of "generic fibers" of morphisms, to prove that a *boundaryless*, naively flat morphism is flat.

After that we prove that the image of a compact strictly analytic space by a flat morphism is an analytic domain of the target and that it admits, when the source is stricly analytic, a compact, flat multisection (i.e., a compact, flat cover of relative dimension zero over which there is a section). This was first proved in the rigid-analytic context by Raynaud, but our proof is completely different: it is based upon Temkin's theory of the reduction of analytic germs and does not make any use of formal models.

In the last part of this work we study where various interesting pointwise relative properties are satisfied. We first prove that the flat locus of a given morphism of analytic spaces is a Zariski-open subset of the source (we follow the method that was introduced by Kiehl for the complex analytic analogue of this statement). We then look at the loci at which a point satisfies various commutative algebra properties *on its fiber*: being geometrically regular, geometrically R_m, complete intersection, or Gorenstein; being S_m or Cohen-Macaulay. We prove that the results we could expect actually hold: these loci are (locally) Zariski-constructible, and Zariski-open under suitable extra assumptions (flatness, and also equidimensionality for S_m and geometric R_m); for that purpose, we first study the general properties of the locally Zariski-constructible subsets of an analytic space.

***Résumé* (Familles d'espaces de Berkovich).** — Ce mémoire est consacré à une étude systématique des propriétés relatives dans le contexte des espaces de Berkovich. Nous commençons par développer une théorie de la platitude dans ce cadre. L'acception naïve de cette notion est inadaptée : nous montrons en effet par un contre-exemple qu'elle n'est pas stable par changement de base, ce qui nous conduit à *imposer* cette stabilité dans la définition. Nous étudions une première classe importante de morphismes plats : celle des morphismes *quasi-lisses*.

Nous montrons ensuite l'existence de *dévissages* locaux (dans l'esprit de Raynaud et Gruson) pour les faisceaux cohérents. Joint à une étude des anneaux locaux des fibres «génériques» des morphismes, cela nous permet de montrer qu'un morphisme *sans bord* qui est plat au sens naïf l'est encore au nôtre.

Puis nous démontrons que l'image d'un espace strictement analytique compact par un morphisme plat est un domaine analytique du but, et qu'elle admet, lorsque la source est strictement analytique, une multisection compacte et plate (*i.e.* un revêtement plat, compact, et de dimension relative nulle sur lequel le morphisme considéré possède une section). Cela avait déjà été établi dans le contexte rigide-analytique par Raynaud, mais notre preuve est complètement différente : elle repose sur la réduction à la Temkin des germes d'espaces analytiques et ne fait pas appel aux schémas formels.

Dans la dernière partie de ce travail, nous étudions les lieux de validité sur la source de certaines propriétés relatives. Nous y démontrons pour commencer que le lieu de platitude d'un morphisme d'espaces analytiques est un ouvert de Zariski de la source (nous suivons la méthode utilisée par Kiehl pour établir l'assertion correspondante en géométrie analytique complexe). Cela nous permet de procéder à l'investigation systématique des ensemble de points satisfaisant *dans leur fibre* les propriétés classiques de l'algèbre commutative : être géométriquement régulier, géométriquement R_m, d'intersection complète ou de Gorenstein ; être S_n ou de Cohen-Macaulay. Nous prouvons que les énoncés auxquels ont peut s'attendre sont effectivement vérifiés : ces lieux de validités sont (localement) Zariski-constructibles, et sont des ouverts de Zariski sous certaines hypothèses supplémentaires (platitude, ainsi qu'équidimensionalité pour les propriétés R_m ou S_m) ; dans ce but, nous procédons tout d'abord à une étude générale des parties localement Zariski-constructibles d'un espace analytique.

CONTENTS

CHAPTER 0

INTRODUCTION

This memoir is, roughly speaking, devoted to a systematic investigation of *families* of objects in Berkovich's non-Archimedean analytic geometry ([**Ber90**], [**Ber93**]). More precisely, let us assume that we are given a morphism $Y \to X$ between analytic spaces, and an object D on Y of a certain kind (think of the space Y itself, or a coherent sheaf, or a complex of coherent sheaves...). Every point x of X gives then rise to an object D_x, living on the fiber Y_x, and we thus get in some sense an analytic family of objects parametrized by the space X. The quite vague problem we would like to address is the following: *how do the object D_x and its relevant properties vary?* Of course, such questions have been intensively studied for a long time in *algebraic* geometry, especially by Grothendieck and his school, and our guideline has been to establish analytic avatars of their results every time it was possible. Let us now give a quick overview of our work.

0.1. First step: flatness in the Berkovich setting

0.1.1. Motivation. — In scheme theory, the key notion upon which the study of families is based is *flatness*. This is a property of families of *coherent sheaves*, which encodes more or less the intuitive idea of a *reasonable* variation (this is why there is almost always a flatness assumption in the description of moduli problems).

The point is that the study of general families is often reduced (typically, through a suitable stratification of the base scheme) to the case where some of the coherent sheaves involved are flat over the parameter space, which is easier to handle.

But we would like to emphasize that flatness is also a crucial technical tool for many other purposes. Let us mention for example:

- descent theory;
- the first occurrence of flatness in algebraic geometry, in the celebrated paper GAGA by Serre [**Ser56**], where the following plays a major role: if X is a

complex algebraic variety and if X^{an} denotes the corresponding analytic space, then for every $x \in X(\mathbf{C})$ the ring $\mathscr{O}_{X^{\mathrm{an}},x}$ is flat over $\mathscr{O}_{X,x}$.

By analogy, our first step toward the understanding of analytic families has been the development of a theory of flatness in Berkovich geometry, which is the core of this memoir; and similarly to what happens for schemes, we hope that it will have many applications beyond the study of families.

In fact, flatness in non-Archimedean geometry had already been considered, but in the *rigid-analytic* setting, following ideas of Raynaud; see the papers [**BL93a**] and [**BL93b**] by Bosch and Lütkebohmert, as well as the more comprehensive recent study by Abbes in [**Abb10**]. We now give a more precise discussion of rigid-analytic flatness, before saying some words concerning our definition in the Berkovich framework.

0.1.2. Flatness in rigid geometry. — The definition of flatness in the rigid setting is as simple as one may hope: if $f : Y \to X$ is a morphism between rigid spaces and if $\mathscr{F}$ is a coherent sheaf on Y, it is rig-flat over X at a point $y \in Y$ if it is flat over X at y *in the sense of the theory of locally ringed spaces*; i.e., the stalk $\mathscr{F}_y$ is a flat $\mathscr{O}_{X,f(y)}$-module (this is the definition by Abbes; the original one by Bosch and Lütkebohmert was slightly different, but both are easily seen to be equivalent using the good properties of the completion of local rings as far as flatness is concerned, cf. [**SGA 1**], Exposé IV, Cor. 5.8).

Flatness in the above sense behaves well: it is stable under base change and ground field extension. But contrary to what happens in scheme theory, this is in no way obvious, because base change and ground field extension are defined using *completed* tensor products. Roughly speaking, the proofs proceed as follows (see [**BL93a**], [**BL93b**] and [**Abb10**]):

- the study of *rigid* flatness is reduced to that of *formal* flatness through formal avatars of Raynaud-Gruson flattening techniques, which are used to build a flat formal model of any given rig-flat coherent sheaf;
- the study of *formal* flatness is reduced to that of *algebraic* flatness, in a more standard way, upon dividing by various ideals of definition and using flatness criteria in the spirit of [**SGA 1**], Exposé IV.

Let us mention that this general strategy (formal flattening and reduction modulo an ideal of definition to replace an analytic problem with an algebraic one) was also used by Raynaud to prove the following fact: if $\varphi : Y \to X$ is a flat morphism between affinoid rigid spaces, $\varphi(Y)$ is a finite union of affinoid domains of X (cf. [**BL93b**], Cor. 5.11).

0.1.3. Flatness in Berkovich geometry. — We fix from now on and for the remaining part of this introduction a complete, non-Archimedean field k. We shall only consider *Berkovich* analytic spaces. Any analytic space X comes with a usual topology, but also with a set-theoretic Grothendieck topology which refines it, the so-called G-*topology* –the corresponding site is denoted by X_{G}; the archetypal example of a G-covering is the covering of an affinoid space by finitely many affinoid domains. The site X_{G} is equipped with a coherent sheaf of rings $\mathscr{O}_{X_{\mathrm{G}}}$, whose restriction $\mathscr{O}_X$ to the category of open subsets makes X a locally ringed space. But $(X, \mathscr{O}_X)$ does not seem to be easily tractable in general: for example, one does not know whether $\mathscr{O}_X$ is coherent, nor whether its stalks are noetherian (one only knows that they are henselian), and the category of coherent $\mathscr{O}_X$-modules is poorly understood – one rather deals with coherent $\mathscr{O}_{X_{\mathrm{G}}}$-modules, which are well-behaved.

Nevertheless, if X is *good*, which means that every point of X has an affinoid neighborhood, then $\mathscr{O}_X$ is coherent, its stalks are noetherian (and even excellent), and the category of coherent $\mathscr{O}_{X_{\mathrm{G}}}$-modules is equivalent through the restriction functor to that of coherent $\mathscr{O}_X$-modules. This is the reason why many properties are first defined and studied for good analytic spaces, and thereafter extended to arbitrary analytic spaces by some "G-gluing" process. Note however that the class of good spaces is quite broad: it contains affinoid domains, analytification of schemes of finite type, generic fibers of both affine and proper formal schemes. . . ; and any open subset of a good space is still good.

From now on we shall simply say "coherent sheaf on X" for "coherent $\mathscr{O}_{X_{\mathrm{G}}}$-module", and write $\mathscr{O}_X$ instead of $\mathscr{O}_{X_{\mathrm{G}}}$. But if X is good, if $\mathscr{F}$ is a coherent sheaf on X, and if $x \in X$, we shall use the notation $\mathscr{F}_x$ to denote the stalk at x of $\mathscr{F}$ *viewed (by restriction to the category of open subsets) as a sheaf on the ordinary topological space underlying X.*

0.1.3.1. *Naive flatness.* — Let $\varphi\colon Y \to X$ be a morphism of *good* k-analytic spaces and let $\mathscr{F}$ be a coherent sheaf on Y. Let us say that $\mathscr{F}$ is *naively* X-flat at a point y of Y is $\mathscr{F}_y$ is a flat $\mathscr{O}_{X,\varphi(y)}$-module, exactly like in the rigid setting (if $\mathscr{F} = \mathscr{O}_Y$ we simply say that Y is *naively X-flat at y*, or that φ is naively flat at y). We immediately face a big problem. Indeed, in this context, the use of completed tensor products does not just make proofs of stability of naive flatness under base change or ground field extension more complicated, as it does in the rigid case: naive flatness is actually *not* stable under base change nor ground field extension.

Let us describe a counter-example, which had been suggested to us by Michael Temkin. Roughly speaking, it is due to a boundary phenomenon: it consists of the embedding into the affine plane of a curve which is drawn on some bi-disc and cannot be extended to a larger disc; the problem occurs at the unique boundary point of the curve. We are now going to give some details; the reader will find proofs of what follows in 4.4. Choose $r > 0$ and $f = \sum a_i T^i \in k[\![T]\!]$ a power series whose radius of convergence is exactly r, and let Y be the closed one-dimensional k-disc of radius r. The Shilov boundary of Y consists of one point y (the one that corresponds to the semi-norm $\sum b_i T^i \mapsto \max |b_i| r^i$). Denote by φ the morphism $(\mathrm{Id}, f) : Y \to \mathbf{A}_k^{2,\mathrm{an}}$ and by X the closed analytic domain of $\mathbf{A}_k^{2,\mathrm{an}}$ defined by the inequality $|T_1| \leqslant r$; note that $\varphi(Y) \subset X$; more precisely, $\varphi(Y)$ is the Zariski-closed subset of X defined by the equation $T_2 = f(T_1)$. One can show that $\mathscr{O}_{\mathbf{A}_k^{2,\mathrm{an}},\varphi(y)}$ is a field: this is due to the fact that $\varphi(Y)$ cannot be extended to a curve defined *around* $\varphi(y)$, because the radius of convergence of f is exactly r. As a consequence, φ is naively flat at the point y. Now:

- $Y = \varphi^{-1}(X) \to X$ is a closed immersion of a one-dimensional space in a purely two-dimensional space, hence is not naively flat at y;
- if L is any complete extension of k such that Y_L has an L-rational point y' lying above y, then $\varphi(y')$ belongs to the topological interior of X_L in $\mathbf{A}_L^{2,\mathrm{an}}$ (because $\varphi(y')$ is a rigid point); therefore $\varphi_L : Y_L \to \mathbf{A}_L^{2,\mathrm{an}}$ is, around y', a closed immersion of a one-dimensional space in a purely two-dimensional space, hence is not naively flat at y'.

The above counter-example is, in some sense, archetypal, because boundary phenomena are actually the only obstruction for naive flatness to be stable under base change: if $Y \to X$ is a morphism between good k-analytic spaces and if y is a point of Y at which $Y \to X$ is inner, then every coherent sheaf $\mathscr{F}$ on Y which is naively X-flat at y remains so after any good base change (Theorem 8.3.4). Our proof is based upon an analytic variant of Raynaud-Gruson's *dévissages* [**RG71**], which is developed in Chapter 8; let us mention that this result had already been proved by Berkovich (in a completely different way) in some unpublished work about flatness. It is now clear why such problems cannot occur in the rigid setting: this is because boundary points are *never* rigid.

0.1.3.2. *Our definition of flatness.* — To overcome the problem we have just mentioned, we define flatness as follows (4.1.2, 4.1.8 ff.). Let $Y \to X$ be a morphism between k-analytic spaces, let y be a point of Y, and let $\mathscr{F}$ be a coherent sheaf on Y.

- If Y and X are good, we say that $\mathscr{F}$ is X-flat at y if it is naively X-flat at y and *if it remains so after any good base change and any ground field extension.*
- In general, $\mathscr{F}$ is X-flat at y if there exists a good analytic domain V of Y containing y and a good analytic domain U of X containing the image of V such that $\mathscr{F}_{|V}$ is U-flat at y (and if it is the case then this holds for every (V, U) as above).

These definitions might seem somehow ad hoc, and not so easy to check nor to use. But we hope that this memoir will convince the reader that they provide a notion of flatness which behaves exactly as expected and is quite convenient to work with.

0.2. Loci of validity

0.2.1. General presentation of the problem. — We now turn back to the situation we have described at the beginning of the Introduction. That is, we are given a morphism $\varphi\colon Y \to X$ between k-analytic spaces, an object D on Y, and we want to understand the variation of properties of D_x for x going through X. This may be investigated by considering two kinds of problems.

- *Problems on the source.* What can be said about the set of points $y \in Y$ such that D satisfies a given (local or punctual) property P fiberwise at y; i.e., $D_{\varphi(y)}$ satisfies P at y as an object living on the $\mathscr{H}(\varphi(y))$-analytic space $Y_{\varphi(y)}$?
- *Problems on the target.* What can be said about the set of points $x \in X$ such that D_x satisfies a given global property Q, as an object living on the $\mathscr{H}(x)$-analytic space Y_x ?

In scheme theory, both kinds of problems are addressed. More precisely, one first proceeds to a study on the source, and one uses the latter *together with Chevalley's constructibility theorem* in order to understand what happens on the target. Note that Chevalley's theorem itself can be described in a somehow pedantic way as an answer to the "problem on the target" for Q being the *non-emptiness* property (of a scheme).

There is kind of an avatar of Chevalley's theorem in analytic geometry, due to L. Lipshitz and Z. Robinson [**LR00**]. It asserts that the image of a morphism between strictly affinoid spaces is constructible with respect to a broad class of functions, far larger than that of global analytic ones. But unfortunately constructible subsets in the sense of Lipshitz and Robinson are (up to now) poorly understood and not that tractable; for that reason, in this memoir we will mainly *ignore* the problems on the target, and focus on what happens *on the source.*

0.2.2. Remark. — The image of an *overconvergent* morphism between strictly affinoid spaces is constructible with respect to a class of functions which is smaller than that of Lipshitz and Robinson, but still larger than that of global analytic ones; this had been proved by Schoutens in [**Sch94**]. Recently, F. Martin [**Mar15**] has given a purely geometric version of Schoutens's statements and proofs (it involves some particular finite sequences of blow-ups instead of the class of recursively defined functions considered by Schoutens).

0.2.3. Loci of validity in the source space. — Let $Y \to X$ be a morphism of k-analytic spaces, and let $\mathscr{F}$ be a coherent sheaf on Y. We prove the following (assertion (1) is Theorem 10.3.2, and the other ones are part of Theorem 10.7.2 together with the fact that quasi-smoothness is equivalent to flatness and fiberwise regularity, see Th.5.3.4 (1)):

(1) The set of points of Y at which $\mathscr{F}$ is X-flat is Zariski-open.
(2) The set of points of Y at which Y is fiberwise geometrically regular (resp. geometrically reduced, resp. geometrically normal, resp. Gorenstein, resp. complete intersection) is locally constructible.
(3) The set of points of Y at which Y is X-flat and fiberwise geometrically regular (resp. Gorenstein, resp. complete intersection) is Zariski-open; if Y is relatively

equidimensional over X (1.4.13; empty fibers are allowed, see 1.4.9), the set of points of Y at which Y is X-flat and fiberwise geometrically reduced (resp. geometrically normal) is Zariski-open.

(4) The set of points of Y at which $\mathscr{F}$ is fiberwise Cohen-Macaulay (resp. S_m) is locally constructible.

(5) The set of points of Y at which $\mathscr{F}$ is X-flat and fiberwise Cohen-Macaulay is Zariski-open.

Let us give some explanations about our terminology.

0.2.3.1. — A subset E of Y is called *constructible* if it is a finite boolean combination Zariski-open subsets, and *locally constructible* if every point of Y has an open neighborhood U such that $U \cap E$ is constructible in U. But when Y is finite-dimensional, every locally constructible subset of Y is constructible (Proposition 10.1.12; see 10.1.14 for a counter-example in the infinite-dimensional case).

0.2.3.2. — Assume that Y is good. It is said to be regular (resp. reduced, ...) at y if the local ring $\mathscr{O}_{Y,y}$ is regular (resp. reduced,...). The coherent sheaf $\mathscr{F}$ is said to be Cohen-Macaulay at y if the $\mathscr{O}_{Y,y}$-module $\mathscr{F}_y$ is Cohen-Macaulay.

0.2.3.3. — The space Y is no longer assumed to be good. Then Y is said to be regular (resp. reduced, ...) at y if there exists a good analytic domain V of Y containing y which is regular (resp. reduced, ...) at y, and it then holds for *every* good analytic domain containing y (Lemma 2.4.3); and $\mathscr{F}$ is said to be Cohen-Macaulay at y if there exists a good analytic domain V of Y containing y such that the restriction of $\mathscr{F}$ to V is Cohen-Macaulay at y, and it then holds for *every* good analytic domain containing y (Lemma 2.4.3 again).

0.2.3.4. — The properties of being Gorenstein or Complete Intersection (for Y) or Cohen-Macaulay (for $\mathscr{F}$) at a given point are preserved by arbitrary ground field extensions. This is not the case in general for the property of being regular, normal or reduced, and when one of them holds at a point and remains valid after arbitrary ground field extensions it is said to hold *geometrically* (to ensure geometric validity, there is in fact no need to check the property over all possible extensions: it suffices that it holds after a single *perfect* extension). The behavior of algebraic properties under ground field extension is described in full detail in 2.6.

0.2.4. Remark. — The property for a morphism to be flat and fiberwise geometrically regular at a given point is of fundamental importance. It is called *quasi-smoothness*, and can also be defined using an analytic avatar of the Jacobian criterion; this is the main topic of Chapter 5.

0.2.5. Loci of validity in the target space. — As we explained above, the general description of the set of points of the target whose fiber satisfies some given property seems out of reach. But we have been able to address that kind of question in two very specific situations.

0.2.5.1. *The case of a proper map.* — Let Y, X and $\mathscr{F}$ be as above, and assume moreover that $Y \to X$ is *proper*. Then under this assumption we have kind of an analytic Chevalley's theorem: indeed, by Kiehl's result on cohomological finiteness of proper morphisms, the image in X of any Zariski-closed subset of Y is Zariski-closed (1.3.23); and one can deduce from this that the image in X of any locally constructible subset of Y is locally constructible (Theorem 10.1.15). The following then follow formally from the aforementioned results on the source (this is part of Theorem 10.7.5).

(1) The set of points x of X such that $\mathscr{F}$ is X-flat at every point of Y_x is a Zariski-open subset of X.
(2) The set of points x of X such that Y_x is geometrically regular (resp. Gorenstein, resp. Complete Intersection) at all of its points is a locally constructible subset of X.
(3) The set of points x of X such that Y is X-flat at every point of Y_x and Y_x is geometrically regular (resp. Gorenstein, resp. Complete Intersection) at all of its points is a Zariski-open subset of X.
(4) The set of points x of X such that the restriction of $\mathscr{F}$ to Y_x is Cohen-Macaulay at each point of Y_x is a locally constructible subset of X.
(5) The set of points x of X such that $\mathscr{F}$ is X-flat at each point of Y_x and the restriction of $\mathscr{F}$ to Y_x is Cohen-Macaulay at each point of Y_x is a Zariski-open subset of X.
(6) Assume moreover that Y is relatively equidimensional over X. Then the set of points x of X such that Y is X-flat at every point of Y_x and Y_x is geometrically regular (resp. geometrically reduced, resp. geometrically normal) at each of its points is a Zariski-open subset of X.

0.2.5.2. *The image of a morphism between affinoid spaces.* — We prove (Theorem 9.2.1) that if $\varphi\colon Y \to X$ if a flat morphism between affinoid spaces, then $\varphi(Y)$ is a compact analytic domain (otherwise said, a finite union of affinoid domains) of X. In the strictly analytic case, this is nothing but the aforementioned result of Raynaud (cf. [**BL93b**], Cor. 5.11). But our methods are completely different and provide a new proof of his theorem, which does not involve any formal model.

We first give a direct proof when the relative dimension of φ is zero (Proposition 9.1.1); it is based upon Temkin's theory of reduction of analytic germs and kind of Chevalley theorem in the framework of (graded) Riemann-Zariski spaces (Theorem 7.2.5). We then handle the general case (with no assumption on the relative dimension of φ) by reducing through a suitable ground field extension to the case where k is non-trivially valued and both Y and X are strict, and then by showing the following, which seems us to be of independent interest: there exist a strictly k-affinoid space Z, a flat morphism $\psi\colon Z \to X$ of relative dimension 0, and an X-map $Z \to Y$, such that $\psi(Z) = \varphi(Y)$ (Theorem 9.1.3). Otherwise said, the image of any flat map between strictly k-affinoid spaces is covered by a flat, quasi-finite multisection of the map.

0.2.6. From the target and the fibers to the source. — Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of noetherian schemes. If $\mathscr{X}$ is regular (resp. reduced, resp. normal, resp. Gorenstein, resp. Complete Intersection), $\mathscr{Y} \to \mathscr{X}$ is flat, and its fibers are regular (resp. reduced, ...), then $\mathscr{Y}$ is regular (resp. reduced, ...). Analogously, let $\mathscr{F}$ be a coherent sheaf on $\mathscr{Y}$. If $\mathscr{X}$ is Cohen-Macaulay, $\mathscr{F}$ is flat over $\mathscr{X}$, and the restriction of $\mathscr{F}$ to every fiber of $\mathscr{Y} \to \mathscr{X}$ is Cohen-Macaulay, then $\mathscr{F}$ is Cohen-Macaulay.

At the end of this memoir (Chapter 11), we develop a general and systematic method to transfer such theorems from algebraic geometry to analytic geometry; in particular, the analytic counterparts of the above statements all hold (see Theorem 11.3.3).

0.3. About our proofs

0.3.1. Technical obstacles in analytic geometry. — Our general strategy is of course to adapt to the Berkovich setting what had been done about relative properties in algebraic geometry, and sometimes in complex analytic geometry. For instance, our investigation of quasi-smooth morphism is inspired by the study of smooth maps by Bosch, Lütkebohmert and Raynaud in [**BLR90**], our theory of dévissages follows that of Raynaud and Gruson in [**RG71**], and our proof of the Zariski-openness of the flat locus is mutatis mutandis the same as that of Kiehl in the complex analytic setting [**Kie67b**].

But most of the time (and that is the case for quasi-smoothness as well as for the dévissages), techniques coming from the Grothendieck school *cannot* be applied straightforwardly in our setting. Let us now give some examples of obstacles we had to face, and then quickly explain the way we overcame them – we hope that the techniques and methods developed for this purpose will be useful in other circumstances.

(1) If Y is an analytic space, the Zariski topology of an analytic domain V of Y is in general strictly finer than the one inherited from the Zariski-topology of Y – even if V is a Zariski-open subset of Y (note that any infinite, discrete and closed subset of $\mathbf{A}_k^{1,\mathrm{an}}$ consisting of rigid points is Zariski-closed in $\mathbf{A}_k^{1,\mathrm{an}}$, but it does not come from a Zariski-closed subset of $\mathbf{P}_k^{1,\mathrm{an}}$). And analogously, if $Y \to X$ is a morphism of k-analytic spaces, the Zariski-topology of a fiber Y_x at a *non-rigid* point is also in general finer than the one inherited from the Zariski-topology of Y.

(2) If y and z are two points of a good analytic space Y and if y belongs to the Zariski closure of z in Y, there is no way to relate in a simple way the local rings $\mathscr{O}_{Y,y}$ and $\mathscr{O}_{Y,z}$ (on a scheme, the latter would be a localization of the former).

(3) If $Y \to X$ is a morphism of good analytic spaces and if y is a point of Y whose image in X is denoted by x, the comparison between $\mathscr{O}_{Y_x,y}$ and $\mathscr{O}_{Y,y}$ is not that clear: for instance, even if $\mathscr{O}_{X,x}$ is a field (which means that Y_x should be thought of as kind of a generic fiber), the local ring $\mathscr{O}_{Y_x,y}$ is in general a huge $\mathscr{O}_{Y,y}$-algebra, though on a scheme it would be equal to $\mathscr{O}_{Y,y}$.

(4) If $\varphi\colon Y \to X$ is a morphism between two k-affinoid spaces, we have already explained in 0.2.1 that $\varphi(Y)$ is not easily understandable in general. In particular, when Y is of dimension d and φ of pure relative dimension δ for some d and δ, there is no reason why $\varphi(Y)$ should be contained in a $(d-\delta)$-dimensional Zariski-closed subset of an analytic domain of X (in scheme theory, one would simply take $\overline{\varphi(Y)}$).

0.3.1.1. — Obstacle (1) is not so big a problem: one can more or less overcome it because Zariski-closedness (or openness) is a G-local property and the irreducible components behave reasonably with respect to analytic domains. Moreover, some work had already been carried out by the author in [**Duc07b**] to remedy the fact that the Zariski-topology on a fiber is "too fine"; see for instance section 4 of op. cit., whose results are used repeatedly in this memoir.

0.3.1.2. — Most of the time, we shall overcome obstacle (2) by working with the respective images η and ζ of y and z on the scheme $\mathscr{Y} := \operatorname{Spec} \mathscr{O}_Y(Y)$, which have the required property (i.e., $\mathscr{O}_{\mathscr{Y},\zeta}$ is a localization of $\mathscr{O}_{\mathscr{Y},\eta}$), and then go back to our original space Y by using some GAGA results.

0.3.1.3. — Obstacle (3) is probably a priori the most harmful for our purposes. Indeed, the study of relative properties in EGA, rests, among other things, on the technique of "spreading out from generic fiber" which we describe roughly. One starts from a morphism of noetherian schemes $\mathscr{Y} \to \mathscr{X}$, and with a point η of $\mathscr{Y}$ at which some property P (of the scheme itself, of a coherent sheaf on it, ...) is satisfied fiberwise; let ξ be the image of η. Then if one sets $\mathscr{T} = \mathscr{Y} \times_{\mathscr{X}} \overline{\{\xi\}}_{\mathrm{red}}$, the fiber $\mathscr{Y}_\xi$ is equal to the *generic fiber* of the map $\mathscr{T} \to \overline{\{\xi\}}_{\mathrm{red}}$; hence the local ring of this fiber at η is equal to $\mathscr{O}_{\mathscr{T},\eta}$; *note that* (3) *tells precisely that this step would fail in analytic geometry.*

The property P thus holds now absolutely (and not only fiberwise) on $\mathscr{T}$ at η; moreover the scheme $\mathscr{T}$ itself, as well as the restriction to $\mathscr{T}$ of any of the coherent sheaves possibly involved in our situation, are flat over $\overline{\{\xi\}}_{\mathrm{red}}$ at η. This might help to "spread" fiberwise validity of P from η to a dense open subset of $\overline{\{\eta\}}$.

0.3.1.4. — We remedy obstacle (3) as follows. Let $Y \to X$ denote a morphism of k-affinoid spaces, let x be a point of X whose local ring is a field, and let y be a point of Y_x. We prove (Theorem 6.3.3) that if y does not belong to the relative boundary of Y over X, then the map Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{Y,y}$ is flat (with complete intersection fibers). And flatness suffices to ensure the transfer of all familiar algebraic properties from $\mathscr{O}_{Y_x,y}$ to $\mathscr{O}_{Y,y}$, which is most of the time what is actually needed (for instance in 0.3.1.3 above, the conclusion that P is satisfied on $\mathscr{T}$ at η only requires the transfer of the property P from $\mathscr{O}_{\mathscr{T}_\xi,\eta}$ to $\mathscr{O}_{\mathscr{T},\eta}$, and not the equality of the rings).

0.3.1.5. — At the end of the memoir, we remedy obstacle (4) by showing the following: if $\varphi\colon Y \to X$, d and δ are as in (4), then there exists a non-empty affinoid domain V of Y and a purely $(d-\delta)$-dimensional Zariski-closed subset S of an analytic domain of X such that $\varphi(V) \subset S$ (11.1.5). This is slightly weaker than the "dream property" discussed in (4) (one does not control the whole of $\varphi(Y)$, but only the image of an affinoid domain which can possibly be very small), but this can nevertheless be useful for some purposes, like carrying out an induction on the dimension of the base space (this is the way we use it in the proof of Theorem 11.3.3).

0.3.2. About the previous results by Kiehl. — In [**Kie68**], Kiehl has established some analogous results for a morphism between two affinoid *rigid* spaces, but they do not *a priori* imply our theorems for the following reason.

In the rigid analytic context, one only deals with fibers over *rigid* points. To be sure, any point of an analytic space can be made rigid after a suitable ground field extension, and it follows from the author's previous work [**Duc09**] that all properties involved can be checked after any ground field extensions. But the combination of these remarks and of Kiehl's theorems does not yield directly the corresponding statements in the Berkovich setting, *unless one knows that the formation of the fiberwise validity locus of a given property in rigid-analytic geometry commutes with scalar extension.* And this has not been addressed by Kiehl, whose methods do not seem to apply straightforwardly to such questions: indeed, these methods are very "algebraic", and it is not clear – at least to the author – how one could use them to deal with scalar extension, which involves completion operations.

Because of that, and also for the reader's convenience, we have chosen to write self-contained and purely "Berkovich" proofs; hence we *recover* and extend Kiehl's results.

Acknowledgements

I started to think about the topics this paper is devoted to when I was asked some questions about flatness by Brian Conrad and Michael Temkin, for their work [**CT**]. I realized quite quickly that answering them would take much more time than I had originally expected... and it eventually gave rise to the present work. I thus would like express my gratitude to Conrad and Temkin for having given to me the initial inspiration – and also for a lot of fruitful discussions since then.

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CHAPTER 1

BACKGROUND MATERIAL

This chapter presents our general conventions in topology, algebra and algebraic geometry, and then provides some basic reminders in analytic geometry. The reader who already knows well Berkovich's theory may skip most of it, but should nevertheless have a look at 1.2.7 ff. for our conventions about analytic spaces without mention of a ground field, and at section 1.2.7 and especially 1.3.7 for our conventions and notation about coherent sheaves and their stalks. Be aware that we depart from Berkovich's terminology in two cases:

- We say "boundaryless" instead of "closed" (1.3.20).
- We say "finite at a point" instead of "quasi-finite at a point", "locally finite" instead of "quasi-finite" (1.2.17), and we have our own definition for "quasi-finite" (1.4.15, Remark 1.4.16).

Let us also mention that we shall make much use in this memoir of *graded* commutative algebra, after Temkin [**Tem04**]. Since this theory essentially consists of a somehow tedious transcription of classical commutative algebra with the words "graded" or "homogeneous" added almost everywhere, we have chosen to write the corresponding reminders (almost without proofs, though some of them are sketched) not in this chapter, but in Appendix A at the end of the memoir. The reader may refer to it if needed. Let us simply say here that for us, "graded" will always mean "$\mathbf{R}_+^\times$-graded", with multiplicative graduation.

1.1. Prerequisites and basic conventions

1.1.1. Prerequisites. — The understanding of this memoir requires of course a robust general knowledge of commutative algebra, algebraic geometry and non-archimedean analytic geometry; let us make this precise.

In commutative algebra, we will use freely the "usual" notions about commutative rings (and especially local noetherian rings) and modules over the latter: flatness, Krull dimension, depth and codepth, regularity, properties like R_m and S_m (due to Serre), Cohen-Macaulay, Gorenstein, or (local) complete intersection; and also Grothendieck's crucial concept of excellence. Possible references on these topics are Matsumura's textbook [**Mat86**] and part of EGA and SGA: [**EGA IV**$_1$], Chapter 0, §15, §16 and §17; [**EGA IV**$_2$], §6 and §7; and [**SGA 1**], Exposé V. We will also assume the reader to be familiar with elementary valuation theory; see for instance [**Bou85**], Chapitre VI.

In algebraic geometry, we will use scheme theory in the spirit of EGA and of the French school in algebraic geometry, and the reader will be assumed to master the corresponding language. Parts of this memoir have been more specifically inspired by the treatment of smoothness in the book [**BLR90**] on Néron models by Bosch, Lütkebohmert and Raynaud, by the first part of Raynaud and Gruson's seminal work on flatness [**RG71**], and by [**EGA IV**$_3$] §12 which is devoted to relative properties. Familiarity with these texts might therefore be helpful, but is not strictly needed.

In analytic geometry, the basics of Berkovich's approach as exposed in [**Ber90**] and [**Ber93**] §1 will be considered known. We will also make much use of the works by the author on dimension theory [**Duc07b**] and on "commutative algebra properties" of analytic spaces [**Duc09**], as well as of Temkin's theory of the reduction of germs [**Tem04**]. Some acquaintance with these topics is then recommended; nonetheless, we will recall all related definitions and statements that are needed for our purposes.

1.1.2. Conventions in algebra. — In this memoir, all rings (and algebras) are commutative and have a unit element; morphisms of rings (and algebras) always respect the unit elements. The group of invertible elements of a ring A will be denoted by $A^\times$. The acronyms CM and CI will stand respectively for "Cohen-Macaulay" and "complete intersection".

If M is a module over a ring A and if I denotes the annihilator of M, the Krull dimension of M will be by definition the Krull dimension of the ring A/I; in more geometric terms, this is the dimension of the support of M on Spec A.

We shall often use for short the *multi-index* notation, which consists of the following. Let Λ be a multiplicative commutative monoid, let n be a non-negative integer and let $\tau = (\tau_1, \ldots, \tau_n)$ be an n-uple of elements of Λ (the reader should have two examples in mind: the case where $\Lambda = \mathbf{R}_+^\times$ and the case where Λ is the multiplicative monoid of an algebra of polynomials or power series over some ground ring in the indeterminates τ_j). Let $I = (i_1, \ldots, i_n) \in \mathbf{Z}^n$ be such that $i_j \geqslant 0$ as soon as τ_j is not invertible. Then the product $\prod \tau_j^{i_j}$ will be simply denoted by τ^I. If all τ_j's are invertible, we will set $\tau^{-1} = (\tau_1^{-1}, \ldots, \tau_n^{-1})$.

1.1.3. Conventions in topology. — We shall use the terminology of Bourbaki [**Bou71**]. A topological space X will be called *quasi-compact* if every open covering of X admits a finite sub-covering; it will be called *compact* if it is quasi-compact *and Hausdorff.*

Let X be a topological space. It will be called:

- *locally compact* if X is Hausdorff and every point of X has a compact neighborhood in X;
- *countable at infinity* if X is Hausdorff and X is the union of (at most) countably many compact subsets;
- *paracompact* if X is Hausdorff and every open covering of X can be refined into a locally finite open covering.

If X is locally compact, it is paracompact if and only if it is the disjoint union of open subsets that are countable at infinity ([**Bou71**], Chapitre I, §9, n° 10, Th 5). If this is the case, then for every basis of neighborhoods $\mathscr{B}$ of X, every open covering of X can be refined into a locally finite covering consisting of elements of $\mathscr{B}$ (this is what the proof of [**Bou71**] actually shows).

A continuous map $p\colon Y \to X$ between two arbitrary topological spaces is said to be *proper* if it is universally closed; i.e., for every continuous map $Z \to X$ and every closed subset F of $Y \times_X Z$, the image of F in Z (by the second projection) is closed. (Be aware that p is *not* required to be separated; this is Bourbaki's convention and we have chosen to follow it despite the inconsistency with the definition of properness in algebraic geometry, because some of our results actually hold for maps between analytic spaces that are topologically proper in Bourbaki's sense without any separatedness assumption; see Theorem 9.2.1 and Theorem 9.2.2). If p is proper, $p^{-1}(K)$ is quasi-compact for every quasi-compact subset K of X ([**Bou71**], Chapitre I, §10, n° 2, Prop. 6). If Y is Hausdorff and X is locally compact, p is proper if and only if $p^{-1}(K)$ is compact for every compact subset K of X ([**Bou71**], Chapitre I, §10, n° 3, Prop. 7).

If E is any subset of a topological space X, the closure of E inside X will be denoted by $\overline{E}^X$.

1.1.4. Conventions in algebraic geometry. — The word "scheme" will be understood here *without particular assumption.* We shall always make precise when the scheme we are working with is separated, noetherian, excellent, reduced, of finite type over some ground ring, etc.

If X is a scheme and if x is a point of X, the maximal ideal of $\mathscr{O}_{X,x}$ will be denoted by $\mathfrak{m}_x$, and its residue field by $\kappa(x)$. If $\varphi\colon Y \to X$ is a morphism of schemes, the *scheme-theoretic* fiber of φ over x will be denoted by $\varphi^{-1}(x)$ or Y_x ; this is a $\kappa(x)$-scheme.

A morphism of schemes $Y \to X$ is called *regular* if it is flat and if for every point x of X the fiber Y_x is locally noetherian and *geometrically* regular; i.e., $Y_x \otimes F$ is regular for any finite, purely inseparable extension F of $\kappa(x)$. A morphism of rings $A \to B$ is said to be regular if Spec $B \to$ Spec A is regular.

If $\mathscr{F}$ is a coherent sheaf on a noetherian scheme X, and if x is a point of X, we shall denote by $\mathscr{F}_x$ the stalk of $\mathscr{F}$ at x, and by $\mathscr{F}_{\kappa(x)}$ the tensor product $\kappa(x) \otimes_{\mathscr{O}_{X,x}} \mathscr{F}$. If Y is any X-scheme, the pull-back of $\mathscr{F}$ to Y will be denoted by $\mathscr{F}_Y$.

1.2. Analytic geometry: basic definitions

1.2.1. Definition. — An *analytic field* is a field endowed with an $\mathbf{R}_+^\times$-valuation for which it is *complete*; unless otherwise stated, the structure valuation of an analytic field will be denoted by $|\cdot|$.

1.2.2. Example. — Any field endowed with the trivial valuation is an analytic field.

1.2.3. — If k is an analytic field, we shall denote by $\widetilde{k}$ its *graded* reduction (A.4.7 and also A.4.8); i.e.,

$$\widetilde{k} = \bigoplus_{r>0} \{x \in k, |x| \leqslant r\}/\{x \in k, |x| < r\}.$$

For every positive r, the r-th summand of $\widetilde{k}$ will be denoted by $\widetilde{k}^r$; note than the usual residue field of $\widetilde{k}$ is nothing but $\widetilde{k}^1$.

If x is any element of k and r any positive number $\geqslant |x|$, we shall denote by $\widetilde{x}^r$ the image of x in $\widetilde{k}^r$; if $r = |x|$ we shall simply write $\widetilde{x}$.

1.2.4. — An *analytic extension* of an analytic field k is an analytic field L together with an isometric embedding $k \hookrightarrow L$; for such an L we shall denote by $d_k(L)$ the transcendance degree of the graded extension $\widetilde{k} \hookrightarrow \widetilde{L}$ (see A.4.11 for a "classical" interpretation of this invariant).

1.2.5. Convention. — The notion of a *k-analytic space* will always be understood in the sense of Berkovich [**Ber93**] §1.

1.2.6. Topologies. — Let k be an analytic field and let X be a k-analytic space. The space X comes with a topology, which enjoys very nice properties (this is one of the distinguished features of Berkovich's theory): every point of X has a basis of open neighborhoods that are Hausdorff, locally compact, path-connected, and countable at infinity.

The space X is also equipped with (set-theoretic) *Grothendieck* topology. Before describing it, let us introduce some terminology. If E is any subset of X and if (E_i) is a family of subsets of E, we shall say that (E_i) is a G-*covering* of E if every point x of E has a neighborhood *in* E of the form $\bigcup_{i\in I} E_i$ for some finite set I such that $x \in \bigcap_{i\in I} E_i$. A subset V of X is called an *analytic domain of* X if V is G-covered by the affinoid domains of X that are contained in V. Affinoid domains and open subsets of X are analytic domains; any analytic domain V of X inherits a canonical structure of a k-analytic space. If x is a point of X, an *analytic neighborhood* of x in X will be an analytic domain of X which is a neighborhood of x in X.

The site X_{G} is then defined as follows:

- Its objects are the analytic domains of X.
- Its morphisms are the inclusion maps.
- Its topology is the so-called G-*topology*, whose covering families are exactly the families $(V_i \subset V)_i$ for (V_i) a G-covering of V.

Any locally finite covering of a Hausdorff analytic domain of X by compact analytic domains is a G-covering. Any open covering of an open subset of X is a G-covering: the G-topology is finer than the usual topology. This can be rephrased in a somehow pedantic way by saying that the forgertful functor induces a morphism of sites from X_{G} to X.

If X is quasi-compact, any G-covering of X admits a finite sub-covering; in particular, X admits a finite affinoid G-covering. Conversely, if X admits a finite set-theoretic affinoid covering, it is quasi-compact because any affinoid domain is compact.

If X is Hausdorff, it is paracompact if and only if it admits a locally finite affinoid covering. Indeed, the direct implication is due to the comments following the definition of paracompactness given in 1.1.3 (by taking for $\mathscr{B}$ the set of finite unions of affinoid domains). For the converse implication, suppose we are given a locally finite affinoid covering $(X_i)_{i\in I}$ of X; we may assume that every X_i is non-empty. Let $\mathscr{R}$ be the finest equivalence relation on I such that $i\mathscr{R}j$ as soon as $X_i \cap X_j \neq \varnothing$ (if the latter holds we shall say that i and j are elementary equivalent). Fix $C \in I/\mathscr{R}$ and fix $i \in C$. For every $n \geqslant 1$, the set C_n of indices $j \in C$ that are linked to i by a chain of at most n elementary equivalences is finite, because (X_i) is locally finite. Hence the union X_{C_n} of all X_j's for j running through C_n is compact; therefore $X_C := \bigcup_{i\in C} X_i = \bigcup_n X_{C_n}$ is countable at infinity. Since $X = \coprod_{C\in I/\mathscr{R}} X_C$ and since every X_C is open in X (again by local finiteness of the covering (X_i)), the space X is paracompact.

Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces. If φ is *topologically* proper (see 1.1.3) then $\varphi^{-1}(V)$ is quasi-compact for every quasi-compact subset V of X, and in particular for every affinoid domain V of X. Conversely, assume that $\varphi^{-1}(V)$ is quasi-compact for every affinoid domain V of X ; the morphism φ is then topologically proper. Indeed, its fibers are quasi-compact, so it suffices to prove that it is closed. But this can be checked G-locally on X, which allows us to assume that X is affinoid. The space Y is then quasi-compact, hence can be written as a finite union $\bigcup Y_i$ with Y_i affinoid for all i. Now $Y_i \to X$ is closed for every i, hence φ is closed.

1.2.7. — If X is a k-analytic space, and if L is an analytic extension of k, we shall denote by X_L the L-analytic space deduced from X by extending the ground field to L. There is a natural map $X_L \to X$ which is surjective, cf. [**Duc07b**], §0.5. If A is a k-affinoid algebra, we shall denote by A_L the L-affinoid algebra $A\widehat{\otimes}_k L$.

1.2.8. — An analytic space *without mention of any ground field* is a pair (k, X) where k is an analytic field and X a k-analytic space; a morphism between two analytic spaces (L, Y) and (k, X) consists of an isometric embedding $k \hookrightarrow L$ and a morphism $Y \to X_L$ of L-analytic spaces. Similarly we define an affinoid algebra (resp. space) as a pair (k, A) (resp. (k, X)) where k is an analytic field and A a k-affinoid algebra (resp. and X a k-affinoid space). While speaking about analytic spaces and morphisms between them, we shall of course most of the time omit mention of the fields and the isometric embeddings involved. Hence instead of saying "let (k, X) be an analytic space" we shall say "let X be an analytic space" and we shall refer to k as to the *field of definition* of X.

1.2.9. — If x is a point of an analytic space, its completed residue field will be denoted by $\mathscr{H}(x)$. We note that if V is an analytic domain of X containing x, then $\mathscr{H}(x)$ does not depend whether x is viewed as belonging to V or to X.

1.2.10. — Let k be an analytic field. A point x of a k-analytic space is called *rigid* if $\mathscr{H}(x)$ is a finite extension of k.

Assume that $|k^{\times}| \neq \{1\}$. By the analytic *Nullstellensatz* ([**BGR84**], 6.1.2, Cor. 3), any non-empty strictly k-affinoid space has a rigid point; this implies that every non-empty strictly k-analytic space (hence in particular any non-empty boundaryless k-analytic space) has a rigid point.

1.2.11. — An analytic space X is said to be *good* if every point of X has an affinoid neighborhood, hence a basis of affinoid neighborhoods.

1.2.12. — Let X be an analytic space, and let k be its field of definition. An *X-analytic space* is a k-analytic space Y together with a morphism $Y \to X$ of k-analytic spaces; we emphasize that *Y and X have the same field of definition.* For such a space Y and for x a point of X, we shall denote by Y_x the fiber of Y over x; this is an $\mathscr{H}(x)$-analytic space. If the morphism $Y \to X$ has been given a name, say φ, this fiber will also be denoted by $\varphi^{-1}(x)$.

1.2.13. — Be aware that the category of analytic spaces in the sense of 1.2.8 does not admit fiber products in general. For instance, $\mathscr{M}(\mathbf{C}_p) \times_{\mathscr{M}(\mathbf{Q}_p)} \mathscr{M}(\mathbf{C}_p)$ does not exist as an analytic space. But if

$$\begin{tikzcd} Y \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

is a diagram in the category of analytic spaces and if Y or Z is X-analytic, then $Y \times_X Z$ does exist in the category of analytic spaces (and is Z-analytic in the first case, and Y-analytic in the second case). Moreover, the natural continuous map from $Y \times_X Z$ to the fiber product of Y and Z over X *in the category of topological spaces* is surjective. Indeed, let x be a point of X, let y be a pre-image of x on Y and let z be a pre-image of x on Z. We want to prove that there exists a point in $Y \times_X Z$ lying above both y and z. For that purpose we may assume that X, Y and Z are affinoid, say respectively equal to $\mathscr{M}(A), \mathscr{M}(B)$ and $\mathscr{M}(C)$. The completed tensor product $\mathscr{H}(y)\widehat{\otimes}_{\mathscr{H}(x)}\mathscr{H}(z)$ is non-zero because it contains $\mathscr{H}(y) \otimes_{\mathscr{H}(x)} \mathscr{H}(z)$ by a result of Gruson ([**Gru66**], §3.2 Thm. 1 (4)); hence its Banach spectrum $\mathscr{M}(\mathscr{H}(y)\widehat{\otimes}_{\mathscr{H}(x)}\mathscr{H}(z))$ is non-empty ([**Ber90**], Thm. 1.2.1). Now if t is a point of $\mathscr{M}(\mathscr{H}(y)\widehat{\otimes}_{\mathscr{H}(x)}\mathscr{H}(z))$, its image on $\mathscr{M}(B\widehat{\otimes}_A C) = Y \times_X Z$ lies above both y and z, and we are done.

1.2.14. Definition. — A *polyradius* is a finite family of positive real numbers.

1.2.15. — Let $r = (r_1, \ldots, r_n)$ be a polyradius and let $T = (T_1, \ldots, T_n)$ be a family of indeterminates. For every analytic field k we denote by k_r the k-algebra of power series $\sum_{I \in \mathbf{Z}^n} a_I T^I$ with coefficients in k such that $|a_I|\, r^I \to 0$ as $|I| \to \infty$. The map $\sum a_I T^I \mapsto \max |a_I|\, r^I$ is a multiplicative norm on k_r (cf. [**Duc07b**], 1.2.1), which makes k_r a k-affinoid algebra. Its analytic spectrum $\mathscr{M}(k_r)$ is the affinoid domain of the analytic n-dimensional affine space $\mathbf{A}_k^{n,\mathrm{an}}$ ([**Ber93**], p. 25) defined by the conditions $|T_i| = r_i$ for $i = 1, \ldots, n$.

The norm of k_r being multiplicative, it defines a point $\eta_{k,r}$ on $\mathscr{M}(k_r) \subset \mathbf{A}_k^{n,\mathrm{an}}$, which is by its very definition the unique point at which every function belonging to k_r achieves its maximum; otherwise said, $\eta_{k,r}$ is the unique element of the Shilov boundary of $\mathscr{M}(k_r)$ and we call it the *Shilov point* of $\mathscr{M}(k_r)$. If there is no ambiguity with the gorund field involved, we shall often write simply η_r instead of $\eta_{k,r}$. Note that by construction, the field $\mathscr{H}(\eta_r)$ is nothing but the completion of the valued field considered in Example A.4.10; in particular, $\widetilde{\mathscr{H}(\eta_r)}$ is isomorphic to $\widetilde{k}(\tau/r)$ where $\tau = (\tau_1, \ldots, \tau_n)$ is a family of indeterminates, and $d_k(\eta_r) = n$.

If r is *k-free*, i.e., r is free when viewed as a family of elements of the $\mathbf{Q}$-vector space $\mathbf{R}_+^\times/|k^\times|^{\mathbf{Q}}$, then k_r is an analytic field (cf. [**Duc07b**], 1.2.2). Hence $\mathscr{M}(k_r) = \{\eta_r\}$ and $\mathscr{H}(\eta_r) = k_r$ in such cases.

Conversely if $\mathscr{M}(k_r) = \{\eta_r\}$ then r is k-free. Indeed, suppose that r is *not* k-free. Up to renumbering the r_i's we can assume that there is some $j < n$ such that $(r_1, \ldots, r_j)$ is k-free and every r_i with $i > j$ is torsion modulo $|k^\times|\, r_1^{\mathbf{Z}} \cdot \ldots \cdot r_j^{\mathbf{Z}}$. Set $L = k_{(r_1,\ldots,r_j)}$. Since $k_r = L_{(r_{j+1},\ldots,r_n)}$, the non-empty affinoid space $\mathscr{M}(k_r)$ is strictly L-affinoid, hence has an L-rigid point x (1.2.10; we can also see this directly by choosing a finite extension F of L such that there exists $(a_i)_{j+1 \leqslant i \leqslant n}$ in $F^{\{j+1,\ldots,n\}}$ satisfying the equality $|a_i| = r_i$ for all $i \in \{j+1, \ldots, n\}$). On the other hand, the equality $k_r = L_{(r_{j+1},\ldots,r_n)}$ implies that $\eta_r = \eta_{L,(r_{j+1},\ldots,r_n)}$. Hence $d_L(\eta_r) = n - j > 0$, and η_r is thus not L-rigid; in particular, $x \neq \eta_r$.

1.2.16. — Let r be a polyradius. If X is an analytic space with field of definition k, we set $X_r = X \times_k \mathscr{M}(k_r)$; analogously, if A is an affinoid algebra with field of definition k, we set $A_r = A\widehat{\otimes}_k k_r$ The *Shilov section* of the continuous arrow $X_r \to X$ is the map that sends a point x to the Shilov point of its fiber $\mathscr{M}(\mathscr{H}(x)_r)$. The Shilov section is continuous by Lemma 3.3.2 (i) of [**Ber90**].

1.2.17. Finite morphisms (see [Ber93], §3.1). — Let X be an analytic space and let Y be an X-analytic space. If X is affinoid, say $X = \mathscr{M}(A)$, then $Y \to X$ is said to be *finite* if Y is X-isomorphic to $\mathscr{M}(B)$ for some finite Banach A-algebra B. In general, $Y \to X$ is said to be finite if $Y \times_X V \to V$ is finite for every affinoid domain V of X, and this can be checked on a given affinoid G-covering of X.

Let y be a point of Y. We shall say that $Y \to X$ is finite *at* y if there exists an open neighborhood V of y in Y and an open neighborhood U of the image of y on X such that $Y \to X$ induces a finite morphism $V \to U$. We shall sometimes say that Y is *finite over* X, resp. *finite over* X *at* y, instead of saying that $Y \to X$ is finite, resp. finite at y.

A morphism will be called *locally finite* if it is finite at every point of the source space.

1.3. Coherent sheaves, Zariski topology and closed immersions

1.3.1. The structure sheaves. — Let X be an analytic space. The site X_{G} inherits a sheaf of rings $\mathscr{O}_{X_{\mathrm{G}}}$. The restriction $\mathscr{O}_X$ of $\mathscr{O}_{X_{\mathrm{G}}}$ to the category of open subsets of X makes X a locally ringed space. The sheaf of rings $\mathscr{O}_{X_{\mathrm{G}}}$ is coherent, and so is $\mathscr{O}_X$ when X is good (for proofs [1] see [**Duc09**], Lemme 0.1).

1.3.2. — If X is an affinoid space, say $X = \mathscr{M}(A)$, the global section functor induces an equivalence between the category of coherent $\mathscr{O}_{X_{\mathrm{G}}}$-modules and that of finitely generated A-modules; if M is a finitely generated A-module the corresponding coherent sheaf on X assigns to any affinoid domain V of X the module $M \otimes_A \mathscr{O}_X(V)$ (this essentially follows from “Tate acyclicity theorem” and a theorem by Kiehl; see [**Ber93**],§1.2).

1.3.3. — If X is good analytic space, the forgetful functor induces an equivalence between the category of coherent $\mathscr{O}_{X_{\mathrm{G}}}$-modules and that of coherent $\mathscr{O}_X$-modules, which preserves the cohomology groups and maps locally free $\mathscr{O}_{X_{\mathrm{G}}}$-modules to locally free $\mathscr{O}_X$-modules ([**Ber93**], Prop. 1.3.4 and Prop. 1.3.6).

1. It was pointed out to the author by Jérôme Poineau that there is a gap in both proofs. Indeed, in each of them one starts with a *surjection* $\mathscr{O}^n \to \mathscr{O}$ and proves that its kernel is locally finitely generated, though in order to get the coherence one should establish such a finiteness result for any, i.e., not necessarily surjective, map $\mathscr{O}^n \to \mathscr{O}$; but the proofs do not make any use of the surjectivity assumptions.

1.3.4. Convention. — Let X be an analytic space. In the sequel, it will be sufficient for us to work with sheaves on X_{G}, and we shall not need to pay special attention, in the good case, to the restriction of such a sheaf to the category of open subsets of X (except for stalks, see 1.3.7 below). For that reason, and for the sake of simplicity, a coherent $\mathscr{O}_{X_{\mathrm{G}}}$-module will simply be called a *coherent sheaf on* X, and we shall write $\mathscr{O}_X$ instead of $\mathscr{O}_{X_{\mathrm{G}}}$.

1.3.5. — If $\mathscr{F}$ is a coherent sheaf on an analytic space X and if $Y \to X$ is a morphism of analytic spaces we shall often denote by $\mathscr{F}_Y$ the pullback of $\mathscr{F}$ on Y; in particular if V is a analytic domain of X we shall often write for short $\mathscr{F}_V$ instead of $\mathscr{F}|_V$; if L is analytic extension of k, we shall write $\mathscr{F}_L$ instead of $\mathscr{F}_{X_L}$; if r is a polyradius, we shall write for short $\mathscr{F}_r$ instead of $\mathscr{F}_{X_r}$.

1.3.6. — Let $\varphi\colon Z \to Y$ and $\psi\colon Z \to X$ be two morphisms of locally noetherian schemes (resp. analytic spaces), let $\mathscr{F}$ be a coherent sheaf on Y and let $\mathscr{G}$ be a coherent sheaf on X. If φ and ψ are clearly understood from the context, we shall denote by $\mathscr{F} \boxtimes \mathscr{G}$ the tensor product $\varphi^*\mathscr{F} \otimes_{\mathscr{O}_Z} \psi^*\mathscr{G}$.

1.3.7. Stalks. — Assume that X is good, let $\mathscr{F}$ be a coherent sheaf on X, and let x be a point of X. We shall denote by $\mathscr{F}_x$ the stalk at x of $\mathscr{F}$ *viewed as a sheaf on the underlying ordinary topological space of* X. In other words,

$$\mathscr{F}_x := \varinjlim_{U \text{ open neighborhood of } x} \mathscr{F}(U).$$

Note that this convention applies in particular for $\mathscr{F} = \mathscr{O}_X$: in this text, $\mathscr{O}_{X,x}$ will always denote the stalk at x of $\mathscr{O}_X$ viewed as a sheaf of rings on the underlying ordinary topological space of X. We shall denote by $\mathfrak{m}_x$ its maximal ideal, and by $\kappa(x)$ its residue field, which is a dense henselian subfield of the valued field $\mathscr{H}(x)$; see [**Ber93**], Thm. 2.3.3 – note that he uses the terminology "quasi-complete" instead of "henselian". The tensor product $\kappa(x) \otimes_{\mathscr{O}_{X,x}} \mathscr{F}_x$ will be denoted by $\mathscr{F}_{\kappa(x)}$.

If X is affinoid, it follows from 1.3.2 that $\mathscr{F}_x = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_X(X)} \mathscr{F}(X)$; we thus also have $\mathscr{F}_{\kappa(x)} = \kappa(x) \otimes_{\mathscr{O}_X(X)} \mathscr{F}(X)$.

1.3.8. Remark. — Be aware that the notation $\mathfrak{m}_x$ and $\kappa(x)$ might be ambiguous, because they do not mention the ambiant space X, though the object they denote actually depends on it (contrary to $\mathscr{H}(x)$). Most of the time this will not cause any trouble; but we will sometimes write x_V instead of x in order to indicate that we think of x as a point of a given good analytic domain V of X: we shall for instance write $\kappa(x_V)$ for the residue field of $\mathscr{O}_{V,x}$, or $\mathscr{F}_{\kappa(x_V)}$ for $\mathscr{F}_{V,x} \otimes_{\mathscr{O}_{V,x}} \kappa(x_V)$. Analogously, if T is a good X-analytic space and t denotes a point of T_x, then we shall write t_x whenever it is important to make precise that t is seen as a point of T_x; e.g., $\kappa(t_x)$ will denote the residue field of $\mathscr{O}_{T_x,t}$.

1.3.9. The Zariski topology. — Let X be an analytic space. A *Zariski-closed* subset of X is a subset of X that is equal to the zero-locus of some coherent sheaf of ideals $\mathscr{I}$ on X; i.e., it consists of points x such that $f(x) = 0$ for every analytic domain V of X containing x and every function f belonging to $\mathscr{I}(V)$. As suggested by the terminology, Zariski-closed subsets of X are exactly the closed subsets of a topology, the so-called *Zariski-topology.* If E is any subset of X, we shall denote its closure for the Zariski topology of X by $\overline{E}^{X_{\mathrm{Zar}}}$.

1.3.10. Remark. — Let V be an analytic domain of X. The reader should be aware that the Zariski topology of V is in general *strictly finer* than the topology induced by the Zariski topology of X. Analogously, if Y is an X-analytic space and if x is a point of X, the Zariski-topology on the fiber Y_x is in general finer than the topology induced by the Zariski-topology of X (nonetheless, they coincide as soon as x is rigid). This "non-transitivity" of Zariski topology is one of the technical subtleties that often prevent from transfering verbatim to the analytic framework what is done in scheme theory.

1.3.11. — Let X be an affinoid space, say $X = \mathscr{M}(A)$. A subset Y of X is Zariski-closed if and only if there exists an ideal I of A such that Y is equal to the set $\{x \in X, (\forall f \in I, f(x) = 0)\}$; in other words, the Zariski topology on X is nothing but the pre-image of the Zariski topology of Spec A under the natural map $X \to \operatorname{Spec} A$.

Let Y be a Zariski-closed subset of X and let I be as above. The morphism $\mathscr{M}(A/I) \to \mathscr{M}(A)$ induced by the quotient map $A \to A/I$ establishes a homeomorphism $\mathscr{M}(A/I) \simeq Y$, which endows Y with the structure of an affinoid space.

1.3.12. — Let X be an analytic space. Let Y be a Zariski-closed subset of X, and let $\mathscr{I}$ be a coherent sheaf of ideals whose zero locus is Y. By performing G-locally the construction described in 1.3.11 above, one gets a structure of an X-analytic space on Y, and the structure morphism $\iota \colon Y \to X$ satisfies the following property.

1. The map underlying ι is the inclusion, and $\iota_* \mathscr{O}_Y = \mathscr{O}_X/\mathscr{I}$.
2. If Z is an analytic space and if $f \colon Z \to X$ is a morphism such that $\mathscr{I}$ is contained in Ker $(\mathscr{O}_X \to f_* \mathscr{O}_Z)$ then f factors uniquely through ι.

The Zariski-closed subset Y together with this analytic structure is called the *closed analytic subspace of* X *defined by* $\mathscr{I}$. Its Zariski topology only depends on the set Y, and not on $\mathscr{I}$: this is simply the restriction to Y of the Zariski-topology of X, and we shall call it the Zariski topology of Y.

1.3.13. Remark. — It follows immediately from Gerritzen-Grauert theorem (more precisely from Temkin's version of it for Berkovich spaces, see [**Tem05**], Thm. 3.1) that if Y is a closed analytic subspace of X, and if V is any analytic domain of Y, then V can be G-covered by affinoid domains of the form $U \cap Y$ with U an affinoid domain of X.

1.3.14. — Let k be an analytic field, let X be a k-analytic space and let Y be a Zariski-closed subset of X. We shall denote by Y_L the preimage of Y on X_L; this is a Zariski-closed subset of X_L.

1.3.15. — A morphism $Z \to X$ of analytic spaces is called a *closed immersion* if it induces an isomorphism between Z and a closed analytic subspace of X.

1.3.16. — Let X be an analytic space and let Z be an X-analytic space. We say that the morphism $Z \to X$ is *separated* (or that Z is separated over X) if the diagonal morphism $Z \to Z \times_X Z$ is a closed immersion; it is called *locally separated* if every point of Z has an open neighborhood U such that $U \to X$ is separated.

If k is an analytic field, a k-analytic space X is called separated (resp. locally separated) if the structure map $X \to \mathscr{M}(k)$ is separated (resp. locally separated).

Every embedding of an analytic domain is separated. Every good k-analytic space is locally separated. Every morphism between locally separated k-analytic spaces is locally separated.

A separated analytic space is Hausdorff ([**Ber93**], Prop. 1.4.2) but the converse is not true in general: for instance, the space Y described in Example 3.4.4 below is Hausdorff (and even compact) and not separated.

1.3.17. Reduced analytic spaces. — An analytic space X is said to be *reduced* if $\mathscr{O}_X(V)$ is reduced for every analytic domain V of X; it suffices to check it on every *affinoid* domain of X.

1.3.18. — If A is an affinoid algebra, then $\mathscr{M}(A)$ is reduced if and only if A is reduced ([**BGR84**], 7.3.2 Cor. 10 in the strict case; one can easily reduce to this by using Lemma 1.3 of [**Duc07b**]).

Let us also mention that in general, an element f of A is nilpotent if and only if $f(x) = 0$ for all $x \in \mathscr{M}(A)$.

1.3.19. G-local nature of Zariski topology and the notion of reduced structure. — Let X be an anaytic space and let Y be subset of X which is G-locally a Zariski-closed subset; i.e., there exists a G-covering (X_i) of X by analytic domains such that $Y \cap X_i$. Let $\mathscr{I}$ be the sheaf of ideals defined by the assignment

$$U \mapsto \{f \in \mathscr{O}_X(U), f(y) = 0 \;\forall\, y \in Y \cap U\}.$$

The sheaf $\mathscr{I}$ is coherent ([**Duc09**], Prop. 4.2 (i); the proof rests in the crucial way on the result recalled in 1.3.18), and its zero-locus is equal to Y; hence Y is Zariski-closed. As a consequence, *being Zariski-closed is of G-local nature.*

It follows from the definition that $\mathscr{I}$ is the greatest coherent sheaf of ideals on X with zero-locus Y; if $\mathscr{J}$ another such sheaf, then $\mathscr{I}$ is the radical of $\mathscr{J}$; i.e., the sheaf of functions that are G-locally nilpotent modulo $\mathscr{J}$ (if $Y = X$ we can take $\mathscr{J} = 0$, hence $\mathscr{I}$ is the sheaf of G-locally nilpotent functions).

The closed analytic subspace defined by $\mathscr{I}$ is denoted by Y_{red}. It is reduced and it is the final object of the category of *reduced* analytic spaces Z equipped with a morphism $\varphi\colon Z \to X$ such that $\varphi(Z) \subset Y$.

Instead of writing Y_{red}, we shall sometimes simply write Y after having made precise that we endow it with its *reduced* analytic structure.

1.3.20. Boundary. — Let X be an analytic space and let Y be an X-analytic space. There is a well-defined notion of *boundary* of the morphism $Y \to X$, which is also called the *relative boundary of Y over X* ([**Ber90**], Def. 2.5.7 for the affinoid case, and [**Ber93**], Def. 1.5.4 in general). This is a closed subset of Y which is denoted by $\partial(\varphi)$ or $\partial(Y/X)$; its complement is called the *interior* of φ, or the *relative interior of Y over X*, and is denoted by $\mathrm{Int}(\varphi)$ or $\mathrm{Int}(Y/X)$. The morphism φ is said to be *inner* or *boundaryless* at some point y of Y if $y \in \mathrm{Int}(Y/X)$; it is said to be inner or boundaryless if it is so at every point of Y; i.e., if $\partial(Y/X) = \varnothing$ (Berkovich calls such a map a *closed* morphism, but we shall not use this terminology here: we shall reserve *closed* for denoting morphisms that are *topologically* closed).

We shall write ∂X and $\mathrm{Int}(X)$ instead of $\partial(X/\mathscr{M}(k))$ and $\mathrm{Int}(X/\mathscr{M}(k))$, for k the field of definition of X; we shall call them respectively the boundary and the interior of X. The space X will be called boundaryless if $\partial X = \varnothing$; any boundaryless space is good.

1.3.21. — Let us list here some useful, basic properties of the boundary that will be useful.

(1) If $Y \to X$ is finite, it is boundaryless. Conversely, if $Y \to X$ is boundaryless *and if both Y and X are affinoid*, then $Y \to X$ is finite ([**Ber90**], Cor. 2.5.13).
(2) If Y is an analytic domain of X then $\partial(Y/X)$ is nothing but the *topological* boundary of Y inside X ([**Ber93**], Prop. 1.5.5 (i)).
(3) Assume that $Y \to X$ is locally separated (1.3.16; note that this assumption holds if both Y and X are good, or if Y is an analytic domain of X), let Z be an X-analytic space and let $\sigma\colon Z \to Y$ be an X-morphism. We then have the equality
$$\mathrm{Int}(Z/X) = \mathrm{Int}(\sigma) \cap (\sigma^{-1}(\mathrm{Int}(Y/X))$$
([**Tem04**], Cor. 5.7). In particular, $\sigma(Z) \subset \mathrm{Int}(Y/X)$ as soon as σ is boundaryless; e.g., σ is finite, see (1).
(4) The map $Y \to X$ is no longer assumed to be locally separated. If X' is an arbitrary analytic space (not necessarily k-analytic) and if $X' \to X$ is a morphism, then $\mathrm{Int}(Y/X) \times_X X' \subset \mathrm{Int}(Y \times_X X' \to X')$ (this rests on Prop. 3.1.3 of [**Ber90**]).
(5) The property for a map of being boundaryless is G-local on the target ([**Tem04**], Cor. 5.6; be aware that Temkin uses the word “closed” instead of “boundaryless”).

1.3.22. Proper morphisms. — Let k be an analytic field and let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces. We shall say that φ is *proper* (or that Y is proper over X, or proper X-analytic space) if it satisfies the two following conditions:

1. φ is boundaryless;
2. φ is proper and separated *as a continuous map between topological spaces*, see 1.1.3.

Properness can be checked G-locally on the target; any proper morphism is closed; any finite morphism is proper.

1.3.23. Kiehl's Theorem. — Let $\varphi\colon Y \to X$ be a *proper* morphism of analytic spaces (1.3.22). If $\mathscr{F}$ is any coherent sheaf on Y, then $\mathrm{R}^q\varphi_*\mathscr{F}$ is a coherent sheaf on X for every q; in particular, $\varphi_*\mathscr{F}$ is coherent. This is essentially due to Kiehl who proved it in the rigid-analytic setting, [**Kie67a**]; for the details about the transfer of this result in Berkovich's theory, see [**Duc15**], section 2.

Let Z be a Zariski-closed subset of Y and let $\mathscr{I}$ be the sheaf of ideals on Y that defines the reduced structure on Z. By definition, a section of $\mathscr{O}_Y$ belongs to $\mathscr{I}$ if and only if it vanishes pointwise on Z. Hence a section of $\mathscr{O}_X$ belongs to the kernel $\mathscr{J}$ of $\mathscr{O}_X \to \varphi_*(\mathscr{O}_Y/Z)$ if and only if it vanishes pointwise on $\varphi(Z)$.

By (the Berkovich avatar of) Kiehl's theorem, $\varphi_*(\mathscr{O}_Y/\mathscr{I})$ is a coherent sheaf on X, hence $\mathscr{J}$ is a coherent sheaf of ideals on X. By the above, $\varphi(Z)$ is contained in the zero-locus of $\mathscr{J}$.

On the other hand, since φ proper, it is in particular topologically proper and $\varphi(Z)$ is thus a closed subset of X. Since $\mathscr{J}$ is the sheaf of functions vanishing pointwise on $\varphi(Z)$, we have the equality $\mathscr{J}_{X\setminus\varphi(Z)} = \mathscr{O}_{X\setminus\varphi(Z)}$. Hence the zero-locus of $\mathscr{J}$ is contained in $\varphi(Z)$, and thus equal to $\varphi(Z)$. As a consequence, $\varphi(Z)$ *is a Zariski-closed subset of* X.

1.4. Dimension theory

We recall here some basic facts about the dimension theory of analytic spaces. References are Berkovich's foundational work on the topic [**Ber90**], §2 and the author's paper [**Duc07b**].

1.4.1. Analytic dimension of an affinoid algebra. — Let k be an analytic field and let A be a k-affinoid algebra. Let L be an analytic extension of k such that A_L is strictly L-affinoid. The Krull dimension of A_L does not depend on the choice of L, and is called the *k-analytic dimension* of A; we shall denote it by $\dim_k A$ (it is finite unless $A = 0$, in which case we have $\dim_k A = -\infty$).

1.4.2. — The analytic dimension may actually depend on k, and not only on the ring A. For instance, let $r = (r_1, \ldots, r_n)$ be a k-free polyradius and let L be any analytic extension of k such that $r_i \in |L^\times|$ for every i (e.g., $L = k_r$). The L-algebra $L\widehat{\otimes}_k k_r = L_r$ is then strictly L-affinoid, and of Krull dimension n; one has therefore $\dim_k k_r = n$. On the other hand, k_r is an analytic field and $\dim_{k_r} k_r = 0$.

1.4.3. — We have $\dim_{\mathrm{Krull}} A \leqslant \dim_k A$. Indeed, choose a k-free polyradius r such that $|k_r^\times| \neq \{1\}$ and such that A_r is strictly k_r-affinoid. If B is any k-affinoid algebra that is a domain, then B_r is a domain too ([**Duc07b**], Lemme 1.3). It follows that the pre-image of an irreducible Zariski-closed subset of Spec A under the natural map Spec $A_r \to$ Spec A is still irreducible. On the other hand, Spec $A_r \to$ Spec A is surjective (it is even faithfully flat, [**Ber93**], Lemma 2.1.2), which implies that two distinct subsets of Spec A have distinct pre-images on Spec A_r. As a consequence, $\dim_{\mathrm{Krull}} A \leqslant \dim_{\mathrm{Krull}} A_r = \dim_k A$.

1.4.3.1. — The k-analytic dimension of A is zero if and only if A is a non-zero finite k-algebra ([**Duc07b**], Lemme 1.7).

1.4.4. Dimension of an affinoid space. — If X is a k-affinoid space, its k-analytic dimension $\dim_k X$ is by definition the k-analytic dimension of the corresponding k-affinoid algebra (hence when X is strictly k-affinoid, it coincides with the Krull dimension of X equipped with its Zariski topology). One has $\dim_k V \leqslant \dim_k X$ for every affinoid domain V of X. It follows that $\dim_k X$ is equal to the supremum of $\dim_k V$ for V going through the set of all affinoid domains of X.

1.4.5. Dimension of an arbitrary space. — If X is an arbitrary k-analytic space, we *define* $\dim_k X$ as the supremum of $\dim_k V$ for V going through the set of all affinoid domains of X (this is compatible with the definition given in 1.4.4 in the affinoid case). If V is any analytic domain of such an X then $\dim_k V \leqslant \dim_k X$.

1.4.6. — Let X be a k-analytic space. If $x \in X$, we set $d_k(x) = d_k(\mathscr{H}(x))$ (see 1.2.4). This invariant plays a role similar to that of the transcendence degree in classical dimension theory; indeed, one has

$$\dim_k X = \sup_{x \in X} d_k(x).$$

1.4.7. — Let X be a non-empty k-analytic space. If X consists only of rigid points it follows from 1.4.6 above that $\dim_k X = 0$. Conversely if $\dim_k X = 0$ it follows from 1.4.3.1 (applied to every non-empty affinoid domain of X) that X only consists of rigid points and is topologically discrete.

1.4.8. Behavior of d_k with respect to the Shilov section. — Let X be a k-analytic space and let $r = (r_1, \ldots, r_n)$ be a polyradius. Let $\mathfrak{s} : X \to X_r$ denote the Shilov section (see 1.2.16) and let x be a point of X. It follows from 1.2.15 that $d_{\mathscr{H}(x)}(\mathfrak{s}(x)) = n$; we thus have $d_k(\mathfrak{s}(x)) = d_k(x) + n$.

Let us now assume that r is k-free. We can then write

$$d_k(\mathfrak{s}(x)) = d_{k_r}(\mathfrak{s}(x)) + d_k(k_r) = d_{k_r}(\mathfrak{s}(x)) + n$$

(because $d_k(k_r) = n$ by 1.2.15), whence the equality

$$d_{k_r}(\mathfrak{s}(x)) = d_k(x).$$

1.4.9. Local dimension. — There is also a notion of *local* k-analytic dimension: if X is a k-analytic space, then for every $x \in X$ one defines the k-analytic dimension of X *at* x as the infimum of $\dim_k V$ for V running through the set of analytic domains of X containing x; we denote it by $\dim_{k,x} X$. If V is any analytic domain of X containing x then $\dim_{k,x} V = \dim_{k,x} X$. For $n \in \mathbf{Z}_{\geqslant 0}$, we will say that X is *of pure dimension* n if $\dim_{k,x} X = n$ for every $x \in X$; if X is of pure dimension n, so is any analytic domain of X.

We shall say that X is *equidimensional* if it is of pure dimension n for some n. Such an n is necessarily equal to $\dim_k X$ if $X \neq \varnothing$; but $\varnothing$ is equidimensional, of pure dimension n for all n, and of dimension $-\infty$.

1.4.10. Abhyankar points. — Let X be a k-analytic space and let x be a point of X. In view of 1.4.6 we have $d_k(x) \leqslant \dim_{k,x} X$. It can be seen as an analytic avatar of the classical Abhyankar inequality stated in A.4.11 (2). For that reason, we shall say that x is an *Abhyankar point* if $d_k(x) = \dim_{k,x} X$.

1.4.11. — Let X be a k-analytic space and let Y be a Zariski-closed subset of X. Let $\mathscr{I}$ be any coherent sheaf of ideals on X whose zero locus is equal to Y. The k-analytic dimension of the closed analytic subspace of X defined by $\mathscr{I}$ only depends on Y, and not of the chosen ideal sheaf $\mathscr{I}$; the same holds for its k-analytic dimension at a given point x of Y. We will then use the expressions "k-analytic dimension of Y" and "k-analytic dimension of Y at x", and the notation $\dim_k Y$ and $\dim_{k,x} Y$, without fixing any k-analytic structure on Y. One has $\dim_k Y \leqslant \dim_k X$ and $\dim_{k,x} Y \leqslant \dim_{k,x} X$.

Let (X_i) be any *set-theoretic* covering of X by Zariski-closed subsets of analytic domains. One has

$$\dim X = \sup_{x \in X} d_k(x) = \sup_i \sup_{x \in X_i} d_k(x) = \sup_i \dim X_i.$$

1.4.12. — Analytic dimension behaves well under ground field extension: if X is a k-analytic space and L is an analytic extension of k, then $\dim_L X_L = \dim_k X$, and $\dim_{L,y} X_L = \dim_{k,x} X$ for every $x \in X$ and every pre-image y of x on X_L.

1.4.13. — When the ground field is clearly understood from the context, we shall omit it in the notation. For instance, if X is an analytic space and if x is a point of X, then $\dim X$ and $\dim_x X$ will denote analytic dimensions over the analytic field that is implicitly part of the definition of X. If Y is an X-analytic space and if y is a point of Y, then $\dim Y_x$ and $\dim_y Y_x$ will denote $\mathscr{H}(x)$-analytic dimensions. The integer $\dim_y Y_x$ will be called the *relative dimension of Y over X at y*; if the morphism $Y \to X$ has been given a name, say φ, we shall also write $\dim_y \varphi$ instead of $\dim_y Y_x$, and call it the *dimension of φ at y*. The $\mathbf{Z}_{\geqslant 0}$-valued function $y \mapsto \dim_y \varphi$ is upper semi-continuous for the Zariski topology of T ([**Duc07b**], Thm. 4.9). If n is a non-negative integer, we shall say that Y is *of pure relative dimension n over X*, or that φ is *of pure dimension n*, if $\dim_y \varphi = n$ for every $y \in Y$ or, which amounts to the same, if all fibers of φ are of pure dimension n. We shall say that Y is *relatively equidimensional over X*, or that φ *is equidimensional*, if φ is of pure dimension n for some $n \in \mathbf{Z}_{\geqslant 0}$; the integer n is then uniquely determined as soon as $Y \neq \varnothing$: this is the common dimension of all non-empty fibers of φ.

1.4.14. — The formula stated in 1.4.6 enables to relate, to some extent, the dimensions of the source, of the target, and of the fibers of a given map. For instance, let let $\varphi \colon Y \to X$ be a morphism of k-analytic spaces. For every $x \in X$, the equality $\sup_{y \in \varphi^{-1}(x)} d_{\mathscr{H}(x)}(y) = \dim \varphi^{-1}(x)$ implies that

$$\dim Y \geqslant \sup_{y \in \varphi^{-1}(x)} d_k(y) = \sup_{y \in \varphi^{-1}(x)} \left[d_{\mathscr{H}(x)}(y) + d_k(x)\right] = \dim \varphi^{-1}(x) + d_k(x).$$

This has the following consequences:

(1) For every $x \in X$, one has $\dim \varphi^{-1}(x) \leqslant \dim Y$.

(2) If d is a non-negative integer such that $\dim \varphi^{-1}(x) \leqslant d$ for every $x \in X$, then

$$\dim Y = \sup_{y \in Y} d_k(y) \leqslant d + \sup_{x \in \varphi(Y)} d_k(x) \leqslant d + \dim X.$$

(3) If d is a non-negative integer such that $\dim \varphi^{-1}(x) = d$ for every $x \in \varphi(Y)$, and $\varphi(Y)$ is a Zariski-closed subset of an analytic domain of X, then

$$\dim Y = d + \sup_{x \in \varphi(Y)} d_k(x) = d + \dim \varphi(Y)$$

(the assumption that $\varphi(Y)$ is Zariski-closed in an analytic domain of X simply ensures that $\dim \varphi(Y)$ *makes sense*; of course, the above formula remains valid without any assumption on $\varphi(Y)$ if we *define* $\dim \varphi(Y)$ as $\sup_{x \in \varphi(Y)} d_k(x)$).

(4) If y is a point of Y such that $d_k(y) = \dim Y$ (such a point exists if and only if Y is finite-dimensional), and if x denotes its image in X, then

$$d_k(x) = \dim Y - \dim \varphi^{-1}(x).$$

1.4.15. Quasi-finite morphisms. — Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces. If y is a point of y, we shall say that φ is *quasi-finite* at y (or that Y is quasi-finite at y over X) if $\dim_y \varphi = 0$. The morphism φ is then finite at y if and only if it is quasi-finite and boundaryless at y ([**Ber93**], Cor. 3.1.10).

We shall say that φ is *quasi-finite* (or that Y is quasi-finite over X) if φ is topologically proper and quasi-finite at every point of y; i.e., φ is topologically proper and of pure relative dimension zero.

A quasi-finite map is finite if and only if it is proper (which means that it is boundaryless and topologically separated, since it is already topologically proper by definition).

1.4.16. Remark. — The reader should be aware that our definition of quasi-finiteness *differs from Berkovich's.* Indeed, Berkovich uses the expression "quasi-finite at the point y" for "finite at the point y" in our sense (1.2.17). We have chosen to depart from Berkovich's definition because we want a quasi-étale map (see Definition 5.2.6) to be quasi-finite at every point of the source space, and also for the sake of analogy with scheme theory (see for instance Lemma 8.4.5, Theorem 8.4.6, Theorem 9.1.2 or Theorem 9.1.3).

1.5. Irreducible components

The Zariski topology of an analytic space is far from being noetherian in general, but there is nevertheless a reasonable theory of irreducible components in this settting. Brian Conrad developped in it the rigid analytic framework in [**Con99**]. The author suggested another approach in [**Duc09**], §4, which works for arbitrary analytic spaces and is perhaps more direct (it does not make any use of the normalization, contrary to [**Con99**]); we are now going to describe it.

1.5.1. Irreducible analytic spaces. — Let X be an analytic space. A Zariski-closed subset of X will be called *irreducible* if it is irreducible for the Zariski topology of X, and X will be called *integral* if it is both irreducible and reduced. If Y is an irreducible Zariski-closed subset of X, it is purely d-dimensional for some d and if Z is any Zariski-closed subset of Y with $Z \neq Y$ then $\dim Z < d$.

1.5.2. Definition. — Let X be an analytic space. There exists a set E of irreducible Zariski-closed subsets of X having the following properties:

- The set E is G-locally finite; i.e., any affinoid domain of X intersects only finitely many elements of E.
- One has $X = \bigcup_{Z \in E} Z$.
- If Y and Z are two elements of E with $Y \subset Z$ then $Y = Z$.

The set E is uniquely determined by those properties. Its elements are exactly the maximal irreducible Zariski-closed subsets of X; moreover, every irreducible Zariski-closed subset of X is contained in one element of E. The elements of E are called the *irreducible components* of X.

1.5.3. — If $X = \mathscr{M}(A)$ is an affinoid space, an irreducible component of X is nothing but the pre-image of an irreducible component of Spec A. In general, i.e., if X is no longer assumed to be affinoid, for every Zariski-closed subset Y of X the following are equivalent:

(i) Y is an irreducible component of X.
(ii) There exist an affinoid domain V of X and an irreducible component Z of V such that $Y = \overline{Z}^{X_{\mathrm{Zar}}}$.

1.5.4. — Let X be an analytic space and let Y be a Zariski-closed subset of X; let $\mathscr{I}$ be a coherent sheaf of ideals with zero locus Y. The irreducible components of the closed analytic subspace of X defined by $\mathscr{I}$ can be characterized purely in terms of the Zariski topology of Y; therefore they only depend on the set Y, and not on $\mathscr{I}$; we shall call them the irreducible components of Y.

For instance, if E is a subset of the set of irreducible components of X, then then $\bigcup_{Z \in E} Z$ is a Zariski-closed subset of X, whose irreducible components are precisely the elements of E.

1.5.5. — Let X be an analytic space, let V be an analytic domain of X, and let d be a non-negative integer.

- If Y is an irreducible component of X of dimension d, then $Y \cap V$ is a union (possibly empty, possibly infinite) of irreducible components of V, each of which has dimension d.
- If Z is an irreducible component of V of dimension d, then $\overline{Z}^{X_{\mathrm{Zar}}}$ is an irreducible component of X, of dimension d.

1.5.6. — Let k be an analytic field, let L be an analytic extension of k and let Y be an irreducible component of a k-analytic space X. The Zariski-closed subset Y_L of X_L has finitely many irreducible components. For every such component Z we have $\dim Z = \dim Y$, the natural map $Z \to Y$ is surjective, and Z is equal to $T \times_F L$ for some finite separable extention F of k inside L and some irreducible component T of Y_F; moreover Z is an irreducible component of X_L.

Conversely, if Z is an irreducible component of X_L then its image Y in X is an irreducible component of X and Z is an irreducible component of Y_L.

1.5.7. — If X is an analytic space, then $\dim X = \sup_Z \dim Z$ for Z running through the set of irreducible components of X. For every $x \in X$ the local dimension $\dim_x X$ is equal to $\max_Z \dim Z$ for Z running through the (finite) set of irreducible components of X containing x.

1.5.8. Remark. — Let x be a point of an analytic space X and let E (resp. F) be the set of irreducible components of X that contain (resp. avoid) x; let Y (resp. Z) be the union of all components belonging to E (resp. F). Both Y and Z are Zariski-closed subsets of X. The dimension of Y being the supremum of the dimensions of all components belonging to E, it is equal to $\dim_x X$. Now $U := Y\backslash Z$ is a Zariski open subset of X that contains x, that is contained in Z, and that intersects every irreducible component of Z. It follows that $\dim U = \dim Y = \dim_x X$.

1.5.9. Remark. — Let X be an analytic space and met x be a point of X. Assume that x lies on a Zariski-closed subset Y of X and is Abhyankar *in* Y (1.4.10). Since $\dim \overline{\{x\}}^{X_{\mathrm{Zar}}} \geqslant d_k(x)$ and since $d_k(x) = \dim_x Y$ is the maximum of the dimensions of the irreducible components of Y containing x, we see that $\overline{\{x\}}^{X_{\mathrm{Zar}}}$ is an irreducible component of Y of dimension $\dim_x Y = d_k(x)$; note that as x is Zariski-dense in $\overline{\{x\}}^{X_{\mathrm{Zar}}}$, the latter is even the *only* irreducible component of Y containing x.

1.5.10. — Let $\varphi\colon Y \to X$ be a finite morphism of analytic spaces. Let (Y_i) be the family of irreducible components of Y. For every i, the image $\varphi(Y_i)$ is an irreducible Zariski-closed subset of X, and $\dim \varphi(Y_i) = \dim Y_i$ by 1.4.14. Since φ is topologically proper, the family $(\varphi(Y_i))$ is G-locally finite; it follows that the irreducible components of $\varphi(Y)$ are precisely the maximal elements among the $\varphi(Y_i)$'s.

Now let x be a point of X having exactly one pre-image y on Y; let J be the set of indices i such that $y \in Y_i$. By the above, the irreducible components of $\varphi(Y)$ that contain x are exactly the maximal elements among the $\varphi(Y_i)$'s for i running through J. Since $\dim \varphi(Y_i) = \dim Y_i$ for every i, this implies that

$$\dim_x \varphi(Y) = \max_{i \in J} \dim Y_i = \dim_y Y.$$

1.5.11. Lemma. — *Let n and m be two integers, and let $Y \to X$ be a morphism of k-analytic spaces, with Y of pure dimension m and X of dimension n. Let x be a point of X such that $d_k(x) = n$. The fiber Y_x is then purely of dimension $m - n$.*

Proof. — We may assume that Y and X are affinoid. Let T be an irreducible component of Y_x, let d be its dimension, and let y be an Abhyankar point of T. This condition implies that T is the only irreducible component of Y_x that contains y (Remark 1.5.9); as a consequence, $\dim_y Y_x = d$. One has $d_k(y) = d_{\mathscr{H}(x)}(y) + d = n + d$, whence the inequality $n + d \leqslant m$. It suffices to prove the reverse inequality.

Since $\dim_y Y_x = d$, the map $Y \to X$ admits by Thm. 4.6 of [**Duc07b**] a factorization $Y \to \mathbf{A}^d_X \to X$ whose first step is quasi-finite at y. By Thm. 4.9 (or, more simply, Thm. 3.2) of [**Duc07b**], there exists an affinoid neighborhood V of y in Y such that $V \to \mathbf{A}^d_X$ is quasi-finite.

Since Y is purely m-dimensional, its non-empty affinoid domain V is m-dimensional. Therefore, there exists $v \in V$ with $d_k(v) = m$. Let z be the image of v on $\mathbf{A}^d_X$, and let t be the image of z on X. The morphism $V \to \mathbf{A}^d_X$ being quasi-finite, one has

$$m = d_k(v) = d_{\mathscr{H}(z)}(v) + d_k(z) = d_k(z) = d_{\mathscr{H}(t)}(z) + d_k(t) \leqslant d + n.$$

□

1.5.12. Lemma. — *Let X be an analytic space, let U be a* Zariski-*open subset of X and let x be a point of X ; set $F = X \backslash U$. The following are equivalent:*

(i) *The point x belongs to $\overline{U}^X$.*
(ii) *The point x belongs to $\overline{U}^{X_{\mathrm{Zar}}}$.*
(iii) *The Zariski-open subset U intersects at least one of the irreducible components of X that contain x;*
(iv) *There exists an irreducible component Z of X that contains x and satisfies* $\dim(F \cap Z) < \dim Z$.

Proof. — It is clear that (i)⇒(ii). Assume that (ii) is true. Let X' be the union of all irreducible components of X that do not contain x. It is a Zariski-closed subset of X. Its complement $X \backslash X'$ is then a Zariski-open neighborhood of x, hence it instersects U by assumption (ii), whence (iii).

Suppose that (iii) is true. Let Z be an irreducible component of X containing x and intersecting U. The intersection $F \cap Z$ is then a proper Zariski-closed subspace of the irreducible analytic space Z; it follows that $\dim(F \cap Z) < \dim Z$.

Assume that (iv) is true and let V be an open neighborhood of x. The intersection $Z \cap V$ cannot be contained in $F \cap Z$ because it is of dimension $\dim Z$. Therefore $V \cap U \neq \varnothing$, whence (i). □

1.5.13. Corollary. — *Let X be an analytic space, let x be a point of X, let V be an analytic domain of X containing x, and let U be a Zariski-open subset of X. The point x belongs to $\overline{U}^X$ if and only if $x \in \overline{(U \cap V)}^V$.*

Proof. — If $x \in \overline{(U \cap V)}^V$ it is obvious that $x \in \overline{U}^X$. Assume that $x \in \overline{U}^X$, and let Z be as in assertion (iv) of Lemma 1.5.12 above. Let T be an irreducible component of $Z \cap V$ containing x. It is of dimension $\dim Z$. Therefore $\dim(F \cap T) < \dim T$, and $x \in \overline{U \cap V}^V$ by Lemma 1.5.12. □

1.5.14. Corollary. — *Let X be a k-analytic space, let L be an analytic extension of k, let x be a point of X and let y be a point of X_L lying above x. Let U be a*

Zariski-open subset of X. One has the equivalence

$$x \in \overline{U}^X \iff y \in \overline{U_L}^{X_L}.$$

Proof. — If y belongs to $\overline{U_L}^{X_L}$, then x obviously belongs to $\overline{U}^X$. Assume conversely that $x \in \overline{U}^X$, and let Z be as in assertion (iv) of Lemma 1.5.12 above. Let T be an irreducible component of Z_L containing y. Since Z is equidimensional, T is of dimension $\dim Z$. Therefore $\dim(F_L \cap T) < \dim T$, and $y \in \overline{U_L}^{X_L}$ by Lemma 1.5.12. □

1.5.15. Codimension. — Let X be a k-analytic space and let Y be a Zariski-closed subset of X. The *codimension* $\operatorname{codim}(Y, X)$ of Y in X is defined as follows.

- If both Y and X are irreducible, $\operatorname{codim}(Y, X) = \dim X - \dim Y$.
- If Y is irreducible, $\operatorname{codim}(Y, X) = \sup_Z \operatorname{codim}(Y, Z)$, where Z varies through the set of irreducible components of X that contain Y.
- In the general case, $\operatorname{codim}(Y, X) = \inf_Z \operatorname{codim}(Z, X)$ where Z varies through the set of irreducible components of Y.

It is trivially checked that these definitions are consistent with each other. If X and Y are non-empty and equidimensional, then $\operatorname{codim}(Y, X) = \dim X - \dim Y$, generalizing the formula which defined codimension when X and Y are each irreducible.

If $x \in X$, we define the codimension of Y in X *at* x as being equal to $\inf_Z \operatorname{codim}(Z, X)$ where Z varies through the set of irreducible components of Y *that contain* x; it is denoted by $\operatorname{codim}_x(Y, X)$. Note that $\operatorname{codim}_x(Y, X)$ does make sense even if $x \notin Y$, and that one has in this case $\operatorname{codim}_x(Y, X) = +\infty$ by the definition.

1.5.16. — Let us now list some basic properties of the codimension.

(1) Let X be an analytic space, let Y be a Zariski-closed subset of X and let x be a point of X. If V is any analytic domain of X containing x, it follows from 1.5.5 that $\operatorname{codim}_x(V \cap Y, V) = \operatorname{codim}_x(Y, X)$.

(2) Let $X = \mathscr{M}(A)$ be an affinoid space and let Y be a Zariski-closed subset of X defined by an ideal I of A. Let x be a point of X and let ξ denote its image in Spec A. By [**Duc07b**], Prop. 1.11 we have the following equalities:

  (2a) $\operatorname{codim}(Y, X) = \operatorname{codim}(\operatorname{Spec}(A/I), \operatorname{Spec} A)$.

  (2b) $\operatorname{codim}_x(Y, X) = \operatorname{codim}_\xi(\operatorname{Spec}(A/I), \operatorname{Spec} A)$

CHAPTER 2

ALGEBRAIC PROPERTIES IN ANALYTIC GEOMETRY

This chapter is devoted to a general study of algebraic properties (like being regular, Gorenstein, Cohen-Macaulay...) in analytic geometry. Section 2.1 provides some reminders about the analytification of a scheme of finite type over an affinoid agebra, and about algebraic properties of (local and global) rings of analytic functions. As a first application, it describes an elementary procedure (2.1.6) which we shall use quite often to reduce the algebraic study of local analytic rings to that of affinoid spaces.

Section 2.2–2.6 may appear slightly unattractive. Their motivation is the following: since we will have to deal with several kind of objects "living on an analytic space" (like the analytic space itself, coherent sheaves, diagrams in the category of coherent sheaves, etc.) and with several properties, we have chosen to introduce a rather abstract framework, consisting of objects and properties satisfying some axioms. From our viewpoint, this offers three advantages:

- This allows us to write proofs once for all, and not to repeat them for every kind of object and/or property of interest.
- This emphasizes which arguments are actually needed for proofs.
- This could be potentially applied to other objects and properties.

But it might of course be unpleasant or boring to read. For that reason, every important definition and statement has been given a concrete counterpart involving only explicit objects and properties, to which readers can directly refer if they prefer to avoid considering our dry formalism.

More precisely, sections 2.2 and 2.3 are essentially devoted to the presentation of the abstract framework alluded to above. Then in section 2.4, we explain what it means for one of the properties we consider to hold at a point of an analytic space; e.g., see the "concrete" Lemma-Definition 2.4.3; the point is that one cannot use local rings of arbitrary (i.e., not necessarily good) analytic spaces, hence the definition has to be given a G-local flavor, which requires checking some compatibilities of restriction to analytic domains. Thereafter we establish GAGA results about those properties;

see the "concrete" Lemma 2.4.6. And we show that some of them have a Zariski-open locus of validity, sometimes automatically non-empty whenever the ambient space is reduced; see the "concrete" Lemma 2.4.9.

In section 2.5, we investigate the validity at a point of some usual properties of a morphism of coherent sheaves (like injectivity, surjectivity, and bijectivity), and it almost does not involve our general abstract setting. We prove that surjectivity can be checked at the level of fibers (2.5.4; this is a straightforward consequence of Nakayama's Lemma), and that our notions of injectivity, surjectivity and bijectivity are compatible with the usual ones in sheaf theory (2.5.5).

In section 2.6 we go back to our general formalism in order to study the behavior of algebraic properties under ground field extension; some of them are preserved by arbitrary such extensions, some of them (essentially, those that involve regularity) only by *analytically separable* extensions; see the "concrete" Proposition 2.6.7, and Definition 2.6.1 for the notion of an analytically separable extension.

The final section of this chapter (2.7) is essentially independent of the preceding ones. It aims at extending some GAGA results which are known for affinoid spaces, but not for arbitrary finitely generated scheme over an affinoid algebra (or at least, they are not available in the literature in such generality). For instance, we get GAGA principles for local dimension (Lemma 2.7.6; note that it only works in the strict case), for codimension (Lemma 2.7.10 (3)), for normalization (Lemma 2.7.15 (2)), and for irreducible components (Proposition 2.7.16).

2.1. Analytification of schemes, algebraic properties of analytic rings

2.1.1. — Let X be an affinoid space, say $X = \mathscr{M}(A)$. We shall denote the scheme $\operatorname{Spec} A$ by X^{al} (here "al" stands for *algebraic*). Let $\mathscr{X}$ be an X^{al}-scheme locally of finite type. The category of good X-analytic spaces Y endowed with a morphism *of locally ringed spaces* $Y \to \mathscr{X}$ making the diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathscr{X} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^{\mathrm{al}} \end{array}$$

commute admits a final object, which is denoted by $\mathscr{X}^{\mathrm{an}}$ and is called the *analytification* of $\mathscr{X}$; the canonical map $\mathscr{X}^{\mathrm{an}} \to \mathscr{X}$ is surjective, and the analytic space $\mathscr{X}^{\mathrm{an}}$ is relatively boundaryless over X ([**Ber93**], Prop. 2.6.2). The space $\mathscr{X}^{\mathrm{an}}$ is Hausdorff, resp. compact, if and only if the X^{al}-scheme $\mathscr{X}$ is separated, resp. proper ([**Ber93**], Cor. 2.6.7 and Prop. 2.6.9). The assignment $\mathscr{X} \mapsto \mathscr{X}^{\mathrm{an}}$ is functorial in $\mathscr{X}$. Note that $(X^{\mathrm{al}})^{\mathrm{an}} = X$. This construction commutes with affinoid base change: if Z is an affinoid space and if $Z \to X$ is a morphism, then $\mathscr{X}^{\mathrm{an}} \times_X Z = (\mathscr{X} \times_{X^{\mathrm{al}}} Z^{\mathrm{al}})^{\mathrm{an}}$.

If x is a point of $\mathscr{X}^{\mathrm{an}}$, its image in $\mathscr{X}$ will be denoted by x^{al}; if F is a subset of $\mathscr{X}^{\mathrm{an}}$, its image in $\mathscr{X}$ will be denoted by F^{al}. If $\mathscr{Y}$ is a closed (resp. open) subscheme of $\mathscr{X}$, then $\mathscr{Y}^{\mathrm{an}}$ is closed analytic subspace (resp. an open subspace) of $\mathscr{X}^{\mathrm{an}}$, which is set-theoretically equal to the pre-image of $\mathscr{Y}$ on $\mathscr{X}^{\mathrm{an}}$. For that reason, we shall more generally denote by E^{an} the pre-image on $\mathscr{X}^{\mathrm{an}}$ of any subset E of $\mathscr{X}$. By surjectivity of $\mathscr{X}^{\mathrm{an}} \to \mathscr{X}$, we have $(E^{\mathrm{an}})^{\mathrm{al}} = E$ for every subset E of $\mathscr{X}$, so the assignment $E \mapsto E^{\mathrm{an}}$ is injective. If $\mathscr{F}$ is a coherent sheaf on $\mathscr{X}$, its pull-back on $\mathscr{X}^{\mathrm{an}}$ will be denoted by $\mathscr{F}^{\mathrm{an}}$.

If $\mathscr{X}$ is proper over X^{al}, then non-Archimedean GAGA holds (cf. for instance [**Poi10**], Annexe A; the case where $\mathscr{X} = \operatorname{Spec} A$ is essentially due to Tate and Kiehl, see 1.3.2): the functor $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}}$ induces an equivalence, which moreover preserves cohomology, between the category of coherent sheaves on $\mathscr{X}$ and that of coherent sheaves on $\mathscr{X}^{\mathrm{an}}$; we shall denote by $\mathscr{G} \to \mathscr{G}^{\mathrm{al}}$ a quasi-inverse of the latter. Therefore $\mathscr{Y} \mapsto \mathscr{Y}^{\mathrm{an}}$ induces a bijection between the set of closed closed subschemes of $\mathscr{X}$ and the set of closed analytic subspaces of $\mathscr{X}^{\mathrm{an}}$. The inverse bijection will be denoted by $Y \mapsto Y^{\mathrm{al}}$. This implies that any Zariski-closed subset of $\mathscr{X}$ is of the form E^{an} for some Zariski-closed subset E of $\mathscr{X}$. By injectivity of $E \mapsto E^{\mathrm{an}}$, it follows that $E \mapsto E^{\mathrm{an}}$ induces a bijection between the set of Zariski-closed subsets of $\mathscr{X}$ and that of Zariski-closed subsets of $\mathscr{X}^{\mathrm{an}}$; the converse bijection is induced by the assignment $F \mapsto F^{\mathrm{al}}$.

2.1.2. — Let X be an analytic space, let x be a point of X and let $\mathscr{F}$ be a coherent sheaf on X. Let $T \to X$ be a morphism of analytic spaces, with T affinoid.

(1) We shall write $\mathscr{F}_T^{\mathrm{al}}$ instead of $(\mathscr{F}_T)^{\mathrm{al}}$; this should not cause any confusion.
(2) If V is an affinoid domain of X containing x we shall denote by x_V^{al} the image of x on V^{al} (this is consistent with the notation x_V introduced in Remark 1.3.8 to indicate that x is viewed as belonging to V).
(3) If X is affinoid and the affinoid space T is X-analytic, we shall write T_x^{al} instead of $(T_x)^{\mathrm{al}}$; this should not cause any confusion. We shall denote by t_x^{al} the image of t on T_x^{al} (this is consistent with the notation t_x introduced in Remark 1.3.8 to indicate that t is viewed as belonging to the fiber T_x). We shall use (in acccordance with our general conventions in scheme theory) the notation $T_{x^{\mathrm{al}}}^{\mathrm{al}}$ for the scheme-theoretic fiber of $T^{\mathrm{al}} \to X^{\mathrm{al}}$ at x^{al}.

2.1.3. Algebraic properties of analytic rings. — Let k be an analytic field and let L be an analytic extension of k. Let A be a k-affinoid algebra, let B be the algebra of analytic functions on some affinoid domain of $\mathscr{M}(A)$. Let X be a good k-analytic space, let V be a good analytic domain of X, and let x be a point of V.

(1) The ring A is excellent ([**Duc09**], Thm. 2.13; the strictly affinoid case is due to Kiehl [**Kie69**]).

(2) The A-algebra B is regular ([**Duc09**], Thm. 3.3; flatness follows from [**Ber90**], Prop. 2.2.4 and from [**BGR84**], Cor. 6 of §7.3.2 in the strict case).
(3) The A-algebra A_L is faithfully flat ([**Ber93**], Lemma 2.1.2).
(4) The local ring $\mathscr{O}_{X,x}$ is noetherian, henselian ([**Ber93**], Thm. 2.1.4 and Thm. 2.1.5), and excellent ([**Duc09**], Thm. 2.13).
(5) The morphism $\mathscr{O}_{X,x} \to \mathscr{O}_{V,x}$ is regular ([**Duc09**], Thm. 3.3; flatness is a straightforward consequence of statement (2) above).
(6) The morphism $X_L \to X$ is flat when viewed as a morphism of locally ringed spaces ([**Ber93**], Cor. 2.1.3).

2.1.4. Regularity of analytification. — Let $\mathscr{X}$ be a scheme locally of finite type over an affinoid algebra. For every $x \in \mathscr{X}^{\mathrm{an}}$, the morphism $\mathscr{O}_{\mathscr{X},x^{\mathrm{al}}} \to \mathscr{O}_{\mathscr{X}^{\mathrm{an}},x}$ is regular ([**Duc09**], Thm. 3.3; flatness is due to Berkovich, [**Ber93**], Prop. 2.6.2).

2.1.5. — Let X be an affinoid space and let x be a point of X. Faithful flatness of $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}} \to \mathscr{O}_{X,x}$ has the following consequences.

(1) Assume that $\mathscr{O}_{X,x}$ is a domain. Being a subring of $\mathscr{O}_{X,x}$, the local ring $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$ is a domain too. This implies that x^{al} lies on a unique irreducible component $\mathscr{X}$ of X^{al}; hence x lies on a unique irreducible component of X, namely $\mathscr{X}^{\mathrm{an}}$ (for another proof of this fact, see [**Duc09**], Lemme 0.11).
(2) Assume moreover that the domain $\mathscr{O}_{X,x}$ is a field. By surjectivity of the map $\operatorname{Spec} \mathscr{O}_{X,x} \to \operatorname{Spec} \mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$, the scheme $\operatorname{Spec} \mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$ consists of one point; therefore the domain $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$ is a field and x^{al} is the generic point of $\mathscr{X}$.

2.1.6. Approximation of finite algebras. — Let X be a good analytic space and let x be a point of X. Let B be a finite $\mathscr{O}_{X,x}$-algebra. Since $\mathscr{O}_{X,x}$ is noetherian, B is finitely presented. Therefore there exists an affinoid neighborhood X' of x in X and a finite $\mathscr{O}_X(X')$-algebra R such that $B = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_X(X')} R$. The affinoid space $Z := \mathscr{M}(R)$ is finite over X'. We shall say that B is *induced* by the finite X'-analytic space Z; if B is a quotient of $\mathscr{O}_{X,x}$, we can chose X' and R so that R is a quotient of $\mathscr{O}_X(X')$, and Z is then a closed analytic subspace of X'. If $z_1, \ldots, z_r$ denote the pre-images of x on Z, one has $B = \prod_i \mathscr{O}_{Z,z_i}$ because $Z \to X'$ is topologically closed (for details see [**Ber93**], Lemma 2.1.6, which is the key point for proving that $\mathscr{O}_{X,x}$ is henselian).

Assume moreover that B is a domain. The point x has thus only one pre-image z on Z, and since $\mathscr{O}_{Z,z} = B$ is a domain, it follows from 2.1.5 (1) that z lies on a unique irreducible component T of Z. Let $\mathscr{I}$ be the sheaf of ideals on Z that defines T_{red}. Since T is the unique irreducible component of Z containing z, this is a neighborhood of z in Z; therefore, any section of $\mathscr{I}$ vanishes pointwise in a neighborhood of z, hence is nilpotent in $\mathscr{O}_{Z,z}$, which is a domain. As a consequence, $\mathscr{I}_z = 0$ and the closed analytic subspace T_{red} of Z induces the identity map $\mathrm{Id}_{\mathscr{O}_{Z,z}}$. Hence the finite X'-space T_{red} still induces the finite $\mathscr{O}_{X,x}$-algebra B.

Therefore any finite $\mathscr{O}_{X,x}$-algebra which is a domain (resp. any quotient of $\mathscr{O}_{X,x}$ by a prime ideal) can be induced by an *integral* finite X'-space (resp. an *integral* closed analytic subspace of X') for some affinoid neighborhood X' of X.

2.2. A rather abstract categorical framework

In this memoir we deal with with various kinds of geometric objects: analytic spaces equipped with their G-topology and the corresponding structure sheaf, schemes of finite type over an affinoid algebra, (spectra of) local rings of good analytic spaces, etc., and the goal of this section is to define a category $\mathfrak{T}$ that encompass all of them.

Technically speaking, all aforementioned objects are locally ringed toposes, and all relevant morphisms between them are morphisms of locally ringed toposes (the reader should not be afraid about that: one does not need to know what a topos is, let alone a morphism of toposes; we will only use these concepts here as unifying terminology). This leads to the following definition of $\mathfrak{T}$.

2.2.1. Definition. — We denote by $\mathfrak{T}$ the smallest subcategory of the category of locally ringed toposes such that the following hold (we allow ourselves to write for short that a given object, resp. arrow, *belongs* to $\mathfrak{T}$):

(1) If X is an analytic space, then X (viewed as equipped with its G-topology and the corresponding structure sheaf) belongs to $\mathfrak{T}$.
(2) Let A be a ring which is either an affinoid algebra or of the form $\mathscr{O}_{X,x}$ with X a good analytic space and x a point of X. If B is any A-algebra essentially of finite type, the affine scheme Spec B belongs to $\mathfrak{T}$.
(3) Let $\mathscr{X}$ be a scheme. If it admits a covering by open subschemes that belong to $\mathfrak{T}$, then $\mathscr{X}$ belongs to $\mathfrak{T}$.
(4) Any morphism $Y \to X$ between analytic spaces belongs to $\mathfrak{T}$.
(5) Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of schemes. If both $\mathscr{Y}$ and $\mathscr{X}$ belong to $\mathfrak{T}$, then $\mathscr{Y} \to \mathscr{X}$ belongs to $\mathfrak{T}$.
(6) Let $\mathscr{X}$ be a scheme of finite type over an affinoid algebra. The morphism of locally ringed toposes $\mathscr{X}^{\mathrm{an}} \to \mathscr{X}$ belongs to $\mathfrak{T}$.

2.2.2. Remark. — By construction, all objects of $\mathfrak{T}$ have a coherent structure sheaf, hence admit a nice theory of coherent sheaves.

2.2.3. Remark. — The only schemes that are required to belong to $\mathfrak{T}$ are those mentioned in (2) and (3). Therefore a given scheme belongs to $\mathfrak{T}$ if and only if it admits an open covering by affine schemes of the form described in (2).

Note that any field can be seen as an affinoid algebra (once equipped with the trivial absolute value); hence any scheme of finite type over a field belongs to $\mathfrak{T}$.

Let A be a complete equicharacteristic local notherian ring. It is a quotient of a formal power series ring $k[\![T_1, \ldots, T_n]\!]$ (cf. [**Mat86**], Thm. 28.3 and the discussion at the beginning of §29). Since the latter can be seen as an affinoid algebra over the trivially valued field k (it is isomorphic to $k\{T_1/r_1, \ldots, T_n/r_n\}$ as soon as all r_i's are smaller than 1), the affine scheme Spec A belongs to $\mathfrak{T}$.

2.2.4. — Let us list some consequences of Remark 2.2.3 above.

(1) Any scheme belonging to $\mathfrak{T}$ is excellent, and even locally embeddable into a *regular* excellent scheme. Indeed if k is an analytic field and X a k-affinoid space, then X admits a closed immersion into some compact polydisc Y over k. Since the local rings of Y as well as those of Y^{al} are regular (for an elementary proof, see [**Duc09**], Lemme 2.1), the scheme X^{al} admits a closed immersion into a regular excellent scheme, and every local ring of X is the quotient of an excellent regular local ring, whence our claim.

(2) Let $\mathscr{X}$ be a scheme. If $\mathscr{X}$ belongs to $\mathfrak{T}$, every $\mathscr{X}$-scheme locally of finite type belongs to $\mathfrak{T}$, and Spec $\mathscr{O}_{\mathscr{X},x}$ belongs to $\mathfrak{T}$ for every point x of $\mathscr{X}$.

(3) Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of schemes. If both $\mathscr{Y}$ and $\mathscr{X}$ belong to $\mathfrak{T}$, then $\mathscr{Y}_x$ and $\mathscr{Y} \times_{\mathscr{X}}$ Spec $\mathscr{O}_{\mathscr{X},x}$ belong to $\mathfrak{T}$ for every $x \in \mathscr{X}$.

2.2.5. — We fix a fibered category $\mathfrak{F}$ over $\mathfrak{T}$; if $Y \to X$ is an arrow of $\mathfrak{T}$, the corresponding pull-back functor $\mathfrak{F}_X \to \mathfrak{F}_Y$ will be denoted by $D \mapsto D_Y$. If A is a ring whose spectrum belongs to $\mathfrak{T}$, we shall sometimes write for short $\mathfrak{F}_A$ instead of $\mathfrak{F}_{\mathrm{Spec}\,A}$, and D_A instead of $D_{\mathrm{Spec}\,A}$. If $\mathscr{X}$ is any scheme belonging to $\mathfrak{T}$, x is a point of $\mathscr{X}$, and D is an object of $\mathfrak{F}_{\mathscr{X}}$, we shall write D_x instead of $D_{\mathscr{O}_{\mathscr{X},x}}$. If $\mathscr{X}$ is a scheme of finite type over an affinoid algebra, the pull-back functor $\mathfrak{F}_{\mathscr{X}} \to \mathfrak{F}_{\mathscr{X}^{\mathrm{an}}}$ will be denoted by $D \mapsto D^{\mathrm{an}}$. If k is an analytic field, X is a k-analytic space, and L is a complete extension of k, the pull-back functor $\mathfrak{F}_X \to \mathfrak{F}_{X_L}$ will be denoted by $D \mapsto D_L$. We assume moreover that $\mathfrak{F}$ satisfies the following property:

GAGA axiom for $\mathfrak{F}$. — *For every affinoid space, the pull-back functor $D \mapsto D^{\mathrm{an}}$ from $\mathfrak{F}_{X^{\mathrm{al}}}$ to $\mathfrak{F}_X$ is an equivalence; we denote by $D \mapsto D^{\mathrm{al}}$ a quasi-inverse of the latter.*

2.2.6. — Let X be an analytic space and let x be a point of X. Let us first assume that X is good. Let V be an affinoid neigborhood of x in X. The composition functor

$$\mathfrak{F}_X \longrightarrow \mathfrak{F}_V \xrightarrow[D \mapsto D^{\mathrm{al}}]{} \mathfrak{F}_{V^{\mathrm{al}}} \longrightarrow \mathfrak{F}_{\mathscr{O}_{X,x}}$$

only depends on x, and not on V. It will be denoted by $D \mapsto D_x$. If $\mathscr{X}$ is any $\mathscr{O}_{X,x}$-scheme belonging to $\mathfrak{T}$ and if D is an object of $\mathfrak{F}_X$ we shall often write $D_{\mathscr{X}}$ instead of $(D_x)_{\mathscr{X}}$, if there is no ambiguity; for example, we shall use the notation $D_{\kappa(x)}$ and $D_{\mathscr{H}(x)}$.

2.2.7. — Let $\mathfrak{L}$ be the category whose objects are local noetherian rings A such that Spec A belongs to $\mathfrak{T}$, and whose arrows are *local* maps. The contravariant functor Spec anti-identifies $\mathfrak{L}$ with a subcategory Spec $\mathfrak{L}$ of $\mathfrak{T}$. The fibered category $\mathfrak{F}$ gives rise by restriction to a fibered category over Spec $\mathfrak{L}$; we shall denote it by $\mathfrak{F}_{\mathfrak{L}}$.

We are now going to give three examples the reader should keep in mind while working with this general, abstract fibered category $\mathfrak{F}$. For each of them, we shall only describe the corresponding fiber categories; the definition of pull-back functors is straightforward and left to the reader.

2.2.8. Example. — We may take for $\mathfrak{F}$ the category $\mathfrak{T}$, viewed as fibered category over itself in the obvious way: for every object X of $\mathfrak{T}$, the fiber category $\mathfrak{T}_X$ simply consists of the single object X with Id_X as unique endomorphism.

The fibered category $\mathfrak{T}_{\mathfrak{L}}$ can then be anti-identified with $\mathfrak{L}$. This allows us to view objects of $\mathfrak{T}_{\mathfrak{L}}$ as objects of $\mathfrak{L}$.

2.2.9. Example. — We may take for $\mathfrak{F}$ the category $\mathfrak{Coh} \to \mathfrak{T}$ of coherent sheaves, defined as follows: for every object X of $\mathfrak{T}$, the fiber category $\mathfrak{Coh}_X$ is the category of coherent sheaves on X.

In particular, objects of $\mathfrak{Coh}_{\mathfrak{L}}$ can be seen as pairs (A, M) with $A \in \mathfrak{L}$ and M a finitely generated A-module.

2.2.10. Example. — Let $\mathfrak{I}$ be a small category. We may take for $\mathfrak{F}$ the fibered category $\mathfrak{Coh}^{\mathfrak{I}} \to \mathfrak{T}$, defined as follows: for every object X of $\mathfrak{T}$, the fiber category $\mathfrak{Coh}_X^{\mathfrak{I}}$ is the category of $\mathfrak{I}$-diagrams of coherent sheaves on X; i.e., of covariant functors from $\mathfrak{I}$ to $\mathfrak{Coh}_X$ (morphisms are natural transformations of functors).

In particular, objects of $\mathfrak{Coh}_{\mathfrak{L}}^{\mathfrak{I}}$ can be seen as pairs (A, D) with $A \in \mathfrak{L}$ and D an $\mathfrak{I}$-diagram of finitely generated A-modules.

2.2.11. Remark. — Examples 2.2.8 and 2.2.9 can actually been interpreted as particular cases of Example 2.2.10.

Indeed, let us begin with Example 2.2.8. For any category $\mathfrak{C}$, there is a unique functor from $\varnothing$ to $\mathfrak{C}$. It simply does *nothing*, because the source category has no object (it is analogous to the empty map from $\varnothing$ to an arbitrary set); and it has a unique endomorphism. Hence we have for every object X of $\mathfrak{T}$ an equivalence $\mathfrak{Coh}_X^{\varnothing} \simeq \mathfrak{T}_X$, both categories involved having a single object and a single morphism. Those equivalences essentially commute with pull-back functors, whence an equivalence $\mathfrak{Coh}^{\varnothing} \simeq \mathfrak{T}$.

Let us now deal with Example 2.2.9. Let $\{*\}$ be the category with one single element and one single morphism. For any category $\mathfrak{C}$, the assignment $F \mapsto F(*)$ induces an equivalence between the category of covariant functors from $\{*\}$ to $\mathfrak{C}$ and $\mathfrak{C}$ itself. This yields for every object X of $\mathfrak{T}$ an equivalence $\mathfrak{Coh}_X^{\{*\}} \simeq \mathfrak{Coh}_X$. Those equivalences essentially commute with pull-back functors, whence eventually an equivalence $\mathfrak{Coh}^{\{*\}} \simeq \mathfrak{Coh}$.

2.2.12. Convention. — If $\mathfrak{F}$ is one of fibered categories considered in Examples 2.2.8, 2.2.9 and 2.2.10, objects of $\mathfrak{F}_{\mathfrak{L}}$ can be interpreted as objects from commutative algebra. We shall always use this viewpoint here. In other words, we shall consider that objects of $\mathfrak{T}_{\mathfrak{L}}$ (resp. $\mathfrak{Coh}_{\mathfrak{L}}$, resp. $\mathfrak{Coh}_{\mathfrak{L}}^{\mathfrak{I}}$) actually *are* objects of $\mathfrak{L}$ (resp. pairs (A, M) as in Example 2.2.9, resp. pairs (A, D) as in Example 2.2.10).

2.3. Formalisation of algebraic properties

2.3.1. — We fix from now on until the end of section 2.6 a property P whose validity makes sense for every object of $\mathfrak{F}_{\mathfrak{L}}$.

2.3.2. Example. — If $\mathfrak{F} = \mathfrak{T}$ then objects of $\mathfrak{F}_{\mathfrak{L}}$ are local noetherian rings (belonging to $\mathfrak{L}$). We can therefore take for P the property of being regular, Gorenstein, CI, or R_m for some specified m (for a definition of the R_m property, see [**EGA IV**$_2$], Def. 5.8.2).

2.3.3. Example. — If $\mathfrak{F} = \mathfrak{Coh}$ then objects of $\mathfrak{F}_{\mathfrak{L}}$ are pairs (A, M) with A a local noetherian ring (belonging to $\mathfrak{L}$) and M a finitely generated A-module. We can therefore take for P the property of being CM, or free of given rank, or of given residue rank, or of given depth or codepth, or S_m for some specified m (for a definition of the S_m property, see [**EGA IV**$_2$], 5.7.2; note that the zero module is S_m for every $m \geqslant 0$).

The notions of depth and codepth play a crucial role in this memoir for the construction of dévissages. Recall that they are related to each other by the formula

$$\operatorname{codepth}_A(M) = \dim_{\mathrm{Krull}} M - \operatorname{depth}_A(M)$$

if $M \neq 0$ (the Krull dimension of M is defined in 1.1.2); if $M = 0$ we have $\dim_{\mathrm{Krull}} M = -\infty$ and $\operatorname{depth}_A(M) = +\infty$, but $\operatorname{codepth}_A(M) = 0$ *by convention.*

2.3.4. Example. — If $\mathfrak{I}$ is an interval of $\mathbf{Z}$ (viewed as a category through its natural ordering) and if $\mathfrak{F} = \mathfrak{Coh}^{\mathfrak{I}}$, then objects of $\mathfrak{F}_{\mathfrak{L}}$ are pairs (A, D) with A a local noetherian ring (belonging to $\mathfrak{L}$) and D an $\mathfrak{I}$-indexed sequence

$$\ldots \to M_{i-1} \to M_i \to M_{i+1} \to \ldots$$

of A-linear maps between finitely generated A-modules. We can therefore take for P the property of being a complex, or of being exact, or of being exact at some specified position $i \in \mathfrak{I}$.

2.3.5. Remark. — The properties considered in Example 2.3.3 also make sense for $\mathfrak{F} = \mathfrak{T}$, by viewing any local noetherian ring (belonging to $\mathfrak{L}$) as a module over itself.

2.3.6. Remark. — One might also of course consider some relevant combinations of the aforementioned properties. The most important examples the reader should have in mind are: the property of being reduced, which amounts satisfying both R_0 and S_1;

and that of being normal, which amounts satisfying both R_1 and S_2 (cf. [**EGA IV**$_2$], Prop. 5.8.5 and Thm. 5.8.6).

2.3.7. Definition. — Let X be either a good analytic space or a scheme belonging to $\mathfrak{T}$. Let x be a point of X and let D be an object of $\mathfrak{F}_X$. We shall say that D *satisfies* P *at* x if D_x satisfies P.

2.3.8. Remark. — If D satisfies P at every point of X, one could be tempted to say for short that D satisfies P, but the reader should be aware that such terminology might be ambiguous. Indeed, if X belongs to Spec $\mathfrak{L}$, it might happen that D satisfies P in the original meaning (validity of P makes sense for any object of $\mathfrak{F}_{\mathfrak{L}}$), but does not satisfy P everywhere at X.

By definition, this problem cannot occur as soon as P satisfies condition (G) below. Hence under this assumption, we shall actually use the expression "D satisfies P" instead of "D satisfies P at every point of X".

Condition (G). — For every local noetherian ring A belonging to $\mathfrak{L}$, every prime ideal $\mathfrak{p}$ of A and every object D of $\mathfrak{F}_A$, the following implication holds:

$$(D \text{ satisfies } \mathsf{P}) \Rightarrow (D_{A_{\mathfrak{p}}} \text{ satisfies } \mathsf{P}).$$

This amounts requiring that for every scheme $\mathscr{X}$ belonging to $\mathfrak{T}$ and every object D of $\mathfrak{F}_{\mathscr{X}}$, the set of points of $\mathscr{X}$ at which D satisfies P is stable under generization.

2.3.9. Example. — The following properties satisfy (G): if $\mathfrak{F} = \mathfrak{T}$, the property of being CI [**Avr75**], of being Gorenstein (Thm. 18.2 of [**Mat86**]), regular (Serre, see Thm. 19.3 of [**Mat86**]), or R_m by its very definition; if $\mathfrak{F} = \mathfrak{Coh}$, the property of being CM ([**Mat86**], Thm. 17.3), of being S_m (by its very definition) or of being of codepth bounded above by m for some specified m (see [**EGA IV**$_2$], Prop. 6.11.5); if $\mathfrak{F} = \mathfrak{Coh}^{\mathfrak{I}}$ for some interval $\mathfrak{I}$ of $\mathbf{Z}$, the property of being exact at some specified position $i \in \mathfrak{I}$.

As far as the CI property is concerned, one may give an alternative, simpler proof, using the facts that any local ring belonging to $\mathfrak{L}$ is a quotient of a regular local ring; see for instance the discussion at the end of [**Mat86**], §21.

2.3.10. Remark. — Let n be a non-negative integer. The property of being free of rank n satisfies (G), but we shall of course use "locally free of rank n" for "free of rank n at every point", and not "free of rank n" which we reserve for globally free sheaves as usual.

2.3.11. Geometric validity. — Let k be a field and let $\mathscr{X}$ be a k-scheme belonging to $\mathfrak{T}$. Let x be a point of $\mathscr{X}$ and let D be an object of $\mathfrak{F}_{\mathscr{X}}$. We shall say that D satisfies P *geometrically* at x, or for short that D satisfies P_{geo} at x, if for every finite extension F of k, the object $D_{\mathscr{X}\times_k F}$ satisfies P at every point of $\mathscr{X}\times_k F$. If P satisfies condition (G), we shall say that D satisfies $\mathsf{P}_{\mathrm{geo}}$ if it does so at every point of $\mathscr{X}$.

Note that if P satisfies (G), the set of points of $\mathscr{X}$ at which D satisfies $\mathsf{P}_{\mathrm{geo}}$ is stable under generization: this is a formal consequence of the fact that finite morphisms are closed.

2.3.12. Definition. — Let $\mathscr{Y}\to\mathscr{X}$ be a morphism of schemes belonging to $\mathfrak{T}$, and let D be an object of $\mathfrak{F}_{\mathscr{X}}$. Let y be a point of $\mathscr{Y}$ and let x be its image on $\mathscr{X}$. We shall say that D satisfies P (resp. $\mathsf{P}_{\mathrm{geo}}$) *fiberwise* at y if $D_{\mathscr{Y}_x}$ satisfies P at y (resp. satisfies $\mathsf{P}_{\mathrm{geo}}$ at y with respect to the canonical morphism $\mathscr{Y}_x\to\operatorname{Spec}\kappa(x)$). If P satisfies (G), we shall say that D satisfies P (resp. $\mathsf{P}_{\mathrm{geo}}$) fiberwise if it does so at every point.

The study of fiberwise validity of P or $\mathsf{P}_{\mathrm{geo}}$ can often be reduced to that of the usual validity of P on the source space through a standard trick, which is frequently used in SGA.

2.3.13. The standard trick. — Suppose that we are given a morphism $\mathscr{Y}\to\mathscr{X}$ of schemes belonging to $\mathfrak{T}$, a point x on $\mathscr{X}$ and an object D of $\mathfrak{F}_{\mathscr{Y}}$. Let F be a finite extension of $\kappa(x)$ (if we are interested in P we shall only consider the case where $F=\kappa(x)$) and let y be a point of $\mathscr{Y}_x\times_{\kappa(x)}F$. Let us endow $\overline{\{x\}}$ with its reduced structure, and let $\mathscr{Z}$ be any integral finite $\overline{\{x\}}$-scheme whose function field is equal to F; e.g., we can take for $\mathscr{Z}$ the normalization of the Japanese scheme $\overline{\{x\}}$ inside F, or the scheme $\overline{\{x\}}$ itself if $F=\kappa(x)$. Let ξ be the generic point of $\mathscr{Z}$, and set $\mathscr{T}=\mathscr{Y}\times_{\mathscr{X}}\mathscr{Z}$. The scheme $\mathscr{Y}_x\times_{\kappa(x)}F$ is nothing but the generic fiber $\mathscr{T}_\xi$. Therefore $\mathscr{O}_{\mathscr{Y}_x\times_{\kappa(x)}F,y}=\mathscr{O}_{\mathscr{T}_\xi,y}=\mathscr{O}_{\mathscr{T},y}$. Hence $D_{\mathscr{Y}_x\times_{\kappa(x)}F}$ satisfies P at y if and only if so does $D_{\mathscr{T}}$.

2.3.14. Analytic version of the standard trick. — Suppose that we are given a morphism $Y\to X$ of good analytic spaces, a point y on Y whose image in X is denoted by x, and an object D of $\mathfrak{F}_Y$. Let ξ be a point of $\operatorname{Spec}\mathscr{O}_{X,x}$, let F be a finite extension of $\kappa(\xi)$ (if we are interested in P we shall only consider the case where $F=\kappa(\xi)$), and let η be a point of $(\operatorname{Spec}\mathscr{O}_{Y,y})_\xi\times_{\kappa(\xi)}F$. Let B be the quotient of $\mathscr{O}_{X,x}$ by its prime ideal that corresponds to ξ.

Let C be any finite B-subalgebra of F with fraction field F; e.g., we may take for C the integral closure of B in F (the ring $\mathscr{O}_{X,x}$ is universally japanese), or the ring B itself when $F=\kappa(\xi)$. Let us shrink X so that X is affinoid and so that C is induced by a finite integral X-analytic space Z; cf. 2.1.6. Let z be the unique pre-image of x in Z. Set $T=Y\times_X Z$ and let $t_1,\ldots,t_r$ be the pre-images of y on T. Let ζ be the generic point of $\operatorname{Spec}\mathscr{O}_{Z,z}$; note that ζ lies over the generic point of Z^{al} by flatness of $\mathscr{O}_{Z^{\mathrm{al}},z^{\mathrm{al}}} \to \mathscr{O}_{Z,z}$. Let p the natural map $\coprod \operatorname{Spec} \mathscr{O}_{T,t_i} \to \operatorname{Spec} \mathscr{O}_{Z,z}$. One has

$$\begin{aligned}\prod_i \mathscr{O}_{T,t_i} &= \mathscr{O}_T(T) \otimes_{\mathscr{O}_Y(Y)} \mathscr{O}_{Y,y} \\ &= \mathscr{O}_Z(Z) \otimes_{\mathscr{O}_X(X)} \mathscr{O}_{Y,y} \\ &= \mathscr{O}_{Z,z} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Y,y}.\end{aligned}$$

Therefore $(\operatorname{Spec} \mathscr{O}_{Y,y})_\xi \times_{\kappa(\xi)} F$ is nothing but the generic fiber $p^{-1}(\zeta)$. The point η lies on $\operatorname{Spec} \mathscr{O}_{T,t_i}$ for some i, and the local rings at η of $\operatorname{Spec} \mathscr{O}_{T,t_i}$ and of $p^{-1}(\zeta)$ coincide. It follows that $D_{(\operatorname{Spec} \mathscr{O}_{Y,y})_\xi \times_{\kappa(\xi)} F}$ satisfies P at η if and only if D_{T,t_i} satisfies P at η.

2.3.15. — We are now going to introduce various technical conditions that make sense for P.

Condition $(\mathsf{H}_{\mathrm{reg}})$. — For any flat morphism $A \to B$ of $\mathfrak{L}$ and any $D \in \mathfrak{F}_A$ the following implications hold:

- If D_B satisfies P then D satisfies P.
- If D satisfies P and if moreover the fibers of $\operatorname{Spec} B \to \operatorname{Spec} A$ are regular then D_B satisfies P.

Condition $(\mathsf{H}_{\mathrm{CI}})$. — For any flat morphism $A \to B$ of $\mathfrak{L}$ and any $D \in \mathfrak{F}_A$ the following implications hold:

- If D_B satisfies P then D satisfies P.
- If D satisfies P and if moreover the fibers of $\operatorname{Spec} B \to \operatorname{Spec} A$ are CI then D_B satisfies P.

Condition (H). — For any flat morphism $A \to B$ of $\mathfrak{L}$ and any $D \in \mathfrak{F}_A$, the object D satisfies P if and only if D_B satisfies P.

Condition (F). — For every field k, every object D of $\mathfrak{F}_k$ satisfies P.

Condition (O). — For any scheme X belonging to $\mathfrak{T}$ and any object D of $\mathfrak{F}_X$, the subset of X consisting of points at which D satisfies P is open.

2.3.16. Remark. — It follows from the definitions that $(\mathsf{H}) \Rightarrow (\mathsf{H}_{\mathrm{CI}}) \Rightarrow (\mathsf{H}_{\mathrm{reg}})$ and that $(\mathsf{O}) \Rightarrow (\mathsf{G})$.

2.3.17. Example. — Assume that $\mathfrak{F} = \mathfrak{T}$. Let m be an integer.

The following properties satisfy $(\mathsf{H}_{\mathrm{reg}})$: being regular, and being R_m; see [**EGA IV**$_2$], Prop. 6.5.1 Prop. 6.5.3 (ii). The following properties satisfy $(\mathsf{H}_{\mathrm{CI}})$: being Gorenstein, see [**Mat86**], Thm. 23.4; being CI, see [**Avr75**].

The properties of being regular, R_m, Gorenstein, and CI obviously satisfy (F). They also satisfy (O): see [**EGA IV**$_2$], Scholie 7.8.3 (iv) for regularity and the R_m property; see [**GM78**], Cor. 1.5 for the Gorenstein property; and see [**GM78**], Cor. 3.3 for the CI property.

As far as condition (O) for Gorenstein and CI properties is concerned, one may find simpler, alternative proofs using the fact that every scheme belonging to $\mathfrak{T}$ is locally embeddable into a regular scheme; see [**Mat86**], Exercise 24.3 and [**EGA IV**$_4$], Cor. 19.3.3.

2.3.18. Example. — Assume that $\mathfrak{F} = \mathfrak{Coh}$. Let m be an integer.

The properties of being CM, of being S_m and of being of codepth m satisfy (H_{CI}); see [**EGA IV**$_2$], Cor. 6.3.2, Prop. 6.4.1 (i) and (ii). The properties of being CM, of being S_m and of being of codepth $\leqslant m$ obviously satisfy (F). They also satisfy (O); see [**EGA IV**$_2$], Scholie 7.8.3 (iv).

The property of being free satisfies (H), see [**EGA IV**$_2$], Prop. 6.2.1 (ii); the property of being of given residue rank obviously satisfies (H), hence being free of specified rank also satisfies (H).

The property of being free satisfies (F) and (O). The property of being free *of specified rank* still satisfies (O) but it does not satisfy (F).

2.3.19. Example. — Assume that $\mathfrak{F} = \mathfrak{Coh}^{\mathfrak{I}}$ for $\mathfrak{I}$ an interval of $\mathbf{Z}$. We will only list here some properties satisfying (H): the properties of being a complex, of being exact, of being a complex having its i-th homology of a given residue rank for some specified $i \in \mathfrak{I}$. In the particular case where $\mathfrak{I} = \{0, 1\}$ (in which case $\mathfrak{Coh}_X^{\mathfrak{I}}$ is the category of maps between two coherent sheaves on X), let us mention the properties of being an isomorphism, an injection, a surjection.

2.3.20. — Assume that P satisfies ($\mathsf{H}_{\mathrm{reg}}$), and let $\mathscr{C}$ be a class of morphisms between analytic spaces that is stable under base change by inclusions of analytic domains and finite morphisms. Let us consider the following four assertions.

(A) Let $Y \to X$ be an arrow belonging to $\mathscr{C}$ with Y and X affinoid and with X integral. Let D be an object of $\mathfrak{F}_{Y^{\mathrm{al}}}$. The object D satisfies P at every point of Y^{al} lying above the generic point of X^{al}.

(A*) Let $Y \to X$ be an arrow belonging to $\mathscr{C}$, with Y and X affinoid and with X integral. Let y be a point of Y and let x be its image in X. Let D be an object of $\mathfrak{F}_Y$. The object D_y satisfies P at every point of Spec $\mathscr{O}_{Y,y}$ lying above the generic point of X^{al}.

(B) Let $Y \to X$ be an arrow belonging to $\mathscr{C}$, with Y and X affinoid. Let D be an object of $\mathfrak{F}_{Y^{\mathrm{al}}}$. The object D satisfies $\mathsf{P}_{\mathrm{geo}}$ fiberwise at every point of Y^{al}, with respect to the morphism $Y^{\mathrm{al}} \to X^{\mathrm{al}}$.

(B*) Let $Y \to X$ be an arrow belonging to $\mathscr{C}$, with Y and X good. Let y be a point of Y and let x be its image in X. Let D be an object of $\mathfrak{F}_Y$. The object D_y satisfies $\mathsf{P}_{\mathrm{geo}}$ fiberwise at every point of Spec $\mathscr{O}_{Y,y}$, with respect to the map Spec $\mathscr{O}_{Y,y} \to$ Spec $\mathscr{O}_{X,x}$.

Then if (A) is true, so are (A*), (B) and (B*). Indeed, we may perform the "standard trick" described in 2.3.13 in order to reduce assertion (B) to assertion (A), and the one described in 2.3.14 in order to reduce assertion assertion (B*) to assertion (A*) (note that both tricks involve base change by inclusions of affinoid domains and finite maps, which preserve $\mathscr{C}$ by assumption).

It thus remains to ensure that (A)⇒(A*). So let us assume that (A) holds, and let us prove (A*). Let η be a point of Spec $\mathscr{O}_{Y,y}$ lying over the generic point of X^{al}. It follows from (A) that D^{al} satisfies P at the image of η on Y^{al}. Since P satisfies $(\mathsf{H}_{\mathrm{reg}})$ and since $\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}} \to \mathscr{O}_{Y,y}$ is regular, this implies that D_y satisfies P at η, whence (A*).

2.4. Validity in analytic geometry

If X is an analytic space and if D is an object of $\mathfrak{F}_X$, we have explained in Definition 2.3.7 what it means for D to satisfy P at a given point of X, under the assumption that X is *good*. The purpose of what follows is to extend this definition to arbitrary analytic spaces, provided that P satisfies $(\mathsf{H}_{\mathrm{reg}})$.

2.4.1. Lemma-Definition. — *Let X be an analytic space, let x be a point of X and let D be an object of $\mathfrak{F}_X$. Assume that* P *satisfies* $(\mathsf{H}_{\mathrm{reg}})$. *The following are then equivalent:*

(i) *For every good analytic domain U of X containing x, the object D_U satisfies* P *at x;*
(ii) *There exists a good analytic domain U of X containing x such that D_U satisfies* P *at x.*

We say that D satisfies P at x *if equivalent conditions* (i) *and* (ii) *are fulfilled. If D satisfies* P *at every point of X and if* P *satisfies* (G)*, we shall say that D* satisfies P *(cf. Remark 2.3.8).*

Proof. — Implication (i)⇒(ii) follows from the fact that there exists a good analytic domain of X containing x; e.g., an affinoid domain. Now assume that there exists U as in (ii). Let V be a good analytic domain of X contaning x. Choose a good analytic domain W of $U \cap V$ that contains x. The morphisms $\mathscr{O}_{V,x} \to \mathscr{O}_{W,x}$ and $\mathscr{O}_{U,x} \to \mathscr{O}_{W,x}$ are regular by 2.1.3 (2). By choice of U, the object $D_{U,x}$ satisfies P. Using twice the fact that P satisfies $(\mathsf{H}_{\mathrm{reg}})$, we see that $D_{W,x}$ satisfies P and that $D_{V,x}$ satisfies P. □

2.4.2. Remark. — We keep the notation of Lemma-Definition 2.4.1. If V is any analytic domain of X containing x, it follows from the definition that D satisfies P at x if and only if so does D_V.

In view of Examples 2.3.17–2.3.19 (and Remark 2.3.6), Lemma-Definition 2.4.1 above leads to the following more concrete statement.

2.4.3. Lemma-Definition (concrete version of Lemma-Definition 2.4.1)
Let X be a k-analytic space, let $\mathscr{F}$ be a coherent sheaf on X, let $\mathscr{G} \to \mathscr{H}$ be a

morphism of coherent sheaves on X, and let S be a complex of coherent sheaves on X. Let x be a point of X and let n be a non-negative integer.

(1) *The space X is said to be regular (resp R_n, resp. Gorenstein, resp. CI, resp. normal, resp. reduced) at x if there exists a good analytic domain V of X containing x such that the local ring $\mathscr{O}_{V,x}$ is regular (resp R_n, resp. Gorenstein, resp. CI, resp. normal, resp. reduced); and if this is the case, this holds for* every *good analytic domain of X containing x.*
(2) *The coherent sheaf $\mathscr{F}$ is said to be S_n (resp. CM, resp. of codepth n, resp. free of rank n) at x if there exists a good analytic domain V of X containing x such that the $\mathscr{O}_{V,x}$-module $\mathscr{F}_{V,x}$ is S_n (resp. CM, resp. of codepth n, resp. free of rank n); and if this is the case, this holds for* every *good analytic domain of X containing x.*
(3) *The morphism $\mathscr{G} \to \mathscr{H}$ is said to be injective (resp. surjective, resp. bijective) at x if there exists a good analytic domain V of X containing x such that the $\mathscr{O}_{V,x}$-linear map $\mathscr{G}_{V,x} \to \mathscr{H}_{V,x}$ is injective (resp. surjective, resp. bijective); and if this is the case, this holds for* every *good analytic domain of X containing x.*
(4) *The complex S is said to be exact at x if there exists a good analytic domain V of X containing x such that the complex $\mathsf{S}_{V,x}$ of $\mathscr{O}_{V,x}$-modules is exact; and if this is the case, this holds for* every *good analytic domain of X containing x.*

2.4.4. Remark. — According to our conventions, a space X will be called reduced it is reduced at every of its points, in the sense of Example 2.4.3 above. One checks that this is consistent with the former definition of reducedness (1.3.17).

***2.4.5. Lemma* (GAGA Principles).** — *Let $\mathscr{X}$ be scheme locally of finite type over an affinoid algebra and let x be a point of $\mathscr{X}^{\mathrm{an}}$. Assume that P satisfies $(\mathsf{H}_{\mathrm{reg}})$. The following are equivalent for every object D of $\mathfrak{F}_{\mathscr{X}}$:*

(i) *The object D satisfies P at x^{al}.*
(ii) *The object D^{an} satisfies P at x.*

Proof. — Since P satisfies $(\mathsf{H}_{\mathrm{reg}})$, this is an immediate consequence of the regularity of the map $\mathscr{O}_{\mathscr{X},x^{\mathrm{al}}} \to \mathscr{O}_{\mathscr{X}^{\mathrm{an}},x}$. □

In view of Examples 2.3.17–2.3.19 (and Remark 2.3.6), the GAGA principles stated above lead to the the following more concrete statement.

2.4.6. Lemma (GAGA principles, concrete version of Lemma 2.4.5)

Let $\mathscr{X}$ be scheme of finite type over an affinoid algebra and let x be a point of $\mathscr{X}^{\mathrm{an}}$. Let $\mathscr{F}$ be a coherent sheaf on $\mathscr{X}$, let $\mathscr{G} \to \mathscr{H}$ be a morphism of coherent sheaves on $\mathscr{X}$, and let S be a complex of coherent sheaves on $\mathscr{X}$.

(1) *The analytic space $\mathscr{X}^{\mathrm{an}}$ is regular (resp R_n, resp. Gorenstein, resp. CI, resp. normal, resp. reduced) at x if and only the scheme $\mathscr{X}$ is regular (resp R_n, resp. Gorenstein, resp. CI, resp. normal, resp. reduced) at x^{al}.*

(2) *The coherent sheaf* $\mathscr{F}^{\mathrm{an}}$ *is* S_n *(resp. CM, resp. of codepth* n*, resp. free of rank* n*) at* x *if and only if the coherent sheaf* $\mathscr{F}$ *is* S_n *(resp. CM, resp. of codepth* n*, resp. free of rank* n*)* x^{al}*.*
(3) *The morphism of coherent sheaves* $\mathscr{G}^{\mathrm{an}} \to \mathscr{H}^{\mathrm{an}}$ *is injective (resp. surjective, resp. bijective) at* x *if and only if* $\mathscr{G} \to \mathscr{H}$ *is injective (resp. surjective, resp. bijective) at* x^{al}*.*
(4) *The complex* S^{an} *is exact at* x *if and only if the complex* S *is exact at* x^{al}*.*

2.4.7. Remark. — Both proofs of Lemma-Definition 2.4.1 and Lemma 2.4.5 rest in a crucial way on the regularity of some local maps ($\mathscr{O}_{V,x} \to \mathscr{O}_{W,x}$ and $\mathscr{O}_{U,x} \to \mathscr{O}_{W,x}$ in the first one, $\mathscr{O}_{\mathscr{X},x^{\mathrm{al}}} \to \mathscr{O}_{\mathscr{X}^{\mathrm{an}},x}$ in the second one), which is required to apply the axiomatic condition ($\mathsf{H}_{\mathrm{reg}}$). But for some of the explicit properties considered in Lemmas 2.4.3 and 2.4.6, it would be enough to know that the local maps involved are flat: this is the case for the property of being free of rank n (for a coherent sheaf), of being injective or bijective (for a map of coherent sheaves), or of being exact (for a complex of coherent sheaves); note that for surjectivity of a map of coherent sheaves, flatness is even not necessary: we would be done via Nakayama's Lemma.

2.4.8. Lemma. — *Let* X *be a* k*-analytic space and let* D *be an object of* $\mathscr{F}_X$*. Assume that* P *satisfies* (O) *and* ($\mathsf{H}_{\mathrm{reg}}$)*. Let* U *be the set of points of* X *at which* D *satisfies* P*.*

(1) *The set* U *is Zariski-open in* X*.*
(2) *Assume moreover that* P *satisfies* (F) *and the space* X *is reduced; then* U *is dense in* X*.*

Proof. — Both assertions are G-local, hence we can assume that X is affinoid. Since P satisfies (O), the subset $\mathscr{U}$ of X^{al} consisting of points at which D^{al} satisfies P is Zariski-open. Since P satisfies ($\mathsf{H}_{\mathrm{reg}}$), Lemma 2.4.5 ensures that $U = \mathscr{U}^{\mathrm{an}}$, whence (1).

Assume now that P satisfies (F) and X is reduced. Let ξ be the generic point of an irreducible component of X^{al}. Since X is reduced, X^{al} is reduced too, so $\mathscr{O}_{X^{\mathrm{al}},\xi}$ is a field; in view of assumption (F), this implies that $\xi \in \mathscr{U}$. As a consequence $\mathscr{U}$ is dense in X^{al}, so $U = \mathscr{U}^{\mathrm{an}}$ is dense in X. □

In view of Examples 2.3.2–2.3.4 (and Remark 2.3.6), Lemma 2.4.8 leads to the more concrete statement.

2.4.9. Lemma (concrete version of Lemma 2.4.8). — *Let* X *be a* k*-analytic space, let* $\mathscr{F}$ *be a coherent sheaf on* X*, let* $\mathscr{G} \to \mathscr{H}$ *be a morphism of coherent sheaves on* X*, and let* S *be a complex of coherent sheaves on* X*. Let* x *be a point of* X *and let* n *be a non-negative integer.*

(1) *The subset of X consisting of points at which X is regular (resp R_n, resp. Gorenstein, resp. CI, resp. normal, resp. reduced) is Zariski-open, and dense if X is reduced.*
(2) *The subset of X consisting of points at which the coherent sheaf $\mathscr{F}$ is S_n (resp. CM, resp. of codepth $\leqslant n$, resp. free) is Zariski-open, and dense if X is reduced.*
(3) *The subset of X consisting of points at which the morphism $\mathscr{G} \to \mathscr{H}$ is injective (resp. surjective, resp. bijective) is Zariski-open.*
(4) *The subset of X consisting of points at which the complex S is exact is Zariski-open.*

2.5. Fibers of coherent sheaves

2.5.1. Definition. — Let X be an analytic space and let $\mathscr{F}$ be a coherent sheaf on X. Let x be a point of X and let V be a good analytic domain containing x. The tensor product $\mathscr{H}(x) \otimes_{\kappa(x_V)} \mathscr{F}_{V,x}$ does not depend on V, and will be denoted by $\mathscr{F}_{\mathscr{H}(x)}$. This is a finite dimensional $\mathscr{H}(x)$-vector space which is called the *fiber* of $\mathscr{F}$ at x. Its dimension is called the *fiber rank* of $\mathscr{F}$ at x and is denoted by $\mathrm{rk}_x(\mathscr{F})$.

2.5.2. Basic properties. — Let X be an analytic space and let $\mathscr{F}$ be a coherent sheaf on X. The following properties show that the fiber rank behaves like the usual fiber rank on a noetherian scheme (or on a locally ringed space with coherent structure sheaf).

(1) For every good analytic domain V of X and every point x of V, we have the equality $\mathscr{F}_{\mathscr{H}(x)} = \mathscr{H}(x) \otimes_{\kappa(x_V)} \mathscr{F}_{\kappa(x_V)}$.
(2) For every affinoid domain V of X and every point x of V, we have the equalities

$$\begin{aligned}
\mathscr{F}_{\mathscr{H}(x)} &= \mathscr{H}(x) \otimes_{\mathscr{O}_{V,x}} \mathscr{F}_{x_V} \\
&= \mathscr{H}(x) \otimes_{\mathscr{O}_X(V)} \mathscr{F}(V) \\
&= \mathscr{H}(x) \otimes_{\mathscr{O}_{V^{\mathrm{al}}}(V^{\mathrm{al}})} \mathscr{F}_V^{\mathrm{al}}(V^{\mathrm{al}}) \\
&= \mathscr{H}(x) \otimes_{\kappa(x_V^{\mathrm{al}})} (\mathscr{F}_V^{\mathrm{al}})_{\kappa(x_V^{\mathrm{al}})}.
\end{aligned}$$

(3) For every point x of X, the functor $\mathscr{F} \mapsto \mathscr{F}_{\mathscr{H}(x)}$ is right-exact.
(4) For every coherent sheaf $\mathscr{F}$, the function $x \mapsto \mathrm{rk}_x \mathscr{F}$ is upper semi-continuous for the *Zariski* topology of X.
(5) For every point x of X and every coherent sheaf $\mathscr{F}$ on X, the following are equivalent:
   (i) The vector space $\mathscr{F}_{\mathscr{H}(x)}$ is zero.

(ii) The coherent sheaf $\mathscr{F}$ is zero at x (in the sense of Lemma-Definition 2.4.1; it means that $\mathscr{F}$ is "free of rank 0" at x in the sense of the more concrete Lemma 2.4.3).

(iii) There exists an open neighborhood U of x such that $\mathscr{F}_U = 0$.

Indeed, (1) and (2) follow immediately from the definitions and the fact that $\mathscr{F}_{x_V} = \mathscr{O}_{V,x} \otimes_{\mathscr{O}_X(V)} \mathscr{F}(V)$ (1.3.7). Assertion (3) is a consequence of (1), and (4) is a consequence of (2) and of upper semi-continuity of the pointwise rank in the locally ringed space context (which itself comes from Nakayama's Lemma). Concerning (5), let us first assume that $\mathscr{F}_{\mathscr{H}(x)} = 0$. Then for every good analytic domain V of X containing x one has $\mathscr{F}_{\kappa(x_V)} = 0$ in view of (1), and hence $\mathscr{F}_{V,x} = 0$ by Nakayama's Lemma, whence (ii). Assume now that (ii) holds, and let V be an affinoid domain of X containing x. By assumption, one has $\mathscr{F}_{V,x} = 0$, which implies that $\mathscr{F}_U = 0$ for some open neighborhhod U of x *in* V; now (iii) follows from the fact that x has a neighborhood *in* X that is the union of finitely many affinoid domains containing x. The implication (iii)⇒(i) is obvious.

2.5.3. The support of a coherent sheaf. — Let $\mathscr{F}$ be a coherent sheaf on X and let $\mathscr{I}$ be the (coherent) annihilator ideal of $\mathscr{F}$ (on the site X_{G}). The *support* of $\mathscr{F}$ is the closed analytic subspace of X defined by $\mathscr{I}$; it is denoted by $\mathrm{Supp}(\mathscr{F})$. If $i\colon \mathrm{Supp}(\mathscr{F}) \hookrightarrow X$ is the canonical closed immersion, then $\mathscr{F} = i_*i^*\mathscr{F} = i_*i^{-1}\mathscr{F}$.

Let W be the subset of X consisting of points x such that $\mathscr{F}_{\mathscr{H}(x)} = 0$. By assertion (4) of 2.5.2 above, W is a Zariski-open subset of X, and $\mathscr{F}_W = 0$ by assertion (5). The complement $X\backslash W$ is then equal to (the Zariski-closed subset underlying) $\mathrm{Supp}(\mathscr{F})$. Indeed, by arguing G-locally on X we may assume that it is good. Let x be a point of X. By the general theory of locally ringed spaces with coherent structure sheaf

$$(\mathscr{F}_x \neq 0) \iff (\mathscr{I}_x \subset \mathfrak{m}_x) \iff (f(x) = 0 \text{ for all } f \in \mathscr{I}_x),$$

and the latter condition exactly means that x lies in $\mathrm{Supp}(\mathscr{F})$.

In accordance with the usual convention in commutative algebra (1.1.2), the *dimension* of $\mathscr{F}$ (resp. the *dimension* of $\mathscr{F}$ *at* x for x a given point of X) is by definition the dimension of $\mathrm{Supp}(\mathscr{F})$ (resp. the dimension of $\mathrm{Supp}(\mathscr{F})$ at x if x lies in $\mathrm{Supp}(\mathscr{F})$, and $-\infty$ otherwise). We denote it by $\dim \mathscr{F}$ (resp. $\dim_x \mathscr{F}$).

2.5.4. Surjectivity can be checked fiberwise. — Let $\mathscr{F} \to \mathscr{G}$ be a morphism between two coherent sheaves on X; let x be a point of X. Let $\mathscr{Q}$ be the cokernel of $\mathscr{F} \to \mathscr{G}$. Since formation of fibers is a right-exact functor by 2.5.2 (3), the sequence

$$\mathscr{F}_{\mathscr{H}(x)} \to \mathscr{G}_{\mathscr{H}(x)} \to \mathscr{Q}_{\mathscr{H}(x)} \to 0$$

is exact. By assertion (5) of loc. cit., the vector space $\mathscr{Q}_{\mathscr{H}(x)}$ is zero if and only if $\mathscr{Q}$ is zero at x, that is, if and only if $\mathscr{F} \to \mathscr{G}$ is surjective at x. In other words, $\mathscr{F} \to \mathscr{G}$ is surjective at x if and only if $\mathscr{F}_{\mathscr{H}(x)} \to \mathscr{G}_{\mathscr{H}(x)}$ is surjective.

2.5.5. Consistency with the usual sheaf-theoretic notions. — According to our general conventions, a morphism of coherent sheaves on X should be called injective, resp. surjective, resp. bijective if and only if satisfies this property at every point of X. But these notions also make sense in the general sheaf-theoretic context; our purpose is now to ensure that both terminologies are compatible.

Let $\mathscr{F} \to \mathscr{G}$ be a morphism between two coherent sheaves on X. Let us denote its kernel by $\mathscr{K}$, and its cokernel by $\mathscr{Q}$. Let U be the complement of $\mathrm{Supp}(\mathscr{K})$ in X, and let V be that of $\mathrm{Supp}(\mathscr{Q})$. It follows from 2.5.3 that U is the set of points of X at which $\mathscr{K} = 0$, that V is the set of points of X at which $\mathscr{Q} = 0$, and that $\mathscr{K}_U = 0$ and $\mathscr{Q}_V = 0$. This implies that U is the set of points at which $\mathscr{F} \to \mathscr{G}$ is injective, that V is the set of points at which $\mathscr{F} \to \mathscr{Q}$ is surjective, that $\mathscr{F}_U \to \mathscr{G}_U$ is sheaf-theoretically injective and that $\mathscr{F}_V \to \mathscr{G}_V$ is sheaf-theoretically surjective. It follows that $U \cap V$ is the set of points at which $\mathscr{F} \to \mathscr{G}$ is bijective, and that $\mathscr{F}_{U\cap V} \to \mathscr{G}_{U\cap V}$ is sheaf-theoretically bijective. As a consequence, our terminology is compatible with that from sheaf theory.

2.6. Ground field extension

Our aim is now to state some properties of the ground field extension functor in analytic geometry, beyond faithful flatness. In *algebraic* geometry, some properties of schemes of finite type over a field (like being CM, CI, Gorenstein, S_m for some specified m) behave well under *any* ground field extension; but some others (like being regular, or R_m for some specified m) are only preserved in full generality by *separable* extensions of the ground field.

An analogous phenomenon holds in analytic geometry. In order to describe it, we first need to explain what the analytic analogue of a separable extension is.

2.6.1. Definition **(after [Duc09]).** — Let k be an analytic field. An analytic extension L of k is called *analytically separable* if char. $k = 0$ or if char. $k = p > 0$ and the semi-norm

$$L\widehat{\otimes}_k k^{1/p} \to \mathbf{R}_+,\ \sum \ell_i \otimes x_i \mapsto \underbrace{\left|\sum \ell_i x_i\right|}_{\in L^{1/p}}$$

is a norm equivalent to the tensor norm of $L\widehat{\otimes}_k k^{1/p}$.

Let us give some examples (for detailed proofs, see [**Duc09**], §1.2, Lemme 1.8 and Exemple 1.9).

2.6.2. Example. — If k is a perfect analytic field, every analytic extension of k is analytically separable.

2.6.3. Example. — A finite extension of an analytic field is analytically separable if and only if it is separable.

2.6.4. Example. — Let k be an analytic field and let r be a polyradius. The analytic extension $k \hookrightarrow \mathscr{H}(\eta_r)$ (1.2.15) is analytically separable (recall that $\mathscr{H}(\eta_r) = k_r$ when r is k-free).

2.6.5. Theorem. — *Let $X = (k, X)$ be a good analytic space and let L be an analytic extension of k. Let x be a point of X and let y be a point of X_L lying over x.*

(1) *The fibers of the faithfully flat morphism* Spec $\mathscr{O}_{X_L,y} \to$ Spec $\mathscr{O}_{X,x}$ *are CI, and geometrically regular when L is analytically separable over k.*
(2) *If X is affinoid, the fibers of the faithfully flat morphism $(X_L)^{\mathrm{al}} \to X^{\mathrm{al}}$ are CI, and geometrically regular when L is analytically separable over k.*

Proof. — Assume that X is affinoid and integral. By Prop. 2.2 (a) of [**Duc09**], there exists a non-empty open subset $\mathscr{U}$ of X^{al} such that $\mathscr{U} \times_{X^{\mathrm{al}}} (X_L)^{\mathrm{al}}$ is CI, and regular when L is analytically separable over k. The theorem then follows by applying 2.3.20 with $\mathfrak{F} = \mathfrak{T}$, with P being the CI property in general, and the regularity property when L is analytically separable, and with $\mathscr{C}$ being the class of morphisms of the form $Y_L \to Y$ for Y a k-analytic space. □

By the definition of $(\mathsf{H}_{\mathrm{CI}})$ and $(\mathsf{H}_{\mathrm{reg}})$, the above theorem implies the following proposition.

2.6.6. Proposition. — *Let k be an analytic field, let X be a k-analytic space, let D be an object of $\mathfrak{F}_X$, and let L be an analytic extension of k. Let x be a point of X and let y be a pre-image of x in Y. Assume that P satisfies $(\mathsf{H}_{\mathrm{reg}})$.*

(1) *If D_L satisfies P at y then D satisfies P at x;*
(2) *If D satisfies P at x, then D_L satisfies P at y if either L is analytically separable over k or P satisfies $(\mathsf{H}_{\mathrm{CI}})$.*

In view of Examples 2.3.2–2.3.4 (and Remark 2.3.6), Proposition 2.6.6 leads to a more concrete statement:

2.6.7. Proposition **(Concrete version of Proposition 2.6.6)**

Let k be an analytic field, let X be a k-analytic space, let $\mathscr{F}$ be a coherent sheaf on X, let $\mathscr{G} \to \mathscr{H}$ be a morphism of coherent sheaves on X, and let S be a complex of coherent sheaves on X. Let x be a point of X and let n be a non-negative integer. Let L be an analytic extension of k and let y be a pre-image of x on X_L.

(1) *The space X_L is Gorenstein (resp. CI) at y if and only if X is Gorenstein (resp. CI) at x.*
(2) *If X_L is regular (resp. R_n, resp. normal, resp. reduced) at y, then X is regular (resp. R_n, resp. normal, resp. reduced) at x.*
(3) *If X is regular (resp. R_n, resp. normal, resp. reduced) at x and L is analytically separable over k, then X_L is regular (resp. R_n, resp. normal, resp. reduced) at y.*

(4) *The coherent sheaf $\mathscr{F}_L$ is S_n (resp. CM, resp. of codepth n, resp. free of rank n) at y if and only if $\mathscr{F}$ is S_n (resp. CM, resp. of codepth n, resp. free of rank n) at x.*

(5) *The morphism of coherent sheaves $\mathscr{G}_L \to \mathscr{H}_L$ is injective (resp. surjective, resp. bijective) at y if and only if $\mathscr{G} \to \mathscr{H}$ is injective (resp. surjective, resp. bijective) at x.*

(6) *The complex S_L is exact at y if and only if S is exact at x.*

2.6.8. Remark. — Proposition 2.6.6 rests on Theorem 2.6.5 which is needed for using the axiomatic conditions ($\mathsf{H}_{\mathrm{reg}}$) or ($\mathsf{H}_{\mathrm{CI}}$). But for some of the explicit properties considered in Proposition 2.6.7, it would be enough to know that the local maps involved are flat: this is the case for the property of being free of rank n (for a coherent sheaf), of being injective or bijective (for a map of coherent sheaves), or of being exact (for a complex of coherent sheaves); note that for surjectivity of a map of coherent sheaves, flatness is even not necessary: we would be done via Nakayama's Lemma.

2.6.9. Geometric validity. — Let k be an analytic field, let X be a k-analytic space, and let x be a point of X. Let D be an object of $\mathfrak{F}_X$. We shall say that D satisfies P *geometrically* at x if for every analytic extension L of k and every point y of X_L lying above X, the object D_L satisfies P at x_L.

Assume that P satisfies ($\mathsf{H}_{\mathrm{reg}}$). In order for D to satisfy geometrically P at x, it is sufficient that there exists a *perfect* analytic extension L of k and a pre-image y of x on X_L such that the object D_L satisfies P at y. Indeed, assume that it is the case, let F be an analytic extension of k and let z be a pre-image of x on X_F. By Lemma 2.6.10 below, there exists an analytic extension K of k, equipped with two isometric k-embeddings $L \hookrightarrow K$ and $F \hookrightarrow K$, and a point $t \in X_K$ lying above both y and z. Since L is perfect, K is analytically separable over L. It thus follows from Proposition 2.6.6 above that D_K satisfies P at t, and then that D_F satisfies P at z.

2.6.10. Lemma. — *Let k be an analytic field and let L and F be two analytic extensions of k. Let $Y \to X$ be a morphism of k-analytic spaces. Let X' be an F-analytic space, let $X' \to X$ be a morphism of analytic spaces and set*

$$Y' = Y \times_X X' = Y_F \times_{X_F} X'.$$

Let y be a point of Y. Let u (resp. y') be a point of Y_L (resp. Y') lying above y. There exist a complete extension K of k, equipped with two isometric k-embeddings $F \hookrightarrow K$ and $L \hookrightarrow K$, and a point ω of

$$Y'_K := Y' \times_F K \simeq Y \times_X X'_K \simeq Y_K \times_{X_K} X'_K \simeq Y_L \times_{X_L} X'_K \simeq Y_F \times_{X_F} X'_K$$

lying above both y' and u.

Proof. — We immediately reduce to the case where Y and X are k-affinoid and X' is F-affinoid. Let A, A', B and B' be the respective algebras of analytic functions on X, X', Y and Y'. The points u and y' furnish a pair of characters

$$(B_L \to \mathscr{H}(u), B' \to \mathscr{H}(y')),$$

the restriction of which to B go through $B \to \mathscr{H}(y)$. The Banach algebra $\mathscr{H}(u)\widehat{\otimes}_{\mathscr{H}(y)}\mathscr{H}(y')$ is non-zero: a result by Gruson ensures that it contains $\mathscr{H}(u) \otimes_{\mathscr{H}(y)} \mathscr{H}(y')$, [**Gru66**], §3.2, Thm. 1 (4). There exists therefore an analytic field K and a bounded homomorphism $\mathscr{H}(u)\widehat{\otimes}_{\mathscr{H}(y)}\mathscr{H}(y') \to K$ ([**Ber90**], Thm. 1.2.1). This makes K an analytic extension of both $\mathscr{H}(u)$ and $\mathscr{H}(y')$ over $\mathscr{H}(y)$. One thus gets a new pair of characters

$$(B_L \to K, B' \to K)$$

whose restriction to B coincide; that pair induces tautologically a character from $B_L\widehat{\otimes}_B B' = B_L\widehat{\otimes}_A A'$ to K, which extends canonically to a character

$$B_K\widehat{\otimes}_{A_K}(K\widehat{\otimes}_F A') \to K.$$

The latter defines a point ω on $Y_K \times_{X_K} X'_K$ lying by construction above both y' and u. □

2.6.11. Remark. — By Proposition 2.6.6 above, if P satisfies $(\mathsf{H}_{\mathrm{CI}})$ then its validity at a point is equivalent to its geometric validity at the latter. Therefore in practice, the notion of geometric validity will be of specific interest only when $\mathfrak{F} = \mathfrak{T}$ and when the property we are considering involves regularity or R_m for some specified m; e.g., geometric regularity, geometric reducedness, and geometric normality.

2.7. Complements on analytifications

Our next goal is to extend some results that hold for (spectra of) affinoid algebras to schemes locally of finite type over such spectra. We fix an analytic field k and a k-affinoid algebra A; we first consider the strict situation.

2.7.1. Lemma. — *Assume that $|k^\times| \neq \{1\}$ and that A is strict. Let $\mathscr{X}$ be an A-scheme locally of finite type and let x be a point of $\mathscr{X}$. The point x is closed if and only if $\kappa(x)$ is a finite extension of k.*

2.7.2. Remark. — The result is well-known for $\mathscr{X} = \operatorname{Spec} A$: this is nothing but the classical analytic *Nullstellensatz*; our proof essentially consists in reducing to this situation.

Proof of Lemma 2.7.1. — If $\kappa(x)$ is finite over k, then $\operatorname{Spec}\kappa(x) \to \mathscr{X}$ is finite, and x is closed. Conversely, let us assume that x is closed, and let ξ be its image in $\operatorname{Spec} A$. Since x is closed in its fiber $\mathscr{X}_\xi$ which is locally of finite type over $\kappa(\xi)$, it suffices to prove that $\kappa(\xi)$ is a finite extension of k. By Chevalley's theorem, $\{\xi\}$ is a constructible subset of Spec A, which means that ξ is open in its Zariski closure $\mathscr{Y}$. Set $\mathscr{Z} = \mathscr{Y} \backslash \{\xi\}$; this is a proper Zariski-closed subset of $\mathscr{Y}$.

Let us endow $\mathscr{Y}$ with its reduced structure, and let f be a non-zero function on $\mathscr{Y}$ that vanishes pointwise on $\mathscr{Z}$; let r be the spectral norm of f, seen as a function on the strict affinoid space $\mathscr{Y}^{\mathrm{an}}$. The strict affinoid domain of $\mathscr{Y}^{\mathrm{an}}$ defined by the condition $|f| = r$ is non-empty, hence admits a rigid point z. By construction, $f(z^{\mathrm{al}}) \neq 0$, which implies that $z^{\mathrm{al}} = \xi$. Therefore $\kappa(\xi)$ is a finite extension of k. $\square$

2.7.3. Lemma. — *Assume that $|k^\times| \neq \{1\}$ and that A is strict. Let $\mathscr{X}$ be an irreducible A-scheme locally of finite type and let $\mathscr{Y}$ be an irreducible closed subset of $\mathscr{X}$.*

(1) *The scheme $\mathscr{X}$ is finite-dimensional, and we have the equality*

$$\dim \mathscr{Y} + \operatorname{codim}(\mathscr{Y}, \mathscr{X}) = \dim \mathscr{X}.$$

(2) *Every closed point of $\mathscr{X}$ is of codimension* $\dim \mathscr{X}$ *in $\mathscr{X}$.*

2.7.4. Remark. — The result is already known for $\mathscr{X} = \operatorname{Spec} A$: see [**Duc07b**], Prop. 1.11, and more precisely §1.11.1 in its proof, in which the strict case is handled. Our proof essentially consists in reducing to this situation.

Proof. — Note that assertion (2) is a particular case of (1); we have written it down because we wanted to emphasize it. But since A is excellent, hence universally catenary, both statements are actually equivalent, and we shall in fact prove (2).

Let us first assume that $\mathscr{X}$ is affine. By the analytic version of Noether's normalization lemma there exist $n \in \mathbf{Z}_{\geqslant 0}$ and a finite, dominant morphism from Spec A to the scheme $\mathscr{Z} := \operatorname{Spec} k\{T_1, \ldots, T_n\}$. Let us choose a factorization of the composite morphism $\mathscr{X} \to \mathscr{Z}$ through a closed immersion $\mathscr{X} \hookrightarrow \mathbf{A}^m_{\mathscr{Z}}$ (for some m). Let x be a closed point of $\mathscr{X}$. By Lemma 2.7.1 above, $\kappa(x)$ is a finite extension of k; if z denotes the image of x on $\mathscr{Z}$, then $\kappa(z)$ is also a finite extension of k, hence z is closed in $\mathscr{Z}$. By Remark 2.7.4 above, $\{z\}$ is of codimension n in $\mathscr{Z}$. Being a closed point of the fiber $\mathbf{A}^m_z$, the point x is of codimension m in it. By flatness of $\mathbf{A}^m_{\mathscr{Z}} \to \mathscr{Z}$, the point x is then of codimension $m + n$ in $\mathbf{A}^m_{\mathscr{Z}}$. By catenarity of the latter scheme, x is of codimension $d := n + m - \operatorname{codim}(\mathscr{X}, \mathbf{A}^m_{\mathscr{Z}})$ in $\mathscr{X}$. Since this holds for every closed point of $\mathscr{X}$, the integer d coincides with $\dim \mathscr{X}$, which ends the proof when $\mathscr{X}$ is affine.

For the general case, let us chose a covering $(\mathscr{X}_i)$ of $\mathscr{X}$ by non-empty affine open subsets. If i and j are two indices, the intersection $\mathscr{X}_i \cap \mathscr{X}_j$ is non-empty, hence has a closed point x. By Lemma 2.7.1, $\kappa(x)$ is finite over k, and x is therefore closed in both $\mathscr{X}_i$ and $\mathscr{X}_j$. By the affine case already proven we have

$$\dim \mathscr{X}_i = \dim \mathscr{X}_j = \dim_{\mathrm{Krull}}(\mathscr{O}_{\mathscr{X},x}).$$

Hence the $\mathscr{X}_i$'s all have the same dimension d. We then have $\dim \mathscr{X} = d$. If x is a closed point of $\mathscr{X}$, it belongs to $\mathscr{X}_i$ for some i, whence (again by the affine case) we get the equality $\dim_{\mathrm{Krull}}(\mathscr{O}_{\mathscr{X},x}) = \dim \mathscr{X}_i = d$. ☐

2.7.5. Remark. — We still assume that k is non-trivially valued and A is strict. Let $\mathscr{X}$ be an A-scheme locally of finite type, let $\mathscr{U}$ be a locally closed subscheme of $\mathscr{X}$ and let x be a point of $\mathscr{U}$. It follows from Lemma 2.7.1 that x is closed in $\mathscr{U}$ if and only if it is closed in $\mathscr{X}$, because both assertions are equivalent to the finiteness of $\kappa(x)$ over k.

Assume moreover that $\mathscr{X}$ is irreducible, and let $\mathscr{U}$ be a non-empty open subscheme of $\mathscr{X}$. We then have $\dim \mathscr{U} = \dim \mathscr{X}$. Indeed, choose a closed point $x \in \mathscr{U}$. By the above and by Lemma 2.7.3, we then have

$$\dim \mathscr{U} = \dim_{\mathrm{Krull}} \mathscr{O}_{\mathscr{X},x} = \dim \mathscr{X}.$$

2.7.6. Lemma. — *Assume that $|k^\times| \neq \{1\}$ and that A is strict. Let $\mathscr{X}$ be an A-scheme locally of finite type and let x be a point of $\mathscr{X}^{\mathrm{an}}$. One has the equality*

$$\dim_x \mathscr{X}^{\mathrm{an}} = \dim_{x^{\mathrm{al}}} \mathscr{X}.$$

Proof. — Since both sides of the equality are local on $\mathscr{X}$, we can assume (by replacing $\mathscr{X}$ with a projective compactification of any affine neighborhood of x) that $\mathscr{X}$ is projective. Let $(\mathscr{X}_i)$ be the family of irreducible components of $\mathscr{X}$ that contain x. By GAGA (see 2.1.1), the $\mathscr{X}_i^{\mathrm{an}}$ are the irreducible components of $\mathscr{X}^{\mathrm{an}}$ that contain x (this is in fact true without the projectivity assumption, but that is more involved; see Prop. 2.7.16 below). Therefore it suffices to prove the required equality for every $\mathscr{X}_i$; we thus can assume that $\mathscr{X}$ is irreducible, and in particular purely of dimension d for some d. It suffices now to prove that $\mathscr{X}^{\mathrm{an}}$ is purely d-dimensional too.

Let V be a non-empty strict affinoid domain of $\mathscr{X}^{\mathrm{an}}$. Let y be a rigid point of V. We have the equalities

$$\widehat{\mathscr{O}_{V^{\mathrm{al}},y_V^{\mathrm{al}}}} = \widehat{\mathscr{O}_{V,y}} = \widehat{\mathscr{O}_{\mathscr{X}^{\mathrm{an}},y}} = \widehat{\mathscr{O}_{\mathscr{X},y^{\mathrm{al}}}}$$

(the first one and the third one come from Lemma 6.3 of [**Ber93**], and the middle one from the fact that y belongs to $\mathrm{Int}(V/\mathscr{X}^{\mathrm{an}})$ since y is rigid). Since a local noetherian ring and its completion have the same Krull dimensions, one has

$$\dim_{\mathrm{Krull}} \mathscr{O}_{V^{\mathrm{al}},y_V^{\mathrm{al}}} = \dim_{\mathrm{Krull}} \mathscr{O}_{\mathscr{X},y^{\mathrm{al}}} = d,$$

where the last equality comes from Lemma 2.7.3 (2). Since the closed points of the scheme V^{al} are exactly the points of the form y_V^{al} for y a rigid point of V, the dimension of the scheme V^{al} is equal to d. Hence $\dim V = d$ and $\mathscr{X}^{\mathrm{an}}$ is purely d-dimensional. ☐

2.7.7. Proposition. — *(The valuation on k is no longer assumed to be non-trivial, nor the algebra A to be strictly affinoid).*

(1) *Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of A-schemes locally of finite type, and let y be a point of $\mathscr{Y}^{\mathrm{an}}$. The relative dimension of $\mathscr{Y}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ at y is equal to the relative dimension of $\mathscr{Y} \to \mathscr{X}$ at y^{al}.*
(2) *Let $\mathscr{Z}$ be a k-scheme locally of finite type. For every point z of $\mathscr{Z}^{\mathrm{an}}$ one has $\dim_z \mathscr{Z}^{\mathrm{an}} = \dim_{z^{\mathrm{al}}} \mathscr{Z}$.*

Proof. — Note that assertion (2) is a particular case of (1); we have written it down because we wanted to emphasize it. But we shall in fact first prove (2), and deduce (1).

Due to Lemma 2.7.6, assertion (2) holds as soon as $|k^\times| \neq \{1\}$. But since both sides of the equality are preserved by ground field extension (1.4.12) we can always reduce to this case.

Let us now prove (1). If x denotes the image of y on $\mathscr{X}^{\mathrm{an}}$, the fiber $\mathscr{Y}_x^{\mathrm{an}}$ is naturally isomorphic to the analytification $(\mathscr{Y}_{x^{\mathrm{al}}} \times_{\kappa(x^{\mathrm{al}})} \mathscr{H}(x))^{\mathrm{an}}$; hence (1) follows from (2) (and from the invariance of the local dimension on a scheme locally of finite type over a field under ground field extension). □

Our purpose is now to extend Lemma 2.7.3 to the non-strict case. In order to do that, we must understand what happens when one performs a base change from k to k_r for some k-free polyradius $r = (r_1, \ldots, r_n)$. The key point will be the following lemma.

2.7.8. Lemma. — *Let $\mathscr{X}$ is a reduced (resp. irreducible) A-scheme locally of finite type and let r be a k-free polyradius. The scheme $\mathscr{X} \times_A A_r$ is also reduced (resp. irreducible).*

2.7.9. Remark. — When $\mathscr{X} = \operatorname{Spec} A$ this is nothing but Lemma 1.3 of [**Duc07b**], and we have simply adapted its proof to our more general setting. Let us also mention that we shall not need the result about reducedness for extending Lemma 2.7.3, and that it also follows from Lemma 2.4.6 and Proposition 2.6.7; but we have chosen to include it in this lemma because our elementary method for dealing with irreducibility provides it almost for free.

Proof of Lemma 2.7.8. — Let us first suppose that $\mathscr{X}$ is affine, say $\mathscr{X} = \operatorname{Spec} B$ for some A-algebra B of finite type. Set $T = (T_1, \ldots, T_n)$. Any function belonging to $B \otimes_A A_r$ can be written in a unique way as an infinite sum $\sum_{I \in \mathbf{Z}^n} b_I T^I$, with $b_I \in B$ for all I, that converges on every affinoid domain of $(\mathscr{X}^{\mathrm{an}})_r = (\mathscr{X} \times_A A_r)^{\mathrm{an}}$. For such a function f, we denote by $\mathscr{Z}(f)$ the Zariski-closed subset of $\mathscr{X}$ defined by the ideal $(b_I)_I$.

Let us now make a general remark. Let $f = \sum b_I T^I$ and $g = \sum \beta_I T^I$ be two elements of $B \otimes_A A_r$ with $fg = 0$. Let x be a point of $\mathscr{X}^{\mathrm{an}}$. The fiber Y of the base-change map $(\mathscr{X}^{\mathrm{an}})_r \to \mathscr{X}^{\mathrm{an}}$ at x is isomorphic to $\mathscr{M}(\mathscr{H}(x))_r$. Since $fg = 0$, we have $(f|_Y)(g|_Y) = 0$. As $\mathscr{H}(x)_r$ is a domain (its norm is multiplicative, cf. [**Duc07b**], 1.2.1), $f|_Y = 0$ or $g|_Y = 0$; otherwise said, we have $\sum b_I(x)T^I = 0$ or $\sum \beta_I(x)T^I = 0$ in $\mathscr{H}(x)_r$, which means that every b_I vanishes at x (hence at x^{al}) or every β_I vanishes at x (hence at x^{al}). As a consequence, $\mathscr{X} = \mathscr{Z}(f) \bigcup \mathscr{Z}(g)$.

Assume that B is reduced. Let $f = \sum b_I T^I$ be a nilpotent element of $B \otimes_A A_r$. By the above remark, $\mathscr{X} = \mathscr{Z}(f)$. This means that for every I, the function b_I vanishes at every point of $\mathscr{X}$, hence is nilpotent, hence is zero because B is reduced. Therefore $f = 0$ and $B \otimes_A A_r$ is reduced.

Assume that Spec B is irreducible (but B can now have non-trivial nilpotents), and let us prove that Spec $B \otimes_A A_r$ is irreducible. By quotienting by the nilradical of B (which does not modify the topology of the schemes involved), we can assume that B is reduced, hence a domain. Note that $B \to B \otimes_A A_r$ is injective by faithful flatness of A_r over A, hence $B \otimes_A A_r \neq 0$. Let $f = \sum b_I T^I$ and $g = \sum \beta_I T^I$ be two elements of $B \otimes_A A_r$ with $fg = 0$. We have then $\mathscr{X} = \mathscr{Z}(f) \cup \mathscr{Z}(g)$. By irreducibility of $\mathscr{X}$, we have $\mathscr{X} = \mathscr{Z}(f)$ or $\mathscr{X} = \mathscr{Z}(g)$. Suppose that $\mathscr{X} = \mathscr{Z}(f)$. For every I, the function b_I vanishes at every point of $\mathscr{X}$, hence is nilpotent, hence is zero because B is a domain. We thus have $f = 0$. Analogously, $g = 0$ if $\mathscr{X} = \mathscr{Z}(g)$. This ends the proof when $\mathscr{X}$ is affine.

The assertion about reducedness is local, hence holds in fact for arbitrary $\mathscr{X}$. Let us now assume that $\mathscr{X}$ is irreducible, but not necessarily affine. Choose a non-empty covering $(\mathscr{X}_i)$ of $\mathscr{X}$ by non-empty open affine subsets (as $\mathscr{X}$ is irreducible, it is non-empty). By the affine case already proven, $\mathscr{X}_i \times_A A_r$ is irreducible for every i. And for every (i, j) the intersection $(\mathscr{X}_i \times_A A_r) \cap (\mathscr{X}_j \times_A A_r)$ is non-empty, because it surjects onto $\mathscr{X}_i \cap \mathscr{X}_j$ by faithful flatness of $A \to A_r$; it follows that $\mathscr{X} \times_A A_r$ is irreducible. □

2.7.10. Lemma. — *Let $\mathscr{X}$ be an irreducible A-scheme locally of finite type and let $\mathscr{Y}$ be an irreducible closed subset of $\mathscr{X}$.*

1. *The scheme $\mathscr{X}$ is finite-dimensional.*
2. *The analytic space $\mathscr{X}^{\mathrm{an}}$ is finite-dimensional and*

$$\dim \mathscr{X}^{\mathrm{an}} = \dim \mathscr{Y}^{\mathrm{an}} + \operatorname{codim}(\mathscr{Y}, \mathscr{X})$$

3. *Let r be a k-free polyradius. We have the following equalities and inequalities (recall that both* $\dim \mathscr{X} \times_A A_r$ *and* $\mathscr{Y} \times_A A_r$ *are irreducible by Lemma 2.7.8 above):*
   - (3a) $\dim \mathscr{X} \leqslant \dim \mathscr{X} \times_A A_r$.
   - (3b) $\operatorname{codim}(\mathscr{Y} \times_A A_r, \mathscr{X} \times_A A_r) = \operatorname{codim}(\mathscr{Y}, \mathscr{X})$.

Proof. — We begin with (3). Lemma 2.7.8 ensures that the pre-image in $\mathscr{X} \times_A A_r$ of any irreducible closed subset of $\mathscr{X}$ is still irreducible. Moreover, since $\mathscr{X} \times_A A_r \to \mathscr{X}$ is surjective (by faithful flatness of $A \to A_r$), the pre-images in $\mathscr{X} \times_A A_r$ of two distinct subsets of $\mathscr{X}$ are still distinct. These facts imply (3a) as well as the inequality

$$\operatorname{codim}(\mathscr{Y},\mathscr{X}) \leqslant \operatorname{codim}(\mathscr{Y}\times_A A_r, \mathscr{X}\times_A A_r).$$

We want to prove the reverse inequality. By catenarity of the scheme $\mathscr{X}\times_A A_r$ (which is excellent), we can argue by induction on codimension; hence we only have to consider the case where $\operatorname{codim}(\mathscr{Y},\mathscr{X})=1$. Choose a parameter f of the one-dimensional local ring $\mathscr{O}_{\mathscr{X},\eta}$; i.e., f is an element of $\mathscr{O}_{\mathscr{X},\eta}$ whose zero locus on Spec $\mathscr{O}_{\mathscr{X},\eta}$ is set-theoretically equal to the closed point (this existence of such an f comes from Thm. 13.4 of [**Mat86**]). Then there exists an open neighborhood $\mathscr{U}$ of the generic point η of $\mathscr{Y}$ in $\mathscr{X}$ on which f is defined and admits $\mathscr{Y}\cap\mathscr{U}$ as set-theoretical zero locus.

Now $(\mathscr{Y}\cap\mathscr{U})\times_A A_r$ is the zero-locus of f in the irreducible scheme $\mathscr{U}\times_A A_r$, hence the codimension of the proper irreducible closed subset $\mathscr{Y}\times_A A_r$ of $\mathscr{X}\times_A A_r$ is equal to 1 by the *Hauptidealsatz*, whence (3b).

Now take r such that $|k_r^\times|\neq\{1\}$ and A_r is strictly k_r-affinoid. The irreducible scheme $\mathscr{X}\times_A A_r$ is finite-dimensional by Lemma 2.7.3, and (1) follows then from (3a).

Since one has the equalities $\dim_k \mathscr{X}^{\mathrm{an}} = \dim_{k_r}(\mathscr{X}^{\mathrm{an}})_r = \dim_{k_r}(\mathscr{X}\times_A A_r)^{\mathrm{an}}$ and the same for $\mathscr{Y}$, assertion (2) follows from (3b), Lemma 2.7.3. and Lemma 2.7.6. □

2.7.11. Remark. — Assertion (2) was already known in the affinoid case, cf. [**Duc07b**], Prop. 1.11; we have simply adapted the latter's proof to our more general setting.

2.7.12. Remark. — Let $\mathscr{X}$ be a scheme locally of finite type over an affinoid algebra A. By Lemma 2.7.10 above, every irreducible component of $\mathscr{X}$ is finite-dimensional. The scheme $\mathscr{X}$ itself is then finite-dimensional if and only if the dimensions of its irreducible components are uniformly bounded above.

Assume that this is the case. The dimensions of the fibers of the map $\mathscr{X}\to \operatorname{Spec} A$ are then bounded above by some integer d. It follows then from Proposition 2.7.7 (1) that the dimensions of the fibers of the map $\mathscr{X}^{\mathrm{an}}\to\mathscr{M}(A)$ are bounded above by d. This implies in view of 1.4.14 (2) that $\dim\mathscr{X}^{\mathrm{an}}\leqslant\dim\mathscr{M}(A)+d$; in particular, $\mathscr{X}^{\mathrm{an}}$ is finite-dimensional.

2.7.13. Corollary. — *Let $\mathscr{X}$ be a finite-dimensional scheme locally of finite type over an affinoid algebra and let $\mathscr{Y}$ be a Zariski-closed subset of $\mathscr{X}$. Let x be a point of $\mathscr{X}^{\mathrm{an}}$. We have the following equalities:*

(1) $\operatorname{codim}(\mathscr{Y}^{\mathrm{an}},\mathscr{X}^{\mathrm{an}})=\operatorname{codim}(\mathscr{Y},\mathscr{X})$.

(2) $\operatorname{codim}_x(\mathscr{Y}^{\mathrm{an}},\mathscr{X}^{\mathrm{an}})=\operatorname{codim}_{x^{\mathrm{al}}}(\mathscr{Y},\mathscr{X})$.

2.7.14. — In the proof of Lemma 2.7.6, we have used the fact (due to GAGA over an affinoid algebra, see 2.1.1) that in the proper case, the irreducible components of the analytification are the analytifications of the irreducible components. We are now going to explain why this holds without any properness assumption. For that purpose, we first need to establish the compatibility between analytfication and normalization ([**Duc09**], §5); this is achieved by the following lemma (which follows directly from the constructions involved when $\mathscr{X} = \operatorname{Spec} A$).

2.7.15. Lemma. — *Let $\mathscr{X}$ be an A-scheme locally of finite type, and let $\mathscr{Y}$ denote its normalization.*

(1) *The closed immersion $(\mathscr{X}_{\mathrm{red}})^{\mathrm{an}} \hookrightarrow \mathscr{X}^{\mathrm{an}}$ identifies $(\mathscr{X}_{\mathrm{red}})^{\mathrm{an}}$ with $(\mathscr{X}^{\mathrm{an}})_{\mathrm{red}}$.*
(2) *The finite morphism $\mathscr{Y}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ identifies $\mathscr{Y}^{\mathrm{an}}$ with the normalization of $\mathscr{X}^{\mathrm{an}}$.*

Proof. — By construction, the closed immersion $(\mathscr{X}_{\mathrm{red}})^{\mathrm{an}} \hookrightarrow \mathscr{X}^{\mathrm{an}}$ is defined by a nilpotent ideal, and its source is reduced (Lemma 2.4.6), whence (1).

Let us now prove (2). In view of (1), we can replace $\mathscr{X}$ with $\mathscr{X}_{\mathrm{red}}$, hence assume that $\mathscr{X}$ is reduced. The formation of the normalization of an analytic space commutes to restriction to analytic domains ([**Duc09**], Lemme 5.1.11) and is G-local on the target ([**Duc09**], proof of Thm. 5.13). Hence by using compactification of affine charts on $\mathscr{X}$ we reduce to the case where the latter is proper over A. Now let Z be the normalization of $\mathscr{X}^{\mathrm{an}}$. The morphism $\pi\colon Z \to \mathscr{X}^{\mathrm{an}}$ is finite, the image of every irreducible component of Z is an irreducible component of $\mathscr{X}^{\mathrm{an}}$, and the set of point of $\mathscr{X}^{\mathrm{an}}$ at which $\mathscr{O}_{\mathscr{X}^{\mathrm{an}}} \to \pi_*\mathscr{O}_Z$ is *not* an isomorphism does not contain any irreducible component of $\mathscr{X}^{\mathrm{an}}$ (indeed, the latter property is G-local, and is fulfilled by construction of the normalization on any affinoid chart).

By GAGA over affinoid algebras (2.1.1) applied to the coherent $\mathscr{O}_{\mathscr{X}^{\mathrm{an}}}$-algebra $\pi_*\mathscr{O}_Z$, the finite morphism $Z \to \mathscr{X}^{\mathrm{an}}$ arises from a (unique) finite morphism $\mathscr{Z} \to \mathscr{X}$ (which we still denote by π). The scheme $\mathscr{Z}$ is normal because $\mathscr{Z}^{\mathrm{an}}$ is normal (Lemma 2.4.6), the image of every irreducible component of $\mathscr{Z}$ is an irrreducible component of $\mathscr{X}$, and the set of points of $\mathscr{X}$ at which $\mathscr{O}_{\mathscr{X}} \to \pi_*\mathscr{O}_{\mathscr{Z}}$ is *not* an isomorphism does not contain any irreducible component of $\mathscr{X}$. If $(\mathscr{X}_i)$ denotes the family of irreducible of $\mathscr{X}$, we thus may write $\mathscr{Z} = \coprod \mathscr{Z}_i$ where each $\mathscr{Z}_i$ is normal and maps birationally onto $\mathscr{X}_i$ (equipped with its reduced structure). In other words, $\mathscr{Z}_i$ is the normalization of $\mathscr{X}_i$ for all i, and $\mathscr{Z}$ is therefore the normalization of $\mathscr{Y}$ of $\mathscr{X}$, whence the equality $Z = \mathscr{Y}^{\mathrm{an}}$. □

2.7.16. Proposition. — *Let $\mathscr{X}$ be an A-scheme locally of finite type, and let $(\mathscr{X}_i)$ be the family of irreducible components of $\mathscr{X}$. The $\mathscr{X}_i^{\mathrm{an}}$'s are the irreducible components of $\mathscr{X}^{\mathrm{an}}$.*

Proof. — We first note that $(\mathscr{X}_i^{\mathrm{an}})$ is a locally finite family of Zariski-closed subsets of $\mathscr{X}^{\mathrm{an}}$, which are pairwise not comparable with respect to the inclusion relation (by surjectivity of $\mathscr{X}^{\mathrm{an}} \to \mathscr{X}$). It is therefore sufficient to prove that $\mathscr{X}_i^{\mathrm{an}}$ is irreducible for every i. Otherwise said, we have reduced to the case where $\mathscr{X}$ is irreducible. Let $\mathscr{Y}$ be its normalization. Since $\mathscr{Y}^{\mathrm{an}}$ is the normalization of $\mathscr{X}^{\mathrm{an}}$ by Lemma 2.7.15, it is sufficient by Thm. 5.17 of [**Duc09**] to prove that $\mathscr{Y}^{\mathrm{an}}$ is connected. Let us choose a covering $(\mathscr{Y}_i)$ of the irreducible, normal scheme $\mathscr{Y}$ by non-empty affine open subschemes. Since $\mathscr{Y}_i \cap \mathscr{Y}_j \neq \varnothing$ for every(i, j), the intersection $(\mathscr{Y}_i^{\mathrm{an}}) \cap (\mathscr{Y}_j^{\mathrm{an}})$ is non-empty; hence it suffices to prove that $\mathscr{Y}_i^{\mathrm{an}}$ is connected for every i.

Fix i and choose a normal, projective compactification $\mathscr{Z}$ of $\mathscr{Y}_i^{\mathrm{an}}$. The analytic space $\mathscr{Z}^{\mathrm{an}}$ is normal (Lemma 2.4.6) and irreducible by GAGA over an affinoid algebra (2.1.1), and $\mathscr{Y}_i^{\mathrm{an}}$ is a non-empty Zariski-open subset of $\mathscr{Z}^{\mathrm{an}}$. It is then connected by the non-Archimedean avatar of Riemann's extension theorem ([**Ber90**], Prop. 3.3.14; it is based upon the rigid-analytic version due to Lütkebohmert, [**Lü74**] Thm. 1.6). $\square$

CHAPTER 3

GERMS, TEMKIN'S REDUCTION AND Γ-STRICTNESS

Convention. — *We fix from now on and until the end of the memoir an analytic field k.* We do not make any assumption on it*: it is not assumed to be algebraically closed, it can be of any characteristic and residue characteristic and is not necessarily perfect, the value group $|k^\times|$ can be any subgroup of $\mathbf{R}_+^\times$, such as $\{1\}$ or $\mathbf{R}_+^\times$, etc.*

Berkovich's theory makes a distinction between strictly k-affinoid spaces, which are defined by a finite system of equations on a *unit* compact polydisc, and general affinoid spaces, whose definition allows arbitrary radii. In Section 3.1, we consider an intermediate class of k-affinoid spaces, namely those whose definition allows radii belonging to a given subgroup Γ of $\mathbf{R}_+^\times$, which are called Γ-*strict*; these are the building blocks of the category of Γ-*strict k-analytic spaces.* Our motivation for considering such a notion (which is also studied by Conrad and Temkin in the preprint [**CT**]; see Remark 3.1.7) is to keep under control the real parameters that are needed to define our spaces, especially as far as the description of the image of a map is involved (Section 9.2); the reader can ignore it at first reading.

If k is non-trivially valued, the category of strictly k-analytic spaces is a full subcategory of the category of all analytic spaces, but this result is by no way obvious; it was shown by Temkin in [**Tem04**], using the theory of *graded* reduction of (punctual) analytic germs which he introduced for this purpose (and which is based upon graded Riemann-Zariski spaces; i.e., spaces of graded valuations). Using exactly the same method, we shall prove that the category of Γ-strict k-analytic spaces is a full subcategory of the categories of all analytic spaces (3.5.6; this is also proved in the same way in [**CT**]). But we first give a detailed account of Temkin's theory in Sections 3.3 and 3.4 – we shall need it throughout the whole memoir, not only for questions related to Γ-strictness, but also because it is a powerful tool for the local study of analytic spaces, and often a very efficient substitute for the theory of formal models (which is technically more involved, and moreover not available in the non-strict case).

Before doing this, we of course have to say a few words about the notion of a (punctual) analytic germ. This is done in Section 3.2, in which we also introduce the *central dimension* of such a germ (Definition 3.2.2). This turns out to measure the difference between the Krull dimension of a local ring of a good analytic space and the local dimension of the space at the corresponding point (Corollary 3.2.9) and will play an important role in this work.

Let us end this introduction by mentioning that in the strict context, there is no need for considering Temkin's graded reduction: the non-graded reduction (based upon usual Riemann-Zariski spaces) which he had developped in [**Tem00**] and is in some sense the "degree 1" part of the graded reduction, is sufficient. This phenomenon extends to the Γ-strict context: if one works with a Γ-strict k-analytic germ, there is no need to consider its whole graded reduction; it is sufficient to consider its "Γ-graded part", as explained in 3.5.3.

3.1. Γ-strictness

3.1.1. Notation. — We fix for the whole chapter a subgroup Γ of $\mathbf{R}_+^\times$ such that $\Gamma \cdot |k^\times| \neq \{1\}$; otherwise said, Γ is non-trivial whenever k is trivially valued.

3.1.2. Γ-strict affinoid algebras. — Let A be a k-affinoid algebra. We shall say that a A is Γ-*strict* if it is a quotient of $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ for some r_j *belonging to* Γ.

If A is a quotient of $k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ for some r_j *belonging to* $(|k^\times| \cdot \Gamma)^{\mathbf{Q}}$, then A is Γ-strict. Indeed, we can choose a k-free polyradius $s = (s_1, \ldots, s_m)$ such that $|k_s^\times| \neq \{1\}$, every s_i belongs to Γ, and every r_j belongs to $|k_s^\times|^{\mathbf{Q}}$. This implies that A_s is strictly k_s-affinoid; the proof of Cor. 2.1.8 of [**Ber90**] together with an easy induction on m shows then that A is Γ-strict.

3.1.3. Γ-strict k-affinoid spaces. — A k-affinoid space will be said to be Γ-strict if its algebra of analytic functions is Γ-strict. If X is such a space, the spectral semi-norm of any analytic function f on X belongs to $(|k^\times| \cdot \Gamma)^{\mathbf{Q}} \cup \{0\}$. Indeed, as we saw in 3.1.2 above, there exists a k-free polyradius $s = (s_1, \ldots, s_m)$ such that $|k_s^\times| \neq \{1\}$, every s_i belongs to Γ, and X_s is strictly k_s-affinoid. The spectral semi-norm of f can be computed on X_s; hence it follows from [**BGR84**], 6.2.1/4 that it belongs to $|k_s^\times|^{\mathbf{Q}} \cup \{0\} \subset (|k^\times| \cdot \Gamma)^{\mathbf{Q}} \cup \{0\}$.

Conversely, let X be a k-affinoid space such that the spectral semi-norm of every analytic function on X belongs to $(|k^\times| \cdot \Gamma)^{\mathbf{Q}} \cup \{0\}$; then X is Γ-strict. Indeed, let A be the algebra of analytic functions on X, and let us fix an admissible epimorphism $k\{T_1/r_1, \ldots, T_n/r_n\} \to A$. For every i, let s_i be the spectral radius of the image of T_i in A. By assumption, $s_i \in (|k^\times| \cdot \Gamma)^{\mathbf{Q}} \cup \{0\}$. Now set $t_i = s_i$

if $s_i \neq 0$ and take for t_i any element of $(|k^\times| \cdot \Gamma)^{\mathbf{Q}} \cap [0, r_i]$ if $s_i = 0$. The admissible epimorphism $k\{T_1/r_1, \ldots, T_n/r_n\} \to A$ then factors through an admissible epimorphism $k\{T_1/t_1, \ldots, T_n/t_n\} \to A$, whence the Γ-strictness of A.

3.1.4. Γ-strict k-analytic spaces. — The class of Γ-strict affinoid spaces is a *dense* class in the sense of [**Ber93**], §1 (the assumption that $|k^\times| \cdot \Gamma \neq \{1\}$ has benne made precisely to ensure this property). It thus gives rise to a corresponding category of analytic spaces, which is called the category of Γ-*strict* k-analytic spaces.

3.1.5. Remark. — If U and V are two Γ-strict affinoid domains of a *separated* k-analytic space X, then $U \cap V$ is a Γ-strict affinoid domain of X by the standard argument: it is a closed analytic subspace of $U \times_k V$. As a consequence, a separated k-analytic space admits a Γ-strict k-analytic structure if and only if it admits a G-covering by Γ-strict affinoid domains, and this structure is then unique. This is for instance the case for separated boundaryless k-analytic spaces: indeed, every point of such a space has a Γ-strict affinoid neighborhood.

But be aware that a general (i.e., non necessarily separated) k-analytic space could *a priori* admit several Γ-strict k-analytic structures. We shall see in 3.5.4 that this is actually impossible, but to avoid circular reasoning, we shall for the moment say that a k-analytic space X "is" Γ-strict if X admits *some* Γ-strict k-analytic structure. A Γ-strict analytic domain of such a space will mean an analytic domain admitting *some* Γ-strict structure, but not necessarily a Γ-strict analytic domain for the given Γ-strict k-analytic structure on the ambient space.

3.1.6. Remark. — If $|k^\times| \neq \{1\}$ then what what we call $\{1\}$-strictness is nothing but the usual notion of strictness, and we shall of course say strict instead of $\{1\}$-strict.

3.1.7. Remark. — There is a small difference between our definition of Γ-strictness and that of Conrad and Temkin in [**CT**]: they require the group Γ to contain $|k^\times|$, which we do not. Nevertheless, this is more or less irrelevant: indeed, an analytic space is Γ-strict in our sense if and only if it is $\Gamma \cdot |k^\times|$-strict in Conrad and Temkin's sense.

The reason why we have chosen to relax the assumption on Γ is the following. Using our convention, if X is a Γ-strict k-analytic space, then X_L is a Γ-strict L-analytic space for any analytic extension L of k; but such a statement simply *does not make any sense in general* if one uses Conrad and Temkin's convention, because even if $|k^\times| \subset \Gamma$ it may happen that $|L^\times|$ is not contained in Γ.

3.1.8. Remark. — One can also define the notion of a Γ-strict analytic space, without any mention of the ground field: this is a pair (L, X) where L is an analytic field such that $|L^\times| \cdot \Gamma \neq 1$ and where X is a Γ-strict L-analytic space; morphisms between such spaces are defined in the obvious way.

3.2. Analytic germs

3.2.1. — Let us recall briefly the definition of the category of (punctual) germs of k-analytic spaces (which we shall simply call for short *k-analytic germs*) given by Berkovich in [**Ber93**], 3.4. This is the localization of the category of pointed k-analytic spaces by the class of morphisms $\varphi\colon (Y,y) \to (X,x)$ having the following property: *φ induces an isomorphism between an open neighborhood of y and an open neighborhood of x.* A germ (X,x) is said to have a given property (preserved by restriction to open subsets) if x admits a neighborhood in X having this property.

By replacing k-analytic spaces with analytic spaces in the above construction, we get the notion of an analytic germ without mention of the ground field.

If (X,x) is an analytic germ, a germ of the form (V,x) for V an analytic domain of X containing x will simply be called an analytic domain of (X,x).

We can define in a similar way the category of Γ-strict k-analytic germs (as a localization of the category of pointed Γ-strict analytic spaces). Be aware that Remark 3.1.5 applies mutatis mutandis in this context, and we shall say that a germ (X,x) "is" Γ-strict if x admits a Γ-strict analytic neighborhood in X; i.e., (X,x) admits *some* Γ-strict k-analytic structure, which is not a priori compatible with the one on X nor unique unless (X,x) is separated (but it will be unique a posteriori).

3.2.2. Definition. — Let X be an analytic space and let x be a point of X. The infimum of the integers $\dim_k \overline{\{x\}}^{V_{\mathrm{Zar}}}$ for V running through the set of analytic *neighborhoods* of x in X only depends on the germ (X,x); it will be called the *k-analytic central dimension* of the germ (X,x) and will be denoted by $\operatorname{centdim}_k(X,x)$, or usually simply by $\operatorname{centdim}(X,x)$ if there is no ambiguity about the ground field.

3.2.3. Remark. — We obviously have $\operatorname{centdim}(X,x) \leqslant \dim \overline{\{x\}}^{X_{\mathrm{Zar}}}$, and this inequality can be strict; see Remark 4.4.8.

3.2.4. Basic properties of central dimension. — Let X be an analytic space and let x be a point of X. There exists an analytic neighborhood of x in X of dimension $\dim_x X$ (Remark 1.5.8); we thus have $\operatorname{centdim}(X,x) \leqslant \dim_x X$.

For every analytic neighborhood V of x in X, one has $\dim \overline{\{x\}}^{V_{\mathrm{Zar}}} \geqslant d_k(x)$; therefore $\operatorname{centdim}(X,x) \geqslant d_k(x)$.

If Y is a closed analytic subspace of X contaning x, it follows from the definition that $\operatorname{centdim}(Y,x) = \operatorname{centdim}(X,x)$. More generally, if Y is a finite X-analytic space and if y is a pre-image of x on Y, then $\operatorname{centdim}(Y,y) = \operatorname{centdim}(X,x)$. Indeed, by topological properness and topological separatedness of finite morphisms we may choose an open neighborhood V of x in X such that $\overline{\{x\}}^{V_{\mathrm{Zar}}} = \operatorname{centdim}(X,x)$ and such that $\overline{\{y\}}^{W_{\mathrm{Zar}}} = \operatorname{centdim}(Y,y)$, where W is the connected component of y inside

$Y \times_X V$. The image of $\overline{\{y\}}^{W_{\mathrm{Zar}}}$ on V is a Zariski-closed subset of V in which x is Zariski-dense, hence it coincides with $\overline{\{x\}}^{V_{\mathrm{Zar}}}$. One has then $\dim \overline{\{x\}}^{V_{\mathrm{Zar}}} = \dim \overline{\{y\}}^{W_{\mathrm{Zar}}}$ by 1.4.14, whence our claim.

3.2.5. Example. — Since an irreducible k-analytic space is zero-dimensional if and only if it consists of one rigid point (1.4.7), the central dimension of a germ (X, x) is zero if and only if x is rigid, in which case $d_k(x) = 0$.

3.2.6. Example. — Let X be an analytic space, let Y be a Zariski-closed subset of X and let x be a point of Y. Assume that $\dim_x Y = d_k(x)$; i.e., x is an Abhyankar point of Y. By 3.2.4 we have

$$\dim_x Y \geqslant \operatorname{centdim}(Y, x) = \operatorname{centdim}(X, x) \geqslant d_k(x).$$

It follows that $\operatorname{centdim}(X, x) = d_k(x)$.

3.2.7. Example. — Let X be a curve and let x be a point of type 4 of X (according to Berkovich's classification described in [**Ber90**], Chapter 4). Then $d_k(x) = 0$ but x is not rigid; therefore $\operatorname{centdim}(X, x) = 1$.

3.2.8. Lemma. — *Let X be an affinoid space and let x be a point of X. The following are equivalent:*

(i) $\mathfrak{m}_{x^{\mathrm{al}}} \mathscr{O}_{X,x} = \mathfrak{m}_x$.
(ii) $\dim_{\mathrm{Krull}} \mathscr{O}_{X,x} = \dim_{\mathrm{Krull}} \mathscr{O}_{X^{\mathrm{al}}, x^{\mathrm{al}}}$.
(iii) $\operatorname{centdim}(X, x) = \dim \overline{\{x\}}^{X_{\mathrm{Zar}}}$.

Proof. — Let ω be the closed point of $\operatorname{Spec} \mathscr{O}_{X,x}$ and let $p \colon \operatorname{Spec} \mathscr{O}_{X,x} \to \operatorname{Spec} \mathscr{O}_{X^{\mathrm{al}},\mathrm{al}}$ be the canonical map. Let us consider the following assertions:

(a) $p^{-1}(x^{\mathrm{al}}) = \{\omega\}_{\mathrm{red}}$ *scheme-theoretically.*
(b) $p^{-1}(x^{\mathrm{al}}) = \{\omega\}$ *set-theoretically.*
(c) $\dim p^{-1}(x^{\mathrm{al}}) = 0$.

Assertion (i) is tautologically equivalent to (a). Since the fibers of p are (geometrically) regular, and in particular reduced, one has (a) $\iff$ (b), and it is clear that (b) $\iff$ (c). As (c) $\iff$ (ii) by flatness of p, we eventually get the equivalence (i) $\iff$ (ii).

We are now going to prove the equivalence (i) $\iff$ (iii). For that purpose, we may replace X with any of its closed analytic subspaces containing x, and in particular with $\overline{\{x\}}^{X_{\mathrm{Zar}}}_{\mathrm{red}}$. Hence we may assume that X is integral and that x^{al} is the generic point of X^{al}; note that under this assumption $\mathscr{O}_{X^{\mathrm{al}}, x^{\mathrm{al}}}$ is a field and $\mathfrak{m}_{x^{\mathrm{al}}} = 0$. Let d be the dimension of X. Since X is irreducible, it is purely d-dimensional, and so are all of its analytic domains.

Assume that (i) is true, i.e., $\mathscr{O}_{X,x}$ is a field. Let V be an affinoid neighborhood of x in X and let Z be a Zariski-closed subset of V containing x. Let $f_1, \ldots, f_n$ be analytic functions on V that generate the ideal of functions vanishing pointwise on Z. For any i, we have $f_i(x) = 0$; the image of f_i in $\mathscr{O}_{X,x}$ is then not invertible, hence is zero. Therefore Z contains a neighborhood U of x in V. The dimension of U being equal to d, the dimension of Z (which is bounded by d) is equal to d too; therefore, (iii) is proved.

Assume that (iii) is true; i.e., $\mathrm{centdim}(X,x) = d$. Let f be a non-zero element of $\mathscr{O}_{X,x}$, and let V be an affinoid neighborhood of x on which f is defined. Let Y and Z be two irreducible components of V containing x. Both are of dimension d; if $Y \neq Z$, their intersection is a Zariski-closed subset of V containing x and of dimension strictly smaller than d, which contradicts (iii). Hence there is only one irreducible component Y of V containing x. The zero-locus of f on Y is a Zariski-closed subset of Y which is not equal to the whole of Y, because otherwise f would vanish pointwise on a neighborhood of x, hence would vanish in the reduced local ring $\mathscr{O}_{X,x}$; but by assumption, this is not the case. Therefore the dimension of the zero-locus of f is strictly smaller than d, hence this locus can not contain x because of (iii). As a consequence, f is invertible in $\mathscr{O}_{X,x}$. Thus the latter is a field, and (i) is proved. □

3.2.9. Corollary. — *Let X be a good analytic space and let x be a point of X. One has the equality*

$$\mathrm{centdim}(X,x) + \dim_{\mathrm{Krull}} \mathscr{O}_{X,x} = \dim_x X.$$

In particular if x is rigid then $\dim_{\mathrm{Krull}} \mathscr{O}_{X,x} = \dim_x X$.

Proof. — Set $d = \mathrm{centdim}(X,x)$. Up to shrinking X we may assume that it is affinoid and that $\dim \overline{\{x\}}^{X_{\mathrm{Zar}}} = d$. Let $X_1, \ldots, X_n$ be the irreducible components of X that contain x. For every i, set $d_i = \dim X_i$ and $\delta_i = \mathrm{codim}_{\mathrm{Krull}}(\overline{\{x\}}^{X_{\mathrm{Zar}}}, X_i)$.

One has $\dim_x X = \max d_i$, and since $\dim \overline{\{x\}}^{X_{\mathrm{Zar}}} = d$, Lemma 3.2.8 ensures that $\dim_{\mathrm{Krull}} \mathscr{O}_{X,x}$ coincides with $\dim_{\mathrm{Krull}} \mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}} = \mathrm{codim}_{\mathrm{Krull}}(\overline{\{x\}}^{X_{\mathrm{Zar}}}, X) = \max \delta_i$. But $d_i = \delta_i + d$ for every i (see [**Duc07b**], Prop. 1.11), whence we get the equality $\dim_x X = \dim_{\mathrm{Krull}} \mathscr{O}_{X,x} + d$. □

3.2.10. Example. — Let X be a good analytic space and let Y be a Zariski-closed subset of X. Let x be a point of Y, and assume that $d_k(x) = \dim_x Y$; i.e., x is Abhyankar in Y. We then have $\mathrm{centdim}(X,x) = d_k(x) = \dim_x Y$ (Example 3.2.6) and $\dim \overline{\{x\}}^{X_{\mathrm{Zar}}} = d_k(x)$ (Remark 1.5.9). It follows then from Corollary 3.2.9 that

$$\dim_{\mathrm{Krull}} \mathscr{O}_{X,x} = \dim_x X - \dim_x Y$$

(in particular, $\mathscr{O}_{X,x}$ is artinian as soon as x is Abhyankar in X; e.g., $Y = X$).

If X is moreover assumed to be affinoid, we deduce from Lemma 3.2.8 that $\dim_{\mathrm{Krull}} \mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$ is also equal to $\dim_x X - \dim_x Y$ and that $\mathfrak{m}_{x^{\mathrm{al}}} \mathscr{O}_{X,x} = \mathfrak{m}_x$.

3.3. Around graded Riemann-Zariski spaces

Our purpose is now to give a short account of Temkin's theory of (graded) reduction of analytic germs; our reference is the foundational article [**Tem04**]. In this section, we shall introduce all the required notions about graded Riemann-Zariski spaces; the application to analytic germs will be explained in Section 3.4.

3.3.1. — Let K be a graded field. If L is any graded extension of K, we shall denote by $\mathbf{P}_{L/K}$ the "graded Riemann-Zariski space of L over K"; i.e., the set of equivalence classes of graded valuations on L whose restriction to K is trivial (or, in other words, whose graded ring contains K). For any set S of homogeneous elements of L, we denote by $\mathbf{P}_{L/K}\{S\}$ the subset of $\mathbf{P}_{L/K}$ that consists of graded valuations $|\cdot|$ such that $|f| \leqslant 1$ for every $f \in S$; note that $\mathbf{P}_{L/K}\{S\} = \mathbf{P}_{L/K}\{S \backslash (S \cap K)\}$ (in particular if $0 \in S$ then it can be removed without modifying $\mathbf{P}_{L/K}\{S\}$). We endow $\mathbf{P}_{L/K}$ with the topology generated by the sets of the form $\mathbf{P}_{L/K}\{S\}$ for S a *finite* set of homogeneous elements of L, which are called *affine* open subsets of $\mathbf{P}_{L/K}$. Note that $\mathbf{P}_{L/K} = \mathbf{P}_{L/K}\{\varnothing\}$ is itself affine, and that the intersection of two affine open subsets is affine. Note also that every affine open subset of $\mathbf{P}_{L/K}$ contains the trivial graded valuation; the latter is thus a generic point of $\mathbf{P}_{L/K}$, and is easily seen to be the only one. Any affine open subset of $\mathbf{P}_{L/K}$ (especially, $\mathbf{P}_{L/K}$ itself) is quasi-compact ([**Tem04**], 5.3.6); since the intersection of two affine open subsets of $\mathbf{P}_{L/K}$ is affine, $\mathbf{P}_{L/K}$ is quasi-separated.

3.3.2. Functoriality. — If F is any graded extension of L, and if E is a graded subfield of F such that $E \cap L \supset K$, restriction of graded valuations defines a map $r \colon \mathbf{P}_{F/E} \to \mathbf{P}_{L/K}$. For every set S of homogeneous elements of L we have $r^{-1}(\mathbf{P}_{L/K}\{S\}) = \mathbf{P}_{F/E}\{S\}$; by applying this when S is finite we see that r is continuous and quasi-compact. Morever, r is surjective as soon as $E = K$.

3.3.3. Γ-strictness. — If Γ is a subgroup of $\mathbf{R}_+^\times$, a quasi-compact open subset U of $\mathbf{P}_{L/K}$ is said to be Γ-*strict* if U is the pre-image of some (possibly empty) quasi-compact open subset of $\mathbf{P}_{L^\Gamma/K^\Gamma}$; equivalently, U is Γ-strict if and only if it is a finite union of affine open subsets whose definition only involve homogeneous elements of L^Γ. We shall simply say *strict* instead of $\{1\}$-strict.

Let S be a finite set of homogeneous elements of L. If $S \subset L^{(\mathfrak{D}(K)\cdot\Gamma)^{\mathbf{Q}}}$, then there exists a finite set S' of homogeneous elements of L^Γ such that $\mathbf{P}_{L/K}\{S\} = \mathbf{P}_{L/K}\{S'\}$, and $\mathbf{P}_{L/K}\{S\}$ is therefore Γ-strict. Indeed, for every $f \in S$ there exists a non-zero homogeneous element a_f of K and a positive integer n_f such that $a_f f^{n_f} \in L^\Gamma$, and we can set $S' = \{a_f f^{n_f}\}_{f \in S}$. Conversely, if $\mathbf{P}_{L/K}\{S\}$ is Γ-strict, then $S \subset L^{(\mathfrak{D}(K)\cdot\Gamma)^{\mathbf{Q}}}$ by Prop. 2.5 (i) of [**Tem04**].

3.3.4. Remark. — Our definition of Γ-strictness is not exactly that of Temkin in [**Tem04**]. Indeed, Temkin requires Γ to contain $\mathfrak{D}(K)$, which we do not. More precisely, Γ-strictness in our sense is equivalent to $\Gamma \cdot \mathfrak{D}(K)$-strictness in Temkin's sense. We have made this choice for the sake of consistency with our definition of strictness in analytic geometry, see Remark 3.1.7.

3.3.5. Remark. — Because of this difference between our definition and Temkin's, there are some results of [**Tem04**] that cannot be applied directly in our setting. But we shall remedy this by using the following fact: *the natural continuous map* $\mathbf{P}_{L^{(\mathfrak{D}(K)\cdot\Gamma)^{\mathbf{Q}}}/K} \to \mathbf{P}_{L^\Gamma/K^\Gamma}$ *is a homeomorphism.*

To see this, we first note that in view of 3.3.3, it is sufficient to prove that this map is bijective. Now let $|\cdot|$ be an element of $\mathbf{P}_{L^\Gamma/K^\Gamma}$ and let f be a homogeneous element of $L^{(\mathfrak{D}(K)\cdot\Gamma)^{\mathbf{Q}}}$. By definition, there exists a non-zero homogeneous element a of K, a homogeneous element g of L^Γ and an integer n such that $f^n = ag$. One checks straightforwardly that the element $|g|^{1/n}$ only depends on f, and not on (a, n, g), and that the assignment $f \mapsto |g|^{1/n}$ is is the unique pre-image of $|\cdot|$ on $\mathbf{P}_{L^{(\mathfrak{D}(K)\cdot\Gamma)^{\mathbf{Q}}}/K}$.

Let us mention an important consequence of the above: $\mathbf{P}_{L/K} \to \mathbf{P}_{L^\Gamma/K^\Gamma}$ is the composition of the surjection $\mathbf{P}_{L/K} \to \mathbf{P}_{L^{(\mathfrak{D}(K)\cdot\Gamma)^{\mathbf{Q}}}/K}$ and of the homeomorphism $\mathbf{P}_{L^{(\mathfrak{D}(K)\cdot\Gamma)^{\mathbf{Q}}}/K} \to \mathbf{P}_{L^\Gamma/K^\Gamma}$, so it is surjective.

3.3.6. The category $\mathscr{S}_{L/K}$. — We denote by $\mathscr{S}_{L/K}$ the full subcategory of the category of topological spaces over $\mathbf{P}_{L/K}$ consisting of objects $\mathsf{X} \to \mathbf{P}_{L/K}$ satisfying the following conditions:

(1) The space X is quasi-compact and quasi-separated.
(2) The morphism $\mathsf{X} \to \mathbf{P}_{L/K}$ is a local homeomorphism.

For instance, any quasi-compact open subset of $\mathbf{P}_{L/K}$ is an object of $\mathscr{S}_{L/K}$ through its inclusion into $\mathbf{P}_{L/K}$.

Let X be an object of $\mathscr{S}_{L/K}$. A *chart* of X is a quasi-compact open subset U of X such that $\mathsf{U} \to \mathbf{P}_{L/K}$ is an open immersion. An *atlas* of X is a finite covering of X by charts.

Let η be the unique generic point of $\mathbf{P}_{L/K}$ and let X_η be the set of pre-images of η on X. Every non-empty chart U of X has a unique intersection point with X_η, which is the unique generic point of U. For $\xi \in \mathsf{X}_\eta$, let us denote by U_ξ the union of all charts of X that contain ξ. By the above, ξ is the unique generic point of U_ξ, and X is the disjoint union of the U_ξ's for ξ running through X_η; it follows by quasi-compactness that X_η is finite and each U_ξ is quasi-compact. Note that the U_ξ's are the connected components of X.

Let F be a graded extension of L, let E be a graded subfield of F containing K, let $r\colon \mathbf{P}_{F/E} \to \mathbf{P}_{L/K}$ be the natural map, and let

$$\begin{tikzcd} \mathsf{Y} \arrow[r] \arrow[d] & \mathbf{P}_{F/E} \arrow[d, "r"] \\ \mathsf{X} \arrow[r] & \mathbf{P}_{L/K} \end{tikzcd}$$

be a commutative diagram of topological spaces in which $\mathsf{Y} \to \mathbf{P}_{F/E}$ and $\mathsf{X} \to \mathbf{P}_{L/K}$ belong respectively to $\mathscr{S}_{F/E}$ and $\mathscr{S}_{F/K}$. Since r is quasi-compact by 3.3.2, the continuous map $\mathsf{Y} \to \mathsf{X}$ is quasi-compact too. If V is a connected component of Y, its generic point lies above the generic point of a connected component of X: this comes from the fact that r sends the trivial graded valuation on F to the trivial graded valuation on L.

3.3.7. Γ-strict objects of $\mathscr{S}_{L/K}$. — Let Γ be a subgroup of $\mathbf{R}_+^\times$ and let X be an object of $\mathscr{S}_{L/K}$. A chart of X is said to be Γ-strict if its image on $\mathbf{P}_{L/K}$ is Γ-strict. An atlas of X is called Γ-strict if it consists of Γ-strict charts with pairwise Γ-strict intersections. The object X is called Γ-strict if it admits a Γ-strict atlas; this amounts to require that X is isomorphic to $\mathsf{Y} \times_{\mathbf{P}_{L^\Gamma/K^\Gamma}} \mathbf{P}_{L/K}$ for some object Y of $\mathscr{S}_{L^\Gamma/K^\Gamma}$. A non-empty quasi-compact open subset of X will be said to be Γ-strict if it is Γ-strict as an objet of $\mathscr{S}_{L/K}$ (this is consistent with the previous definition when $\mathsf{X} = \mathbf{P}_{L/K}$).

3.3.8. — We are now going to list some properties related to the notion of a Γ-strict object, with references to Temkin's seminal paper [**Tem04**]. Note that Remark 3.3.5 allows us to use Temkin's result in our setting (let us also mention that Temkin only deals with connected non-empty spaces, but this assumption is not seriously needed for what follows because one can argue componentwise due to 3.3.6). Let $\mathsf{X} \to \mathbf{P}_{L/K}$ be a Γ-strict object of $\mathscr{S}_{L/K}$ and let F be a graded extension of L.

(1) *Canonicity.* The object of $\mathscr{S}_{L^\Gamma/K^\Gamma}$ from which X comes is unique up to a unique homeomorphism by [**Tem04**] Prop. 2.6. We shall denote it by X^Γ. Note that in view of Remark 3.3.5, the natural continuous map $\mathsf{X} \to \mathsf{X}^\Gamma$ is surjective.

(2) *Functoriality.* Let

$$\begin{tikzcd} \mathsf{Y} \arrow[r] \arrow[d] & \mathbf{P}_{F/K} \arrow[d] \\ \mathsf{X} \arrow[r] & \mathbf{P}_{L/K} \end{tikzcd}$$

be a commutative diagram of topological spaces with $\mathsf{Y} \to \mathbf{P}_{F/K}$ a Γ-strict object of $\mathscr{S}_{F/K}$. There exists a unique continuous map $\mathsf{Y}^\Gamma \to \mathsf{X}^\Gamma$ making the

diagram

$$\begin{tikzcd}
 & \mathsf{Y} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{P}_{F/K} \arrow[dl] \arrow[dd] \\
\mathsf{Y}^{\Gamma} \arrow[rr] \arrow[dd] & & \mathbf{P}_{F^{\Gamma}/K^{\Gamma}} \arrow[dd] & \\
 & \mathsf{X} \arrow[rr] \arrow[dl] & & \mathbf{P}_{L/K} \arrow[dl] \\
\mathsf{X}^{\Gamma} \arrow[rr] & & \mathbf{P}_{L^{\Gamma}/K^{\Gamma}} &
\end{tikzcd}$$

commute; this follows again from Prop. 2.6 of [**Tem04**] Note that the uniqueness of the map is an obvious consequence of the surjectivity of $\mathsf{Y} \to \mathsf{Y}^{\Gamma}$.

(3) Let U be a Γ-strict, quasi-compact open subset of X. By (2) the open immersion $\mathsf{U} \hookrightarrow \mathsf{X}$ is obtained from a continuous map $\mathsf{U}^{\Gamma} \to \mathsf{X}^{\Gamma}$ in the category $\mathscr{S}_{L^{\Gamma}/K^{\Gamma}}$, through the base-change functor by the map $\mathbf{P}_{L/K} \to \mathbf{P}_{L^{\Gamma}/K^{\Gamma}}$. Since the map $\mathbf{P}_{L/K} \to \mathbf{P}_{L^{\Gamma}/K^{\Gamma}}$ is surjective and since $\mathsf{U} \to \mathsf{X}$ is injective, $\mathsf{U}^{\Gamma} \to \mathsf{X}^{\Gamma}$ is injective, hence is an open immersion. Taking into account the uniqueness part in the assertion of (2), or more directly the surjectivity of $\mathsf{X} \to \mathsf{X}^{\Gamma}$, we get the following: *the map* $\mathsf{V} \mapsto \mathsf{V} \times_{\mathsf{X}^{\Gamma}} \mathsf{X}$ *establishes a bijection (inclusion preserving in both directions) from the set of quasi-compact open subsets of* X^{Γ} *to that of* Γ*-strict quasi-compact open subsets of* X.

(4) Let Δ be a subgroup of $\mathbf{R}_{+}^{\times}$ containing Γ. It follows immediately from the definitions that X is Δ-strict and that $\mathsf{X}^{\Delta} = \mathsf{X}^{\Gamma} \times_{\mathbf{P}_{L^{\Gamma}/K^{\Gamma}}} \mathbf{P}_{L^{\Delta}/K^{\Delta}}$ (and hence X^{Δ} is Γ-strict and $(X^{\Delta})^{\Gamma} = X^{\Gamma}$).

(5) It follows immediately from the definition that $\mathsf{Z} := \mathsf{X} \times_{\mathbf{P}_{L/K}} \mathbf{P}_{F/K} \to \mathbf{P}_{F/K}$ is a Γ-strict object of $\mathscr{S}_{F/K}$ and that $\mathsf{Z}^{\Gamma} = \mathsf{X}^{\Gamma} \times_{\mathbf{P}_{L^{\Gamma}/K^{\Gamma}}} \mathbf{P}_{F^{\Gamma}/K^{\Gamma}}$.

3.4. Temkin's construction

To every k-analytic germ (X, x), Temkin associates a non-empty, connected object of $\mathscr{S}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, which is denoted by $\widetilde{(X, x)}$ and called the *(graded) reduction* of (X, x). Let us first explain how it is defined, and then list some of its basic properties; proofs can be found in Section 4 of [**Tem04**].

3.4.1. Definition of $\widetilde{(X, x)}$: the good case. — Assume that (X, x) is good; i.e., x has an affinoid neighborhood in X. Let V be such a neighborhood, say $V = \mathscr{M}(A)$. We endow A with its spectral semi-norm, which allows to define a residue graded $\widetilde{k}$-algebra $\widetilde{A}$. It is finitely generated (this follows from [**Tem04**], Prop. 3.1 (iii), applied to any presentation of A as an admissible quotient of a Tate algebra). The map

$f \mapsto f(x)$ induces a morphism of graded $\widetilde{k}$-algebras $\widetilde{A} \to \widetilde{\mathscr{H}(x)}$; let B denotes its image. The graded $\widetilde{k}$-algebra B is finitely generated. The subset $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{B\}$ of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ being equal to $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{S\}$ for any finite set S of homogeneous generators of B over $\widetilde{k}$, it is an affine open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ and in particular an object of $\mathscr{S}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$. It only depends on X, and not on V ; it is denoted by $\widetilde{(X,x)}$.

Let $(f_1, \ldots, f_n)$ be invertible functions on (X,x); for every i, set $r_i = |f_i(x)|$. Let (Y,x) be the analytic domain of (X,x) defined by the conjunction of inequalities $|f_i| \leqslant r_i$. Then the germ (Y,x) is good and

$$\widetilde{(Y,x)} = \widetilde{(X,x)} \cap \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{\widetilde{f_1(x)}, \ldots, \widetilde{f_n(x)}\} \subset \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}.$$

Moreover, every good analytic domain of (X,x) is of the above form (this is a consequence of Gerritzen-Grauert theorem, see [**BGR84**], §7.3.5 Thm. 1 Cor. 3 in the strict case, and [**Duc03**], Lemme 2.4 or [**Tem05**], Prop. 3.5 for the general case).

3.4.2. Definition of $\widetilde{(X,x)}$: the general case. — We do not suppose anymore that (X,x) is good. The graded reduction $\widetilde{(X,x)}$ is then defined as the colimit of the $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$-spaces $\widetilde{(Y,x)}$ for (Y,x) running through the set of good analytic domains of (X,x). This has the following concrete meaning:

- For every good analytic domain (Y,x) of (X,x) the space $\widetilde{(X,x)}$ is endowed with an open immersion $\iota_{(Y,x)} \colon \widetilde{(Y,x)} \hookrightarrow \widetilde{(X,x)}$ of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}\widetilde{k}}$-spaces.
- For every good analytic domain (Y,x) of (X,x) and every good analytic domain (Z,x) of (Y,x), the open immersion $\iota_{(Z,x)}$ is equal to the restriction of $\iota_{(Y,x)}$ to $\widetilde{(Z,x)}$ (which is in a natural way an open subset of $\widetilde{(Y,x)}$ as explained in 3.4.1).
- The space $\widetilde{(X,x)}$ is equal to $\bigcup_{(Y,x)} \iota_{(Y,x)}(\widetilde{(Y,x)})$ for (Y,x) running through the set of good analytic domains of (X,x).

3.4.3. Properties of (X,x) than can be seen on $\widetilde{(X,x)}$. — The germ (X,x) is separated, resp. good, resp. boundaryless if and only if the $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$-space $\widetilde{(X,x)}$ is an open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, resp. an affine open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, resp. the whole of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$.

3.4.4. Example. — Let us assume that $X = \mathscr{M}(k\{T\})$ and that x is its Shilov points (in other words, $x = \eta_1$). There is a $\widetilde{k}$-isomorphism $\widetilde{k}(\tau) \simeq \widetilde{\mathscr{H}(x)}$ that sends τ to $\widetilde{T(x)}$. Therefore

$$\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}} \simeq \mathbf{P}_{\widetilde{k}(\tau)/\widetilde{k}} = \mathbf{P}_{\widetilde{k}_1(\tau)/\widetilde{k}_1} = \mathbf{P}^1_{\widetilde{k}_1}$$

(the middle equality comes from Remark 3.3.5 applied with $\Gamma = \{1\}$, together with the fact that $\mathfrak{D}(\widetilde{k}(\tau)) = \mathfrak{D}(\widetilde{k})$). The reduction $\widetilde{(X,x)}$ is equal to the quasi-compact open subset $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{\widetilde{T(x)}\}$ of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, which is identified with $\mathbf{A}^1_{\widetilde{k}_1}$ through the above homeomorphism [1] between $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ and $\mathbf{P}^1_{\widetilde{k}_1}$.

Now let Y be the k-analytic space obtained by gluing $\mathscr{M}(k\{T\})$ and $\mathscr{M}(k\{S\})$ along the isomorphism $\mathscr{M}(k\{T,T^{-1}\}) \simeq \mathscr{M}(k\{S,S^{-1}\})$ given by $S \mapsto T$. The Shilov points of $\mathscr{M}(k\{T\})$ and $\mathscr{M}(k\{S\})$ are identified and give rise to a single point y on Y. By the above, there is a homeomorphism between the Zariski-Riemann space $\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$ and $\mathbf{P}^1_{\widetilde{k}_1}$, modulo which Temkin's reduction $\widetilde{(Y,y)}$ is the *affine line with double origin*, viewed as a $\mathbf{P}^1_{\widetilde{k}_1}$-space through the open immersion of each of the two copies of $\mathbf{A}^1_{\widetilde{k}_1}$ it contains (by design). Hence $\widetilde{(Y,y)} \to \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$ is not one-to-one, which witnesses the fact that (Y,y) is not separated.

3.4.5. Functoriality. — Let L be an analytic extension of k, let (X,x) be a k-analytic germ, and let (Y,y) be an L-analytic germ. Any morphism $(Y,y) \to (X,x)$ gives rise in a natural way to a commutative diagram of topological spaces

$$\begin{tikzcd} \widetilde{(Y,y)} \arrow[r] \arrow[d] & \widetilde{(X,x)} \arrow[d] \\ \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}} \arrow[r] & \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}} \end{tikzcd}$$

in which the bottom arrow is the one induced by the extension $\widetilde{\mathscr{H}(x)} \hookrightarrow \widetilde{\mathscr{H}(y)}$. Note that $\widetilde{(Y,y)} \to \widetilde{(X,x)}$ is quasi-compact by 3.3.6. Let us now list some very useful properties of this construction.

(1) If (X,x) is any analytic germ, then $(Y,x) \mapsto \widetilde{(Y,x)}$ induces a bijection between the set of analytic domains of (X,x) and the set of quasi-compact, non-empty open subsets of $\widetilde{(X,x)}$; moreover, this bijection commutes with finite unions and intersections.

(2) If k is an analytic field, a morphism $(Y,y) \to (X,x)$ of k-analytic germs is boundaryless if and only if the local homeomorphism
$$\widetilde{(Y,y)} \to \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}} \widetilde{(X,x)}$$
is bijective (hence a homeomorphism).

(3) If (X,x) is any analytic germ and if (Y,x) is a closed analytic subgerm of (X,x), then $\widetilde{(Y,x)} \to \widetilde{(X,x)}$ is a homeomorphism.

(4) Let X be a k-analytic space and let Y be an X-analytic space. Let L be an analytic extension of k, let Z be an L-analytic space, and let $Z \to X$ be a

1. Be aware that this homeomorphism does not preserve the notion of an affine open subset: indeed, as remarked above, the whole space $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ is affine, though $\mathbf{P}^1_{\widetilde{k}^1}$ is not affine as a scheme! But this local terminology inconsistency should not cause any trouble in practice.

morphism of analytic spaces. Set $T = Y \times_X Z$, let t be a point of T and let x, y and z denote the images of t in X, Y and Z respectively. Let us set for short

$$\begin{aligned}\mathsf{X} &= \widetilde{(X,x)} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}} \mathbf{P}_{\widetilde{\mathscr{H}(t)}/\widetilde{L}}\\ \mathsf{Y} &= \widetilde{(Y,y)} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}} \mathbf{P}_{\widetilde{\mathscr{H}(t)}/\widetilde{L}}\\ \mathsf{Z} &= \widetilde{(Z,z)} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(z)}/\widetilde{L}}} \mathbf{P}_{\widetilde{\mathscr{H}(t)}/\widetilde{L}}\end{aligned}$$

The natural continuous $\mathbf{P}_{\widetilde{\mathscr{H}(t)}/\widetilde{L}}$-map $\widetilde{(T,t)} \to \mathsf{Y} \times_{\mathsf{X}} \mathsf{Z}$ is then a homeomorphism.

(5) Let $(Y, y) \to (X, x)$ be a morphism of analytic germs, let (V, x) be an analytic domain of (X, x), and set $(W, y) = (Y, y) \times_{(X,x)} (V, x)$. The reduction $\widetilde{(W,y)}$ is equal to the pre-image of $\widetilde{(V,x)}$ in $\widetilde{(Y,y)}$.

3.4.6. Remarks. — Assertion (4) is stated by Temkin only when $L = k$: this is Prop. 4.6 of [**Tem04**] But in view of Prop. 3.1 of op. cit., its proof can be straightforwardly adapted to work for arbitrary L. Assertion (3) can be seen as a particular case of (2), but it can be checked directly from the definition after reduction to the affinoid case. Assertion (5) is a particular case of (4).

3.5. Temkin's reduction and Γ-strictness

3.5.1. Lemma. — *Let (X, x) be a k-analytic germ. The following are equivalent:*

(i) *The point x has a Γ-strict k-affinoid neighborhood in X.*

(ii) *The germ $\widetilde{(X,x)}$ is a Γ-strict affine open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$.*

Proof. — Let us assume that (i) holds, and let us choose a Γ-strict affinoid neighborhood V of x, say $V = \mathscr{M}(A)$. By 3.1.3, the spectral semi-norm on A takes values in $(|k^\times|^{\mathbf{Q}} \cdot \Gamma)_0$; the image B of the natural morphism $\widetilde{A} \to \widetilde{\mathscr{H}(x)}$ is thus contained in $\widetilde{\mathscr{H}(x)}^{|k^\times|^{\mathbf{Q}} \cdot \Gamma}$. This implies, in view of the equality $\widetilde{(X,x)} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{S\}$ for any finite set S of homogeneous generators of B, that $\widetilde{(X,x)}$ is affine and Γ-strict.

Let us now assume that (ii) holds. We can then write $\widetilde{(X,x)} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{f_1, \ldots, f_n\}$ where each f_i is a non-zero element of $\widetilde{\mathscr{H}(x)}^{r_i}$ for some r_i in Γ. Since $\widetilde{(X,x)}$ is an affine open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, the germ (X, x) is good (3.4.3); in other words, x has an affinoid neighborhood V in X. By shrinking V if needed, we may and do assume that there exist invertible analytic functions $h_1, \ldots, h_n$ on V satisfying for every i the equalities $|h_i(x)| = r_i$ and $\widetilde{h_i(x)} = f_i$. Let $h\colon V \to \mathbf{A}_k^{n,\mathrm{an}}$ be the morphism induced by the h_i's; set $t = h(x)$. Let W be the affinoid domain of $\mathbf{A}_k^{n,\mathrm{an}}$ defined by the inequalities $|T_i| \leqslant r_i$ for $i = 1, \ldots, n$ (where the T_i's are the coordinate functions on the affine space). Since the quasi-compact open subset $\widetilde{(X,x)} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{f_1, \ldots, f_n\}$

of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ is by construction the pre-image of $\mathbf{P}_{\widetilde{\mathscr{H}(t)}/\widetilde{k}}\{\widetilde{T_1(t)}, \ldots, \widetilde{T_n(t)}\} = \widetilde{(W,t)}$, it follows from 3.4.5 (5) that $(X,x) \to (\mathbf{A}_k^{n,\mathrm{an}}, t)$ goes through (W,t). Hence we can shrink V so that there exist an affinoid neighborhood W' of t in $\mathbf{A}_k^{n,\mathrm{an}}$ such that $h(V)$ is contained in $W \cap W'$. Since $\mathbf{A}_k^{n,\mathrm{an}}$ has no boundary, we may and do assume that W' is Γ-strict. As $\widetilde{(V,x)} = \widetilde{(X,x)}$ is the pre-image of $\widetilde{(W \cap W', t)} = \widetilde{(W,t)}$ inside $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, the morphism $V \to W \cap W'$ is inner at x by 3.4.5 (2). As $W \cap W'$ is Γ-strict, Lemma 2.5.11 of [**Ber90**] immediately implies that x has a Γ-strict affinoid neighborhood in V, hence in X. □

3.5.2. Lemma. — *Let (X,x) be a k-analytic germ. The following are equivalent:*

(i) *The germ (X,x) is Γ-strict.*
(ii) *The reduction $\widetilde{(X,x)}$ is Γ-strict (see 3.3.7).*

Proof. — The implication (i)⇒(ii) follows directly from Lemma 3.5.1. Now let us assume that $\widetilde{(X,x)}$ is Γ-strict, and let (U_i) be a Γ-strict atlas of $\widetilde{(X,x)}$. For every i, let (X_i, x) be the analytic domain of (X,x) that corresponds to U_i; for every (i,j), the analytic domain of (X,x) that corresponds to $\mathsf{U}_i \cap \mathsf{U}_j$ is $(X_i \cap X_j, x)$. In order to prove that (X,x) is Γ-strict, it is sufficient to prove that (X_i,x) and $(X_i \cap X_j, x)$ are Γ-strict for all i,j; hence we reduce to the case where $\widetilde{(X,x)}$ is a Γ-strict, non-empty, quasi-compact open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$.

Under this assumption $\widetilde{(X,x)}$ admits a finite covering (V_j) by Γ-strict *affine* open subsets. For every j, let (V_j, x) denote the analytic domain of (X,x) that corresponds to V_j. Since the intersection of two Γ-strict affine open subsets of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ is still affine and Γ-strict, it follows from Lemma 3.5.1 that (V_j, x) and $(V_j \cap V_\ell, x)$ are Γ-strict (and good) for all j, ℓ. This implies that (X,x) is Γ-strict, which ends the proof. □

3.5.3. The Γ-graded reduction. — Let (X,x) be a Γ-strict k-analytic germ. Its reduction $\widetilde{(X,x)}$ is Γ-strict by Lemma 3.5.2 above; recall that $\widetilde{(X,x)}^{\Gamma}$ then denotes the unique object of $\mathscr{S}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$ from which $\widetilde{(X,x)}$ arises. If Δ is any subgroup of $\mathbf{R}_+^\times$ containing Γ, then (X,x) is Δ-strict as well and

$$\widetilde{(X,x)}^{\Delta} = \widetilde{(X,x)}^{\Gamma} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}} \mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Delta}/\widetilde{k}^{\Delta}}$$

(see 3.3.8 (4)).

Let L be an analytic extension of k, let (Y,y) be a Γ-strict L-analytic germ, and let $(Y,y) \to (X,x)$ be a morphism of analytic germs. There is a unique continuous map $\widetilde{(Y,y)}^{\Gamma} \to \widetilde{(X,x)}^{\Gamma}$ making the diagram

$$\begin{tikzcd}
& \widetilde{(Y,y)} \arrow[rr] \arrow[dl] \arrow[dd] & & \widetilde{(X,x)} \arrow[dl] \arrow[dd] \\
\widetilde{(Y,y)}^{\Gamma} \arrow[rr] \arrow[dd] & & \widetilde{(X,x)}^{\Gamma} \arrow[dd] & \\
& \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}} \arrow[rr] \arrow[dl] & & \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}} \arrow[dl] \\
\mathbf{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{k}^{\Gamma}} \arrow[rr] & & \mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}} &
\end{tikzcd}$$

commute. Indeed, if $L = k$ this is a direct application of 3.3.8 (2); and if $Y = X_L$, this is a consequence of the equalities

$$\widetilde{(X_L,y)} = \widetilde{(X,x)} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}} \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{L}}$$

and

$$\widetilde{(X_L,y)}^{\Gamma} = \widetilde{(X,x)}^{\Gamma} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}} \mathbf{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{L}^{\Gamma}}$$

(the first one comes from 3.4.5 (4), and it implies the second one). The general case now follows formally by combining those two particular cases.

It follows from 3.3.8 (3) that a quasi-compact open subset of $\widetilde{(X,x)}$ is Γ-strict if and only if it is the pre-image of a quasi-compact open subset of $\widetilde{(X,x)}^{\Gamma}$. This implies that a finite union or a finite intersection of Γ-strict quasi-compact open subsets of $\widetilde{(X,x)}$ is a Γ-strict quasi-compact open subset, and that the pre-image in $\widetilde{(Y,y)}$ of any Γ-strict quasi-compact open subset of $\widetilde{(X,x)}$ is a Γ-strict quasi-compact open subset of $\widetilde{(Y,y)}$.

3.5.4. Canonicity of the Γ-strict structure. — Let (X,x) be a Γ-strict k-analytic germ. Let $(V_i,x)_i$ be a Γ-strict affinoid atlas defining a Γ-strict analytic structure on (X,x), and let (V,x) be a Γ-strict analytic domain of (X,x). By Lemma 3.5.2, $\widetilde{(V,x)}$ and the $\widetilde{(V_i,x)}$'s are Γ-strict. Therefore for every index i, the intersection $\widetilde{(V,x)} \cap \widetilde{(V_i,x)}$ is Γ-strict by 3.5.3 above, which implies again by Lemma 3.5.2 that $(V,x) \cap (V_i,x)$ is Γ-strict. Since (V_i,x) is separated, its Γ-strict structure is unique, hence $(V,x) \cap (V_i,x)$ is a Γ-strict analytic domain of (X,x) *for the given* Γ*-strict structure on* (X,x). As this holds for every i, the analytic domain (V,x) is a Γ-strict analytic domain *for the given* Γ*-strict structure on* (X,x).

As a consequence, there exists a *unique* Γ-strict k-analytic structure on (X,x); the corresponding Γ-strict analytic domains are simply the analytic domains (V,x) that are Γ-strict in the sense that was given up to now to this notion (i.e., analytic domains that admit *some* Γ-strict analytic structure); by Lemma 3.5.2, these are precisely the analytic domains (V, x) whose reduction $\widetilde{(V, x)}$ is Γ-strict.

These facts immediately extend to the category of k-analytic spaces: if a k-analytic space Z admits a Γ-strict analytic structure, the latter is unique and the corresponding Γ-strict analytic domains are simply the analytic domains that admit a Γ-strict analytic structure.

Therefore all (possible) ambiguities mentioned in Remark 3.1.5 vanish, so now we can use the notion of Γ-strictness without worrying about such unpleasant subtleties.

3.5.5. Γ-strictness is a local notion. — Let us emphasize an important consequence of 3.5.4: a k-analytic space X is Γ-strict if and only if the germ (X, x) is Γ-strict for every point x of X. Indeed, the direct implication is obvious. Assume now that (X, x) is Γ-strict for all $x \in X$. Then every point of X admits a Γ-strict analytic neighborhood, which can be chosen to be an *open* subset of X, because any open subset of a Γ-strict analytic space is Γ-strict. Therefore X can be covered by Γ-strict open subsets. Since the intersection of two Γ-strict open subsets of X is Γ-strict (again, because Γ-strictness is inherited by open subsets of a Γ-strict space), X is Γ-strict.

3.5.6. Fullness of the Γ-strict subcategory. — Let (X, x) be a Γ-strict k-analytic germ, let L be an analytic extension, and let (Y, y) be a Γ-strict L-analytic germ. Suppose that we are given a morphism $(Y, y) \to (X, x)$ of analytic germs.

Let (V, x) be a Γ-strict analytic domain of (X, x) and let (W, y) be the fiber product $(Y, y) \times_{(X,x)} (V, x)$. By Lemma 3.5.2 and 3.5.3 above, the pre-image of $\widetilde{(V, x)}$ on $\widetilde{(Y, x)}$ is open, quasi-compact and Γ-strict. But this pre-image can be identified with $\widetilde{(W, y)}$ by 3.4.5 (5); hence (W, y) is a Γ-strict analytic domain of (Y, y) by Lemma 3.5.2, and $(Y, y) \to (X, x)$ is thus a morphism of Γ-strict analytic germs (note that here we have used implicitly 3.5.4).

We have thus proved that the category of Γ-strict analytic germs is a *full* subcategory of the category of analytic germs. As a consequence, the category of Γ-strict k-analytic germs (resp. analytic spaces, resp. k-analytic spaces) is a full subcategory of the category of k-analytic germs (resp. analytic spaces, resp. k-analytic spaces).

3.5.7. Preservation of Γ-strictness under boundaryless pullback. — Let $(Y, y) \to (X, x)$ be a boundaryless morphism of k-analytic germs. Assume that (X, x) is Γ-strict. Its reduction $\widetilde{(X, x)}$ is then Γ-strict by Lemma 3.5.2. Since $(Y, y) \to (X, x)$ is boundaryless,

$$\widetilde{(Y, y)} = \widetilde{(X, x)} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}} \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$$

by 3.4.5 (2), hence $\widetilde{(Y, y)}$ is Γ-strict by 3.3.8 (5). Using again Lemma 3.5.2, we see that (Y, y) is Γ-strict.

In view of 3.5.5, this immediately implies that if $Y \to X$ is a boundaryless morphism of k-analytic spaces, then Y is Γ-strict as soon as X is Γ-strict.

3.5.8. Remark. — By 3.5.7 above, if $Y \to X$ is a finite morphism between k-analytic spaces and if X is Γ-strict, then Y is Γ-strict. This can also be seen without using Temkin's reduction, as follows. First of all, it is sufficient to ensure that the pull-back of a given Γ-strict affinoid atlas on X is a Γ-strict affinoid atlas on Y. Hence we reduce to the case where both Y and X are affinoid, say $Y = \mathscr{M}(B)$ and $X = \mathscr{M}(A)$.

Now since A is Γ-strict, there exists a polyradius $r = (r_1, \ldots, r_n)$ consisting of elements of Γ such that $|k_r^\times| \neq \{1\}$ and A_r is k_r-strict. The finite A_r-algebra B_r is then k_r-strict as well. Therefore by [**BGR84**] 6.2.1/4, the spectral radius of every element of B_r belongs to

$$\left|k_r^\times\right|^{\mathbf{Q}} \cup \{0\} \subset \cup(\left|k^\times\right| \cdot \Gamma)^{\mathbf{Q}} \cup \{0\}.$$

This holds in particular for every element of B, whence the Γ-strictness of B by 3.1.3.

3.5.9. — It follows from 3.5.3 that the assignment $(X, x) \mapsto \widetilde{(X, x)}^\Gamma$ is functorial in the Γ-strict analytic germ (X, x). Using straightforward descent arguments (based upon the surjectivity of the continuous map $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}} \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}^\Gamma/\widetilde{k}^\Gamma}$) we deduce from 3.4.3, 3.4.5 and Lemma 3.5.1 that it enjoys the following properties:

(0) A Γ-strict k-analytic germ is separated, resp. good, resp. boundaryless if and only if the $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^\Gamma/\widetilde{k}^\Gamma}$-space $\widetilde{(X, x)}^\Gamma$ is an open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^\Gamma/\widetilde{k}^\Gamma}$, resp. an affine open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^\Gamma/\widetilde{k}^\Gamma}$, resp. the whole of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^\Gamma/\widetilde{k}^\Gamma}$.

(1) If (X, x) is any Γ-strict analytic germ, then $(V, x) \mapsto \widetilde{(V, x)}^\Gamma$ induces a bijection between the set of Γ-strict analytic domains of (X, x) and the set of quasi-compact, non-empty open subsets of $\widetilde{(X, x)}^\Gamma$; moreover, this bijection commutes with finite unions and intersections.

(2) If k is an analytic field, a morphism $(Y, y) \to (X, x)$ of Γ-strict k-analytic germs is boundaryless if and only if the continuous local homeomorphism

$$\widetilde{(Y, y)}^\Gamma \to \mathbf{P}_{\widetilde{\mathscr{H}(y)}^\Gamma/\widetilde{k}^\Gamma} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}^\Gamma/\widetilde{k}^\Gamma}} \widetilde{(X, x)}^\Gamma$$

is bijective (hence a homeomorphism).

(3) If (X, x) is any Γ-strict analytic germ and if (Y, x) is a closed analytic subspace of (X, x), then $\widetilde{(Y, x)}^\Gamma \to \widetilde{(X, x)}^\Gamma$ is a homeomorphism.

(4) Let X be a Γ-strict k-analytic space, and let Y be a Γ-strict X-analytic space. Let L be an analytic extension of k, let Z be a Γ-strict L-analytic space and let $Z \to X$ be a morphism of analytic spaces. Set $T = Y \times_X Z$, let t be a point of T, and let x, y and z denote the images of t in X, Y, and Z respectively. Let us set for short

$$\begin{aligned}
\mathsf{X} &= \widetilde{(X,x)}^{\Gamma} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}} \mathbf{P}_{\widetilde{\mathscr{H}(t)}^{\Gamma}/\widetilde{L}^{\Gamma}}\\
\mathsf{Y} &= \widetilde{(Y,y)}^{\Gamma} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{k}^{\Gamma}}} \mathbf{P}_{\widetilde{\mathscr{H}(t)}^{\Gamma}/\widetilde{L}^{\Gamma}}\\
\mathsf{Z} &= \widetilde{(Z,z)}^{\Gamma} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(z)}^{\Gamma}/\widetilde{L}^{\Gamma}}} \mathbf{P}_{\widetilde{\mathscr{H}(t)}^{\Gamma}/\widetilde{L}^{\Gamma}}
\end{aligned}$$

The natural continuous $\mathbf{P}_{\widetilde{\mathscr{H}(t)}^{\Gamma}/\widetilde{L}^{\Gamma}}$-map $\widetilde{(T,t)}^{\Gamma} \to \mathsf{Y} \times_{\mathsf{X}} \mathsf{Z}$ is then a homeomorphism.

(5) Let $(Y,y) \to (X,x)$ be a morphism of Γ-strict analytic germs, let (V,x) be a Γ-strict analytic domain of (X,x), and set $(W,y) = (Y,y) \times_{(X,x)} (V,x)$. The reduction $\widetilde{(W,y)}^{\Gamma}$ is equal to the pre-image of $\widetilde{(V,x)}^{\Gamma}$ in $\widetilde{(Y,y)}^{\Gamma}$.

3.5.10. Remark. — Assume that $|k^{\times}| \neq \{1\}$. If (X,x) is a strictly k-analytic space, we shall write $\widetilde{(X,x)}^{1}$ instead of $\widetilde{(X,x)}^{\{1\}}$. The $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^{1}/\widetilde{k}^{1}}$-space $\widetilde{(X,x)}^{1}$ is nothing but the *non-graded* Temkin reduction of (X,x) defined in [**Tem00**], and all properties of Γ-strict analytic spaces and Γ-graded reduction that we have just established were already known by Temkin's works [**Tem00**] and [**Tem04**] in the case $\Gamma = \{1\}$. What we have done here simply consists in explaining how Temkin's methods actually apply to an arbitrary subgroup of $\mathbf{R}_{+}^{\times}$.

3.5.11. Remark. — Let (X,x) be a Γ-strict good germ, and let V be a Γ-strict affinoid neighborhood of x in X, say $V = \mathscr{M}(A)$. Let B denote the image of $\widetilde{A}$ in $\widetilde{\mathscr{H}(x)}$ through the evaluation map at x.

Let f be any non-nilpotent function belonging to A. Its spectral semi-norm belongs to $(|k^{\times}|\cdot\Gamma)^{\mathbf{Q}}$; hence there exists $\lambda \in k^{\times}$ and an integer n such that the spectral norm of λf^n belongs to Γ. If $|\cdot|$ belongs to $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$, then $\left|\widetilde{f(x)}\right| \leqslant 1$ if and only if $\left|\widetilde{\lambda f^n(x)}\right| \leqslant 1$. As a consequence,

$$\widetilde{(X,x)} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{B\} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{B^{\Gamma}\}.$$

Since $B^{\Gamma} \subset \widetilde{\mathscr{H}(x)}^{\Gamma}$, it follows that $\widetilde{(X,x)}^{\Gamma} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}\{B^{\Gamma}\}$. Note that B^{Γ} can be described directly as the image of the finitely generated $\widetilde{k}^{\Gamma}$-graded algebra $\widetilde{A}^{\Gamma}$ inside $\widetilde{\mathscr{H}(x)}^{\Gamma}$ through the evaluation map at x.

Let (Y,x) be a good Γ-strict analytic domain of (X,x). It can be described by a finite conjunction of inequalities

$$|f_1| \leqslant r_1 \text{ and } \ldots \text{ and } |f_n| \leqslant r_n$$

where the f_i's are invertible function on (X,x) and $r_i = |f_i(x)|$ for every i. Now by Γ-strictness of (Y,y), every r_i appears as the spectral semi-norm of f_i on some Γ-strict affinoid neighborhood of x in Y, hence belongs to $(|k^{\times}|\cdot\Gamma)^{\mathbf{Q}}$. Therefore by replacing every f_i with $\lambda_i f_i^{n_i}$ for suitable (λ_i, n_i) in $k^{\times} \times \mathbf{Z}_{\geqslant 0}$ we may assume that $r_i \in \Gamma$ for all i, and the equality $\widetilde{(Y,x)} = \widetilde{(X,x)} \cap \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{\widetilde{f_1(x)}, \dots, \widetilde{f_i(x)}\}$ then implies that $\widetilde{(Y,x)}^{\Gamma} = \widetilde{(X,x)}^{\Gamma} \cap \mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}}\{\widetilde{f_1(x)}, \dots, \widetilde{f_n(x)}\}$.

One could use the above to describe directly the Γ-graded reduction of a general Γ-strict germ, analogously to what was done in 3.4.1 and 3.4.2 for general graded reductions (i.e., $\mathbf{R}_{+}^{\times}$-graded reductions), by considering good charts and gluing. When $\Gamma = \{1\}$, this is essentially the way Temkin's ungraded reductions were built in [**Tem00**].

3.5.12. Remark. — Let X be a Γ-strict analytic space. If X is quasi-compact, then it admits a finite G-covering by Γ-strict affinoid domains. If X is paracompact, it admits a locally finite covering by Γ-strict affinoid domains: the arguments are *mutatis mutandi* the same as in 1.2.6.

CHAPTER 4

FLATNESS IN ANALYTIC GEOMETRY

In this chapter, we introduce one of the key notions of this memoir, namely flatness. The definition is given in Section 4.1 (Definition 4.1.8) and as explained in the Introduction, it differs from naive flatness even in the good case: one *requires* stability under arbitrary base change (including ground field extension). The basic properties are then stated, and some simple examples are given: for instance, any k-analytic space is flat over k (Lemma 4.1.13; note that the stability under base change requires some work).

Section 4.2 is devoted to GAGA results about flatness. For instance, let $\mathscr{Y} \to \mathscr{X}$ be a morphism of schemes of finite type over a given affinoid algebra, and let y be a point of $\mathscr{Y}^{\mathrm{an}}$. Then $\mathscr{Y}^{\mathrm{an}}$ is flat over $\mathscr{X}^{\mathrm{an}}$ at y if and only if $\mathscr{Y}$ is flat over $\mathscr{X}$ at the image of y (the "only if" part is the easiest one, see Lemma 4.2.1; but the "if" is still quite straightforward, see Proposition 4.2.4). Let us now consider a morphism $Y \to X$ between k-affinoid spaces, and let y denote a point of Y. If Y is flat over X at y, then Spec $\mathscr{O}_Y(Y)$ is flat over Spec $\mathscr{O}_X(X)$ at the image of y (this is also covered by Lemma 4.2.1), but the converse is false in general: we give a counter-example in 4.4.9. Nevertheless, if Spec $\mathscr{O}_Y(Y)$ is flat over Spec $\mathscr{O}_X(X)$ at the image of y and if y lies on a closed analytic subspace of Y which is finite over X, then $Y \to X$ is naively flat at y: this is Theorem 4.2.5, whose proof rests on *Critères locaux de platitude*, [**SGA 1**] Exposé IV (we shall see later that under these assumptions, $Y \to X$ is even flat at y; see Theorem 8.3.7).

In Section 4.3, we investigate finite flat morphisms. The results we present there are essentially due to Berkovich (see [**Ber93**], 3.2), but we include proofs (sometimes different from Berkovich's) for the reader's convenience. Let us simply mention here one of them: finite flat maps are open (Corollary 4.3.2).

In Section 4.4 we present a counter-example (suggested by Temkin, and described in 4.4.2) showing that naive flatness is not preserved by base-change; note that the detailed study of this counter-example uses some results of section 4.3, which is the reason why we have not carried it out immediately after having given the definition of naive flatness.

We end this Chapter by showing in Section 4.5 that our notion of flatness behaves similarly to flatness in algebraic geometry. For instance: usual algebraic properties descend under flat maps (Lemma 4.5.2); flatness can be checked after flat base change or arbitrary ground field extension (Prop. 4.5.5 and 4.5.6); flatness has the expected properties as far as exactness of complexes of coherent sheaves is concerned (Proposition 4.5.7); it ensures that some properties that hold inside a fiber can be spread out to the whole space (Lemma 4.5.8); and it implies the usual formula relating the local dimensions of the source space, of the target space, and of the fibers (Lemma 4.5.11).

4.1. Naive and non-naive flatness

4.1.1. — Let $Y \to X$ be a morphism of *good* analytic spaces, let y be a point of Y and let x be its image on X. Let $\mathscr{F}$ be a coherent sheaf on Y. One could be tempted to say that $\mathscr{F}$ is X-flat at y if it is the case in the framework of locally ringed spaces; i.e., if $\mathscr{F}_y$ is a flat $\mathscr{O}_{X,x}$-module.

But we have chosen to call the latter property *naive* X-flatness of $\mathscr{F}$ at y, because it turns out that it is not a reasonable candidate to be the analytic avatar of scheme-theoretic flatness. Indeed, as we are going to see below in Section 4.4 through an explicit example, *it is not stable under base change.* Let us now give the "right" definition of flatness.

4.1.2. Definition **(Analytic flatness: the good case)**
Let $Y \to X$ be a morphism between good k-analytic spaces, and let $\mathscr{F}$ be a coherent sheaf on Y. Let y be a point of Y. We shall say that $\mathscr{F}$ is *X-flat at y* if for any any good analytic space X', for any morphism $X' \to X$, and for any point y' lying above y on $Y' := Y \times_X X'$, the pull-back of $\mathscr{F}$ on Y' is naively X'-flat at y'.

4.1.3. Remark. — We emphasize that in Definition 4.1.2 above, the space X' can be any good analytic space in the sense of 1.2.8: naive flatness has to be checked after base change by a space defined over an arbitrary analytic extension of k; in particular, it is required to hold after arbitrary ground field extension.

4.1.4. Remark. — Theoretically, checking X-flatness of $\mathscr{F}$ at y requires to consider all possible base-changes. But we shall see in fact later (Theorem 8.3.6) that it if there exists an analytic extension L of k and an L-rigid point on Y_L over y at which $\mathscr{F}_L$ is naively X_L-flat, then $\mathscr{F}$ is X-flat at y.

4.1.5. Example. — Let X be a good k-analytic space and let V be a good analytic domain of X. For every $x \in V$, the coherent sheaf $\mathscr{O}_V$ is X-flat at x by 2.1.3 (2).

For further investigation about flatness, we shall need the following technical (but very easy) lemma.

4.1.6. Lemma. — *Let*

$$\begin{tikzcd} D & C \arrow[l] \\ B \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

be a commutative diagram of commutative rings such that C, resp. D, is flat over A, resp. faithfully flat over B. If M is a B-module such that $D \otimes_B M$ is C-flat, then M is A-flat.

Proof. — Let $N \hookrightarrow N'$ be an injective linear map between two A-modules. As C is A-flat, $C \otimes_A N \hookrightarrow C \otimes_A N'$. As $M \otimes_B D$ is C-flat,

$$\underbrace{(M \otimes_B D) \otimes_C (C \otimes_A N)}_{(M \otimes_B D) \otimes_A N} \hookrightarrow \underbrace{(M \otimes_B D) \otimes_C (C \otimes_A N')}_{(M \otimes_B D) \otimes_A N'}.$$

In other words, $(N \otimes_A M) \otimes_B D \hookrightarrow (N' \otimes_A M) \otimes_B D$. Faithful flatness of the B-algebra D now implies that $N \otimes_A M \hookrightarrow N' \otimes_A M$. $\square$

4.1.7. — Let $\varphi \colon Y \to X$ be a morphism of good k-analytic spaces, and let $\mathscr{F}$ be a coherent sheaf on Y. Let y be a point of Y.

(1) If $\mathscr{F}$ is X-flat at y, it is in particular naively X-flat at y (by definition).
(2) Let V be a good analytic domain of Y containing y, and let U be a good analytic domain of X containing $\varphi(Y)$. Let us consider the four following properties:
   (i) $\mathscr{F}$ is naively X-flat at y;
   (ii) $\mathscr{F}_{|V}$ is naively U-flat at y.
   (iii) $\mathscr{F}$ is X-flat at y;
   (iv) $\mathscr{F}_{|V}$ U-flat at y.

   It follows straightforwardly from Example 4.1.5 and from Lemma 4.1.6 that (i) $\Longleftrightarrow$ (ii). Applying this after an arbitrary good base change we see that (iii) $\Longleftrightarrow$ (iv) as well.

4.1.8. Definition **(Analytic flatness: the general case)**

Let $\varphi \colon Y \to X$ be a morphism of *non-necessarily good* k-analytic spaces and let $y \in Y$. Let $\mathscr{F}$ be a coherent sheaf on Y. It follows from 4.1.7 (2) that the following are equivalent:

(i) For all pairs (V, U), where V is a good analytic domain of Y containing y and where U is a good analytic domain of X containing $\varphi(V)$, the coherent sheaf $\mathscr{F}_{|V}$ is U-flat at y.
(ii) There exist a good analytic domain V of Y containing y and a good analytic domain U of X containing $\varphi(V)$ such that the coherent sheaf $\mathscr{F}_{|V}$ is U-flat at y.

(iii) There exist an affinoid domain V of Y containing y and an affinoid domain U of X containing $\varphi(V)$ such that the coherent sheaf $\mathscr{F}_{|V}$ is U-flat at y.

We shall say that $\mathscr{F}$ is *X-flat at y* if it satisfies the equivalent assertions (i), (ii) and (iii) above (this definition is obviously equivalent to the previous one when Y and X are good). We shall say that $\mathscr{F}$ is X-flat if it is X-flat at every point of Y. We shall say that Y is X-flat as y or X-flat if so is $\mathscr{O}_Y$. We shall say that $\mathscr{F}$ is flat at y, resp. flat, if it is Y-flat at y, resp. Y-flat (with respect to $\mathrm{Id}\colon Y \to Y$)

We are now going to state some basic facts (4.1.9–4.1.12), each of which can be proved by reduction to the good case (which is allowed by the very definition of flatness). After such a reduction, the first two follow straightforwadly from the definition, the third from Example 4.1.5, and the fourth from 4.1.7 (2). We shall then investigate some situations in which flatness is expected, and actually holds – but some work is needed to prove it.

4.1.9. Stability under base-change. — Let $Y \to X$ be a morphism between k-analytic spaces and let y be a point of Y. Let X' be an analytic space, let $X' \to X$ be a morphism, and let y' be a point of $Y' := Y \times_X X'$ lying over y. If $\mathscr{F}$ is a coherent sheaf on Y that is X-flat at y, its pull-back to Y' is X'-flat at y': this follows directly from the definition, which *requires* that flatness be preserved under base change.

4.1.10. Stability under composition. — Let $Z \to Y$ and $Y \to X$ be morphisms between k-analytic spaces, let z be a point of Z and let y be its image on Y.

(1) If $\mathscr{F}$ is a coherent sheaf on Z that is Y-flat at z, and if Y is X-flat at y, then $\mathscr{F}$ is X-flat at Z.

(2) If $\mathscr{G}$ is a coherent sheaf on Y that is X-flat at y, and if Z is Y-flat at z, then $\mathscr{G}_Z$ is X-flat at z.

Indeed, both assertions are G-local, which allows to reduce to the good case and then to the corresponding naive statements, which are are obvious.

4.1.11. Flatness of analytic domains. — The inclusion of an analytic domain is flat: we see this again by reducing to the good case, and then to the corresponding naive statement, which is Example 4.1.5.

4.1.12. Good behavior by restriction to analytic domains. — Let $Y \to X$ be a morphism of k-analytic spaces, let V be an analytic domain of Y and let U be an analytic domain of X which contains the image of V. Let $\mathscr{F}$ be a coherent sheaf on Y and let y be a point of V. The coherent sheaf $\mathscr{F}$ is X-flat at y if and only if $\mathscr{F}_V$ is U-flat at y: this follows once again by reducing to the good case and then to the corresponding naive statement, which is 4.1.7 (2).

4.1.13. Lemma. — *Let Y be a k-analytic space. The structure map $Y \to \mathscr{M}(k)$ is flat.*

Proof. — We may and do assume that Y is k-affinoid. Let X be an affinoid space, let y be a point of $Y \times_k X$, and let x be its image on X. Let U be an affinoid neighborhood of x in X, and let V be an affinoid neighborhood of y in $Y \times_X U$. Let A, B and C be the respective algebras of analytic functions on U, Y and V. The $A\widehat{\otimes}_k B$-algebra C is flat (2.1.3 (2)); the A-algebra $A\widehat{\otimes}_k B$ is flat ([**Ber93**], Lemma 2.1.2; its statement involves an analytic extension K of k, but its proof works for K any Banach space over k, and we apply it with $K = B$); hence C is A-flat. By a straightforward limit argument, $\mathscr{O}_{Y\times_k X,y}$ is a flat $\mathscr{O}_{X,x}$-algebra, whence the lemma. □

4.1.14. Lemma. — *Let X be a good analytic space and let $\mathscr{F}$ ba a coherent sheaf on X. Let x be a point of X at which $\mathscr{F}$ is naively flat; i.e., $\mathscr{F}_x$ is a flat $\mathscr{O}_{X,x}$-module. There exists an open neighborhood U of x in X such that $\mathscr{F}_U$ is a free $\mathscr{O}_U$-module, and $\mathscr{F}$ is flat at x.*

Proof. — Since $\mathscr{F}_x$ is flat over $\mathscr{O}_{X,x}$, it is a free- $\mathscr{O}_{X,x}$-module. This implies the existence of an open neighborhood U such that $\mathscr{F}_U$ is a free $\mathscr{O}_U$-module. Now for every good analytic space Z and every morphism $Z \to X$, the pre-image of $\mathscr{F}$ on $Z \times_X U$ is free, and thus naively flat at every point of $Z \times_X U$, and in particular at every pre-image of x on Z. The coherent sheaf $\mathscr{F}$ is then flat at x. □

4.1.15. Lemma. — *Let $Y \xrightarrow{\pi} T \to X$ be a diagram of analytic spaces, with π finite. Let t be a point of T and let $y_1, \dots, y_r$ be the pre-images of t in Y. Let $\mathscr{F}$ be a coherent sheaf on Y.*

(1) *Assume that Y and T are good. The $\mathscr{O}_{T,t}$-module $(\pi_*\mathscr{F})_t$ is then naturally isomorphic to $\prod \mathscr{F}_{y_i}$.*
(2) *Assume that Y, T and X are good, and let us consider the following assertions.*
   (i) *The coherent sheaf $\pi_*\mathscr{F}$ is naively X-flat at t.*
   (ii) *The coherent sheaf $\mathscr{F}$ naively X-flat at every y_i.*
   (iii) *The coherent sheaf $\pi_*\mathscr{F}$ is X-flat at t.*
   (iv) *The coherent sheaf $\mathscr{F}$ is X-flat at every y_i.*

   We then have the equivalences
$$\text{(i)} \iff \text{(ii)} \quad \textit{and} \quad \text{(iii)} \iff \text{(iv)}.$$
(3) *If we drop the goodness assumption, the equivalence*
$$\text{(iii)} \iff \text{(iv)}$$
still holds.

Proof. — Let us assume that Y and T are good. The finite morphism $Y \to T$ is in particular closed. Hence for every neighborhood V of $\{y_1, \dots, y_r\}$, there exist an affinoid neighborhood of t in T whose pre-image is included in V and is a *disjoint* union $\coprod V_i$, where V_i is for every i an affinoid neighborhood of y_i in Y. This implies that $(\pi_*\mathscr{F})_t = \prod \mathscr{F}_{y_i}$, and (1) holds.

Now let us prove (2). Let let x denote the image of t in X. By the above,the $\mathscr{O}_{X,x}$-module $(\pi_*\mathscr{F})_t$ is flat if and only if $\mathscr{F}_{y_i}$ is $\mathscr{O}_{X,x}$-flat for every i, whence the equivalence (i) $\iff$ (ii).

Assume that (iii) holds. Let X' be a good analytic space and let $X' \to X$ be a morphism. Set $T' = T \times_X X'$ and $Y' = Y \times_X X'$. Let i be an element of $\{1, \dots, r\}$ and let z be a pre-image of y_i on Y'; let t' and x' denote the images of z on T' and X', and let π' be the natural finite map $Y' \to T'$. Since $\pi_*\mathscr{F}$ is X-flat at t, the coherent sheaf $(\pi_*\mathscr{F})_{T'}$ is naively X'-flat at t'. But $(\pi_*\mathscr{F})_{T'} = \pi'_*(\mathscr{F}_{Y'})$ (to see it, one can assume that T and Y are affinoid, in which case it is obvious by viewing coherent sheaves as modules). By the implication (i) $\iff$ (ii) already proven, $\mathscr{F}_{Y'}$ is naively X'-flat at every primage of t', and in particular at z; hence the coherent sheaf $\mathscr{F}$ is X-flat at y_i, and (iv) holds.

Assume conversely that (iv) holds. Let X' be an affinoid space and let $X' \to X$ be a morphism. Set $T' = T \times_X X'$ and $Y' = Y \times_X X'$, and let π' be the natural finite map $Y' \to T'$. Let t' be a pre-image of t on X'. If z is a pre-image of t' on Y', then the image of z in Y is equal to y_i for some i. By assumption, $\mathscr{F}$ is universally X-flat at y. Hence $\mathscr{F}_{Y'}$ is naively X'-flat at z. Since this holds for any such z, the implication (ii) $\iff$ (i) already proved ensures that $\pi'_*(\mathscr{F}_{Y'}) = (\pi_*\mathscr{F})_{T'}$ is naively X'-flat at t'. Hence $\pi_*\mathscr{F}$ is X-flat at t, and (iii) holds.

Since flatness in the general case can *by definition* be checked on good analytic domains, (3) follows from (2). □

4.2. Algebraic flatness *versus* analytic flatness

We shall first prove some GAGA principles for flatness "in the easy direction" under very weak assumptions; we shall then prove the converse implication in some particular (but nonetheless signifiant) cases.

4.2.1. Lemma. — *Let $A \to B$ be a morphism between k-affinoid algebras; let $\mathscr{Y}$ (resp. $\mathscr{X}$) be a B-scheme of finite type (resp. an A-scheme of finite type). Let $\mathscr{F}$ be a cohrent sheaf on $\mathscr{Y}$, and let $\mathscr{Y} \to \mathscr{X}$ be an A-morphism. Let y be a point of $\mathscr{Y}^{\mathrm{an}}$ at which $\mathscr{F}^{\mathrm{an}}$ is naively $\mathscr{X}^{\mathrm{an}}$-flat. The coherent sheaf $\mathscr{F}$ is then $\mathscr{X}$-flat at y^{al}.*

Proof. — Let x be the image of y on $\mathscr{X}^{\mathrm{an}}$. In the commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{\mathscr{Y}^{\mathrm{an}},y} & \longleftarrow & \mathscr{O}_{\mathscr{X}^{\mathrm{an}},x} \\ \uparrow & & \uparrow \\ \mathscr{O}_{\mathscr{Y},y^{\mathrm{al}}} & \longleftarrow & \mathscr{O}_{\mathscr{X},x^{\mathrm{al}}} \end{array}$$

the vertical arrows are faithfully flat (2.1.4), and naive flatness of $\mathscr{F}^{\mathrm{an}}$ at y means that the $\mathscr{O}_{X^{\mathrm{an}},x}$-module $\mathscr{F}^{\mathrm{an}}_y$ is flat. Lemma 4.1.6 then ensures that $\mathscr{F}_{y^{\mathrm{al}}}$ is flat over $\mathscr{O}_{\mathscr{X},x^{\mathrm{al}}}$; i.e., $\mathscr{F}$ is flat at y^{al}. □

We are now interested in the converse of Lemma 4.2.1. We are first going to mention in 4.2.2 and 4.2.3 two special cases in which it is more or less well-known. Then we shall generalize 4.2.2 to any morphism between schemes of finite type over a *given* affinoid algebra (Proposition 4.2.4), and hence prove (Theorem 4.2.5) a GAGA principle for a morphism between affinoid spaces that extends both 4.2.2 and 4.2.3.

4.2.2. — Let $Y \to X$ be a *finite* morphism between affinoid spaces, let y be a point of Y and let $\mathscr{F}$ is a coherent sheaf on Y. The coherent sheaf $\mathscr{F}$ is naively X-flat at y if and only if $\mathscr{F}^{\mathrm{al}}$ is X^{al}-flat at y^{al}: this is essentially Prop. 3.2.1 of [**Ber93**] – the latter is written only for $\mathscr{F} = \mathscr{O}_Y$, but using Lemma 4.1.15 one can easily easily adapt it so that it works for any coherent sheaf.

4.2.3. — Let $Y \to X$ be a morphism between affinoid spaces, let y be a rigid point of Y, and let x be its image on X; let $\mathscr{F}$ be a coherent sheaf on Y. We claim that $\mathscr{F}$ is naively X-flat at y if and only if $\mathscr{F}^{\mathrm{al}}$ is X^{al}-flat at y^{al}. If $|k^\times| \neq \{1\}$ and if Y and X are strictly k-affinoid, this is a classical assertion of rigid-analytic geometry, but its proof is very simple and immediately extends to our situation: indeed, one knows from [**SGA 1**], Exposé IV Cor. 5.8 that $\mathscr{F}_y$ is flat over $\mathscr{O}_{X,x}$ if and only if its completion $\widehat{\mathscr{F}_y} = \mathscr{F}_y \otimes_{\mathscr{O}_{Y,y}} \widehat{\mathscr{O}_{Y,y}}$ is flat over $\widehat{\mathscr{O}_{X,x}}$, and that $\mathscr{F}^{\mathrm{al}}_{y^{\mathrm{al}}}$ is flat over $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$ if and only if $\widehat{\mathscr{F}^{\mathrm{al}}_{y^{\mathrm{al}}}} = \mathscr{F}^{\mathrm{al}}_{y^{\mathrm{al}}} \otimes_{\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}} \widehat{\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}}$ is flat over $\widehat{\mathscr{O}_{X,x^{\mathrm{al}}}}$; but as x and y are rigid, $\widehat{\mathscr{O}_{X,x}} = \widehat{\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}}$ and $\widehat{\mathscr{O}_{Y,y}} = \widehat{\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}}$ ([**Ber93**], Lemma 2.6.3), whence our claim.

4.2.4. Proposition. — *Let $\mathscr{Y} \to \mathscr{X}$ be a morphism between schemes of finite type over a given affinoid algebra. Let $\mathscr{F}$ be a coherent sheaf on $\mathscr{Y}$ and let y be a point of $\mathscr{Y}^{\mathrm{an}}$. Assume that $\mathscr{F}$ is $\mathscr{X}$-flat at y^{al}. The coherent sheaf $\mathscr{F}^{\mathrm{an}}$ is then $\mathscr{X}^{\mathrm{an}}$-flat at y.*

Proof. — We can assume that $\mathscr{X}$ is affine. Let us first prove that the coherent sheaf $\mathscr{F}^{\mathrm{an}}$ is *naively* flat at y. Let x be the image of y on $\mathscr{X}^{\mathrm{an}}$ and let U be an affinoid neighborhood of x in $\mathscr{X}^{\mathrm{an}}$. There is a natural map from $\mathscr{O}_{\mathscr{X}}(\mathscr{X})$ to $\mathscr{O}_U(U)$ which induces a morphism $U^{\mathrm{al}} \to \mathscr{X}$, and the space $\mathscr{Y}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} U$ can be identified with the analytification $(\mathscr{Y} \times_{\mathscr{X}} U^{\mathrm{al}})^{\mathrm{an}}$ of the U^{al}-scheme of finite type $\mathscr{Y} \times_{\mathscr{X}} U^{\mathrm{al}}$. Since flatness is preserved by any *scheme-theoretic* base change, the pull-back of $\mathscr{F}$ on $(\mathscr{Y} \times_{\mathscr{X}} U^{\mathrm{al}})$ is U^{al}-flat at the image of y. This implies, in view of the fact that $\mathscr{Y}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} U \to \mathscr{Y} \times_{\mathscr{X}} U^{\mathrm{al}}$ is flat as a morphism of locally ringed spaces (2.1.4), that $\mathscr{F}^{\mathrm{an}}_y$ is a flat $\mathscr{O}_{U^{\mathrm{al}},x^{\mathrm{al}}_U}$-module. Since $\mathscr{O}_{X,x}$ is the direct limit of the local rings $\mathscr{O}_{U^{\mathrm{al}},x^{\mathrm{al}}_U}$ for U running through the set of affinoid neighborhoods of x in $\mathscr{X}^{\mathrm{an}}$, we conclude that $\mathscr{F}^{\mathrm{an}}_y$ is a flat $\mathscr{O}_{\mathscr{X}^{\mathrm{an}},x}$-module; otherwise said, $\mathscr{F}^{\mathrm{an}}$ is naively $\mathscr{X}^{\mathrm{an}}$-flat at y.

Let us now prove that $\mathscr{F}^{\mathrm{an}}$ is flat at y. Let V be an affinoid space, let $V \to \mathscr{X}^{\mathrm{an}}$ be a morphism, and let z be a point of $\mathscr{Y}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} V$ lying above y. There is a natural map $\mathscr{O}_{\mathscr{X}}(\mathscr{X}) \to \mathscr{O}_V(V)$ which induces a morphism $V^{\mathrm{al}} \to \mathscr{X}$, and $\mathscr{Y}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} V$ can be identified with the analytification $(\mathscr{Y} \times_{\mathscr{X}} V^{\mathrm{al}})^{\mathrm{an}}$ of the V^{al}-scheme of finite type $\mathscr{Y} \times_{\mathscr{X}} V^{\mathrm{al}}$. As flatness behaves well under *scheme-theoretic* base change, the pull-back of $\mathscr{F}$ on $\mathscr{Y} \times_{\mathscr{X}} V^{\mathrm{al}}$ is V-flat at the image of z. It follows then from the naive version of the proposition (which we have proved above) that the pull-back of $\mathscr{F}^{\mathrm{an}}$ on $\mathscr{Y}^{\mathrm{an}} \times_{\mathscr{X}^{\mathrm{an}}} V$ is naively V-flat at z, which ends the proof. $\square$

4.2.5. Theorem. — *Let $Y \to X$ be a morphism between k-affinoid spaces, let $\mathscr{F}$ be a coherent sheaf on Y and let y be a point of Y. Assume that there exists a closed analytic subspace Z of Y containing y such that $Z \to X$ is finite, and that $\mathscr{F}^{\mathrm{al}}$ is X^{al}-flat at y^{al}. The coherent sheaf $\mathscr{F}$ is then naively X-flat at y.*

4.2.6. Remark. — In Theorem 8.3.7 we shall in fact prove that $\mathscr{F}$ is actually X-flat at y.

Proof of Theorem 4.2.5. — We denote by A, resp. B, the algebra of analytic functions on X, resp. Y; and we set $M = \mathscr{F}(Y)$. Let x be the image of y in X. Our purpose is now to prove that the $\mathscr{O}_{X,x}$-module $M \otimes_B \mathscr{O}_{Y,y}$ is flat. Let V be an affinoid neighborhood of x in X and set $W = Y \times_X V$. We denote by A_V, resp. B_W, the algebra of analytic functions on V, resp. W. Let $\mathfrak{p}$ the prime ideal of A that corresponds to x^{al}, and let I be the ideal of B that corresponds to Z.

The core of the proof consists in showing that $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}$-module $M \otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}$ is flat. By [**SGA 1**], Exposé IV, Thm. 5.6 the latter is true if and only if the two following conditions are satisfied:

(A) $M \otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}/\mathfrak{p}$ is $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$-flat.

(B) For every $d > 0$, the natural map

$$M \otimes_B (\mathfrak{p}^d \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}/\mathfrak{p}^{d+1}\mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}) \to \mathfrak{p}^d(M \otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}})/\mathfrak{p}^{d+1}(M \otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}})$$

is an isomorphism.

We first prove (A). For that purpose, let us begin with a general remark. Let Λ be an arbitrary B/I-module. As $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}/\mathfrak{p}$ is a field, $\Lambda \otimes_A \mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}/\mathfrak{p}$ is a flat $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}/\mathfrak{p}$-module. It follows that $\Lambda \otimes_A \mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$ is a flat $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$-module, which on the other hand can be rewritten as

$$\Lambda \otimes_{B/I} (B/I) \otimes_A A_V \otimes_{A_V} \mathscr{O}_{V^{\mathrm{al}},x^{\mathrm{al}}}/\mathfrak{p} = \Lambda \otimes_{B/I} (B_W/I) \otimes_{A_V} \mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p},$$

where the equality comes from the fact that $B_W/I = (B/I)\widehat{\otimes}_A A_V = (B/I) \otimes_A A_V$ by finiteness of B/I over A. Since $\mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}/(\mathfrak{p}+I)$ is a localization of $(B_W/I) \otimes_{A_V} \mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}$, the $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$-module

$$\Lambda \otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}/\mathfrak{p} = \Lambda \otimes_{B/I} \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}/(\mathfrak{p} + I)$$

is flat as well.

Set $N = M/\mathfrak{p}M$, and let n be an element of $\mathbf{Z}_{>0}$. By applying the above to the B/I-module $\Lambda = I^nN/I^{n+1}N$, we see that

$$(I^nN/I^{n+1}N)\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}} = (I^nN/I^{n+1}N)\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}/\mathfrak{p}$$

is a flat $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$-module. As $\mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}$ is a flat B-algebra (indeed, it is a localization of B_W which is itself B-flat; see 2.1.3 (2)), we have the equality

$$(I^nN/I^{n+1}N)\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}} = I^n(N\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_w^{\mathrm{al}}})/I^{n+1}(N\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}).$$

The $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$-module $I^n(N\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}})/I^{n+1}(N\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}})$ is thus flat for any non-negative n. It obviously implies that for any such n, the $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$-module

$$(N\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}})/I^{n+1}(N\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}})$$

is flat. By [**EGA III**$_1$], Chapter 0, §10.2.6, $M\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}/\mathfrak{p} = N\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}$ is then $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}/\mathfrak{p}$-flat; hence (A) is true.

Let us now prove (B). Let d be a positive integer. By assumption, $\mathscr{F}^{\mathrm{al}}$ is X^{al}-flat at y^{al}, which means that $M\otimes_B \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}$ is $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$-flat. Therefore the natural map

$$(M\otimes_B \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}})\otimes_{\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}} \mathfrak{p}^d\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}} \to \mathfrak{p}^dM\otimes_B \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}$$

is an isomorphism. But it can be written as the composition

$$(M\otimes_B \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}})\otimes_{\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}} \mathfrak{p}^d\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}} \to M\otimes_B \mathfrak{p}^d\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}} \to \mathfrak{p}^dM\otimes_B \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}.$$

The left arrow being surjective and the composition of the two arrows being injective, the right arrow is injective. As it is also surjective, it is an isomorphism. It follows that

$$M\otimes_B (\mathfrak{p}^d\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}/\mathfrak{p}^{d+1}\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}) \to \mathfrak{p}^d(M\otimes_B \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}})/\mathfrak{p}^{d+1}(M\otimes_B \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}})$$

is an isomorphism as well. As $\mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}$ is a flat $\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}$-algebra by 2.1.3 (2), assertion (B) follows by tensoring with $\mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}$ over $\mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}$. Therefore $M\otimes_B \mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}$ is flat over $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}$, as announced.

Now let T be any affinoid neighborhood of y in W. As $\mathscr{O}_{T^{\mathrm{al}},y_T^{\mathrm{al}}}$ is a flat $\mathscr{O}_{W^{\mathrm{al}},y_W^{\mathrm{al}}}$-algebra by 2.1.3 (2), the $\mathscr{O}_{V^{\mathrm{al}},v^{\mathrm{al}}}$-module $M\otimes_B \mathscr{O}_{T^{\mathrm{al}},y_T^{\mathrm{al}}}$ is flat.

We have thus shown the following: if V is any neighborhood of x in X and if T is any affinoid neighborhood of y in the pre-image of V inside Y, then $M\otimes_B \mathscr{O}_{T,y_T^{\mathrm{al}}}$ is $\mathscr{O}_{V^{\mathrm{al}},x_V^{\mathrm{al}}}$-flat. A straightforward limit argument then ensures that $M\otimes_B\mathscr{O}_{Y,y}$ is $\mathscr{O}_{X,x}$-flat. □

4.2.7. Remark. — In the above proof, the existence of a closed analytic subspace of Y containing y and finite over X was used only while proving assertion (A), and X^{al}-flatness of $\mathscr{F}^{\mathrm{al}}$ at y^{al} was used only while proving assertion (B).

4.3. The flat, (locally) finite morphisms

We shall use what we have just done to show some results which were already proven in [**Ber93**], §3.2 when $\mathscr{F} = \mathscr{O}_Y$; we include the proofs (which partially differ of those of [**Ber93**]) for the convenience of the reader.

4.3.1. Proposition. — *Let $Y \to X$ be a morphism between good k-analytic spaces and let y be a point of Y at which this morphism is finite; let x be the image of y on X. Let $\mathscr{F}$ be a coherent sheaf on Y. The following are equivalent:*

(i) *The coherent sheaf $\mathscr{F}$ is naively X-flat at y.*

(ii) *There exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $\pi_* \mathscr{F}_{|T}$ is flat at x.*

(iii) *There exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $\pi_*(\mathscr{F}_T)$ is a free $\mathscr{O}_S$-module.*

(iv) *There exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $\mathscr{F}(T)$ is a flat $\mathscr{O}_S(S)$-module.*

(v) *There exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ and such that $(\mathscr{F}_T)^{\mathrm{al}}$ is S^{al}-flat at y_T^{al}.*

(vi) *The coherent sheaf $\mathscr{F}$ is X-flat at y.*

Proof. — Suppose that (i) is true. As $Y \to X$ is finite at y, there exist an affinoid neighborhood T of y in Y and an affinoid neighborhood S of x in X such that $T \to X$ goes through a finite map $\pi : T \to S$ for which y is the only pre-image of x. As $\mathscr{F}$ is naively X-flat at y, Lemma 4.1.15 applied to the diagram $T \to S \xrightarrow{\mathrm{Id}} S$ yields the naive flatness of $\pi_*(\mathscr{F}_T)$ at x; by Lemma 4.1.14, $\pi_*(\mathscr{F}_T)$ is then flat at x, whence (ii). If (ii) is true, then by Lemma 4.1.14 we can shrink S (and T) so that $\pi_*(\mathscr{F}_T)$ is a free $\mathscr{O}_S$-module, whence (iii). If (iii) holds then $\mathscr{F}(T)$ is a free, thus flat, $\mathscr{O}(S)$-module, whence (iv). The implication (iv)$\Rightarrow$(v) is obvious. Now if (v) holds, Proposition 4.2.4 ensures that $\mathscr{F}_T$ is S-flat y, whence (vi). Implication (vi)$\Rightarrow$(i) is tautological. $\square$

4.3.2. Corollary. — *Let $Y \to X$ be a morphism of k-analytic spaces, and let $\mathscr{F}$ be a coherent sheaf on Y. Let y be a point of* $\mathrm{Supp}(\mathscr{F})$ *at which* $\mathrm{Supp}(\mathscr{F}) \to X$ *is finite and at which $\mathscr{F}$ is X-flat, and let x be the image of y in X. The image of* $\mathrm{Supp}(\mathscr{F})$ *on X is then a neighborhood of x in X.*

Proof. — By arguing G-locally on X, shrinking Y around y, and replacing Y with $\mathrm{Supp}(\mathscr{F})$ we may assume that both Y and X are good and $Y = \mathrm{Supp}(\mathscr{F})$, so $Y \to X$ is finite. Let us now choose T and S as in (iii) above. Since $\mathscr{F}_y \neq 0$, it follows from Lemma 4.1.15 (1) that $(\pi_*(\mathscr{F}_T))_x \neq 0$; hence the free $\mathscr{O}_S$-module $\pi_*(\mathscr{F}_T)$ has positive rank. It follows that $(\pi_*(\mathscr{F}_T))_z \neq 0$ for every point z of S; using again Lemma 4.1.15 (1), we conclude that S is contained in the image of $\mathrm{Supp}(\mathscr{F})$. □

4.3.3. Corollary. — *Let $Y \to X$ be a morphism of k-analytic spaces, let $y \in Y$ be such that $Y \to X$ is finite at y and let x be the image of y in Y. Let $\mathscr{F}$ be a coherent sheaf on Y. If y belongs to* $\mathrm{Supp}(\mathscr{F})$ *and $\mathscr{F}$ is X-flat at y, then* $\dim_x X = \dim_y \mathscr{F}$ *(2.5.3).*

Proof. — By arguing G-locally on X and shrinking Y around y we may assume that both Y and X are affinoid, that $Y \to X$ is finite, and that y is the only pre-image of x in Y. The image of $\mathrm{Supp}(\mathscr{F})$ in X is then a Zariski-closed subset T of X, and one has $\dim_x T = \dim_y \mathrm{Supp}(\mathscr{F}) = \dim_y \mathscr{F}$ (the first equality comes from 1.5.10 and the second one holds by definition of $\dim_y \mathscr{F}$); on the other hand, T is a neighborhood of x in X by Corollary 4.3.2 above, hence $\dim_x T = \dim_x X$. □

4.4. Naive flatness is not preserved by base change

As we have already mentioned, the reason why we have introduced (in the good case) a sophisticated definition for flatness instead of dealing with naive flatness is the fact that the latter is not preserved by base change. The purpose of this section is to discuss in full detail a counter-example that was initially suggested by Michael Temkin.

4.4.1. — Before introducing the counter-example, let us describe a situation that we shall encouter several times, in which we can conclude that a given morphism between good spaces is *not* naively flat at a given point; the key argument will be the violation of the dimension equality provided by Corollary 4.3.3.

Let $\varphi \colon Y \to \Omega$ be a morphism between good k-analytic spaces; we assume that φ factorizes through a closed immersion $Y \hookrightarrow X$ for X a good analytic domain of Ω and that Y and Ω are respectively of pure dimension d and d' with $d < d'$ (in all specific examples considered below, $d = 1$ and $d' = 2$). Let y be a point of Y at which φ is inner; this can be the case for instance if $X = \Omega$ (because then φ is boundaryless) or if y is rigid (because y is then an inner point of Y). Since $Y \hookrightarrow X$ is boundaryless, $\mathrm{Int}(\varphi)$ is equal to $\varphi^{-1}(\mathrm{Int}(X/\Omega))$; as a consequence φ induces a closed immersion $\mathrm{Int}(\varphi) \hookrightarrow \mathrm{Int}(X/\Omega)$, and since $\mathrm{Int}(X/\Omega)$ is an open subset of Ω (this is the *topological* interior of X inside Ω), the morphism φ is finite at every point of $\mathrm{Int}(\varphi)$. It is particular finite at y, and since $\dim_y Y = d$ and $\dim_{\varphi(y)} \Omega = d' > d$, it follows from Corollary 4.3.3 that φ is not naively flat at y (strictly speaking, it only follows from Corollary 4.3.3 that φ is not flat at y; but as φ is finite at y, it is flat at y if and only if it is naively flat at y, by Prop. 4.3.1).

4.4.2. Presentation of the counter-example. — Let r be a positive real number and let $f = \sum \alpha_i T^i$ be a power series with coefficients in k whose radius of convergence is exactly r; i.e.,

$$|\alpha_i|\, r^i \xrightarrow[i\to+\infty]{} 0$$

and $(|\alpha_i|\, s^i)_i$ is non-bounded as soon as $s > r$. We denote by $p : \mathbf{A}_k^{2,\mathrm{an}} \to \mathbf{A}_k^{1,\mathrm{an}}$ the first projection. Let X be the analytic domain of $\mathbf{A}_k^{2,\mathrm{an}}$ defined by the inequality $|T_1| \leqslant r$, and let Y be the one-dimensional closed disc of radius r; note that $X = p^{-1}(Y)$; i.e., X can be identified with $Y \times_k \mathbf{A}_k^{1,\mathrm{an}}$. The map $\varphi := (\mathrm{Id}, f)$ from Y to $Y \times_k \mathbf{A}_k^{1,\mathrm{an}}$ induces a closed immersion $Y \hookrightarrow X$, and more precisely an isomorphism between Y and the Zariski-closed subspace Z of X defined by the sheaf of ideals $(T_2 - f(T_1))\mathscr{O}_X$; the inverse isomorphism is nothing but $p|_Z$. Set $x = \varphi(\eta_r)$. We are going to show that φ is naively flat at η_r, but not flat; i.e., naive flatness of φ at η_r does not hold universally.

4.4.3. — The easiest part of our study is the negative one; i.e., the fact that φ is *not* flat at y. Let us give two examples of base-change functors that witness it.

(1) The base-change of the morphism φ by the inclusion $X \hookrightarrow \mathbf{A}_k^{2,\mathrm{an}}$ is the closed immersion $Y \hookrightarrow X$, and it follows from the general discussion in 4.4.1 that $Y \hookrightarrow X$ is not naively flat at η_r.

(2) Let L be any analytic extension of k such that η_r has an L-rational pre-image y in X_L; e.g., $L = \mathscr{H}(\eta_r)$. The morphism $\varphi_L \colon Y_L \to \mathbf{A}_L^{2,\mathrm{an}}$ is then inner at y, and it follows again from the general discussion in 4.4.1 that φ_L is not naively flat at y.

4.4.4. — We are now going to prove that φ is naively flat at η_r. We shall in fact prove the following stronger result: *the local ring $\mathscr{O}_{\mathbf{A}_k^{2,\mathrm{an}},x}$ is a field.*

Before giving the rigorous proof (see Proposition 4.4.6 below), let us roughly explain what is going on. We want to prove that any analytic function defined in a neighborhood of x and that vanishes at x vanishes around x; or otherwise said, that any Zariski-closed subgerm (W, x) of $(\mathbf{A}_k^{2,\mathrm{an}}, x)$ is equal to the whole of $(\mathbf{A}_k^{2,\mathrm{an}}, x)$. So let us consider such a (W, x). The point x is not rigid, so the dimension of (W, x) is $\geqslant 1$. Assume that it is equal to 1. Using again the fact that x is not rigid, we see that the one-dimensional Zariski-closed subgerms $(X \cap W, x)$ and (Z, x) of (X, x) coincide; hence by gluing the germ (W, x) to the curve Z (whose boundary is $\{x\}$), we can in some sense extend Z beyond x in $\mathbf{A}^{2,\mathrm{an}}$, and we get a contradiction with the fact that the radius of convergence of f is exactly r. Therefore $\dim(W, x) = 2$ and $(W, x) = (\mathbf{A}_k^{2,\mathrm{an}}, x)$.

4.4.5. Lemma. — *Let T be a reduced one-dimensional good analytic space and let t be a non-rigid point of T. The local ring $\mathscr{O}_{T,t}$ is a field.*

Proof. — By Corollary 3.2.9, $\mathrm{centdim}(T,t) + \dim_{\mathrm{Krull}} \mathscr{O}_{T,t} = \dim_t T$. As t is not rigid, $\mathrm{centdim}(T,t) > 0$; as T is one-dimensional, $\dim_t T \leqslant 1$. Therefore $\dim_{\mathrm{Krull}} \mathscr{O}_{T,t} = 0$; being reduced, $\mathscr{O}_{T,t}$ is thus a field. □

4.4.6. Proposition. — *The local ring $\mathscr{O}_{\mathbf{A}_k^{2,\mathrm{an}},x}$ is a field.*

Proof. — As the analytic space $\mathbf{A}_k^{2,\mathrm{an}}$ is reduced, it is sufficient to prove that $\dim_{\mathrm{Krull}} \mathscr{O}_{\mathbf{A}_k^{2,\mathrm{an}},x} = 0$, which is equivalent, in view of Corollary 3.2.9, to the fact that $\mathrm{centdim}(\mathbf{A}_k^{2,\mathrm{an}}, x) = 2$. Since x is not a rigid point (because one has by construction $\mathscr{H}(x) = \mathscr{H}(\eta_r)$, and because $d_k(\eta_r) = 1$ by 1.2.15, $\mathrm{centdim}(\mathbf{A}_k^{2,\mathrm{an}}, x) > 0$.

Assume that $\mathrm{centdim}(\mathbf{A}_k^{2,\mathrm{an}}, x) = 1$. Then there exists an affinoid neighborhood V of x in $\mathbf{A}_k^{2,\mathrm{an}}$ and an irreducible one-dimensional Zariski-closed subset W of V that contains x. Both $W \cap X = W \cap (V \cap X)$ and $Z \cap V = Z \cap (V \cap X)$ are purely one-dimensional Zariski-closed subsets of $V \cap X$ containing x. As x is not a rigid point, it belongs to a unique irreducible component of $(W \cap X) \cup (Z \cap V)$. Therefore it belongs to a unique irreducible component of $W \cap X$ and to a unique irreducible component of $Z \cap V$, so those two components coincide. One can hence shrink V so that $W \cap X = Z \cap V$; by endowing W with its reduced structure, this equality becomes an equality of closed analytic subspaces of $X \cap V$.

If w is any point of W such that $p(w) = \eta_r$, then in view of the equality $d_k(\eta_r) = 1$, the inequality $d_k(w) \leqslant 1$ (due to the fact that W is one-dimensional) forces $d_{\mathscr{H}(\eta_r)}(w)$ to be equal to zero. Therefore $(p|_W)^{-1}(\eta_r)$ is zero-dimensional. In particular $p|_W$ is quasi-finite at x; moreover, since x belongs to the topological interior of V in $\mathbf{A}_k^{2,\mathrm{an}}$, the map $p|_W$ is inner at x; Prop. 3.1.4 of [**Ber93**] then ensures that $p|_W$ is finite at x; as $\mathscr{O}_{\mathbf{A}_k^{1,\mathrm{an}},\eta_r}$ is a field by Lemma 4.4.5, $p|_W$ is naively flat at x. It follows then from Proposition 4.3.1 that there exists an affinoid neighborhood W_0 of x in W and an affinoid neighborhood U of η_r in $\mathbf{A}_k^{1,\mathrm{an}}$ such that $p(W_0) \subset U$, and such that $p|_{W_0} : W_0 \to U$ is finite and makes $\mathscr{O}_{W_0}(W_0)$ a free $\mathscr{O}_U(U)$-module of finite positive rank, say r (note that we thus have $p(W_0) = U$). By restricting to Y and using the fact that $X = p^{-1}(Y)$, one sees that $p|_{W_0 \cap X} : W_0 \cap X \to U \cap Y$ is finite and makes $\mathscr{O}_{W_0 \cap X}(W_0 \cap X)$ a free $\mathscr{O}_{U \cap Y}(U \cap Y)$-module of rank r; we thus have $p(W_0 \cap X) = U \cap Y$.

It follows from the inclusion $W_0 \cap X \subset W \cap X = Z \cap V$ that $W_0 \cap X$ is an analytic domain of Z. Since $p|_Z$ induces an isomorphism $Z \simeq Y$ whose inverse isomorphism is induced by φ, the morphism $p|_{W_0 \cap X}$ induces an isomorphism from $W_0 \cap X$ to $p(W_0 \cap X) = U \cap Y$ whose inverse is $\varphi_{|U \cap Y}$. As $\mathscr{O}_{W_0 \cap X}(W_0 \cap X)$ is a free $\mathscr{O}_{U \cap Y}(U \cap Y)$-module of rank r, we have $r = 1$, which means that p induces an isomorphism $W_0 \simeq U$. The inverse isomorphism defines a section σ of the first projection $U \times_k \mathbf{A}_k^{1,\mathrm{an}} \to U$; we have $\sigma_{|U \cap Y} = \varphi_{|U \cap Y}$. We can thus glue σ and φ to obtain a section of the first projection $(U \cup Y) \times_k \mathbf{A}_k^{1,\mathrm{an}} \to (U \cup Y)$ that coincides with φ on Y i.e., an analytic function g on $U \cup Y$ that coincides with f on Y. As U

is a neighborhood of η_r in $\mathbf{A}_k^{1,\mathrm{an}}$, the analytic domain $U \cup Y$ of $\mathbf{A}_k^{1,\mathrm{an}}$ contains a closed disc centered at the origin and whose radius is $> r$; but on the other hand the radius of convergence of f is exactly r, hence f does not extend to any disc of radius $> r$, contradiction. As a consequence, $\mathrm{centdim}(\mathbf{A}_k^{2,\mathrm{an}}, x) \neq 1$ and thus $\mathrm{centdim}(\mathbf{A}_k^{2,\mathrm{an}}, x) = 2$. □

*4.4.7. **Remark.*** — The above counter-example rests on a boundary phenomenon. In fact, it turns out that such phenomena are the only obstructions for naive flatness to be preserved by base-change; indeed, we shall see later that naive flatness implies flatness in the boundaryless case (Theorem 8.3.4).

*4.4.8. **Remark.*** — Let ρ be an element of $(0, r)$ and set $z = \varphi(\eta(\rho))$. We have $\mathscr{H}(z) = \mathscr{H}(\eta_\rho)$, and $d_k(\eta_\rho) = 1$ by 1.2.16. Therefore $\mathrm{d}_k(z) = 1$; since z lies on the one-dimensional irreducible Zariski-closed subset Z of X, it follows that $\overline{\{z\}}^{X_{\mathrm{Zar}}} = Z$ (1.5.9). Hence $\overline{\{z\}}^{\mathbf{A}^{2,\mathrm{an}}_{k,\mathrm{Zar}}}$ contains x, and is thus of dimension 2 since $\mathrm{centdim}(\mathbf{A}_k^{2,\mathrm{an}}, x) = 2$, as seen in the proof of Proposition 4.4.6. This implies that $\overline{\{z\}}^{\mathbf{A}^{2,\mathrm{an}}_{k,\mathrm{Zar}}}$ is the whole of $\mathbf{A}_k^{2,\mathrm{an}}$ because the latter is irreducible; see Proposition 2.7.16 or simply note that $\mathbf{A}_k^{2,\mathrm{an}}$ is non-empty, connected, and normal. Note that the same reasoning would more generally show that $\overline{\{z\}}^{D_{\mathrm{Zar}}} = D$ for any irreducible analytic domain D of $\mathbf{A}_k^{2,\mathrm{an}}$ containing z and such that $\mathrm{centdim}(D, x) = 2$; e.g., D is a neighborhood of x.

Choose $s \in (\rho, r)$. Let X' be the open subset of $\mathbf{A}_k^{2,\mathrm{an}}$ defined by the inequality $|T_1| < s$. The intersection $Z \cap X'$ is a closed analytic subspace of X' containing z which is isomorphic through p_1 to the open disc of radius s, and is thus one-dimensional. Since $d_k(z) = 1$, it follows from Example 3.2.6 that $\mathrm{centdim}(X', z) = 1$. We thus have

$$\mathrm{centdim}(\mathbf{A}_k^{2,\mathrm{an}}, z) = \mathrm{centdim}(X', z) = 1 < 2 = \dim \overline{\{z\}}^{\mathbf{A}^{2,\mathrm{an}}_{k,\mathrm{Zar}}}.$$

We are now going to explain how the above construction also provides a counter-example to general GAGA-principle for naive flatness, and another one to stability of scheme-theoretic flatness under analytic ground field extension.

Let D be a closed two-dimensional polydisc centered at the origin such that Z is contained in the corresponding open polydisc. Let ψ denote the morphism $Y \to D$ induced by φ. Note that $\mathscr{O}_{D,x} = \mathscr{O}_{\mathbf{A}^{2,\mathrm{an}},x}$; since the latter is a field, ψ is naively flat at η_r. Note also that ψ factorizes by construction through a closed immersion $Y \hookrightarrow D \cap X$.

4.4.9. — Since η_r is a norm, it lies above the generic point ξ of Y^{al}. As ψ is naively flat at η_r, it follows from Lemma 4.2.1 that the induced map $Y^{\mathrm{al}} \to D^{\mathrm{al}}$ is flat at ξ. Now let y be a non-rigid point of $Y \backslash \{\eta_r\}$ (e.g., $y = \eta_{r'}$ for some $r' \in (0, r)$). The point y does not lie on any proper Zariski-closed subset of Y, which means that $y^{\mathrm{al}} = \xi$.

But since $y \in \mathrm{Int}(Y/k)$, it follows from the general discussion in 4.4.1 that ψ is not naively flat at y.

4.4.10. — Let us now assume that $r \notin |k^\times|^{\mathbf{Q}}$. In this case, $\{y\}$ is an affinoid domain of Y (defined by the equality $|T| = r$) whose corresponding k-affinoid algebra is nothing but k_r. In view of Remark 4.4.8, the Zariski-closure $\overline{\{\psi(y)\}}^{D_{\mathrm{Zar}}}$ is equal to the whole of D, which implies that $\psi(y)$ lies above ξ. As a consequence, the morphism Spec $k_r \to D^{\mathrm{al}}$ induced by $\psi|_{\{y\}}$ is flat.

Let L be any analytic extension of k such that r belongs to the group $|L^\times|^{\mathbf{Q}}$ (e.g., $L = k_r$). The space $\mathscr{M}(L\widehat{\otimes}_k k_r)$ is strictly L-affinoid and non-empty; it has thus an L-rigid point, say t. Since t is rigid, it belongs to $\mathrm{Int}(\mathscr{M}(L\widehat{\otimes}_k k_r))$, and since $\psi|_{\{y\}}$ factorizes through a closed immersion from $\{y\}$ to the affinoid domain of D defined by the equality $|T_1| = r$, it follows again from the general discussion in 4.4.1 that $\mathscr{M}(L\widehat{\otimes}_k k_r) \to D_L$ is not naively flat at t. We deduce then from Theorem 4.2.5 (for a direct and simpler proof, see 4.2.3) that Spec $(L\widehat{\otimes}_k k_r) \to D_L^{\mathrm{al}}$ is not flat at t^{al}.

4.5. Analytic flatness has the expected properties

In this section, we show that flatness in our sense behaves reasonably; i.e., the analogues of classical results from algebraic geometry hold in our setting. We begin with the descent of algebraic properties. We shall first write a statement that holds in the abstract settings of 2.2 and 2.3, where we deal with general objects and properties of the latter satisfying various axioms; and then we shall then write down what it means for some *explicit* properties of interest. For the notion of validity of a property at a point, the reader may refer to Lemma-Definition 2.4.1 in our general abstract setting and to Lemma-Definition 2.4.3 for a more concrete version.

4.5.1. Lemma. — *Let $Y \to X$ be a morphism of analytic spaces, let y be a point of Y at which Y is X-flat, and let x be the image of y in X. Let $\mathfrak{F}$ be a fibered category as in 2.2 and let* P *be a property as in 2.3.1. Let D be an object of $\mathfrak{F}_X$.*

1. *If* P *satisfies condition* (H) *of 2.3.15 and if D satisfies* P *at x, then D_Y satisfies* P *at y.*
2. *If* P *satisfies condition* $(\mathsf{H}_{\mathrm{reg}})$ *of 2.3.15 and if D_Y satisfies* P *at y, then D satisfies* P *at x.*

Proof. — We may and do assume that Y and X are good. Being X-flat at y, the space Y is in particular naively X-flat at y, which means that $\mathscr{O}_{Y,y}$ is a flat $\mathscr{O}_{X,x}$-algebra. The lemma follows then immediately from the definitions of conditions (H) and $(\mathsf{H}_{\mathrm{reg}})$. □

4.5.2. Lemma **(A concrete version of Lemma 4.5.1)**

Let $Y \to X$ be a morphism between k-analytic spaces, let y be a point of X at *which Y is X-flat, and let x be its image on X. Let $\mathscr{F}$ be a coherent sheaf on X, and let m be an element of $\mathbf{Z}_{\geqslant 0}$. Let $\mathsf{S} = (\mathscr{E}' \to \mathscr{E} \to \mathscr{E}'')$ be a short complex of coherent sheaves on X.*

(1) *If Y is regular, resp. R_m, resp. Gorenstein, resp. CI, at y, so is X at x. If $\mathscr{F}_Y$ is CM, resp. S_m, resp. free of rank m, at y, so is $\mathscr{F}$ at x. If S_Y is exact at y, so is S at x.*

(2) *If $\mathscr{F}$ is free of rank m at x, so is $\mathscr{F}_Y$ at y. If S is exact at x, so is S_Y at y.*

We are now going to prove that flatness can be checked after flat base change (Proposition 4.5.5) and ground field extension (Proposition 4.5.6).

4.5.3. Lemma. — *Let*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & T \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

be a commutative diagram of good analytic spaces, let z be a point of Z and let t, resp. y, be its image in T, resp. Y. Let $\mathscr{F}$ be a coherent sheaf on Y and let $\mathscr{G}$ be its pull-back on Z. Suppose that T is naively X-flat at t and that Z is naively Y-flat at z. If $\mathscr{G}$ is naively T-flat at z then $\mathscr{F}$ is naively X-flat at y.

Proof. — This follows straightforwardly from Lemma 4.1.6. □

4.5.4. Remark. — We emphasize that in Lemma 4.5.3 above, the spaces involved are *not* assumed to be k-analytic; indeed, we want typically to apply it to diagrams arising from ground field extension.

4.5.5. Proposition. — *Let*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & T \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

be a commutative diagram of k-analytic spaces, let z be a point of Z and let t, resp. y, be its image in T, resp. Y. Let $\mathscr{F}$ be a coherent sheaf on Y. Suppose that T is X-flat at t, that Z is Y-flat at z and that $\mathscr{F}_Z$ is T-flat at z; under those assumptions $\mathscr{F}$ is X-flat at y.

Proof. — One immediately reduces to the case where all spaces are affinoid. Let X' be a good analytic space and let $X' \to X$ be a morphism. We set $Y' = Y \times_X X'$ and so on. Let y' be a point on Y' lying above y, and let z' be a point on Z' lying above both z and y' (such a point always exists by Lemma 2.6.10); denote by t' the image of z' on T'. Since $\mathscr{F}_Z$ is T-flat at z, the sheaf $\mathscr{F}_{Z'}$ is naively T'-flat at z'. Since T, resp. Z, is X-flat at t, resp. Y-flat at z, the space T', resp. Z', is naively X'-flat at t', resp. naively Y'-flat at z'. Lemma 4.5.3 above now implies that $\mathscr{F}_{Y'}$ is naively X'-flat at y'. □

4.5.6. Proposition. — *Let $Y \to X$ be a morphism of k-analytic spaces and let L be an analytic extension of k. Let $y \in Y$ and let $\mathscr{F}$ be a coherent sheaf on Y. Let u be a point of Y_L lying above y. Suppose that $\mathscr{F}_L$ is X_L-flat at u; the coherent sheaf $\mathscr{F}$ is then X-flat at y.*

Proof. — One can assume that both Y and X are affinoid. Let X' be a good F-analytic space for some analytic extension F of k. We set $Y' = Y \times_X X'$. Let y' be a point on Y' lying above y; we are going to show that $\mathscr{F}_{Y'}$ is naively X'-flat at y; by shrinking X', one can assume that it is F-affinoid.

By Lemma 2.6.10 there exists an analytic extension K of both F and L and a point ω on $Y'_K := Y_K \times_{X_K} X'_K$ lying above both u and y'. By X_L-flatness of $\mathscr{F}_L$ at u the coherent sheaf $\mathscr{F}_{Y'_K}$ is naively X'_K-flat at ω. Applying Lemma 4.5.3 above to the diagram

$$\begin{tikzcd} Y'_K \arrow[r] \arrow[d] & X'_K \arrow[d] \\ Y' \arrow[r] & X' \end{tikzcd}$$

(which is possible in view of 2.1.3 (3)), one immediately gets the naive X'-flatness of $\mathscr{F}_{Y'}$ at y'. □

Proposition 4.5.7 below describes some consequences of flatness on the homology of complexes of coherent sheaves (for the notion of exactness, injectivity, bijectivity, etc. at a given point, see Lemma-Definition 2.4.3). Assertions (1) and (2) are analogues of well-known results in scheme theory. Assertions (3) and (4) are stated and proved for further use in the study of the loci of validity of various properties (Chapter 10).

4.5.7. Proposition. — *Let $Y \to X$ be a morphism between k-analytic spaces, let y be a point of Y and let x be its image in X. Let L be an analytic extension of k, let X' be an L-analytic space and let $X' \to X$ be a morphism of analytic spaces. Let y' be a pre-image of y on $Y' := Y \times_X X'$ and let x' denote the image of y' in X'. Let $\mathscr{H}$ be a coherent sheaf on X'.*

(1) *Let $\mathsf{S} = (\mathscr{E} \to \mathscr{E}' \to \mathscr{E}'')$ be a sequence of coherent sheaves on Y which is exact at y. If $\mathscr{H}$ is X_L-flat at x', the sequence $\mathsf{S} \boxtimes \mathscr{H}$ is exact at y'.*

(2) *Let $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{E} \to 0$ be a sequence of coherent sheaves on Y which is exact at y; moreover, assume that $\mathscr{E}$ is X-flat at y. The sequence*

$$0 \to \mathscr{G} \boxtimes \mathscr{H} \to \mathscr{F} \boxtimes \mathscr{H} \to \mathscr{E} \boxtimes \mathscr{H} \to 0$$

of coherent sheaves on Y' is then exact at y'.

(3) *Let* $\mathsf{S} = (\mathscr{F}_n \to \mathscr{F}_{n-1} \to \ldots \to \mathscr{F}_0 \to 0)$ *be a sequence of coherent sheaves on* Y. *Assume that all the* $\mathscr{F}_i$*'s are* X*-flat at* y, *and that* S *is exact at* y. *The sequence* $\mathsf{S}\boxtimes\mathscr{H}$ *on* Y' *is then exact at* y', *and if* $n \geqslant 1$ *the kernel of* $\mathscr{F}_n \to \mathscr{F}_{n-1}$ *is* X*-flat at* y.

(4) *Let* n *be a positive integer and let* $\mathsf{S} = (\mathscr{F}_n \to \mathscr{F}_{n-1} \to \ldots \to \mathscr{F}_0 \to 0)$ *be a complex of coherent sheaves on* Y. *Asssume that* $\mathscr{F}_i$ *is* X*-flat at* y *for every* $i \leqslant n-1$, *and that* S *is exact at* y *except possibly at* $\mathscr{F}_{n-1}$. *The natural map*

$$\mathrm{H}_{n-1}(\mathsf{S}) \boxtimes \mathscr{H} \to \mathrm{H}_{n-1}(\mathsf{S} \boxtimes \mathscr{H})$$

of coherent sheaves on Y' *is an isomorphism at* y'.

Proof. — For all assertions one can assume that X, Y, X', and Y' are affinoid. Let us first prove (1). As $\mathscr{H}$ is X_L-flat at x', the coherent sheaf $\mathscr{H}_{Y'}$ is naively Y_L-flat at y', hence is naively Y-flat at y' since Y_L is naively flat over Y. In other words, $\mathscr{H}_{Y',y'}$ is a flat $\mathscr{O}_{Y,y}$-module. By assumption, S_y is exact. Tensoring with the flat $\mathscr{O}_{Y,y}$-module $\mathscr{H}_{Y',y'}$ then yields the exactness of the sequence $\mathsf{S}_{y'}$, whence (1).

Let us prove (2). As X' is affinoid, it can be identified with a Zariski-closed subspace of $X_L \times_L D$ where D is some closed polydisc over the field L. Right-exactness of the tensor product ensures that

$$\mathscr{G}_{Y_L \times_L D, y'} \to \mathscr{F}_{Y_L \times_L D, y'} \to \mathscr{E}_{Y_L \times_L D, y'} \to 0$$

is exact. Since $X_L \times_L D \to X_L$ is flat by Lemma 4.1.13, it follows from assertion (1) already proven (and applied with $X' = X_L \times_L D$ and $Y' = Y_L \times_L D$) that the arrow $\mathscr{G}_{Y_L \times_L D, y'} \to \mathscr{F}_{Y_L \times_L D, y'}$ is injective; hence

$$0 \to \mathscr{G}_{Y_L \times_L D, y'} \to \mathscr{F}\otimes_{Y_L \times_L D, y'} \to \mathscr{E}_{Y_L \times_L D, y'} \to 0$$

is exact. As X' is a Zariski-closed subspace of $X_L \times_L D$, the local ring $\mathscr{O}_{Y',y'}$ is naturally isomorphic to $\mathscr{O}_{Y_L \times_L D, y'} \otimes_{\mathscr{O}_{X_L \times_L D, x'}} \mathscr{O}_{X',x'}$. Therefore the sequence

$$0 \to (\mathscr{G} \boxtimes \mathscr{H})_{y'} \to (\mathscr{F} \boxtimes \mathscr{H})_{y'} \to (\mathscr{E} \boxtimes \mathscr{H})_{y'} \to 0$$

is simply deduced from the exact sequence

$$0 \to \mathscr{G}_{Y_L \times_L D, y'} \to \mathscr{F}_{Y_L \times_L D, y'} \to \mathscr{E}_{Y_L \times_L D, y'} \to 0$$

by applying the functor $\otimes_{\mathscr{O}_{X_L \times_L D, x'}} \mathscr{H}_{x'}$. As the coherent sheaf $\mathscr{E}$ is X-flat at y, the $\mathscr{O}_{X_L \times_L D, x'}$-module $\mathscr{E}_{Y_L \times_L D, y'}$ is flat; it follows then immediately from the $\mathrm{Tor}_\bullet$ exact sequence that

$$0 \to (\mathscr{G} \boxtimes \mathscr{H})_{y'} \to (\mathscr{F} \boxtimes \mathscr{H})_{y'} \to (\mathscr{E} \boxtimes \mathscr{H})_{y'} \to 0$$

is exact, whence (2).

Let us prove (3). We argue by induction on n. For $n = 0$ there is nothing to prove. Let us now assume that $n \geqslant 1$ and that the required assertion is true for all integers $< n$. Let $\mathscr{N}$ be the kernel of $\mathscr{F}_1 \to \mathscr{F}_0$. The two sequences

$$\mathscr{F}_n \to \mathscr{F}_{n-1} \to \ldots \to \mathscr{F}_2 \to \mathscr{N} \to 0 \text{ and } 0 \to \mathscr{N} \to \mathscr{F}_1 \to \mathscr{F}_0 \to 0$$

are exact at y.

Let us prove that $\mathscr{N}$ is X-flat at y. Let $Z \to X$ be any morphism with Z affinoid, let $t \in T := Y \times_X Z$ be a point lying above y, and let z be the image of t on Z. Since the coherent sheaf $\mathscr{F}_0$ is X-flat at y, assertion (2) applied with $X' = Z$, $Y' = T$, $x' = z$, $y' = t$, and $\mathscr{H} = \mathscr{O}_Z$ ensures that the sequence

$$0 \to \mathscr{N}_{T,t} \to \mathscr{F}_{1,T,t} \to \mathscr{F}_{0,T,t} \to 0$$

is exact. As $\mathscr{F}_1$ and $\mathscr{F}_0$ are X-flat at y, the $\mathscr{O}_{Z,z}$-modules $\mathscr{F}_{1,T,t}$ and $\mathscr{F}_{0,T,t}$ are flat. By a Tor computation, it follows that $\mathscr{N}_{T,t}$ is also flat over $\mathscr{O}_{Z,z}$; therefore, $\mathscr{N}$ is X-flat at y.

It follows from (2) that the sequence

$$0 \to \mathscr{N} \boxtimes \mathscr{H} \to \mathscr{F}_1 \boxtimes \mathscr{H} \to \mathscr{F}_0 \boxtimes \mathscr{H} \to 0$$

is exact at y'. Since we have just seen that $\mathscr{N}$ is X-flat at y, the case $n = 1$ is settled and it follows from the induction hypothesis that the kernel of $\mathscr{F}_n \to \mathscr{F}_{n-1}$ is X-flat at y if $n \geqslant 2$ and that

$$\mathscr{F}_n \boxtimes \mathscr{H} \to \mathscr{F}_{n-1} \boxtimes \mathscr{H} \to \ldots \to \mathscr{F}_2 \boxtimes \mathscr{H} \to \mathscr{N} \boxtimes \mathscr{H} \to 0$$

is exact at y', which yields the exactness of

$$\mathscr{F}_n \boxtimes \mathscr{H} \to \mathscr{F}_{n-1} \boxtimes \mathscr{H} \to \ldots \to \mathscr{F}_0 \boxtimes \mathscr{H} \to 0$$

at y' and ends the proof of (3).

Let us prove (4). If $n = 1$ then (4) simply means that

$$\operatorname{Coker}(\mathscr{F}_1 \to \mathscr{F}_0) \boxtimes \mathscr{H} \to \operatorname{Coker}(\mathscr{F}_1 \boxtimes \mathscr{H} \to \mathscr{F}_0 \boxtimes \mathscr{H})$$

is an isomorphism at y', which is true; indeed, it is an isomorphism at every point of Y' by right-exactness of the tensor product. Now assume that $n \geqslant 2$, and let $\mathscr{N}$ be the kernel of $\mathscr{F}_{n-1} \to \mathscr{F}_{n-2}$. By assertion (3) the coherent sheaf $\mathscr{N}$ is X-flat at y. Applying (3) to the complex

$$0 \to \mathscr{N} \to \mathscr{F}_{n-1} \to \mathscr{F}_{n-2} \to \ldots \to \mathscr{F}_0 \to 0$$

(which is exact at y), we see that

$$\mathscr{N} \boxtimes \mathscr{H} \to \operatorname{Ker}(\mathscr{F}_{n-1} \boxtimes \mathscr{H} \to \mathscr{F}_{n-2} \boxtimes \mathscr{H})$$

is an isomorphism at y'. On the other hand, $\mathrm{H}_{n-1}(\mathsf{S})$ is the cokernel of $\mathscr{F}_n \to \mathscr{N}$. By right-exactness of $\bullet \boxtimes \mathscr{H}$, the coherent sheaf $\mathrm{H}_{n-1}(\mathsf{S}) \boxtimes \mathscr{H}$ is then the cokernel of $\mathscr{F}_n \boxtimes \mathscr{H} \to \mathscr{N} \boxtimes \mathscr{H}$. Hence we see that $\mathrm{H}_{n-1}(\mathsf{S}) \boxtimes \mathscr{H} \to \mathrm{H}_{n-1}(\mathsf{S} \boxtimes \mathscr{H})$ is an isomorphism at y'. $\square$

In scheme theory, flatness is often useful to spread out some properties from a given fiber across the ambiant space. As an application of the preceding proposition, we give a first example of such a phenomenon in analytic geometry.

4.5.8. Lemma. — *Let $Y \to X$ be a morphism between k-analytic spaces, let y be a point of Y and let x be its image on X. Let $\mathscr{G} \to \mathscr{F}$ be a morphism between coherent sheaves on Y. If $\mathscr{G}_{Y_x} \to \mathscr{F}_{Y_x}$ is an isomorphism at y and $\mathscr{F}$ is X-flat at y, then $\mathscr{G} \to \mathscr{F}$ is an isomorphism at y.*

Proof. — We may and do assume that both Y and X are k-affinoid. Let $\mathscr{N}$ be the kernel of $\mathscr{G} \to \mathscr{F}$. Since $\mathscr{G}_{Y_x} \to \mathscr{F}_{Y_x}$ is surjective at y, the map $\mathscr{G}_{\mathscr{H}(y)} \to \mathscr{F}_{\mathscr{H}(y)}$ is surjective, which implies that $\mathscr{G} \to \mathscr{F}$ is surjective at y by 2.5.4; hence the sequence

$$0 \to \mathscr{N} \to \mathscr{G} \to \mathscr{F} \to 0$$

is exact at y. The coherent sheaf $\mathscr{F}$ being X-flat at y, Prop. 4.5.7 (2) applied to the morphism $\mathscr{M}(\mathscr{H}(x)) \to X$ yields the exactness of

$$0 \to \mathscr{N}_{Y_x} \to \mathscr{G}_{Y_x} \to \mathscr{F}_{Y_x} \to 0$$

at y. Since $\mathscr{G}_{Y_x} \to \mathscr{F}_{Y_x}$ is by assumption an isomorphism at y, this implies that $\mathscr{N}_{Y_x,y} = 0$; therefore $\mathscr{N}_{\mathscr{H}(y)} = 0$ and $\mathscr{N}_y = 0$ by 2.5.4. As a consequence, $\mathscr{G} \to \mathscr{F}$ is an isomorphism at y. □

We are now going to give two flatness criteria. The first one describes the behavior of flatness with respect to extensions. The second one might look somehow specific, but it will be crucial for the study of quasi-smooth morphisms in the next chapter and of fiberwise regular sequences in 10.6.

4.5.9. Lemma. — *Let $Y \to X$ be a morphism of k-analytic spaces, let y be a point of Y, and let $0 \to \mathscr{E} \to \mathscr{E}' \to \mathscr{E}'' \to 0$ be a sequence of coherent sheaves on Y. Assume that this sequence is exact at y.*

1. *If $\mathscr{E}$ and $\mathscr{E}''$ are X-flat at y, so is $\mathscr{E}'$.*
2. *If $\mathscr{E}'$ and $\mathscr{E}''$ are X-flat at y, so is $\mathscr{E}$.*

Proof. — Let us prove (1) (resp. 2). So we assume that $\mathscr{E}''$ and $\mathscr{E}$ (resp. $\mathscr{E}'$) is X-flat. We immediately reduce to the case where X and Y are good. Let $Z \to X$ be any morphism of analytic spaces with Z good, let t be a pre-image of y on $T := Y \times_X Z$, and let z be the image of t in Z. We have to show that $\mathscr{E}'_{T,t}$ (resp. $\mathscr{E}_{T,t}$) is flat over $\mathscr{O}_{Z,z}$.

By Proposition 4.5.7 (2), the sequence

$$0 \to \mathscr{E}'_{T,t} \to \mathscr{E}_{T,t} \to \mathscr{E}''_{T,t} \to 0$$

is still exact. It follows from our flatness assumptions that $\mathscr{E}''_{T,t}$ and $\mathscr{E}_{T,t}$ (resp. $\mathscr{E}'_{T,t}$) are $\mathscr{O}_{Z,z}$-flat, hence so is $\mathscr{E}'_{T,t}$ (resp. $\mathscr{E}_{T,t}$) by a straightforward Tor computation. □

4.5.10. Lemma. — *Let $Y \to X$ be a morphism of k-analytic spaces, let y be a point of Y and let x be its image in X. Let $\mathscr{G} \to \mathscr{F}$ be a morphism of coherent sheaves on Y, and let $\mathscr{E}$ be its cokernel. Assume that $\mathscr{G}$ and $\mathscr{F}$ are X-flat at y, and that $\mathscr{G}_{Y_x} \to \mathscr{F}_{Y_x}$ is injective at y. Under this assumption, $\mathscr{E}$ is X-flat at y.*

Proof. — We can assume that Y and X are k-affinoid. Let L be an analytic extension of k, let X' be an L-affinoid space, and let $X' \to X$ be a morphism. Let y' be a pre-image of y on $Y' := Y \times_X X'$ and let x' be its image on X'. We have to prove that $\mathscr{E}_{Y'}$ is naively flat over X' at y'. By Prop. 4.5.6, it suffices to do this after some ground field extension. Hence we may assume that x' is L-rational.

Since $\mathscr{F}$ and $\mathscr{G}$ are X-flat by assumption, the sequence

$$\mathscr{G}_{Y',y'} \to \mathscr{F}_{Y',y'} \to \mathscr{E}_{Y',y'} \to 0$$

is the truncation of a flat resolution of the $\mathscr{O}_{X',x'}$-module $\mathscr{E}_{Y',y'}$. The map $\mathscr{G}_{Y_x} \to \mathscr{F}_{Y_x}$ is injective at y; by naive flatness of ground field extension, $\mathscr{G}_{Y'_{x'},y'} \to \mathscr{F}_{Y'_{x'},y'}$ is injective too. But since x' is L-rational, the local ring $\mathscr{O}_{Y'_{x'},y'}$ is equal to the quotient $\mathscr{O}_{Y',y'}/\mathfrak{m}_{x'}\mathscr{O}_{Y',y'}$. Hence by passing to the quotient of the truncated flat resolution

$$\mathscr{G}_{Y',y'} \to \mathscr{F}_{Y',y'} \to \mathscr{E}_{Y',y'} \to 0$$

modulo $\mathfrak{m}_{x'}$, one gets an exact sequence whose first arrow is injective. This immediately implies that $\mathrm{Tor}_1^{\mathscr{O}_{X',x'}}(\mathscr{E}_{Y',y'}, \mathscr{O}_{X',x'}/\mathfrak{m}_{x'}) = 0$. As a consequence, $\mathscr{E}_{Y',y'}$ is a flat $\mathscr{O}_{X',x'}$-module ([**SGA 1**], Exposé IV, Thm. 5.6). □

We end this section by showing that flatness ensures a reasonable behavior of local dimension (this is a generalization of Cor. 4.3.3). We use the notion of dimension for modules and coherent sheaves (1.1.2,2.5.3).

4.5.11. Lemma. — *Let $Y \to X$ be a morphism between k-analytic spaces, let $\mathscr{F}$ be a coherent sheaf on Y, let y be a point of Y and let x be its image in X. Assume that y belongs to* $\mathrm{Supp}(\mathscr{F})$ *(this is the case if and only it belongs to* $\mathrm{Supp}(\mathscr{F}_{Y_x})$*, because by 2.5.2 both properties are equivalent to the fact that $\mathscr{F}_{\mathscr{H}(y)} \neq 0$) and that $\mathscr{F}$ is X-flat at y. One has then the equality* $\dim_y \mathscr{F} = \dim_y \mathscr{F}_{Y_x} + \dim_x X$.

Proof. — We immediately reduce to the case where Y and X are affinoid. By extending the ground field, we may assume that y is a k-point (hence, so is x). By assumption, $\mathscr{F}_y$ is a flat $\mathscr{O}_{X,x}$-module; moreover, the quotient $\mathscr{O}_{Y,y}/\mathfrak{m}_x\mathscr{O}_{Y,y}$ is equal to $\mathscr{O}_{Y_x,y}$ because x is a rigid point. By Cor. 6.1.2 of [**EGA IV**$_2$] one has therefore the equality

$$\dim_{\mathrm{Krull}} \mathscr{F}_{Y,y} = \dim_{\mathrm{Krull}} \mathscr{F}_{Y_x,y} + \dim_{\mathrm{Krull}} \mathscr{O}_{X,x}.$$

Since y and x are rigid, Corollary 3.2.9 and GAGA for the support of a coherent sheaf allow us to rewrite it as $\dim_y \mathrm{Supp}(\mathscr{F}) = \mathscr{F} = \dim_y \mathscr{F}_{Y_x} + \dim_x X$. □

CHAPTER 5

QUASI-SMOOTH MORPHISMS

One of the most important example of flat morphism in algebraic geometry, both conceptually and technically, is that of a smooth morphism: this is a locally finitely presented flat morphism with geometrically regular fibers. The purpose of this chapter is to introduce and study the corresponding class of maps in analytic geometry, which are said to be *quasi-smooth*.

In fact, the definition of a scheme-theoretic smooth morphism we have just given is not the usual one: one classically defines smooth morphisms using the sheaf of relative Kähler differentials, and then proves that they can be characterized by the aforementioned property. This is what we shall do here (we have more precisely been inspired by the approach on smoothness of Bosch, Lütkebohmert and Raynaud in [**BLR90**]). Thus after having recalled the definition and the basic properties of the coherent sheaf $\Omega_{Y/X}$ attached to a morphism of k-analytic spaces $Y \to X$ (Section 5.1), we use kind of a Jacobian criterion to define what it means for $Y \to X$ to be quasi-smooth at a given point y of Y (Definition 5.2.4); we say that $Y \to X$ is *quasi-étale* at y if it is quasi-smooth and quasi-finite at y (a former definition of quasi-étaleness had been given by Berkovich in [**Ber94**]; it is consistent with ours by 5.4.11).

Then in Section 5.3 we prove the expected characterization of quasi-smooth morphism (Theorem 5.3.4): if y is a point of Y lying above a point x of X, then $Y \to X$ is quasi-smooth at y if and only if Y is X-flat at y and Y_x is geometrically regular at y.

Section 5.4 explains the links between quasi-smoothness and quasi-étaleness on one hand, and smoothness and étaleness in the sense of Berkovich ([**Ber93**] 3.3 and 3.5) on the other hand. We prove more precisely the following: $Y \to X$ is étale at y if and only it it is quasi-étale and boundaryless at y; *if morevoer Y and X are good*, then $Y \to X$ is smooth at y if and only if it is quasi-smooth and boundaryless at y

(see Corollary 5.4.8 for smoothness, and Remark 5.4.9 for some comments about the goodness assumption and the étale case).

The chapter ends with Section 5.5, in which we prove that all usual algebraic properties are preserved by quasi-smooth maps (Proposition 5.5.5). This rests on the following fact: if Y and X are good and y is a point of Y lying over a point x of X and at which $Y \to X$ is quasi-smooth, then $\operatorname{Spec} \mathscr{O}_{Y,y} \to \operatorname{Spec} \mathscr{O}_{X,x}$ is regular; i.e., flat with geometrically regular fibers (Theorem 5.5.3).

5.1. Reminders about the sheaf of relative differentials

We begin with some reminders about the sheaf of Kähler differentials in analytic geometry; a general reference for the results of this section is [**Ber93**], §3.3.

5.1.1. — Let $Y \to X$ be a morphism between k-analytic spaces. The diagonal map $\delta : Y \to Y \times_X Y$ is G-locally a closed immersion and it has therefore a *conormal sheaf* (see the paragraph before Remark 1.3.8 in [**Ber93**]); this is a coherent sheaf on Y which is denoted [1] by $\Omega_{Y/X}$ and is called the *sheaf of relative (Kähler) differentials of Y over X*.

5.1.2. Contravariant functoriality. — Let

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

be a commutative diagram of analytic spaces. It gives rise to a natural morphism $(\Omega_{Y/X})_{Y'} \to \Omega_{Y'/X'}$, which is an isomorphism if (at least) one of the two following conditions is fullfilled:

– The above diagram is cartesian (see [**Ber93**], Prop. 3.3.2).
– Both maps $Y' \to X'$ and $Y \to X$ are inclusion of analytic domains.

In particular if V is an analytic domain of Y, then $\Omega_{V/X} = (\Omega_{Y/X})_V$; if U is an analytic domain of X, then $\Omega_{U/X} = 0$. If x is a point of X, then $(\Omega_{Y/X})_{Y_x} = \Omega_{Y_x/\mathscr{H}(x)}$ and $(\Omega_{Y/X})_{\mathscr{H}(y)} = (\Omega_{Y_x/\mathscr{H}(x)})_{\mathscr{H}(y)}$ for every $y \in Y_x$.

5.1.3. The universal differential. — Let V be an analytic domain of Y and let p_1 and p_2 be the two projections from $V \times_X V$ to X. If f is an analytic function on V, then $p_1^* f - p_2^* f$ vanishes on the diagonal $V \hookrightarrow V \times_X V$, hence defines an element $\mathrm{d}f$ of $\Omega_{V/X}(V) = \Omega_{Y/X}(V)$. The map $\mathrm{d}\colon \mathscr{O}_Y \to \Omega_{Y/X}$ is an X-derivation, and $(\Omega_{Y/X}, \mathrm{d})$ is the initial object of the category of coherent sheaves $\mathscr{F}$ on Y equipped

1. Berkovich denotes it by $\Omega_{Y_{\mathrm{G}}/X_{\mathrm{G}}}$; for the sake of simplicity, and according to our general conventions, we have decided to simply denote it by $\Omega_{Y/X}$.

with an X-derivation $\mathscr{O}_Y \to \mathscr{F}$. In the situation of 5.1.2 above, the natural morphism $(\Omega_{Y/X})_{Y'} \to \Omega_{Y'/X'}$ commutes with the derivations.

5.1.4. GAGA principle. — If A is a k-affinoid algebra and if $\mathscr{Y} \to \mathscr{X}$ is a morphism between A-schemes of finite type, there is a natural isomorphism

$$(\Omega_{\mathscr{Y}/\mathscr{X}})^{\mathrm{an}} \simeq \Omega_{\mathscr{Y}^{\mathrm{an}}/\mathscr{X}^{\mathrm{an}}},$$

which commutes with the derivations.

5.1.5. — Let

$$Z \longrightarrow Y \longrightarrow X$$

be a diagram in the category of k-analytic spaces. The natural sequence

$$(\Omega_{Y/X})_Z \to \Omega_{Z/X} \to \Omega_{Z/Y} \to 0$$

is then exact ([**Ber93**], Prop. 3.3.2 (i)).

5.1.6. — Let X be a k-analytic space. For every non-negative integer n, the coherent $\mathscr{O}_{\mathbf{A}^n_X}$-module $\Omega_{\mathbf{A}^n_X/X}$ is free with basis $\mathrm{d}T_1, \ldots, \mathrm{d}T_n$.

5.1.7. — Let $Y \to X$ be a morphism of k-analytic spaces, let (f_i) be a family of analytic functions on Y, and let Z be the closed analytic subspace of Y defined by the sheaf of ideals $(f_i)_i$. The morphism $(\Omega_{Y/X})_Z \to \Omega_{Z/X}$ is a surjection whose kernel is generated by the pullbacks of the $\mathrm{d}f_i$'s ([**Ber93**], Prop. 3.3.2 (ii)).

5.1.8. — Let X be a good k-analytic space, let x be a point of X such that $\mathscr{H}(x) = k$, and let $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}_{X,x}$. The map $f \mapsto f - f(x)$ from $\mathscr{O}_{X,x}$ to $\mathfrak{m}$ is then a k-derivation which induces an isomorphism

$$(\Omega_{X/k})_{\kappa(x)} = (\Omega_{X/k})_x/\mathfrak{m}(\Omega_{X/k})_x \simeq \mathfrak{m}/\mathfrak{m}^2,$$

whose inverse isomorphism is induced by the derivation d (this can be checked by direct computation, as in algebraic geometry).

5.1.9. — Let X be an analytic space and let x be a point of X. If follows respectively from Lemma 6.2 and Prop. 6.3 of [**Duc09**] that $\mathrm{rk}_x(\Omega_{X/k}) \geqslant \dim_x X$ and that the following are equivalent:

(i) One has the equality $\mathrm{rk}_x(\Omega_{X/k}) = \dim_x X$.
(ii) The space X is geometrically regular at x.

Moreover if $\mathscr{H}(x) = k$ or if k is perfect, then (i) and (ii) hold if and only if X is regular at x. Indeed, this also follows from Lemma 6.2 and Prop. 6.3 of [**Duc09**] except possibly when $\mathscr{H}(x) = k$ and $|k^\times| = 1$. But in this situation we can first perform a scalar extension to k_r for some arbitrary positive r; since k_r is analytically separable over r, this operation has no effect on regularity, hence allows us to reduce to the non-trivially valued case.

5.1.10. — Assume that (i) and (ii) above hold, and let d be the dimension of X at x. There exists a purely d-dimensional open neighborhood U of x in X such that the coherent sheaf $\Omega_{U/k}$ is free of rank d ([**Duc09**], Prop. 6.6; note that this implies that (i) and (ii) hold at *every* point of U).

5.1.11. Remark. — Let $Y \to X$ be a morphism of k-analytic spaces, let y be a point of Y and let x be its image in X. It follows from 5.1.9 that

$$\mathrm{rk}_y(\Omega_{Y/X}) \leqslant \dim_y Y_x$$

with equality if and only if the fiber Y_x is geometrically regular at y.

5.2. Quasi-smoothness: definition and first properties

We begin with a technical lemma, which will be needed for dealing with the Jacobian criterion we have in mind.

5.2.1. Lemma. — *Let X be a k-analytic space, let n be a non-negative integer, and let V be an affinoid domain of $\mathbf{A}^n_X$. Let Y be a Zariski-closed subset of V, let I be the corresponding ideal of the ring of analytic functions on V, and let $g_1, \ldots, g_r$ be a generating family of I. For every $z \in Y$, let $s(z)$ denote the rank of the family $((\mathrm{d}g_1)(z), \ldots, (\mathrm{d}g_r)(z))$ in the vector space $(\Omega_{V/X})_{\mathscr{H}(z)}$.*

(1) *For every $z \in Y$ the rank $\mathrm{rk}_z(\Omega_{Y/X})$ is equal to $n - s(z)$; in particular,*

$$\mathrm{rk}_z(\Omega_{Y/X}) \geqslant n - r.$$

(2) *Assume that there exists $y \in Y$ with $s(y) = r$ (which is equivalent, by (1), to the fact that $\mathrm{rk}_y(\Omega_{Y/X}) = n - r$). Then:*
   - *Every generating family of I has cardinality at least r.*
   - *There exists an affinoid neighborhood U of y in V such that the morphism $U \cap Y \to X$ is purely of relative dimension $n - r$, and such that $s(z) = r$ for every z belonging to $U \cap Y$; in particular, every fiber of $U \cap Y \to X$ is geometrically regular.*

Proof. — Let us first prove (1). By 5.1.6, the $\mathscr{H}(z)$-vector space $(\Omega_{V/X})_z$ is n-dimensional; and it follows from 5.1.7 that $(\Omega_{Y/X})_{\mathscr{H}(z)}$ is naturally isomorphic to

$$(\Omega_{V/X})_{\mathscr{H}(z)}/((\mathrm{d}g_1)(z), \ldots, (\mathrm{d}g_r)(z)),$$

whence (1).

Now let us come to assertion (2). If $(h_1, \ldots, h_t)$ is a a generating family of I, applying (1) to it yields the inequality $n - r \geqslant n - t$; i.e., $t \geqslant r$, as required. Let x be a point of X. Being an affinoid domain of $\mathbf{A}^{n,\mathrm{an}}_{\mathscr{H}(x)}$, the fiber V_x is purely n-dimensional. As the ideal of Y_x in V_x is generated by r functions, it follows from the *Hauptidealsatz* applied on the noetherian scheme $(V_x)^{\mathrm{al}}$ that the Krull codimension of any irreducible component of Y_x in V_x is at most r. Therefore, the dimension of such a component is at least $n-r$, and it follows that $\dim_z(Y \to X) \geqslant n-r$ for every $z \in Y$.

By upper-semi-continuity of the rank of the stalks of a given coherent sheaf (2.5.2), there exists an affinoid neighborhood U of y in V such that $\mathrm{rk}_z(\Omega_{Y/X})$ is bounded by $n-r$ for every $z \in U \cap Y$; note that this rank is then actually *equal* to $n-r$ in view of (1). Let z be a point of $U \cap Y$ and let x be its image on X. The $\mathscr{H}(x)$-analytic space Y_x if of dimension *at least* $n-r$ at z; and $\mathrm{rk}_z \Omega_{Y_x/\mathscr{H}(x)} = \mathrm{rk}_z(\Omega_{Y/X}) = n-r$. We thus deduce from 5.1.9 that Y_x is of dimension $n-r$ at z, which ends the proof (the claim about geometric regularity comes from 5.1.9). $\square$

5.2.2. Definition. — Let $Y \to X$ be a morphism of k-analytic spaces and let y be a point of Y. Let W be an affinoid domain of Y containing y, let n be an element of $\mathbf{Z}_{\geqslant 0}$, and let V be an affinoid domain of $\mathbf{A}^n_X$ such that $W \to X$ goes through a closed immersion $W \hookrightarrow V$; let us denote by I the ideal defining the latter (in the ring of analytic functions on V), and set $r = n - \mathrm{rk}_y(\Omega_{Y/X})$. We say that the diagram $W \hookrightarrow V \subset \mathbf{A}^n_X$ is a *Jacobian presentation of* $Y \to X$ *at* y if I can be generated by r elements.

5.2.3. — We use the notation of Definition 5.2.2 above. If $W \hookrightarrow V \subset \mathbf{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y, it follows from Lemma 5.2.1 that r is the minimal cardinality of a generating family of I, that $Y \to X$ is of dimension $n-r$ at y, and that the fiber of $Y \to X$ containing y is geometrically regular at y. Lemma 5.2.1 also ensures that there exists an affinoid neighborhood V' of y inside V such that $W \times_V V' \hookrightarrow V'$ is a Jacobian presentation of $Y \to X$ at *each of its points*, and such that $W \times_V V'$ is purely of relative dimension $n-r$ over X.

5.2.4. Definition. — Let $Y \to X$ be a morphism of k-analytic spaces, and let y be a point of Y. We say that $Y \to X$ is *quasi-smooth* at y if there exists a Jacobian presentation of $Y \to X$ at y. We say that it is *quasi-smooth* if it is quasi-smooth at every point of Y.

5.2.5. — Let $Y \to X$ be a morphism of k-analytic spaces, let y be a point of Y, and let x be its image in X. If $Y \to X$ is quasi-smooth at y, it follows from 5.2.3 that Y_x is geometrically regular at y.

5.2.6. Definition. — A morphism of k-analytic spaces $Y \to X$ is *quasi-étale* at a point y of Y if it is quasi-smooth and quasi-finite at y; and it is *quasi-étale* if it is quasi-étale at every point of Y.

5.2.7. Remark. — An analytic space (k, X) is called quasi-smooth, resp. quasi-étale, at a given point x of X if $X \to \mathscr{M}(k)$ is.

5.2.8. Comments on the terminology. — Berkovich has defined ([**Ber93**], §3) the notions of étale and smooth maps. We shall see below (Theorem 5.4.6 and Remark 5.4.9) that a map is étale at a given point if and only if it is quasi-étale and boundaryless at that point; and that a map between *good* k-analytic spaces is smooth at a given point if and only if it is quasi-smooth and boundaryless at that point. (For some comments about the need for a goodness assumption, see Remark 5.4.9).

There is already a notion of quasi-étale morphism, which was defined by Berkovich ([**Ber94**], §3); we shall see below that his definition is equivalent to ours (Lemma 5.4.11).

In [**Duc09**], §6, an analytic space was said to be quasi-smooth (*quasi-lisse* in French) at x if it is geometrically regular at x; this turns out to be consistent with our current definition of quasi-smoothness (Corollary 5.3.5.)

If $|k^{\times}| \neq \{1\}$, if Y and X are strictly k-analytic spaces and if y is a rigid point of Y, quasi-smoothness of $Y \to X$ at y is nothing but *rig-smoothness* of $Y \to X$ at y; we nevertheless have chosen to use "quasi-smooth" instead of "rig-smooth" to be consistent with the terminology "quasi-étale".

5.2.9. — Let X be an analytic space and let n be a non-negative integer. The space $\mathbf{A}^n_X$ is then quasi-smooth over X of relative dimension n: indeed, for every $y \in \mathbf{A}^n_X$ and every affinoid domain W of $\mathbf{A}^n_X$ containing the point y, the diagram $W \simeq W \subset \mathbf{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y.

5.2.10. — Let $Y \to X$ be a morphism of k-analytic spaces, let y be a point of Y, let V be an analytic domain of Y containing y, and let U be an analytic domain of X containing the image of V. Then $Y \to X$ is quasi-smooth at y if and only if $V \to U$ is quasi-smooth at y.

Indeed, let us first assume that $V \to U$ is quasi-smooth at y. Then if $Z \hookrightarrow T \subset \mathbf{A}^n_U$ is a Jacobian presentation of $V \to U$ at y, it follows immediately from the definition that $Z \hookrightarrow T \subset \mathbf{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y.

Conversely, let us assume that $Y \to X$ is quasi-smooth at y, and let us choose a Jacobian presentation $Z \hookrightarrow T \subset \mathbf{A}^n_X$ of $Y \to X$ at y; the image of y in $\mathbf{A}^n_X$ belongs to $\mathbf{A}^n_U$. Let T' be an affinoid domain of $\mathbf{A}^n_U$ that contains the image of y. The fiber product $Z' := Z \times_T T'$ is an affinoid domain of Y which contains y, and $Z' \to T'$ is a closed immersion. By Remark 1.3.13 there exists an affinoid domain T'' of T' containing the image of y such that $Z'' := Z' \times_{T'} T''$ is included in $V \cap Z'$. It follows from the construction that $Z'' \hookrightarrow T'' \subset \mathbf{A}^n_U$ is a Jacobian presentation of $V \to U$ at y.

5.2.11. — If X is an analytic space, the morphism Id_X is quasi-mooth (5.2.9 with $n = 0$); it follows by 5.2.10 that if Y is an analytic domain of X then $Y \to X$ is quasi-smooth.

5.2.12. Behavior with respect to base change. — Let X' be an analytic space and let $X' \to X$ be a morphism. If y is a point of Y, y' is a point of $Y' := Y \times_X X'$ lying over y, and $Y \to X$ is quasi-smooth at y then $Y' \to X'$ is quasi-smooth at y'. Indeed, let $W \hookrightarrow V \subset \mathbf{A}^n_X$ be a Jacobian presentation of $Y \to X$ at y, and let V' (resp. W') denote the fiber product $V \times_X X'$ (resp. $W \times_X X'$). Let V'' be any affinoid domain of V' which contains the image of y' by the closed immersion $V' \hookrightarrow W'$; the fiber product $W'' := W' \times_{V'} V''$ is an affinoid domain of Y', and it is easily seen that $W'' \hookrightarrow V'' \subset \mathbf{A}^n_{X'}$ is a Jacobian presentation of $Y' \to X'$ at y'.

5.2.13. Behavior with respect to composition. — Let Z be an analytic space, let $Z \to Y$ be a morphism, let z be a point of Z, and let y be its image in Y. If $Z \to Y$ is quasi-smooth at z and if $Y \to X$ is quasi-smooth at y then the composite map $Z \to Y \to X$ is quasi-smooth at z.

Indeed, let $W \hookrightarrow V \subset \mathbf{A}^n_X$ be a Jacobian presentation of $Y \to X$ at y. As $Z \to Y$ is quasi-smooth at z, the map $Z \times_Y W \to W$ is quasi-smooth at z too by 5.2.10 or 5.2.12. Let $T \hookrightarrow S \subset \mathbf{A}^m_W$ be a Jacobian presentation of $Z \times_Y W \to W$ at z. As S is an affinoid domain of the Zariski-closed subspace $\mathbf{A}^m_W$ of $\mathbf{A}^m_V$, it follows from Remark 1.3.13 that there exists an affinoid domain S' of $\mathbf{A}^m_V$ such that $S' \cap \mathbf{A}^m_W$ is contained in S and contains the image of z; set $T' = T \times_S (S' \cap \mathbf{A}^m_W)$. The morphism $T' \hookrightarrow S'$ is equal to the composition of $T' \hookrightarrow S' \cap \mathbf{A}^m_W$ and $S' \cap \mathbf{A}^m_W \hookrightarrow S'$, hence is a closed immersion. Being an affinoid domain of $\mathbf{A}^m_V$, which is itself an analytic domain of $\mathbf{A}^{n+m}_X$, the space S' is an affinoid domain of $\mathbf{A}^{n+m}_X$. Set $d = \mathrm{rk}_y(\Omega_{Y/X})$ and $\delta = \mathrm{rk}_z(\Omega_{Z/Y})$. It follows from 5.1.7 that $\mathrm{rk}_z(\Omega_{Z/X}) \leqslant d + \delta$.

Now, as $W \hookrightarrow V \subset \mathbf{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y, the Zariski-closed subspace W of V can be defined by $n - d$ equations; hence the Zariski-closed subspace $S' \cap \mathbf{A}^m_W$ of S' can also be defined by $n - d$ equations. And as $T \hookrightarrow S \subset \mathbf{A}^m_W$ is a Jacobian presentation of $Z \times_Y W \to W$ at z, the closed analytic subspace T of S can be defined by $m - \delta$ equations; hence the closed analytic subspace T' of $S' \cap \mathbf{A}^m_W$ can be defined by $m - \delta$ equations. It follows that the closed analytic subspace T' of S' can be defined using $m + n - d - \delta$ equations. We deduce then from Lemma 5.2.1 (1) that $\mathrm{rk}_z(\Omega_{Z/X}) \geqslant d + \delta$. On the other hand, we have proven above that $\mathrm{rk}_z(\Omega_{Z/X}) \leqslant d + \delta$, whence the equality

$$\mathrm{rk}_z(\Omega_{Z/X}) = d + \delta.$$

Therefore $T' \hookrightarrow S' \subset \mathbf{A}^{n+m}_X$ is a Jacobian presentation of $Z \to X$ at z, so $Z \to X$ is quasi-smooth at z.

5.2.14. — Let A be a k-affinoid algebra and let $\mathscr{Y} \to \mathscr{X}$ be a morphism between A-schemes of finite type. Let y be a point of $\mathscr{Y}^{\mathrm{an}}$. If $\mathscr{Y} \to \mathscr{X}$ is smooth at y^{al} then $\mathscr{Y}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ is quasi-smooth at y. Indeed, as $\mathscr{Y} \to \mathscr{X}$ is smooth at y^{al}, there exists an integer n, an affine open neighborhood $\mathscr{V}$ of y^{al}, and an affine open subset $\mathscr{U}$ of $\mathbf{A}^n_{\mathscr{X}}$ so that $\mathscr{V} \to \mathscr{X}$ goes through a closed immersion $\mathscr{V} \hookrightarrow \mathscr{U}$ whose ideal can be generated by r elements, where $r = n - \mathrm{rk}_{y^{\mathrm{al}}}(\Omega_{\mathscr{Y}/\mathscr{X}})$. Now if U is any affinoid domain of $\mathscr{U}^{\mathrm{an}}$ containing the image of y and if we set $V = \mathscr{V}^{\mathrm{an}} \times_{\mathscr{U}^{\mathrm{an}}} U$ then $V \hookrightarrow U \subset \mathbf{A}^n_{\mathscr{X}^{\mathrm{an}}}$ is a Jacobian presentation of $\mathscr{Y}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ at y (due to 5.1.4).

5.2.15. — By obvious relative dimension arguments, the conclusions in 5.2.10-5.2.14 remain true with "quasi-smooth" replaced by "quasi-étale".

5.2.16. — Let $Y \to X$ be a map between k-analytic spaces, let y be a point of Y and let x be its image on X. Assume that $Y \to X$ is étale at y. Under this assumption, there exists an affinoid domain U of X containing x and an affinoid domain V of Y containing y such that $V \to X$ goes through a finite étale map $V \to U$. But saying that $V \to U$ is finite étale simply means that $V^{\mathrm{al}} \to U^{\mathrm{al}}$ is finite étale, which implies that $V \to U$ is quasi-étale (5.2.14 and 5.2.15); as a consequence, $Y \to X$ is quasi-étale at y by 5.2.10.

5.2.17. — Let $Y \to X$ be a map between k-analytic spaces and let y be a point of Y. Assume that $Y \to X$ is smooth at y. By definition, there exists an open neighborhood V of y in Y such that $V \to X$ goes through an étale map $V \to \mathbf{A}^n_X$ for some n. It follows from 5.2.16 above that $V \to \mathbf{A}^n_X$ is quasi-étale. Since $\mathbf{A}^n_X \to X$ is quasi-smooth by 5.2.9, one deduces from 5.2.13 and 5.2.10 that $Y \to X$ is quasi-smooth at y.

5.3. Quasi-smoothness, flatness and fiberwise geometric regularity

Our goal is to establish some expected properties of quasi-smoothness: the fact that a morphism $\varphi \colon Y \to X$ is quasi-smooth at a point y of Y if and only if $Y \to X$ is flat at y and $Y_{\varphi(y)}$ is geometrically regular at y; and the fact that if this is the case, then $\Omega_{Y/X}$ is free of rank $\dim_y \varphi$ at y. We begin with a slightly technical lemma that has no interest by itself − especially in view of the announced results − but which we shall need to argue by induction on the number of equations in a given Jacobian presentation.

5.3.1. Lemma. — *Let d be a non-negative integer, let $Y \to X$ be a morphism of k-analytic spaces, and let y be a point of Y at which $Y \to X$ is quasi-smooth of relative dimension d. Assume that both $\mathscr{O}_Y$ and $\Omega_{Y/X}$ are X-flat at y.*

(1) *The coherent sheaf $\Omega_{Y/X}$ is free of rank d at y.*
(2) *Let f be an analytic function on Y such that the element $(\mathrm{d}f)(y)$ of $(\Omega_{Y/X})_{\mathscr{H}(y)}$ is non-zero, and let Z be the closed analytic subspace of Y defined by the ideal (f).*
   (2a) *The morphism $Z \to X$ is quasi-smooth of relative dimension $d-1$ at y.*
   (2b) *The coherent sheaves $\mathscr{O}_Z$ and $\Omega_{Z/X}$ are X-flat at y.*

Proof. — All properties involved can be checked G-locally, hence we may and do assume that both Y and X are k-affinoid. Let x denote the image of y in X.

As $Y \to X$ is quasi-smooth of relative dimension d at y, the dimension of $(\Omega_{Y/X})_{\mathscr{H}(y)}$ is equal to d. Let us choose global forms $\omega_1, \ldots, \omega_d$ belonging to $\Omega_{Y/X}(Y)$ such that $(\omega_i(y))_i$ is a basis of $(\Omega_{Y/X})_{\mathscr{H}(y)}$, which is possible in view of 2.5.2 (2). The ω_i's define a morphism $\mathscr{O}_Y^d \to \Omega_{Y/X}$. Since Y_x is geometrically regular at y, the sheaf $\Omega_{Y_x/\mathscr{H}(x)}$ is free of rank d at y; it follows therefore from Nakayama's Lemma that $\mathscr{O}_{Y_x}^d \to \Omega_{Y_x/\mathscr{H}(x)}$ is an isomorphism at y. Since $\Omega_{Y/X}$ is X-flat at y by assumption, Lemma 4.5.8 implies that $\mathscr{O}_Y^d \to \Omega_{Y/X}$ is an isomorphism at y, whence (1).

The morphism $Y \to X$ being quasi-smooth at y, it admits a Jacobian presentation $W \hookrightarrow V \subset \mathbf{A}_X^n$ at y. There exists a finite family $(g_1, \ldots, g_{n-d})$ of analytic functions on V such that the ideal $(g_1, \ldots, g_{n-d})$ defines the closed immersion $W \hookrightarrow V$, and such that the family $(\mathrm{d}g_i(y))_i$ of elements of the vector space $(\Omega_{V/X})_{\mathscr{H}(y)}$ is free. As

$$(\Omega_{Y/X})_{\mathscr{H}(y)} \simeq (\Omega_{V/X})_{\mathscr{H}(y)}/((\mathrm{d}g_1)(y), \ldots, (\mathrm{d}g_{n-d})(y)),$$

the fact that $(\mathrm{d}f)(y)$ is non-zero in $(\Omega_{Y/X})_{\mathscr{H}(y)}$ simply means that the family $((\mathrm{d}g_1)(y), \ldots, (\mathrm{d}g_{n-d})(y), (\mathrm{d}f)(y))$ is free in $(\Omega_{V/X})_{\mathscr{H}(y)}$. As the ideal $(g_1, \ldots, g_{n-d}, f)$ defines precisely the closed immersion $W \cap Z \hookrightarrow V$, the diagram $(W \cap Z) \hookrightarrow V \subset \mathbf{A}_X^n$ is a Jacobian presentation of $Z \to X$ at y, and $Z \to X$ is quasi-smooth at y of relative dimension $d-1$, whence (2a).

Denote by ι the closed immersion $Z \hookrightarrow Y$. We have an exact sequence

$$\mathscr{O}_Y \xrightarrow{\times f} \mathscr{O}_Y \longrightarrow \iota_*\mathscr{O}_Z \longrightarrow 0\ .$$

By assumption, $\mathscr{O}_Y$ is X-flat at y. As Y is quasi-smooth over X at y, the fiber Y_x is (geometrically) regular at y. Since we have made the hypothesis that $(\mathrm{d}f)(y) \neq 0$ in $(\Omega_{Y/X})_{\mathscr{H}(y)}$, the element f of the regular local ring $\mathscr{O}_{Y_x,y}$ is non-zero. As a regular local ring is a domain, the multiplication by f from $\mathscr{O}_{Y_x}$ to itself is injective at y. It now follows from Lemma 4.5.10 that $\iota_*\mathscr{O}_Z$ is X-flat at y ; in other words, $\mathscr{O}_Z$ is X-flat at y.

Consider now the exact sequence

$$\mathscr{O}_Z \xrightarrow{u} (\Omega_{Y/X})_Z \longrightarrow \Omega_{Z/X} \longrightarrow 0\ ,$$

where u is the multiplication by the pullback of $\mathrm{d}f$. We have just proven that $\mathscr{O}_Z$ is X-flat at y. Since $\Omega_{Y/X}$ has been seen to be free at y, its pull-back $(\Omega_{Y/X})_Z$ is free at y as well, hence is X-flat at y because so is $\mathscr{O}_Z$. Since $(\mathrm{d}f)(y) \neq 0$ in $(\Omega_{Y/X})_{\mathscr{H}(y)}$, the image of $\mathrm{d}f$ in $(\Omega_{Y/X})_{Z_x,y}$ is non-zero. As Z is quasi-smooth over X at y, the fiber Z_x is (geometrically) regular at y; in particular, $\mathscr{O}_{Z_x,y}$ is a domain, and the free $\mathscr{O}_{Z_x,y}$ -module $(\Omega_{Y/X})_{Z_x,y}$ is thus torsion-free. It follows that u induces an *injection* from $\mathscr{O}_{Z_x,y}$ into $(\Omega_{Y/X})_{Z_x,y}$. We then deduce from Lemma 4.5.10 that $\Omega_{Z/X}$ is X-flat at y, whence (2b). □

5.3.2. Corollary. — *Let $Y \to X$ be a morphism of k-analytic spaces, let d be an element of $\mathbf{Z}_{\geqslant 0}$, and let y be a point of Y at which $Y \to X$ is quasi-smooth of relative dimension d. The sheaf $\mathscr{O}_Y$ is X-flat at y, and $\Omega_{Y/X}$ is free of rank d at y.*

Proof. — It is sufficient to prove that both coherent sheaves $\mathscr{O}_Y$ and Ω_Y are X-flat at y; it will then follow from Lemma 5.3.1 (1) that $\Omega_{Y/X}$ is free of rank d at y. Let us choose a Jacobian presentation $W \hookrightarrow V \subset \mathbf{A}^n_X$ of $Y \to X$ at y, and set $r = n - \dim_{\mathscr{H}(y)}(\Omega_{Y/X})_{\mathscr{H}(y)}$. By definition of such a presentation, there exists a family $(g_1, \ldots, g_r)$ of analytic functions on V such that the ideal $(g_1, \ldots, g_r)$ defines the closed immersion $W \hookrightarrow V$, and such that the family $(\mathrm{d}g_1)(y), \ldots, (\mathrm{d}g_r)(y))$ of elements of $(\Omega_{V/X})_{\mathscr{H}(y)}$ is free. For every $i \in \{0, \ldots, r\}$, denote by V_i the closed analytic subspace of V defined by the ideal $(g_1, \ldots, g_i)$; note that $V_0 = V$ and that $V_r = W$. The map $V \to X$ is quasi-smooth at y; moreover, $\mathscr{O}_V$ and $\Omega_{V/X}$ are X-flat. Indeed, $\mathbf{A}^n_X$ is flat over X: this comes from Lemma 4.1.13, or from the fact that $\mathbf{A}^n_Z$ is flat over Z for every affinoid space Z in view of Proposition 4.2.4, because $\mathbf{A}^n_{Z^{\mathrm{al}}}$ is flat over Z^{al}; as a consequence, V is flat over X. And since $\Omega_{V/X}$ is a free $\mathscr{O}_V$-module (with basis $(\mathrm{d}T_i)_i$), it is flat over X too.

Now Lemma 5.3.1 together with a straightforward induction on i shows that for every $i \in \{0, \ldots, r\}$, the space V_i is quasi-smooth at y and the coherent sheaves $\mathscr{O}_{V_i}$ and $\Omega_{V_i/X}$ are X-flat at y. By taking $i = r$ we get the expected statements. □

5.3.3. — Let $Y \to X$ be a morphism of k-*affinoid* spaces, let y be a point of Y and let x be its image on X. Let us assume that Y_x is geometrically regular, and that $Y \to X$ is flat at y. There exists $n \in \mathbf{Z}_{\geqslant 0}$ so that the morphism $Y \to X$ goes through a closed immersion $Y \hookrightarrow D \times_k X$, where D is a closed n-dimensional polydisc. Set $r = n - \mathrm{rk}_y(\Omega_{Y/X})$ (note that $Y \to X$ is then of relative dimension $n - r$ at y), and let $\mathscr{I}$ be the sheaf of ideals of $\mathscr{O}_{D \times_k X}$ that defines the closed immersion $Y \hookrightarrow D \times_k X$. Let us choose global sections $g_1, \ldots, g_r$ of $\mathscr{I}$ such that

$$(\Omega_{Y/X})_{\mathscr{H}(y)} = (\Omega_{D \times_k X/X})_{\mathscr{H}(y)}/((\mathrm{d}g_1)(y), \ldots, (\mathrm{d}g_r)(y)),$$

which is possible by 5.1.7 and 2.5.2 (2); note that $((\mathrm{d}g_1)(y), \ldots, (\mathrm{d}g_r)(y))$ is then a free family of elements of the vector space $(\Omega_{D \times_k X/X})_{\mathscr{H}(y)}$. Let Z be the Zariski-closed subspace of $D \times_k X$ defined by the ideal sheaf $\mathscr{J} := (g_1, \ldots, g_r)$; by construction, Y is a Zariski-closed subspace of Z, and $Z \to X$ is quasi-smooth at y of relative dimension $n - r$.

As $Z \to X$ is quasi-smooth at y of relative dimension $n - r$, the $\mathscr{H}(x)$-space Z_x is geometrically regular at y of relative dimension $n - r$. In particular, there exists a connected affinoid neighborhood U of y in Z_x that is normal, connected and of dimension $n - r$. As the fiber Y_x is geometrically regular at y and as we have the equalities $\mathrm{rk}_y(\Omega_{Y_x/\mathscr{H}(x)}) = \mathrm{rk}_y(\Omega_{Y/X}) = n - r$, we can shrink U so that $U \cap Y_x$ is of dimension $n - r$. The intersection $U \cap Y_x$ being a closed analytic subspace of the reduced, irreducible, $(n - r)$-dimensional space U, it coincides with U.

The natural surjection $\mathscr{O}_{Z,y} \to \mathscr{O}_{Y,y}$ is bijective. Indeed, we have proven above that $U = U \cap Y_x$, which implies that $\mathscr{O}_{Z_x,y} \to \mathscr{O}_{Y_x,y}$ is a bijection; and since $\mathscr{O}_Y$ is X-flat at y by assumption, it follows then from Lemma 4.5.8 that $\mathscr{O}_{Z,y} \to \mathscr{O}_{Y,y}$ is bijective.

The bijectivity of $\mathscr{O}_{Z,y} \to \mathscr{O}_{Y,y}$ is equivalent to the bijectivity of $\mathscr{J}_y \to \mathscr{I}_y$. As a consequence, there exists an affinoid neighborhood V of y in $D \times_k X$ such that the closed immersion $V \cap Y \hookrightarrow V \cap Z$ is an isomorphism; note that $V \cap Y$ is an affinoid neighborhood of y in Y, and that $V \cap Y \hookrightarrow V \subset \mathbf{A}^n_X$ is a Jacobian presentation of $Y \to X$ at y.

5.3.4. Theorem. — *Let $Y \to X$ be a morphism of k-analytic spaces, let y be a point of Y and let x be its image on X.*

(1) *The following are equivalent:*
   (i) *The morphism $Y \to X$ is quasi-smooth at y.*
   (ii) *The morphism $Y \to X$ is flat at y, and the fiber Y_x is geometrically regular at y.*
   (iii) *The morphism $Y \to X$ is flat at y, and $\mathrm{rk}_y(\Omega_{Y/X}) = \dim_y Y_x$.*

(2) *If moreover Y and X are good, then those properties hold if and only if there exists a Jacobian presentation $W \hookrightarrow V \subset \mathbf{A}^n_X$ of $Y \to X$ at y with W being an affinoid* neighborhood *of y in Y.*

Proof. — We first remark that (ii) $\iff$ (iii) by Remark 5.1.11. If $Y \to X$ is quasi-smooth at x, we already know that Y_x is geometrically regular at y, and flatness of $Y \to X$ at y is part of Corollary 5.3.2; so (i)$\Rightarrow$(ii).

Assume now that $Y \to X$ is flat at y, and that Y_x is geometrically regular at y. In order to prove that $Y \to X$ is quasi-smooth at y, we may assume that both Y and X are k-affinoid. But under that assumption, we have seen in 5.3.3 that there exists a Jacobian presentation $W \hookrightarrow V \subset \mathbf{A}^n_V$ of $Y \to X$ at y with W being an affinoid neighborhood of y in Y, which at the same time ends the proof of (ii)$\Rightarrow$(i) and proves (2). $\square$

5.3.5. Corollary. — *If X is a k-analytic space and if x is a point of X, then X is quasi-smooth at x if and only if it is geometrically regular at x.*

Proof. — This is an immediate consequence of Theorem 5.3.4 above, together with the fact that $X \to \mathscr{M}(k)$ is flat (Lemma 4.1.13). $\square$

5.3.6. Corollary. — *Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of schemes of finite type over a given affinoid algebra, and let y be point of $\mathscr{Y}^{\mathrm{an}}$. The map $\mathscr{Y}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ is quasi-smooth at y if and only if $\mathscr{Y} \to \mathscr{X}$ is smooth at y^{al}.*

Proof. — Let x be the image of y on $\mathscr{X}^{\mathrm{an}}$. By Theorem 5.3.4, the natural morphism $\mathscr{Y}^{\mathrm{an}} \to \mathscr{X}^{\mathrm{an}}$ is quasi-smooth at y if and only if it is flat at y and the dimension of the

$\mathscr{H}(y)$-vector space $(\Omega_{\mathscr{Y}^{\mathrm{an}}/\mathscr{X}^{\mathrm{an}}})_{\mathscr{H}(y)}$ is equal to the relative dimension of $\mathscr{Y}^{\mathrm{an}}$ over $\mathscr{X}^{\mathrm{an}}$ at y.

On the other hand, $\mathscr{Y} \to \mathscr{X}$ is smooth at y^{al} if and only if it is flat at y^{al} and the dimension of the $\kappa(y^{\mathrm{al}})$-vector space $(\Omega_{\mathscr{Y}/\mathscr{X}})_{\kappa(y^{\mathrm{al}})}$ is equal to the relative dimension of $\mathscr{Y}$ over $\mathscr{X}$ at y^{al}.

The corollary now follows from GAGA principles about flatness (Lemma 4.2.1 and Proposition 4.2.4), about the sheaf of differential forms (5.1.4), and about relative dimension (Prop. 2.7.7). $\square$

5.3.7. Corollary. — *Let $Y \to X$ be a morphism between k-analytic spaces and let d be an integer. The set of points of Y at which $Y \to X$ is quasi-smooth of relative dimension d is an open subset of Y.*

Proof. — We immediately reduce to the case where both Y and X are affinoid. Let y be a point of Y at which Y is quasi-smooth of relative dimension d over X. By Theorem 5.3.4, there exists an affinoid neighborhood W of y in Y and a Jacobian presentation $W \hookrightarrow V \subset \mathbf{A}^m_X$ of $Y \to X$ at y. Now by 5.2.3 there exists an affinoid neighborhood V' of y in V such that $W \times_V V' \hookrightarrow V'$ is a Jacobian presentation of $Y \to X$ at each point of $W \times_V V'$, and such that $Y \to X$ is of relative dimension d at each point of $W \times_V V'$. Hence $Y \to X$ is quasi-smooth of relative dimension d at each point of the affinoid neighborhood $W' \times_V V'$ of y. $\square$

5.3.8. Remark. — We shall see later that the set of points of Y at which $Y \to X$ is quasi-smooth of relative dimension d is even *Zariski*-open in Y (Theorem 10.7.2).

5.4. Links with étale and smooth morphisms

Our purpose is now to investigate the link between quasi-smooth morphisms, and smooth morphisms in the sense of Berkovich [**Ber93**], 3.5. As we shall see, *as far as the spaces involved are good*, the situation is very pleasant: a morphism is smooth (at a given point of the source) if and only if it is quasi-smooth and boundaryless; and it is quasi-smooth if and only if it is (locally) the composition of an inclusion of an analytic domain and of a smooth map. We shall discuss thereafter this goodness assumption, and see that it is not needed in the case of relative dimension zero; i.e., for the comparison between quasi-étaleness and étaleness.

5.4.1. Definition. — Let $Y \to X$ be a morphism of k-analytic spaces and let y be a point of Y. The morphism $Y \to X$ is said to be *unramified at y* (we shall also say that Y is *unramified over* X at y) if $(\Omega_{Y/X})_{\mathscr{H}(y)} = 0$; i.e., if y does not belong to the support of $\Omega_{Y/X}$ (2.5.2). The morphism $Y \to X$ is called *unramified* if it is unramified at every point of Y; i.e., if $\Omega_{Y/X} = 0$ (we shall also say "Y is *unramified over X*").

5.4.2. — Let $Y \to X$ be a morphism of k-analytic spaces and let y be a point of X. If $Y \to X$ is unramified at y, then the inequality $\mathrm{rk}_y(\Omega_{Y/X}) \geqslant \dim_y(Y \to X)$ implies that Y is of relative dimension zero over X at y. As a consequence, it follows from Theorem 5.3.4 that $Y \to X$ is quasi-étale at y if and only if it is flat and unramified at y.

5.4.3. Lemma. — *Let $Y \to X$ be a morphism of k-analytic spaces. It is unramified if and only if the diagonal $Y \to Y \times_X Y$ is G-locally on Y an open immersion with closed (and open) image; i.e., there exists a G-covering (Y_i) of Y such that for every i, the diagonal $Y_i \to Y_i \times_X Y_i$ is an open immersion with closed image.*

Proof. — If $Y \to Y \times_X Y$ is G-locally on Y an open immersion, its conormal sheaf is trivial and $\Omega_{Y/X} = 0$. Conversely, assume that $\Omega_{Y/X} = 0$, and let us prove that the diagonal $Y \to Y \times_X Y$ is G-locally on Y an open immersion with closed image. By arguing G-locally we reduce to the case were both Y and X are affinoid. The diagonal map $Y \to Y \times_X Y$ is then a closed immersion, inducing a closed immersion $Y^{\mathrm{al}} \hookrightarrow (Y \times_X Y)^{\mathrm{al}}$ at the scheme-theoretic level. Since $\Omega_{Y/X} = 0$ the conormal sheaf of the closed immersion $Y \hookrightarrow Y \times_X Y$ is trivial, hence the conormal sheaf of the closed immersion $Y^{\mathrm{al}} \hookrightarrow (Y \times_X Y)^{\mathrm{al}}$ is trivial as well, which means that this closed immersion is also an open immersion. Therefore $Y \hookrightarrow Y \times_X Y$ is an open immersion with closed image. ☐

5.4.4. Corollary. — *Let*

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

be a commutative diagram in the category of k-analytic spaces. Let z be a point of Z and let y be its image on Y. Assume that Z is quasi-étale over X at z, and that Y is unramified over X at y. Then Z is quasi-étale over Y at z.

Proof. — The quasi-étaleness locus of a morphism is an open subset of the source by Corollary 5.3.7, and its unramified locus also (indeed, this is the complement of the support of a coherent sheaf). Hence we may shrink Z and Y around z and y respectively so that Z becomes quasi-étale over X and Y becomes unramified over X.

The morphism $Z \to Y$ is the composition of its graph $Z \hookrightarrow Z \times_X Y$ and the second projection $Z \times_X Y \to Y$. Since $Z \to X$ is quasi-étale, $Z \times_X Y \to Y$ is quasi-étale too. And the graph $Z \hookrightarrow Z \times_X Y$ arises from the diagonal $Y \hookrightarrow Y \times_X Y$ through the base-change functor by $Z \to Y$ (the product $Y \times_X Y$ being seen as a Y-space through the first projection). Since $Y \to X$ is unramified, the diagonal $Y \to Y \times_X Y$

is G-locally on Y an open immersion by Lemma 5.4.3 above; in particular, it is quasi-étale. Therefore the graph $Z \to Z \times_X Y$ is quasi-étale too, and $Z \to X$ is quasi-étale as the composition of two quasi-étale maps. $\square$

5.4.5. Lemma. — *Let $Y \to X$ be a morphism between k-analytic spaces and let y be a point of Y. Assume that $Y \to X$ is quasi-smooth at y of relative dimension d, and let $f_1, \ldots, f_\ell$ be analytic functions on Y such that $((\mathrm{d}f_i)(y))_i$ is a free family of elements of $(\Omega_{Y/X})_{\mathscr{H}(y)}$ (note that one then has $\ell \leqslant d$). The map $Y \to \mathbf{A}^\ell_X$ defined by the f_i's is quasi-smooth of relative dimension $d - \ell$ at y.*

Proof. — One immediately reduces to the case where Y and X are affinoid. Under that assumption, the map $Y \to \mathbf{A}^\ell_X$ goes through $D \times_k X$ for some ℓ-dimensional compact polydisc D, and it is sufficient to prove that $Y \to D \times_k X$ is quasi-smooth of relative dimension $d - \ell$ on an analytic neighborhood of y.

Both spaces Y and $D \times_k X$ are k-affinoid, hence $Y \to D \times_k X$ goes through a closed immersion $Y \hookrightarrow \Delta \times_k D \times_k X$ where Δ is a closed polydisc. Let $\mathscr{I}$ be the corresponding ideal sheaf on $\Delta \times_k D \times_k X$ and let δ be the dimension of Δ.

By the choice of the f_i's, and in view of 5.1.5 applied to the diagram

$$Y \to D \times_k X \to X,$$

the rank $\mathrm{rk}_y(\Omega_{Y/D\times_k X})$ is equal to $d - \ell$. As a consequence and in view of 1.3.7 and 5.1.7, we can find $\delta - d + \ell$ global sections $g_1, \ldots, g_{\delta-d+\ell}$ of $\mathscr{I}$ such that $((\mathrm{d}g_j)(y))_j$ is a free family of elements of $(\Omega_{\Delta\times_k D\times_k X/D\times_k X})_{\mathscr{H}(y)}$; it remains free when viewed as a family of vectors of $(\Omega_{\Delta\times_k D\times_k X/X})_{\mathscr{H}(y)}$, because the former vector space is a quotient of the latter by 5.1.5. The quotient of $(\Omega_{\Delta\times_k D\times_k X/X})_{\mathscr{H}(y)}$ by the $(\mathrm{d}g_j)(y)$'s has then dimension d; the natural surjection

$$(\Omega_{\Delta\times_k D\times_k X/X})_{\mathscr{H}(y)}/((\mathrm{d}g_j)(y))_j \to (\Omega_{Y/X})_{\mathscr{H}(y)}$$

is thus an isomorphism.

On the other hand, $Y \to X$ is by assumption quasi-smooth at y; by Corollary 5.3.2, this implies that $Y \to X$ is flat at y, and that Y_x is geometrically regular at y.

By the above and in view of 5.3.3, there exists an affinoid neighborhood V of y in $\Delta \times_k D \times_k X$ such that the closed analytic subspace $Y \cap V$ of V is defined by the ideal $(g_j)_j$; since $((\mathrm{d}g_j(y))_j$ is a free family of elements $(\Omega_{\Delta\times_k D\times_k X/D\times_k X})_{\mathscr{H}(y)}$, the map $Y \to D \times_k X$ is quasi-smooth of relative dimension $d - \ell$ at y. $\square$

5.4.6. Theorem. — *Let $Y \to X$ be a morphism of* good *k-analytic spaces, and let y be a point of Y. The following are equivalent :*

(i) *There exists an affinoid neighborhood Y_0 of y in Y and a smooth X-space Z such that Y_0 is X-isomorphic to an affinoid domain of Z.*
(ii) *The morphism $Y \to X$ is quasi-smooth at y.*

Proof. — As embeddings of analytic domains and smooth morphisms are quasi-smooth (5.2.11, 5.2.17), assertion (i) implies assertion (ii). Let us now assume that (ii) is true. In order to prove (i), we may and do assume that Y is affinoid. Let d be the dimension of $(\Omega_{Y/X})_{\mathscr{H}(y)}$ and let $f_1, \ldots, f_d$ be analytic functions on Y such that the family $((\mathrm{d}f_i)(y))_i$ is a basis of $(\Omega_{Y/X})_{\mathscr{H}(y)}$; let $\varphi\colon Y \to \mathbf{A}^d_X$ be the morphism induced by the f_i's. By Lemma 5.4.5, the morphism φ is quasi-étale at y. Set $\xi = \varphi(y)$. As φ is quasi-finite at y, the analytic Zariski's Main Theorem ([**Duc07b**], Thm. 3.2) ensures that Y can be shrunken so that φ admits a factorization $Y \to T_0 \to T \to \mathbf{A}^d_X$ where T is finite étale over an open neighborhood U of ξ, T_0 is an affinoid domain of T, and $Y \to T_0$ is finite.

The finite morphism $Y \to T_0$ is étale at y. Indeed, T_0 is quasi-étale over $\mathbf{A}^d_X$ because both maps $T_0 \to T$ and $T \to \mathbf{A}^d_X$ are quasi-étale (the first one is the embedding of an analytic domain, and the second one is étale). Since $Y \to \mathbf{A}^d_X$ is also quasi-étale at y, it follows from Corollary 5.4.4 that $Y \to T_0$ is quasi-étale at y. This implies that $Y \to T_0$ is flat at y and that $(\Omega_{Y/T_0})_{\mathscr{H}(y)} = 0$; the map $Y \to T_0$ being finite, those conditions exactly mean that it is étale at y.

Let t be the image of y in T. The categories of finite étale covers of the germ (T_0, t) and (T, t) are naturally equivalent (both are equivalent to the category of finite étale $\mathscr{H}(t)$-algebras, [**Ber93**] Thm. 3.4.1). Therefore there exists:

- an open neighborhood T_1 of t in T;
- a finite étale map $Z \to T_1$;
- an isomorphism between $Z \times_{T_1} (T_1 \cap T_0)$ and an open neighborhood Y_1 of y in Y.

All morphisms in the diagram

$$Z \to T_1 \to U \to \mathbf{A}^d_X \to X$$

are smooth; hence their composition $Z \to X$ is smooth. Now one can take Y_0 equal to any affinoid neighborhood of y inside Y_1. ☐

5.4.7. Remark. — In the strictly k-analytic case, such a result has already been proved by Berkovich ([**Ber99**], Remark 9.7).

5.4.8. Corollary. — *Let $Y \to X$ be a morphism between* good *k-analytic spaces and let y be a point of Y. The following are equivalent:*

(1) *The morphism $Y \to X$ is smooth at y.*
(2) *The morphism $Y \to X$ is quasi-smooth and boundaryless at y.*

Proof. — If $Y \to X$ is smooth at y, it is quasi-smooth and boundaryless at y (without a goodness assumption). Assume conversely that $Y \to X$ is quasi-smooth and boundaryless at y. Since $Y \to X$ is quasi-smooth at y and since Y and X are good, it follows from Theorem 5.4.6 that there exists an affinoid neighborhood Y_0 of y in Y such that Y_0 can be identified with an affinoid domain of a smooth X-analytic space

Z. Since $Y \to X$ is boundaryless at Y, the morphism $Y_0 \to X$ is also boundaryless at y. This imply that $Y_0 \hookrightarrow Z$ is boundaryless as y; since Y_0 is an affinoid domain of Z, this means that y belongs to the topological interior U of Y_0 in Z. But U is a neighborhood of y in Y and is smooth over X (as an open subset of Z); hence $Y \to X$ is smooth at y. □

5.4.9. Remark. — The author does not know if Corollary 5.4.8 above is true without the goodness assumption. By looking carefully at what happens, the reader should be convinced that the main problem to face in the non-good case is the following: if $Y \to X$ be a morphism of analytic spaces and if $y \in Y$, there is no reason why there should exist analytic functions $f_1, \ldots, f_r$ defined in a *neighborhood* of y such that $((\mathrm{d}f_1)(y)), \ldots, (\mathrm{d}f_r)(y))$ generate $(\Omega_{Y/X})_{\mathscr{H}(y)}$.

However, in the case where the relative dimension is zero (where the aforementioned problem vanishes), Corollary 5.4.8 is true without any goodness assumption. Indeed, let us assume that $Y \to X$ is quasi-étale and boundaryless at y. Being zero-dimensional and boundaryless at y, it is finite at y ([**Ber93**], Prop. 3.14), hence we can shrink Y and X so that $Y \to X$ is finite, and so that y is the only pre-image of its image x in X. Now choose a compact analytic neighborhood of x in X of the form $V_1 \cup \ldots \cup V_m$ where the V_i's are affinoid domains of X containing x. For every i the map $Y \times_X V_i \to V_i$ is finite; being quasi-étale at y, it is in particular flat and unramified at y, hence étale at y, which is the only pre-image of x. As a consequence, there exists an affinoid neighborhood W_i of x in V_i such that $Y \times_X W_i \to W_i$ is étale. If one sets $W = \bigcup W_i$, then W is a compact analytic neighborhood of x and $Y \times_X W \to W$ is étale, whence our claim.

In his work [**Ber94**] on vanishing cycles for formal schemes, Berkovich has defined quasi-étale morphisms. We are going to check that our notion of quasi-étaleness coincides with Berkovich's.

5.4.10. Berkovich's definition ([Ber94], §3). — Let $Y \to X$ be a morphism of k-analytic spaces and let y be a point of Y. The morphism $Y \to X$ is quasi-étale at y in the sense of Berkovich if y has a compact analytic neighborhood in Y of the form $V_1 \cup \ldots \cup V_n$ where every V_i is an affinoid domain of Y that is X-isomorphic to an affinoid domain of an étale X-analytic space.

5.4.11. Lemma. — *Let $Y \to X$ be a morphism of k-analytic spaces and let y be a point of Y. The following are equivalent:*

(i) *The morphism $Y \to X$ is quasi-étale at y in the sense of Berkovich.*

(ii) *The morphism $Y \to X$ is quasi-étale at y in our sense.*

Proof. — In what follows, "quasi-étale" will mean "quasi-étale in our sense", and we will write "quasi-étale in the sense of Berkovich" when needed. Let us assume (i). There exists in particular an affinoid domain V of Y containing y such that V can be identified with an affinoid domain of an analytic space X' which is étale over X. As $V \hookrightarrow X'$ and $X' \to X$ are quasi-étale, $V \to X$ is quasi-étale; in particular, $V \to X$ is quasi-étale at y, and $Y \to X$ is therefore quasi-étale at y.

Let us now assume (ii), and let x denote the image of y in X. Let us choose a compact analytic neighborhood of Y which is a finite union $\bigcup V_i$ of affinoid domains of Y containing y; we may assume that there exists for every i an affinoid domain U_i of X such that $V_i \to X$ goes through U_i. Fix i. As $Y \to X$ is quasi-étale at y, the morphism $V_i \to U_i$ is quasi-étale at y too (5.2.10). Hence it follows from Theorem 5.4.6 that there exists an affinoid neighborhood V_i' of y in V_i and an étale U_i-space U_i' such that V_i' is U_i-isomorphic to an affinoid domain of U_i'. The categories of finite étale covers of the germs (X, x) and (U_i, x) are naturally equivalent (both are equivalent to the category of finite étale $\mathscr{H}(x)$-algebras by [**Ber93**], Thm. 3.4.1). Therefore there exists an open neighborhood X_i of x in X and a finite étale morphism $X_i' \to X_i$ such that $X_i' \times_X U_i$ can be identified with an open neighborhood of y in U_i'; let us choose an affinoid neighborhood V_i'' of y in V_i' such that $V_i'' \subset X_i' \times_X U_i \subset X_i'$. The union of the V_i'''s is a neighborhood of y, and for every i one can identify V_i'' with an affinoid domain of the X-étale space X_i'; therefore $Y \to X$ is quasi-étale at y in the sense of Berkovich. □

5.5. Transfer of algebraic properties

Let $Y \to X$ be a morphism of k-affinoid spaces, let y be a point of Y and let x be its image in X. Assume that Y is quasi-smooth over X at y. By Theorem 5.3.4, Y is X-flat at y, and Y_x is geometrically regular at y. The purpose of what follows is to establish algebraic avatars of this result. Namely, we shall prove that both morphisms $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}} \to \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}$ and $\mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ are regular. In fact, the general results established in 2.3.20 will enable us to deduce both statements from a particular case of the first one, which is the object of Lemma 5.5.1 below.

5.5.1. Lemma. — *Let $Y \to X$ be a quasi-smooth morphism between k-affinoid spaces, let y be a point of Y and let x be its image in X. Assume that X is integral and that x^{al} is the generic point of X. The scheme Y^{al} is then regular at y^{al}.*

Proof. — By flatness of the map of locally ringed spaces $Y \to Y^{\mathrm{al}}$ it is enough to prove that Y is regular at y. For that purpose we may shrink Y around y; hence using Theorem 5.4.6 we reduce to the case where Y is X-isomorphic to an affinoid domain of some X-smooth space X', and it suffices then to prove that X' is regular at every point lying above x.

Let x' be such a point. There exists an open neighborhood U of x' in X' and an étale X-morphism $U \to \mathbf{A}^n_X$ for some n; let z be the image of x' in $\mathbf{A}^n_X$. Since x^{al} is the generic point of the integral scheme X^{al}, the local ring $\mathscr{O}_{\mathbf{A}^n_{X^{\mathrm{al}}},z^{\mathrm{al}}}$ coincides with $\mathscr{O}_{\mathbf{A}^n_{\kappa(x^{\mathrm{al}})},z^{\mathrm{al}}}$, hence is regular. In view of Lemma 2.4.6, this implies that $\mathscr{O}_{\mathbf{A}^n_X,z}$ is regular. Since $\mathscr{O}_{X',x'}$ is a finite étale $\mathscr{O}_{\mathbf{A}^n_X,z}$-algebra, it is regular too. $\square$

5.5.2. Lemma. — *Let*

$$\begin{tikzcd} \mathscr{Z} \arrow[d, "f"'] \arrow[dr, "g"] & \\ \mathscr{Y} \arrow[r, "h"'] & \mathscr{X} \end{tikzcd}$$

be a commutative diagram of locally noetherian schemes. If g is regular and if f is faithfully flat then h is regular.

Proof. — The faithful flatness of f and the flatness of g imply the flatness of h (one can check it directly, or see it as a particular case of Lemma 4.1.6). It remains to show that the fibers of h are geometrically regular.

Let x be a point of $\mathscr{X}$ and let L be a finite extension of $\kappa(x)$. Since the map g is regular, the scheme $\mathscr{Z}_{x,L}$ is regular. The morphism $\mathscr{Z}_{x,L} \to \mathscr{Y}_{x,L}$ is faithfully flat because it is deduced from f by base change along the map $\mathscr{Y}_{x,L} \to \mathscr{Y}$; as a consequence, $\mathscr{Y}_{x,L}$ is regular ([**EGA IV**$_2$], Prop. 6.5.3 (i)). $\square$

5.5.3. Theorem. — *Let $Y \to X$ be a morphism between good k-analytic spaces, let y be a point of Y and let x be its image in X. Assume that $Y \to X$ is quasi-smooth at y.*

(1) *The morphism $\mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ is regular.*
(2) *If moreover Y and X are affinoid, then $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}} \to \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}$ is regular.*

Proof. — We shall use the general abstract results of 2.3.20 by taking for $\mathfrak{F}$ the categeory $\mathfrak{T}$ itself (see 2.2, and especially Example 2.2.8), for P the property of being regular, and for $\mathscr{C}$ the class of quasi-smooth morphisms. With this convention, Lemma 5.5.1 is nothing but assertion (A) of 2.3.20, which implies assertions (A*), (B) and (B*) of loc. cit., as explained there. Now (B*) implies that the fibers of the map Spec $\mathscr{O}_{Y,y} \to$ Spec $\mathscr{O}_{X,x}$ are geometrically regular; on the other hand, $\mathscr{O}_{Y,y}$ is a flat $\mathscr{O}_{X,x}$-algebra because $Y \to X$ is quasi-smooth at y, hence flat at y by Corollary 5.3.2; this ends the proof of (1).

Let us now come to (2). Since the quasi-smooth locus of $Y \to X$ is open by Corollary 5.3.7, there exists an affinoid neighborhood V of y in Y such that the arrow $V \to Y$ is quasi-smooth. We have mentioned above that assertion (B) of 2.3.20 holds (with our choices of $\mathfrak{F}, \mathsf{P}$, and $\mathscr{C}$); this implies that he morphism $V^{\mathrm{al}} \to X^{\mathrm{al}}$ has geometrically regular fibers, and on the other hand it is flat because $V \to X$ is flat as a quasi-smooth morphism, and in view of Lemma 4.2.1: moreover the morphism $V^{\mathrm{al}} \to Y^{\mathrm{al}}$ is regular by 2.1.3 (2). We can thus apply Lemma 5.5.2 to the commutative

diagram

$$\begin{tikzcd} \operatorname{Spec} \mathscr{O}_{V^{\mathrm{al}}, y_V^{\mathrm{al}}} \arrow[d] \arrow[dr] & \\ \operatorname{Spec} \mathscr{O}_{Y^{\mathrm{al}}, y^{\mathrm{al}}} \arrow[r] & \operatorname{Spec} \mathscr{O}_{X^{\mathrm{al}}, x^{\mathrm{al}}} \end{tikzcd}$$

and it yields the regularity of $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}} \to \mathscr{O}_{Y^{\mathrm{al}},y^{\mathrm{al}}}$. □

Lemmas 4.5.1 and 4.5.2 provide some descent and transfer results for flat morphisms. Due to Theorem 5.5.3, the transfer results alluded to above can be strengthened in the case of a quasi-smooth map. Again, we shall first write a statement that holds in the abstract settings of 2.2 and 2.3, where we deal with general objects and properties of the latter satisfying various axioms; and then we shall write down what it means for some *explicit* properties of interest. For the notion of validity of a property at a point, the reader may refer to Lemma-Definition 2.4.1 in our general abstract setting and to Lemma-Definition 2.4.3 for a more concrete version.

5.5.4. Proposition. — *Let $\mathfrak{F}$ be a fibered category as in 2.2, and let* P *be a property as in 2.3.1. Let us assume that* P *satisfies* $(\mathsf{H}_{\mathrm{reg}})$ *(2.3.15). Let $Y \to X$ be a morphism between k-analytic spaces, let y be a point of Y, let x be its image in X, and let D be an object of the fiber category $\mathfrak{F}_X$. Assume that Y is quasi-smooth over X at x. If D satisfies* P *at x, then D_Y satisfies* P *at y.*

Proof. — For both assertions, we can assume that the spaces Y and X are affinoid; now $\operatorname{Spec} \mathscr{O}_{Y,y} \to \operatorname{Spec} \mathscr{O}_{X,x}$ is flat with (geometrically) regular fibers in view of Theorem 5.5.3.The proposition then follows immediately from the fact that the property P satisfies $(\mathsf{H}_{\mathrm{reg}})$ by assumption. □

5.5.5. Proposition **(A concrete version of Proposition 5.5.4)**

Let $Y \to X$ be a morphism between k-analytic spaces, let y be a point of X and let x be its image in X. Let $\mathscr{F}$ be a coherent sheaf on X, and let m be an element of $\mathbf{Z}_{\geqslant 0}$. Let $\mathsf{S} = \mathscr{E} \to \mathscr{E}' \to \mathscr{E}''$ be a complex of coherent sheaves on X. Assume that Y is quasi-smooth over X at y. If X is regular, resp. R_m, resp. Gorenstein, resp. CI at x, so is Y at y. If $\mathscr{F}$ is CM, resp. S_m, resp. free of rank m at x, so is $\mathscr{F}_Y$ at y. Is S *is exact at x, so is S_Y at y.*

CHAPTER 6

GENERIC FIBERS IN ANALYTIC GEOMETRY

In the study of relative properties in scheme theory, a key role is played by the technique of "spreading out from the generic fiber", which is based upon the following obvious remark. Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of schemes and let ξ be a point of $\mathscr{X}$ such that $\mathscr{O}_{\mathscr{X},\xi}$ is a field (if $\mathscr{X}$ is noetherian and reduced, this amounts to requiring that ξ is the generic point of an irreducible component of $\mathscr{X}$). Then for every point y of the fiber $\mathscr{Y}_\xi$, the natural map $\mathscr{O}_{\mathscr{Y},y} \to \mathscr{O}_{\mathscr{Y}_\xi,y}$ is an isomorphism.

Let us now consider an analytic analogue of our situation: namely, $Y \to X$ is a morphism between good k-analytic spaces, x is a point of X such that $\mathscr{O}_{X,x}$ is a field, and y is a point of Y_x. Except in some very particular cases (e.g., if x is rigid and $X = \{x\}$), the natural map $\mathscr{O}_{Y,y} \to \mathscr{O}_{Y_x,y}$ cannot be expected to be an isomorphism, because the formation of Y_x involves several completion operations: indeed, if X and Y are affinoid, say $X = \mathscr{M}(A)$ and $Y = \mathscr{M}(B)$, then $Y_x = \mathscr{M}(B\widehat{\otimes}_A\mathscr{H}(x))$, and $\mathscr{H}(x)$ is itself the completion of $\kappa(x)$.

But nevertheless, it does not prevent us from implementing a technique of "spreading out from the generic fiber" in analytic geometry. The point is the following: in the scheme-theoretic situation we have described above, what actually matters for carrying out this technique (as far as one is only interested in the locus of validity of the usual properties) is not the equality $\mathscr{O}_{\mathscr{Y},y} \to \mathscr{O}_{\mathscr{Y}_\xi,y}$ by itself, but the slightly weaker fact that every algebraic property of interest that holds over $\mathscr{O}_{\mathscr{Y}_\xi,y}$ descends to $\mathscr{O}_{\mathscr{Y},y}$. And in this section, we prove the following (Theorem 6.3.3): if Y, X, y, and x are as above, *and if moreover y belongs to* $\mathrm{Int}(Y/X)$, then $\mathrm{Spec}\ \mathscr{O}_{Y_x,y} \to \mathrm{Spec}\ \mathscr{O}_{Y,y}$ is flat (with CI fibers, and even with regular fibers whenever char. $k = 0$); this will be sufficient to ensure descent of algebraic properties that descent along flat local maps (e.g., all properties that satisfy condition $(\mathsf{H}_{\mathrm{reg}})$ of 2.3.15 and more concretely, all properties mentioned in Definition-Lemma 2.4.3), hence to perform spreading out from the generic fiber – and we shall use it repeatedly throughout the rest of this memoir. Let us make some additional comments about this result.

- The assumption that y belongs to $\mathrm{Int}(Y/X)$ cannot be dropped: we give a counter-example in 6.3.4 where y belongs to $\partial(Y/X)$ and the morphism Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{Y,y}$ is not flat. This means that for spreading out properties from generic fibers, one always will have to reduce to the inner case.
- The fact that the fibers of Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{Y,y}$ are CI, and regular if moreover char. $k = 0$ (as soon as $y \in \mathrm{Int}(Y/X)$), is not used in this memoir; but we get it almost for free by the method we use for proving flatness, and it seems us to be of independent interest. Let us mentions that the characteristic assumption on k cannot be dropped as far as regularity is concerned, as witnessed by the counter-example 6.3.5 (which was communicated to the author by Temkin).
- If x is moreover an Abhyankar point (1.4.10), the map Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{Y,y}$ is regular without having to assume that $y \in \mathrm{Int}(Y/X)$ or char. $k = 0$ (Theorem 6.3.7). This ensures that all properties satisfying $(\mathsf{H}_{\mathrm{reg}})$ satisfy descent and transfer between $\mathscr{O}_{Y,y}$ and $\mathscr{O}_{Y_x,y}$, and illustrates a general phenomenon: the best analogue of a generic fiber in analytic geometry (for a map between good spaces) is a fiber over an *Abhyankar* point whose local ring is a field; i.e., an Abhyankar point at which the target space is reduced (because the local ring of any Abhyankar point is artinian, cf. Example 3.2.10).

Before investigating local rings of generic fibers, we shall need some preparatory work. Section 6.1 collects slightly technical (but easy) lemmas involving the completed residue field of a point of an analytic space. Section 6.2 is devoted to a local study of smooth morphisms, which can be of independent interest. Let us say a few words about it.

In complex analytic geometry, a morphism if smooth if and only if it is, locally on the source and on the target, of the form $D \times X \to X$ for some open polydisc D. There is no hope for so nice a description in the non-archimedean setting, even in the absolute context. Indeed, if k is algebraically closed and if X is an irreducible, smooth, projective curve over k of positive genus, there always exists a point x of X^{an} that has *no* neighborhood isomorphic to an open disc; nevertheless if k is moreover non-trivially valued, any non-empty smooth k-analytic space has a k-point, hence contains an open polydisc (choose a suitable open neighborhood of this point). In Section 6.2, we extend the latter absolute result to a relative situation. We prove the following (Proposition 6.2.4), which holds over an arbitrary analytic field k. Let $Y \to X$ be a smooth morphism of good k-analytic spaces. Let x be a point of X such that $|\mathscr{H}(x)^\times| \neq \{1\}$ and $Y_x \neq \varnothing$; there exists an étale map $X' \to X$ whose image contains x and an open subset of $Y' := Y \times_X X'$ that is X'-isomorphic to $D \times_k X'$ for some open polydisc D (Proposition 6.2.4 also gives a similar, but slightly more complicated result, when $\mathscr{H}(x)$ is trivially valued).

This almost immediately implies the openness of quasi-smooth boundaryless morphisms (Corollary 6.2.5), which had already been proved by Berkovich with more sophisticated tools (see detailed comments at Remark 6.2.6), and also (at least when the ground field is non-trivially valued), the fact that a surjective smooth morphisms between good k-analytic spaces admits sections locally for the étale topology on the target (Corollary 6.2.7).

6.1. Preliminary lemmas

We begin with some general results about extensions of valued fields.

6.1.1. Lemma. — *Let F be a field and let L be a finite, separable extension of $F(\!(t)\!)$. There exists a finite extension K of F such that $L \otimes_F K$ admits a quotient isomorphic to $K(\!(\tau)\!)$.*

Proof. — Let us consider $F(\!(t)\!)$ as the completion of the function field of $\mathbf{P}^1_F$ at the origin. Krasner's lemma ensures that there exists a projective, normal, irreducible F-curve Y equipped with a finite, generically étale map to $\mathbf{P}^1_F$, such that L can be identified with the completion of $F(Y)$ at a closed point P lying above the origin. There exists a finite extension K of F such that the normalization Z of $Y \times_F K$ is smooth and admits a K-rational point Q over P. Since the completion of $K(Z)$ at Q is isomorphic to $K(\!(\tau)\!)$, the extension K satisfies the required property. □

6.1.2. Lemma. — *Let K be a real valued field and let V be a subgroup of K. Assume that there exists $\rho \in (0,1)$ such that for every $\lambda \in K$ there exists $\mu \in V$ with $|\lambda - \mu| \leqslant \rho\,|\lambda|$. The group V is dense in K.*

Proof. — For every $\lambda \in K$ we choose an element $\varphi(\lambda)$ in V such that $|\lambda - \varphi(\lambda)| \leqslant \rho\,|\lambda|$. Now let λ be an element of K. Define inductively the sequence $(\lambda_i)_i$ by setting $\lambda_0 = 0$ and $\lambda_{i+1} = \lambda_i + \varphi(\lambda - \lambda_i)$. By induction, one sees that $\lambda_i \in V$ and that $|\lambda - \lambda_i| \leqslant \rho^i\,|\lambda|$ for every i; hence $\lambda_i \to \lambda$. □

6.1.3. Lemma. — *Let K be a real valued field such that $|K^\times|$ is free of rank 1, and let F be a complete subfield of K such that $|F^\times| \neq \{1\}$ and the* classical *residue extension $\widetilde{F}^1 \hookrightarrow \widetilde{K}^1$ is finite. The field K is a finite extension of F.*

Proof. — The assumptions on the value groups ensures that $|K^\times|/|F^\times|$ is finite; hence the *graded* extension $\widetilde{F} \hookrightarrow \widetilde{K}$ is finite too. Let $\lambda_1, \ldots, \lambda_n$ be elements of $K^\times$ such that $(\widetilde{\lambda_i})_i$ is a basis of $\widetilde{K}$ over $\widetilde{F}$. Let us call V the F-vector subspace of K generated by the λ_i's. Let λ be an element of K; there exist $a_1, \ldots, a_n \in F$ such that $\widetilde{\lambda} = \sum \widetilde{a_i}\widetilde{\lambda_i}$, which exactly means, if $\lambda \neq 0$, that $|\lambda - \sum a_i\lambda_i| < |\lambda|$. As $|K^\times|$ is free of rank one, $|K^\times| \cap (0,1)$ has a maximal element ρ. By the above, for every $\lambda \in K$ there exists $\mu \in V$ with $|\lambda - \mu| \leqslant \rho\,|\lambda|$. By Lemma 6.1.2, the group V is dense in K. Since V is a finite dimensional F-vector space, it is complete; hence $V = K$. □

We are now going to use the above lemmas to establish some results related to the completed residue field of a point of an analytic space.

6.1.4. Lemma. — *Let F be a trivially valued field and let X be a non-empty, boundaryless F-analytic space. There exists $x \in X$ such that $\mathscr{H}(x)$ is either a finite extension of F or a finite extension of F_r for some $r \in]0;1[$.*

Proof. — Choose an arbitrary $s \in]0;1[$. As X_s is a non-empty, boundaryless space over the non-trivially valued field F_s, it has an F_s-rigid point y (1.2.10); let x be the image of y on X. Note that $\widetilde{\mathscr{H}(x)}^1$ is a subfield of $\widetilde{\mathscr{H}(y)}^1$, which is itself finite over $\widetilde{F_s}^1 = F$; hence $\widetilde{\mathscr{H}(x)}^1$ is finite over F.

If $|\mathscr{H}(x)^\times| = \{1\}$, then as $\mathscr{H}(x) = \widetilde{\mathscr{H}(x)}^1$, it is finite over F and we are done.

If $|\mathscr{H}(x)^\times| \neq \{1\}$, choose $r \in |\mathscr{H}(x)^\times| \cap (0,1)$, and choose $\lambda \in \mathscr{H}(x)^\times$ such that $|\lambda| = r$. The complete subfield E generated by λ over F in $\mathscr{H}(x)$ is isomorphic to F_r. As $\mathscr{H}(x)$ is a subfield of $\mathscr{H}(y)$, the non-trivial group $|\mathscr{H}(x)^\times|$ is free of rank 1; together with the fact that $\widetilde{E}^1 = \widetilde{F_r}^1 = F$ this implies, in view of Lemma 6.1.3, that $\mathscr{H}(x)$ is a finite extension of $E \simeq F_r$. □

6.1.5. Lemma. — *Let $r = (r_1, \ldots, r_n)$ be a k-free polyradius and let $S_1, \ldots, S_n$ be elements of k_r such that $|S_i| = r_i$ for every i. The complete subfield L of k_r generated by k and the S_i's over k is equal to k_r; in other words, $S_1, \ldots, S_n$ are coordinate functions of k_r.*

Proof. — Let $T_1, \ldots, T_n$ be coordinate functions of k_r; note that there is a well-defined k-isometry $\varphi : \sum a_I T^I \mapsto \sum a_I S^I$ between k_r and L.

For every i one can write $S_i = \alpha_i T_i + u_i$ where $\alpha_i \in k$ and $u_i \in k_r$, and where $|\alpha_i| = 1$ and $|u_i| < r_i$. By replacing S_i with $\alpha_i^{-1} S_i$, we may assume that $\alpha_i = 1$ for all i. Therefore, there exists $\rho \in (0,1)$ such that $|T_i - S_i| \leqslant \rho r_i$ and $\left|T_i^{-1} - S_i^{-1}\right| \leqslant \rho r_i^{-1}$ for every i; it follows that $|\lambda - \varphi(\lambda)| \leqslant \rho\,|\lambda|$ for every $\lambda \in k_r$. Lemma 6.1.2 then ensures that L is dense in k_r; as L is complete we get $L = k_r$, as required. □

6.1.6. Lemma. — *Let Y be a quasi-smooth k-analytic space and let y be a point of Y such that $\mathscr{H}(y) \simeq k_r$ for some k-free polyradius $r = (r_1, \ldots, r_m)$ (as an analytic extension of k). Let $(g_1, \ldots, g_m)$ be analytic functions on Y such that $|g_i(y)| = r_i$ for every i. The $(\mathrm{d}g_i)(y)$'s are $\mathscr{H}(y)$-linearly independent elements of $(\Omega^1_{Y/k})_{\mathscr{H}(y)}$.*

Proof. — One can assume that Y is k-affinoid and of pure dimension, say, n. Let V be the affinoid domain of Y defined as the locus of simultaneous validity of the equalities $|g_i| = r_i$. Its k-affinoid structure factorizes through a k_r-analytic structure provided by the g_i's, for which y is k_r-rational by Lemma 6.1.5 above. For every $x \in V$ we have $d_{k_r}(x) = d_k(x) - d_k(k_r) = d_k(x) - m$ (1.2.15). Therefore

$$\dim_{k_r} V = \sup_{x \in V} d_{k_r}(x) = \sup_{x \in V} d_k(x) - m = \dim_k V - m = n - m.$$

As V is quasi-smooth, $\mathscr{O}_{V,y}$ is regular; since y is k_r-rational, this implies that $(\Omega^1_{V/k_r})_{\mathscr{H}(y)}$ is of dimension $n - m$ (5.1.9); otherwise said, V is quasi-smooth *over* k_r at y. On the other hand, by quasi-smoothness of Y (hence of V) over k, the $\mathscr{H}(y)$-vector space $(\Omega^1_{V/k})_{\mathscr{H}(y)}$ is of dimension n.

By 5.1.5, we have an exact sequence $(\Omega^1_{k_r/k})_V \to \Omega_{V/k} \to \Omega_{V/k_r} \to 0$. Let (T_i) denotes the family of coordinates functions of k_r. By 5.1.6 (and 5.1.2) the family $(\mathrm{d}T_i)_i$ is a basis of the k_r-vector space Ω_{k_r}/k. Since the k_r-analytic structure on V arises from the morphism $(T_i \mapsto g_i)_i$, it follows from the above that

$$(\Omega_{V/k_r})_{\mathscr{H}(y)} = (\Omega_{V/k})_{\mathscr{H}(y)}/((\mathrm{d}g_1)(y), \ldots, (\mathrm{d}g_m)(y)).$$

By considering the dimensions of both sides, we see that $((\mathrm{d}g_1)(y), \ldots, (\mathrm{d}g_m)(y))$ is free as a family of elements of $(\Omega_{V/k_r})_{\mathscr{H}(y)}$, as announced. □

6.2. Relative polydiscs inside relative smooth spaces

6.2.1. Lemma. — *Let X be a good k-analytic space, let x be a point of X and let n be an element of $\mathbf{Z}_{\geqslant 0}$. Let m be a non-negative integer $\leqslant n$, let $r = (r_1, \ldots, r_m)$ be an $\mathscr{H}(x)$-free polyradius, and set $r_i = 0$ for $m < i \leqslant n$. Let ξ be the point of $\mathbf{A}^n_X$ lying above x and defined by the semi-norm*

$$\sum a_I T^I \mapsto \max |a_I|\, r^I$$

on the ring $\mathscr{H}(x)[T]$, and let V be an open neighborhood of ξ in $\mathbf{A}^n_X$. The open subset V of $\mathbf{A}^n_X$ contains an open neighborhood of ξ of the form

$$U \times_k D_1 \times_k \ldots \times_k D_n$$

where U is an open neighborhood of x in X and D_i is for every integer $i \leqslant m$ (resp. $i > m$) a one-dimensional open annulus (resp. disc) with coordinate function T_i.

Proof. — Through a straightforward induction argument one immediately reduces to the case where $n = 1$; in that situation r is either zero or an $\mathscr{H}(x)$-free positive number. Let X_0 be an affinoid neighborhood of x in X and set $A = \mathscr{O}_X(X_0)$; let X'_0 be the topological interior of X_0 in X. By the explicit description of the topology of the analytification of an A-scheme of finite type (cf. for instance [**Duc07a**], §1.4), there exist a finite family $P_1, \ldots, P_\ell$ of elements of $A[T]$, and a finite family $I_1, \ldots, I_\ell$ of (relatively) open intervals of $\mathbf{R}_+$ such that the open subset of $\mathbf{A}^1_{X_0}$ defined by the conditions $|P_j| \in I_j$ (for $j = 1, \ldots, \ell$) contains ξ and is included in V; write $P_j = \sum a_{i,j} T^i$.

Assume that $r = 0$. In that case one has $|P_j(\xi)| = |a_{0,j}(x)|$ for every j. There exists for every j an open neighborhood I'_j of $|a_{0,j}(x)|$ in I_j and a positive number R_j such that $|P_j(\eta)| \in I_j$ as soon as $|a_{0,j}(\eta)| \in I'_j$ and $|T(\eta)| < R_j$. Let us denote by U the set of points $y \in X'_0$ such that $|a_{0,j}(y)| \in I'_j$ for every j, and let R be any positive number smaller than all R_j's. The product of U and of the open disc centered at the origin with radius R is then included in V and contains ξ, which ends the proof when $r = 0$.

Assume that r is an $\mathscr{H}(x)$-free positive number. In that case there exists for every j an index i_j such that $|a_{i_j,j}(x)|\, r^{i_j} > |a_{i,j}(x)|\, r^i$ for all $i \neq i_j$. One can find for every j two positive numbers S_j and R_j with $S_j < r < R_j$ and a family $(I'_{ij})_{i \leqslant \deg P_j}$ of (relatively) open intervals of $\mathbf{R}_+$, each of which contains $|a_{ij}(x)|$, such that $|P_j(\eta)|$ is equal to $|a_{i_j,j}(\eta)| \cdot |T(\eta)|^{i_j}$ and belongs to I_j as soon as $|a_{i,j}(\eta)| \in I'_{ij}$ for all $i \leqslant \deg P_j$ and $S_j < |T(\eta)| < R_j$. Let us denote by U the set of points $y \in X'_0$ such that $|a_{i,j}(y)| \in I'_{i,j}$ for every j and every $i \leqslant \deg P_j$, and let R and S be two positive number such that $S < r < R$ and such that $S_j < S$ and $R < R_j$ for every j. The product of U and of the open annulus with bi-radius (S, R) is then included in V and contains ξ, which ends the proof. □

6.2.2. Lemma. — *Let X be a good k-analytic space, let x be a point of X, and let n be an element of $\mathbf{Z}_{\geqslant 0}$; let m be a non-negative integer $\leqslant n$ and let $r = (r_1, \ldots, r_m)$ be an $\mathscr{H}(x)$-free polyradius. Let $Y \to X$ be a smooth morphism of relative dimension n, and assume that Y_x contains a point y with $\mathscr{H}(y) \simeq \mathscr{H}(x)_r$ as analytic extensions of $\mathscr{H}(x)$. There exists an open subset V of Y which is X-isomorphic to $U \times_k D \times_k \Delta$, where U is an open neighborhood of x in X, D is an m-dimensional open poly-annulus, and Δ is an $(n-m)$-dimensional open polydisc.*

6.2.3. Remark. — We emphasize the fact that (contrary to Lemma 6.2.1 with ξ), we do *not* require the open subset V to contain y, and our proof actually does not enable us to achieve it: as the reader will see, we need at some point to replace y with a suitable approximation z, and the neighborhood we seek will be built around z, and it might avoid y. (We shall encounter a similar restriction while dealing with étale multisections; see Remark 6.2.8 below.)

But this should not cause any trouble. Indeed, as explained in the introduction of this chapter, we are interested in exhibiting nice open subsets inside relative smooth spaces, but there is no hope for having a *basis* of such open subsets.

Proof of Lemma 6.2.2. — Let us choose analytic functions $f_1, \ldots, f_m$ defined in an open neighborhood of y such that $|f_i(y)| = r_i$ for every i. According to Lemma 6.1.6 (which one applies to the $\mathscr{H}(x)$-analytic space Y_x), the elements $(\mathrm{d}f_1)(y), \ldots, (\mathrm{d}f_m)(y)$ are linearly independent in $(\Omega_{Y/X})_{\mathscr{H}(y)}$; one can hence choose $f_{m+1}, \ldots, f_n$ in $\mathscr{O}_{Y,y}$ so that $((\mathrm{d}f_1)(y), \ldots, (\mathrm{d}f_m)(y), (\mathrm{d}f_{m+1})(y), \ldots, (\mathrm{d}f_n)(y))$ is a *basis* of $(\Omega_{Y/X})_{\mathscr{H}(y)}$.

The X-morphism $Y \to \mathbf{A}^n_X$ induced by the f_i's is quasi-étale at y by Lemma 5.4.5; as $Y \to X$ is boundaryless (it is smooth), $Y \to \mathbf{A}^n_X$ is boundaryless at y, and thus étale (Remark 5.4.9). Lemma 6.1.5 ensures that the complete subfield of $\mathscr{H}(y)$ generated over $\mathscr{H}(x)$ by the $f_i(y)$'s for $i = 1, \ldots, m$ is equal to $\mathscr{H}(y)$ itself. Therefore, if y' denotes the image of y on $\mathbf{A}^n_X$, one has $\mathscr{H}(y') = \mathscr{H}(y)$; as $Y \to \mathbf{A}^n_X$ is étale at y, this implies that $Y \to X$ induces an isomorphism between an open neighborhood of y in Y and an open neighborhood of y' in $\mathbf{A}^n_X$ ([**Ber93**], Thm. 3.4.1). We thus may reduce to the case where Y is an open subset of $\mathbf{A}^n_X$, and where the following holds: for the projection $p\colon \mathbf{A}^n_X \to \mathbf{A}^m_X$ defined by $(T_1, \ldots, T_m)$, the point $p(y)$ of $\mathbf{A}^{m,\mathrm{an}}_{\mathscr{H}(x)}$ is the one that corresponds to the Gauß norm $\sum a_I T^I \mapsto \max |a_I| r^I$, and y is an $\mathscr{H}(p(y))$-rational point of the fiber $p^{-1}(p(y))$.

Let κ be the residue field of $\mathscr{O}_{\mathbf{A}^m_X, p(y)}$; by density of κ inside $\mathscr{H}(p(y))$, the fiber $(p_{|Y})^{-1}(p(y))$ possesses an $\mathscr{H}(p(y))$-point z such that $T_i(z) \in \kappa$ for every integer $i \in \{m+1, \ldots, n\}$. Let V be an open neighborhood of $p(y)$ in $\mathbf{A}^m_X$ on which the $T_i(z)$'s are defined; translation by $(0, \ldots, 0, T_{m+1}(z), \ldots, T_n(z))$ identifies over V the space $Y \times_{\mathbf{A}^m_X} V$ with an open subset of $\mathbf{A}^n_X \times_{\mathbf{A}^m_X} V$ whose fiber over $p(y)$ contains the origin of $\mathbf{A}^{m,an}_{\mathscr{H}(p(y))}$. It follows then from Lemma 6.2.1 that there exists an open subset W of Y which is V-isomorphic to $V' \times \Delta$ where V' is an open neighborhood of $p(y)$ in V and where Δ is a $(n-m)$-dimensional open polydisc.

Now by applying once again Lemma 6.2.1, but this time to the map $\mathbf{A}^m_X \to X$ and at the point $p(y)$, one sees that there exists an open neighborhood V'' of $p(y)$ in V' that is X-isomorphic to $U \times_k D$ for some open neighborhood U of x in X and some m-dimensional open poly-annulus D. Now $W \times_{V'} V''$ is an open subset of Y that is X-isomorphic to $U \times_k D \times_k \Delta$, as required. □

We are now ready to investigate the local structure of arbitrary smooth morphisms between good analytic spaces.

6.2.4. Proposition. — *Let n be an element of $\mathbf{Z}_{\geqslant 0}$ and let $Y \to X$ be a smooth map of pure relative dimension n between good k-analytic spaces. Let x be a point of X, and let W be a non-empty open subset of Y_x. There exist:*

- *a flat, locally finite morphism $X' \to X$ which can be chosen to be étale if $|\mathscr{H}(x)^\times| \neq \{1\}$;*
- *a pre-image x' of x on X', such that the morphism* $\mathrm{Spec}\ \mathscr{O}_{X',x'} \to \mathrm{Spec}\ \mathscr{O}_{X,x}$ *has a reduced closed fiber (note that this is kind of a weak substitute to étaleness in the case where $\mathscr{H}(x)$ is trivially valued);*
- *an open subset V of $Y' := Y \times_X X'$ whose intersection with $Y'_{x'}$ is contained in $W_{\mathscr{H}(x')}$, and which is X'-isomorphic to $X' \times_k D$ where D is:*
  - ◇ *an n-dimensional open polydisc if W has an $\mathscr{H}(x)$-rigid point, which is always the case if $|\mathscr{H}(x)^\times| \neq \{1\}$ or if $n = 0$;*
  - ◇ *the product of a 1-dimensional open annulus and an $(n-1)$-dimensional open polydisc if W has a no $\mathscr{H}(x)$-rigid point, which can occur only if $\mathscr{H}(x)$ is trivially valued and $n > 0$.*

Proof. — By replacing Y with an open subset of Y whose intersection with Y_x is equal to W, we may assume that $W = Y_x$. By assumption, $Y_x \neq \varnothing$.

Let us assume that $|\mathscr{H}(x)^\times| \neq \{1\}$. Let us choose $y \in Y_x$. As $Y \to X$ is smooth, there exists a neighborhood Z of y in Y such that $Z \to X$ goes through an étale map $Z \to \mathbf{A}^n_X$; the image of Z on $\mathbf{A}^n_X$ is an open subset U of the latter, and U_x is non-empty. Let K be the completion of an algebraic closure of $\mathscr{H}(x)$. Since $\mathscr{H}(x)$ is not trivially valued, the analytic *Nullstellensatz* ensures that $U_x(K) \neq \varnothing$(1.2.10); as the separable closure of $\mathscr{H}(x)$ in K is dense (again because $|\mathscr{H}(x)^\times| \neq \{1\}$), there exists $u \in U_x$ with $\mathscr{H}(u)$ a finite, separable extension of $\mathscr{H}(x)$. Now let us choose a pre-image z of u on Z; as $Z \to U$ is étale, $\mathscr{H}(z)$ is a finite, separable extension of $\mathscr{H}(u)$, and hence a finite, separable extension of $\mathscr{H}(x)$ too. The categories of finite étale covers of the germ (X, x) and of finite étale $\mathscr{H}(x)$-algebras being naturally equivalent ([**Ber93**], Thm. 3.4.1), there exists an étale morphism $X' \to X$ and a pre-image x' of x on X such that z has a pre-image z' on $Y' := Y \times_X X'$ with $\mathscr{H}(z') = \mathscr{H}(x')$. This implies, in view of Lemma 6.2.2, that one can shrink X' around x' so that Y' possesses an open subset which is X'-isomorphic to the product of X' and an n-dimensional open polydisc. This ends the proof in the case where $|\mathscr{H}(x)^\times| \neq \{1\}$.

Let us assume that $|\mathscr{H}(x)^\times| = \{1\}$ and that Y_x has an $\mathscr{H}(x)$-rigid point y. As $\mathscr{H}(x)$ is trivially valued, it coincides with the residue field κ of $\mathscr{O}_{X,x}$. Therefore, there exists a finite, flat, local $\mathscr{O}_{X,x}$-algebra A with $A \otimes_{\mathscr{O}_{X,x}} \kappa \simeq \mathscr{H}(y)$ (over κ). Since X is good, one can find a locally finite, flat map $X' \to X$ and a pre-image x' of x on X' such that $\mathscr{O}_{X',x'} \simeq A$; note that $\mathscr{H}(x') \simeq \mathscr{H}(y)$ and that the closed fiber of Spec $\mathscr{O}_{X',x'} \to$ Spec $\mathscr{O}_{X,x}$ is reduced. By construction, y has a pre-image y' on $Y' := Y \times_X X'$ lying above x' and satisfying $\mathscr{H}(y') = \mathscr{H}(x')$. This implies, in view of Lemma 6.2.2, that one can shrink X' around x' so that Y' possesses an open subset which is X'-isomorphic to the product of X' and an n-dimensional open polydisc. This ends the proof in the case where $|\mathscr{H}(x)^\times| = \{1\}$ and where Y_x has a rigid point.

Let us assume that $|\mathscr{H}(x)^\times| = \{1\}$ and that Y_x has no $\mathscr{H}(x)$-rigid point. In this case, there exists $t \in Y_x$ and $r \in (0, 1)$ such that $\mathscr{H}(t)$ is a finite extension of $\mathscr{H}(x)_r$ (Lemma 6.1.4). Due to Lemma 6.1.3, there exists a finite $\mathscr{H}(x)$-extension F such that $F \otimes_{\mathscr{H}(x)} \mathscr{H}(t)$ admits a quotient $\mathscr{H}(x)$-isomorphic (as a valued extension of F) to F_s for some $s \in (0, 1)$. As $\mathscr{H}(x)$ is trivially valued, it coincides with the residue field $\kappa(x)$ of $\mathscr{O}_{X,x}$. Therefore, there exists a finite, flat, local $\mathscr{O}_{X,x}$-algebra A with $A \otimes_{\mathscr{O}_{X,x}} \kappa(x) \simeq F$. One can find a locally finite, flat map $X' \to X$ and a pre-image x' of x on X' such that $\mathscr{O}_{X',x'} \simeq A$; note that $\mathscr{H}(x') \simeq F$ and that the closed fiber of Spec $\mathscr{O}_{X',x'} \to$ Spec $\mathscr{O}_{X,x}$ is reduced. By construction, y has a pre-image y' on $Y' := Y \times_X X'$ lying above x' and such that $\mathscr{H}(y')$ is $\mathscr{H}(x')$-isomorphic (as a valued field) to $\mathscr{H}(x')_s$ for some $s \in (0, 1)$. This implies, in view of Lemma 6.2.2 that one can shrink X' around x' so that Y' possesses an open subset which is X'-isomorphic to the product of X' and an $(n-1)$-dimensional open polydisc and a 1-dimensional open annulus, which ends the proof. □

6.2.5. Corollary. — *Any quasi-smooth, boundaryless map is open.*

Proof. — Let $Y \to X$ be a quasi-smooth, boundaryless map. To prove that it is open, one may argue G-locally on X, hence assume that X is good. Now Y is also good because $Y \to X$ is boundaryless, and it follows from Corollary 5.4.8 that $Y \to X$ is smooth. Its openness then follows immediately from Proposition 6.2.4 above together with openness of flat, locally finite morphisms (Corollary 4.3.2). □

6.2.6. Remark. — Corollary 6.2.5 had already been proven by Berkovich ([**Ber93**], Cor. 3.7.4). Strictly speaking, Berkovich proved the openness of smooth maps, and not of all quasi-smooth, boundaryless maps – and the latter class is likely broader than the former (see Remark 5.4.9); but openness can be checked G-locally on the target, hence in the good case where both notions coincide (Corollary 5.4.8).

Our proof is different from Berkovich's. The latter used the dévissage of a smooth morphism in "elementary" curve fibrations, whose existence comes from the semi-stable reduction theorem. Ours involves less sophisticated tools: it is essentially based upon easy explicit computations (Lemma 6.2.1) and the fact that if $(Y, y) \to (X, x)$ is an étale morphism of analytic germs such that $\mathscr{H}(x) \simeq \mathscr{H}(y)$, it is an isomorphism ([**Ber93**], Thm. 3.4.1).

6.2.7. Corollary. — *Let $Y \to X$ be a smooth morphism between good k-analytic spaces and let x be a point of X such that $|\mathscr{H}(x)^\times| \neq \{1\}$. If $Y_x \neq \varnothing$, there exists an étale morphism $X' \to X$ whose image contains x and an X-map $X' \to Y$.*

6.2.8. Remark. — Let us keep using the notation of Corollary 6.2.7 above. Let y be any rigid point of the fiber Y_x such that $\mathscr{H}(y)$ is separable over $\mathscr{H}(x)$. We do *not* claim that we can build an étale multisection of $Y \to X$ around x that goes through y. And in fact, such a multisection *does not exist in general.* Indeed, let X' be as in Corollary 6.2.7, let x' be a pre-image of x in X', and let y be the image of x' in Y. The morphism $X' \to Y$ is quasi-finite and boundaryless at x', hence finite at x' ([**Ber93**], Cor. 3.1.10). Therefore

$$\mathrm{centdim}(Y, y) = \mathrm{centdim}(X', x') = \mathrm{centdim}(X, x)$$

(both equalities follow from 3.2.4).

Now let f be any power series belonging to $k[\![t]\!]$ whose radius of convergence is a positive real number r, let D be the closed k-analytic disc of radius r, and let φ be the morphism (Id, f) from D to $\mathbf{A}_k^{2,\mathrm{an}}$. Let x be the Shilov point of D, and set $y = \varphi(x)$. If p denotes the first projection $\mathbf{A}^{2,\mathrm{an}} \to \mathbf{A}^{1,\mathrm{an}}$, then $p(y) = x$ and $\mathscr{H}(y) = \mathscr{H}(x)$. By the proof of Proposition 4.4.6 (be aware that our current point y is denoted by x in loc. cit.), $\mathrm{centdim}(\mathbf{A}^{2,\mathrm{an}}, y) = 2$. Therefore $\mathrm{centdim}(\mathbf{A}^{2,\mathrm{an}}, y) \neq \mathrm{centdim}(\mathbf{A}^{1,\mathrm{an}}, x)$ because the latter is $\leqslant 1$ (it is in fact equal to 1 because x is not rigid, but we do not need that). Hence by the above, p does not admit any étale multisection around x and going through y.

6.3. Local rings of generic fibers

We begin this section with a lemma that ensures kind of a "spreading out from the generic fiber" of a topological property, in the particular case of a smooth morphism; it will play a key role in our study of local rings of generic fibers of an arbitrary map at an inner point (Theorem 6.3.3).

6.3.1. Proposition. — *Let $Y \to X$ be a map between good k-analytic spaces. Let y be a point of Y and let x be its image in X. Assume that $Y \to X$ is smooth at y and the local ring $\mathscr{O}_{X,x}$ is artinian. Let Z be a Zariski-closed subset of Y that contains a neighborhood of y in Y_x. Under those assumptions, Z is a neighborhood of y in Y.*

Proof. — We can shrink Y around y so that $Y \to X$ is smooth. The required property being purely topological, one may assume that X is reduced; in this case, $\mathscr{O}_{X,x}$ is a field, and is in particular normal. The normality locus of X is (Zariski-)open by Lemma 2.4.9 (1). We thus may shrink X so that it is itself normal. Now the X-quasi-smooth space Y is normal too, in view of Proposition 5.5.5; by shrinking Y (and Z, accordingly) we eventually reduce to the case where Y is connected, hence irreducible (and equi-dimensional) and where Z is the zero-locus of a finite family $(f_1, \ldots, f_n)$ of analytic functions on Y. We shall prove that Z contains a non-empty open subset of Y, which will force it to coincide with Y, and end the proof.

By Proposition 6.2.4, there exists a flat, locally finite map $X' \to X$, a point x' on X' lying above x, and a k-analytic space D such that:

- $\mathscr{O}_{X',x'}$ is a field ;
- D is an open polydisc, or the product of an open polydisc and a 1-dimensional open annulus;
- if one sets $Y' = Y \times_X X'$ and $Z' = Z \times_X X'$, there exists an open subset V of Y' which is X'-isomorphic to $D \times_k X'$ and such that $V_{x'} \subset Z'_{x'}$.

As $\mathscr{O}_{X',x'}$ is a field, we may shrink X' so that it is connected and normal, by the same reasoning as above. Let us still denote by $f_1, \ldots, f_n$ the pull-backs of the f_i's on Y'. Analytic functions on $V \simeq D \times_k X'$ consist of power series $\sum a_I T^I$ where the a_I's are analytic functions on X'. For any j, let us write $f_{j|V} = \sum a_{I,j} T^I$. By construction, $V_{x'} \subset Z'_{x'}$. Therefore $a_{I,j}(x') = 0$ for every (I, j). As $\mathscr{O}_{X',x'}$ is a field, $a_{I,j}$ vanishes in a neighborhood of x' in the normal connected space X' for every (I, j); therefore $a_{I,j} = 0$ for every (I, j). This implies that $V \subset Z'$; hence Z' contains a non-empty open subset of Y'. As $Y' \to Y$ is flat and locally finite, it is open by Corollary 4.3.2. Therefore Z contains a non-empty open subset of Y, which we have seen is sufficient. □

6.3.2. — Let $\mathscr{X}$ be a locally noetherian scheme over a field κ. For technical purposes, we shall have to consider the following property $\mathfrak{R}$: *there exist a subfield κ_0*

of κ *with* $[\kappa:\kappa_0]<+\infty$ *and a* regular κ_0*-scheme* $\mathscr{X}_0$ *such that* $\mathscr{X}\simeq\mathscr{X}_0\otimes_{\kappa_0}\kappa$. Let us mention some basic facts about this property.

- It follows from the definition that if the κ-scheme $\mathscr{X}$ satisfies $\mathfrak{R}$, then so does the λ-scheme $\mathscr{X}\otimes_\kappa\lambda$ for every finite extension λ of κ.
- If κ is of char. 0, then it is immediate that $\mathscr{X}$ satisfies $\mathfrak{R}$ if and only if it is regular.
- Assume that the κ-scheme $\mathscr{X}$ satisfies $\mathfrak{R}$. Let us choose a closed immersion of Spec κ into $\mathbf{A}^m_{\kappa_0}$ for some m. Since κ is CI, this closed immersion is a regular embedding. By taking the fiber product with the flat κ_0-scheme $\mathscr{X}_0$, we get a regular embedding of $\mathscr{X}$ into the regular scheme $\mathbf{A}^m_{\mathscr{X}_0}$.

6.3.3. Theorem. — *Let* $Y\to X$ *be a morphism between good* k*-analytic spaces. Let* y *be a point of* Y *and let* x *be its image in* X*. Assume that* y *belongs to* $\mathrm{Int}(Y/X)$ *and that* $\mathscr{O}_{X,x}$ *is a field. The morphism* Spec $\mathscr{O}_{Y_x,y}\to$ Spec $\mathscr{O}_{Y,y}$ *is then flat, and its fibers satisfy property* $\mathfrak{R}$ *of 6.3.2, In particular, its fibers are CI, and are regular if* char. $k=0$.

Proof. — Set $n=\dim_y Y_x$. We argue by induction on n; let us begin with some preparation.

According to Thm. 4.6 of [**Duc07b**], one can shrink Y around y such that $Y\to X$ goes through a map $Y\to\mathbf{A}^n_X$ that is quasi-finite at y. By assumption, y belongs to $\mathrm{Int}(Y/X)$, so it belongs to $\mathrm{Int}(Y\to\mathbf{A}^n_X)$; hence $Y\to X$ it is finite at y by [**Ber93**], Prop. 3.1.4. Denote by t the image of y in $\mathbf{A}^n_X$. One can shrink Y around y so that it is finite over an affinoid neighborhood V of t in $\mathbf{A}^n_X$ (note that $V\to X$ is then boundaryless at t) and so that y is the only preimage of t in Y. Let A (resp. B) be the algebra of analytic functions on V (resp. Y). Then $\mathscr{O}_{Y,y}=B\otimes_A\mathscr{O}_{V,t}$ and $\mathscr{O}_{Y_x,y}=B\otimes_A\mathscr{O}_{V_x,t}$; hence $\mathscr{O}_{Y_x,y}=\mathscr{O}_{V_x,t}\otimes_{\mathscr{O}_{V,t}}\mathscr{O}_{Y,y}$. It is thus sufficient (by finiteness of $\mathscr{O}_{Y,y}$ over $\mathscr{O}_{V,t}$) to prove that Spec $\mathscr{O}_{V_x,t}\to$ Spec $\mathscr{O}_{V,t}$ is flat and that its fibers satisfy $\mathfrak{R}$. Let us list some facts that will be useful for the proof.

(A) As $\mathscr{O}_{X,x}$ is a field, it is regular; hence $\mathscr{O}_{V,t}$ is regular by Proposition 5.5.5.
(B) The local ring $\mathscr{O}_{V_x,t}$ is regular by 5.1.9.
(C) The ring $\mathscr{O}_{V,t}$ being regular by (A), it is in particular reduced; this implies that if f is a non-zero element of it, its zero-locus (which is a Zariski-closed subset of a suitable neighborhood of t) contains no neighborhood of t in V. By Proposition 6.3.1, it follows that this zero-locus contains no neighborhood of t in V_x.

Let us now go back to the proof by induction on n. If $n=0$ then $t=x$, and both $\mathscr{O}_{V,t}=\mathscr{O}_{X,x}$ and $\mathscr{O}_{V_x,t}=\mathscr{H}(x)$ are fields, hence we are done. Assume that $n>0$ and that the theorem has been proved for any integer $<n$.

Let us first prove that $\mathscr{O}_{V_x,t}$ is a flat $\mathscr{O}_{V,t}$-algebra. By [**SGA 1**], Exposé IV, Th. 5.6, it is sufficient to prove that $\mathscr{O}_{V_x,t}/\mathfrak{m}_t^d\mathscr{O}_{V_x,t}$ is a flat $\mathscr{O}_{V,t}/\mathfrak{m}_t^d$-algebra for any $d>0$.

If $\mathfrak{m}_t = 0$, then $\mathscr{O}_{V,t}$ is a field and we are done. Suppose now that $\mathfrak{m}_t \neq 0$ and let d be a positive integer. Due to assertion (C) above, the closed analytic subspace Z defined in a neighborhood of t by the finitely generated ideal $\mathfrak{m}_t^d$ contains no neighborhood of t in V_x ; therefore $Z \to X$ is of dimension $< n$ at t (and also boundaryless at t, because $Z \hookrightarrow V$ is a closed immersion). By induction, $\mathscr{O}_{Z_x,t}$ is a flat $\mathscr{O}_{Z,t}$-algebra. But $\mathscr{O}_{Z_x,t}$ (resp. $\mathscr{O}_{Z,t}$) is nothing but $\mathscr{O}_{V_x,t}/\mathfrak{m}_t^d\mathscr{O}_{V_x,t}$ (resp. $\mathscr{O}_{V,t}/\mathfrak{m}_t^d$), whence the desired claim is proved.

Now, let us prove that any fiber of Spec $\mathscr{O}_{V_x,t} \to$ Spec $\mathscr{O}_{V,t}$ satisfies $\mathfrak{R}$. Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_{V,t}$. If $\mathfrak{p} = 0$, the fiber of Spec $\mathscr{O}_{V_x,t}$ over $\mathfrak{p}$ is the spectrum of a localization of $\mathscr{O}_{V_x,t}$; but the latter is regular by assertion (B) above, hence we are done. Suppose now that $\mathfrak{p} \neq 0$. By assertion (C) above, the closed analytic subspace Z defined in a neighborhood of t by the finitely generated ideal $\mathfrak{p}$ contains no neighborhood of t in V_x ; therefore $Z \to X$ is of dimension $< n$ at t (and also boundaryless at t, because $Z \hookrightarrow V$ is a closed immersion). The fiber of Spec $\mathscr{O}_{V_x,t}$ over $\mathfrak{p}$ is nothing but the generic fiber of the map Spec $\mathscr{O}_{Z_x,t} \to$ Spec $\mathscr{O}_{Z,t}$. By the induction hypothesis, the latter satisfies $\mathfrak{R}$, which ends the proof. □

We are now going to show that the assumptions of Theorem 6.3.3 are probably not far from being optimal; we still denote by $Y \to X$ a morphism of good k-analytic spaces, by x a point of X whose local ring is a field, and by y a point of Y_x.

6.3.4. — One cannot expect in general flatness of Spec $\mathscr{O}_{Y_x,y}$ over Spec $\mathscr{O}_{Y,y}$ if y belongs to $\partial(Y/X)$. Indeed, let r be a positive real number let and let f be a power series in one variable with coefficients in k whose radius of convergence is equal to r. Let V be the analytic domain of $\mathbf{A}_k^{2,\mathrm{an}}$ defined by the condition $|T_1| = r$. There is a natural closed immersion $\varphi : D \to V$ given by (Id, f), where D is the closed disc of radius r; let x denote the image under φ of the Shilov point of D. Proposition 4.4.6 ensures that $\mathscr{O}_{\mathbf{A}_k^{2,\mathrm{an}},x}$ is a field. The fiber of $V \hookrightarrow \mathbf{A}_k^{2,\mathrm{an}}$ at x is nothing but $\mathscr{M}(\mathscr{H}(x))$, and $\mathscr{O}_{V_x,x}$ is then simply the field $\mathscr{H}(x)$. As x lies on a one-dimensional Zariski-closed subset (namely, $\varphi(D)$) of the purely 2-dimensional space V, the local ring $\mathscr{O}_{V,x}$ *cannot* be a field (Corollary 3.2.9). As a consequence, Spec $\mathscr{H}(x) \to$ Spec $\mathscr{O}_{V,x}$ is not flat.

6.3.5. — One cannot expect in general regularity of the fibers of the morphism Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{Y,y}$ if k is of positive characteristic. Indeed, let us give the following counter-example, which was communicated to the author by M. Temkin. Assume that k is a non-algebraically closed field of char. $p > 0$ such that $|k^\times| \neq \{1\}$. Let k^a be an algebraic closure of k, and let k^s be the separable closure of k inside k^a; assume moreover that k^s is of countable dimension over k (for instance, we can take for k the field $\mathbf{F}_p((t))$ equipped with an arbitrary t-adic absolute value). By assumption, there exists an increasing sequence $(k_n)_{n\in\mathbf{Z}_{\geqslant 0}}$ of subfields of k^s that are finite over k and whose union is equal to k^s. For any n, the complement of a finite union of proper k-vector subspaces of the k-Banach space k_n

is a dense subset of it. Therefore there exists a sequence (λ_n) of elements of k^s and a decreasing sequence (r_n) of positive real numbers such that the following hold:

(a) for any n, one has $k[\lambda_n] = k_n$;
(b) for any n and any non-trivial k-conjugate μ of λ_n in k^a, one has $|\mu - \lambda_n| > r_n$;
(c) for any $m > n$, one has $|\lambda_m - \lambda_n| < r_n$;
(d) one has $r_n \to 0$ as $n \to \infty$.

For any n let us denote by D_n the affinoid domain of $\mathbf{A}^{1,\mathrm{an}}_{k_n}$ defined by the inequality $|T - \lambda_n| \leqslant r_n$. It follows from (a) and (b) that the natural map $\mathbf{A}^{1,\mathrm{an}}_{k_n} \to \mathbf{A}^{1,\mathrm{an}}_{k}$ induces an isomorphism beween D_n and an affinoid domain Δ_n of $\mathbf{A}^{1,\mathrm{an}}_{k}$. It follows from (c) and (d) that $(D_n(\widehat{k^a}))$ is a decreasing sequence of (naive) closed discs of $\widehat{k^a}$ whose intersection consists of a single element $\lambda \in \widehat{k^a}$. Let $x \in \mathbf{A}^{1,\mathrm{an}}_{k}$ be the point that corresponds to λ. We have $x \in \bigcap \Delta_n$. Therefore, k_n embeds into $\mathscr{H}(x) \subset \widehat{k^a}$ for every n. Hence $\mathscr{H}(x)$ is a closed subfield of $\widehat{k^a}$ containing k^s; the latter being dense in $\widehat{k^a}$, one has $\mathscr{H}(x) = \widehat{k^a}$; in particular, x is not a rigid point.

Let φ be the (finite, flat) morphism $\mathbf{A}^{1,\mathrm{an}}_{k} \to \mathbf{A}^{1,\mathrm{an}}_{k}$ induced by the morphism from $k[T]$ to itself that sends T to T^p, and let y be the unique preimage of x by φ. As x in non-rigid, y is non-rigid and $\mathscr{O}_{\mathbf{A}^{1,\mathrm{an}}_{k},y}$ is a field due to Lemma 4.4.5. Now $\mathscr{O}_{\varphi^{-1}(x),y} = \mathscr{H}(x)[\tau]/(\tau^p - T(x))$. Since $\mathscr{H}(x) = \widehat{k^a}$, the local ring $\mathscr{O}_{\varphi^{-1}(x),y}$ is non-reduced, and in particular, non-regular. Hence the map

$$\operatorname{Spec} \mathscr{O}_{\varphi^{-1}(x),y} \to \operatorname{Spec} \mathscr{O}_{\mathbf{A}^{1,\mathrm{an}}_{k},y}$$

is not regular.

6.3.6. — Let X be a reduced good k-analytic space, and let x be an Abhyankar point of X (1.4.10). The local ring $\mathscr{O}_{X,x}$ is then artinian (Example 3.2.10), hence it a field by reducedness. If Y is any good X-analytic space, we can thus apply Theorem 6.3.3 to the map $Y \to X$ over the point x. But in fact, due to the Abhyankar property of x, we have the following slightly stronger result, whose proof is totally different.

6.3.7. Theorem. — *Let $Y \to X$ be a morphism between good k-analytic spaces, with X reduced. Let x be an Abhyankar point of X (1.4.10), and let y be a point of the fiber Y_x. The morphism*

$$\operatorname{Spec} \mathscr{O}_{Y_x,y} \to \operatorname{Spec} \mathscr{O}_{Y,y}$$

is regular (note that we do not assume $y \in \operatorname{Int}(Y/X)$ nor $\operatorname{char.} k = 0$).

Proof. — Set $n = \dim_x X$. By shrinking X and Y we immediately reduce to the case where both are affinoid and where $\dim X = n$ (we can proceed to such a reduction without modifying $\mathscr{O}_{Y,y}$ because n is the infimum of the dimensions of all analytic *neighborhoods* of x in X, by 1.5.8); we denote by A (resp. B) the algebra of analytic functions on X (resp. Y).

Let us first assume that X is a closed n-dimensional polydisc centered at the origin of $\mathbf{A}_k^{n,\mathrm{an}}$ and $x = \eta_r$ for some polyradius $r = (r_1, \ldots, r_n)$. The point x has a canonical $\mathscr{H}(x)$-rational pre-image t on the $\mathscr{H}(x)$-analytic disc $X_{\mathscr{H}(x)}$. Choose $\varepsilon \in \mathbf{R}_+^\times$ such that $\varepsilon < r_i$ for every i. Let V be the affinoid domain of $X_{\mathscr{H}(x)}$ defined by the inequalities $|T_i - T_i(t)| \leqslant \varepsilon$ for $i = 1, \ldots, n$ (where the T_i's are the coordinate functions on X) and let C be the algebra of analytic functions on V. For every point v of V and every i, we have $\widetilde{T_i(v)} = \widetilde{T_i(t)}$ (the latter equality makes sense because $\mathscr{H}(t) = \mathscr{H}(x) \subset \mathscr{H}(v)$), and $|T_i(v)| = r_i$. Since t lies above x, the elements $\widetilde{T_1(t)}, \ldots, \widetilde{T_n(t)}$ of $\widetilde{\mathscr{H}(t)} = \widetilde{\mathscr{H}(x)}$ are algebraically independent over the graded field $\widetilde{k}$; therefore $\widetilde{T_1(v)}, \ldots, \widetilde{T_n(v)}$ are algebraically independent over $\widetilde{k}$, which implies (together with the fact that $|T_i(v)| = r_i$ for every i) that the image of v in X is equal to $\eta_r = x$; hence V is contained in the fiber of $X_{\mathscr{H}(x)}$ over x. As a consequence, for every $a \in A$ the image of a under the composite map $A \to A_{\mathscr{H}(x)} \to C$ is equal to the element $a(x)$ of $\mathscr{H}(x)$ (indeed, since C is reduced, it is sufficient to check this equality pointwise on V). In other words, both maps

$$A \to A_{\mathscr{H}(x)} \to C \quad \text{and} \quad A \xrightarrow{a \mapsto a(x)} \mathscr{H}(x) \longrightarrow C$$

coincide. Let W be the preimage of V in $Y_{\mathscr{H}(x)}$. This is an affinoid domain of $Y_{\mathscr{H}(x)}$, and

$$\mathscr{O}_{Y_x}(W) = B \widehat{\otimes}_A C = (B \widehat{\otimes}_A \mathscr{H}(x)) \widehat{\otimes}_{\mathscr{H}(x)} C.$$

Hence the morphism $W \to Y$ goes through Y_x, and the Y_x-analytic space W is isomorphic to $Y_x \times_{\mathscr{H}(x)} V$. As $\mathscr{H}(t) = \mathscr{H}(x)$, the natural map from $Y_{\mathscr{H}(x),t}$ to Y_x is an isomorphism; let y' be the unique point of $Y_{\mathscr{H}(x),t}$ lying above y. Since V is a neighborhood of t in $X_{\mathscr{H}(x)}$, the affinoid domain W is a neighborhood of y' in $Y_{\mathscr{H}(x)}$, and we thus have $\mathscr{O}_{W,y'} = \mathscr{O}_{Y_{\mathscr{H}(x)},y'}$. As $\mathscr{H}(x)$ is analytically separable over k, the morphism from Spec $\mathscr{O}_{W,y'} =$ Spec $\mathscr{O}_{Y_{\mathscr{H}(x)},y'}$ to Spec $\mathscr{O}_{Y,y}$ is regular (Theorem 2.6.5). And by flatness of V over $\mathscr{H}(x)$ (Lemma 4.1.13) and in view of the Y_x-isomorphism $W \simeq Y_x \times_{\mathscr{H}(x)} V$, the map Spec $\mathscr{O}_{W,y'} \to$ Spec $\mathscr{O}_{Y_x,y}$ is (faithfully) flat (in fact, by quasi-smoothness of V and in view of Theorem 5.3.4, it is even regular but we shall not need that). Lemma 5.5.2 then applies to the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathscr{O}_{W,y'} \arrow[d] \arrow[dr] & \\
\operatorname{Spec} \mathscr{O}_{Y_x,y} \arrow[r] & \operatorname{Spec} \mathscr{O}_{Y,y}
\end{tikzcd}$$

and yields the regularity of the map Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{Y,y}$, which ends the proof in the special case considered.

Let us now prove the theorem for arbitrary X. Since $d_k(x) = n$, there exist analytic functions $f_1, \ldots, f_n$ on X, invertible at x and such that the $\widetilde{f_i(x)}$'s are algebraically independent over $\widetilde{k}$. Let R be the supremum of the spectral norms of the f_i's. Those functions induce a morphism from X to the polydisc $Z := \mathscr{M}(k\{T_1/R, \ldots, T_n/R\})$; let z denote the image of x in Z. If we set $r_i = |f_i(x)|$ for every i, then our assumption on the f_i's means that $z = \eta_r$ for $r = (r_1, \ldots, r_n)$. Let t be a point of X_z. By assumption, $\dim X = n$ and we know that $d_k(\mathscr{H}(z)) = n$ (cf. Example A.4.10). As a consequence, $n \geqslant d_k(t) = d_{\mathscr{H}(z)}(t) + n$, whence the equality $d_{\mathscr{H}(z)}(t) = 0$. It follows that the fiber X_z is zero-dimensional. It thus consists of finitely many points $x_1 = x, x_2, \ldots, x_m$. By assumption, the space X is reduced. By the particular case proven above, Spec $\mathscr{O}_{X_z,x_j} \to$ Spec $\mathscr{O}_{X,x_j}$ is regular for every j, which implies that the $\mathscr{O}_{X_z,x_j}$'s are reduced (this follows for instance from [**EGA IV**$_2$], Prop. 6.4.1 (ii) and Prop. 6.5.3 (ii)). Therefore the fiber X_z is isomorphic to $\coprod \mathscr{M}(\mathscr{H}(x_j))$; it follows that

$$Y_z \simeq \coprod Y_{x_j}.$$

In particular, Y_x is an open subspace of Y_z, and $\mathscr{O}_{Y_x,y}$ is thus equal to $\mathscr{O}_{Y_z,y}$. Using again the particular case proven above (now applied to $Y \to Z$) we get the regularity of the map Spec $\mathscr{O}_{Y_z,y} \to$ Spec $\mathscr{O}_{Y,y}$; hence Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{Y,y}$ is regular. $\square$

CHAPTER 7

IMAGES OF MORPHISMS: LOCAL RESULTS

The general purpose of this chapter is the study of the "image" of a morphism between analytic germs. Most of the work is in fact carried out in the realm of (graded) Riemann-Zariski spaces, and is transferred thereafter to the analytic world through Temkin's theory.

Our main result about Riemann-Zariski spaces is kind of a "Chevalley theorem" in this setting (Theorem 7.2.5). It tells the following (we use the definitions and notation introduced in 3.3.1). Let K be a graded field, let F be a graded extension of K, and let L be a graded extension of F. Let U be a quasi-compact open subset of $\mathbf{P}_{L/K}$. The image of U in $\mathbf{P}_{F/K}$ is a quasi-compact open subset of $\mathbf{P}_{F/K}$.

The proof goes as follows. Using scalar extension to $K(T/r)$ for suitable polyradius r (this notation is introduced in A.1.10), one reduces to the non-graded case; one can clearly assume that $\mathsf{U} = \mathbf{P}_{L/K}\{f_1, \ldots, f_n\}$ for some elements $f_1, \ldots, f_n$ of L. One first considers the case where L is algebraic over F, for which one proves the required assertion by performing explicit computations based on Newton polygons (Proposition 7.1.3). For handling the general case, one sets $A = F[f_1, \ldots, f_n] \subset L$, and shows (Theorem 7.1.4) that there exist finitely many closed point $\mathsf{y}_1, \ldots, \mathsf{y}_m$ of Spec A such that the image of U in $\mathbf{P}_{F/K}$ is the union of the images of the sets $\mathbf{P}_{\kappa(\mathsf{y}_j)/K}\{f_1(\mathsf{y}_j), \ldots, f_n(\mathsf{y}_j)\}$ in $\mathbf{P}_{F/K}$ for j varying between 1 and m; it is thus a quasi-compact open subset of $\mathbf{P}_{F/K}$ by the algebraic case already investigated. Beside some quasi-compactness arguments, the proof of Theorem 7.1.4 rests on the so-called *quantifier elimination in the theory of non-trivially valued algebraically closed fields*, which can be seen as a valuative avatar of Chevalley's theorem. For the reader's convenience, Section 7.4 at the end of the chapter explains what it consists of, and also provides a way to replace it (for our purposes) by a scheme-theoretic argument.

Applications of the above to analytic geometry are carried out in Section 7.3. The main result is the following: if $(Y, y) \to (X, x)$ is a morphism of analytic germs, then there exists a smallest analytic domain (Z, x) of (X, x) through which the map

$(Y, y) \to (X, x)$ factorizes (Theorem 7.3.1 (1)). Moreover, if Γ is a subgroup of $\mathbf{R}_+^\times$ such that $\Gamma \cdot |k^\times| \neq \{1\}$, then (Z, x) is Γ-strict if (Y, y) is Γ-strict and (X, x) is separated (loc. cit., (3)). The latter fact has a global consequence: if $\varphi \colon Y \to X$ is a morphism between affinoid spaces and Y is Γ-strict, then $\varphi(Y)$ is contained in a Γ-strict analytic domain of X (Proposition 7.3.6).

7.1. Maps between Riemann-Zariski Spaces: the non-graded case

The purpose of what follows is to prove an avatar of Chevalley's constructibility theorem in the setting of (non-graded) Riemann-Zariski spaces.

7.1.1. Lemma. — *Let K be a field and let $|\cdot|$ be a valuation on it. Let L be an algebraic extension of K ; let $|\cdot|'$ be a valuation on L extending $|\cdot|$. Let $S_1, \ldots, S_n$ be indeterminates, and let $|\cdot|''$ be an extension of $|\cdot|'$ to $L(S_1, \ldots, S_n)$ whose restriction to $K(S_1, \ldots, S_n)$ is equal to $|\cdot|_{\text{Gauß}}$. Then $|.|'' = |.|'_{\text{Gauß}}$.*

Proof. — We denote by k, resp. ℓ, the residue field of $|\cdot|$, resp. $|\cdot|'$. For every i one has $|S_i|_{\text{Gauß}} = 1$, and the images of the S_i's in the residue field of $|\cdot|_{\text{Gauß}}$ are algebraically independent over k. As L is algebraic over F, the field ℓ is algebraic over k. Therefore, the images of the S_i's in the residue field of $|\cdot|''$ are algebraically independent over ℓ, whence the required equality $|\cdot|'' = |.|'_{\text{Gauß}}$. □

7.1.2. Lemma. — *Let F be a field and let $P = T^n + a_{n-1}T^{n-1} + \ldots + a_0$ be a monic polynomial belonging to $F[T]$; set $a_n = 1$. Assume that P is totally split. Let $|\cdot|$ be a valuation on F. The following are equivalent:*

(i) $|\lambda| > 1$ *for every root* λ *of* P *;*
(ii) $|a_0| > |a_j|$ *for every* $j > 0$.

Proof. — If (i) is true then (ii) follows immediately from the usual relations between the coefficients and the roots of P. Suppose that (ii) is true, and let λ be a root of P. As $P(\lambda) = 0$ there exists $j > 0$ such that $|a_0| \leqslant |a_j \lambda^j|$. Since $|a_0| > |a_j|$, this implies that $|\lambda| > 1$. □

7.1.3. Proposition. — *Let k be a field, let K be an extension of k, and let L be an algebraic extension of K. Let U be a quasi-compact open subset of $\mathbf{P}_{L/k}$. Its image V on $\mathbf{P}_{K/k}$ is a quasi-compact open subset of the latter.*

Proof. — We can assume that U is equal to $\mathbf{P}_{L/k}\{f_1, \ldots, f_\ell\}$ for suitable elements $f_1, \ldots, f_\ell$ of L. Let $S = (S_1, \ldots, S_\ell)$ be a family of indeterminates, and let f be the element $f_1S_1 + \ldots f_\ell S_\ell$ of $L(S)$. Let $P = T^n + a_{n-1}T^{n-1} + \ldots + a_0$ be the minimal polynomial of f over $K(S)$. Let Λ be a finite extension of $K(S)$ over which P splits. Let $|\cdot|$ be a valuation on K whose restriction to k is trivial. We fix an extension $|\cdot|_0$ of $|\cdot|_{\text{Gauß}}$ to Λ.

We are going to prove that the valuation $|\cdot|$ belongs to V if and only if $|\cdot|_{\mathrm{Gauß}}$ extends to a valuation $|\cdot|''$ on $L(S)$ such that $|f|'' \leqslant 1$. Let us first assume that $|\cdot| \in \mathsf{V}$. This means that it extends to a valuation $|\cdot|'$ on L such that $|f_i|' \leqslant 1$ for every i, so $|\cdot|'_{\mathrm{Gauß}}$ is then an extension of $|\cdot|_{\mathrm{Gauß}}$ to $L(S)$ satisfying the required properties. Conversely, assume that $|\cdot|_{\mathrm{Gauß}}$ extends to a valuation $|\cdot|''$ on $L(S)$ such that $|f|'' \leqslant 1$ and let $|\cdot|'$ denotes the restriction of $|\cdot|''$ to L. By Lemma 7.1.1, one has $|\cdot|'' = |\cdot|'_{\mathrm{Gauß}}$; therefore, the inequality $|f|'' \leqslant 1$ simply means that $|f_i|' \leqslant 1$ for every i, and we are done.

On the other hand, the valuation $|\cdot|_{\mathrm{Gauß}}$ admits an extension $|\cdot|''$ to $L(S)$ such that $|f|'' \leqslant 1$ if and only if there exists a root λ of P in Λ such that $|\lambda|_0 \leqslant 1$; according to Lemma 7.1.2, the latter condition is equivalent to the existence of $j > 0$ such that $|a_0|_{\mathrm{Gauß}} \leqslant |a_j|_{\mathrm{Gauß}}$ (note that if there exists such j, it can always be chosen so that $a_j \neq 0$: this is obvious if $a_0 \neq 0$, and if $a_0 = 0$ we have $n = 1$ and we take $j = 1$).

We thus have proved that V is equal to the preimage under $|\cdot| \mapsto |\cdot|_{\mathrm{Gauß}}$ of $\bigcup_{a_j \neq 0} \mathbf{P}_{K(S)/k}\{a_0/a_j\}$; and this preimage is easily seen, by the very definition of $|\cdot|_{\mathrm{Gauß}}$ for a given $|\cdot|$, to be a quasi-compact open subset of $\mathbf{P}_{K/k}$. □

7.1.4. Theorem. — *Let K be a field, let F be an extension of K, and let L be an extension of F. Let $f_1, \dots, f_n$ be finitely many elements of L and let A be the F-subalgebra of L generated by the f_i's. For any $y \in \operatorname{Spec} A$, let p_y denote the map $\mathbf{P}_{\kappa(y)/K} \to \mathbf{P}_{F/K}$.*

(1) *The image V of $\mathbf{P}_{L/K}\{f_1, \dots, f_n\}$ in $\mathbf{P}_{F/K}$ is a quasi-compact open subset of the latter.*

(2) *There exist finitely many* closed *points $y_1, \dots, y_m$ of* Spec A *such that*
$$\mathsf{V} = \bigcup_j p_{y_j}(\mathbf{P}_{\kappa(y_j)/K}\{f_1(y_j), \dots, f_n(y_j)\}\ ).$$

Proof. — If y is any closed point of Spec A, its residue field $\kappa(y)$ is finite over F. Proposition 7.1.3 above then ensures that $p_y(\ \mathbf{P}_{\kappa(y)/K}\{f_1(y), \dots, f_n(y)\}\ )$ is a quasi-compact open subset of $\mathbf{P}_{F/K}$. As V is a quasi-compact topological space, it is therefore enough, to establish both (1) and (2), to prove that
$$\mathsf{V} = \bigcup_{y \in \mathsf{C}} p_y(\mathbf{P}_{\kappa(y)/K}\{f_1(y), \dots, f_n(y)\}\ )$$
where C is the set of *all* closed points of Spec A.

Let us first prove that
$$\bigcup_{y \in \mathsf{C}} p_y(\ \mathbf{P}_{\kappa(y)/K}\{f_1(y), \dots, f_n(y)\}\ ) \subset \mathsf{V}.$$
We shall even show that
$$\bigcup_{y \in \operatorname{Spec} A} p_y(\ \mathbf{P}_{\kappa(y)/K}\{f_1(y), \dots, f_n(y)\}\ ) \subset \mathsf{V}.$$

Let y be any point of Spec A and let $|\cdot|$ be a valuation on F that is trivial on K and that belongs to $p_y(\mathbf{P}_{\kappa(y)/K}\{f_1(y),\dots,f_n(y)\})$; i.e., $|\cdot|$ extends to a valuation $|\cdot|'$ on $\kappa(y)$ which satisfies the inequality $|f_j(y)|' \leqslant 1$ for every j. Let $|\cdot|''$ be a valuation on L whose ring dominates $\mathscr{O}_{\mathrm{Spec}\,A,y}$. The residue field Λ of $|\cdot|''$ is an extension of $\kappa(y)$; we choose an extension $|\cdot|'''$ of $|\cdot|'$ to Λ. The composition of $|\cdot|''$ and $|\cdot|'''$ is a valuation on L whose restriction to F is equal to $|\cdot|$ and whose ring contains the f_j's. Hence $|\cdot| \in \mathsf{V}$.

Let us now prove that

$$\mathsf{V} \subset \bigcup_{y\in\mathsf{C}} p_y(\mathbf{P}_{\kappa(y)/K}\{f_1(y),\dots,f_n(y)\}).$$

Let $|\cdot|$ be a valuation on F that is trivial on K and that belongs to V; i.e., it extends to a valuation $|\cdot|'$ on L that satisfies the inequalities $|f_i|' \leqslant 1$ for all i.

Let L^{a} be an algebraic closure of L and let F^{a} be the algebraic closure of F inside L^{a}. We choose an extension $|\cdot|''$ of $|\cdot|'$ to L^{a}. Let $(P_1,\dots,P_m)$ be polynomials that generate the ideal of relations between the f_i's over the field F, so $A = F[T_1,\dots,T_n]/(P_1,\dots,P_m)$. The system of equations and inequalities (in variables $x_1,\dots,x_n$)

$$\{P_j(x_1,\dots,x_n)=0\}_{j=1,\dots,m} \text{ and } \{|x_i|''\leqslant 1\}_{i=1,\dots,n}$$

has a solution in L^{a}, provided by the f_i's. This implies that it has a solution $(g_1,\dots,g_n)$ in F^{a}. Indeed, if $|\cdot|$ is trivial this comes from the *Nullstellensatz*, because the inequality $|x| \leqslant 1$ is then satisfied by every $x \in F$. And if $|.|$ is non-trivial, this is a particular case of the so-called *model completeness* of the theory of algebraically closed, non-trivially valued fields, which itself follows from *quantifier elimination*; but this can also be given a direct, algebro-geometric proof. Reminders on model-completeness and quantifier elimination have been postponed to Section 7.4 (see Theorems 7.4.4 and 7.4.5), as well as the direct proof alluded to (Proposition 7.4.7 and Theorem 7.4.8).

Now evaluation at $(g_1,\dots,g_n)$ defines a map from $A = F[T_1,\dots,T_n]/(P_1,\dots,P_m)$ to F^{a}, which sends f_i to g_i for every i. Its kernel is a closed point y of Spec A, and we have an F-embedding ι of $\kappa(y)$ into F^{a} mapping $f_i(y)$ to g_i for every i. The composition $|\cdot|''\circ\iota$ is then a valuation on $\kappa(y)$ that extends $|.|$ and whose ring contains the $f_i(y)$'s, and we are done. $\square$

7.2. Maps between Riemann-Zariski spaces: the general case

We are now going to give a graded version of Theorem 7.1.4 (1). We refer the reader to A.1 for our general conventions in graded commutative algebra, to A.2 for graded linear algebra (and especially to Definition A.2.8 ff. for graded tensor products and flatness in the graded context), and to A.4 for graded valuations.

7.2.1. Lemma. — *Let K be a graded field and let E and L be two graded field extensions of K. Let A, resp. B, resp. C be a graded valuation ring of K, resp. E, resp. L; assume that*

$$B \cap K = C \cap K = A.$$

1. *Let F be a graded field equipped with an injective morphism $E \otimes_K L \hookrightarrow F$, making F a graded extension of both E and L. There exists a graded valuation ring D of F such that $D \cap E = B$ and $D \cap L = C$.*
2. *Assume that E and L are algebraic over K (A.3.3). There exists a common graded extension Λ of E and L over K and a graded valuation ring Δ of Λ such that $\Delta \cap E = B$ and $\Delta \cap L = C$.*

Proof. — We denote by a, b and c the respective residue graded fields of A, B and C. We choose a maximal homogeneous ideal of the non-zero graded ring $b \otimes_a c$ and denote by d be the corresponding quotient. Note that since B and C have no non-zero homogeneous A-torsion, both are flat as graded A-modules: this follows from the flatness criterion given at the end of A.2.11 and from the fact that every ideal I of A generated by a finite set S of homogeneous elements is principal (if $S = \varnothing$ one has $I = 0$, and if $S \neq \varnothing$ one has $I = (s)$ for any element s of S of maximal valuation).

Let us show (1). By A-flatness of C, the natural map

$$u \colon B \otimes_A C \to E \otimes_A C = E \otimes_K (K \otimes_A C)$$

is injective. As $K \otimes_A C$ is simply a graded localization of C by a homogeneous multiplicative subset which does not contain zero, it embeds into L; the natural map $v \colon E \otimes_A C \to E \otimes_K L \hookrightarrow F$ is thus injective. As a consequence, the natural map $v \circ u \colon B \otimes_A C \to F$ induces an isomorphism between $B \otimes_A C$ and the graded subring $B \cdot C$ of F generated by B and C. Hence there exists a (unique) map $B \cdot C \to d$ extending both $B \to b \to d$ and $C \to c \to d$. The kernel of this map is a homogeneous prime ideal (because its target is a graded field); by Zorn's Lemma the corresponding graded localization of $B \cdot C$ is dominated by a graded valuation ring D of F. By construction, $D \cap E = B$ and $D \cap L = C$, whence (1).

Let us now prove (2). Let $\mathfrak{p}$ be the kernel of the morphism $B \otimes_A C \to b \otimes_a c \to d$ and let R be the graded localization $(B \otimes_A C)_{\mathfrak{p}}$. Since B and C are flat over A, the graded tensor product $B \otimes_A C$ is flat over A, hence has no non-zero homogeneous A-torsion. The graded A-algebra R is then also torsion-free, and $R \otimes_A K$ is consequently non-zero. In particular, it has a homogeneous prime ideal, which corresponds (as in classical commutative algebra) to a homogeneous prime ideal $\mathfrak{q}$ of $B \otimes_A C$ such that $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \cap A = 0$. Let Λ denote the graded fraction field of $(B \otimes_A C)/\mathfrak{q}$.

Let r be a positive real number and let x be an element of E^r. By assumption, x is algebraic over K; therefore there exists a unitary homogeneous element P of $K[T/r]$ such that $P(x) = 0$. Choose a non-zero homogeneous element a of A such that $ab \in A$ for every coefficient b of P. The element ax of E is then easily seen to be the root of a unitary homogeneous element of $K[T/(r \cdot \deg a)]$ *with coefficients in* A. By a straightforward valuation computation, this implies that $ax \in B$. As a consequence, the localization $B \otimes_A K$ of B is equal to E, and $C \otimes_A K = L$ analogously, whence the equality

$$(B \otimes_A C) \otimes_A K = E \otimes_K L.$$

The latter implies, together with the fact that $\mathfrak{q} \cap A = 0$, that $B \otimes_A C \to \Lambda$ admits a factorization

$$B \otimes_A C \to E \otimes_K L \to \Lambda.$$

Hence Λ can be seen as common graded extension of E and L over K so that the image of $B \otimes_A C \to \Lambda$ is the graded subring $B \cdot C$ of Λ generated by B and C. Since $\mathfrak{q} \subset \mathfrak{p}$, the map $B \otimes_A C \to b \otimes_a c \to d$ induces a morphism $B \cdot C \to d$, extending both $B \to b \to d$ and $C \to c \to d$. The kernel of this latter map is the homogeneous prime ideal $\mathfrak{p}/\mathfrak{q}$; by Zorn's Lemma the corresponding graded localization of $B \cdot C$ is dominated by a graded valuation ring Δ of Λ. By construction, $\Delta \cap E = B$ and $\Delta \cap L = C$, whence (2). □

7.2.2. Corollary. — *Let*

$$\begin{tikzcd} E \arrow[r] & F \\ K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

be a commutative diagrams of graded fields such that $E \otimes_K L \to F$ is injective. Let ℓ be a graded subfield of L; set $k = K \cap \ell$. Define π, ρ, φ and ψ by the commutative diagram

$$\begin{tikzcd} \mathbf{P}_{F/\ell} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \mathbf{P}_{E/k} \arrow[d, "\psi"] \\ \mathbf{P}_{L/\ell} \arrow[r, "\rho"] & \mathbf{P}_{K/k} \end{tikzcd}$$

and let U be any subset of $\mathbf{P}_{E/k}$. One has $\varphi(\pi^{-1}(\mathsf{U})) = \rho^{-1}(\psi(\mathsf{U}))$. In particular, $\mathbf{P}_{F/\ell} \to \mathbf{P}_{E/k} \times_{\mathbf{P}_{K/k}} \mathbf{P}_{L/\ell}$ is surjective.

Proof. — The inclusion $\varphi(\pi^{-1}(\mathsf{U})) \subset \rho^{-1}(\psi(\mathsf{U}))$ follows formally from the commutativity of the diagram. Now, let $|\cdot|$ be a valuation belonging to $\rho^{-1}(\psi(\mathsf{U}))$. This means that $|\cdot|$ is a graded valuation on L, trivial over ℓ, and that there exists a graded valuation $|\cdot|' \in \mathsf{U}$ such that $|\cdot|'_{|K} = |\cdot|_{|K}$. By Lemma 7.2.1 (1), there exists a graded valuation $|\cdot|''$ on F whose restriction to E is equal to $|\cdot|'$, and whose restriction to L is equal to $|\cdot|$. The latter fact implies that the restriction of $|\cdot|''$ to ℓ is trivial; therefore $|\cdot|'' \in \pi^{-1}(\mathsf{U})$ and $|\cdot| \in \varphi(\pi^{-1}(\mathsf{U}))$; as a consequence $\varphi(\pi^{-1}(\mathsf{U})) = \rho^{-1}(\psi(\mathsf{U}))$, as required. The last assertion follows by applying this equality for U a singleton. □

Corollary 7.2.2 above can be used every time one has a commutative diagram of graded fields

$$\begin{tikzcd} E \arrow[r] & F \\ K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

such that $E \otimes_K L \to F$ is injective. Let us give two examples of such a diagram, which will play a role in the sequel.

7.2.3. Example. — Let K be a graded field, let L be a graded extension of K and let Γ be a subgroup of $\mathbf{R}_+^\times$. *The natural map $L^\Gamma \otimes_{K^\Gamma} K \to L$ is injective.* Indeed, for every class c of $\mathbf{R}_+^\times$ modulo Γ, set $K^c = \bigoplus_{r\in c} K^r$, and define L^c analogously. Let $\mathscr{C}$ be the susbet of $\mathbf{R}_+^\times$ consisting of classes c such that $K^c \neq 0$. One has $K = \bigoplus\limits_{c\in\mathscr{C}} K^c$. For every $c \in \mathscr{C}$, the summand K^c is a one-dimensional graded vector space over K^Γ, and L^c is a one-dimensional graded vector space over L^Γ. Therefore $L^\Gamma \otimes_{K^\Gamma} K^c \simeq L^c$ for every $c \in \mathscr{C}$ and

$$L^\Gamma \otimes_{K^\Gamma} K \simeq \bigoplus_{c\in\mathscr{C}} L^c \subset L,$$

whence the claim.

7.2.4. Example. — Let K be a graded field, let $s = (s_1, \dots, s_n)$ be a polyradius, and let $S = (S_1, \dots, S_n)$ be a finite family of indeterminates. Let L be a graded extension of K. *The natural map $L \otimes_K K(S/s) \to L(S/s)$ is injective.* Indeed, it follows directly from the definition that

$$L \otimes_K K[S/s] \simeq L[S/s].$$

Therefore $L \otimes_K K(S/s)$ appears as a graded localization of the graded domain $L[S/s]$ by a homogeneous multiplicative system which does not contain zero; hence it embeds in the graded fraction field $L(S/s)$ of $L[S/s]$.

7.2.5. Theorem. — *Let K be a graded field, let F be a graded extension of K and let L be a graded extension of F. Let Γ be a subgroup of $\mathbf{R}^\times$, and let V be a Γ-strict quasi-compact open subset of $\mathbf{P}_{L/K}$. Its image in $\mathbf{P}_{F/K}$ is a Γ-strict quasi-compact open subset of the latter.*

Proof. — Let us first assume that $\Gamma = \{1\}$. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{P}_{L/K} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & \mathbf{P}_{L^1/K^1} \arrow[d, "\psi"] \\ \mathbf{P}_{F/K} \arrow[r, "\rho"] & \mathbf{P}_{F^1/K^1} \end{tikzcd}$$

in which ρ, π, φ and ψ are the obvious maps. By assumption, there exists a quasi-compact open subset U of $\mathbf{P}_{L^1/K^1}$ such that $\mathsf{V} = \pi^{-1}(\mathsf{U})$. By Corollary 7.2.2 and

Example 7.2.3, one has $\varphi(\mathsf{V}) = \rho^{-1}(\psi(\mathsf{U}))$. By Theorem 7.1.4 (1), the image $\psi(\mathsf{U})$ is a quasi-compact open subset of $\mathbf{P}_{F^1/K^1}$, whence the result.

Let us treat now the general case. We may and do assume that V is equal to $\mathbf{P}_{L/K}\{f_1, \ldots, f_n\}$ for some non-zero homogeneous elements f_i of L^Γ. For every i, we denote by s_i the degree of f_i; set $s = (s_1, \ldots, s_n)$. Let us consider the commutative diagram

$$\begin{tikzcd} \mathbf{P}_{L(s^{-1}S)/K(s^{-1}S)} \arrow[r, "\mu"] \arrow[d, "\theta"'] & \mathbf{P}_{L/K} \arrow[d, "\varphi"] \\ \mathbf{P}_{F(s^{-1}S)/K(s^{-1}S)} \arrow[r, "\nu"] & \mathbf{P}_{F/K} \arrow[l, bend left, "\sigma"] \end{tikzcd}$$

in which μ, θ, ν and φ are the obvious maps and in which σ is the map $|\cdot| \mapsto |\cdot|_{\mathrm{Gauß}}$. The quasi-compact open subset $\mu^{-1}(\mathsf{V})$ of $\mathbf{P}_{L(s^{-1}S)/K(s^{-1}S)}$ is equal to

$$\mathbf{P}_{L(s^{-1}S)/K(s^{-1}S)}\{f_1, \ldots, f_n\} = \mathbf{P}_{L(s^{-1}S)/K(s^{-1}S)}\{f_1/S_1, \ldots, f_n/S_n\},$$

hence is strict. It follows therefore from the case $\Gamma = \{1\}$ already proven that $\theta(\mu^{-1}(\mathsf{V}))$ is a strict quasi-compact open subset of $\mathbf{P}_{F(s^{-1}S)/K(s^{-1}S}$. By Corollary 7.2.2 and Example 7.2.4, one has $\theta(\mu^{-1}(\mathsf{V})) = \nu^{-1}(\varphi(\mathsf{V}))$, which formally implies that $\varphi(\mathsf{V}) = \sigma^{-1}(\theta(\mu^{-1}(\mathsf{V}))$. Therefore in order to conclude, it suffices to show that $\sigma^{-1}(\mathsf{W})$ is a Γ-strict quasi-compact open subset of $\mathbf{P}_{F/K}$ for every *strict* quasi-compact open subset W of $\mathbf{P}_{F(s^{-1}S)/K(s^{-1}S)}$.

So, let us consider such a W. We may and do assume that $\mathsf{W} = \mathbf{P}_{L(s^{-1}S)/K(s^{-1}S)}\{g\}$ for some homogeneous element g of degree 1. Let us write $g = \sum a_I S^I / \sum b_J S^J$ (with $\sum b_I S^J \neq 0$) and let $|\cdot|$ be a graded valuation belonging to $\mathbf{P}_{F/K}$. One has the equivalence

$$|\cdot| \in \sigma^{-1}(\mathsf{W}) \iff \frac{\max_I |a_I|}{\max_J |b_J|} \leqslant 1.$$

As a consequence,

$$\sigma^{-1}(\mathsf{W}) = \bigcap_I \bigcup_{J, b_J \neq 0} \mathbf{P}_{F/K}\{a_I/b_J\}.$$

Fix I and J with $b_J \neq 0$. Since g is homogeneous of degree 1, the elements $a_I s^I$ and $b_J S^J$ of $L(s^{-1}S)$ are homogeneous of the same degree. It follows that a_I/b_J is homogeneous of degree $s^J/s^I = s^{J-I}$, which belongs to Γ. Hence $\sigma^{-1}(\mathsf{W})$ is a Γ-strict quasi-compact open subset of $\mathbf{P}_{F/K}$, which ends the proof. $\square$

7.2.6. The case of a normal graded extension. — Let K be a graded field, let F be a graded extension of K and let L be an algebraic graded algebraic extension of F (A.3.3). We assume moreover that L is normal over F and we set $G = \mathrm{Gal}(L/F)$ (A.3.6). The group G acts on $\mathbf{P}_{L/K}$ and $\mathbf{P}_{L/K} \to \mathbf{P}_{F/K}$ is G-equivariant. We are going to prove that $\mathbf{P}_{L/K}/G \to \mathbf{P}_{F/K}$ is a homeomorphism.

This map is surjective, and it is open by Theorem 7.2.5. It suffices therefore to prove that it is injective. Let $|\cdot|_1$ and $|\cdot|_2$ be two graded valuations on L that have the same restriction to F, and let us show that they are conjugate under G. By Lemma 7.2.1, there exists a graded extension M of F, a graded valuation $|\cdot|$ on M, and two F-embeddings j_1 and j_2 from L to M such that $|\cdot|_1 = |\cdot| \circ j_1$ and $|\cdot|_2 = |\cdot| \circ j_2$. Let r be a positive real number. Since $F \hookrightarrow L$ is normal, $j_1(L)^r$ is the subset of M^r consisting of elements x such that there exists an element of L^r whose minimal polynomial over F vanishes at x, and the same holds for $j_2(L)^r$. As a consequence, $j_1(L) = j_2(L)$ and $|\cdot|_1 = |\cdot|_2 \circ j_2^{-1} \circ j_1$; otherwise said, $|\cdot|_1$ and $|\cdot|_2$ are conjugate by the element $j_2^{-1} \circ j_1$ of G.

7.2.7. Remark. — Assume that L is radicial over F (A.3.6). The group G is then trivial, and 7.2.6 thus ensures that $\mathbf{P}_{L/K} \to \mathbf{P}_{F/K}$ is a homeomorphism. But this can easily be seen directly. Indeed, let $|\cdot|$ be a graded valuation on L, and let a be a homogeneous element of L. As L is radicial over K, there exists $N \in \mathbf{Z}_{\geqslant 0}$ such that $a^N \in F$; since $|a| \leqslant 1 \iff |a^N| \leqslant 1$, we see that $|\cdot|$ is uniquely determined by its restriction to F, whence our claim.

7.3. Applications to analytic geometry

We fix an analytic field k and a subgroup Γ of $\mathbf{R}_+^\times$ such that $|k^\times| \cdot \Gamma \neq \{1\}$. The purpose of what follows is to give some consequences of Theorem 7.2.5 concerning morphisms between k-analytic germs. We shall use freely Temkin's reduction of analytic germs, as well as its Γ-graded avatar; see Sections 3.4 and 3.5.

7.3.1. Theorem. — *Let $(Y, y) \to (X, x)$ be a morphism of k-analytic germs.*

1. *There exists a smallest analytic domain (Z, x) of (X, x) through which the morphism $(Y, y) \to (X, x)$ factorizes; the reduction $\widetilde{(Z, x)} \subset \widetilde{(X, x)}$ is equal to the image of the map $\widetilde{(Y, y)} \to \widetilde{(X, x)}$.*
2. *If both germs (Y, y) and (X, x) are Γ-strict, then (Z, x) is Γ-strict too, and $\widetilde{(Z, x)}^{\Gamma} \subset \widetilde{(X, x)}^{\Gamma}$ is equal to the image of the map $\widetilde{(Y, y)}^{\Gamma} \to \widetilde{(X, x)}^{\Gamma}$.*
3. *If (Y, y) is Γ-strict and if (X, x) is separated (but not necessarily Γ-strict), then (Z, x) is Γ-strict.*

Proof. — Let p denote the natural map $\widetilde{(Y, y)} \to \widetilde{(X, x)}$. Let us choose an atlas $\mathscr{U}$ of $\widetilde{(X, x)}$ (3.3.6) and an atlas $\mathscr{V}$ of $\widetilde{(Y, y)}$ such that the covering $\mathscr{V}$ is a refinement of $p^{-1}(\mathscr{U})$ (this is possible because p is quasi-compact, see 3.3.2); if moreover (Y, y) and (X, x) are Γ-strict, we require that $\mathscr{U}$ and $\mathscr{V}$ be Γ-strict. Let V be an element of $\mathscr{V}$, and let U be an element of $\mathscr{U}$ containing $p(\mathsf{V})$. By considering the diagram

$$\begin{tikzcd} \mathsf{V} \arrow[r, hook] \arrow[d, "p"'] & \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}} \arrow[d] \\ \mathsf{U} \arrow[r, hook] & \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}} \end{tikzcd}$$

and applying Theorem 7.2.5 we see that $p(\mathsf{V})$ is a quasi-compact open subset of $\widetilde{(X,x)}$, which is Γ-strict whenever (Y,y) and (X,x) are Γ-strict (because then so is V by construction). In view of the equality

$$p(\widetilde{(Y,y)}) = \bigcup_{\mathsf{V}\in\mathscr{V}} p(\mathsf{V}),$$

this implies that $p(\widetilde{(Y,y)})$ is a (necessarily non-empty) quasi-compact open subset of $\widetilde{(X,x)}$, which is Γ-strict as soon as (Y,y) and (X,x) are.

This non-empty quasi-compact open subset of $\widetilde{(X,x)}$ is equal to $\widetilde{(Z,x)}$ for a uniquely determined analytic domain (Z,x) of (X,x), which is Γ-strict as soon as (Y,y) and (X,x) are (3.4.5 (1) and Lemma 3.5.2). In view of assertion (5) of 3.4.5, (Z,x) is the smallest analytic domain of (X,x) through which p factorizes, which ends the proof of (1), and of the first statement of (2). If both (Y,y) and (X,x) are Γ-strict, we have a commutative diagram

$$\begin{tikzcd}
\widetilde{(Y,y)} \arrow[r] \arrow[d] & \widetilde{(Y,y)}^{\Gamma} \arrow[dd, bend left] \\
\widetilde{(Z,x)} \arrow[r] \arrow[d, hook] & \widetilde{(Z,x)}^{\Gamma} \arrow[d, hook] \\
\widetilde{(X,x)} \arrow[r] & \widetilde{(X,x)}^{\Gamma}
\end{tikzcd}$$

in which $\widetilde{(Y,y)} \to \widetilde{(Z,x)}$ is surjective by definition of (Z,x), and in which all horizontal arrows are surjective. It follows that $\widetilde{(Y,y)}^{\Gamma} \to \widetilde{(X,x)}^{\Gamma}$ factorizes through a surjection $\widetilde{(Y,y)}^{\Gamma} \to \widetilde{(Z,z)}^{\Gamma}$, which ends the proof of (2).

Now we assume that (Y,y) is Γ-strict and that (X,x) is separated. Let $\mathscr{V}$ be any Γ-strict atlas of $\widetilde{(Y,y)}$. Let V be a chart belonging to $\mathscr{V}$. By considering the commutative diagram

$$\begin{tikzcd}
\mathsf{V} \arrow[r, hook] \arrow[d, "p"'] & \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}} \arrow[d] \\
\widetilde{(X,x)} \arrow[r, hook] & \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}
\end{tikzcd}$$

and applying Theorem 7.2.5 we see that $p(\mathsf{V})$ is a Γ-strict quasi-compact open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$. In view of the equality

$$p(\widetilde{(Y,y)}) = \bigcup_{\mathsf{V}\in\mathscr{V}} p(\mathsf{V}),$$

this implies that $\widetilde{(Z,x)} = p(\widetilde{(Y,y)})$ is a Γ-strict quasi-compact open subset of $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$; therefore (Z,x) is Γ-strict, which proves (3). $\square$

7.3.2. — The following facts follow straightforwardly from the characterization of (Z,x) by its reduction $\widetilde{(Z,x)}$:

(1) If $(Y_1,y),\ldots,(Y_n,y)$ are analytic domains of (Y,y) such that $(Y,y) = \bigcup(Y_i,y)$ and if (Z_i,x) denotes (for every i) the smallest analytic domain of (X,x) through which $(Y_i,y) \to (X,x)$ factorizes, then $(Z,x) = \bigcup(Z_i,x)$.

(2) If $(X,x) \to (T,t)$ is another morphism of germs and if (W,t) denotes the smallest analytic domain of (T,t) through which $(Z,x) \to (T,t)$ factorizes, then (W,t) is also the smallest analytic domain of (T,t) through which the composite map

$$(Y,y) \to (X,x) \to (T,t)$$

factorizes.

7.3.3. Example. — Let X be a k-analytic space and let V be an analytic domain of X. If x is a point of V, it follows immediately from the definition that (V,x) is the smallest analytic domain of (X,x) through which the map $(V,x) \hookrightarrow (X,x)$ factorizes.

7.3.4. Example. — Let X and Y be two k-analytic spaces and let $\varphi\colon Y \to X$ be a boundaryless morphism (e.g., a closed immersion). Let y be a point of Y and let x be its image in X. Since $\widetilde{(Y,y)} = \widetilde{(X,x)} \times_{\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}} \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$, the continuous map $\widetilde{(Y,y)} \to \widetilde{(X,x)}$ is surjective. Hence (X,x) is the smallest analytic domain of (X,x) through which $(Y,y) \to (X,x)$ factorizes.

This has the following concrete meaning: if V is any analytic domain of X containing x such that $\varphi^{-1}(V)$ is a neighborhood of y in Y, then V is a neighborhood of x in X.

7.3.5. Example. — In view of 7.3.2 one may combine Examples 7.3.3 and 7.3.4 and get the following. Let X and Y be two k-analytic spaces, let V be an analytic domain of X, and let $Y \to X$ be the composition of a boundaryless morphism $Y \to V$ and of the inclusion $V \hookrightarrow X$. Then (V,x) is the smallest analytic domain of (X,x) through which $(Y,y) \to (X,x)$ factorizes.

We are now going to give the first *global* consequence of Theorem 7.3.1. Note that the source is not assumed to be Hausdorff, but that the target is assumed to be separated, which is stronger than Hausdorff.

7.3.6. Proposition. — *Let $\varphi\colon Y \to X$ be morphism between k-analytic spaces. Assume that Y is quasi-compact and Γ-strict and X is separated. The image $\varphi(Y)$ is contained in a compact Γ-strict analytic domain of X.*

Proof. — Let y be a point of Y, and set $x = \varphi(y)$. It follows from Theorem 7.3.1 that there exists a smallest analytic domain (Z, x) of (X, x) through which the map $(Y, y) \to (X, x)$ factorizes, and that it is Γ-strict. Using simply the fact that the map $(Y, y) \to (X, x)$ factorizes through (Z, x), we get the existence of an analytic neighborhood V_y of y in Y and a compact Γ-strict analytic neighborhood U_x of $x \in X$ such that $\varphi(V_y) \subset U_x$. By quasi-compactness of Y there is a finite subset E of U such that $Y = \bigcup_{y \in E} V_y$. We then have $\varphi(Y) \subset \bigcup_{y \in E} U_{\varphi(y)}$. Since X is separated and every $U_{\varphi(y)}$ for $y \in E$ is a compact, Γ-strict analytic domain of X, $\bigcup_{y \in E} U_{\varphi(y)}$ is a compact Γ-strict analytic domain of X (Remark 3.1.5). This ends the proof. □

7.4. Complement: around quantifier elimination in the theory ACVF

In the proof of Theorem 7.1.4 we have referred at one point to *quantifier elimination* in the theory of non-trivially valued algebraically closed fields. The aim of this section is to quickly recall what it consists of, to explain why it implies the statement that was needed for showing Theorem 7.1.4, and to give another proof of this statement, based upon classical arguments of algebraic geometry.

7.4.1. — Let us first define recursively what a *formula* [1] is; for the moment, it should be seen as a purely formal syntactic object, without any meaning. We fix a countable set of symbols which are called the *variables*. Every formula Φ will involve two disjoint [2] finite sets of variables: the set $\mathscr{V}_{\mathrm{b}}(\Phi)$ of *bound* variables, and the set $\mathscr{V}_{\mathrm{f}}(\Phi)$ of *free* variables (the status – free or bound – of a variable is not an absolute notion: a variable is free or bound *in a given formula*).

(1) Let Φ be an inequality of the form $|P| \bowtie |Q|$, where P and Q belong to $\mathbf{Z}[x_1, \ldots, x_n]$ for some variables $x_1, \ldots, x_n$ and where $\bowtie$ is a symbol belonging to $\{<, >, \leqslant, \geqslant\}$. Then Φ is a formula, $\mathscr{V}_{\mathrm{b}}(\Phi) = \varnothing$ and $\mathscr{V}_{\mathrm{f}}(\Phi)$ is the set of x_i's that actually occur in P or Q.

(2) If Φ is a formula, its negation Ψ is a formula too, and one has
$$\mathscr{V}_{\mathrm{b}}(\Psi) = \mathscr{V}_{\mathrm{b}}(\Psi) \text{ and } \mathscr{V}_{\mathrm{f}}(\Psi) = \mathscr{V}_{\mathrm{f}}(\Psi).$$

(3) If Φ is a formula and if x belongs to $\mathscr{V}_{\mathrm{f}}(\Phi)$, then $\Psi := (\forall x, \Phi)$ is a formula and one has
$$\mathscr{V}_{\mathrm{f}}(\Psi) = \mathscr{V}_{\mathrm{f}}(\Phi) \backslash \{x\} \text{ and } \mathscr{V}_{\mathrm{b}}(\Psi) = \mathscr{V}_{\mathrm{b}}(\Phi) \cup \{x\}.$$

1. The formulas we define here are formulas *in the language of valued fields*; but since we shall not use any other language in this section, we do not need such precise terminology.

2. The general definition of a formula is much more complicated: in a given formula the same variable may occur at some places with the "free" status and at some other places with the "bound" status. But any formula in this sense is, up to renaming some bound variables, equivalent (at least as far as the "concrete" interpretation is concerned) to a formula in our sense; that is why we have chosen this simpler definition.

(4) If Φ and Ψ are two formulas with $(\mathscr{V}_b(\Phi) \cup \mathscr{V}_b(\Psi)) \cap (\mathscr{V}_f(\Phi) \cup \mathscr{V}_f(\Psi) = \varnothing$ then $\Theta := (\Phi \text{ and } \Psi)$ is a formula and

$$\mathscr{V}_{\mathrm{b}}(\Theta) = \mathscr{V}_{\mathrm{b}}(\Phi) \cup \mathscr{V}_{\mathrm{b}}(\Psi) \text{ and } \mathscr{V}_{\mathrm{f}}(\Theta) = \mathscr{V}_{\mathrm{f}}(\Phi)) \cup \mathscr{V}_{\mathrm{f}}(\Psi).$$

For instance,

$$\exists x, \forall y, (\left|3x^2 - 8y + 7\right| \leqslant |-x + z + t|) \text{ or } (\left|5xy - y^2\right| > \left|w + x + 2y^3 - zt\right|)$$

is a formula with free variables z, t and w.

7.4.2. — Let Φ be a formula and let $x_1, \dots, x_r$ be the free variables of Φ. Let K be a valued field and let $a_1, \dots, a_r$ be elements of K. By replacing x_i with a_i for every i, one gets a statement $\Phi(a_1, \dots, a_r)$ whose truth value in K, and more generally in any valued extension of K, makes sense.

7.4.3. Remark. — Assume that Φ is quantifier-free (equivalently, it only involves free variables). Then for every valued extension L of K and every r-uple $(a_1, \dots, a_r)$ of elements of K, the statement $\Phi(a_1, \dots, a_r)$ holds in L if and only if it holds in K.

We are now going to state the *quantifier elimination* theorem for the theory of algebraically closed (non-trivially) valued field, ACVF for short. It is usually attributed to Robinson, though the result proved in his book [**Rob77**] is weaker – but the main ideas are there; for a complete proof, cf. [**Pre86**] or [**Wei84**].

***7.4.4. Theorem* (quantifier elimination in ACVF).** — *Let Φ be any formula with free variables $(x_1, \dots, x_r)$. There exists a* quantifier-free *formula Ψ with free variables $(x_1, \dots, x_r)$ such that for every algebraically closed, non-trivially valued field K and any r-tuple $(a_1, \dots, a_r)$ of elements on K, the statement $\Phi(a_1, \dots, a_r)$ holds in K if and only if $\Psi(a_1, \dots, a_r)$ holds in K.*

In view of Remark 7.4.3 above, quantifier elimination in ACVF implies the so-called *model-completeness* of ACVF:

***7.4.5. Theorem* (model-completeness of ACVF).** — *Let Φ be any formula with free variables $(x_1, \dots, x_r)$, let K be an algebraically closed, non-trivially valued field and let L be an algebraically closed valued extension of K. For every r-uple $(a_1, \dots, a_r)$ of elements of K, the statement $\Phi(a_1, \dots, a_r)$ holds in L if and only if it holds in K.*

7.4.6. Remark. — The assumption that K is non-trivially valued cannot be removed. Indeed, let K be a trivially valued algebraically closed field; choose an algebraically closed, non-trivially valued extension L of K. The statement

$$\exists x, (x \neq 0) \text{ and } |x| < 1$$

then holds in L, but not in K.

The *Nullstellensatz* in the trivially valued case and model-completeness of ACVF (Theorem 7.4.5 above) in the non-trivially valued case immediately imply the following proposition, which was the key ingredient in the proof of Theorem 7.1.4.

7.4.7. Proposition. — *Let K be an algebraically closed valued field, and let L be a valued extension of K; let $(P_1, \ldots, P_m)$ be elements of $K[T_1, \ldots, T_n]$ for some n. If the system of equations $\{P_i = 0\}_{1 \leqslant i \leqslant m}$ has a solution $(x_1, \ldots, x_n)$ in L^n such that $|x_i| \leqslant 1$ for all i, then it already has a solution $(y_1, \ldots, y_n)$ in K^n such that $|y_i| \leqslant 1$ for all i.*

We now aim at giving a direct, algebro-geometric proof of Proposition 7.4.7 above. We shall in fact prove the following slightly stronger result.

7.4.8. Theorem. — *Let K be an algebraically closed valued field and let A be its valuation ring. Let $\mathscr{X}$ be a finitely presented A-scheme and let B be a faithfully flat A-algebra. If $\mathscr{X}(B) \neq \varnothing$, then $\mathscr{X}(A) \neq \varnothing$.*

Before proving it, let us explain why this implies Proposition 7.4.7. We use the notation of loc. cit. and call A, resp. B, the valuation ring of K, resp. L. By multiplying every P_i by a suitable element of $K^\times$, we may and do assume that P_i belongs to $A[T_1, \ldots, T_n]$ for all i. Let $\mathscr{X}$ be the A-scheme Spec $A[T_1, \ldots, T_n]/(P_1, \ldots, P_m)$. The A-algebra B is faithfully flat, and $\mathscr{X}(B) \neq \varnothing$ by assumption. Therefore if we assume that Theorem 7.4.8 holds, we can conclude that $\mathscr{X}(A) \neq \varnothing$, which is exactly the required assertion.

Proof of Theorem 7.4.8. — By assumption, $\mathscr{X}$ has a B-point. Since $\mathscr{X}$ is finitely presented, this B-point is induced by a B'-point of $\mathscr{X}$ for some finitely generated subalgebra B' of B. Since being A-flat simply means having no non-zero A-torsion, the A-algebra B' is flat; being finitely generated, it is finitely presented by a result of Nagata [**Nag66**] (this holds more generally for A any *domain*; see [**RG71**], Cor. 3.4.7); moreover, since Spec $B \to$ Spec A goes through Spec B', the map Spec $B' \to$ Spec A is surjective, which means that B' is faithfully flat over A. Hence by replacing B with B', we may and do assume that B is finitely presented over A. It is now sufficient to prove that Spec $B \to$ Spec A has a section, because by composing the latter with any B-point Spec $B \to \mathscr{X}$ we will get an A-point Spec $A \to \mathscr{X}$.

As B is finitely presented over A, there exists a subring A_0 of A finitely generated over $\mathbf{Z}$, and a flat A_0-algebra of finite type B_0 such that $B \simeq A \otimes_{A_0} B_0$. Let x be the closed point of Spec A, and let x_0 be the image of x on Spec A_0. Since the fiber $(\mathrm{Spec}\ B)_x$ is non-empty, the finitely generated $\kappa(x_0)$-scheme $(\mathrm{Spec}\ B_0)_{x_0}$ is non-empty; its CM locus is thus non-empty (it contains the maximal points), and therefore has a closed point y. Let $(f_1, \ldots, f_r)$ be a finite family of elements of B_0 that lifts a maximal regular sequence of $(\mathrm{Spec}\ B_0)_{x_0}$ at y, and let Z be the closed subscheme of Spec B_0 defined by the f_i's. At the point y, the A_0-scheme Z is quasi-finite by construction, and flat by [**EGA IV**$_3$], Thm. 11.3.8 (c). By Zariski's Main Theorem, there is an open neighborhood U of y in Z and an open immersion $U \hookrightarrow V$ over A_0 for some finite A_0-scheme V.

Set $U' = U \times_{A_0} A$ and $V' = V \times_{A_0} A$. By construction, U' is a locally closed subscheme of Spec B, flat over A at some point y' lying over x; the scheme V' is finite over A, and there is an open immersion of A-schemes $U' \hookrightarrow V'$.

Since K is algebraically closed, the valuation ring A is henselian. The finite A-scheme V' is therefore a disjoint union of finitely many finite, local A-schemes. This implies that the connected component W of y' in U' is a finite, local A-scheme. Since U' is flat over Spec A at y', so is W; as a consequence, the image of $W \to \operatorname{Spec} A$ contains the generic point of Spec A. Let us choose a point w on the generic fiber of W, and let endow the Zariski-closed subset $\overline{\{w\}}$ of Spec B with its reduced structure. One has $\overline{\{w\}} = \operatorname{Spec} C$ for some finite A-algebra C; the ring C is a domain and the map $A \to C$ is injective. The tensor product $C \otimes_A K$ is a localization of the domain C by a mutliplicative subset which does not contain 0, so $C \otimes_A K$ is a domain. Since it is finite over K, this is a field and thus a finite extension of K. As K is algebraically closed, it follows that $C \otimes_A K = K$. Since A is normal, this implies that $C = A$, and the closed embedding $\overline{\{w\}} \hookrightarrow \operatorname{Spec} B$ then defines a section of $\operatorname{Spec} B \to \operatorname{Spec} A$. $\square$

CHAPTER 8

DÉVISSAGES À LA RAYNAUD-GRUSON

Most of this chapter is inspired by the celebrated work of Raynaud and Gruson on flatness [**RG71**]. In that paper, they consider a finitely presented morphism of schemes $X \to S$ and a quasi-coherent $\mathscr{O}_X$-module of finite type $\mathscr{M}$. The key notion they introduce is that of a *dévissage* of $\mathscr{M}$ over S at a given point of $\mathrm{Supp}(\mathscr{M})$ ([**RG71**], Def. 1.2.2); they prove that it always exists after some Nisnevich localization on the target and on the source; see Prop. 1.2.3 of [**RG71**] for the precise statement. They use dévissages for studying the S-flat locus of $\mathscr{M}$, describing the local structure of $\mathscr{M}$ at a point at which it is S-flat, or flattening it through a blow-up in general...

Now, let $Y \to X$ be a morphism of good k-analytic spaces, and let $\mathscr{F}$ be a coherent sheaf on Y. We define the notion of an X-dévissage of $\mathscr{F}$ at a given point of $\mathrm{Supp}(\mathscr{F})$ (Definition 8.2.4), and prove that such a dévissage always exists (see Theorem 8.2.5); note that there is no need for Nisnevich localization here: it suffices to work on a small enough affinoid neighborhood of y in Y – this ultimately relates to the henselian property of local rings of good analytic spaces.

We then give two applications of dévissages. We first use them to prove that if $y \in \mathrm{Int}(Y/X)$ and $\mathscr{F}$ is naively X-flat at y, then $\mathscr{F}$ is X-flat at y (Theorem 8.3.4). Otherwise said, naive flatness at a relatively inner point is automatically universal; i.e., it remains true after arbitrary good base change (including ground field extension). More precisely, in the spirit of Cor. 2.3 of [**RG71**], Theorem 8.3.4 characterizes naive X-flatness and X-flatness of $\mathscr{F}$ at y in terms of an X-dévissage of $\mathscr{F}$ at y (provided $y \in \mathrm{Supp}(\mathscr{F})$; if not, $\mathscr{F}$ is obviously X-flat at y). It turns out that both characterizations are equivalent when $y \in \mathrm{Int}(Y/X)$, which ultimately rests on the nice properties of local rings of generic fibers at inner points (Theorem 6.3.3).

The second application of dévissages concerns the local structure of relatively CM (i.e., flat and fiberwise CM) coherent sheaves. There are two basic examples of such sheaves:

(a) If $Y \to X$ is finite and $\mathscr{F}$ is X-flat at y, then $\mathscr{F}$ is relatively CM at y.

(b) If $Y \to X$ is quasi-smooth at y, then $\mathscr{O}_Y$ is relatively CM at y (this follows from Theorem 5.3.4).

Now Theorem 8.4.6 essentially states that every coherent sheaf relatively CM at a point arises around this point as a "combination" of a relatively CM coherent sheaf of the kind described in (a), and of another one of the kind described in (b).

Though the two aforementioned applications of dévissages are the only ones in this memoir, we hope that they will be useful in the future for other purposes, such as for development of flattening techniques in the non-archimedean setting.

8.1. Universal injectivity and flatness

The purpose of this section is to introduce a technical notion, namely that of *universal injectivity*, and to study how it interacts with (naive and non-naive) flatness. We shall use freely and repeatedly the notions of exactness (of a complex of coherent sheaves), and injectivity, surjectitvity and bijectivity (of a morphism between coherent sheaves) *at a given point of an analytic space*, and the related affinoid GAGA principles; see Lemma-Definition 2.4.3 (3) and (4), and Lemma 2.4.6 (3) and (4).

We shall also use freely basic results about the fibers of coherent sheaves (2.5), and especially the fact that the property of being zero at a point (for a coherent sheaf) or of being surjective at a point (for a map of coherent sheaves) can be checked fiberwise; this follows basically from Nakayama's Lemma, see 2.5.2 and 2.5.4 for details.

8.1.1. Definition. — Let $Y \to X$ be a morphism between k-analytic spaces, and let $\mathscr{G} \to \mathscr{F}$ be a linear map between coherent sheaves on Y. Let y be a point of Y. We say that $\mathscr{G} \to \mathscr{F}$ is *X-universally injective at y* if for every analytic space X', for every morphism $X' \to X$, and for every point y' lying above y on $Y' := Y \times_X X'$, the map $\mathscr{G}_{Y'} \to \mathscr{F}_{Y'}$ is injective at y'.

8.1.2. Remark. — Definition 8.1.1 of universal injectivity is equivalent to the same in which one (apparently) weakens the condition by taking for X' a good space, or even an affinoid one.

8.1.3. Basic properties of universal injectivity. — It follows from its definition that universal injectivity is preserved by any base change (including ground field extensions).

Let $Y \to X$ be a morphism of k-analytic spaces, and let $\mathscr{G} \to \mathscr{F}$ be a morphism between coherent sheaves on Y. Let U be an analytic domain of X and let V be an analytic domain of $Y \times_X U$. For every $y \in V$, the map $\mathscr{G} \to \mathscr{F}$ is X-universally injective at y if and only if $\mathscr{G}_V \to \mathscr{F}_V$ is U-universally injective at y: this comes from the fact that the validity of injectivity at a given point is insensitive to the restriction to an analytic domain.

8.1.4. Remark. — We could also have defined universal surjectivity and universal bijectivity at a point in the same way, and obtained analogous properties. But there is no need for such notions, because surjectivity and bijectivity (of a morphism of coherent sheaves) at a point are *automatically universal*: indeed, after reduction to the good case this simply follows from the fact that surjectivity and bijectivity (of a morphism of modules) are preserved by tensor product.

8.1.5. About the bijectivity locus. — If X is an analytic space and if $\mathscr{F} \to \mathscr{G}$ is a morphism of coherent sheaves on X, we shall denote by $\mathrm{Bij}(\mathscr{F} \to \mathscr{G})$ the set of points of X at which $\mathscr{F} \to \mathscr{G}$ is bijective. By 2.5.5, $\mathrm{Bij}(\mathscr{F} \to \mathscr{G})$ is a Zariski-open subset of X (hence $\overline{\mathrm{Bij}(\mathscr{F} \to \mathscr{G})}^{X} = \overline{\mathrm{Bij}(\mathscr{F} \to \mathscr{G})}^{X_{\mathrm{Zar}}}$ by Lemma 1.5.12) and $\mathscr{F}_{\mathrm{Bij}(\mathscr{F}\to\mathscr{G})} \to \mathscr{G}_{\mathrm{Bij}(\mathscr{F}\to\mathscr{G})}$ is an isomorphism.

Let $Y \to X$ be a morphism of analytic spaces. It follows from the above that

$$(\mathrm{Bij}(\mathscr{F} \to \mathscr{G}))_Y \subset \mathrm{Bij}(\mathscr{F}_Y \to \mathscr{G}_Y).$$

Moreover, if y is a point of Y at which Y is X-flat (e.g., Y is an analytic domain of X, or the space X_L for some analytic extension L of k), then y belongs to $\mathrm{Bij}(\mathscr{F}_Y \to \mathscr{G}_Y)$ *if and only if* it belongs to $(\mathrm{Bij}(\mathscr{F} \to \mathscr{G}))_Y$: this is a particular case of Lemma 4.5.1 (2) or of Lemma 4.5.2 (2).

8.1.6. — In this chapter, we shall often encounter the following situation. We are given a quasi-smooth morphism $T \to X$ of k-affinoid spaces and a point t of T whose image in X is denoted by x, and our investigation requires to choose a point z whose image z_x^{al} on T_x^{al} is the generic point of the connected component of t_x^{al} (we use the notation described in 2.1.2). Let us now make some basic remarks.

(1) Since T_x is a quasi-smooth $\mathscr{H}(x)$-analytic space, it is geometrically regular by Lemma 5.2.1 (2); therefore T_x^{al} is a regular scheme by affinoid GAGA, see Lemma 2.4.6 (1). Hence $\mathscr{O}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}}$ is the fraction field of the regular local ring $\mathscr{O}_{T_x^{\mathrm{al}},t_x^{\mathrm{al}}}$.

(2) The point z can be chosen in any given open subset U of T_x which intersects the connected component of t in T_x. Indeed, let n be the dimension of T_x at t and let V be the intersection of U and the connected component of t in T_x. The dimension of V is also n and as a consequence, there exists z in V such that $d_{\mathscr{H}(x)}(z) = n$, which fulfills our requirement (Remark 1.5.9).

(3) If U is a Zariski-open subset of T_x such that t belongs to $\overline{U}^{T_x}$, then U has a non-empty intersection with the connected component of t in T_x, hence contains z.

(4) If V is a Zariski-open subset of T containing t, then it contains z: apply (3) above with $U = V \cap T_x$. As a consequence, t^{al} belongs to $\overline{\{z^{\mathrm{al}}\}}^{T^{\mathrm{al}}}$, so $\mathscr{O}_{T^{\mathrm{al}},z^{\mathrm{al}}}$ is a localization of $\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}$.

8.1.7. — We are now going to prove two technical results. The first one is Proposition 8.1.8 below, which is the analogue of [**RG71**], Lemme 2.2; its statement is not very enlightening, and it will only be used as an intermediate step for the second technical result, namely Proposition 8.1.10. The latter is the analogue of Thm. 2.1 of [**RG71**], and its statement is designed for our purposes: it provides a general flatness criterion, and also a criterion for naive flatness in the inner case, which we shall apply to some of the coherent sheaves that appear in a dévissage (of course, the distinction between flatness and naive flatness has no counterpart in the work of Raynaud and Gruson).

8.1.8. Proposition. — *Let $T \to X$ be a quasi-smooth morphism between k-analytic spaces. Let $\mathscr{L}$ be a free $\mathscr{O}_T$-module of finite rank and let $\mathscr{N}$ be a coherent sheaf on T. Let t be a point of T, and let x be its image on X. Let $\mathscr{L} \to \mathscr{N}$ be a map such that t belongs to $\overline{\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{N}_{T_x})}^{T_x}$. The following are equivalent:*

(i) *The map $\mathscr{L} \to \mathscr{N}$ is X-universally injective at t.*
(ii) *The map $\mathscr{L} \to \mathscr{N}$ is injective at t.*
(iii) *The point t belongs to $\overline{\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_x}^{T_x}$.*

8.1.9. Remark. — Let X' be an analytic space, and let $X' \to X$ be a morphism; set $T' = T \times_X X'$. Let t' be a pre-image of t on T', and let x' be the image of t' on X'. The coherent sheaf $\mathscr{L}_{T'}$ is then a free $\mathscr{O}_{T'}$-module. Moreover, t' belongs to $\overline{\mathrm{Bij}(\mathscr{L}_{T'_{x'}} \to \mathscr{N}_{T'_{x'}})}^{T'_{x'}}$. Indeed,

$$\mathrm{Bij}(\mathscr{L}_{T'_{x'}} \to \mathscr{N}_{T'_{x'}}) = (\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{N}_{T_x}))_{\mathscr{H}(x')}$$

by 8.1.5, and our claim thus follows from Corollary 1.5.14. Hence the data $(T', X', t', x', \mathscr{L}_{T'} \to \mathscr{N}_{T'})$ also fulfills the assumptions of the proposition (possibly over a ground field larger than k).

Now assume that assertion (iii) is satisfied. Corollary 1.5.14 then ensures that t' belongs to

$$\overline{(\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_x)_{\mathscr{H}(x')}}^{T'_{x'}} = \overline{(\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_{T'})_{x'}}^{T'_{x'}},$$

and since $\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_{T'} \subset \mathrm{Bij}(\mathscr{L}_{T'} \to \mathscr{N}_{T'})$ by 8.1.5, the point t' belongs to $\overline{\mathrm{Bij}(\mathscr{L}_{T'} \to \mathscr{N}_{T'})_{x'}}^{T'_{x'}}$; otherwise said, the analogue of assertion (iii) with respect to the data $(T', X', t', x', \mathscr{L}_{T'} \to \mathscr{N}_{T'})$ also holds.

Proof of Proposition 8.1.8. — We begin with the equivalence (ii) $\iff$ (iii). We can assume that T and X are affinoid. We chose z in T such that z_x^{al} is the generic point of the connected component of T_x^{al} containing t_x^{al}.

Assume that (ii) holds. This means that the arrow $\mathscr{L}_t \to \mathscr{N}_t$ is injective. Therefore:

(a) The map $\mathscr{L}^{\mathrm{al}}_{t^{\mathrm{al}}} \to \mathscr{N}^{\mathrm{al}}_{t^{\mathrm{al}}}$ is injective.
(b) The map $\mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}} \to \mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}}$ is injective by (a) and 8.1.6 (4).

By assumption, t belongs to $\overline{\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{N}_{T_x})}^{T_x}$; therefore $\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{N}_{T_x})$ contains z by 8.1.6 (3) ; as a consequence, $\mathscr{L}_{\mathscr{H}(z)} \to \mathscr{N}_{\mathscr{H}(z)}$ is an isomorphism, and is in particular surjective. Therefore:

(c) The map $\mathscr{L} \to \mathscr{N}$ is surjective at z.
(d) The map $\mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}} \to \mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}}$ is surjective by (c).
(e) The map $\mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}} \to \mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}}$ is bijective by (b) and (d).

By (e), the point z belongs to $\mathrm{Bij}(\mathscr{L} \to \mathscr{N})$; since t belongs the Zariski-closure of z in T_x, it belongs to $\overline{\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_x}^{T_x}$; hence (iii) holds.

Assume conversely that (iii) holds, and let us prove (ii). We can do it after an arbitrary scalar extension: Remark 8.1.9 ensures that our hypotheses (including (iii)) will remain the same; and Proposition 2.6.7 (5) ensures that (ii) will descent to the original ground field. We can thus assume that x is a rational point.

By assumption, t belongs to $\overline{\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_x}^{T_x}$; hence $\mathrm{Bij}(\mathscr{L} \to \mathscr{N})$ contains z by 8.1.6 (3); therefore $\mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}} \simeq \mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}}$.

We want to prove that the top horizontal arrow of the following commutative diagram

$$\begin{array}{ccc} \mathscr{L}^{\mathrm{al}}_{t^{\mathrm{al}}} & \longrightarrow & \mathscr{N}^{\mathrm{al}}_{t^{\mathrm{al}}} \\ \downarrow & & \downarrow \\ \mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}} & \longrightarrow & \mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}} \end{array}$$

is an injection; the bottom horizontal arrow being an isomorphism, it is enough to establish the injectivity of the left vertical arrow. The $\mathscr{O}_T$-module $\mathscr{L}$ is free of finite rank; therefore, it suffices to prove that the map $\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}} \to \mathscr{O}_{T^{\mathrm{al}},z^{\mathrm{al}}}$ is injective. We denote by S be the multiplicative subset of $\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}$ that consists of all elements a such that $a(z^{\mathrm{al}}) \neq 0$; by 8.1.6 (4), we have $\mathscr{O}_{T^{\mathrm{al}},z^{\mathrm{al}}} = S^{-1}\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}$.

Let a be an element of S. Since $a(z^{\mathrm{al}})$ is non-zero, the image of a in $\kappa(z^{\mathrm{al}}_x)$ is non-zero. But $\kappa(z^{\mathrm{al}}_x)$ coincides with $\mathscr{O}_{T^{\mathrm{al}}_x,z^{\mathrm{al}}_x}$; i.e., with $\mathrm{Frac}(\mathscr{O}_{T^{\mathrm{al}}_x,t^{\mathrm{al}}_x})$. Therefore the image of a in the domain $\mathscr{O}_{T^{\mathrm{al}}_x,t^{\mathrm{al}}_x}$ is non-zero, and hence is not a zero divisor. On the other hand, $\mathscr{O}_{T^{\mathrm{al}}_x,t^{\mathrm{al}}_x} = \mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}/\mathfrak{m}_{x^{\mathrm{al}}}\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}$ because x is a k-point; and since T is quasi-smooth over X, the ring $\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}$ is flat over $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$ by Theorem 5.5.3 (2) (or more directly by Corollary 5.3.2 and Lemma 4.2.1). The above properties imply that the multiplication by a in $\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}$ is injective ([**Mat86**], Thm. 22.5). The set S thus only consists of elements that are not zero divisors. As a consequence, the localization map $\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}} \to S^{-1}\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}} = \mathscr{O}_{T^{\mathrm{al}},z^{\mathrm{al}}}$ is injective, and (ii) holds, whence the equivalence (ii) $\iff$ (iii).

Let us now prove that (i) $\iff$ (ii), the spaces T and X being no longer assumed to be affinoid. The direct implication is tautological; it thus remain to show that (ii)$\Rightarrow$(i). So let us assume that $\mathscr{L} \to \mathscr{N}$ is injective at t. Let $X' \to X$ be an arbitrary morphism, and set $T' = T \times_X X'$. Let t' be a pre-image of t in T' and let x' be the image of t' in X'. By the implication (ii)$\Rightarrow$(iii) already proven, the point t belongs to $\overline{\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_x}^{T_x}$. Remark 8.1.9 then ensures that t' belongs to $\overline{\mathrm{Bij}(\mathscr{L}_{T'} \to \mathscr{N}_{T'})_{x'}}^{T'_{x'}}$; since it also ensures that $(T', X', t', x', \mathscr{L}_{T'} \to \mathscr{N}_{T'})$ fulfills the assumptions of the proposition, the implication (iii)⇒(ii) already proven yields injectivity of $\mathscr{L}_{T'} \to \mathscr{N}_{T'}$ at t'. Therefore $\mathscr{L} \to \mathscr{N}$ is X-universally injective at t. □

8.1.10. Proposition. — *Let $T \to X$ be a quasi-smooth morphism between k-analytic spaces. Let $\mathscr{L}$ be a free $\mathscr{O}_T$-module of finite rank and let $\mathscr{N}$ be a coherent sheaf on T. Let t be a point of T, and let x be its image in X. Let $\mathscr{L} \to \mathscr{N}$ be a map such that t belongs to $\overline{\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{N}_{T_x})}^{T_x}$. Let $\mathscr{P}$ be the cokernel of $\mathscr{L} \to \mathscr{N}$.*

(1) *The following are equivalent:*
  (i) *The coherent sheaf $\mathscr{N}$ is X-flat at t.*
  (ii) *The map $\mathscr{L} \to \mathscr{N}$ is injective at t and $\mathscr{P}$ is X-flat at t.*
(2) *Assume that X and T are good and that t belongs to $\mathrm{Int}(T/X)$. The following are equivalent:*
  (iii) *The coherent sheaf $\mathscr{N}$ is naively X-flat at t.*
  (iv) *The map $\mathscr{L} \to \mathscr{N}$ is injective at t and $\mathscr{P}$ is naively X-flat at t.*

Proof. — We begin with (2). By shrinking X and T, we may assume that both are affinoid, and that the maximal ideal $\mathfrak{m}_x$ is generated by an ideal of $\mathscr{O}_X(X)$; we denote by Y the corresponding closed analytic subspace of X, and by S the fiber product $T \times_X Y$. We have by construction $\mathscr{O}_{Y,x} = \mathscr{O}_{X,x}/\mathfrak{m}_x = \kappa(x)$; hence $\mathscr{O}_{Y,x}$ is a field, and $\mathscr{O}_{Y^{\mathrm{al}},x^{\mathrm{al}}}$ is thus a field as well by 2.1.5; we therefore have $\mathscr{O}_{Y^{\mathrm{al}},x^{\mathrm{al}}} = \mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}/\mathfrak{m}_{x^{\mathrm{al}}}$, and thus $\mathscr{O}_{S^{\mathrm{al}},\tau} = \mathscr{O}_{T^{\mathrm{al}},\tau}/\mathfrak{m}_{x^{\mathrm{al}}}\mathscr{O}_{T^{\mathrm{al}},\tau}$ for every point τ of T^{al} lying above x^{al}. We choose a point z in the open neighborhood $\mathrm{Int}(T/X)_x$ of t in T_x such that z_x^{al} is the generic point of the connected component of T_x^{al} containing t_x^{al}, which is possible by 8.1.6 (2). Note that z then also belongs to $\mathrm{Int}(S/Y)$, by base change.

Assume that (iii) holds. Let $\mathscr{R}$ be the kernel of the map $\mathscr{L} \to \mathscr{N}$. The point t lies on $\overline{\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{N}_{T_x})}^{T_x}$. By 8.1.6 (3), this implies that z belongs to $\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{N}_{T_x})$. In particular, $\mathscr{P}_{\mathscr{H}(z)} = 0$; hence $\mathscr{P}_z = 0$ and $\mathscr{P}^{\mathrm{al}}_{z^{\mathrm{al}}} = 0$. The sequence

$$0 \to \mathscr{R}^{\mathrm{al}}_{z^{\mathrm{al}}} \to \mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}} \to \mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}} \to 0$$

is thus exact.

By assumption, $\mathscr{N}_t$ is $\mathscr{O}_{X,x}$-flat. By Lemma 4.2.1, the $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$-module $\mathscr{N}^{\mathrm{al}}_{t^{\mathrm{al}}}$ is therefore flat; in view of 8.1.6 (4), the $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$-module $\mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}}$ is also flat. Hence the sequence

$$0 \to \mathscr{R}^{\mathrm{al}}_{z^{\mathrm{al}}}/(\mathfrak{m}_{x^{\mathrm{al}}}\mathscr{R}^{\mathrm{al}}_{z^{\mathrm{al}}}) \to \mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}}/(\mathfrak{m}_{x^{\mathrm{al}}}\mathscr{L}^{\mathrm{al}}_{z^{\mathrm{al}}}) \to \mathscr{N}^{\mathrm{al}}_{z^{\mathrm{al}}}/(\mathfrak{m}_{x^{\mathrm{al}}}\mathscr{N}_{z^{\mathrm{al}}}) \to 0$$

is exact; note that it can be rewritten

$$0 \to \mathscr{R}^{\mathrm{al}}_{S^{\mathrm{al}},z^{\mathrm{al}}} \to \mathscr{L}^{\mathrm{al}}_{S^{\mathrm{al}},z^{\mathrm{al}}} \to \mathscr{N}^{\mathrm{al}}_{S^{\mathrm{al}},z^{\mathrm{al}}} \to 0.$$

As a consequence, the sequence

$$0 \to \mathscr{R}_{S,z} \to \mathscr{L}_{S,z} \to \mathscr{N}_{S,z} \to 0$$

is exact too. Since $\mathscr{O}_{Y,x}$ is a field and z belongs to $\mathrm{Int}(S/Y)$, Theorem 6.3.3 tells us that $\mathscr{O}_{S_x,z}$ is a flat $\mathscr{O}_{S,z}$-algebra; this yields the exactness of the sequence

$$0 \to \mathscr{R}_{S_x,z} \to \mathscr{L}_{S_x,z} \to \mathscr{N}_{S_x,z} \to 0,$$

which can also be written

$$0 \to \mathscr{R}_{T_x,z} \to \mathscr{L}_{T_x,z} \to \mathscr{N}_{T_x,z} \to 0.$$

As z belongs to $\mathrm{Bij}(\mathscr{L}_{|T_x} \to \mathscr{N}_{|T_x})$, we have $\mathscr{R}_{T_x,z} = 0$; hence $\mathscr{R}_{\mathscr{H}(z)} = 0$ and $\mathscr{R}_z = 0$. Since $\mathscr{P}_z = 0$ too, the point z belongs to the Zariski-open subset $\mathrm{Bij}(\mathscr{L} \to \mathscr{N})$; being a Zariski-specialization of z inside T_x, the point t belongs to $\overline{\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_x}^{T_x,\mathrm{Zar}} = \overline{\mathrm{Bij}(\mathscr{L} \to \mathscr{N})_x}^{T_x}$ (the equality comes from Lemma 1.5.12). By Proposition 8.1.8, this implies that $\mathscr{L} \to \mathscr{N}$ is injective at t, and even X-*universally* injective at t. In particular, it remains injective after base-change by the map $\mathscr{M}(\mathscr{H}(x)) \to X$ induced by x; this means that $\mathscr{L}_{T_x} \to \mathscr{N}_{T_x}$ is injective at t. The map $\mathscr{L}_{T_x,t} \to \mathscr{N}_{T_x,t}$ is thus injective, and it can also be written $\mathscr{L}_{S_x,t} \to \mathscr{N}_{S_x,t}$. Since $\mathscr{O}_{X,x}$ is a field and t lies in $\mathrm{Int}(S/Y)$, Theorem 6.3.3 ensures that $\mathscr{O}_{S_x,t}$ is a flat $\mathscr{O}_{S,t}$-algebra. As a consequence, the map $\mathscr{L}_{S,t} \to \mathscr{N}_{S,t}$ is injective; it can be rewritten $\mathscr{L}_t/(\mathfrak{m}_x\mathscr{L}_t) \to \mathscr{N}_t/(\mathfrak{m}_x\mathscr{N}_t)$.

Now since $\mathscr{N}_t$ is a flat $\mathscr{O}_{X,x}$-module by assumption, the exact sequence

$$0 \to \mathscr{L}_t \to \mathscr{N}_t \to \mathscr{P}_t \to 0$$

induces a long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathscr{O}_{X,x}}(\mathscr{P}_t, \kappa(x)) \to \mathscr{L}_t/(\mathfrak{m}_x\mathscr{L}_t) \to \mathscr{N}_t/(\mathfrak{m}_x\mathscr{N}_t) \to \mathscr{P}_t/(\mathfrak{m}_x\mathscr{P}_t) \to 0$$

The injectivity of $\mathscr{L}_t/(\mathfrak{m}_x\mathscr{L}_t) \to \mathscr{N}_t/(\mathfrak{m}_x\mathscr{N}_t)$ thus yields the equality

$$\mathrm{Tor}_1^{\mathscr{O}_{X,x}}(\mathscr{P}_t, \kappa(x)) = 0,$$

which implies that $\mathscr{P}_t$ is a flat $\mathscr{O}_{X,x}$-module ([**SGA 1**], Exposé IV, Thm. 5.6). Otherwise said, $\mathscr{P}$ is naively X-flat at t, and (iv) holds.

Assume conversely that (iv) holds. The $\mathscr{O}_{X,x}$-module $\mathscr{P}_t$ is then flat. Since $T \to X$ is quasi-smooth, it is flat (Corollary 5.3.2) and $\mathscr{O}_{T,t}$ is thus flat over $\mathscr{O}_{X,x}$. As $\mathscr{L}$ is a free $\mathscr{O}_T$-module, $\mathscr{L}_t$ is flat over $\mathscr{O}_{X,x}$ too. The map $\mathscr{L} \to \mathscr{N}$ being injective at t, the sequence $0 \to \mathscr{L}_t \to \mathscr{N}_t \to \mathscr{P}_t \to 0$ is exact. Since both $\mathscr{P}_t$ and $\mathscr{L}_t$ are flat over $\mathscr{O}_{X,x}$, an easy Tor computation shows that $\mathscr{N}_t$ is flat over $\mathscr{O}_{X,x}$; i.e., $\mathscr{N}$ is naively X-flat at t, and (iii) holds.

We can now prove (1). We may assume that T and X are affinoid. Let us suppose that (i) holds. Let L be a complete extension of k, let X' be an L-affinoid space, let t' be a pre-image of t on $T' := X' \times_X T$, and let x' be the image of t' in X'. It suffices to show that the map $\mathscr{L}_{T'} \to \mathscr{N}_{T'}$ is injective at t' (this will yield the injectivity of $\mathscr{L} \to \mathscr{N}$ at t by taking $X' = X$) and that $\mathscr{P}_{T'}$ is naively X'-flat at t'. It suffices to show both properties after enlarging the field L (Proposition 4.5.6 and Proposition 2.6.7 (5)), which allows us to assume that t' is L-rational; in particular, t' belongs to $\mathrm{Int}(T'/X')$. Since $\mathscr{N}$ is universally X-flat at t, the coherent sheaf $\mathscr{N}_{T'}$

is X'-flat at t'. Moreover, by Remark 8.1.9, the data $(T', X', t', x', \mathscr{L}_{T'} \to \mathscr{N}_{T'})$ fulfill the assumptions of our proposition (which are the same as those of Proposition 8.1.8). Hence we can apply the assertion (2) already proven; it ensures that $\mathscr{L}_{T'} \to \mathscr{N}_{T'}$ is injective at t', and that $\mathscr{P}_{T'}$ is naively X'-flat at t', whence (ii).

Let us now suppose that (ii) holds and prove (i). Let L be a complete extension of k, let X' be an L-affinoid space, let t' be a pre-image of t on $T' := X' \times_X T$ and let x' be the image of t' in X'. It suffices to show that the map $\mathscr{N}_{T'}$ is naively X'-flat at t'. It is harmless to enlarge the field L (Lemma 4.5.3), which allows to assume that t' is L-rational; in particular, t' belongs to $\mathrm{Int}(T'/X')$. Since $\mathscr{L} \to \mathscr{N}$ is injective at t, it follows from Proposition 8.1.8 that it is in fact X-*universally* injective at t; hence $\mathscr{L}_{T'} \to \mathscr{N}_{T'}$ is injective at t'. Since $\mathscr{P}$ is universally X-flat at t, the coherent sheaf $\mathscr{P}_{T'}$ is naively X'-flat at t'. Moreover, by Remark 8.1.9, the data $(T', X', t', x', \mathscr{L}_{T'} \to \mathscr{N}_{T'})$ fulfill the assumptions of our proposition (which are the same as those of Proposition 8.1.8). Hence we can apply the assertion (2) already proven; it ensures that $\mathscr{N}_{T'}$ is naively X'-flat at t', whence (i). □

8.2. Dévissages: definition and existence

In this section we will make much use of the notions of dimension, depth and codepth of a finitely generated module over a local noetherian ring (1.1.2, 2.3.3). We shall also need the notions of dimension and codepth of a coherent sheaf at a given point of the ambient space; dimension in this setting is defined in 2.5.3, and we are now going to define codepth.

8.2.1. Definition. — Let Y be an analytic space, let y be a point of Y, and let $\mathscr{F}$ be a coherent sheaf on Y. In view of Lemma-Definition 2.4.1 and Example 2.3.18, for a good analytic domain V of Y containing y, the codepth of the $\mathscr{O}_{V,y}$-module $\mathscr{F}_{V,y}$ only depends on y, and not on V (Lemma-Definition 2.4.1 together with Example 2.3.18). It is called the *codepth* of $\mathscr{F}$ at y, and we denote it by $\mathrm{codepth}_y \mathscr{F}$.

8.2.2. — Let Y be an analytic space, let y be a point of Y and let $\mathscr{F}$ be a coherent sheaf on Y.

(1) Since the codepth of the zero-module is equal to zero by convention, we have $\mathrm{codepth}_y \mathscr{F} = 0$ as soon as $y \notin \mathrm{Supp}(\mathscr{F})$.
(2) If y belongs to $\mathrm{Supp}(\mathscr{F})$, then $\mathrm{codepth}_y \leqslant \dim_y \mathscr{F}$. Indeed, one immediately reduces to the good case, and one can then write

$$\mathrm{codepth}_y \mathscr{F} = \mathrm{codepth}_{\mathscr{O}_{Y,y}} \mathscr{F}_y \leqslant \dim_{\mathrm{Krull}} \mathscr{F}_y \leqslant \dim_y \mathscr{F},$$

where the last inequality comes from Corollary 3.2.9. In particular we see that $\mathrm{codepth}_y \mathscr{F} = 0$ if $\dim_y Y = 0$.
(3) If Y is regular at y, then $\mathrm{codepth}_y \mathscr{O}_Y = 0$: this follows from the fact that any regular local ring has codepth zero.

8.2.3. Convention. — Let Y be an analytic space and let $\mathscr{F}$ be a coherent sheaf on Y. The unique coherent sheaf on $\mathrm{Supp}(\mathscr{F})$ that induces $\mathscr{F}$ will be also denoted by $\mathscr{F}$, if there is no risk of confusion.

8.2.4. Definition. — Let $Y \to X$ be a morphism between good k-analytic spaces. Let $\mathscr{F}$ be a coherent sheaf on Y, let y be a point of $\mathrm{Supp}(\mathscr{F})$ and let x be its image on X. Let r be a positive integer, and let $n_1 > n_2 > \ldots > n_r$ be a decreasing sequence of non-negative integers. A *Γ-strict X-dévissage of $\mathscr{F}$ at y in dimensions* $n_1, \ldots, n_r$ is a list of data $(V, \{T_i, \pi_i, t_i, \mathscr{L}_i, \mathscr{P}_i\}_{i \in \{1,\ldots,r\}})$, where:

- V is a Γ-strict affinoid neighborhood of y in Y;
- T_i is for every i a Γ-strict k-affinoid domain of a smooth X-space of pure relative dimension n_i and t_i is a point of T_i lying over x;
- for every i, $\mathscr{L}_i$ and $\mathscr{P}_i$ are coherent $\mathscr{O}_{T_i}$-modules and $\mathscr{L}_i$ is free;
- $t_i \in \mathrm{Supp}(\mathscr{P}_i)$ if $i < r$, and $\mathscr{P}_r = 0$;
- π_1 is a finite X-map from $\mathrm{Supp}(\mathscr{F}_V)$ to T_1 such that we have $\pi_1^{-1}(t_1) = \{y\}$ *set-theoretically*;
- π_i for any $i \in \{2, \ldots, r\}$ is a finite X-map from $\mathrm{Supp}(\mathscr{P}_{i-1})$ to T_i such that we have $\pi_i^{-1}(t_i) = \{t_{i-1}\}$ *set-theoretically*;
- $\mathscr{L}_1$ is endowed with a morphism $\mathscr{L}_1 \to \pi_{1*}\mathscr{F}_V$ whose cokernel is $\mathscr{P}_1$ and such that $t_1 \in \overline{\mathrm{Bij}((\mathscr{L}_1)_{T_{1,x}} \to (\pi_{1*}\mathscr{F}_V)_{T_{1,x}})}^{T_{1,x}}$;
- for any $i \in \{2, \ldots, r\}$, $\mathscr{L}_i$ is endowed with a morphism $\mathscr{L}_i \to \pi_{i*}\mathscr{P}_{i-1}$ whose cokernel is $\mathscr{P}_i$ and such that $t_i \in \overline{\mathrm{Bij}((\mathscr{L}_i)_{T_{i,x}} \to (\pi_{i*}\mathscr{P}_{i-1})_{T_{i,x}})}^{T_{i,x}}$.

If we do not care about Γ (i.e., if we take $\Gamma = \mathbf{R}_+^\times$), we shall simply say *$X$-dévissage*.

The following commutative diagram of pointed spaces will hopefully make things easier to understand; at the beginning of every line, we have put the corresponding exact sequence of coherent sheaves (they live on the space T_i that lies on the line).

$$
\begin{array}{c}
\mathscr{L}_1 \to \pi_{1*}\mathscr{F}_V \to \mathscr{P}_1 \to 0 \\
\mathscr{L}_2 \to \pi_{2*}\mathscr{P}_1 \to \mathscr{P}_2 \to 0 \\
\vdots \\
\mathscr{L}_{r-1} \to \pi_{r-1,*}\mathscr{P}_{r-2} \to \mathscr{P}_{r-1} \to 0 \\
\mathscr{L}_r \to \pi_{r*}\mathscr{P}_{r-1} \to \mathscr{P}_r = 0
\end{array}
$$

$$
\begin{tikzcd}
 & & & & (\mathrm{Supp}(\mathscr{F}_V), y) \arrow[r, hook] \arrow[d, "\pi_1"] & (V, y) \arrow[dddd, two heads] \\
 & & & (\mathrm{Supp}(\mathscr{P}_1), t_1) \arrow[r, hook] \arrow[d, "\pi_2"] & (T_1, t_1) \arrow[dddr, two heads] & \\
 & & (\mathrm{Supp}(\mathscr{P}_2), t_2) \arrow[r, hook] & (T_2, t_2) \arrow[ddrr, two heads] & & \\
(\mathrm{Supp}(\mathscr{P}_{r-1}), t_{r-1}) \arrow[r, hook] \arrow[d, "\pi_r"] & (T_{r-1}, t_{r-1}) \arrow[drrrr, two heads] & & & & \\
(T_r, t_r) \arrow[rrrrr, two heads] & & & & & (X, x)
\end{tikzcd}
$$

8.2.5. Theorem. — *Let $Y \to X$ be a map between good k-analytic spaces, let $\mathscr{F}$ be a coherent sheaf on Y, let y be a point of* $\mathrm{Supp}(\mathscr{F})$*, and let x be its image in X. Assume that the germ (Y, y) is Γ-strict. Let $c = \mathrm{codepth}_y \mathscr{F}_{Y_x}$ and let $n = \dim_y \mathscr{F}_{Y_x}$. There exists a Γ-strict X-dévissage of $\mathscr{F}$ at y in dimensions belonging to $[n - c\ ;\ n]$ (recall that $c \leqslant n$ by 8.2.2 (2)).*

Proof. — According to Cor. 4.7 of [**Duc07b**], there exist an affinoid neighborhood Z of y in $\mathrm{Supp}(\mathscr{F})$, an affinoid domain T of a smooth X-analytic space of pure relative dimension n, and a finite map $\pi\colon Z \to T$ through which $Z \to X$ factorizes. Let us set $t = \pi(y)$. We can first assume, by shrinking T, that y is the only pre-image of t.

By Example 7.3.4, the smallest analytic domain of (T, t) through which the map $(Z, y) \to (T, t)$ factorizes is the whole of (T, t); since (Y, y) is Γ-strict, so is (Z, y) and Theorem 7.3.1 (3) then ensures that (T, t) is Γ-strict as well. By shrinking again T and Z, we can thus assume that T is Γ-strict; the k-affinoid space Z is then Γ-strict too by 3.5.7 (see also Remark 3.5.8).

Let V be any Γ-strict affinoid neighborhood of y in Y such that $V \cap \mathrm{Supp}(\mathscr{F}) \subset Z$. By the very definition of the topology on T, every point of T has a basis of open neighborhoods that are finite intersections of subsets of T defined by inequalities of the form $|g| \in I$, for g an analytic function on T and I an interval of $\mathbf{R}_+$ open in $\mathbf{R}_+$. It follows that every point also has a basis of *compact* neighborhoods that are finite intersections of subsets of T defined by inequalities of the form $|g| \in I$, for g an analytic function on T and I an compact interval of $\mathbf{R}_+$ with non-empty interior in $\mathbf{R}_+$; by density of $(\Gamma \cdot |k^\times|)^{\mathbf{Q}}$ in $\mathbf{R}_+^\times$, we can even restrict to such intervals with endpoints in $(\Gamma \cdot |k^\times|)^{\mathbf{Q}} \cup \{0\}$. By the above and by topological properness (and topological separatedness) of $T \to V$, there exists analytic functions $g_1, \ldots, g_m$ on T, and elements $r_1, \ldots, r_m, s_1, \ldots, s_m$ of $(\Gamma \cdot |k^\times|)^{\mathbf{Q}} \cup \{0\}$ with $r_i < s_i$ for every i, such that the Γ-strict affinoid domain T' of T defined by the inequalities $r_i \leqslant |g_i| \leqslant s_i$ for $i = 1, \ldots, m$ is a neighborhood of t and satisfies the inclusion $\pi^{-1}(T') \subset V \cap \mathrm{Supp}(\mathscr{F})$. For every i, choose a lifting h_i in $\mathscr{O}_Y(V)$ of the pull-back of g_i in $\mathscr{O}_Z(V \cap Z) = \mathscr{O}_Z(V \cap \mathrm{Supp}(\mathscr{F}))$; let V' be the Γ-strict affinoid domain of V defined by the inequalities $r_i \leqslant |h_i| \leqslant s_i$ for $i = 1, \ldots, m$. We then have $\pi^{-1}(T') = V' \cap Z$. Hence by replacing T with T', Z with $V' \cap Z$ and V with V', we may assume that $Z = V \cap \mathrm{Supp}(\mathscr{F})$. (Note that in the construction above, we may also require that T' be contained in any given neighborhood of t; this allows us if needed to modify T, Z and V so that T becomes arbitrary small.) We set $\mathscr{G} = \pi_*(\mathscr{F}_V)$, and $\delta = \dim_{\mathrm{Krull}} \mathscr{O}_{Z_x, y} = \dim_{\mathrm{Krull}} \mathscr{F}_{Y_x, y}$.

By finiteness of $Z_x \to T_x$, one has $\mathrm{centdim}(T_x, t) = \mathrm{centdim}(Z_x, y)$ (3.2.4). Since $\dim_y Z_x = \dim_t T_x = n$, Corollary 3.2.9 then ensures that

(a) $$\dim_{\mathrm{Krull}} \mathscr{O}_{T_x, t} = \dim_{\mathrm{Krull}} \mathscr{O}_{Z_x, y} = \delta.$$

The support of $\mathscr{G}$ is equal to $\pi(Z)$. It follows thus from 1.5.10 that

(b) $$\dim_t \mathscr{G}_{T_x} = \dim_t \pi(Z_x) = n$$

because Z_x is of dimension n at y. The fiber T_x being a smooth $\mathscr{H}(x)$-analytic space of pure dimension n, equality (b) implies that $(\mathrm{Supp}(\mathscr{G}))_x$ contains the connected component of t in T_x, and the annihilator of $\mathscr{G}_{T_x}$ is thus zero at t. Therefore

(c) $$\dim_{\mathrm{Krull}} \mathscr{G}_{T_x,t} = \dim_{\mathrm{Krull}} \mathscr{O}_{T_x,t} = \delta.$$

As $\pi^{-1}(t) = \{y\}$, one has $\mathscr{G}_{T_x,t} = \mathscr{F}_{Y_x,y}$ (Lemma 4.1.15 (1)). It follows then from [**EGA IV**$_1$], Chapt. 0, §16.4.8 that

(d) $$\mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{G}_{T_x,t} = \mathrm{depth}_{\mathscr{O}_{Y_x,y}} \mathscr{F}_{Y_x,y}.$$

In view of (c) this yields the equality

(e) $$\mathrm{codepth}_t \mathscr{G}_{T_x} = \mathrm{codepth}_y \mathscr{F}_{Y_x} = c.$$

We now argue by induction on c. Assume that $c = 0$. Then $\mathscr{G}_{T_x,t}$ is a finitely generated module of codepth 0 and of maximal Krull dimension over the regular local ring $\mathscr{O}_{T_x,t}$. It is thus free ([**EGA IV**$_1$], Chapt. 0, 17.3.4). Let $(f_i)_{1\leqslant i\leqslant r}$ be a family of sections of $\mathscr{G}$ over the affinoid space T such that $(f_i(t))$ is a basis of $\mathscr{G}_{\mathscr{H}(t)}$; set $\mathscr{L} = \mathscr{O}_T^r$ and consider the map $\mathscr{L} \to \mathscr{G}$ that sends $(a_1, \ldots, a_r)$ to $\sum a_i f_i$. By Nakayama's Lemma, this map is surjective at t; moreover, its restriction to T_x is *bijective* at t, because its stalk at t is a surjective map between free modules of the same finite rank over $\mathscr{O}_{T_x,t}$. Hence by suitably shrinking T (and all other data) we may assume that $\mathscr{L} \to \mathscr{G}$ is surjective, and that its restriction to T_x is bijective. We get this way a Γ-strict X-dévissage of $\mathscr{F}$ at y in dimension n.

Suppose now that $c > 0$, and that the theorem has been proved in codepth $< c$. Choose a point $z \in T_x$ such that z_x^{al} is the generic point of the connected component of T_x^{al} containing t_x^{al}. Since the support of $\mathscr{G}_{T_x}$ contains the connected component of t, the vector space $\mathscr{G}^{\mathrm{al}}_{T_x^{\mathrm{al}},\kappa(z_x^{\mathrm{al}})}$ is of positive dimension; let us call it r. Let $(f_i)_{1\leqslant i\leqslant r}$ be a family of sections of $\mathscr{G}$ over T such that $(f_i(z_x^{\mathrm{al}}))_i$ is a basis of the $\kappa(z_x^{\mathrm{al}})$-vector space $\mathscr{G}^{\mathrm{al}}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}}$. Set $\mathscr{L} = \mathscr{O}_T^r$ and consider the map $\mathscr{L} \to \mathscr{G}$ that sends $(a_1, \ldots, a_r)$ to $\sum a_i f_i$. Since $\mathscr{O}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}} = \kappa(z_x^{\mathrm{al}})$, the map $\mathscr{L}^{\mathrm{al}}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}} \to \mathscr{G}^{\mathrm{al}}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}}$ is an isomorphism; this implies that z belongs to $\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{G}_{T_x})$. Being a Zariski-specialization of z, the point t belongs to $\overline{\mathrm{Bij}(\mathscr{L}_{T_x} \to \mathscr{G}_{T_x})}^{T_x}$.

The scheme T_x^{al} being regular, $\mathscr{O}_{T_x^{\mathrm{al}},t_x^{\mathrm{al}}}$ is a domain whose fraction field is $\mathscr{O}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}}$. As $\mathscr{L}$ is a free $\mathscr{O}_T$-module, this implies that the map $\mathscr{L}^{\mathrm{al}}_{T_x^{\mathrm{al}},t_x^{\mathrm{al}}} \to \mathscr{L}^{\mathrm{al}}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}}$ is injective. Hence in the commutative diagram

$$\begin{tikzcd}
\mathscr{L}^{\mathrm{al}}_{T_x^{\mathrm{al}},t_x^{\mathrm{al}}} \arrow[r] \arrow[d] & \mathscr{G}^{\mathrm{al}}_{T_x^{\mathrm{al}},t_x^{\mathrm{al}}} \arrow[d] \\
\mathscr{L}^{\mathrm{al}}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}} \arrow[r] & \mathscr{G}^{\mathrm{al}}_{T_x^{\mathrm{al}},z_x^{\mathrm{al}}}
\end{tikzcd}$$

the bottom horizontal arrow is an isomorphism and the left vertical arrow is injective. It follows that the top horizontal arrow $\mathscr{L}^{\mathrm{al}}_{T_x^{\mathrm{al}},t_x^{\mathrm{al}}} \to \mathscr{G}^{\mathrm{al}}_{T_x^{\mathrm{al}},t_x^{\mathrm{al}}}$ is injective. The map $\mathscr{L}_{T_x,t} \to \mathscr{G}_{T_x,t}$ is thus injective. But it is not surjective. Indeed, if it were surjective it would be bijective and the codepth of $\mathscr{G}_{T_x,t}$ would then be equal to zero (the local ring $\mathscr{O}_{T_x,t}$ being regular), which would contradict the fact that $c > 0$.

Set $\mathscr{P} = \mathrm{Coker}\ (\mathscr{L} \to \mathscr{G})$. Since $\mathscr{L}_{T_x,t} \to \mathscr{G}_{T_x,t}$ is not surjective, t lies in $\mathrm{Supp}(\mathscr{P})_x$. Since $\mathrm{Supp}(\mathscr{P})_x$ is obviously included in $T_x \backslash \mathrm{Bij}(\mathscr{L}_{|T_x} \to \mathscr{G}_{T_x})$, we have $\dim_t \mathscr{P}_{T_x} < n$. Hence there exists a global section a on T_x of the annihilator ideal of $\mathscr{P}_{T_x}$ whose zero-locus contains no neighborhood of t. Since the annihilator of $\mathscr{P}_{T_x,t}$ contains the germ of a, it is non-zero, which yields the inequality

$$\text{(f)} \qquad \dim_{\mathrm{Krull}} \mathscr{P}_{T_x,t} < \delta.$$

By Thm. 16.7 of [**Mat86**], the depth of any non-zero finitely generated $\mathscr{O}_{T_x,t}$-module M is the smallest integer i such that $\mathrm{Ext}^i_{\mathscr{O}_{T_x,t}}(\kappa(t_x), M) \neq 0$. Since $r > 0$ and $\mathscr{O}_{T_x,t}$ is regular, $\mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{L}_{T_x,t} = \delta$; in particular, $\mathrm{Ext}^i_{\mathscr{O}_{T_x,t}}(\kappa(t_x), \mathscr{L}_{T_x}) = 0$ as soon as $i < \delta$. By considering the $\mathrm{Ext}^\bullet_{\mathscr{O}_{T_x,t}}(\kappa(t_x), \cdot)$ exact sequence associated with

$$0 \to \mathscr{L}_{T_x,t} \to \mathscr{G}_{T_x,t} \to \mathscr{P}_{T_x,t} \to 0,$$

we deduce from the above that $\mathrm{Ext}^i_{\mathscr{O}_{T_x,t}}(\kappa(t_x), \mathscr{G}_{T_x,t}) = \mathrm{Ext}^i_{\mathscr{O}_{T_x,t}}(\kappa(t_x), \mathscr{P}_{T_x,t})$ for every $i < \delta$. Now (f) implies that $\mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{P}_{T_x,t} < \delta$. Using again the caracterization of depth in terms of the Ext functors, it follows that

$$\text{(g)} \qquad \mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{P}_{T_x,t} = \mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{G}_{T_x,t} = \delta - c,$$

where the second equality comes from (e). In view of (f), we get the inequality

$$\text{(h)} \qquad \mathrm{codepth}_{\mathscr{O}_{T_x,t}} \mathscr{P}_{T_x,t} < c.$$

This allows us to apply the induction hypothesis. It ensures that $\mathscr{P}$ admits a Γ-strict X-dévissage at t, in dimensions belonging to

$$I := [\dim_t \mathscr{P}_{T_x} - \mathrm{codepth}_t \mathscr{P}_{T_x} \ ; \ \dim_t \mathscr{P}_{T_x}].$$

We are going to show show that $I \subset [n - c, n)$; by shrinking suitably V, Z, and T, the dévissage of $\mathscr{P}$ together with $V, T, \pi, \mathscr{L}, \mathscr{P}, \mathscr{L} \to \mathscr{G} = \pi_* \mathscr{F}$ will then provide a Γ-strict dévissage of $\mathscr{F}$ at y in dimensions belonging to $[n - c \ ; \ n]$.

As $\dim_t \mathscr{P}_{T_x} < n$, the interval I is strictly bounded above by n. Let us now prove that it is bounded below by $n - c$. One has

$$\begin{array}{rll}
& \dim_t \mathscr{P}_{T_x} - \mathrm{codepth}_t \mathscr{P}_{T_x} & \\
= & \dim_t \mathscr{P}_{T_x} - \dim_{\mathrm{Krull}} \mathscr{O}_{\mathrm{Supp}(\mathscr{P})_x, t} + \mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{P}_{T_x,t} & \\
= & \mathrm{centdim}(\mathrm{Supp}(\mathscr{P})_x, t) + \mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{P}_{T_x,t} & \text{by Corollary 3.2.9} \\
= & \mathrm{centdim}(T_x, t) + \mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{P}_{T_x,t} & \text{by 3.2.4} \\
= & \mathrm{centdim}(Z_x, y) + \mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{P}_{T_x,t} & \text{by 3.2.4} \\
= & \mathrm{centdim}(Z_x, y) + \mathrm{depth}_{\mathscr{O}_{T_x,t}} \mathscr{G}_{T_x,t} & \text{by (g)} \\
= & \mathrm{centdim}\ (Z_x, y) + \mathrm{depth}_{\mathscr{O}_{Y_x,y}} \mathscr{F}_{Y_x,y} & \text{by (d)} \\
= & \dim_y Z_x - \dim_{\mathrm{Krull}} \mathscr{O}_{Z_x,y} + \mathrm{depth}_{\mathscr{O}_{Y_x,y}} \mathscr{F}_{Y_x,y} & \text{by Corollary 3.2.9} \\
= & n - c. &
\end{array}$$

□

8.3. Flatness can be checked naively in the inner case

Let $Y \to X$ be a map between good k-analytic spaces and let $\mathscr{F}$ be a coherent module on Y. We want to give some criteria (in terms of a dévissage at y) for $\mathscr{F}$ to be X-flat at a given point y of Y, and to use them to show that in the boundaryless (and, more generally, overconvergent) case, naive X-flatness at y is equivalent to X-flatness at y (this means that it is preserved under arbitrary good base change, including ground field extension); this fact had already been proved by Berkovich, using a completely different method, in some unpublished work.

8.3.1. Definition. — Let $Y \to X$ be a map between k-analytic spaces and let $\mathscr{F}$ be a coherent module on Y. Let y be a point of Y. We shall say that $\mathscr{F}$ is *X-overconvergent at y* if there exist an analytic neighborhood W of y in Y, an X-isomorphism between W and an analytic domain of a *boundaryless* X-space W', and a coherent sheaf $\mathscr{G}$ on W' such that $\mathscr{F}_W \simeq \mathscr{G}_W$.

8.3.2. Remark. — If y belongs to $\mathrm{Int}(Y/X)$, then $\mathscr{F}$ is automatically X-overconvergent at y.

8.3.3. Remark. — We shall only use this notion when Y and X are good. In this case, W and W' can be chosen to be affinoid (note that a boundaryless space over a good space is itself good *by definition*, see [**Ber93**], page 34).

8.3.4. Theorem. — *Let $Y \to X$ be a morphism between good k-analytic spaces, let y be a point of Y, and let $\mathscr{F}$ be a coherent sheaf at Y.*

(1) *If $y \notin \mathrm{Supp}(\mathscr{F})$, then $\mathscr{F}$ is X-flat at y.*
(2) *Assume that y belongs to $\mathrm{Supp}(\mathscr{F})$ and let $(V, \{T_i, \pi_i, t_i, \mathscr{L}_i, \mathscr{P}_i\}_{i \in \{1,\dots,r\}})$ be an X-dévissage of $\mathscr{F}$ at y (such a dévissage always exists by Theorem 8.2.5). The following are equivalent.*
    (i) *The coherent sheaf $\mathscr{F}$ is X-flat at y.*

(ii) *The arrow* $\mathscr{L}_1 \to \pi_{1*}\mathscr{F}_V$ *is injective at* t_1 *and for every* $i \geqslant 2$, *the arrow* $\mathscr{L}_i \to \pi_{i*}\mathscr{P}_{i-1}$ *is injective at* t_i.

(3) *Under the assumptions of* (2), *suppose moreover that* $\mathscr{F}$ *is* X*-overconvergent at* y*; e.g.,* $y \in \mathrm{Int}(Y/X)$. *Assertions* (i) *and* (ii) *are then equivalent to:*

(iii) *The coherent sheaf* $\mathscr{F}$ *is naively* X*-flat at* y.

Proof. — Assertion (1) is obvious (and was written down only for the sake of completeness). By Lemma 4.1.15, naive X-flatness (resp. X-flatness) of $\mathscr{F}$ at y is equivalent to that of $\pi_{1*}\mathscr{F}_V$ at t_1 ; for the same reason, if $i \leqslant r-1$, then naive X-flatness (resp. X-flatness) of $\mathscr{P}_i$ at t_i is equivalent to that of $\pi_{(i+1)*}\mathscr{P}_i$ at t_{i+1}. Hence the equivalence (i) $\iff$ (iii), and the equivalence (i) $\iff$ (ii) $\iff$ (iii) *when* $y \in \mathrm{Int}(Y/X)$, follow from a repeated application of Proposition 8.1.10, once one has remarked that since $\mathscr{P}_r = 0$, it is X-flat at t_r.

It remains to show that (iii)$\Rightarrow$(i) under the assumption that $\mathscr{F}$ is X-overconvergent at y. By shrinking Y, we may assume that it is an affinoid domain of a (relatively) boundaryless X-analytic space Y', and $\mathscr{F} = \mathscr{G}_Y$ for some coherent sheaf $\mathscr{G}$ on Y'. Assume that $\mathscr{G}_Y$ is naively X-flat at y. This implies that $\mathscr{G}$ is naively X-flat at y (4.1.7). By the boundaryless case already established, $\mathscr{G}$ is X-flat at y. Therefore, $\mathscr{G}_Y$ is X-flat at y (4.1.12). □

8.3.5. Remark. — If properties (i) and (ii) are satisfied, it turns out that the map $\mathscr{L}_r \to \pi_{r*}\mathscr{P}_{r-1}$ (or $\mathscr{L}_1 \to \pi_{1*}\mathscr{F}_V$ when $r = 1$) is *bijective at* t_r, because it is injective by (ii) and surjective since its cokernel $\mathscr{P}_r$ is zero.

Let us now give three easy (but important) consequences of Theorem 8.3.4. The first one shows that checking flatness at a given point of a good analytic space does not actually require to consider all possible base changes; the second one is an improvement of Theorem 4.2.5; the third one explains how flatness can be checked fiberwise in some cases (this is kind of an analytic analogue of [**SGA 1**], Exposé IV, Cor. 5.7, and our proof consists in reducing straightforwardly to the latter result).

8.3.6. Theorem. — *Let* $Y \to X$ *be a morphism of good* k*-analytic spaces, let* $\mathscr{F}$ *be a coherent sheaf on* Y *and let* y *be a point of* Y. *Let* L *be an analytic extension of* k, *and let* z *be an* L*-rigid point of* Y_L *lying above* y. *If* $\mathscr{F}_L$ *is naively* X_L*-flat at* z, *then* $\mathscr{F}$ *is* X*-flat at* y.

Proof. — Since z is an L-rigid point, it belongs to $\mathrm{Int}(Y_L/X_L)$. As $\mathscr{F}_L$ is naively X_L-flat at z by assumption, Theorem 8.3.4 ensures that $\mathscr{F}_L$ is X_L-flat at z; the coherent sheaf $\mathscr{F}$ is then X-flat at y by Proposition 4.5.6. □

8.3.7. Theorem. — *Let* $Y \to X$ *be a morphism between* k*-affinoid spaces and let* Z *be a closed analytic subspace of* Y *such that* $Z \to X$ *is finite. Let* y *be a point of* Z *and let* $\mathscr{F}$ *be a coherent sheaf on* Y. *Assume that* $\mathscr{F}^{\mathrm{al}}$ *is* X^{al}*-flat at* y^{al}. *The coherent sheaf* $\mathscr{F}$ *is then* X*-flat at* y.

Proof. — Theorem 4.2.5 tells us that $\mathscr{F}$ is naively X-flat at y. As $Z \to X$ is finite, it is inner at y, hence $Y \to X$ is inner at y too by 1.3.21 (3); in view of Theorem 8.3.4, $\mathscr{F}$ is X-flat at y. □

8.3.8. Theorem. — *Let X be an analytic space, and let $Z \to Y$ be a morphism of X-analytic spaces. Let z be a point of Z, and let y and x denote the images of z in Y and X, respectively; we assume that Y is X-flat at y. Let $\mathscr{F}$ be a coherent sheaf on Z. The following are equivalent:*

(i) *$\mathscr{F}$ is Y-flat at z;*

(ii) *$\mathscr{F}$ is X-flat at z, and $\mathscr{F}_{Z_x}$ is Y_x-flat at z.*

Proof. — By arguing G-locally on X, Y, and Z, we may and do assume that all of them are good. Proposition 4.5.6 allows us to enlarge the ground field before proving the theorem; we thus can assume that x, y, and z are rigid. In view of Theorem 8.3.6 and since

$$\mathscr{O}_{Y_x,y} = \mathscr{O}_{Y,y}/\mathfrak{m}_x\mathscr{O}_{Y,y} \text{ and } \mathscr{O}_{Z_x,z} = \mathscr{O}_{Z,z}/\mathfrak{m}_x\mathscr{O}_{Z,z}$$

because z is rigid, it suffices to prove that the following are equivalent:

(iii) $\mathscr{F}_z$ is flat over $\mathscr{O}_{Y,y}$;

(iv) $\mathscr{F}_z$ is flat over $\mathscr{O}_{X,x}$ and $\mathscr{F}_z/\mathfrak{m}_x\mathscr{F}_z$ is flat over $\mathscr{O}_{Y,y}/\mathfrak{m}_x\mathscr{O}_{Y,y}$.

As Y is X-flat at y, it is naively X-flat at y, which means that $\mathscr{O}_{Y,y}$ is flat over $\mathscr{O}_{X,x}$. The equivalence (iii) $\Longleftrightarrow$ (iv) then comes from a direct application of Cor. 5.9 of [**SGA 1**], Exposé IV with $A = \mathscr{O}_{X,x}$, $B = \mathscr{O}_{Y,y}$, $C = \mathscr{O}_{Z,z}$, and $M = \mathscr{F}_z$. □

8.4. The relative CM property

In this section we shall use the CM property of a coherent sheaf at a given point of an analytic space; this has to be understood in the sense of Lemma-Definition 2.4.3. Note that being CM at a point amounts to being of codepth zero at it, in the sense of Definition 8.2.1.

8.4.1. Definition. — Let $Y \to X$ be a morphism of k-analytic spaces, and let $\mathscr{F}$ be a coherent sheaf on Y. Let y be a point of Y and let x be its image on X. We say that $\mathscr{F}$ is CM *over* X at y if $\mathscr{F}_{Y_x}$ is CM at y and $\mathscr{F}$ is X-flat at y. We say that Y is CM over X at y if $\mathscr{O}_Y$ is. We say "CM over X" to mean "CM over X at every point of Y".

8.4.2. Remark. — Let $Y \to X$ be a morphism between k-analytic spaces and let $\mathscr{F}$ be a coherent sheaf on Y. Let y be a point of Y, let V be an analytic domain of Y containing y, and let U be an analytic domain of X containing the image of V. Then $\mathscr{F}$ is CM over X at y if and only if $\mathscr{F}_V$ is CM over U at y: this follows from the good behavior of flatness and of the CM property with respect to the restriction to analytic domains (see 4.1.12 for flatness and Remark 2.4.2 for the CM property).

8.4.3. Example. — Let $Y \to X$ be a morphism of k-analytic spaces, let y be a point of Y, and let x be its image in X. Let $\mathscr{F}$ be a coherent sheaf on Y. Assume that $Y \to X$ is quasi-finite at y. We then have $\dim_y Y_x = 0$, which implies that $\mathscr{F}_{Y_x}$ is of codepth 0 at y (8.2.2 (2)); therefore $\mathscr{F}$ is CM over X at y if and only if it is X-flat at y.

8.4.4. Example. — Let $Y \to X$ be a morphism of k-analytic spaces. Let y be a point of Y. If Y is quasi-smooth over X at y, then $\mathscr{O}_Y$ is CM at y: this follows from 8.2.2 (3) together with Theorem 5.3.4.

Our purpose is now to show by using dévissages that a coherent sheaf is relatively CM at a given point if and only if it can be written around this point as a "combination" of a relatively CM coherent sheaf as in Example 8.4.3 (with $Y \to X$ even *finite* at y) and of another one of the kind described in Example 8.4.4.

8.4.5. Lemma. — *Let $Y \to T$ be a finite morphism and let $T \to X$ be a quasi-smooth morphism. Let $\mathscr{F}$ be a T-flat coherent sheaf on Y. The sheaf $\mathscr{F}$ is CM over X.*

Proof. — One can assume that Y, T, and X are k-affinoid. Since $T \to X$ is quasi-smooth, it is X-flat, whence the X-flatness of $\mathscr{F}$. Let x be a point of X. We are going to show that $\mathscr{F}_{Y_x}$ is CM. Let y be a pre-image of x in Y and let t be its image on T_x. The ring $\mathscr{O}_{T_x,t}$ is regular by quasi-smoothness of T_x,so it is CM. The morphism Spec $\mathscr{O}_{Y_x,y} \to$ Spec $\mathscr{O}_{T_x,t}$ is finite, hence has zero-dimensional fibers; the finite module $\mathscr{F}_{Y_x,y}$ is $\mathscr{O}_{T_x,t}$-flat by T-flatness of $\mathscr{F}$. It follows then from Prop. 6.4.1 (ii) of [**EGA IV**$_2$] that $\mathscr{F}_{Y_x,y}$ is CM. $\square$

8.4.6. Theorem. — *Let X be a good k-analytic space, let Y be a good X-analytic space, and let y be a point of Y. Assume that the germ (Y, y) is Γ-strict. Let n be an integer. Let $\mathscr{F}$ be a coherent sheaf on Y such that $y \in \mathrm{Supp}(\mathscr{F})$ and let U be the set of points of Y at which $\mathscr{F}$ is CM over X. Assume that $y \in U$ and $\dim_y \mathscr{F}_{Y_x} = n$. There exist:*

- *a Γ-strict k-affinoid neighborhood V of y in Y that is contained in U;*
- *a Γ-strict k-affinoid domain T of a smooth X-space of pure relative dimension n;*
- *a finite X-morphism $\pi : \mathrm{Supp}(\mathscr{F}_V) \to T$ with respect to which $\mathscr{F}_V$ is T-flat.*

Proof. — Let x be the image of y in X. Since $\mathscr{F}$ is CM over X at y, one has codepth$\mathscr{F}_{Y_x,y} = 0$. By Theorem 8.2.5, the coherent sheaf $\mathscr{F}$ admits a Γ-strict X-dévissage $(V, T, \pi, t, \mathscr{L}, \mathscr{P} = 0)$ at y in dimension n. As $\mathscr{F}$ is X-flat at y, Remark 8.3.5 ensures that $\mathscr{L} \to \pi_*\mathscr{F}_V$ is bijective at t. We can hence shrink the data so that $\pi_*\mathscr{F}_V$ is a free $\mathscr{O}_T$-module, which implies T-flatness of $\mathscr{F}_V$ by Proposition 4.3.1. And it follows from Lemma 8.4.5 that $V \subset U$. $\square$

8.4.7. — Let X be a k-analytic space, let Y be an X-analytic space and let $\mathscr{F}$ be a coherent sheaf on Y. Let U be the set of points of Y at which $\mathscr{F}$ is CM over X.

(1) It follows from Theorem 8.4.6 that U is an open subset of Y. We shall see later that it is even *Zariski*-open (Theorem 10.7.2).

(2) Assume that $\mathscr{F}$ is X-flat, and let x be a point of X. By our flatness assumption on $\mathscr{F}$, the intersection $U \cap Y_x$ is the CM locus of $\mathscr{F}_{Y_x}$. It is a Zariski-open subset of Y_x (Lemma 2.4.9 (2)), which is dense. Indeed, to show this one can assume that Y is affinoid. Now if η denotes the generic point of an irreducible component of Y_x^{al}, then $\mathscr{F}^{\mathrm{al}}_{Y_x^{\mathrm{al}},\eta}$ is CM because $\mathscr{O}_{Y_x^{\mathrm{al}},\eta}$ is artinian; this fact together with affinoid GAGA (Lemma 2.4.6) implies our claim.

CHAPTER 9

QUASI-FINITE MULTISECTIONS AND IMAGES OF MAPS

A celebrated theorem by Raynaud asserts the following, in our language [(1)]: if $|k^\times| \neq \{1\}$ and if $\varphi \colon Y \to X$ is a flat morphism between strictly k-affinoid spaces, then $\varphi(Y)$ is a strict analytic domain of X, cf. [**BL93b**], Cor. 5.11. In this chapter, we slighlty generalize this result and prove the following (Theorem 9.2.1): if $\varphi \colon Y \to X$ is map between k-affinoid spaces, and if Y is the support of an X-flat coherent sheaf $\mathscr{F}$, then $\varphi(Y)$ is an analytic domain of X, which is Γ-strict whenever Y is Γ-strict (as usual, Γ denotes a subgroup of $\mathbf{R}_+^\times$ which is non-trivial if k is trivially valued). Moreover, our methods differ from Raynaud's (we replace the use of formal schemes by that of Temkin's graded reduction, and we do not perform any flattening), hence we provide a new proof of Raynaud's theorem.

Let us now roughly explain how we proceed. We first consider the case where φ is quasi-finite (Proposition 9.1.1). The coherent sheaf $\mathscr{F}$ is then CM over X, and Theorem 8.4.6 enables us to reduce to the case where φ is a quasi-étale map. Then by arguing locally and using Theorem 7.3.1 (which ensures the existence of a smallest analytic domain containing the image of a morphism of analytic germs, described through Temkin's reduction), we reduce to the situation where φ is finite and étale, in which case $\varphi(Y)$ is a union of connected components of X, and we are done.

To handle the general case we first reduce, by performing a ground field extension to k_r for some suitable k-free polyradius r and using the corresponding Shilov section, to the case where $|k^\times| \neq \{1\}$ and Y is strict. We then prove (Theorem 9.1.3) that under these assumptions, there exists a strictly k-affinoid space X', a quasi finite map $\psi \colon X' \to X$, and an X-morphism $\sigma \colon X' \to Y$ such that:

(a) $\mathscr{F}$ is CM over X at every point of $\sigma(X')$;
(b) $\sigma^*\mathscr{F}$ is X-flat;
(c) $\psi(X') = \varphi(Y)$.

1. Raynaud's result is written in the rigid-analytic language, with the corresponding notion of flatness; the consistency with our notion will be established later; see Corollary 10.3.3

Now assertion (c) together with the quasi-finite case already proven ensures that $\varphi(Y)$ is a strictly k-analytic domain of X.

Let us say a few words about the existence of a quasi-finite multisection of φ satisfying (a), (b) and (c), which seems to us of independent interest. It is the analogue of a classical scheme-theoretic result (see [**EGA IV**$_4$], Prop. 19.2.9), but it is more involved, because of boundary phenomena, and also because in analytic geometry, the Zariski topology of a fiber is in general finer than the topology induced by the Zariski topology of the ambient space. The core of our proof is a local construction (which can afterward easily be globalized, by compactness of Y); it is the object of an independent theorem (Theorem 9.1.2), whose proof uses the following ingredients:

- Once again, the existence of a smallest analytic domain containing the image of a morphism of analytic germs and its description through Temkin's reduction (Theorem 7.3.1). In fact when the source germ is strict (which is the case here, by strictness of Y), one can get a more precise description, involving finitely many closed points of some "residue scheme" attached to the situation (Theorem 7.1.4), and this is absolutely crucial for our purposes: these closed points precisely indicate in which "directions" one has to draw multisections if one wants to be sure that they will cover the whole image of our germ.
- The local structure of coherent sheaves CM over the ground space (Theorem 8.4.6).
- The "quasi-finite version of Raynaud's theorem", already proven (this is the aforementioned Proposition 9.1.1).
- The fact that over a non-trivially valued field, a smooth morphism admits étale multisections locally on its image (Corollary 6.2.7).

We end the chapter by recording the following extra-results about the images of maps, which we deduce from (our version of) Raynaud's theorem; other ingredients are the coincidence of the topological and analytic interiors for the inclusion of an analytic domain (this is used for (1), and for deducing (2b) from (2a)), and our local description of morphisms of relative dimension d (Cor. 4.7 of [**Duc07b**]) for (2a).

(1) Let $\varphi\colon Y \to X$ be a boundaryless morphism. If Y is the support of an X-flat sheaf, then φ is open (Theorem 9.2.3; this had been proved by Berkovich in some unpublished work).

(2) Let n and d be two non-negative integers, and let $\varphi\colon Y \to X$ be a morphism between analytic spaces. Assume that X is normal and purely n-dimensional, φ is of pure relative dimension d, and Y is purely $(n+d)$-dimensional. Then:
   - (2a) If Y and X are affinoid, $\varphi(Y)$ is a compact analytic domain of X, which is Γ-strict whenever Y is Γ-strict (Theorem 9.2.2).
   - (2b) If φ is boundaryless, it is open.

9.1. Flat, quasi-finite multisections of flat maps

9.1.1. Proposition. — *Let Y be a Γ-strict quasi-compact k-analytic space, let X be a separated k-analytic space, and let $\varphi\colon Y \to X$ be a morphism. Let $\mathscr{F}$ be a coherent sheaf on Y which is X-flat and whose support is quasi-finite over X. The image $\varphi(\operatorname{Supp}(\mathscr{F}))$ is a Γ-strict compact analytic domain of X.*

Proof. — We can replace Y with the support of $\mathscr{F}$; i.e., we can assume that the support of $\mathscr{F}$ is equal to Y. We first make some reductions, which are allowed because we can argue G-locally on Y since the latter is quasi-compact.

(a) By Proposition 7.3.6, $\varphi(Y)$ is contained in a compact Γ-strict analytic domain of X; hence we can assume that both Y and X are Γ-strict and affinoid.
(b) Since $\mathscr{F}$ is X-flat with quasi-finite support, it is CM over X (Example 8.4.3). By Theorem 8.4.6, we can thus assume that there exist a Γ-strict k-affinoid quasi-étale X-space T, and a factorization of $Y \to X$ through a finite map $\pi\colon Y \to T$ such that $\pi_*\mathscr{F}$ is a free $\mathscr{O}_T$-module of positive rank. The latter condition implies that $\pi(Y) = T$. Replacing Y with T, we can assume that $Y \to X$ is quasi-étale.
(c) By Theorem 5.4.6, we can suppose that the quasi-étale map $Y \to X$ can be written as a composition $Y \hookrightarrow X' \to X$ where $Y \hookrightarrow X'$ identifies Y with a Γ-strict affinoid domain of X', where X' is connected and where $X' \to X$ factorizes through a finite étale map from X' to a connected Γ-strict affinoid domain Z of X. Let X'' be a connected finite Galois covering of Z dominating X'. One can replace X by Z and Y by its preimage on X''; the union of all Galois conjugates of Y is then a Γ-strict compact analytic domain of X'' (possibly non-affinoid) whose image on X coincides with that of Y.

We thus now assume that X is a Γ-strict connected k-affinoid space and that Y is a (possibly non-affinoid) Galois-invariant Γ-strict compact analytic domain of a finite connected Galois cover X'' of X. Let y be a point of Y and let $x = \varphi(y)$. Let G be the set-theoretic stabilizer of y inside $\operatorname{Gal}(X''/X)$; since Y is Galois invariant, G stabilizes the germ (Y, y). By usual Galois theory, $\mathscr{H}(y)$ is a Galois extension of $\mathscr{H}(x)$ with group G. It follows then from A.4.12 that $\widetilde{\mathscr{H}(y)}^{\Gamma}$ is a normal graded extension of $\widetilde{\mathscr{H}(x)}^{\Gamma}$ (A.3.6) and the natural map $G \to \operatorname{Gal}(\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{\mathscr{H}(x)}^{\Gamma})$ is surjective. In view of 7.2.6, the continuous map $\mathbf{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{k}^{\Gamma}} \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$ identifies topologically $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$ with $\mathbf{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{k}^{\Gamma}}/\operatorname{Gal}(\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{\mathscr{H}(x)}^{\Gamma})$.

Let (U, x) be the smallest analytic domain of the germ (X, x) through which the map $(Y, y) \to (X, x)$ factorizes; it exists by Theorem 7.3.1, which also ensures that (U, x) is Γ-strict and its reduction $\widetilde{(U,x)}^{\Gamma}$ is the image of $\widetilde{(Y,y)}^{\Gamma}$ on $\mathbf{P}_{\widetilde{\mathscr{H}(x)}^{\Gamma}/\widetilde{k}^{\Gamma}}$. As G stabilizes (Y, y) and surjects onto $\operatorname{Gal}(\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{\mathscr{H}(x)}^{\Gamma})$, the group $\operatorname{Gal}(\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{\mathscr{H}(x)}^{\Gamma})$ stabilizes $\widetilde{(Y,y)}^{\Gamma}$; therefore the pre-image of $\widetilde{(U,x)}^{\Gamma}$

inside $\mathbf{P}_{\widetilde{\mathscr{H}(y)}^{\Gamma}/\widetilde{k}^{\Gamma}}$ is precisely $\widetilde{(Y,y)}^{\Gamma}$. As a consequence, the map $(Y,y)\to(U,x)$ is boundaryless by the criterion 3.5.9 (2). Being quasi-étale, it is then étale by Remark 5.4.9. Therefore, there exist a Γ-strict compact analytic neighborhood V of x in U and a compact analytic neighborhood W of y in Y such that φ induces a finite étale map $W\to V$ (note that W is Γ-strict by 3.5.7, but we do not need this). The image $\varphi(W)$ is now a finite union of connected components of V; in particular, it is a Γ-strict compact analytic domain of X. This ends the the proof due to the compactness of Y. □

We are now going to state and prove our results on the existence of flat, quasi-finite multisections (for maps whose source space is the support of a coherent sheaf flat over the target).

9.1.2. Theorem **(Existence of flat, quasi-finite multisections: the local case)** *Assume that $|k^{\times}|\neq\{1\}$, and let $\varphi\colon Y\to X$ be a morphism of k-analytic spaces, with Y strict and X separated. Let $\mathscr{F}$ be an X-flat coherent sheaf on Y; let y be a point of* $\mathrm{Supp}(\mathscr{F})$ *and let x be its image in X. Denote by Z the set of points of Y at which $\mathscr{F}$ is CM over X (this is an open subset of Y by 8.4.7) and by (U,x) the smallest analytic domain of (X,x) through which $(\mathrm{Supp}(\mathscr{F}),y)\to(X,x)$ factorizes (see Theorem 7.3.1).*

There exist $r\geqslant 1$, quasi-finite maps

$$\psi_1\colon X_1\to X,\dots,\psi_r\colon X_r\to X,$$

and X-morphisms

$$\sigma_1\colon X_1\to Z\cap\mathrm{Supp}(\mathscr{F}),\dots,\sigma_r\colon X_r\to Z\cap\mathrm{Supp}(\mathscr{F})$$

such that:

(1) *For every j, the space X_j is compact and strictly k-affinoid, and the point x has a unique pre-image x_j on X_j.*
(2) *For every j, the coherent sheaf $\sigma_j^*\mathscr{F}$ is X-flat, and $\psi_j(X_j)$ is thus a compact strictly k-analytic domain of X (by Proposition 9.1.1).*
(3) *One has $(U,x)=\bigcup(\psi_j(X_j),x)$.*

Moreover:

(A) *If $Y\to X$ is quasi-smooth at y and if $\mathscr{F}=\mathscr{O}_Y$, the ψ_j's can be chosen to be quasi-étale.*
(B) *If $Y\to X$ is boundaryless at y (which implies that $(U,x)=(X,x)$; see Example 7.3.4) and if the germs (Y,y) and (X,x) are good, one can take $r=1$, and ψ_1 inner, hence finite, at x_1.*

Proof. — By replacing Y with $\mathrm{Supp}(\mathscr{F})$, we may assume that $Y=\mathrm{Supp}(\mathscr{F})$. We are first going to reduce all assertions to the case where both Y and X are strictly affinoid, by arguing locally or G-locally (hence we shall implicitly use the good behavior of flatness and of the "relative" 'CM property with respect to restriction to analytic domains; see 4.1.12 and Remark 8.4.2).

Let us begin with assertion (B). So, we assume that Y and X are good and $Y \to X$ is inner at y. As noted in the statement of the theorem, this implies in view of Example 7.3.4 that $(U, x) = (X, x)$; strictness of (Y, y) then implies that of (X, x) by Theorem 7.3.1. We can thus shrink Y and X so that both are strictly k-affinoid.

Let us now come to the other assertions (so, we do not assume anymore that Y and X are good nor that φ is inner at y). By replacing Y with a strictly analytic compact neighborhood of y, one can assume that it is compact. Now, as X is separated, by Proposition 7.3.6 $\varphi(Y)$ is contained in a compact, strictly analytic domain X_0 of X; as $(U, x) \subset (X_0, x)$ we can replace X with X_0, and hence reduce to the case where X itself is strict. The point x thus has a neighborhood in X that is a finite union of strictly affinoid domains containing x; all assertions involved are G-local on the germ (X, x), so we can assume that X itself is strictly k-affinoid. The point y has a neighborhood in Y that is a finite union of strictly affinoid domains containing it; all assertions involved being G-local on the germ (Y, y), one can assume that Y itself is strictly k-affinoid.

Local convention. — *As the proof will involve from now on only strictly k-analytic spaces, it will be sufficient to consider* non-graded *reductions; therefore, in order to simplify notation, we shall write* for the rest of the proof $\widetilde{(Y, y)}, \widetilde{k}$*, etc., to denote* $\widetilde{(Y, y)}^1, \widetilde{k}^1$*, etc.*

9.1.2.1. — Let A (resp. B) be the algebra of analytic functions on X (resp. Y). Let A° be the subring of A that consists of functions whose spectral semi-norm is bounded above by 1, let $A^{\circ\circ}$ be the ideal of A° that consists of functions whose spectral semi-norm is *strictly* bounded above by 1, and let $\widetilde{A}$ be the quotient $A^\circ/A^{\circ\circ}$; we define $B^\circ, B^{\circ\circ}$ and $\widetilde{B}$ analogously; both $\widetilde{k}$-algebras $\widetilde{A}$ and $\widetilde{B}$ are finitely generated ([**BGR84**], 6.3.4, Cor. 3). We denote by A' (resp. B') the image of the natural evaluation map $\widetilde{A} \to \widetilde{\mathscr{H}(x)}$ (resp. $\widetilde{B} \to \widetilde{\mathscr{H}(y)}$); we denote by B'' the subring of $\widetilde{\mathscr{H}(y)}$ generated by $\widetilde{\mathscr{H}(x)}$ and B'. By Temkin's definition of the (non-graded) reduction of a strict analytic germ ([**Tem00**]; see also Remark 3.5.11), one has

$$\widetilde{(X, x)} = \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}\{A'\} \text{ and } \widetilde{(Y, y)} = \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{B'\}.$$

Let $f_1, \ldots, f_n$ be elements of B° whose images generate the $\widetilde{k}$-algebra $\widetilde{B}$. The $\widetilde{k}$-algebra B' is then generated by $\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}$, whence the equality

$$\widetilde{(Y, y)} = \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{\widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)}\}.$$

The inner case. If φ is inner at y, then $\widetilde{(Y, y)}$ is equal to the pre-image of $\widetilde{(X, x)}$ in $\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$; in other words, $\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{B'\} = \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}\{A'\}$, which implies that B' is integral over A'; in this case, B'' is a fortiori algebraic over $\widetilde{\mathscr{H}(x)}$, hence is a field.

9.1.2.2. — By Theorem 7.1.4, there exist finitely many closed points $\mathsf{y}_1, \ldots, \mathsf{y}_m$ of Spec B'' such that

$$\widetilde{(U,x)} = \bigcup p_j \left(\mathbf{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}} \{ \widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_n(y)}(\mathsf{y}_j) \} \right),$$

where p_j denotes the natural continuous map $\mathbf{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}} \to \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ for every j. Set $\mathsf{U}_j = p_j \left(\mathbf{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}} \{ \widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_n(y)}(\mathsf{y}_j) \} \right) \subset \mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ for every j; by Proposition 7.1.3, U_j is open and quasi-compact. For every j, choose a compact strictly analytic domain U_j of X that contains x and satisfies the equalities $\widetilde{(U_j,x)} = \mathsf{U}_j$. Since $\widetilde{(U,x)}$ is the union of the U_j's, we have the equality

$$(U,x) = \bigcup_j (U_j, x).$$

The inner case. If φ is inner at y, then as B'' is a field, $m = 1$ and y_1 is the only point of Spec B''. It follows that $\mathbf{P}_{\kappa(\mathsf{y}_1)/\widetilde{k}} \{ \widetilde{f_1(y)}(\mathsf{y}_1), \ldots, \widetilde{f_n(y)}(\mathsf{y}_1) \}$ is nothing but $\mathbf{P}_{B''/\widetilde{k}} \{ \widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)} \}$. But since $\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}} \{ \widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)} \}$ is, under our innerness assumption, the pre-image of $\widetilde{(X,x)}$ in $\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{k}}$, the open subset $\mathbf{P}_{B''/\widetilde{k}} \{ \widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)} \}$ of $\mathbf{P}_{B''/\widetilde{k}}$ is the pre-image of $\widetilde{(X,x)}$ in $\mathbf{P}_{B''/\widetilde{k}}$. One thus has

$$\mathbf{P}_{\kappa(\mathsf{y}_1)/\widetilde{k}} \{ \widetilde{f_1(y)}(\mathsf{y}_1), \ldots, \widetilde{f_n(y)}(\mathsf{y}_1) \} = p_1^{-1}(\widetilde{(X,x)}).$$

9.1.2.3. — We fix an integer j belonging to $\{1, \ldots, m\}$. Let R be the subring of $\mathscr{O}_{X,x}$ consisting of functions f such that $|f(x)| \leqslant 1$. For any $i \in \{1, \ldots, n\}$, let P_i be a polynomial in $R[T_1, \ldots, T_i]$ that is monic in T_i and is such that $\widetilde{P}_i(\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_{i-1}(y)}(\mathsf{y}_j), T)$ is the minimal polynomial of $\widetilde{f_i(y)}(\mathsf{y}_j)$ over $\widetilde{\mathscr{H}(x)}[\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_{i-1}(y)}(\mathsf{y}_j)]$ (by $\widetilde{P}_i$ we denote of course the image of P_i under the natural map $R[T_1, \ldots, T_i] \to \widetilde{\mathscr{H}(x)}[T_1, \ldots, T_i]$). Let D be a strictly affinoid neighborhood of x in X on which all the coefficients of the P_i's are defined. Let Ω be the open subset of $Y \times_X D$ defined as the locus of simultaneous validity of the inequalities

$$|P_1(f_1)| < 1, |P_2(f_1, f_2)| < 1, \ldots, |P_n(f_1, \ldots, f_n)| < 1.$$

Let us prove by contradiction that $\Omega_x \neq \varnothing$. Suppose that $\Omega_x = \varnothing$. Let I be the subset of $\{1, \ldots, n\}$ consisting of integers i such that $|P_i(f_1(y), \ldots, f_i(y))| = 1$ (a priori, this absolute value is *at most* 1). For every $i \in I$, let Y_i be the affinoid domain of Y_x defined by the condition $|P_i(f_1, \ldots, f_i)| = 1$. Under our assumption that $\Omega_x = \varnothing$, the union of the Y_i's for $i \in I$ is a neighborhood of y in Y_x. We thus have $\widetilde{(Y_x, y)} = \bigcup_{i \in I} \widetilde{(Y_i, y)}$. Let us describe both terms of this equality.

- By 3.5.9, one has $\widetilde{(Y_x, y)} = \mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{\mathscr{H}(x)}} \{ \widetilde{f_1(y)}, \ldots, \widetilde{f_n(y)} \}$.

- If i is any element of I, then $\widetilde{(Y_i,y)}$ is equal to

$$\mathbf{P}_{\widetilde{\mathscr{H}(y)}/\widetilde{\mathscr{H}(x)}}\{\widetilde{f_1(y)},\ldots,\widetilde{f_n(y)},\widetilde{P}_i(\widetilde{f_1(y)},\ldots,\widetilde{f_i(y)}),\widetilde{P}_i(\widetilde{f_1(y)},\ldots,\widetilde{f_i(y)})^{-1}\}.$$

There exists a valuation v on $\widetilde{\mathscr{H}(y)}$ that is trivial on $\widetilde{\mathscr{H}(x)}$ and whose ring $\mathscr{O}_{\mathsf{v}}$ dominates $\mathscr{O}_{\mathrm{Spec}\,B'',\mathsf{y}_j}$. As $\widetilde{f_1(y)},\ldots,\widetilde{f_n(y)}$ belong to B'', they belong to $\mathscr{O}_{\mathsf{v}}$; as $\widetilde{P}_i(\widetilde{f_1(y)}(\mathsf{y}_j),\ldots,\widetilde{f_i(y)}(\mathsf{y}_j))=0$ for all i, the element $\widetilde{P}_i(\widetilde{f_1(y)},\ldots,\widetilde{f_i(y)})$ belongs to the maximal ideal of $\mathscr{O}_{\mathsf{v}}$ for all i. It now follows from the above explicit descriptions of $\widetilde{(Y_x,y)}$ and of the $\widetilde{(Y_i,y)}$'s that v belongs to $\widetilde{(Y_x,y)}$ but not to $\bigcup_{i\in I}\widetilde{(Y_i,y)}$, contradiction.

9.1.2.4. — As $\Omega_x\neq\varnothing$, it follows from 8.4.7 (2) that there exists a point ω in Ω_x lying on Z. By Theorem 8.4.6, there exists a strictly affinoid neighborhood V of ω in $\Omega\cap Z$ such that $V\to X$ admits a factorization $V\to T\to X$, where T is a strictly affinoid domain of a smooth X-space S and $V\to T$ is a finite map with respect to which $\mathscr{F}_V$ is T-flat. By Theorem 5.4.6, if φ is quasi-smooth at y and $\mathscr{F}=\mathscr{O}_Y$, one can suppose that $V=T$.

Let ϖ be the image of ω in T. By Corollary 4.3.2, the image of $\mathrm{Int}(V/Y)$ in T contains an open neighborhood W of ϖ. As W_x is a non-empty strictly $\mathscr{H}(x)$-analytic space, it has an $\mathscr{H}(x)$-rigid point (1.2.10), which automatically belongs to $\mathrm{Int}(T_x/S_x)$. This implies the existence of an open subset of S whose fiber at x is non-empty and is included in W_x. Since $|k^\times|\neq\{1\}$, applying Corollary 6.2.7 to this open subset provides an étale X-space X' and an X-morphism $X'\to S$ whose image intersects W_x.

We fix a pre-image x' of x in X' whose image in S belongs to W and is denoted by t. We choose a pre-image v of t in $\mathrm{Int}(V/Y)$. We denote by T' the analytic domain $T\times_S X'$ of X', and by V' the fiber product $V\times_T T'$. We choose a point $v'\in V'$ lying above both v and x'. We then have the following commutative diagram of pointed spaces,

$$\begin{tikzcd}
 & (V,v) \arrow[r, hook] \arrow[d] & (Z,v) \arrow[dd, bend left] \\
(V',v') \arrow[ur] \arrow[d] & (T,t) \arrow[r, hook] & (S,t) \arrow[d] \\
(T',x') \arrow[ur] \arrow[r, hook] & (X',x') \arrow[ur] \arrow[r] & (X,x)
\end{tikzcd}$$

in which both squares are cartesian. Since $X'\to X$ is étale, it is boundaryless. As it factorizes through $X'\to S$, the latter map is boundaryless as well; thus $T'\to T$ and $V'\to V$ are also boundaryless.

As $v\in\mathrm{Int}(V/Y)$, the germ (V,v) coincides with (Y,v); in other words, we have $\widetilde{(V,v)}=\mathbf{P}_{\widetilde{\mathscr{H}(v)}/\widetilde{k}}\{\widetilde{f_1(v)},\ldots,\widetilde{f_n(v)}\}$. Since $V'\to V$ is boundaryless,

$\widetilde{(V',v')} = \mathbf{P}_{\widetilde{\mathscr{H}(v')}/\widetilde{k}}\{\widetilde{f_1(v')}, \ldots, \widetilde{f_n(v')}\}$ (one still writes f_i for the pull-back of f_i into the ring of functions on V'); note that this implies that V' is good.

By the choice of V, the point v belongs to Ω_x. We hence have for every i the inequality $|P_i(f_1, \ldots, f_i)(v)| < 1$. It implies that $\widetilde{P}_i(\widetilde{f_1(v')}, \ldots, \widetilde{f_i(v')}) = 0$ for every i. By the very definition of the P_i's (resting on j as chosen at the beginning of 9.1.2.3), it follows that there exists an $\widetilde{\mathscr{H}(x)}$-isomorphism between $\kappa(\mathsf{y}_j)$ and $\widetilde{\mathscr{H}(x)}[\widetilde{f_1(v')}, \ldots, \widetilde{f_n(v')}]$ that sends $\widetilde{f_i(y)}(\mathsf{y}_j)$ to $\widetilde{f_i(v')}$ for any i. The image of $\widetilde{(V',v')}$ inside $\mathbf{P}_{\widetilde{\mathscr{H}(x)}/\widetilde{k}}$ therefore coincides with $p_j\left(\mathbf{P}_{\kappa(\mathsf{y}_j)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_j), \ldots, \widetilde{f_n(y)}(\mathsf{y}_j)\}\right) = \mathsf{U}_j$. As a consequence, (U_j, x) is the smallest analytic domain of (X, x) through which $(V', v') \to (X, x)$ factorizes.

The morphism $V' \to X$ is quasi-finite. The space T' is quasi-étale, and in particular quasi-smooth, over X, and $\mathscr{F}_{V'}$ is flat over T' because $\mathscr{F}_V$ is flat over T. Moreover, if $Y \to X$ is quasi-smooth and $\mathscr{F} = \mathscr{O}_Y$ then V' is quasi-étale over X (because in this situation $V = W$).

The inner case. If φ is inner at y we have seen at the end of 9.1.2.2 that $j = 1$ and that

$$\mathbf{P}_{\kappa(\mathsf{y}_1)/\widetilde{k}}\{\widetilde{f_1(y)}(\mathsf{y}_1), \ldots, \widetilde{f_n(y)}(\mathsf{y}_1)\} = p_1^{-1}(\widetilde{(X,x)}).$$

Therefore $\widetilde{(V',v')}$ is the pre-image of $\widetilde{(X,x)}$ in $\mathbf{P}_{\widetilde{\mathscr{H}(v')}/\widetilde{k}}$, which exactly means that $(V', v') \to (X, x)$ is boundaryless.

9.1.2.5. *Conclusion.* — As $V' \to X$ is quasi-finite and V' is good, there exists a strictly k-affinoid neighborhood X_j of v' in V' such that v' is the only pre-image of x inside X_j. To emphasize the dependance on j, let us denote now by x_j the point v', by ψ_j the natural map $X_j \to X$, and by σ_j the natural X-map $X_j \to Z$.

The following follow from what was done in 9.1.2.4.

- The morphism ψ_j is quasi-finite.
- The coherent sheaf $\sigma_j^*\mathscr{F}$ is X-flat.
- The smallest analytic domain of (X, x) through which (X_j, x_j) factorizes is (U_j, x).
- If $\mathscr{F} = \mathscr{O}_Y$ and φ is quasi-smooth at y then ψ_j is quasi-étale.
- If φ is inner at y then $j = 1$ and ψ_1 is inner, hence finite, at x_1.

As the coherent sheaf $\sigma_j^*\mathscr{F}$ is X-flat (and has support X_j because $\mathscr{F}$ has support Y), Proposition 9.1.1 ensures that $\psi_j(X_j)$ is an analytic domain of X. We can shrink X_j so that $\psi_j(X_j) \subset U_j$; thus $(\psi_j(X_j), x) = (U_j, x)$ by minimality of (U_j, x). Since (U, x) is the union of the (U_j, x)'s (9.1.2.2), the data $(X_j, \psi_j, \sigma_j)_j$ satisfy the conclusions of the theorem. □

9.1.3. Theorem. — *Assume that $|k^\times| \neq \{1\}$. Let Y be a quasi-compact, strictly k-analytic space and let X be a separated k-analytic space. Let $\varphi : Y \to X$ be a morphism and let $\mathscr{F}$ be an X-flat coherent sheaf on Y. Denote by Z the set of points of Y at which $\mathscr{F}$ is CM over X (this is an open subset of Y by 8.4.7).*

There exist a strictly k-affinoid space X', a quasi-finite map $\psi\colon X' \to X$, and an X-morphism $\sigma\colon X' \to Z \cap \mathrm{Supp}(\mathscr{F})$ such that the following hold:

(1) *The coherent sheaf $\sigma^*\mathscr{F}$ is X-flat (so $\psi(X')$ is a compact strictly analytic domain of X by Proposition 9.1.1).*
(2) *The image $\varphi(\mathrm{Supp}(\mathscr{F}))$ is equal to $\psi(X')$.*

If moreover $Y \to X$ is quasi-smooth and $\mathscr{F} = \mathscr{O}_Y$, then ψ can be chosen to be quasi-étale.

Proof. — By replacing Y with $\mathrm{Supp}(\mathscr{F})$ we may assume that $\mathrm{Supp}(\mathscr{F}) = Y$. Let y be a point of Y. Using the notation of Theorem 9.1.2, and setting

$$X^y = \coprod X_j,\ \psi^y = \coprod \psi_j, \sigma^y = \coprod \sigma_j,$$

one gets the existence of a strictly k-affinoid space X^y, a morphism $\psi^y : X^y \to X$ which is quasi-finite and even quasi-étale if φ is quasi-smooth and $\mathscr{F} = \mathscr{O}_Y$, and an X-map $\sigma^y : X^y \to Z$ such that the following are satisfied:

(a) The coherent sheaf $(\sigma^y)^*\mathscr{F}$ is X-flat, so $\psi^y(X^y)$ is a compact strictly k-analytic domain of X by Proposition 9.1.1.
(b) The germ $(\psi^y(X^y), x)$ is equal to the smallest analytic domain (U^y, x) of (X, x) through which $(Y, y) \to (X, x)$ factorizes.
(c) As $(Y, y) \to (X, x)$ factorizes through $(\psi^y(X^y), x)$, there exists an analytic neighborhood V^y of y in Y such that $\varphi(V^y) \subset \psi^y(X^y)$.

By quasi-compactness of Y, there exist finitely many points $y_1, \dots, y_n$ on Y such that $Y = \bigcup_i V^{y_i}$. Now set $X' = \coprod X^{y_i}, \psi = \coprod \psi^{y_i}$, and $\sigma = \coprod \sigma^{y_i}$. By construction, (1) is satisfied and ψ is quasi-étale if $\mathscr{F} = \mathscr{O}_Y$ and φ is quasi-smooth; it thus remains to show (2). For every i, one has $\varphi(V^{y_i}) \subset \psi^{y_i}(X^{y_i})$. As $Y = \bigcup_i V^{y_i}$, this implies that $\varphi(Y) \subset \psi(X')$; but the existence of σ provides the reverse inclusion, whence (2). □

9.1.4. Remark. — The strictness assumption on Y cannot be dropped from the statement of Theorem 9.1.3 (even if one does not require anymore in the conclusion that X' be strict). Indeed, let Y be a compact k-analytic space. The coherent sheaf $\mathscr{O}_Y$ is k-flat (Lemma 4.1.13); hence we can apply Theorem 9.1.3 with $X = \mathscr{M}(k)$ and $\mathscr{F} = \mathscr{O}_Y$; and even if one drops the strictness requirement on X', it then simply states that if Y is strict and non-empty, it has a rigid point. This is nothing but the analytic Nullstellensatz, and this does not hold in general if Y is non-strict (e.g., $Y = \mathscr{M}(k_r)$ for some non-empty k-free polyradius r).

(Let us emphasize that we do not claim to have given a new proof of the analytic Nullstellensatz, because we used it at several places in the proof of Theorem 9.1.2, either directly or indirectly; e.g., it is used for the local existence of étale multisections on the image of a smooth map.)

9.2. Images of maps

We are now going to deduce general results on the images of maps from Theorem 9.1.3. Theorems 9.2.1 and 9.2.2 are stated under slightly general assumptions (for instance, the spaces involved are not assumed to be Hausdorff), but for each of them the case of interest is mainly that of affinoid source and target (and in fact, the proof first reduces to this case for both of them).

9.2.1. Theorem. — *Let Y be a Γ-strict k-analytic space, let X be a k-analytic space, and let $\varphi\colon Y \to X$ be a morphism. Assume that φ is* topologically *proper (1.1.3; this amounts to requiring that $\varphi^{-1}(V)$ is quasi-compact for every affinoid domain V of X, see 1.2.6; we emphasize that we do not assume that φ is topologically separated). Let $\mathscr{F}$ be a coherent sheaf on Y which is X-flat. Assume that at least one of the two conditions below is satisfied:*

1. *The space X is Γ-strict.*
2. *The space X is separated and Y admits a locally finite covering by Γ-strict affinoid domains (if Y is Hausdorff, the latter condition is equivalent to paracompactness of Y; see 1.2.6 and Remark 3.5.12).*

Then $\varphi(\mathrm{Supp}(\mathscr{F}))$ is a closed Γ-strict analytic domain of X.

Proof. — By replacing Y with $\mathrm{Supp}(\mathscr{F})$ we reduce to the case where $\mathrm{Supp}(\mathscr{F}) = Y$. Since φ is topologically proper, $\varphi(Y)$ is a closed subset of X and $\varphi^{-1}(E)$ is quasi-compact for every quasi-compact subset E of X. We are now going to reduce to the case where both Y and X are Γ-strict and k-affinoid.

Let us first consider the case where (1) is fulfilled. One can check the result G-locally on X, which allows to assume that X is Γ-strict and k-affinoid. In this case, Y is quasi-compact, hence admits a finite covering by Γ-strict, affinoid domains; one therefore immediately reduces to the case where Y is also Γ-strict and affinoid.

Let us now consider the case where (2) is fulfilled. Choose a locally finite Γ-strict affinoid covering (Y_i) of Y. It is sufficient to prove that $\varphi(Y_i)$ is a Γ-strict compact analytic domain of X for any i. Indeed, assume that it is the case. Then for every affinoid domain V of X, the pre-image $\varphi^{-1}(V)$ is quasi-compact, hence intersects only finitely many Y_i's; this implies that $(\varphi(Y_i))_i$ is a locally finite covering of $\varphi(Y)$ by Γ-strict compact analytic domains of X, which can be refined into a locally finite covering by Γ-strict affinoid domains. As X is separated, the intersection of two such domains will still be affinoid and Γ-strict; hence our covering is a Γ-strict affinoid atlas on $\varphi(Y)$, and $\varphi(Y)$ is a closed Γ-strict analytic domain of X. We thus reduce to the case where Y is compact. By Proposition 7.3.6, we can then assume X is compact and Γ-strict, and even, since one can check the result G-locally on X, that it is affinoid and Γ-strict. And as Y admits a finite covering by Γ-strict k-affinoid domains, we eventually reduce to the case where Y is also Γ-strict and k-affinoid.

So let us prove the theorem when both spaces Y and X are affinoid and Γ-strict. Let $r = (r_1, \ldots, r_n)$ be a k-free polyradius such that the r_i's belong to Γ, the valuation of k_r is non-trivial, and X_r and Y_r are strictly k_r-affinoid. Let $\mathfrak{s} : X \to X_r$ be the Shilov section (1.2.16). By Theorem 9.1.3, the image $\varphi_r(Y_r)$ is a compact strictly k_r-analytic domain of X_r. The subset $\varphi(Y)$ of X is nothing but $\mathfrak{s}^{-1}(\varphi_r(Y_r))$. By the Gerritzen-Grauert theorem, $\varphi_r(Y_r)$ is a finite union of strictly k_r-rational domains; it is then easily seen (using the explicit formula for $\mathfrak{s}$, see the proof of Lemma 2.4 of [**Duc03**]) that $\mathfrak{s}^{-1}(\varphi_r(Y_r))$ is itself a finite union of rational domains whose definitions only involve elements of $\mathbf{R}_+^\times$ that belong to Γ; therefore, $\varphi(Y)$ is a compact Γ-strict analytic domain of X. □

9.2.2. Theorem. — *Let n and d be two non-negative integers, let Y be a Γ-strict k-analytic space, and let φ be a topologically proper morphism from Y to a normal k-analytic space X. Assume that X is purely d-dimensional, Y is purely $(n+d)$-dimensional, and the fibers of φ are purely n-dimensional. If X is Γ-strict, or if X is separated and Y admits a locally finite Γ-strict affinoid covering, then $\varphi(Y)$ is a Γ-strict closed k-analytic domain of X.*

Proof. — Exactly as at the beginning of the proof of Theorem 9.2.1, we reduce to the case where both Y and X are Γ-strict k-affinoid spaces. By compactness, one can argue locally on Y. Hence, by combining Cor. 4.7 of [**Duc07b**] with 3.5.7 (or Remark 3.5.8), we can assume that there exists a factorization $Y \to T \to X$ where the map $Y \to T$ is finite and where T is a Γ-strict affinoid domain of a smooth X-space of pure relative dimension n.

By flatness of quasi-smooth morphisms (Corollary 5.3.2) and Theorem 9.2.1, the image of any non-empty compact analytic domain T' of T in X is a non-empty compact analytic domain of X, hence is of dimension d. Since the fibers of $T \to X$ are purely n-dimensional, it follows from 1.4.14 (3) that T' is of dimension $n+d$; therefore T is purely $(n+d)$-dimensional. As $Y \to T$ is finite and Y is purely $(n+d)$-dimensional, the image of any irreducible component of Y in T is an irreducible Zariski-closed subset of T of dimension $n+d$ (again by 1.4.14); i.e., this is an irreducible component of T. Therefore the image Z of Y in T is a union of irreducible components of T. Since X is normal and $T \to X$ is quasi-smooth, T is normal by Proposition 5.5.5. Therefore Z is a union of connected components of T, hence is a Γ-strict affinoid domain of T. By Theorem 9.2.1 the image of Z in X, which coincides with that of Y, is a compact Γ-strict analytic domain of X. □

9.2.3. Theorem. — *Let $\varphi\colon Y \to X$ be a morphism between k-analytic spaces and let $\mathscr{F}$ be a coherent sheaf on Y which is X-flat. Let y be a point of* $\mathrm{Supp}(\mathscr{F})$ *at which φ is inner, and let x be its image on X. The image* $\varphi(\mathrm{Supp}(\mathscr{F}))$ *is a neighborhood of x.*

Proof. — By shrinking X around x (and by shrinking Y accordingly) we may assume that X is Hausdorff. By replacing Y with a compact analytic neighborhood of y, we can further assume that Y is compact. Then it follows from Theorem 9.2.1 that $\varphi(\mathrm{Supp}(\mathscr{F}))$ is an analytic domain U of X. As φ is inner at y, $\varphi_{|\mathrm{Supp}(\mathscr{F})}$ is inner at y too. Therefore x belongs to $\mathrm{Int}(U/X)$; i.e., to the *topological* interior of U in X, whence the result. □

9.2.4. Remark. — The openness of flat, boundaryless morphisms between good k-analytic spaces has already been proved by Berkovich, in a slightly different way, in unpublished work.

9.2.5. Theorem. — *Let n and d be two non-negative integers and let $\varphi\colon Y \to X$ be a morphism between k-analytic spaces. Assume that X is normal and purely d-dimensional, φ is purely of relative dimension n, and Y is of pure dimension $n+d$. Let y be a point of Y at which φ is inner, and let x be its image on X. The image $\varphi(Y)$ is a neighborhood of x.*

Proof. — By shrinking X around x (and by shrinking Y accordingly) we may assume that X is Hausdorff. By replacing Y with a compact analytic neighborhood of y, we can further assume that Y is compact. Then it follows from Theorem 9.2.2 that $\varphi(Y)$ is an analytic domain U of X. As φ is inner at y, x belongs to $\mathrm{Int}(U/X)$; i.e., to the *topological* interior of U in X, whence the result. □

CHAPTER 10

CONSTRUCTIBLE LOCI

This quite long chapter is devoted to the study of "loci of validity". In order to describe them, we need to fix some terminology. We shall say that a subset of an analytic space X is *constructible* if it is a finite boolean combination of Zariski-open subsets of X (Definition 10.1.1 below). Though the properties of being Zariski-open or Zariski-closed are G-local, that of being constructible is *not*: we sketch a counter-example (10.1.14). But it is nonetheless "almost" G-local: if X is a *finite-dimensional* analytic space (which simply means that the dimensions of the irreducible components of X are uniformly bounded above), then every G-locally constructible subset of X is actually constructible (Proposition 10.1.12).

We now aim to establish that various loci of validity of relative properties are locally constructible (by the above, they will even be constructible as soon as the source space is finite-dimensional), and sometimes Zariski-open when the property involved encapsulates some flatness condition.

For that purpose we develop in Section 10.2, in a rather abstract categorical setting, a general strategy inspired by Kiehl's paper [**Kie67b**] and by the technique of "spreading out from the generic fiber" in algebraic geometry – the latter is not directly available here, but as explained in the general Introduction (see 0.3.1.3), we bypass this obstacle by using Theorem 6.3.3 that applies to local rings of generic fibers. This strategy will enable us to reduce various constructibility and Zariski-openness statements to simpler ones (see the short introduction of Section 10.2 for some more specificity). One may skip those quite formal considerations and go directly to Section 10.3, but one will then have to accept repeatedly arguments of the form "in view of this and that result of 10.2, we may assume…".

We do not give further details here about the results of this chapter; we refer the reader to the local introduction of each section, from 10.2 to 10.7.

10.1. Constructibility in analytic spaces

10.1.1. Definition. — Let X be an analytic space. We say that a subset E of X is *constructible* if it can be written as a finite union $\bigcup(U_i \cap F_i)$ where U_i (resp. F_i) is a Zariski-open (resp. a Zariski-closed) subset of X for every i.

We say that a subset E of X is *locally constructible* (resp. *G-locally constructible*) if there exists an open covering (resp. a G-covering) (X_i) of X such that $E \cap X_i$ is a constructible subset of X_i for all i.

10.1.2. Remark. — The set of constructible subsets of X is a Boolean sub-algebra of $\mathscr{P}(X)$; i.e., it is stable under finite unions, finite intersections, and complements. We could also have defined it as the Boolean sub-algebra of $\mathscr{P}(X)$ generated by all Zariski-open subsets (or all Zariski-closed subsets).

The set of locally constructible (resp. G-locally constructible) subsets of X is also a Boolean sub-algebra of $\mathscr{P}(X)$.

10.1.3. Remark. — In EGA, the Boolean algebra of constructible subsets of an arbitrary topological space X is defined as the Boolean sub-algebra of $\mathscr{P}(X)$ generated by *retrocompact* open subsets; i.e., open subsets whose intersection with any quasi-compact open subset of X is quasi-compact. If X is a locally noetherian topological space, then every open subset of X is retrocompact.

Now if X is a *quasi-compact* analytic space, its Zariski topology is noetherian; therefore every Zariski-open subset of X is retrocompact with respect to the Zariski topology, hence a subset of X is constructible in our sense if and only if it is constructible in the sense of EGA for the Zariski topology. We do not know whether this is the case for an arbitrary analytic space, since it is unclear (at least to the author) which Zariski open subsets of an arbitrary analytic space are retrocompact with respect to the Zariski topology.

10.1.4. Example. — Let $\mathscr{X}$ be a scheme locally of finite type over an affinoid algebra A. If E is a constructible (resp. locally constructible) subset of $\mathscr{X}$, then its pre-image E^{an} in $\mathscr{X}^{\mathrm{an}}$ is a constructible (resp. locally constructible) subset of $\mathscr{X}^{\mathrm{an}}$; it is closed, resp. open, if and only if E is closed, resp. open ([**Ber93**], Cor. 2.6.6; this is written for a constructible subset, but since the problem is local for the Zariski topology on $\mathscr{X}$, one reduces immediately to this case). If $\mathscr{X}$ is proper over A it follows from GAGA (2.1.1) that $E \mapsto E^{\mathrm{an}}$ establishes a bijection between the set of constructible subsets of $\mathscr{X}$ and that of constructible subsets of $\mathscr{X}^{\mathrm{an}}$, whose converse bijection maps a constructible subset F of $\mathscr{X}^{\mathrm{an}}$ to its image F^{al} in $\mathscr{X}$.

10.1.5. — Let X be analytic space, and let Y be a Zariski-closed subset of X. Endow Y with any structure of a closed analytic subspace of X. Then a subset E of Y is constructible (with respect to this given structure) if and only if it is constructible as a subset of X. In particular, this does not depend on the chosen structure of Y, and we shall often simply speak about constructible subsets of Y, without fixing any structure of closed analytic subspace on it. The same also holds for locally constructible and G-locally constructible sets (for the latter, this rests on Remark 1.3.13).

10.1.6. — Let $Y \to X$ be a morphism of analytic spaces. If E is a constructible (resp. locally constructible, resp. G-locally constructible) subset of X, it follows from the definition that the pre-image of E in Y is a constructible (resp. locally constructible, resp. G-locally constructible) subset of Y. We shall often apply it for Y an analytic domain of X, in which case the pre-image of E in Y is nothing but the intersection $E \cap Y$.

10.1.7. — Let X be a k-analytic space. Assume that its Zariski-topology is noetherian; e.g., X is quasi-compact. Since every irreducible Zariski-closed subset Y of X has a Zariski-generic point (pick y in Y such that $d_k(y) = \dim Y$), it follows from [**EGA III**$_1$], Chapt. 0, Cor. 9.2.4 that X is quasi-compact for the constructible topology. If X is affinoid, this can also be deduced through the assignement $E \mapsto E^{\mathrm{an}}$ from the compactness of X^{al} for the constructible topology. But be aware that the Hausdorff property of the constructible topology of X^{al} does not transfer to the constructible topology of X in general: indeed, two distinct points of X lying over the same point of X^{al} belong to the same constructible subsets of X, hence cannot be separated using such subsets.

We are now going to show that some other basic properties of the constructible subsets of a noetherian topological space (like Prop. 9.2.2 and Prop. 9.2.5 of [**EGA III**$_1$], Chapt. 0) actually hold for the constructible subsets of an arbitrary analytic space. Our proofs are basically the same as those of the results in EGA alluded to, except that we replace noetherian induction by some arguments involving dimension theory and the decomposition into irreducible components.

10.1.8. Lemma. — *Let X be an analytic space and let E be a constructible subset of X.*

(1) *The following are equivalent:*
 (i) *E contains a Zariski-dense open subset of X;*
 (ii) *E is Zariski-dense in X.*
(2) *The following are equivalent:*
 (iii) *E is Zariski-closed in X;*
 (iv) *every irreducible Zariski-closed subset Z of X such that $E \cap Z$ contains a non-empty Zariski-open subset of Z is contained in E.*
(3) *The following are equivalent:*
 (v) *E is Zariski-open in X;*
 (vi) *for every pair (T, Z) of irreducible Zariski-closed subsets of X with $T \subset Z$, if $T \cap E$ is Zariski-dense in T then $Z \cap E$ is Zariski-dense in Z.*

Proof. — Let us begin with (1). We obviously have (i)⇒(ii). Let us now assume that (ii) holds. Write $E = \bigcup_{i\in I} U_i \cap F_i$, where I is a finite set and where U_i is Zariski-open and F_i Zariski-closed for all i. Let (X_j) be the family of irreducible components of X. Fix j. Any non-empty Zariski-open subset U of X_j contains a non-empty Zariski-open subset of X, namely $U \cap \bigcup_{\ell\neq j} X_\ell$. Therefore the intersection $E \cap X_j$ is a Zariski-dense subset of X_j. This implies that there exists some index i such that $F_i \cap X_j = X_j$ and $U_i \cap X_j \neq \varnothing$. Let V_j be the intersection of $U_i \cap X_j$ with the complement of $\bigcup_{\ell\neq j} X_\ell$. By construction, $\coprod_{j\in J} V_j$ is a Zariski-dense open subset of X contained in E, and (i) holds.

Let us prove (2). The implication (iii)⇒(iv) is obvious. Assume that (iv) holds. In order to prove that E is Zariski-closed in X, it suffices to prove that $E \cap Y$ is Zariski-closed for every irreducible component Y of X (because the set of irreducible components is G-locally finite and the property of being Zariski-closed is G-local); we can thus assume that X is irreducible, and we argue by induction on $\dim X$. There is nothing to prove if $\dim X = 0$; assume that $\dim X > 0$ and the assertion is true in smaller dimensions. Write $E = \bigcup_{i\in I} U_i \cap F_i$, where I is a finite set and where U_i is Zariski-open; we can of course assume that $U_i \cap F_i \neq \varnothing$ for all i. If there exists i such that $F_i = X$ then E contains the non-empty Zariski-open subset U_i of X, hence $E = X$ by assumption (iv). If $F_i \neq X$ for all i then $F := \bigcup_i F_i$ is a Zariski-closed subset of X containing E and of dimension $< \dim X$; then by arguing componentwise on F and using the induction hypothesis we see that E is Zariski-closed in F (hence in X).

Let us prove (3). The implication (v)⇒(vi) is obvious. Assume that (iv) holds, and set $F = X\backslash E$. Let Z be an irreducible subset of X such that $F \cap Z$ contains a non-empty Zariski-open subset U of Z. We are going to prove by contradiction that F contains Z; this will ensure in view of (2) that F is Zariski-closed, and thus that E is Zarsiki-open. Assume that there exists $z \in Z$ such that $z \notin F$. Then $z \in E$ and $E \cap \overline{\{z\}}^{X_{\mathrm{Zar}}}$ is Zariski-dense in $\overline{\{z\}}^{X_{\mathrm{Zar}}}$; by assumption (3) this implies that $E \cap Z$ is Zariski-dense in Z, which contradicts the fact that $U \subset F$. □

10.1.9. Remark. — If X is affinoid, Lemma 10.1.8 can of course also be deduced through the assignement $E \mapsto E^{\mathrm{an}}$ from the corresponding statement on the constructible subsets of X^{al}.

10.1.10. Lemma. — *Let X be a k-analytic space, let E be a constructible subset of X, and let V be an analytic domain of X. Let L be an analytic extension of k and let E_L be the pre-image of E on X_L.*

(1) $\overline{E}^{X} = \overline{E}^{X_{\mathrm{Zar}}}$.
(2) $\overline{E \cap V}^{V_{\mathrm{Zar}}} = \overline{E}^{X_{\mathrm{Zar}}} \cap V$.
(3) $\overline{E_L}^{X_{L,\mathrm{Zar}}} = (\overline{E}^{X_{\mathrm{Zar}}})_L$.
(4) *E is open (resp. closed) if and only if it is Zariski-open (resp. Zariski-closed).*

Proof. — Assertion (4) is an immediate consequence of (1). Let us now prove (1), (2), and (3). All terms involved in these equalities commute with finite unions; one can therefore assume that $E = U \cap F$, where U (resp. F) is a Zariski-open (resp. Zariski-closed) subset of X; by replacing X with F one can assume that E is Zariski-open in X. The required equalities now follow from Lemma 1.5.12, Corollary 1.5.13, and Corollary 1.5.14. $\square$

10.1.11. Corollary. — *Let X be a k-analytic space, let E be a subset of X and let (X_i) be a G-covering of X by analytic domains such that the intersection $E \cap X_i$ is a constructible subset of X_i for every i. Under those assumptions $\overline{E}^{X_{\mathrm{Zar}}} = \overline{E}^{X}$ and $\overline{E}^{X} \cap X_i = \overline{E \cap X_i}^{X_i}$ for every i.*

Proof. — Let i and j be two indices. By Lemma 10.1.10

$$\overline{E \cap X_i}^{X_i} \cap X_j = \overline{E \cap X_j}^{X_j} \cap X_i = \overline{E \cap X_i \cap X_j}^{(X_i \cap X_j)}.$$

Therefore if one sets $F = \bigcup_i \overline{E \cap X_i}^{X_i}$, one has $F \cap X_i = \overline{E \cap X_i}^{X_i}$ for every i. As a consequence, F is a Zariski-closed subset of X; it follows from the definitions that if Z is a closed subset of X containing E then $Z \supset F$. One has thus $F = \overline{E}^{X} = \overline{E}^{X_{\mathrm{Zar}}}$. $\square$

10.1.12. Proposition. — *Let X be an analytic space and let E be a subset of X. The following are equivalent:*

(i) *E is G-locally constructible.*
(ii) *E is locally constructible.*

If moreover X is finite-dimensional, those assertions are also equivalent to

(iii) *E is constructible.*

Proof. — It is clear that (iii)⇒(ii)⇒(i).

Assume that X is finite-dimensional and satisfies (i); choose a G-covering (X_i) of X such that $E \cap X_i$ is a constructible subset of X_i for all i, and let us prove that X satisfies (iii). We argue by induction on $\dim X \in \{-\infty\} \cup \mathbf{Z}_{\geqslant 0}$. If $X = \varnothing$ there is nothing to prove. Assume that X is non-empty, and that the proposition is true in dimensions $< \dim X$.

Set $Y = \overline{E}^{X_{\mathrm{Zar}}}$ and $F = Y \backslash E$; for every i, we denote by Y_i (resp. E_i, resp. F_i) the intersection of X_i with Y (resp. E, resp. F).

Fix i. It follows from Corollary 10.1.11 that E_i is a Zariski-dense constructible subset of Y_i and $\overline{F}^{Y_{\mathrm{Zar}}} \cap Y_i = \overline{F_i}^{Y_{i,\mathrm{Zar}}}$. Since E_i is a Zariski-dense constructible subset of Y_i, Lemma 10.1.8 implies that E_i contains a Zariski-dense open subset of Y_i; as a consequence, $\overline{F_i}^{Y_{i,\mathrm{Zar}}}$ contains no irreducible components of Y_i. We thus have $\dim \overline{F_i}^{Y_{i,\mathrm{Zar}}} < \dim Y_i$ as soon as $Y_i \neq \varnothing$.

By the above, $\dim(\overline{F}^{Y_{\mathrm{Zar}}} \cap Y_i) < \dim Y_i$ for every i such that $Y_i \neq \varnothing$. This implies that $\overline{F}^{Y_{\mathrm{Zar}}}$ is of dimension $< \dim X$. The induction hypothesis then ensures that F

is a constructible subset of $\overline{F}^{Y_{\mathrm{Zar}}}$, and therefore a constructible subset of X. As a consequence, $E = Y \backslash F$ is a constructible subset of X, whence (iii).

It remains to show that (i)⇒(ii). Assume that E is G-locally constructible. Let x be a point of X, and let V be a finite-dimensional open neighborhood of x in X (e.g., V is the topological interior of a compact analytic neighborhood of x). Since E is G-locally constructible, $E \cap V$ is a G-locally constructible subset of V. Since (i)⇒(iii) for a finite-dimensional ambient space, it follows that $E \cap V$ is a constructible subset of V. Hence (ii) holds (by varying x). □

10.1.13. Remark. — If $\mathscr{X}$ is a finite-dimensional locally noetherian scheme, every locally constructible subset of $\mathscr{X}$ is constructible; the proof is mutatis mutandis the same as that of implication (i)⇒(iii) in Proposition 10.1.12 above. But note that this is obvious if $\mathscr{X}$ is of finite type (by transitivity of the Zariski topology in the scheme-theoretic setting).

10.1.14. Counter-example. — In Proposition 10.1.12 above, the assumption that X is finite-dimensional cannot be dropped. Indeed, let us denote by X the closed unit disc over k. For every $n \in \mathbf{Z}$, let j_n be the closed immersion $X^n \simeq X^n \times \{0\} \hookrightarrow X^{n+1}$. We define inductively a constructible subset E_n of X^n by the following conditions:

- $E_0 = X^0 = \{0\}$.
- $E_{n+1} = X^{n+1} \backslash j_n(E_n)$ for every n.

By construction, the subset $\coprod_n E_n$ of $\coprod_n X^n$ is locally constructible; but it is not constructible (exercise left to the reader).

We end this section by proving a kind of analytic Chevalley theorem for proper maps; the key point will be Kiehl's theorem on the direct image of a coherent sheaves (cf. 1.3.23).

***10.1.15. Theorem* (Proper Chevalley theorem).** — *Let $\varphi \colon Y \to X$ be a proper morphism of k-analytic spaces and let E be a locally constructible subset of Y. The image $\varphi(E)$ is a locally constructible subset of X.*

Proof. — Let us first mention that since φ is proper, $\varphi(T)$ is a Zariski-closed subset of X for every Zariski-closed subset T of Y (by Kiehl's theorem on the direct images of coherent sheaves, cf. 1.3.23); we shall use it repeatedly throughout the proof.

By Proposition 10.1.12, the assertion is G-local on X; we can thus asume that it is affinoid. The space Y is then compact by topological properness and topological separatedness, and E is thus constructible by Proposition 10.1.12. We can therefore write $E = \bigcup_{i \in I} U_i \cap F_i$ where I is a finite set and U_i (resp. F_i) is for every i a Zariski-closed (resp. Zariski-open) subset of Y; we can moreover assume that F_i is irreducible and $U_i \cap F_i \neq \varnothing$ for all i.

We are going to prove by noetherian induction on X that $\varphi(E)$ is a constructible subset of X. We thus assume that the intersection of $\varphi(E)$ with every proper Zariski-closed subset of X is constructible, and we shall prove that $\varphi(E)$ is constructible. Let (X_j) be the family of irreducible components of X.

If X is not irreducible, then $X_j \subsetneq X$ for all j, and $\varphi(E) = \bigcup \varphi(E) \cap X_j$ is thus a constructible subset of X. Assume now that X is irreducible. For every i, the image $\varphi(F_i)$ is an irreducible Zariski-closed subset of X, and $\varphi(E) \subset \bigcup_i \varphi(F_i)$. As a consequence, if $\bigcup_i \varphi(F_i) \subsetneq X$, then $\varphi(E)$ is constructible by the induction hypothesis. It remains to consider the case where there exists i such that $\varphi(F_i) = X$; we write F and U instead of F_i and U_i, respectively. It suffices now to prove that there exists a proper Zariski-closed subset Z of X such that $\varphi(F \cap U)$ contains $X \backslash Z$. indeed, if this is the case, one will have $\varphi(E) = (X\backslash Z) \cup (\varphi(E) \cap Z)$, and since $\varphi(E) \cap Z$ is contructible by the induction hypothesis, we shall be done.

Set $n = \dim X$ and $m = \dim F$. Let ξ be an Abhyankar point of X (1.4.10). By Lemma 1.5.11, the fiber F_ξ is of pure dimension $n-m$. As it is non-empty (recall that $\varphi(F) = X$ by assumption), there exists a point $y \in F_\xi$ such that $d_{\mathscr{H}(\xi)}(y) = n - m$. We then have $d_k(y) = n - m + d_k(\xi) = n$. The point y is thus an Abhyankar point of F, hence is Zariski-dense in F (Remark 1.5.11). Since the relative dimension is upper semi-continuous for the Zariski topology ([**Duc07b**], Thm. 4.9), the minimal relative dimension of $\varphi|_F$ is equal to $n - m$, and the set of points z of F such that $\dim_z \varphi > m - n$ is a proper Zariski-closed subset of F, whose image in X does not contain ξ (because F_ξ is of pure dimension $n - m$), hence is a proper Zariski-closed subset of X.

Let G be the complement of $F \cap U$ in F. This is a proper Zariski-closed subset of F; let $(G_\ell)_{\ell\in\Lambda}$ be the family (possibly empty if $G = \varnothing$) of irreducible components of G, and let Λ_0 be the subset of Λ consiting of indexes ℓ such that $\varphi(G_\ell) = X$.

Let ℓ be an element of Λ_0. By the same reasoning as above, the minimal relative dimension of $\varphi|_{G_\ell}$ is equal to $\dim G_\ell - n$, the subset H_ℓ of G_ℓ consisting of points z such that $\dim_z \varphi|_{G_\ell} > \dim G_\ell - n$ is a proper Zariski-closed subset of G_ℓ, and $\varphi(H_\ell)$ is a proper Zariski-closed subset of X.

Set $H = (\bigcup_{\ell\in\Lambda\backslash\Lambda_0} G_\ell) \cup \bigcup_{\ell\in\Lambda_0} H_\ell$. By construction, H is a proper Zariski-closed subset of F, and $Z := \varphi(H)$ is a proper Zariski-closed subset of X. It suffices now to prove that $X\backslash Z \subset \varphi(F \cap U)$.

Let x be a point of X that does not lie on Z. The fiber G_x does not intersect $\bigcup_{\ell\in\Lambda\backslash\Lambda_0} G_\ell$, nor $\bigcup_{\ell\in\Lambda_0} H_\ell$. Since $\dim_y G_{\ell,\varphi(y)} = \dim G_\ell - n < m - n$ for every $\ell \in \Lambda_0$ and every $y \in G_\ell\backslash H_\ell$ (by definition of H_ℓ), it follows that $\dim G_x < m - n$. But every fiber of $\varphi|_F$ is non-empty and of dimension at least $m-n$; as a consequence, $G_x \subsetneq F_x$, which exactly means that F_x intersects $F \cap U$; i.e., $x \in \varphi(F \cap U)$. $\square$

10.2. The diagonal trick

We are going to describe here a general method which will be useful for establishing the (local) constructibility or the Zariski-openness of some loci. It is inspired by Kiehl's [**Kie67b**], in which he proved the Zariski-openness of the flat locus of a *complex-analytic* morphism.

Roughly speaking, it consists of the following. One wants to understand the locus of validity of some relative property of a morphism $Y \to X$. If the formation of this locus commutes with the "tautological" base-change by $Y \to X$, one can replace $Y \to X$ by the second projection $Y \times_X Y \to Y$ and only investigate what happens on the diagonal. One thus reduces to the case where $Y \to X$ has a section σ and where it suffices to understand the intersection of our locus of validity with $\sigma(X)$.

Considering only points lying of $\sigma(X)$ will bring two advantages, say, in the case where both Y and X are affinoid (as can most of the time be assumed without loss of generality by arguing G-locally):

- The scheme $\sigma(X)^{\mathrm{al}}$ is of finite type over X^{al} (it is even isomorphic to it!), which is of course in general not the case for Y^{al}; this finiteness condition plays a key role in the study of the flatness locus, through an intermediate theorem of Kiehl on morphisms between noetherian schemes (see Theorem 10.3.1 below).
- Every point of $\sigma(X)$ belongs to $\mathrm{Int}(Y/X)$ by 1.3.21 (3). This will allow us to apply Theorem 6.3.3 on the local rings of analytic fibers, and to only deal with *naive* flatness in view of Theorem 8.3.4.

10.2.1. Our general axiomatic setting. — We fix a subcategory $\mathfrak{C}$ of the category of analytic spaces such that for every object X of $\mathfrak{C}$, the category of analytic spaces over X (which contains all X-analytic spaces, but also more generally all X_L-analytic spaces for every complete extension L of k) embeds fully faithfully in $\mathfrak{C}$. We still use the notations $\mathfrak{T}$, $\mathfrak{L}$ and and $\mathfrak{F}$ of Section 2.2, and the notation $\mathfrak{Coh}$ and $\mathfrak{Coh}^{\mathfrak{I}}$ introduced in Examples 2.2.9 and 2.2.10.

We denote by Q a property whose validity at a given point of an object X of $\mathfrak{C}$ makes sense for every object of $\mathfrak{F}_X$, and which satisfies the following conditions:

(1) For every $X \in \mathfrak{C}$, every $x \in X$, every analytic domain V of X containing x, and every object D of $\mathfrak{F}_X$, the object D satisfies Q at x if and only if D_V satisfies Q at x.

(2) For every $X \in \mathfrak{C}$, every $x \in X$, every analytic extension L of k, every point x' of X_L lying above x, and every $D \in \mathfrak{F}_X$, the object D satisfies Q at x if and only if D_L satisfies Q at x' (we insist that one only needs to check Q at *one* pre-image of x on X_L, and not at all of them).

10.2.2. Example. — We can take for $\mathfrak{C}$ the category of all analytic spaces, and for $\mathfrak{F}$ any fibered category as in 2.2.5. Let P be a property making sense for any object of $\mathfrak{F}_{\mathfrak{L}}$; assume moreover that P satisfies $(\mathsf{H}_{\mathrm{reg}})$ (2.3.15). By Remark 2.4.2 and Proposition

2.6.6, we can take for Q the property of satisfying *geometrically* the property P at a point of an analytic space (2.6.9); if P satisfies the stronger condition $(\mathsf{H}_{\mathrm{CI}})$, we do not need to require the validity to be geometric (Remark 2.6.11).

10.2.3. Example. — Let Y be an analytic space and let E be a locally constructible subset of Y. We can take for $\mathfrak{C}$ the category of all analytic spaces over Y, for $\mathfrak{F}$ the category $\mathfrak{T}$ itself, and for Q the property defined as follows: if Z is an object of $\mathfrak{C}$ and if $z \in Z$, then Z satisfies Q at z if z belongs to the Zariski-closure of the pre-image of E under the structure map $Z \to Y$.

10.2.4. — We also fix a functor $\mathscr{S}$ from $\mathfrak{F}_{\mathfrak{C}}$ to $\mathfrak{Coh}_{\mathfrak{C}}$ which is compatible with the fibered structures over $\mathfrak{C}$; i.e., for every arrow $p\colon Y \to X$ in $\mathfrak{C}$ and every object $D \in \mathfrak{F}_X$, one has a canonical isomorphism $\mathscr{S}(p^*D) \simeq p^*\mathscr{S}(D)$.

10.2.5. Example. — Let $\mathfrak{I}$ be an interval of $\mathbf{Z}$ (viewed as a category) and assume that $\mathfrak{F} = \mathfrak{Coh}^{\mathfrak{I}}$ (and let $\mathfrak{C}$ and Q be arbitrary). For every $Y \in \mathfrak{C}$, the fiber category $\mathfrak{F}_Y$ is the category of diagrams of $\mathscr{O}_Y$-linear maps $\ldots \to \mathscr{F}_i \to \mathscr{F}_{i+1} \to \mathscr{F}_{i+2} \to \ldots$ where i runs through $\mathfrak{I}$ and where the $\mathscr{F}_i$'s are coherent sheaves on Y. We can then take for $\mathscr{S}$ the functor sending such a diagram to the i-th coherent sheaf involved (for given $i \in \mathfrak{I}$), or more generally to the direct sum $\bigoplus_{i\in J} \mathscr{F}_i$ for some finite subset J of I.

10.2.6. Notation. — Let $Y \to X$ be a morphism of k-analytic spaces and let $\mathscr{E}$ be a coherent sheaf on Y. The X-flat locus of $\mathscr{E}$ will be denoted by $\mathrm{Flat}(\mathscr{E}/X)$.

10.2.7. Fiberwise validity. — Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces, with $Y \in \mathfrak{C}$. We shall say that a given object D of $\mathfrak{F}_Y$ satisfies Q *fiberwise* at some point $y \in Y$ if $D_{Y_{\varphi(y)}}$ satisfies Q at y. The set of points of Y at which D satisfies Q fiberwise will be denoted by $\mathsf{Q}_{\mathrm{fib}}(D/X)$; if $\mathscr{F}$ is a coherent sheaf on Y, we shall denote by $\mathrm{Flat}(\mathscr{F}/X)$ the X-flatness locus of $\mathscr{F}$.

10.2.8. Three statements. — The main purpose of this section is to develop general methods for proving the three following statements:

(α) For every morphism $Y \to X$ between k-analytic spaces with $Y \in \mathfrak{C}$ and every object $D \in \mathfrak{F}_Y$, the set $\mathsf{Q}_{\mathrm{fib}}(D/X)$ is a locally constructible subset of Y.

(β) For every morphism $Y \to X$ between k-analytic spaces with $Y \in \mathfrak{C}$ and every object $D \in \mathfrak{F}_Y$, the intersection $\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)$ is a Zariski-open subset of Y.

(γ) For every morphism $Y \to X$ between k-analytic spaces and each coherent sheaf $\mathscr{F}$ on Y, the set $\mathrm{Flat}(\mathscr{F}/X)$ is a Zariski-open subset of Y.

10.2.9. Remark. — Assertion (γ) is nothing but assertion (β) when we take for $\mathfrak{F}$ the category $\mathfrak{Coh}$, for $\mathfrak{C}$ the category of all analytic spaces, for $\mathscr{S}$ the identity functor, and for Q the property TRUE.

10.2.10. Remark. — Suppose that we are given a finite family $(\mathscr{S}_i)$ of $\mathfrak{C}$-functors from $\mathfrak{F}_{\mathfrak{C}}$ to $\mathfrak{Coh}_{\mathfrak{C}}$, and that $\mathscr{S} = \bigoplus \mathscr{S}_i$. Then the subset of Y that is involved in (β) is nothing but the intersection of $\mathsf{Q}_{\mathrm{fib}}(D/X)$ with $\bigcap_i \mathrm{Flat}(\mathscr{S}_i(D)/X)$.

10.2.11. Remark. — We emphasize that in statements (α) and (β), the space Y is assumed to be an object of $\mathfrak{C}$, but not the space X. And we will actually apply the results of this section in some cases where $X \notin \mathfrak{C}$, for instance while working in the situation of Example 10.2.3.

10.2.12. Some auxiliary statements. — We shall need some (apparently) weaker versions of (α), (β) and (γ), which we are now going to list.

- (α') For every morphism $Y \to X$ between k-affinoid spaces with $Y \in \mathfrak{C}$, every section $\sigma : X \to Y$, and every $D \in \mathfrak{F}_Y$, the preimage $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X))$ is a constructible subset of X.
- (α'') For every morphism $Y \to X$ between k-affinoid spaces with *integral* X and with $Y \in \mathfrak{C}$, every section $\sigma\colon X \to Y$, and every object D of $\mathfrak{F}_Y$, either $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(X/D))$ or $X\backslash\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(X/D))$ contains a non-empty Zariski-open subset of X; i.e., $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(X/D))$ either contains or is disjoint from a non-empty Zariski-open subset of X.
- $(\beta^\flat)$ For every morphism $Y \to X$ between k-affinoid spaces with $Y \in \mathfrak{C}$ and every object $D \in \mathfrak{F}_Y$ such that $\mathscr{S}(D)$ is X-flat, $\mathsf{Q}_{\mathrm{fib}}(D/X)$ is a Zariski-open subset of Y.
- (β') For every morphism $Y \to X$ between k-affinoid spaces with $Y \in \mathfrak{C}$, every section $\sigma\colon X \to Y$, and every $D \in \mathfrak{F}_Y$, the pre-image
$$\sigma^{-1}\left(\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)\right)$$
is a Zariski-open subset of X.
- (β'') For every *flat* morphism $Y \to X$ between k-affinoid spaces with $Y \in \mathfrak{C}$, every section $\sigma\colon X \to Y$ and every object $D \in \mathfrak{F}_Y$, the pre-image
$$\sigma^{-1}\left(\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)\right)$$
is a Zariski-open subset of X.
- (γ') For every morphism $Y \to X$ between k-affinoid spaces, every section $\sigma\colon X \to Y$, and every coherent sheaf $\mathscr{F}$ on Y, the pre-image $\sigma^{-1}(\mathrm{Flat}(\mathscr{F}/X))$ is a Zariski-open subset of X.

We have the following hierarchy between our statements:

- (α') is a particular case of (α), and (α'') is a particular case of (α');
- $(\beta^\flat)$ and (β') are particular cases of (β), and (β'') is a particular case of (β');
- (γ') is a particular case of (γ).

10.2.13. Remark. — Assertion (γ') is nothing but assertion (β') when we take for $\mathfrak{F}$ the category $\mathfrak{Coh}$, for $\mathfrak{C}$ the category of all analytic spaces, for $\mathscr{S}$ the identity functor, and for Q the property TRUE.

Our purpose is now to explain how various combinations of the above "auxiliary statements" (with possibly some extra assumptions) imply (α), (β), or (γ) (see Lemmas 10.2.16 – 10.2.23).

10.2.14. Lemma. — *Let $Y \to X$ be a morphism of k-affinoid spaces with $Y \in \mathfrak{C}$. Let p_1 and p_2 be the two projections from $Y \times_X Y$ to Y, and let σ be the diagonal immersion $Y \hookrightarrow Y \times_X Y$. Let D be an object of $\mathfrak{F}_Y$, let y be a point of Y, and let $\mathscr{F}$ be a coherent sheaf on Y.*

(1) *The following are equivalent:*
   (i) *D satisfies Q fiberwise at y;*
   (ii) *p_1^*D satisfies Q fiberwise at $\sigma(y)$ with respect to p_2.*

(2) *If moreover Y is flat over X, the following are equivalent:*
   (iii) *$\mathscr{F}$ is X-flat at y;*
   (iv) *$p_1^*\mathscr{F}$ is Y-flat at $\sigma(y)$ with respect to p_2.*

Proof. — Let x denote the image of y on X. We have $p_2(\sigma(y)) = y$, and p_1 induces an isomorphism $p_2^{-1}(y) \simeq Y_x \times_{\mathscr{H}(x)} \mathscr{H}(y)$, which sends $\sigma(y)$ to y. The equivalence (i) $\iff$ (ii) thus follows from the good behavior of Q with respect to ground field extension, see axiom (2) in 10.2.1.

The implication (iii)$\Rightarrow$(iv) is true without the flatness assumption on Y because flatness at a point is by definition preserved by any base change. If Y is flat over X, the implication (iv)$\Rightarrow$(ii) follows from Proposition 4.5.5. $\square$

10.2.15. Lemma. — *Let $Y \to X$ be a morphism between k-affinoid spaces with X integral, and let $\sigma\colon X \to Y$ be a section of $Y \to X$. Let $\mathscr{F}$ be any coherent sheaf on Y. The pre-image $\sigma^{-1}(\mathrm{Flat}(\mathscr{F}/X))$ contains each pre-image of the generic point of X^{al}, and is in particular non-empty.*

Proof. — Let x be a point of X such that x^{al} is the generic point of X^{al}. Since $\mathscr{O}_{X^{\mathrm{al}},x^{\mathrm{al}}}$ is a field, $\mathscr{F}^{\mathrm{al}}$ is X^{al}-flat at $\sigma(x)^{\mathrm{al}}$. The closed immersion σ makes $\sigma(X)$ a closed analytic subspace of Y, which is finite over X (the map $\sigma(X) \to X$ is even an isomorphism). It follows then from Theorem 8.3.7 that $\mathscr{F}$ is X-flat at $\sigma(x)$. As a consequence, $x \in \sigma^{-1}(\mathrm{Flat}(\mathscr{F}/X))$. $\square$

10.2.16. Lemma. — *Assume that (α') holds. Then (α) holds.*

Proof. — Let Y, X and D be as in (α); since (α) is G-local, we can assume that Y and X are affinoid. Let p_1 and p_2 be the two projections from $Y \times_X Y$ to Y, and let σ be the diagonal immersion $Y \hookrightarrow Y \times_X Y$. By Lemma 10.2.14 (1), one has

$$\mathsf{Q}_{\mathrm{fib}}(D/X) = \sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(p_1^*D/Y)),$$

where fiberwise validity on the right hand side has to be understood with respect to p_2. By applying (α') (which holds by assumption) to $(Y \times_X Y \xrightarrow{p_2} Y, p_1^* D, \sigma)$, we see that $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(p_1^* D/Y))$ is a constructible subset of Y. Therefore $\mathsf{Q}_{\mathrm{fib}}(D/X)$ is a constructible subset of Y. □

10.2.17. Lemma. — *Assume that* (α'') *hold. Then* (α) *holds.*

Proof. — We shall prove that (α') holds, which will imply that (α) holds in view of Lemma 10.2.16. Let Y, X and D be as in (α'). The affinoid space X is quasi-compact for the constructible topology (10.1.7). It thus suffices to prove that for every point x of X, there exists a constructible subset of X containing x which is either included in $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X))$ or in its complement $X\backslash\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X))$.

So, let x be a point of X. Since we are interested in a fiberwise property, we may replace X with the reduced Zariski closure of $\{x\}$; hence we can assume that X is integral and that x^{al} is the generic point of X^{al}. As (α'') holds, either $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X))$ or its complement contains a non-empty Zariski-open subset of X, which is a constructible subset of X containing x. □

10.2.18. Lemma. — *Assume that* (β) *and* (γ) *hold. Then* (α) *holds.*

Proof. — We shall prove that (α'') holds, which will imply that (α) holds in view of Lemma 10.2.17. Let $Y \to X$ be as in (α''). Since (β) and (γ) are assumed to hold, the sets $\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)$ and $\mathrm{Flat}(\mathscr{S}(D)/X)$ are Zariski-open subsets of Y. It follows from Lemma 10.2.15 (applied to $\mathscr{F} = \mathscr{S}(D)$) that the Zariski-open subset $\sigma^{-1}(\mathrm{Flat}(\mathscr{S}(D)/X))$ of X is non-empty. We now distinguish two cases:

- If $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)) \neq \varnothing$, this is a non-empty Zariski-open subset in X which is contained in $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X))$, and we are done.
- If $\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)) = \varnothing$, then $\sigma^{-1}(\mathrm{Flat}(\mathscr{S}(D)/X))$ is a non-empty Zariski-open subset of X which is contained in $X\backslash\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(D/X))$, and we are done.

□

10.2.19. Lemma. — *We make the following assumptions:*

(a) *The fibered category* $\mathfrak{F}$ *is equal to* $\mathfrak{Coh}^{\mathfrak{I}}$ *for some small category* $\mathfrak{I}$ *(this includes the cases* $\mathfrak{F} = \mathfrak{T}$ *and* $\mathfrak{F} = \mathfrak{Coh}$*; cf. Remark 2.2.11).*

(b) *The functor* $\mathscr{S}$ *commutes (as a functor between fibered categories; i.e., incorporating a natural compatibility with pullback isomorphisms) with push-forwards by closed immersions.*

(c) *For every* $D \in \mathfrak{F}_Y$ *and every closed immersion* $\iota\colon Y \hookrightarrow Z$ *of* X*-analytic spaces,* D *satisfies fiberwise* Q *at a given point* $y \in Y$ *if and only if* $\iota_* D$ *satisfies* Q *fiberwise at* $\iota(y)$.

Assume moreover that (β'') *holds. Then* (β) *holds.*

Proof. — Let Y, X, and D be as in (β); since (β) is G-local, we can assume that Y and X are affinoid. The morphism $Y \to X$ then factorizes through a closed immersion $\iota\colon Y \hookrightarrow \Delta \times_k X$ for a suitable compact k-polydisc Δ. Set

$$U = \mathsf{Q}_{\mathrm{fib}}(\Delta \times_k X/X) \cap \mathrm{Flat}(\mathscr{S}(\iota_* D)/X).$$

By the assumptions of the lemma, $U \cap Y = \mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)$; it is therefore sufficient to prove that U is a Zariski-open subset of $\Delta \times_k X$. Hence by replacing Y with $\Delta \times_k X$ and D with $\iota_* D$, we reduce to the case where Y is X-flat.

Let p_1 and p_2 be the two projections from $Y \times_X Y$ to Y, and let σ be the diagonal immersion $Y \hookrightarrow Y \times_X Y$. By Lemma 10.2.14 (note that flatness of Y over X is needed to apply statement (2) of loc. cit.), one has

$$\mathsf{Q}_{\mathrm{fib}}(D/X) = \sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(p_1^* D/Y))$$

and

$$\mathrm{Flat}(\mathscr{S}(D)/X) = \sigma^{-1}(\mathrm{Flat}(p_1^*\mathscr{S}(D)/Y)) = \sigma^{-1}(\mathrm{Flat}(\mathscr{S}(p_1^* D)/Y)),$$

where fiberwise validity and flatness over Y are understood to be with respect to p_2. By applying (β'') (which holds by assumption) to $(Y \times_X Y \xrightarrow{p_2} Y, p_1^* D, \sigma)$, we see that

$$\sigma^{-1}(\mathsf{Q}_{\mathrm{fib}}(p_1^* D/Y) \cap \mathrm{Flat}(\mathscr{S}(p_1^* D)/Y))$$

is a Zariski-open subset of Y (note that we use once again the flatness of Y over X, because this is one of the assumptions of (β'')); therefore $\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)$ is a Zariski-open subset of Y. □

10.2.20. Remark. — If we take for $\mathfrak{F}$ the category $\mathfrak{Coh}$, for $\mathfrak{C}$ the category of all analytic spaces, for $\mathscr{S}$ the identity functor, and for Q the property of being CM, or S_m for some specified m, or of a given codepth, or more simply the property TRUE, then the assumptions (a), (b) and (c) of Lemma 10.2.19 above are fulfilled.

10.2.21. Lemma. — *Assume that* (γ') *holds. Then* (γ) *holds.*

Proof. — Take for $\mathfrak{F}$ the category $\mathfrak{Coh}$, for $\mathfrak{C}$ the category of all analytic spaces, for $\mathscr{S}$ the identity functor, and for Q the property TRUE. By Lemma 10.2.19 and Remark 10.2.20, we have $(\beta'') \Rightarrow (\beta)$, and thus also $(\beta') \Rightarrow (\beta)$ since (β') is stronger that (β''). But in our context, $(\beta) = (\gamma)$ and $(\beta') = (\gamma')$, whence the claim. □

10.2.22. Lemma. — *Assume that* (γ) *and* (β') *hold. Then* (α) *holds.*

Proof. — We shall prove that (β) holds, which will imply that (α) holds in view of Lemma 10.2.18. Let Y, X, and D be as in (β); since (β) is G-local, we can assume that Y and X are affinoid. Let p_1 and p_2 be the two projections from $Y \times_X Y$ to Y, and let σ be the diagonal immersion $Y \hookrightarrow Y \times_X Y$. Set

$$E = \mathsf{Q}_{\mathrm{fib}}(p_1^* D/Y) \cap \mathrm{Flat}(p_1^*\mathscr{S}(D)/Y) = \mathsf{Q}_{\mathrm{fib}}(p_1^* D/Y) \cap \mathrm{Flat}(\mathscr{S}(p_1^* D)/Y),$$

where fiberwise validity and flatness over Y are understood to be with respect to p_2. Since $\sigma(\mathrm{Flat}(\mathscr{S}(D)/X)) \subset \mathrm{Flat}(p_1^*\mathscr{S}(D))/Y)$, we have

$$\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X) = \sigma^{-1}(E) \cap \mathrm{Flat}(\mathscr{S}(D)/X),$$

due to Lemma 10.2.14. Since we assume that (γ) holds, the set $\mathrm{Flat}(\mathscr{S}(D)/X)$ is Zariski-open, and it is thus sufficient to prove that $\sigma^{-1}(E)$ is Zariski-open. But the latter follows by applying (β') (which holds by assumption) to the list of data $(Y \times_X Y \xrightarrow{p_2} Y, p_1^*D, \sigma)$. □

10.2.23. Lemma. — *Assume that* (α), $(\beta^\flat)$, *and* (γ) *hold. Then* (β) *holds.*

Proof. — Let Y, X, and D be as in (β); since (β) is G-local, we can assume that Y and X are affinoid. Since (α) and (γ) hold, $\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)$ is a constructible subset of Y. In order to prove (β), it thus suffices to prove that the intersection $\mathsf{Q}_{\mathrm{fib}}(D/X)\cap\mathrm{Flat}(\mathscr{S}(D)/X)$ is open (Lemma 10.1.10 (4)). Let y be a point of $\mathsf{Q}_{\mathrm{fib}}(D/X) \cap \mathrm{Flat}(\mathscr{S}(D)/X)$. Since $y \in \mathrm{Flat}(\mathscr{S}(D)/X)$, it follows from (γ) that there exists an affinoid neighborhood V of y in Y which is included in $\mathrm{Flat}(\mathscr{S}(D)/X)$. Since we assume that $(\beta^\flat)$ holds, the set $\mathsf{Q}_{\mathrm{fib}}(D_V/X) = \mathsf{Q}_{\mathrm{fib}}(D/X) \cap V$ is a Zariski-open subset of V; in particular, it contains a neighborhood of y. □

10.3. The flat locus

The main theorem of this section (Theorem 10.3.2) says the following: if $Y \to X$ is a morphism of k-analytic spaces and $\mathscr{F}$ is a coherent sheaf on Y, the X-flat locus of $\mathscr{F}$ is a Zariski-open subset of Y; our proof follows Kiehl's strategy developed in [**Kie67b**].

Zariski-openness of the flat locus has the following consequences, in view of the Nullstellensatz: if $|k^\times| \neq \{1\}$ and if Y and X are strict, then $\mathscr{F}$ is X-flat if and only if it is X-flat at every rigid point of Y; and $\mathscr{F}$ is X-flat at every point lying over a given point x of X if and only if it is X-flat at every rigid point of Y_x.

We use the first consequence together with Theorem 8.3.7 to get the compatibility between our notion of flatness and that of rigid flatness (Corollary 10.3.3) as well as that of formal flatness (Corollary 10.3.5).

We use the second one to prove that flatness holds automatically over any Abhyankar point of the target space – provided the latter is reduced (Theorem 10.3.7). Let us make this precise. Once having reduced to the strict good case in a standard way, we only have to check flatness at rigid points of the fiber under investigation (by the above). We then use a “deboundarization” result (Lemma 10.3.6) for such a rigid point, which rests on the notion of smallest analytic domain containing the image of a morphism of analytic germs (Theorem 7.3.1); this enables us to reduce to the inner case, for which it suffices to check naive flatness (Theorem 8.3.4). But the latter holds for free because the local ring of an Abhyankar point of a good analytic space is artinian (Example 3.2.10), hence a field whenever the space is reduced.

For the reader's convenience, we state and prove the following theorem of Kiehl, which will be crucial for our description of the flat locus.

***10.3.1. Theorem* (Kiehl, [Kie67b] Satz 1).** — *Let $Y \to X$ be a morphism of noetherian schemes, let $\mathscr{E}$ be a coherent sheaf on Y, and let Z be a closed subscheme of Y of finite type over X. The intersection*

$$Z \cap \mathrm{Flat}(\mathscr{E}/X)$$

is a Zariski-open subset of Z.

Proof. — If Y is itself of finite type over X and if $Z = Y$, this is [**EGA IV**$_3$], Thm. 11.1.1. For the general case, one can follow *mutatis mutandis* the proof of loc. cit., except that classical "generic flatness" ([**EGA IV**$_2$], Lemme 6.9.2) has to be replaced with the following stronger statement: *let $A \to B$ be a morphism of noetherian rings with A a* domain*, let M be a finite B-module, and let J be an ideal of B such that the A-algebra B/J is finitely generated; there exists $a \neq 0$ in A such that $M_{\mathfrak{p}}$ is A-flat for every prime ideal $\mathfrak{p}$ of B with $J \subset \mathfrak{p}$ and $a \notin \mathfrak{p}$.*

Let us prove this claim. The ring $C := \bigoplus_n J^n/J^{n+1}$ is finitely generated over B/J (since J is finitely generated by noetherianity of B), hence it is also finitely generated over A. By classical generic flatness, there exists $a \neq 0$ in A such that $(C \otimes_B M)_a$ is A-flat. Let $\mathfrak{p}$ be a prime ideal of B with $J \subset \mathfrak{p}$ and $a \notin \mathfrak{p}$. Since $a \notin \mathfrak{p}$, the B-module $(C \otimes_B M)_{\mathfrak{p}}$ is a localization of $(C \otimes_B M)_a$, hence is A-flat. This means that $J^n M_{\mathfrak{p}}/J^{n+1}M_{\mathfrak{p}}$ is A-flat for every n. Therefore $M_{\mathfrak{p}}/J^n M_{\mathfrak{p}}$ is A-flat for every n. Since $J \subset \mathfrak{p}$ this implies by [**EGA III**$_1$], Chapitre 0, Prop. 10.2.6 that the A-module $M_{\mathfrak{p}}$ is flat. □

10.3.2. Theorem. — *Let $Y \to X$ be a morphism of k-analytic spaces and let $\mathscr{E}$ be a coherent sheaf on Y. The subset* $\mathrm{Flat}(\mathscr{E}/Y)$ *of Y is Zariski-open.*

Proof. — We want to prove assertion (γ) of 10.2.8. By Lemma 10.2.21, it is sufficient to prove assertion (γ') of 10.2.12. We thus may assume that Y and X are k-affinoid and $Y \to X$ has a section σ, and it is then sufficient to prove that $\sigma^{-1}(\mathrm{Flat}(\mathscr{E}/X))$ is a Zariski-open subset of X; or, what amounts to the same, that $Z \cap \mathrm{Flat}(\mathscr{E}/X)$ is a Zariski-open subset of Z, where Z is the closed analytic subspace of Y defined by the closed immersion σ.

By Theorem 8.3.7, the intersection $\mathrm{Flat}(\mathscr{E}/X) \cap Z$ is nothing but the pre-image of $\mathrm{Flat}(\mathscr{E}^{\mathrm{al}}/X^{\mathrm{al}}) \cap Z^{\mathrm{al}}$ in Z. Both Y^{al} and X^{al} are noetherian schemes, and Z^{al} is a Zariski-closed subscheme of Y^{al} that is of finite type over X^{al}. By Kiehl's theorem stated above (Theorem 10.3.1), $\mathrm{Flat}(\mathscr{E}^{\mathrm{al}}/X^{\mathrm{al}}) \cap Z^{\mathrm{al}}$ is a Zariski-open subset of Z^{al}; as a consequence, $\mathrm{Flat}(\mathscr{E}/X) \cap Z$ is a Zariski open subset of Z. □

Due to this theorem we recover the fact that in the rigid setting, global algebraic flatness implies global analytic flatness:

10.3.3. Corollary. — *Let $Y \to X$ be a morphism of* strict *k-affinoid spaces. Let $\mathscr{E}$ be a coherent sheaf on Y. Then $\mathscr{E}$ is X-flat if and only if $\mathscr{E}(Y)$ is a flat $\mathscr{O}_X(X)$-module.*

Proof. — If $\mathscr{E}$ is X-flat, it is in particular naively X-flat, and Lemma 4.2.1 then ensures that $\mathscr{E}(Y)$ is flat over $\mathscr{O}_X(X)$. Conversely, assume that $\mathscr{E}(Y)$ is flat over $\mathscr{O}_X(X)$. Then by Theorem 8.3.7, $\mathscr{E}$ is X-flat at any rigid point of Y. Since $\mathrm{Flat}(\mathscr{E}/X)$ is a Zariski-open subset of Y by Theorem 10.3.2, it follows from the analytic (resp. algebraic) Nullstellensatz if $|k^\times| \neq \{1\}$ (resp. $|k^\times| = \{1\}$), that $\mathrm{Flat}(\mathscr{E}/X) = Y$. □

10.3.4. Remark. — Corollary 10.3.3 above is false in general without any strictness assumption; for a counter-example, see 4.4.10.

10.3.5. Corollary. — *Let $\mathfrak{Y} \to \mathfrak{X}$ be a morphism between topologically finitely presented* Spf *k°-formal schemes, and let $\mathscr{E}$ be a coherent sheaf on $\mathfrak{Y}$ which is $\mathfrak{X}$-flat. The associated coherent sheaf $\mathscr{E}_\eta$ on $\mathfrak{Y}_\eta$ is $\mathfrak{X}_\eta$-flat.*

Proof. — We can assume that both $\mathfrak{Y}$ and $\mathfrak{X}$ are affine formal schemes. By assumption, $\mathscr{E}(\mathfrak{Y})$ is a flat $\mathscr{O}_{\mathfrak{X}}(\mathfrak{X})$-module; therefore, $\mathscr{E}_\eta(\mathfrak{Y}_\eta) = \mathscr{E}(\mathfrak{Y}) \otimes_{k^\circ} k$ is flat over $\mathscr{O}_{\mathfrak{X}_\eta}(\mathfrak{X}_\eta) = \mathscr{O}_{\mathfrak{X}}(\mathfrak{X}) \otimes_{k^\circ} k$. In view of the preceding corollary, this implies that $\mathscr{E}_\eta$ is $\mathfrak{X}_\eta$-flat. □

Our purpose is now to prove that flatness holds automatically over Abhyankar points (1.4.10) of reduced spaces. This is Theorem 10.3.7 below, which witnesses the fact that the best analytic analogue of a scheme-theoretic generic fiber is a fiber over an Abhyankar point of a reduced space.

Theorem 10.3.7 rests on the following "deboundarization" lemma, which is of independent interest.

10.3.6. Lemma. — *Let $Y \to X$ be a morphism of separated Γ-strict k-analytic spaces, let y be a point of Y and let x be its image in X. If $\mathscr{H}(y)$ is finite over $\mathscr{H}(x)$, there exists a Γ-strict k-analytic space X', a quasi-étale morphism $X' \to X$, and a pre-image y' of y on $Y' := Y \times_X X'$ such that $Y' \to X'$ is inner at y'.*

Proof. — Let L be the separable closure of $\mathscr{H}(x)$ inside $\mathscr{H}(y)$, and let Z be a finite étale cover of an analytic neighborhood of x in X such that x has one pre-image x' on Z with $\mathscr{H}(x)' \simeq L$ (such a Z exists by [**Ber93**], Thm. 3.4.1).

There is a canonical point y' of $T := Y \times_X Z$ lying above y and x' such that $\mathscr{H}(y')$ is a finite radicial extension of $\mathscr{H}(x')$, which implies that $\widetilde{\mathscr{H}(y')}$ is a finite radicial extension of $\widetilde{\mathscr{H}(x')}$ (A.4.12).

Let (X',x') be the smallest analytic domain of (Z,x') through which (T,y') factorizes (Theorem 7.3.1); it is Γ-strict by loc. cit., and the quasi-compact open subset $\widetilde{(X',x')}^{\Gamma}$ of $\mathbf{P}_{\widetilde{\mathscr{H}(x')}^{\Gamma}/\widetilde{k}^{\Gamma}}$ is the image of $\widetilde{(T,y')}^{\Gamma}$ under $\mathbf{P}_{\widetilde{\mathscr{H}(y')}^{\Gamma}/\widetilde{k}^{\Gamma}} \to \mathbf{P}_{\widetilde{\mathscr{H}(x')}^{\Gamma}/\widetilde{k}^{\Gamma}}$, which is a homeomorphism because $\widetilde{\mathscr{H}(y')}^{\Gamma}$ is radicial over $\widetilde{\mathscr{H}(x')}^{\Gamma}$ (Remark 7.2.7). Hence the inverse image of $\widetilde{(X',x')}^{\Gamma}$ in $\mathbf{P}_{\widetilde{\mathscr{H}(y')}^{\Gamma}/\widetilde{k}^{\Gamma}}$ is exactly $\widetilde{(T,y')}^{\Gamma}$. This implies by the criterion 3.5.9 (2) that the map $(T,y') \to (X',x')$ is boundaryless, and the X-analytic space X' satisfies the required conditions. □

10.3.7. Theorem. — *Let $Y \to X$ be a morphism of k-analytic spaces, with X reduced. Let x be an Abhyankar point of X (1.4.10), and let $\mathscr{E}$ be a coherent sheaf on Y. The sheaf $\mathscr{E}$ is X-flat above x.*

Proof. — We set $n = d_k(x) = \dim_x X$. We may assume that Y and X are k-affinoid. Let r be a k-free polyradius such that $|k_r^{\times}| \neq \{1\}$ and such that both Y_r and X_r are strictly k_r-affinoid. The k_r-analytic space X_r is n-dimensional and is reduced (as follows from the fact that k_r is analytically separable over k, or in a more elementary way from Lemma 2.7.8 applied with $\mathscr{X} = \operatorname{Spec} A$). If $\mathfrak{s} : X \to X_r$ denotes the Shilov section, one has $d_{k_r}(\mathfrak{s}(x)) = d_k(x) = n$ (1.4.8). Hence, due to Proposition 4.5.6, we can replace the field k by k_r, the spaces X and Y by X_r and Y_r, the sheaf $\mathscr{E}$ by $\mathscr{E}_{Y_r}$, and the point x by $\mathfrak{s}(x)$; i.e., we can assume that $|k^{\times}| \neq \{1\}$ and that Y and X are strictly k-affinoid.

Theorem 10.3.2 ensures that $\operatorname{Flat}(\mathscr{E}/X)$ is a Zariski-open subset of Y. We want to prove that it contains Y_x. Since Y_x is strictly affinoid over the non-trivially valued field $\mathscr{H}(x)$, it is sufficient to prove that it contains every rigid point of Y_x.

Let y be a rigid point of Y_x. By Lemma 10.3.6 above, there exists a strictly k-analytic space X', a quasi-étale map $X' \to X$, and a pre-image y' of y in $Y' := Y \times_X X'$ such that $Y' \to X'$ is inner at y'; we can assume (by shrinking it if necessary) that X' is strictly k-affinoid, in which case so is Y'.

Since X' is quasi-étale over the reduced analytic space X, it is reduced by Proposition 5.5.5; moreover, it is of dimension $\leqslant n$ by 1.4.14. Since $\mathscr{H}(x')$ is finite over $\mathscr{H}(x)$, we have $d_k(x') = n$ (hence $\dim_{x'} X' = n$). It follows from Example 3.2.6 that $\operatorname{centdim}(X',x') = n$, which implies in view of Corollary 3.2.9 that $\mathscr{O}_{X',x'}$ is artinian, hence a field because it is also reduced.

Therefore $\mathscr{E}_{Y',y'}$ is flat over $\mathscr{O}_{X',x'}$. As $(Y',y') \to (X',x')$ is boundaryless, this implies in view of Theorem 8.3.4 that $\mathscr{E}_{Y'}$ is X'-flat at y'. Since $X' \to X$ is quasi-étale, it is flat (Corollary 5.3.2). Therefore $\mathscr{E}$ is X-flat at y by Proposition 4.5.5; otherwise said, $y \in \operatorname{Flat}(\mathscr{E}/X)$. □

10.4. The fiberwise closure of a locally constructible set

In Section 10.7, we shall reduce our study of the locus of fiberwise validity of the S_n property (resp. geometric R_n property) to that of fiberwise validity of the CM property (resp. the quasi-smooth) property, through a result that expresses the former in terms of the latter and of some codimension considerations (Lemma 10.7.1).

We thus have to investigate, being given a morphism $\varphi\colon Y \to X$ of k-analytic spaces and a locally constructible subset E of Y such that E_x is a Zariski-closed subset of Y_x for every $x \in X$, the *fiberwise codimension function* $y \mapsto \operatorname{codim}_y(E_{\varphi(y)}, Y_{\varphi(y)})$. For that purpose we prove the following results in this section, with $\varphi\colon Y \to X$ as above.

(1) Let E be a locally constructible subset of Y, and set $\overline{E}^{\varphi} = \bigcup_{x\in X} \overline{E_x}^{Y_x}$. Then $\overline{E}^{\varphi}$ is a locally constructible subset of Y (Theorem 10.4.3).
(2) For every non-negative integer d, the set of points $y \in \overline{E}^{\varphi}$ such that $\dim_y(\overline{E}^{\varphi})_{\varphi(y)} = d$ is locally constructible (Theorem 10.4.3).
(3) Let F be a locally constructible subset of Y contained in E. For every non-negative integer d, the set of points $y \in \overline{E}^{\varphi}$ such that $\operatorname{codim}_y(\overline{F}^{\varphi}, \overline{E}^{\varphi})_{\varphi(y)} = d$ is locally constructible (Proposition 10.4.4).

Note that the situation adressed by these statements might look slightly more general than the one we have described as motivation, which involves only the fiberwise codimension in Y of a locally constructible subset which is fiberwise Zariski-closed, but in fact even for proving (3) in this particular case, we do not see how to avoid using (1) and (2) in their full generality (and we think moreover that they are of independent interest).

Let us mention that our results are quite analogous to the ones proved in [**EGA IV**$_3$] 9.5 but our proofs are more involved, because the Zariski topology of a fiber is in general strictly finer than the one induced by the global Zariski topology. The key tool to bypass this problem is Theorem 6.3.3 about the local rings of generic fibers.

10.4.1. — Let $\varphi : Y \to X$ be a morphism of k-analytic spaces, and let E be a locally constructible subset of Y. We set

$$\overline{E}^{\varphi} = \bigcup_{x\in X} \overline{E \cap Y_x}^{Y_x} = \bigcup_{x\in X} \overline{E \cap Y_x}^{Y_x,\mathrm{Zar}}$$

(the second equality comes from Corollary 10.1.11). We say that $\overline{E}^{\varphi}$ is the *fiberwise closure of E with respect to φ (or over X).* We obviously have $\overline{E}^{\varphi} \subset \overline{E}^{Y}$.

Let y be a point of $\overline{E}^{\varphi}$ and let x be its image in X. The dimension of the Zariski-closed subset $\overline{E}^{\varphi}_x$ of Y_x at y is called the *relative dimension of $\overline{E}^{\varphi}$ at y.*

We shall prove below that $\overline{E}^{\varphi}$ is locally constructible (Theorem 10.4.3). For that purpose, we need the following technical proposition.

10.4.2. Proposition. — *Let $\varphi\colon Y \to X$ be a morphism between k-affinoid spaces, let Z be a closed analytic subspace of Y with dense (Zariski-open) complement U, and let t be a point of Z. Assume that $t \in \operatorname{Int}(Y/X)$ and both Y and Z are X-flat at t. The point t then belongs to $\overline{U}^{\varphi}$.*

Proof. — Let s be the image of t in X. We can perform any base change consisting of replacing X with an affinoid neighborhood of s without modifying our assumptions (as far as density of U in Y is concerned, this rests on Corollary 1.5.13). We can therefore assume that $\mathfrak{m}_s$ is generated by an ideal of $\mathscr{O}_X(X)$, and we denote by S the corresponding closed analytic subspace of X; we set $T = Y \times_X S$. The local ring $\mathscr{O}_{S,s} = \mathscr{O}_{X,s}/\mathfrak{m}_s\mathscr{O}_{X,s}$ is then a field, and $\mathscr{O}_{S^{\mathrm{al}},s^{\mathrm{al}}}$ is thus also a field by 2.1.5. We use the conventions of 2.1.2 (3); be aware that $t_s^{\mathrm{al}} := (t_s)^{\mathrm{al}}$ is a point of the scheme $T_s^{\mathrm{al}} := (T_s)^{\mathrm{al}}$, while t^{al} is a point of T^{al} lying in the fiber $T_{s^{\mathrm{al}}}^{\mathrm{al}} := (T^{\mathrm{al}})_{s^{\mathrm{al}}}$ of T^{al} over the point s^{al} of S^{al}. We are now going to prove the two following statements.

(1) The point t^{al} belongs to the closure of $U_{s^{\mathrm{al}}}^{\mathrm{al}}$ in $Y_{s^{\mathrm{al}}}^{\mathrm{al}}$.
(2) The scheme Y_s^{al} is flat over $Y_{s^{\mathrm{al}}}^{\mathrm{al}}$ at t_s^{al}.

Let us prove (1) by contradiction. Assume that it does not hold. The (support of) the closed subscheme $Z_{s^{\mathrm{al}}}^{\mathrm{al}}$ of $Y_{s^{\mathrm{al}}}^{\mathrm{al}}$ is then a neighborhood of t^{al} in $Y_{s^{\mathrm{al}}}^{\mathrm{al}}$, and we thus have

$$\text{(a)} \qquad \dim_{\mathrm{Krull}} \mathscr{O}_{Y_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}} = \dim_{\mathrm{Krull}} \mathscr{O}_{Z_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}}.$$

By assumption, the affinoid spaces Y and Z are X-flat at t; therefore the schemes Y^{al} and Z^{al} are X^{al}-flat at t^{al} (Lemma 4.2.1), whence the equations

$$\begin{aligned} &\text{(b)} \qquad \dim_{\mathrm{Krull}} \mathscr{O}_{Y^{\mathrm{al}},t^{\mathrm{al}}} &= \dim_{\mathrm{Krull}} \mathscr{O}_{X^{\mathrm{al}},s^{\mathrm{al}}} + \dim_{\mathrm{Krull}} \mathscr{O}_{Y_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}} \\ &\text{(c)} \qquad \dim_{\mathrm{Krull}} \mathscr{O}_{Z^{\mathrm{al}},t^{\mathrm{al}}} &= \dim_{\mathrm{Krull}} \mathscr{O}_{X^{\mathrm{al}},s^{\mathrm{al}}} + \dim_{\mathrm{Krull}} \mathscr{O}_{Z_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}} \end{aligned}$$

which together with (a) yield the equality $\dim_{\mathrm{Krull}} \mathscr{O}_{Y^{\mathrm{al}},t^{\mathrm{al}}} = \dim_{\mathrm{Krull}} \mathscr{O}_{Z^{\mathrm{al}},t^{\mathrm{al}}}$. This implies that (the support of) Z^{al} contains at least one irreducible component of Y^{al} going through t^{al} and contradicts the assumption that $U^{\mathrm{al}} = Y^{\mathrm{al}} \backslash Z^{\mathrm{al}}$ is a dense open subset of Y^{al}, whence (1).

Let us now prove (2). The local ring $\mathscr{O}_{S,s}$ is a field, and t lies in $\operatorname{Int}(T/S)$ since it lies in $\operatorname{Int}(Y/X)$ by assumption; therefore Theorem 6.3.3 implies that $\mathscr{O}_{Y_s,t} = \mathscr{O}_{T_s,t}$ if flat over $\mathscr{O}_{T,t}$. The latter is flat over $\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}}$ (by 2.1.4), and since $\mathscr{O}_{S^{\mathrm{al}},s^{\mathrm{al}}}$ is a field we have

$$\mathscr{O}_{T^{\mathrm{al}},t^{\mathrm{al}}} = \mathscr{O}_{T_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}} = \mathscr{O}_{Y_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}}.$$

As a consequence, $\mathscr{O}_{Y_s,t}$ is flat over $\mathscr{O}_{Y_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}}$. On the other hand, $\mathscr{O}_{Y_s,t}$ is flat over $\mathscr{O}_{Y_s^{\mathrm{al}},t^{\mathrm{al}}}$ (again by 2.1.4). The vertical and horizontal arrows of the commutative

diagram

$$\begin{tikzcd}
\mathscr{O}_{Y_s,t} & \mathscr{O}_{Y_s^{\mathrm{al}},t_s^{\mathrm{al}}} \arrow[l] \\
\mathscr{O}_{Y_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}} \arrow[u] \arrow[ur] &
\end{tikzcd}$$

are thus flat; hence $\mathscr{O}_{Y_{s^{\mathrm{al}}}^{\mathrm{al}},t^{\mathrm{al}}} \to \mathscr{O}_{Y_s^{\mathrm{al}},t_s^{\mathrm{al}}}$ is flat too and (2) is proven.

Due to (1), there exists $\omega \in U_{s^{\mathrm{al}}}^{\mathrm{al}}$ which specializes to t^{al}. This implies in view of (2) that there exists a point ω' on Y_s^{al} lying above ω and specializing to t_s^{al}. Since ω' lies above ω, it belongs to U_s^{al}; we thus have shown that $t_s^{\mathrm{al}} \in \overline{U_s^{\mathrm{al}}}^{Y_s^{\mathrm{al}}}$; but this means that $t \in \overline{U_s}^{Y_s,\mathrm{Zar}} = \overline{U_s}^{Y_s}$ or, in other words, that t belongs to $\overline{U}^{\varphi}$. □

10.4.3. Theorem. — *Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces and let E be a locally constructible subset of Y. Let N be a subset of $\mathbf{Z}_{\geqslant 0}$. The subset of $\overline{E}^{\varphi}$ consisting of points at which the relative dimension of $\overline{E}^{\varphi}$ belongs to N is a locally constructible subset of Y. In particular, $\overline{E}^{\varphi}$ is locally constructible (take $N = \mathbf{Z}_{\geqslant 0}$).*

Proof. — We first consider the case where $N = \mathbf{Z}_{\geqslant 0}$; i.e., we first prove that $\overline{E}^{\varphi}$ is locally constructible.

By arguing locally on Y (which is possible due to Corollary 1.5.13), we can assume that E is constructible. Write $E = \bigcup U_i \cap F_i$, where (U_i), resp. (F_i), is a finite family of Zariski-open, resp. Zariski-closed, subsets of Y. Since $\overline{E}^{\varphi} = \bigcup_i \overline{U_i \cap F_i}^{\varphi}$, it suffices to treat the case where $E = U \cap F$, with U, resp. F, a Zariski-open, resp. Zariski-closed, subset of Y. By replacing Y with F (equipped with any structure of a closed analytic subspace; e.g., its reduced structure) we can assume that $E = U$.

We are now going to apply some of the general results in 10.2, which involve two categories $\mathfrak{F}$ and $\mathfrak{C}$ and a property Q as in 10.2.1. We take $\mathfrak{F}, \mathfrak{C}$ and Q as in Example 10.2.3 (with $E = U$). What we want to prove is assertion (α) of 10.2.8. By Lemma 10.2.17, it is sufficient to prove assertion (α'') of 10.2.12. We thus reduce to the following situation: Y and X are affinoid, X is integral, and φ admits a section σ; and we have to prove that there exists a non-empty Zariski-open subset of X which is either contained in $\sigma^{-1}(\overline{U}^{\varphi})$ or disjoint from it. This amounts to proving that there exists a non-empty Zariski-open subset of $\sigma(X)$ which is either contained in $\overline{U}^{\varphi}$ or disjoint of it. We choose $x \in X$ such that x^{al} is the generic point of X^{al} and we distinguish three cases.

If $\sigma(x) \in U$, then $U \cap \sigma(X)$ is a non-empty Zariski-open subset of $\sigma(X)$ contained in $\overline{U}^{\varphi}$, and we are done. If $\sigma(x) \notin \overline{U}^{Y}$, then $\sigma(X)\backslash\overline{U}^{Y}$ is a non-empty Zariski-open subset of $\sigma(X)$ disjoint from $\overline{U}^{Y}$ (hence disjoint from $\overline{U}^{\varphi}$) and we are done.

We thus can assume that $\sigma(x)$ lies on $\overline{U}^{Y}$ but not on U. Under this assumption, σ goes through $\overline{U}^{Y}_{\mathrm{red}}$; therefore by replacing Y with $\overline{U}^{Y}_{\mathrm{red}}$, we can assume that U is Zariski-dense in Y. Let $Z = (Y\backslash U)_{\mathrm{red}}$; this is closed analytic subspace of Y through which σ factorizes. According to Theorem 10.3.2, the X-flat loci $\mathrm{Flat}(Y/X)$ and $\mathrm{Flat}(Z/X)$ are Zariski-open subsets of Y and Z respectively, and both of them contain $\sigma(x)$ by Lemma 10.2.15. Therefore $\mathrm{Flat}(Y/X) \cap \mathrm{Flat}(Z/X) \cap \sigma(X)$ is a Zariski-open subset of $\sigma(X)$ which contains $\sigma(x)$, and it suffices to prove that it is contained in $\overline{U}^{\varphi}$. But this follows from Proposition 10.4.2 since $\sigma(X) \subset \mathrm{Int}(Y/X)$ by 1.3.21 (3); this ends the proof when $N = \mathbf{Z}_{\geqslant 0}$.

Let us consider now the case of an arbitrary subset N of $\mathbf{Z}_{\geqslant 0}$. We go back to the general assumptions of the theorem. By arguing locally on Y (which is possible due to Corollary 1.5.13), we may assume that Y is finite-dimensional and E is constructible. Since the dimension of any fiber of φ is bounded by $\dim Y$, the subset of $\overline{E}^{\varphi}$ we are interested in is a finite Boolean combination of sets of the form

$$\overline{E}^{\varphi}_{\geqslant\delta} := \{y \in \overline{E}^{\varphi} \,|\, \dim_y \overline{E}^{\varphi} \geqslant \delta\}.$$

It thus suffices to prove that $\overline{E}^{\varphi}_{\geqslant\delta}$ is constructible for every $\delta \in \mathbf{Z}_{\geqslant 0}$.

The constructible subset E can be written $\bigcup U_i \cap F_i$, where (U_i), resp. (F_i), is a finite family of Zariski-open, resp. Zariski-closed, subsets of Y; the set $\overline{E}^{\varphi}_{\geqslant\delta}$ is then equal to $\bigcup_i \overline{U_i \cap F_i}^{\varphi}_{\geqslant\delta}$. It is thus suffices to prove the theorem for each of the $U_i \cap F_i$'s; i.e., we may assume that $E = U \cap F$ with U, resp. F, a Zariski-open, resp. Zariski-closed, subset of Y. By replacing Y with F (equipped with any structure of a closed analytic subspace of Y; e.g., its reduced structure), we reduce to the case where $E = U$.

For every integer $\delta \geqslant 0$, let us denote by $G_{\geqslant\delta}$ be the subset of Y consisting of points at which φ is of dimension $\geqslant \delta$; since $y \mapsto \dim_y \varphi$ is upper semi-continuous for the Zariski topology ([**Duc07b**], Thm. 4.9), $G_{\geqslant\delta}$ is Zariski-closed, and $G_{\geqslant\delta} \cap U$ is thus constructible.

Let x be a point of X and let (Y_i) be the family of irreducible components of Y_x. For every i, we denote by d_i the dimension of Y_i. Let I be the set of indices i such that Y_i intersects U. Fix $\delta \in \mathbf{Z}_{\geqslant 0}$. We have the following equalities:

$$\text{(a)} \qquad \overline{U}^{\varphi}_x = \bigcup_{i \in I} Y_i$$

$$\text{(b)} \qquad (\overline{U}^{\varphi}_{\geqslant\delta})_x = \bigcup_{i \in I, d_i \geqslant \delta} Y_i$$

$$\text{(c)} \qquad (G_{\geqslant\delta} \cap U)_x = \bigcup_{i \in I, d_i \geqslant \delta} (Y_i \cap U)$$

$$\text{(d)} \qquad \overline{G_{\geqslant\delta} \cap U}^{\varphi}_x = \bigcup_{i \in I, d_i \geqslant \delta} Y_i$$

(note that (a) and (d) rest on density of $Y_i \cap U$ in Y_i for every $i \in I$). We deduce from (b) and (d) (which hold for every $x \in X$) that

$$\overline{U}^{\varphi}_{\geqslant\delta} = \overline{G_{\geqslant\delta} \cap U}^{\varphi}.$$

By the case $N = \mathbf{Z}_{\geqslant 0}$ already proven, $\overline{G_{\geqslant\delta} \cap U}^{\varphi}$ is constructible, whence the constructibility of $\overline{U}^{\varphi}_{\geqslant\delta}$. □

We now come to our original motivation, namely the fiberwise codimension function.

10.4.4. Proposition. — *Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces, and let E and F be two locally constructible subsets of Y such that $F \subset E$ (recall that by Proposition 10.1.12, a locally constructible subset of Y is constructible as soon as Y is finite-dimensional). Let N be a subset of $\mathbf{Z}_{\geqslant 0} \cup \{+\infty\}$. The set*

$$\{y \in \overline{E}^{\varphi} | \mathrm{codim}_y(\overline{F}^{\varphi}_{\varphi(y)}, \overline{E}^{\varphi}_{\varphi(y)}) \in N\}$$

is a locally constructible subset of Y.

10.4.5. Remark. — We recall that a a locally constructible subset of Y is constructible as soon as Y is finite-dimensional, by Proposition 10.1.12. We also recall that for y a point of $\overline{E}^{\varphi}$, the codimension $\mathrm{codim}_y(\overline{F}^{\varphi}_{\varphi(y)}, \overline{E}^{\varphi}_{\varphi(y)})$ is equal to $+\infty$ if and only if $y \notin \overline{F}^{\varphi}$ (1.5.15).

Proof of Proposition 10.4.4. — By arguing locally (which is possible in view of 1.5.16 (1)), we can assume that Y is finite dimensional. Let $D \in \mathbf{Z}_{\geqslant 0}$ be such that all fibers of $Y \to X$ are of dimension bounded by D (if $Y \neq \varnothing$ one may take $D = \dim Y$). For every (n, m) in $\{0, \dots, D\}^2$, we denote by $\Lambda_{n,m}$ the set of $y \in \overline{F}^{\varphi}$ for which there exist an irreducible component T of $\overline{F}^{\varphi}_{\varphi(y)}$ of dimension m and an irreducible component Z of $\overline{E}^{\varphi}_{\varphi(y)}$ of dimension n with $y \in T \subset Z$ (note that $\Lambda_{n,m} = \varnothing$ if $m > n$).

Fix $(n, m) \in \{0, \dots, D\}^2$; we are going to give an alternative description of $\Lambda_{n,m}$ from which we shall deduce that it is constructible. For that purpose, let us denote by G (resp. H) the subset of $\overline{E}^{\varphi}$ (resp. $\overline{F}^{\varphi}$) consisting of points at which $\overline{E}^{\varphi}$ (resp. $\overline{F}^{\varphi}$) is of relative dimension n (resp. m) over X. By Theorem 10.4.3 and Proposition 10.1.12, the subsets H and G of Y are constructible, and thus so are $\overline{G}^{\varphi}$ and $\overline{H}^{\varphi}$.

Let x be a point of X. By definition,

$$G_x = \{y \in \overline{E}^{\varphi}_x | \dim_y \overline{E}^{\varphi}_x = n\}$$

and $\overline{G}^{\varphi}_x$ is the closure of G_x inside Y_x. As a consequence, $\overline{G}^{\varphi}_x$ is the union of all n-dimensional irreducible components of $\overline{E}^{\varphi}_x$. Analogously, $\overline{H}^{\varphi}_x$ is the union of all m-dimensional irreducible components of $\overline{F}^{\varphi}_x$. Now let T be an m-dimensional irreducible component of $\overline{F}^{\varphi}_x$. By the above, T is contained in an n-dimensional irreducible component of $\overline{E}^{\varphi}_x$ if and only if it is contained in $\overline{G}^{\varphi}_x$ or, what amounts to the same, if and only if it is an irreducible component of $\overline{F}^{\varphi}_x \cap \overline{G}^{\varphi}_x$; conversely, any m-dimensional irreducible component of $\overline{F}^{\varphi}_x \cap \overline{G}^{\varphi}_x$ is contained in $\overline{G}^{\varphi}_x$ and is an irreducible component of $\overline{F}^{\varphi}_x$ by a dimension argument. Therefore $\Lambda_{n,m,x}$ is the union of all m-dimensional irreducible components of $\overline{F}^{\varphi}_x \cap \overline{G}^{\varphi}_x$, whence we get the equality

$$\Lambda_{n,m,x} = \{y \in \overline{F}^{\varphi}_x \cap \overline{G}^{\varphi}_x | \dim_y \overline{F}^{\varphi}_x \cap \overline{G}^{\varphi}_x = m\}$$

(because $\overline{F}^{\varphi}_x \cap \overline{G}^{\varphi}_x$ is contained in $\overline{F}^{\varphi}_x$ and is thus of dimension $\leqslant m$).

We have thus proved that $\Lambda_{n,m}$ is the subset of $\overline{G}^{\varphi} \cap \overline{H}^{\varphi}$ consisting of points at which $\overline{G}^{\varphi} \cap \overline{H}^{\varphi}$ is of relative dimension m; using again Theorem 10.4.3, we get the constructibility of $\Lambda_{n,m}$.

For any $y \in \overline{E}^{\varphi}$, let us denote by $\urcorner(y)$ the set of pairs $(n,m) \in \{0,\ldots,D\}^2$ such that $y \in \Lambda_{(n,m)}$. If P is a subset of $\{0,\ldots,D\}^2$, then the set of $y \in \overline{E}^{\varphi}$ such that $\urcorner(y) = P$ is a boolean combination of some of the $\Lambda_{(n,m)}$'s and is therefore a constructible subset of Y. Now the function from $\overline{E}^{\varphi}$ to $\mathbf{Z}_{\geqslant 0} \cup \{+\infty\}$ that sends y to $\operatorname{codim}_y(\overline{F}^{\varphi}_{\varphi(y)}, \overline{E}^{\varphi}_{\varphi(y)})$ is constant on every fiber of $\urcorner$ (note that the set of points at which it takes the value $+\infty$ is exactly $\urcorner^{-1}(\varnothing)$), whence the proposition. $\square$

10.5. The fiberwise exactness locus

Let $Y \to X$ be a morphism of k-analytic spaces, and let $D = (\mathscr{G} \to \mathscr{F} \to \mathscr{E})$ be a sequence of coherent sheaves on Y. In this section we prove the following.

(1) The set of points of Y at which D is fiberwise a complex, or fiberwise exact, or more generally at which it is a complex with fiberwise homology at $\mathscr{F}$ having a given fiber rank, is locally constructible (Theorem 10.5.3).
(2) The set of points of Y at which D is fiberwise exact and $\mathscr{E}$ is X-flat is Zariski-open (Theorem 10.5.4).
(3) The set U of points of Y at which D is fiberwise exact and $\mathscr{E}$ and $\mathscr{F}$ are X-flat is Zariski-open, and D_U is exact (Proposition 10.5.5).
(4) The set of points of Y at which $\mathscr{E}$ is fiberwise locally free is locally constructible (Theorem 10.5.6 (1)).
(5) The set U of points of Y at which $\mathscr{E}$ is fiberwise locally free and X-flat is locally constructible, and $\mathscr{E}_U$ is locally free (Theorem 10.5.6 (2)).

Those are more or less analogues of classical results in scheme theory (with often slightly different proofs): see [**EGA IV**$_3$], Prop. 9.4.4 for (1); Prop. 12.3.3 for (2) and (3); and Prop. 9.4.7 for (4). But note that this analogy (and the intrinsic interest of these statements) was not our only motivation: we shall also need (1) and (2) for studying the fiberwise codepth and the fiberwise Gorenstein property, which we will investigate by homological methods.

10.5.1. Notation. — If Y is an analytic space and if $D = (\mathscr{G} \to \mathscr{F} \to \mathscr{E})$ is a sequence of morphisms of coherent sheaves on Y, we shall denote by $\mathscr{Z}(D)$ (resp. $\mathscr{B}(D)$) the kernel (resp. the image) of $\mathscr{F} \to \mathscr{E}$ (resp. $\mathscr{G} \to \mathscr{F}$); note that both $\mathscr{Z}(D)$ and $\mathscr{B}(D)$ are coherent subsheaves of $\mathscr{F}$; their internal sheaf-theoretic sum inside $\mathscr{F}$ will be denoted by $\mathscr{C}(D)$. If $y \in Y$, then D is a complex at y if and only if $(\mathscr{C}(D)/\mathscr{Z}(D))_{\mathscr{H}(y)} = 0$, and it is exact at y if and only if

$$(\mathscr{C}(D)/\mathscr{Z}(D))_{\mathscr{H}(y)} = 0 \text{ and } (\mathscr{C}(D)/\mathscr{B}(D))_{\mathscr{H}(y)} = 0.$$

10.5.2. Remark. — The functors that send D to $\mathscr{C}(D), \mathscr{B}(D)$, and $\mathscr{Z}(D)$ commute with flat base change and with ground field extension: this is an immediate consequence of Proposition 4.5.7 (1).

10.5.3. Theorem. — *Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces, let N and M be two subsets of $\mathbf{Z}_{\geqslant 0}$ and let $D = (\mathscr{G} \to \mathscr{F} \to \mathscr{E})$ be a sequence of morphisms of coherent sheaves on Y.*

(1) *The set C of points $y \in Y$ such that*

$$\mathrm{rk}_y(\mathscr{C}(D_{Y_{\varphi(y)}})/\mathscr{Z}(D_{Y_{\varphi(y)}})) \in N$$

and

$$\mathrm{rk}_y(\mathscr{C}(D_{Y_{\varphi(y)}})/\mathscr{B}(D_{Y_{\varphi(y)}})) \in M$$

is a locally constructible subset of Y.

(2) *The set of points $y \in Y$ such that D is fiberwise a complex at y (i.e., $D_{Y_{\varphi(y)}}$ is a complex at y) is a locally constructible subset of Y.*

(3) *The set of points $y \in Y$ such that D is fiberwise exact at y (i.e., $D_{Y_{\varphi(y)}}$ is exact at y) is a locally constructible subset of Y.*

Proof. — We first note that (2) is nothing but (1) for $N = \{0\}$ and $M = \mathbf{Z}_{\geqslant 0}$, and that (3) is nothing but (1) for $N = M = \{0\}$. It suffices thus to prove (1). We introduce the following notation: if Z is an analytic space and if Δ is a 3-term sequence of coherent sheaves on Z, then for any point z of Z we set $\lambda(\Delta, z) = \mathrm{rk}_z(\mathscr{C}(\Delta)/\mathscr{B}(\Delta))$ and $\mu(\Delta, z) = \mathrm{rk}_z(\mathscr{C}(\Delta)/\mathscr{Z}(\Delta))$.

We are now going to apply some of the general results described in 10.2, which involve two categories $\mathfrak{F}$ and $\mathfrak{C}$ and a property Q as described in 10.2.1. We take for $\mathfrak{F}$ the category of 3-term sequences of coherent sheaves (i.e., $\mathfrak{Coh}^{\{0,1,2\}}$ with the notation of Example 2.2.10), for $\mathfrak{C}$ the category of all analytic spaces, and for Q the property defined as follows: if Z is an analytic space and if Δ is an object of $\mathfrak{F}_Z$, then Δ satisfies Q at a point z of Z if $\lambda(\Delta, z) \in N$ and $\mu(\Delta, z) \in M$ (axioms (1) and (2) of 10.2.1 are fulfilled due to Remark 10.5.2).

What we want to prove is assertion (α) of 10.2.8. By Lemma 10.2.17, it is sufficient to prove assertion (α'') of 10.2.12. We thus reduce to the following situation: Y and X are affinoid, X is integral, and φ admits a section σ; and we have to prove that there exists a non-empty Zariski-open subset of X which is either contained in $\sigma^{-1}(C)$ or disjoint from it. Let us introduce some notation. We denote by $\mathscr{K}$ the kernel of $\mathscr{G} \to \mathscr{F}$ and by $\mathscr{F}'$ its cokernel; we denote by $\mathscr{I}$ the image of $\mathscr{F} \to \mathscr{E}$ and by $\mathscr{E}'$ its cokernel; we denote by $\mathscr{L}$ the cokernel of $\mathscr{C}(D) \to \mathscr{F}$. We have the following exact sequences:

$$0 \to \mathscr{K} \to \mathscr{G} \to \mathscr{B}(D) \to 0,\ 0 \to \mathscr{B}(D) \to \mathscr{F} \to \mathscr{F}' \to 0,$$

$$0 \to \mathscr{Z}(D) \to \mathscr{F} \to \mathscr{I} \to 0,\ 0 \to \mathscr{I} \to \mathscr{E} \to \mathscr{E}' \to 0,$$

$$0 \to \mathscr{C}(D) \to \mathscr{F} \to \mathscr{L} \to 0\ ,\ \mathscr{B}(D) \oplus \mathscr{Z}(D) \to \mathscr{C}(D) \to 0.$$

Let U be the X-flat locus of $\mathscr{B}(D) \oplus \mathscr{F}' \oplus \mathscr{I} \oplus \mathscr{E}' \oplus \mathscr{L}$. By Theorem 10.3.2, U is a Zariski-open subset of Y; and by Lemma 10.2.15, the Zariski-open subset $\sigma^{-1}(U)$ of X is non-empty. As $\mathscr{F}'_U$ and $\mathscr{B}(D)_U$ are X-flat, so is $\mathscr{F}_U$ and hence $\mathscr{C}_D(U)$ since $\mathscr{L}_U$ is X-flat too (Lemma 4.5.9).

Let x be a point of X. By construction, the restriction to U of each of the above exact sequences has an X-flat right term, hence remains exact after arbitray base change $X' \to X$ by Proposition 4.5.7. In particular, they remain exact after restriction to the fiber U_x; therefore the coherent sheaves $\mathscr{B}(D_{U_x})$, $\mathscr{Z}(D_{U_x})$ and $\mathscr{C}(D_{U_x})$ are respectively naturally isomorphic to $\mathscr{B}(D)_{U_x}$, $\mathscr{Z}(D)_{U_x}$, and $\mathscr{C}(D)_{U_x}$, whence the equalities

$$\text{(a)} \qquad \lambda(D,y) = \lambda(D_{U_{\varphi(y)}}, y)$$

$$\text{(b)} \qquad \mu(D,y) = \mu(D_{U_{\varphi(y)}}, y)$$

for all $y \in U$.

Let E be the set of points $y \in Y$ such that $\lambda(D,y)$ and $\mu(D,y)$ belong respectively to N and M. Since the pointwise rank function of a given coherent sheaf on the affinoid space Y takes only finitely many values, E is a constructible subset of Y. Moreover we have in view of (a) and (b) the equality $C \cap U = E \cap U$.

The pre-image $\sigma^{-1}(E)$ is a constructible subset of the integral affinoid space X. Therefore, there exists a non-empty Zariski-open subset V of X such that $V \subset \sigma^{-1}(E)$ or $V \subset X \backslash \sigma^{-1}(E)$. Now $W := V \cap \sigma^{-1}(U)$ is a non-empty Zariski-open subset of X. From the equality $C \cap U = E \cap U$ it follows that $W \subset \sigma^{-1}(C)$ if $V \subset \sigma^{-1}(E)$ and that $W \subset X \backslash \sigma^{-1}(C)$ otherwise. □

10.5.4. Theorem. — *Let $\varphi\colon Y \to X$ be a morphism between k-analytic spaces and let $D = (\mathscr{G} \to \mathscr{F} \to \mathscr{E})$ be a complex of coherent sheaves on Y. The set A of points of Y at which $\mathscr{E}$ is X-flat and D is fiberwise exact is a Zariski-open subset of Y.*

Proof. — We are going to apply some of the general results described in 10.2, which involve two categories $\mathfrak{F}$ and $\mathfrak{C}$ and a property Q as described in 10.2.1, and a functor $\mathscr{S}$ as described in 10.2.4. We take for $\mathfrak{F}$ the category of 3-term complexes of coherent sheaves, for $\mathfrak{C}$ the category of all analytic spaces, and for Q the exactness property. We take for $\mathscr{S}$ the functor that sends a 3-term complex to its right term. We want to prove assertion (β) of 10.2.8. Since assertion (γ) of loc. cit. holds (this is Theorem 10.3.2), Lemma 10.2.22 ensures that it suffices to prove assertion (β') of 10.2.12; i.e., we can assume that Y and X are affinoid and $Y \to X$ has a section σ, and we only need to prove that $\sigma^{-1}(A)$ is a Zariski-open subset of X.

Let A_0 be the subset of Y consisting of points at which D is fiberwise exact. By Theorem 10.5.3 (3), A_0 is a constructible subset of Y and since Y is affinoid (hence finite-dimensional), A_0 is constructible by Proposition 10.1.12. By Theorem 10.3.2, $\mathrm{Flat}(\mathscr{E}/X)$ is Zariski-open subset of Y. As a consequence, $A = A_0 \cap \mathrm{Flat}(\mathscr{E}/X)$ is a constructible subset of Y, and $\sigma^{-1}(A)$ is thus a constructible subset of X.

Let (T, Z) be a pair of irreducible Zariski-closed subsets of X with $T \subset Z$ and $T \cap \sigma^{-1}(A)$ Zariski-dense in T; we are going to prove that $Z \cap \sigma^{-1}(A)$ is Zariski-dense in Z, which will yield to the Zariski-openness of A (Lemma 10.1.8 (3); see also Remark 10.1.9).

By 1.5.16 (2) there exists a chain $T_0 = T \subset T_1 \ldots \subset T_{\mathrm{codim}(T,Z)} = Z$ where every T_i is an irreducible Zariski-closed subset of X and where we have $\mathrm{codim}(T_i, T_{i+1}) = 1$ for all $i < \mathrm{codim}(T, Z)$. Hence we reduce by induction on codimension to the case where $\mathrm{codim}(T, Z) = 1$. Note that since $\sigma^{-1}(A) \cap T$ is a Zariski-dense constructible subset of T, it contains a non-empty Zariski-open subset of T (Lemma 10.1.8 (1); see also Remark 10.1.9). Let $\widetilde{Z}$ be the normalization of Z_{red}. Choose an irreducible component $\widetilde{T}$ of $\widetilde{Z} \times_Z T$ that dominates T (this exists by surjectivity of the scheme-theoretic normalization map $\widetilde{Z}^{\mathrm{al}} \to Z^{\mathrm{al}}$). Set $\widetilde{Y} = Y \times_X \widetilde{Z}$, let $\widetilde{A}_0$ and $\widetilde{A}$ be the pre-images of A_0 and A on $\widetilde{Y}$, and let $\widetilde{\sigma}\colon \widetilde{Z} \to \widetilde{Y}$ denote the map induced by σ. Let B be the set of points of $\widetilde{Y}$ at which $\mathscr{E}_{\widetilde{Y}}$ is $\widetilde{Z}$-flat and at which $D_{\widetilde{Y}}$ is fiberwise exact; since the formation of the fiberwise exactness locus commutes to arbitrary base change, we have $\widetilde{A} \subset \widetilde{A}_0 \cap \mathrm{Flat}(\mathscr{E}_{\widetilde{Y}}/\widetilde{Z}) = B$. As the intersection $\sigma^{-1}(A) \cap T$ contains a non-empty Zariski-open subset of T, its pre-image $\widetilde{\sigma}^{-1}(\widetilde{A}) \cap \widetilde{T}$ under the finite dominant map $\widetilde{T} \to T$ is Zariski-dense in $\widetilde{T}$; hence $\widetilde{\sigma}^{-1}(B) \cap \widetilde{T}$ is a fortiori Zariski-dense in $\widetilde{T}$, and it suffices to prove that this implies Zariski-density of the intersection $\widetilde{\sigma}^{-1}(B) \cap \widetilde{Z}$ in $\widetilde{Z}$. Indeed, assume that the latter holds. Then $\widetilde{\sigma}^{-1}(\widetilde{A}_0) \cap \widetilde{Z}$ is a Zariski-dense subset of $\widetilde{Z}$; since $\widetilde{Z} \to Z$ is birational, it follows that $\sigma^{-1}(A_0) \cap Z$ is Zariski dense in Z. On the other hand, $\sigma^{-1}(\mathrm{Flat}(\mathscr{E}/X)) \cap T$ contains $\sigma^{-1}(A) \cap T$, hence is Zariski-dense in T and in particular non-empty; since $\sigma^{-1}(\mathrm{Flat}(\mathscr{E}/X))$ is a Zariski-open subset of X, the intersection $\sigma^{-1}(\mathrm{Flat}(\mathscr{E}/X)) \cap Z$ is a non-empty Zariski open subset of the irreducible space Z, so $\sigma^{-1}(A) \cap Z = \sigma^{-1}(A_0) \cap \sigma^{-1}(\mathrm{Flat}(\mathscr{E}/X)) \cap Z$ is Zariski-dense in Z.

As a consequence, by performing base change from X to $\widetilde{Z}$ and by considering the pair $(\widetilde{T}, \widetilde{Z})$ we reduce to the case where $Z = X$ and X is integral and normal. Set $n = \dim X$; we then have $\dim T = n - 1$.

The set $T \cap \sigma^{-1}(A)$ contains a non-empty Zariski-open subset of T, so there exists $t \in T$ such that $d_k(t) = n - 1$. By Example 3.2.10, both local rings $\mathscr{O}_{X,t}$ and $\mathscr{O}_{X^{\mathrm{al}},t^{\mathrm{al}}}$ are of dimension 1, hence are discrete valuation rings as X is normal. Let $\varpi \in \mathscr{O}_X(X)$ be a function that generates $\mathfrak{m}_{t^{\mathrm{al}}}$; i.e., it is a uniformizing parameter of $\mathscr{O}_{X^{\mathrm{al}},t^{\mathrm{al}}}$. As $\mathfrak{m}_{t^{\mathrm{al}}}\mathscr{O}_{X,t} = \mathfrak{m}_t$ by Example 3.2.10, ϖ is a uniformizing parameter of $\mathscr{O}_{X,t}$ as well.

Let us endow T with its reduced structure, and let us choose a point x in X with $d_k(x) = n$. By Remark 1.5.9, the point x is Zariski-dense in X and the point t is Zariski-dense in T; i.e., x^{al} is the generic point of X^{al} and t^{al} is the generic point of T^{al}. By Example 3.2.10, both local rings $\mathscr{O}_{X,x}$ and $\mathscr{O}_{T,t}$ are artinian, hence are fields because X and T are reduced.

In order to prove that $\sigma^{-1}(A)$ is Zariski-dense, it suffices to prove that $x \in \sigma^{-1}(A)$. Since t belongs to $\sigma^{-1}(A)$, it belongs to $\sigma^{-1}(\mathrm{Flat}(\mathscr{E}/X))$; the latter is thus a non-empty Zariski-open subset of X, hence it contains x. It remains only to show that x belongs to $\sigma^{-1}(A_0)$; i.e., D is fiberwise exact at $\sigma(x)$. We shall need the following flatness assertions:

(a) The local ring $\mathscr{O}_{Y_x,\sigma(x)}$ is flat over $\mathscr{O}_{Y^{\mathrm{al}},\sigma(x)^{\mathrm{al}}}$.
(b) The local ring $\mathscr{O}_{Y_t,\sigma(t)}$ is flat over $\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}/\varpi\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$.

Let us first prove (a). The point $\sigma(x)$ belongs to $\mathrm{Int}(Y/X)$ by 1.3.21 (3) and $\mathscr{O}_{X,x}$ is a field; therefore $\mathscr{O}_{Y_x,\sigma(x)}$ is flat over $\mathscr{O}_{Y,\sigma(x)}$ by Theorem 6.3.3. On the other hand, $\mathscr{O}_{Y,\sigma(x)}$ is flat over $\mathscr{O}_{Y^{\mathrm{al}},\sigma(x)^{\mathrm{al}}}$ (2.1.4), so $\mathscr{O}_{Y_x,\sigma(x)}$ is flat over $\mathscr{O}_{Y^{\mathrm{al}},\sigma(x)^{\mathrm{al}}}$.

Let us now prove (b). The point $\sigma(t)$ belongs to $\mathrm{Int}(Y/X)$ by 1.3.21 (3), hence also to $\mathrm{Int}(Y \times_X T/T)$. Since the local ring $\mathscr{O}_{T,t}$ is a field, Theorem 6.3.3 ensures that $\mathscr{O}_{(Y\times_X T)_t,\sigma(t)} = \mathscr{O}_{Y_t,\sigma(t)}$ is flat over $\mathscr{O}_{Y\times_X T,\sigma(t)}$. The latter is itself flat over $\mathscr{O}_{Y^{\mathrm{al}}\times_{X^{\mathrm{al}}}T^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$, which is nothing but $\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}/\varpi\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$ because since T^{al} is reduced, it is defined by the equation $\varpi = 0$ around its generic point t^{al}. Hence $\mathscr{O}_{Y_t,\sigma(t)}$ is flat over $\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}/\varpi\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$.

By assumption, $\sigma(t) \in A$. Therefore, D is fiberwise exact at $\sigma(t)$ and $\mathscr{E}$ is X-flat at $\sigma(t)$. Fiberwise exactness of D at $\sigma(t)$ means that $D_{Y_t,\sigma(t)}$ is exact. This implies in view of (b) that

$$D^{\mathrm{al}}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}} \otimes_{\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}} \mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}/\varpi\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$$

is exact too.

Since $\mathscr{E}$ is X-flat at $\sigma(t)$, Lemma 4.2.1 ensures that $\mathscr{E}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$ is flat over $\mathscr{O}_{X^{\mathrm{al}},t^{\mathrm{al}}}$. Applying Lemma 12.3.3.1 of [**EGA IV**$_3$] with $B = \mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$, $t = \varpi$, and M, N, and P respectively equal to $\mathscr{G}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}, \mathscr{F}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$, and $\mathscr{E}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$ (which is possible since $\mathscr{E}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$ has no non-zero ϖ-torsion because it is flat over $\mathscr{O}_{X^{\mathrm{al}},t^{\mathrm{al}}}$), we get the exactness of the complex $D_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$. Moreover the local ring $\mathscr{O}_{Y^{\mathrm{al}},\sigma(x)^{\mathrm{al}}}$ is a localization of $\mathscr{O}_{Y^{\mathrm{al}},\sigma(t)^{\mathrm{al}}}$ because x^{al} is a generalization of t^{al}. It follows that the complex $D_{Y^{\mathrm{al}},\sigma(x)^{\mathrm{al}}}$ is exact. This implies in view of (a) that $D_{Y_x,\sigma(x)}$ is exact. □

10.5.5. Proposition. — *Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces and let $D = (\mathscr{G} \to \mathscr{F} \to \mathscr{E})$ be a complex of coherent sheaves on Y. The set A of points of Y at which $\mathscr{E}$ and $\mathscr{F}$ are X-flat and at which D is fiberwise exact is a Zariski-open subset of Y on which D is exact.*

Proof. — It follows from Theorem 10.5.4 above and from Zariski-openness of the X-flat locus of $\mathscr{F}$ (Theorem 10.3.2) that A is Zariski-open. It remains to prove that D is exact on A.

We can assume that Y and X are affinoid. Let y be a point of A and let x be its image in X. We shall prove that $D_{Y,y}$ is exact. This can be proved after enlarging the ground field, which allows us to assume that x is rigid. The local ring $\mathscr{O}_{Y_x,y}$ is then equal to $\mathscr{O}_{Y,y}/\mathfrak{m}_x\mathscr{O}_{X,x}$.

By our assumptions, the complex $D_{Y,y} = (\mathscr{G}_{Y,y} \to \mathscr{F}_{Y,y} \to \mathscr{E}_{Y,y})$ enjoys the following properties:

- The $\mathscr{O}_{X,x}$-modules $\mathscr{F}_{Y,y}$ and $\mathscr{E}_{Y,y}$ are flat.
- The complex $D_{Y,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{Y,y}/\mathfrak{m}_x\mathscr{O}_{Y,y}$ is exact.

A repeated application of Lemma 12.3.3.5 of [**EGA IV**$_3$] ensures that the complex $D_{Y,y} \otimes_{\mathscr{O}_{Y,y}} /\mathfrak{m}_x^n\mathscr{O}_{Y,y}$ is exact for every $n > 0$. As the complex $D_{Y,y}$ only involves finitely generated modules over the noetherian ring $\mathscr{O}_{Y,y}$ and as $\mathrm{H}(D_{Y,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{Y,y}/\mathfrak{m}_x^n\mathscr{O}_{Y,y}) = 0$ for all $n > 0$, it follows from [**EGA III**$_2$], 7.4.7.2 that

$$\mathrm{H}(D_{Y,y}) \otimes_{\mathscr{O}_{Y,y}} \varprojlim \mathscr{O}_{Y,y}/\mathfrak{m}_x^n\mathscr{O}_{Y,y} = 0.$$

Being finitely generated over $\mathscr{O}_{Y,y}$, the module $\mathrm{H}(D_{Y,y})$ is separated for the $\mathfrak{m}_y$-adic topology and a fortiori for the $\mathfrak{m}_x$-adic topology, so it is itself zero, which ends the proof. □

10.5.6. Theorem. — *Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces and let $\mathscr{E}$ be a coherent sheaf on Y.*

(1) *The set A of points of Y at which $\mathscr{E}$ is fiberwise locally free is a constructible subset of Y.*
(2) *The set B of points of Y at which $\mathscr{E}$ is X-flat and fiberwise locally free is a Zariski-open subset of Y over which $\mathscr{E}$ is locally free.*

Proof. — We may assume that Y and X are affinoid. Let y be a point of Y. We set $r = \mathrm{rk}_y(\mathscr{E})$ and we choose r global sections $f_1, \dots, f_r$ of $\mathscr{E}$ on Y such that $(f_1(y), \dots, f_r(y))$ is a basis of $\mathscr{E}_{\mathscr{H}(y)}$; let ℓ be the morphism $\mathscr{O}_Y^r \to \mathscr{E}$ that is induced by the f_i's. Let E be the set of $z \in Y$ such that $\ell_{Y_{\varphi(z)}}$ is an isomorphism at z. By assertion (3) of Theorem 10.5.3, E is a constructible subset of Y.

If $y \in A$, the constructible subset E of Y contains y, and it is included in A by definition. Now assume that $y \notin A$. Let F be the set of points z of Y such that $\mathrm{rk}_z(\mathscr{E}) = r$. It is a constructible subset of Y which contains y. Let V be the complement of $\mathrm{Supp}(\mathrm{Coker}\,\ell)$; it is a Zariski-open subset which contains y. Let G be the set of $z \in Y$ at which $\ell_{Y_{\varphi(z)}}$ is *not* an isomorphism. By assertion 10.5.3 (3) G is a constructible subset of Y, and it contains y since $y \notin A$. Now consider $z \in F \cap V \cap G$. The rank $\mathrm{rk}_z(\mathscr{E})$ is equal to r and $\ell_{Y_{\varphi(z)}}$ is surjective at z; if $\mathscr{E}_{Y_{\varphi(z)}}$ were locally free at z, then $\ell_{Y_{\varphi(z)}}$ would be an isomorphism at z, which would contradict the assumption that $z \in G$. Therefore, the set $F \cap V \cap G$ is a constructible subset of Y which contains y and is included in $Y \backslash A$. Assertion (1) then follows by quasi-compactness of Y for the constructible topology (10.1.7).

Assume that $y \in B$. Let U be the X-flat locus of $\mathscr{E}$; it is a Zariski-open subset of Y by Theorem 10.3.2. It follows from Lemma 4.5.8 that ℓ is an isomorphism at every point of $U \cap E$. This shows that the constructible set $U \cap E$ is open, hence Zariski-open by [**Ber93**], Cor. 2.6.6, and that $\mathscr{E}$ is locally free on $U \cap E$. Hence $U \cap E$ is a Zariski-open subset of Y containing y, contained in B, and on which $\mathscr{E}$ is locally free; this proves (2). □

10.6. Regular sequences

Motivated by the study of the fiberwise CI property, we introduce in this section the notion of a *regular sequence* with respect to a coherent sheaf (but we shall in fact mainly use it for the structure sheaf). The main result is Proposition 10.6.4 below, which investigates the locus of "fiberwise regularity" of a given sequence. Its proof is quite easy modulo our preceding work on fiberwise homology (Theorem 10.5.3 and Proposition 10.5.5), and some basic results on flatness (Proposition 4.5.7 and Lemma 4.5.10).

10.6.1. Definition. — Let X be a k-analytic space, let $\mathscr{F}$ be a coherent sheaf on X, and let $(g_1, \ldots, g_n)$ be a family of analytic functions on X. For every i, we set $\mathscr{F}_i = \mathscr{O}_X/(g_1, \ldots, g_{i-1})$, and we denote by h_i the endomorphism $a \mapsto g_i a$ of $\mathscr{F}_i$. Let x be a point of X at which all the f_i's vanish. The sequence $(g_1, \ldots, g_n)$ is called *$\mathscr{F}$-regular* at x if every h_i is injective at x. We shall say often say *regular* instead of $\mathscr{O}_X$-regular.

10.6.2. Basic properties. — We keep the notation of Definition 10.6.1. If V is an analytic domain of X containing x, then (g_i) is regular at x if and only if $(g_{i|V})$ is regular at x. If V is good, this is equivalent to requiring that (f_i) is a regular sequence of the module $\mathscr{F}_{V,x}$ over the noetherian local ring $\mathscr{O}_{V,x}$; since the latter property is invariant under any permutation of the g_i's, the property of $(g_1, \ldots, g_n)$ being regular at x is invariant under any permutation of the g_i's.

10.6.3. — Let $Y \to X$ be a morphism between k-analytic spaces, let $\mathscr{F}$ be a coherent sheaf on Y. Let y be a point of Y and let x be its image in X. Let $(f_1, \ldots, f_n)$ be a family of analytic functions on Y vanishing at y. We shall say that (f_i) is *fiberwise $\mathscr{F}$-regular* at y (or fiberwise regular at y if $\mathscr{F} = \mathscr{O}_Y$) if the family $(f_{i|Y_x})$ of analytic functions on the $\mathscr{H}(x)$-analytic space Y_x is $\mathscr{F}_{Y_x}$-regular at y.

Let $\mathscr{I}$ be a coherent sheaf of ideals on Y containing the f_i's. It gives rise to two a priori different objects on the fiber Y_x: its pull-back $\mathscr{I}_{Y_x}$ on Y_x *as an abstract* *coherent sheaf*, and a coherent sheaf of *ideals* on Y_x, namely the image of the natural map $\mathscr{I}_{Y_x} \to \mathscr{O}_{Y_x}$, which is not injective in general.

We say that the f_i's generate $\mathscr{I}$ fiberwise at y *as a coherent sheaf* if the $f_{i|Y_x}$'s generate $\mathscr{I}_{|Y_x}$ at y. This is equivalent to the fact that the f_i's generate $\mathscr{I}$ at y (consider the morphism $(a_i) \mapsto \sum a_i f_i$ from $\mathscr{O}_Y^n$ to $\mathscr{I}$ and apply 2.5.4).

We say that the f_i's generate $\mathscr{I}$ fiberwise at y *as an ideal sheaf* if the $f_{i|Y_x}$'s generate the image of $\mathscr{I}_{Y_x} \to \mathscr{O}_{Y_x}$ at y. Of course, if the f_i's generate $\mathscr{I}$ fiberwise at y as a coherent sheaf they generate it fiberwise at y as an ideal sheaf, but the converse is not true in general.

10.6.4. Proposition. — *Let $Y \to X$ be a morphism of k-analytic spaces, and let $\mathscr{I}$ be a coherent sheaf of ideals on Y. Let $(f_1, \ldots, f_n)$ be a family of analytic functions on Y that belong to $\mathscr{I}$, and let Z denote the closed analytic subspace of Y defined by $\mathscr{I}$.*

(1) *The set of points of Z at which (f_i) is fiberwise regular and generates $\mathscr{I}$ fiberwise as an ideal sheaf is a locally constructible subset of Z.*

(2) *Let y be a point of Z at which Y is X-flat. The following are equivalent:*
   (i) *(f_i) generates $\mathscr{I}$ fiberwise as an ideal sheaf at y, is fiberwise regular at y, and Z is X-flat at y.*
   (ii) *(f_i) generates $\mathscr{I}$ at y and is fiberwise regular at y.*

(3) *If Y is X-flat, the locus of validity in Z of the equivalent properties* (i) *and* (ii) *of* (2) *is a Zariski-open subset of Z.*

Proof. — For every i, we denote by $\mathscr{F}_i$ the coherent sheaf $\mathscr{O}_Y/(f_1, \ldots, f_{i-1})$ and by h_i the endomorphism $a \mapsto f_i a$ of $\mathscr{F}_i$.

Let us prove (1). Denote by D the complex

$$\mathscr{O}_Y^n \xrightarrow{(a_i)\mapsto\sum a_i f_i} \mathscr{O}_Y \longrightarrow \mathscr{O}_Y/\mathscr{I} \longrightarrow 0$$

of coherent sheaves on Y. Let y be a point of Z. The family $(f_1, \ldots, f_n)$ generates $\mathscr{I}$ fiberwise as an ideal sheaf at y if and only if D is fiberwise exact at y, and $(f_1, \ldots, f_n)$ is fiberwise regular at y if and only if every h_i is fiberwise injective at y. Therefore assertion (1) comes from Theorem 10.5.3.

Let us now prove (2). Suppose that (i) holds. Since Z is X-flat at y, the coherent sheaf $\mathscr{O}_Y/\mathscr{I}$ is X-flat at y. It follows then from Proposition 4.5.7 (2) that the exact sequence

$$0 \to \mathscr{I} \to \mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{I} \to 0$$

is fiberwise exact at y; hence the natural map $\mathscr{I}_{Y_{\varphi(y)}} \to \mathscr{O}_{Y_{\varphi(y)}}$ is injective at y. Therefore the f_i's generate $\mathscr{I}_{|Y_{\varphi(y)}}$ at y; this implies that they generate $\mathscr{I}$ at y (10.6.3), so (ii) holds. Suppose conversely that (ii) holds. Since the f_i's generate $\mathscr{I}$ at y, they generate it fiberwise as an ideal sheaf at y. Since all h_i's are fiberwise injective at y and s$\mathscr{O}_Y$ is X-flat at y, a repeated application of Lemma 4.5.10 ensures that $\mathscr{O}_Y/(f_1, \ldots, f_n)$ is X-flat at y. But since the f_i's generate $\mathscr{I}$ at y, this means that Z is X-flat at y, so (i) holds.

Let us now prove (3). We denote by $\mathscr{K}$ the cokernel of

$$\mathscr{O}_Y^n \xrightarrow{(a_i)\mapsto\sum a_i f_i} \mathscr{I} \ .$$

The locus of validity of (ii) on Z is the set of points of Z at which the following properties are satisfied:

(a) $\mathscr{K}$ is zero;

(b) every h_i is fiberwise injective.

Since $\mathscr{O}_Y$ is X-flat, a repeated application Lemma 4.5.10 shows that at every point of Z at which (b) is true, the coherent sheaves $\mathscr{F}_i$ are X-flat. Therefore (b) is equivalent to

(c) every $\mathscr{F}_i$ is X-flat and every h_i is fiberwise injective.

The locus of validity of (a) on Z is Zariski-open, and that of (c) is also Zariski-open by Proposition 10.5.5. Therefore the set of points of Z at which the equivalent conditions (i) and (ii) are fulfilled is Zariski-open. □

10.7. The main theorem

This section is essentially devoted to the (lengthy) proof of Theorem 10.7.2. The latter establishes the local constructibility of the loci of fiberwise validity of the usual algebraic properties, as well as Zariski-openness results under extra assumptions (always involving flatness), thus providing analytic counterparts of classical scheme-theoretic results (which we *do not use* in the proof); cf. the following statements of [**EGA IV**$_3$]:

- Prop. 9.9.2 (iv) (v) (vii) (viii) (ix);
- Prop. 9.9.4 (i) (ii) (iii) (iv);
- Thm. 12.1.1 (v) (vi) (vii);
- Thm. 12.1.6 (i) (ii) (iii) (iv);
- Cor. 12.1.7.

We then prove two additional theorems. The first one is Theorem 10.7.4, which essentially asserts that some of the constructible loci exhibited at various parts of this memoir (Theorem 10.4.3, Proposition 10.4.4 Theorem 10.7.2) are algebraizable as soon as the source space Y and all data living on it are algebraizable and the map $Y \to X$ under investigation is not too "widely analytic". The proof is quite easy and consists in reducing to Theorem 10.7.2 by using GAGA and the extension of coherent sheaves (for dense open immersions in scheme theory).

The second one is Theorem 10.7.5, which roughly speaking turns all local constructibility or Zariski-openness assertions of Theorem 10.7.2 into local constructibility or Zariski-openness assertions *on the target* when the map involved is proper. It rests on Kiehl's theorem on the direct images of coherent sheaves by a proper map (which ensures that a proper map is closed for the Zariski topologies involved; see 1.3.23) and on our "proper Chevalley theorem" (Theorem 10.1.15).

10.7.1. Lemma. — *Let X be a k-analytic space, let Y be its non-regular locus and let $\mathscr{E}$ be a coherent sheaf on X. Let x be a point of X and let m be a non-negative integer.*

(1) *For any n, the subset U_n of X consisting of points at which codepth of $\mathscr{E}$ at x is bounded above by n is a Zariski-open subset of X.*
(2) *The coherent sheaf $\mathscr{E}$ is S_m at x if and only if*

$$\operatorname{codim}_x(X\backslash U_n, \operatorname{Supp}(\mathscr{E})) > n+m$$

for every n.

(3) *The space X is R_m at x if and only if* $\operatorname{codim}_x(Y, X) > m$.

Proof. — We can assume that X is affinoid. By GAGA principles (Lemma 2.4.6 and 1.5.16 (2)) we reduce to the corresponding scheme-theoretic statements on X^{al}.

Now (1) comes from the fact that X^{al} is isomorphic to a closed subscheme of a regular scheme (see 2.2.4 (1)) and from a theorem by Auslander; cf. [**EGA IV**$_2$], Prop. 6.11.2 (i). Assertion (2) comes from from Prop. 5.7.4 (i) of [**EGA IV**$_2$], and assertion (3) from the definition of an R_m-scheme. □

10.7.2. Theorem. — *Let X be a k-analytic space, let Y be an X-analytic space, let $\mathscr{E}$ be a coherent sheaf on Y, let n and d be two non-negative integers. Let us consider the following subsets of Y (fiberwise notions are understood with respect to X).*

- *The set A_n of points at which $\mathscr{E}$ is fiberwise of codepth n.*
- *The set A'_n of points at which $\mathscr{E}$ is X-flat and fiberwise of codepth $\leqslant n$.*
- *The set A_∞ of points at which $\mathscr{E}$ is fiberwise CM.*
- *The set A'_∞ of points at which $\mathscr{E}$ is X-flat and fiberwise CM; i.e., CM over X in the sense of Definition 8.4.1.*
- *The set B_n of points at which $\mathscr{E}$ is fiberwise S_n.*
- *The set B'_n of points at which $\mathscr{E}$ is X-flat and fiberwise S_n.*
- *The set C of points at which Y is fiberwise Gorenstein.*
- *The set C' of points at which Y is X-flat and fiberwise Gorenstein.*
- *The set D of points at which Y is fiberwise CI.*
- *The set D' of points at which Y is X-flat and fiberwise CI.*
- *The set E_n of points at which Y is fiberwise geometrically R_n.*
- *The set E'_n of points at which Y is X-flat and fiberwise geometrically R_n*
- *The set E_∞ of points at which Y is fiberwise quasi-smooth.*
- *The set $E'_{\infty,d}$ of points at which Y is quasi-smooth of relative dimension d over X.*

- *The set E'_∞ of points at which Y is quasi-smooth over X.*
- *The set Δ of points at which Y is fiberwise geometrically reduced.*
- *The set Δ' of points at which Y is X-flat and fiberwise geometrically reduced.*
- *The set Θ of points at which Y is fiberwise geometrically normal.*
- *The set Θ' of points at which Y is X-flat and fiberwise geometrically normal.*

The sets A_n, A_∞, B_n, C, D, E_n, E_∞, Δ, and Θ are locally constructible (hence constructible if Y is finite dimensional, by Proposition 10.1.12), and the sets A'_n, A'_∞, C', D', $E'_{\infty,d}$, and E'_∞ are Zariski-open. If $\operatorname{Supp}(\mathscr{E})$ *is purely of relative dimension d over X, then B'_n is Zariski-open too. If Y is purely of relative dimension d over X, then E'_n, Δ', and Θ' are Zariski-open too.*

Proof. — By Remark 2.3.6, the G-local constructibility of Δ will follow from that of B_1 and E_0, and the G-local constructibility of Θ will follow from that of B_2 and E_1; analogously, the Zariski-openness of Δ' and Θ' in the equidimensional case will respectively follow from that of B'_1 and E'_0, and from that of B'_2 and E'_1.

10.7.2.1. *Study of A_n, A'_n, C, and C': first reductions.* — Let us denote by $A_{\leqslant n}$ the set of points of Y at which $\mathscr{E}$ is fiberwise of codepth $\leqslant n$. Instead of proving directly the constructibility of A_n, we shall prove that of $A_{\leqslant n}$; this will clearly imply the former one because $A_n = A_{\leqslant n} \backslash A_{\leqslant n-1}$.

We are going to apply some of the general results in 10.2, which involve two categories $\mathfrak{F}$ and $\mathfrak{C}$ and a property Q as in 10.2.1, and a functor $\mathscr{S}$ as in 10.2.4. For the study of $A_{\leqslant n}$ and A'_n, we take $\mathfrak{F}$ to be the category $\mathfrak{Coh}$, $\mathfrak{C}$ to be the category of all analytic spaces, and for Q to be the property of being of codepth $\leqslant n$, and $\mathscr{S}$ to be the identity functor. For the study of C and C', we take $\mathfrak{F}$ to be the category $\mathfrak{T}$, $\mathfrak{C}$ to be the category of all analytic spaces, Q to be the property of being of Gorenstein, and $\mathscr{S}$ to be the functor that sends any analytic space to its structure sheaf.

In both cases, we are interested in assertions (α) and (β) of 10.2.8. Since assertion (γ) of loc. cit. holds (this is Theorem 10.3.2), Lemma 10.2.22 ensures that it suffices to prove assertion (β') of 10.2.12. And as far as $A_{\leqslant n}$ and A'_n are concerned, it even suffices to prove assertion $(\beta^\flat)$ of loc. cit., in view of Lemma 10.2.19 and Remark 10.2.20.

More explicitly, those reductions mean that we can assume that Y and X are affinoid and the morphism $Y \to X$ has a section σ, and we have to prove the following:

(1) If Y is X-flat then $\sigma^{-1}(A'_n)$ is a Zariski-open subset of X.
(2) The subset $\sigma^{-1}(C')$ of X is Zariski-open (without any flatness assumption on Y).

10.7.2.2. *Some homological computations.* — We fix a non-negative integer d such that the relative dimension of $\operatorname{Supp}(\mathscr{E})$ over X is bounded above by d (if $\operatorname{Supp}(\mathscr{E}) \neq \varnothing$ one can take $d = \dim \operatorname{Supp}(\mathscr{E})$). Choose a resolution

$$\mathscr{F}_{d+1} \to \mathscr{F}_d \to \ldots \to \mathscr{F}_1 \to \mathscr{F}_0 \to \sigma_* \mathscr{O}_X \to 0,$$

where the $\mathscr{F}_i$'s are free $\mathscr{O}_Y$-modules of finite rank, and let F denote the complex

$$\mathscr{F}_{d+1} \to \mathscr{F}_d \to \ldots \to \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{F}_{-1} = 0.$$

Let x be a point of X such Y is X-flat at $\sigma(x)$. For any $i \in \{0, \ldots, d\}$ we have natural isomorphisms

$$\begin{array}{llcl}
\text{(a)} & \mathrm{Ext}^i_{\mathscr{O}_{Y_x,\sigma(x)}}(\mathscr{H}(x), \mathscr{E}_{Y_x,\sigma(x)}) & \simeq & \mathrm{H}_i\left(\mathrm{Hom}(\mathsf{F}_{Y_x,\sigma(x)}, \mathscr{E}_{Y_x,\sigma(x)})\right) \\
\text{(b)} & & \simeq & \mathscr{H}_i\left(\mathscr{H}om(\mathsf{F}_{Y_x}, \mathscr{E}_{Y_x})\right)_{\sigma(x)} \\
\text{(c)} & & \simeq & \mathscr{H}_i\left(\mathscr{H}om(\mathsf{F}, \mathscr{E})_{Y_x}\right)_{\sigma(x)}
\end{array}$$

Indeed, since Y is X-flat at $\sigma(x)$, so is $\mathscr{F}_i$ for every i (because it is free over $\mathscr{O}_Y$): morevoer, $\sigma_* X$ is X-flat (everywhere). Proposition 4.5.7 then ensures that

$$\mathscr{F}_{d+1,Y_x,\sigma(x)} \to \mathscr{F}_{d,Y_x,\sigma(x)} \to \ldots \to \mathscr{F}_{1,Y_x,\sigma(x)} \to (\sigma_*\mathscr{O}_X)_{Y_x,\sigma(x)} \to 0$$

is exact. Isomorphism (a) now follows since $\mathscr{F}_{i,Y_x,\sigma(x)}$ is for every i a free $\mathscr{O}_{Y_x,\sigma(x)}$-module (because $\mathscr{F}_i$ is free over $\mathscr{O}_Y$) and the $\mathscr{O}_{Y_x,\sigma(x)}$-module $(\sigma_*\mathscr{O}_X)_{Y_x,\sigma(x)}$ is nothing but the residue field $\mathscr{H}(x)$ of $\mathscr{O}_{Y_x,\sigma(x)}$. Isomorphism (b) comes from the coherence of the sheaves involved, and isomorphism (c) is due to the fact that $\mathscr{H}om(\mathscr{F}_i, \mathscr{E})_{Y_x} = \mathscr{H}om(\mathscr{F}_{i,Y_x}, \mathscr{E}_{Y_x})$ for every i, again by freeness of $\mathscr{F}_i$.

We thus have

$$\begin{array}{llcl}
\text{(d)} & \mathrm{Ext}^i_{\mathscr{O}_{Y_x,\sigma(x)}}(\mathscr{H}(x), \mathscr{E}_{Y_x,\sigma(x)}) & \simeq & \mathscr{H}_i\left(\mathscr{H}om(\mathsf{F}, \mathscr{E})_{Y_x}\right)_{\sigma(x)} \\
\text{(e)} & & = & \mathscr{H}_i\left(\mathscr{H}om(\mathsf{F}, \mathscr{E})_{Y_x}\right)_{\mathscr{H}(x)}
\end{array}$$

(equality (e) comes from the fact that the $\mathscr{O}_{Y_x,\sigma(x)}$-module $\mathscr{H}_i\left(\mathscr{H}om(\mathsf{F}, \mathscr{E})_{Y_x}\right)_{\sigma(x)}$ is in fact an $\mathscr{H}(x)$-vector space by (d)). In view of the description of depth through Ext functors (cf. Thm. 16.7 of [**Mat86**]), we deduce from (d) that

$$\text{(f)} \qquad \mathrm{depth}_{\mathscr{O}_{Y_x,\sigma(x)}} \mathscr{E}_{Y_x,\sigma(x)} = \inf\left\{i \mid \mathscr{H}_i\left(\mathscr{H}om(\mathsf{F}, \mathscr{E})_{Y_x}\right)_{\sigma(x)} \neq 0\right\}$$

(note that this is true even if $\mathscr{E}_{Y_x,\sigma(x)} = 0$ because in this case its depth is equal to $+\infty$).

10.7.2.3. — Let a be an element of $\in \{0, \ldots, d+1\}$, and let $\mathsf{F}^{\leqslant a}$ be the truncated complex $\mathscr{F}_a \to \mathscr{F}_{a-1} \to \ldots \to \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{F}_{-1} = 0$. We denote by Λ_a the subset of Y consisting of points at which $\mathscr{H}om(\mathsf{F}^{\leqslant a}, \mathscr{E})$ is fiberwise exact and $\mathscr{E}$ is X-flat.

For every $i \in \{-1, \ldots, d+1\}$, the coherent sheaf $\mathscr{F}_i$ is free over $\mathscr{O}_Y$, hence $\mathscr{H}om(\mathscr{F}_i, \mathscr{E})$ is a direct sum of finitely many copies of $\mathscr{E}$. Moreover, since $\mathscr{F}_0$ surjects onto $\sigma_*\mathscr{O}_X$, it is of positive rank as soon as $X \neq \varnothing$, which is always the case when $Y \neq \varnothing$. We have therefore the equality

$$\mathrm{Flat}(\mathscr{E}/X) = \mathrm{Flat}(\mathscr{H}om(\mathscr{F}_0, \mathscr{E})/X) = \bigcap_{-1 \leqslant i \leqslant a} \mathrm{Flat}(\mathscr{H}om(\mathscr{F}_i, \mathscr{E})/X).$$

We can therefore describe Λ_a as the subset of Y consisting of points at which the complex $\mathscr{H}om(\mathsf{F}^{\leqslant a}, \mathscr{E})$ is fiberwise exact and at which every sheaf involved in this complex is X-flat. It follows then from Theorem 10.5.4 that Λ_a is a Zariski-open subset of Y.

10.7.2.4. *Proof of assertion* (1) *of 10.7.2.1.* — We assume that Y is X-flat, and we are going to prove that $\sigma^{-1}(A'_n)$ is a Zariski-open subset of X. Let us begin with a remark. If y is a point of Y and if x denotes its image in X, we define the fiberwise *depth* of $\mathscr{E}$ at y as the depth of the $\mathscr{O}_{Y_x,y}$-module $\mathscr{E}_{Y_x,y}$. If $y \notin \mathrm{Supp}(\mathscr{E})$, this fiberwise depth is equal to $+\infty$; if $y \in \mathrm{Supp}(\mathscr{E})$, it is bounded by $\dim_{\mathrm{Krull}} \mathscr{O}_{\mathrm{Supp}(\mathscr{E})_x,y}$, hence by d in view of Corollary 3.2.9.

Let m be an integer such that $0 \leqslant m \leqslant d$. We denote by F_m the subset of Y consisting of points at which $\mathscr{E}$ is X-flat and of fiberwise depth $\geqslant m$. Let us first show that $\sigma^{-1}(F_m)$ is a Zariski-open subset of X. Let x be a point of X. By equation (f) of 10.7.2.2, $\sigma(x) \in F_m$ if and only if the two following conditions are fulfilled:

(i) The complex $\mathscr{H}om(\mathsf{F}^{\leqslant m+1}, \mathscr{E})$ is fiberwise exact at $\sigma(x)$.
(ii) The coherent sheaf $\mathscr{E}$ is X-flat at $\sigma(x)$.

Otherwise said, $\sigma^{-1}(F_m) = \sigma^{-1}(\Lambda_{m+1})$. Since Λ_{m+1} is a Zariski-open subset of Y (10.7.2.3), $\sigma^{-1}(F_m)$ is Zariski-open subset of X, as desired.

For every $\delta \in \mathbf{Z}_{\geqslant 0}$, we denote by H_δ, resp. $H_{\leqslant\delta}$, the set of points of $\mathrm{Supp}(\mathscr{E})$ at which the relative dimension of $\mathrm{Supp}(\mathscr{E})$ over X is equal to δ (resp. $\leqslant \delta$). By Zariski-semi-continuity of the relative dimension ([**Duc07b**], Thm. 4.9), the sets $H_{\leqslant\delta}$ and H_δ are respectively Zariski-open and constructible in $\mathrm{Supp}(\mathscr{E})$.

Let $x \in \sigma^{-1}(\mathrm{Supp}(\mathscr{E}))$. Let δ and m be respectively the relative dimension and the fiberwise codepth of $\mathrm{Supp}(\mathscr{E})$ at $\sigma(x)$. Since $\mathscr{H}(\sigma(x)) = \mathscr{H}(x)$, the Krull dimension of $\mathscr{O}_{\mathrm{Supp}(\mathscr{E})_x,\sigma(x)}$ is equal to δ (Corollary 3.2.9); the fiberwise codepth of $\mathscr{E}$ at $\sigma(x)$ is thus equal to $\delta - m$.

As a consequence,

$$\sigma^{-1}(A'_n) = \big(X \backslash \sigma^{-1}(\mathrm{Supp}(\mathscr{E}))\big) \bigcup \bigcap_{\delta \leqslant d, m \leqslant d, \delta - m \leqslant n} \sigma^{-1}(F_m \cap H_{\leqslant\delta}).$$

We have seen that $\sigma^{-1}(F_m)$ is a Zariski-open subset of X for every $m \leqslant d$, and that $H_{\leqslant\delta}$ is a Zariski-open subset of $\mathrm{Supp}(\mathscr{E})$ for every δ. Therefore $\bigcap_{\delta \leqslant d, m \leqslant d, \delta - m \leqslant n} \sigma^{-1}(F_m \cap H_{\leqslant\delta})$ is a Zariski-open subset of $\sigma^{-1}(\mathrm{Supp}(\mathscr{E}))$, so $\sigma^{-1}(A'_n)$ is a Zariski-open subset of X. We have thus proved assertion (1) of 10.7.2.1.

10.7.2.5. *Proof of assertion* (2) *of 10.7.2.1.* — The affinoid space Y is no longer assumed to be X-flat, and we are going to prove that $\sigma^{-1}(C')$ is a Zariski-open subset of X. We will use the results and notation of 10.7.2.1, 10.7.2.2 10.7.2.3, and 10.7.2.4 for $\mathscr{E} = \mathscr{O}_Y$; so d, H_δ, $H_{\leqslant\delta}$, and F_m refer from now on to the fiberwise dimension and depth of Y over X, and Λ_a involves the complex $\mathscr{H}om(\mathsf{F}^{\leqslant a}, \mathscr{O}_Y)$.

We fix an integer $\delta \leqslant d$, and we are going to show that $\sigma^{-1}(C' \cap H_\delta)$ is a Zariski-open subset of X, which will be sufficient since $X = \bigcup_{\delta \leqslant d} \sigma^{-1}(H_\delta)$. Let

U be the subset of X consisting of points x such that $\sigma(x) \in H_{\leqslant\delta}$ and the $\mathscr{H}(x)$-vector space $\operatorname{Ext}^i_{\mathscr{O}_{Y_x,\sigma(x)}}(\mathscr{H}(x), \mathscr{O}_{Y_x,\sigma(x)}) = \mathscr{H}_i(\mathscr{H}om(\mathsf{F}, \mathscr{E})_{Y_x})_{\sigma(x)}$ (see equation (d) of 10.7.2.2) has rank 0 for $i < \delta$ and rank 1 for $i = \delta$.

Let x be a point of U. The depth of $\mathscr{O}_{Y_x,\sigma(x)}$ is equal to δ by equation (f) of 10.7.2.2; therefore the Krull dimension of that ring is at least δ. Since $\dim_{\sigma(x)} Y_x \leqslant \delta$ by definition of U, it follows from Corollary 3.2.9 that

$$\dim_{\mathrm{Krull}} \mathscr{O}_{Y_x,\sigma(x)} = \dim_{\sigma(x)} Y_x = \delta.$$

We thus see that $U \cap \sigma^{-1}(\mathrm{Flat}(Y/X))$ coincides with $\sigma^{-1}(C' \cap H_\delta)$; it remains to show that $U \cap \sigma^{-1}(\mathrm{Flat}(Y/X))$ is a Zariski-open subset of X. By definition of U and equation (f) of 10.7.2.2 we have the equality

$$U \cap \sigma^{-1}(\mathrm{Flat}(Y/X)) = \sigma^{-1}(H_{\leqslant\delta} \cap \mathrm{Flat}(Y/X) \cap V \cap V'),$$

where:

- V is the subset of Y consisting of points at which the complex
$$\mathscr{H}om(\mathsf{F}^{\leqslant\delta+1}, \mathscr{O}_Y)$$
is fiberwise exact in degrees $< \delta$.
- V' is the subset of Y consisting of points y such that
$$\mathrm{rk}_y(\mathscr{H}_\delta(\mathscr{H}om(\mathsf{F}^{\leqslant\delta+1}_{Y_x}, \mathscr{E}_{Y_x}))) = 1,$$
with x the image of y in X.

But we also have

$$U \cap \sigma^{-1}(\mathrm{Flat}(Y/X)) = \sigma^{-1}(H_{\leqslant\delta} \cap \mathrm{Flat}(Y/X) \cap V \cap V'')$$

where V'' is the set of points y of Y such that

$$\mathrm{rk}_y(\mathscr{H}_\delta(\mathscr{H}om(\mathsf{F}^{\leqslant\delta+1}_{Y_x}, \mathscr{E}_{Y_x}))) \leqslant 1,$$

for x the image of y in X.

Indeed, let x be a point of $\sigma^{-1}(H_{\leqslant\delta} \cap \mathrm{Flat}(Y/X) \cap V \cap V'')$. The Krull dimension of $\mathscr{O}_{Y_x,\sigma(x)}$ is bounded above by δ, so its depth is too. Now since $\sigma(x) \in V$, it follows from equation (e) and (f) of 10.7.2.2 that the depth of $\mathscr{O}_{Y_x,\sigma(x)}$ is $\geqslant \delta$. It is therefore equal to δ, which implies by 10.7.2.2 that

$$\mathscr{H}_\delta(\mathscr{H}om(\mathsf{F}^{\leqslant\delta+1}_{Y_x}, \mathscr{E}_{Y_x}))_{\mathscr{H}(x)} \neq 0.$$

As a consequence, $\mathrm{rk}_{\sigma(x)}(\mathscr{H}_\delta(\mathscr{H}om(\mathsf{F}^{\leqslant\delta+1}_{Y_x}, \mathscr{E}_{Y_x}))) = 1$ and the point x belongs to $\sigma^{-1}(H_{\leqslant\delta} \cap \mathrm{Flat}(Y/X) \cap V \cap V')$, whence our claim.

We know that $H_{\leqslant\delta}$ is a Zariski-open subset of Y. The intersection $V \cap \mathrm{Flat}(Y/X)$ is nothing but the subset Λ_δ of Y, which is Zariski-open (10.7.2.3). It follows from Proposition 4.5.7 (4) (and from the fact that $\mathrm{Flat}(Y/X) \subset \bigcap_i \mathrm{Flat}(\mathscr{F}_i/X)$, see 10.7.2.3) that $V'' \cap \mathrm{Flat}(Y/X)$ is the set of points y of $\mathrm{Flat}(Y/X)$ such that $\mathrm{rk}_y(\mathscr{H}_\delta(\mathscr{H}om(\mathsf{F}, \mathscr{O}_Y))) \leqslant 1$. By Zariski-upper semi-continuity of the pointwise rank of a coherent sheaf (and Theorem 10.3.2), $W'' \cap \mathrm{Flat}(Y/X)$ is a Zariski-open subset of Y. This implies that $H_{\leqslant\delta} \cap \mathrm{Flat}(Y/X) \cap W \cap W''$ is a Zariski-open subset of Y; therefore

$$\sigma^{-1}(C' \cap H_\delta) = U \cap \sigma^{-1}(\mathrm{Flat}(Y/X)) = \sigma^{-1}(H_{\leqslant\delta} \cap \mathrm{Flat}(Y/X) \cap V \cap V'')$$

is a Zariski-open subset of X, and assertion (2) of 10.7.2.1 is proved.

10.7.2.6. *Study of A_∞ and A'_∞.* — By arguing G-locally on Y we may assume that it is good and that there is an integer δ such that $\dim Y_x$ is bounded by δ for every $x \in X$; we then also have $\dim_{\mathrm{Krull}} \mathscr{O}_{Y_x,y} \leqslant \delta$ for every $y \in Y_x$ (Corollary 3.2.9). We thus see that $A_m = Y$ for $m > \delta$. As a consequence,

$$A_\infty = \bigcap_{m\leqslant\delta} A_m,\ A'_\infty = \bigcap_{m\leqslant\delta} A'_m.$$

We have already proven that the A_m's are locally constructible, and that the A'_m are Zariski-open. Therefore A_∞ is locally constructible, and A'_∞ is Zariski-open.

10.7.2.7. *Study of B_n and B'_n.* — We begin with the local constructibility of B_n. By arguing locally, we may assume that Y is finite-dimensional. Under this assumption, the subset A_m of Y is empty for all but finitely many m, and any G-locally constructible subset of Y is constructible by Proposition 10.1.12. It follows then from the (G-local) constructibility of A_m for every m (which has already been proved), Lemma 10.7.1 (2), and the (G-local) constructibility of the fiberwise codimension function (Proposition 10.4.4) that B_n is constructible.

Let us assume now that $\mathrm{Supp}(\mathscr{E})$ is of pure relative dimension d over X, and let us prove that B'_n is open. We are going to apply some of the general results in 10.2, which involve two categories $\mathfrak{F}$ and $\mathfrak{C}$ and a property Q as in 10.2.1, and a functor $\mathscr{S}$ as in 10.2.4. We take $\mathfrak{F}$ to be the category $\mathfrak{Coh}$, $\mathfrak{C}$ to be the category of all analytic spaces, Q to be the property of being of S_n, and $\mathscr{S}$ to be the identity functor. We are interested in assertion (β) of 10.2.8. Since assertions (α) and (γ) of loc. cit.already hold ((α) is the G-local constructibility of B_n we have just established, and (γ) is Theorem 10.3.2), Lemma 10.2.23 ensures that it is sufficient to prove assertion $(\beta^\flat)$ of 10.2.12; i.e., we may assume that Y and X are affinoid and $\mathscr{E}$ is X-flat (the proof of Lemma 10.2.23 only involves arguing G-locally, hence it does not modify the dimension of $\mathrm{Supp}(\mathscr{E}) \to X$).

Let m be a non-negative integer. We have already proved that A'_m is Zariski-open; hence $Y\backslash A'_m$ is a Zariski-closed subset of Y which is contained in $\mathrm{Supp}(\mathscr{E})$. Let J_m be the set of points of $\mathrm{Supp}(\mathscr{E})$ at which the fiberwise codimension of $Y\backslash A'_m$ in $\mathrm{Supp}(\mathscr{E})$ is $> n + m$. By Zariski-upper semi-continuity of the dimension of a morphism ([**Duc07b**], Thm. 4.9), and because $\mathrm{Supp}(\mathscr{E}) \to X$ is of pure dimension d, the subset J_m of Y is Zariski-open *in the Zariski-closed subset* $\mathrm{Supp}(\mathscr{E})$; note that it is equal to the whole of $\mathrm{Supp}(\mathscr{E})$ for large enough m. By Lemma 10.7.1 (2) and

X-flatness of $\mathscr{E}$, the set B'_n is equal to

$$(Y\backslash \mathrm{Supp}(\mathscr{E})) \bigcup \bigcap_m J_m$$

and hence is Zariski-open in Y.

10.7.2.8. *Study of D and D'.* — We may assume that Y and X are affinoid. Under this assumption, there exists an affinoid X-space X' which is flat with regular fibers, such that Y can be identified (over X) with a closed analytic subspace of X'; one can take for example X' to be the product of X and a suitable compact polydisc, but we want to emphasize that what follows holds for an arbitrary such X'. Let $\mathscr{I}$ be the ideal sheaf on X' that corresponds to Y; we fix a system of global sections $f_1, \ldots, f_n$ generating $\mathscr{I}$.

For every $J \subset \{1, \ldots, n\}$, let P_J be the set of points of Y at which $(f_i)_{i \in J}$ generates $\mathscr{I}$ fiberwise *as an ideal* (10.6.3) and at which $(f_i)_{i \in J}$ is fiberwise regular (Definition 10.6.1). It follows from Proposition 10.6.4 (1) that P_J is a constructible subset of Y.

Let us prove that D is the union of the P_J's, hence is constructible. Let y be a point of $\bigcup P_J$ and let x be its image in X. By assumption, there exists J such that $(f_i)_{i \in J}$ is a regular sequence of the regular local ring $\mathscr{O}_{X'_x,y}$ and $\mathscr{O}_{Y_x,y} = \mathscr{O}_{X'_x,y}/(f_i)_{i \in J}$. Then $\mathscr{O}_{Y_x,y}$ is CI by definition, hence $y \in D$. Assume conversely that y lies on D. Let J be a minimal subset of $\{1, \ldots, n\}$ such that the f_i's for $i \in J$ generate $\mathscr{I}$ fiberwise at y; i.e., the f_i's for $i \in J$ generate the kernel of $\mathscr{O}_{X'_x,y} \to \mathscr{O}_{Y_x,y}$. Since $y \in D$, the local ring $\mathscr{O}_{Y_x,y}$ is CI. Since $\mathscr{O}_{X'_x,y}$ is regular, the family $(f_i)_{i \in J}$ is a regular sequence of the local ring $\mathscr{O}_{X'_x,y}$, cf. [**Mat86**], 17.4(iii), (1) $\Leftrightarrow$ (3); i.e., $(f_i)_{i \in J}$ is fiberwise regular at y and $y \in P_J$.

Now $D' = D \cap \mathrm{Flat}(Y/X)$. By the above, D' is the union of the $\mathrm{Flat}(Y/X) \cap P_J$'s for $J \subset \{1, \ldots, n\}$. Since $X' \to X$ is flat, $\mathrm{Flat}(Y/X) \cap P_J$ is a Zariski-open subset of Y for every J by Proposition 10.6.4. Therefore D' is a Zariski-open subset of Y.

10.7.2.9. *Study of $E_n, E'_n, E_\infty, E'_\infty$ and $E'_{\infty,d}$.* — For every δ, let $E_{\infty,\delta}$ be the set of points of Y at which Y is fiberwise quasi-smooth of dimension δ. This is the set of points at which $Y \to X$ is of dimension δ and $\Omega_{Y/X}$ has fiber rank δ; it is therefore locally constructible by Theorem 4.9 of [**Duc07b**]. Now E_∞ is the union of all $E_{\infty,\delta}$'s; since any compact analytic domain of Y intersects only finitely many $E_{\infty,\delta}$'s, we see that E_∞ is locally constructible.

The set $E'_{\infty,d}$ is the intersection of two subsets of Y:

- the set $E_{\infty,d}$, which we have just seen is constructible;
- the X-flatness locus of Y, which is Zariski-open by Theorem 10.3.2.

Therefore $E'_{\infty,d}$ is locally constructible. By Corollary 5.3.7, it is also open; it is therefore Zariski-open by Lemma 10.1.10 (4). It follows that $E'_\infty = \bigcup_\delta E'_{\infty,\delta}$ is Zariski-open.

For any $x \in X$, the intersection $E_\infty \cap Y_x$ is nothing but the set E'_∞ understood with respect to the map $Y_x \to \mathscr{M}(\mathscr{H}(x))$: it is thus Zariski-open in Y_x (this was already known; see [**Duc09**], Prop. 6.6). By Lemma 10.7.1 (3), one can describe E_n as the set of points of Y at which the fiberwise codimension of $Y \backslash E_\infty$ in Y is $> n$. By Proposition 10.4.4, this is a constructible subset of Y.

Let us assume now that Y is of pure relative dimension d over X, and let us prove that E'_n is open. We are going to apply some of the general results in 10.2, which involve two categories $\mathfrak{F}$ and $\mathfrak{C}$ and a property Q as in 10.2.1, and a functor $\mathscr{S}$ as in 10.2.4. We take $\mathfrak{F}$ to be the category $\mathfrak{T}$, $\mathfrak{C}$ to be the category of all analytic spaces, Q to be the property of being geometrically S_n, and $\mathscr{S}$ the functor that sends any analytic space to its structure sheaf. We are interested in assertion (β) of 10.2.8. Since assertions (α) and (γ) of loc. cit.already hold ((α) is the G-local constructibility of E_n we have just established, and (γ) is Theorem 10.3.2), Lemma 10.2.23 ensures that it is sufficient to prove assertion $(\beta^\flat)$ of 10.2.12; i.e., we may assume that Y and X are affinoid and Y is X-flat (the proof of Lemma 10.2.23 only involves arguing G-locally, hence it does not modify the dimension of $Y \to X$).

Now as $Y \to X$ is flat, the constructible subset E_∞ is equal to E'_∞, hence is Zariski-open by the above. By Lemma 10.7.1 (3), one can describe E'_n as the set of points of Y at which the fiberwise codimension of $Y \backslash E_\infty$ in Y is $> n$. The relative dimension of $Y \to X$ is equal to d everywhere, and the relative dimension of $Y \backslash E_\infty \to X$ is Zariski-upper semi-continuous by Thm. 4.9 of [**Duc07b**] (we use the fact that $Y \backslash E_\infty$ is a Zariski-closed subset of Y). It follows that E'_n is a Zariski-open subset of Y. $\square$

10.7.3. Remark. — We have proved the Zariski-openness of B'_n, E'_n, Δ', and Θ' only under the assumption that $Y \to X$ is equidimensional. We think that this is not optimal. For instance, in scheme theory, the analogues of Δ' and Θ' (for a finitely presented morphism between noetherian schemes) are Zariski-open without any assumption on the dimension ([**EGA IV**$_3$], Thm. 1.2.4 (iv) and (v)), and so is the analogue of B'_n when $\mathscr{E} = \mathscr{O}_Y$ (Thm. 2.2.6 (i) of op. cit.). Note nevertheless that there is a counter-example to the scheme-theoretic version of Zariski-openness of E'_1 when one allows non-equidimensional fibers (op. cit., Rem. 12.1.8 (ii)), which can be turned into a counter-example to Zariski-openness of E'_1 in our setting by endowing the ground field with the trivial absolute value and applying GAGA principles.

In order to get stronger statements, we should probably investigate carefully the variation of the number of (local) geometric irreducible or embedded components on which a point lies *it its fiber*, which in turn woud require for the "spreading out" process a deeper understanding of the links between the local rings of a generic fiber and those of the source space, far beyond Theorem 6.3.3.

We are now going to show that most constructible loci considered in this memoir tend to be algebraizable when one starts from algebraic data.

10.7.4. Theorem. — *Let V be a k-affioid space, ant let $\mathscr{Y}$ be a V^{al}-scheme locally of finite type; set $Y = \mathscr{Y}^{\mathrm{an}}$. Let $\varphi\colon Y \to X$ be a morphism of k-analytic spaces such that (at least) one of the following assertions holds:*

(A) *The morphism φ is the composition of the structure map $Y \to V$ and an arbitrary morphism $V \to X$.*

(B) *There exists a k-affinoid space U, a morphism $V \to U$, and a scheme $\mathscr{X}$ locally of finite type over U^{al} such that $X = \mathscr{X}^{\mathrm{an}}$ and φ is induced by an U^{al}-map $\mathscr{Y} \to \mathscr{X}$.*

Let $\mathscr{E}$ be a coherent sheaf on Y and let E and F be two locally constructible subsets of Y with $F \subset E$. Assume moreover that $\mathscr{E}$ arises from an algebraic coherent sheaf on $\mathscr{Y}$, and both E and F arise from locally constructible subsets of $\mathscr{Y}$; i.e., E^{al} and F^{al} are locally constructible, $E = (E^{\mathrm{al}})^{\mathrm{an}}$ and $F = (F^{\mathrm{al}})^{\mathrm{an}}$. Let n and d be two integers.

(1) *The following subsets of Y are of the form P^{an}, for P a locally constructible subset of $\mathscr{Y}$ (note that such a P is constructible as soon as $\mathscr{Y}$ is of finite type, or more generally finite-dimensional, see Remark 10.1.13).*
   - (1a) *The set of points y such that $\dim_y \varphi = d$.*
   - (1b) *The subset of $\overline{E}^{\varphi}$ consisting of points at which the fiberwise dimension of $\overline{E}^{\varphi}$ over X belongs to a given subset of $\mathbf{Z}_{\geqslant 0}$; in particular, $\overline{E}^{\varphi}$ iself.*
   - (1c) *The subset of $\overline{E}^{\varphi}$ consisting of points at which the fiberwise codimension of $\overline{F}^{\varphi}$ in $\overline{E}^{\varphi}$ belongs to a given subset of $\mathbf{Z}_{\geqslant 0} \cup \{+\infty\}$.*
   - (1d) *The sets A_n , A_∞, B_n, C,D, E_n, E_∞, Δ, and Θ of Theorem 10.7.2.*

(2) *The following sets are of the form P^{an}, for P a Zariski-open subset of $\mathscr{Y}$:*
   - (2a) *The set of points y such that $\dim_y \varphi \leqslant d$.*
   - (2b) *The set of points at which $\mathscr{E}$ is X-flat.*
   - (2c) *The sets A'_n , A'_∞, C', D', $E'_{\infty,d}$, and E'_∞ of Theorem 10.7.2.*
   - (2d) *The sets E'_n, Δ', and Θ' of Theorem 10.7.2 if φ is of pure relative dimension d.*
   - (2e) *The set B'_n of Theorem 10.7.2 if $\operatorname{Supp}(\mathscr{E}) \to X$ is of pure relative dimension d.*

Proof. — In case (A), the theorem is local on $\mathscr{Y}$, and we can thus assume that the latter admits a proper compactification $\overline{\mathscr{Y}}$ over V^{al}.

In case (B), the theorem is local on $\mathscr{Y}$ and $\mathscr{X}$. We can thus assume that $\mathscr{X}$ admits a proper compactification $\overline{\mathscr{X}}$ over U^{al}, and that $\mathscr{Y} \to \overline{\mathscr{X}} \times_{U^{\mathrm{al}}} V^{\mathrm{al}}$ admits a proper compactification $\overline{\mathscr{Y}}$.

Hence in both cases we can assume that $\mathscr{Y}$ admits a proper compactification $\overline{\mathscr{Y}}$ over V^{al}, and that φ extends to a morphism $\overline{\mathscr{Y}}^{\mathrm{an}} \to X$ (for achieving this in case (B) one first needs to replace $\mathscr{X}$ with $\overline{\mathscr{X}}$); note that E^{al} and F^{al} are now constructible since $\mathscr{Y}$ is of finite type over V^{al}. By [**EGA I**], Cor. 9.4.8, the coherent sheaf on $\mathscr{Y}$ from which $\mathscr{E}$ arises can be extended to a coherent sheaf on $\overline{\mathscr{Y}}$; hence $\mathscr{E}$ can be extended to a coherent sheaf $\overline{\mathscr{E}}$ on $\overline{\mathscr{Y}}^{\mathrm{an}}$. Moreover, E and F remain constructible inside $\overline{\mathscr{Y}}^{\mathrm{an}}$

(in contrast with non-transitivity of the analytic Zariski-topology in general) because E^{al} and F^{al} are constructible in $\overline{\mathscr{Y}}$.

By replacing the scheme $\mathscr{Y}$ with $\overline{\mathscr{Y}}$ and the coherent sheaf $\mathscr{E}$ with $\overline{\mathscr{E}}$ (and still working with E and F), we reduce to the case where $\mathscr{Y}$ is proper, *except possibly for* (2d) *and* (2e), because the relative equidimensionality property they require might be lost. Now, modulo GAGA:

- Cases (1a) and (2a) come from Zariski-upper semi-continuity of the relative dimension ([**Duc07b**], Thm. 4.9), case (1b) from Theorem 10.3.2, and cases (1c) and (2c) from Theorem 10.7.2.
- Cases (1c) and (1d) come respectively from Theorem 10.4.3 and Proposition 10.4.4.

It remains to consider cases (2d) and (2e) (the scheme $\mathscr{Y}$ is no longer assumed to be proper). We already know that the sets considered in (1d) and (2b) are of the form P^{an} for P a locally constructible subset of $\mathscr{Y}$; it follows that the sets considered in (2d) and (2e) are also of this form. Moreover, they are Zariski-open by Theorem 10.7.2; in particular, they are open in $\mathscr{Y}^{\mathrm{an}}$. But if P is a locally constructible subset of $\mathscr{Y}$ such that P^{an} is open in $\mathscr{Y}^{\mathrm{an}}$, then P is open in $\mathscr{Y}$ by [**Ber93**], Cor. 2.6.6 (which is stated for the case of a constructible subset, but extends to the locally constructible case by arguing locally), hence we are done. $\square$

We end this section by showing that if one starts from a proper map, one can get some local constructibility and Zariski-openness results *on the target.*

10.7.5. Theorem. — *Let X be a k-analytic space and let Y be a* proper *X-analytic space. Let $\mathscr{E}$ be a coherent sheaf on Y and let n and d be two non-negative integers. For every subset Π of Y, we denote by $[\Pi]$ the set of points x of X such that $Y_x \subset \Pi$. We use the notation of Theorem 10.7.2.*

(1) *The sets $[A_n]$, $[A_\infty]$, $[B_n]$ $[C]$, $[D]$, $[E_n]$, $[E_\infty]$, $[\Delta]$ and $[\Theta]$ are locally constructible subsets of X (hence are constructible if X is finite-dimensional, see Proposition 10.1.12).*
(2) *The sets $[A'_n]$, $[A'_\infty]$, $[C']$, $[D']$, $[E'_{\infty,d}], [E'_\infty]$ and $[\mathrm{Flat}(\mathscr{E}/X)]$ are Zariski-open subsets of X. The subset Ω of X consisting of points x such that $\dim Y_x \leqslant d$ is Zariski-open.*
(3) *If $\mathrm{Supp}(\mathscr{E})$ is purely of relative dimension d over X, then the set $[B'_n]$ is a Zariski-open subset of X. If Y is purely of relative dimension d over X, then $[E'_n], [\Delta']$ and $[\Theta']$ are Zariski-open subsets of X.*

Proof. — Since Y is proper over X, the image in X of any Zariski-closed subset (resp. locally constructible subset) of Y is a Zariski-closed subset of X (resp. a locally constructible subset of X): this follows from 1.3.23 (resp. Theorem 10.1.15). Hence the theorem is a straightforward consequence of Theorem 10.7.2, Theorem 10.3.2 as far as $[\mathrm{Flat}(\mathscr{E}/X)]$ is concerned, and Thm. 4.9 of [**Duc07b**] as far as Ω is concerned. $\square$

CHAPTER 11

ALGEBRAIC PROPERTIES: TARGET, FIBERS AND SOURCE

In commutative algebra and algebraic geometry, many familiar properties satisfy a principle of the following kind: if one is given a flat map, and if the property under investigation holds on the target and fiberwise, then it holds on the source. This has proved of fundamental importance, and the goal of this chapter is to get similar results in analytic geometry.

Our general strategy is not the same as in Chapt. 10 [1]. In the latter, we established analogues of various algebraic results by writing direct analytic proofs, without using these results. Here we shall start from a property satisfying some principle of the kind described above in the algebraic setting (we use again a rather abstract, axiomatic presentation as we did in 2.2-2.4: see Section 11.2 below), and deduce from this that it satisfies the analogous principle in analytic geometry; this is Theorem 11.3.1, see also its "concrete" version in Theorem 11.3.3.

Section 11.1 is devoted to some preparatory work, and ends with a result that will be crucial for the proof of Theorem 11.3.1, but also has independent interest (Theorem 11.1.5). Let us quickly explain what it consists of. Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of integral schemes of finite type over a field, and let d be the generic dimension of the fibers. It is then obvious that $\mathscr{Y} \to \mathscr{X}$ admits a factorization $\mathscr{Y} \to \mathscr{Z} \hookrightarrow \mathscr{X}$, with $\mathscr{Z}$ an integral closed subscheme of $\mathscr{X}$ of dimension $\dim \mathscr{Y} - d$, and $\mathscr{Y} \to \mathscr{Z}$ dominant: simply take for $\mathscr{Z}$ the reduced closure of the image of the generic point of $\mathscr{Y}$ in $\mathscr{X}$. This is of course very useful for various purposes: induction on dimension of the ground scheme, reduction to the dominant situation in order to use genericity arguments, etc.

Unfortunately, there is no such theorem in analytic geometry, but Theorem 11.1.5 provides a weak (and nonetheless useful) substitute for it. Let $\varphi\colon Y \to X$ be a morphism between k-analytic spaces and let d and δ be two non-negative integers. Assume

1. We will nevertheless *use* two results from Chapt. 10, namely the Zariski-openness of the relative flat locus (Theorem 10.3.2) and our "deboundarization" statement (Lemma 10.3.6).

that Y is of dimension d (so $Y \neq \varnothing$) and that φ is purely of relative dimension δ. Then Theorem 11.1.5 asserts that there exists a non-empty affinoid domain V of Y, an affinoid domain U of X, and a Zariski-closed subset S of U of pure dimension $d-\delta$ such that $\varphi(V) \subset S$. One thus gets a nice factorization, but only after restriction to some affinoid domain of the source. We shall not say here anything about its proof, except that one proceeds in a perhaps unusual way: in some sense, one reduces to the *maximally non-strict case*.

11.1. A weak "dominant factorization theorem"

11.1.1. Notation. — If X is a k-analytic space and if x is a point of X, we denote by $\rho_k(x)$ the dimension of the $\mathbf{Q}$-vector space $|\mathscr{H}(x)^\times|^{\mathbf{Q}}/|k^\times|^{\mathbf{Q}}$.

11.1.2. Remark. — Let X be a k-analytic space of finite dimension $d \geqslant 0$ and let x be a point of X. Since $d_k(x)$ is the sum of $\rho_k(x)$ and of the transcendence degree of $\widetilde{\mathscr{H}(x)}^1$ over $\widetilde{k}^1$, we have $\rho_k(x) \leqslant d_k(x)$, with equality if and only if $\widetilde{\mathscr{H}(x)}^1$ is algebraic over $\widetilde{k}^1$. As $d_k(x) \leqslant d$, we see that $\rho_k(x) = d$ if and only if $d_k(x) = d$ and $\widetilde{\mathscr{H}(x)}^1$ is algebraic over $\widetilde{k}^1$.

11.1.3. Lemma. — *Let X be a k-analytic space of finite dimension $d \geqslant 0$. Assume that the $\mathbf{Q}$-vector space $\mathbf{R}_+^\times/|k^\times|^{\mathbf{Q}}$ is infinite-dimensional. There exists a point $x \in X$ such that $\rho_k(x) = d$.*

Proof. — The problem we are dealing with is insensitive to nilpotents, hence we can assume that X is reduced. The space X has a d-dimensional affinoid domain, and the latter has a d-dimensional irreducible component. We can thus can assume that X is affinoid and integral. Let us argue now by induction on d.

If $d = 0$ the space X consists of one rigid point and we are done. Let us assume now that $d > 0$ and that the lemma has been proven for smaller dimension. We distinguish two cases.

Let us first assume that the Krull dimension of X^{al} is zero. Since X is reduced, this means that $\mathscr{O}_X(X)$ is a field. The generalized affinoid *Nullstellensatz* ([**Duc07b**], Thm. 2.7) then tells that (at least) one of the following assertions holds:

(a) There exists a k-free polyradius r such that $X = \mathscr{M}(k_r)$.

(b) The absolute value of k is trivial, and there exist two real numbers r and s with $0 < r \leqslant s < 1$ such that X is the compact annulus $\{x \in \mathbf{A}_k^{1,\mathrm{an}}, r \leqslant |T(x)| \leqslant s\}$.

If (a) holds then X consists of one point x and $d = d_k(x) = \rho_k(x)$, hence we are done. If (b) holds then $d = 1$ and we may take $x = \eta_r$.

Let us now assume that the Krull dimension of X^{al} is positive. Under this assumption, X^{al} admits a non-empty proper Zariski-closed subset. As a consequence, there exists an analytic function f on X whose zero-locus is non-empty, and not the whole space X. The image $|f|(X)$ is then a compact interval $[0, R]$ with $R > 0$; by our assumption on the value group $|k^\times|$, this interval contains an element r that does not belong to $|k|^{\times \mathbf{Q}}$.

Let V be the affinoid domain of X defined by the equation $|f| = r$. By the choice of r, the domain V is non-empty (hence d-dimensional), and its k-analytic structure factorizes through a k_r-analytic structure induced by f. For any $x \in V$ one has $d_k(x) = d_{k_r}(x) + 1$, which implies that the k_r-analytic dimension of V is $d - 1$ Since $\mathbf{R}_+^\times / |k_r^\times|^{\mathbf{Q}}$ is still infinite dimensional (because $|k_r^\times|^{\mathbf{Q}} / |k^\times|^{\mathbf{Q}} = r^{\mathbf{Q}}$), we may apply the induction hypothesis. It asserts that there exists a point x in V such that $\rho_{k_r}(x) = d - 1$, and one has $\rho_k(x) = \rho_{k_r}(x) + 1 = d$. $\square$

11.1.4. Lemma. — *Let d be a non-negative integer, and let $\varphi\colon Y \to X$ be a morphism of pure relative dimension d between k-analytic spaces. Assume that Y is quasi-compact and d-dimensional. The image $\varphi(Y)$ consists of finitely many rigid points.*

Proof. — Let V be a non-empty affinoid domain of Y, and let V' be an irreducible component of V (say, with its reduced structure). We shall prove that $\varphi(V')$ consists of a single rigid point, which will yield the required result by quasi-compactness of Y. Let r be a k-free polyradius such that $|k_r^\times| \neq \{1\}$ and such that V'_r is strictly k_r-affinoid; note that V'_r remains irreducible (Lemma 2.7.8 applied with $\mathscr{X} = \operatorname{Spec} A$, or [**Duc07b**], Lemma 1.3).

Let V'' be the union of all irreducible components of V that are not equal to V'. The open subset $V'_r \backslash V''_r$ of V'_r is non-empty and strictly k_r-analytic, hence has a k_r-rigid point y (1.2.10); let x be its image on X_r. Since x is a rigid point, $\varphi^{-1}_{|V_r}(x)$ is a Zariski-closed subset of V_r, which is purely d-dimensional by assumption and contains y; since $y \notin V''_r$ and $\dim V' \leqslant d$, this forces $\varphi^{-1}_{|V_r}(x)$ to contain the whole component V'_r (and the dimension of V' to be equal to d: as a by-product of our proof, Y is purely d-dimensional). In other words, $\varphi(V'_r) = \{x\}$.

Let t be the image of x on X. One has $\varphi(V') = \{t\}$; it remains to show that t is rigid. For any pre-image z of t on X_r, the fiber of V'_r over z is naturally isomorphic to $V' \times_{\mathscr{H}(t)} \mathscr{H}(z)$; in particular, it is non-empty, which shows that $z = x$; in other words, the set of pre-images of t on X_r is the singleton $\{x\}$. Since this set can be identified with $\mathscr{M}(\mathscr{H}(t)_r)$, we see that r is $\mathscr{H}(t)$-free and $\mathscr{H}(x) = \mathscr{H}(t)_r$ (1.2.15). As $\mathscr{H}(x)$ is a finite Banach k_r-algebra, $\mathscr{H}(t)$ is a finite Banach k-algebra by Prop. 2.1.11 of [**Ber90**], which means that t is rigid. $\square$

11.1.5. Theorem. — *Let $\varphi\colon Y \to X$ be a morphism between k-analytic spaces and let d and δ be two non-negative integers. Assume that Y is of dimension d (so $Y \neq \varnothing$) and φ is purely of relative dimension δ (so $\delta \leqslant d$). There exists a non-empty affinoid domain V of Y, an affinoid domain U of X, and a Zariski-closed subset S of U of pure dimension $d - \delta$ such that $\varphi(V) \subset S$. (Note that in the case where $\delta = d$, this is an immediate consequence of Lemma 11.1.4).*

11.1.6. Remark. — Even if $|k^\times| \neq \{1\}$ and Y and X are strictly k-analytic, the affinoid domain built by the proof of the theorem is not strict in general: if the analytic field k is topologically of countable type over its prime complete field (e.g., $k = \mathbf{Q}_p, \mathbf{C}_p$ or $\mathbf{F}_p(\!(t)\!)$) and if $d > \delta$, the domain V is k_r analytic for some k-free polyradius $r = (r_1, \dots, r_{d-\delta})$, hence non-strict.

Proof of the theorem 11.1.5. — Choose a d-dimensional irreducible component Y' of Y, and then choose a non-empty affinoid domain Y'' inside the open subset of Y' consisting of points that do not belong to another component; by construction, Y'' is purely d-dimensional. Let X' be an affinoid domain of X intersecting $\varphi(Y'')$. By replacing Y with a non-empty affinoid domain of $\varphi^{-1}(X') \cap Y''$ and X with X'', we can assume that X and Y are affinoid, and that Y is of pure dimension d.

The definition of $\varphi\colon Y \to X$ only involves countably many parameters; therefore there exists a complete subfield k_0 of k topologically of countable type over its prime complete field, and a morphism $Y_0 \to X_0$ between k_0-affinoid spaces such that $Y \to X$ is deduced from $Y_0 \to X_0$ by ground field extension to k. Since the assertion we are interested in is "stable under ground field extension", we can replace k with k_0, and $Y \to X$ with $Y_0 \to X_0$, and hence assume that k is topologically of countable type over its prime complete field.

Since k is of countable type over its prime complete field, $|k^\times|$ is countable and $\mathbf{R}_+^\times/|k^\times|^{\mathbf{Q}}$ is thus infinite-dimensional. It follows then from Lemma 11.1.3 that there exists $y \in Y$ with $\rho_k(y) = d$; set $x = \varphi(y)$. Since Y is d-dimensional, Remark 11.1.2 ensures that $d_k(y) = d$ and that $\widetilde{\mathscr{H}(y)}^1$ is algebraic over $\widetilde{k}^1$. Since φ is purely of relative dimension δ, it follows from 1.4.14 (4) that $d_k(x) = d - \delta$. As $\widetilde{\mathscr{H}(x)}^1 \subset \widetilde{\mathscr{H}(y)}^1$, $\widetilde{\mathscr{H}(x)}^1$ is also algebraic over $\widetilde{k}$; we thus have $\rho_k(x) = d_k(x) = d - \delta$, again by Remark 11.1.2.

Since X is affinoid, the group $|\mathscr{H}(x)^\times|$ is generated by elements of the form $|f(x)|$ where f is an analytic function on X which does not vanish at x. Therefore there exist analytic functions $f_1, \dots, f_{d-\delta}$ on X which do not vanish at x and such that $(|f_i(x)|)_{1\leqslant i\leqslant d-\delta}$ is a basis of $|\mathscr{H}(x)^\times|^{\mathbf{Q}}/|k^\times|^{\mathbf{Q}}$; we set $r_i = |f_i(x)|$ for every i, and $r = (r_1, \dots, r_{d-\delta})$. Note that r consists by construction of positive real numbers which are multiplicatively $\mathbf{Q}$-linearly independent modulo $|k^\times|$; otherwise said, r is k-free.

Let U be the affinoid domain of X defined by the equations $\{|f_i| = r_i\}_{1\leqslant i\leqslant \delta}$ and let V be its pre-image in Y; note that $y \in V$ by construction, so $V \neq \varnothing$. As V is a non-empty affinoid domain of Y, it is d-dimensional and compact; by construction, $y \in V$. Now U and V inherit through the f_i's a compatible k_r-analytic structure. For every $z \in V$ one has $d_k(z) = d_{k_r}(z) + d - \delta$; therefore V is of k_r-analytic dimension $d - (d - \delta) = \delta$. Since the k_r-analytic map $V \to U$ is purely of relative dimension δ, Lemma 11.1.4 above ensures that $S := \varphi(V)$ consists of finitely many

k_r-rigid points of U; in particular, S is Zariski-closed in U and $d_k(t) = d - \delta$ for every $t \in S$; therefore S is of pure k-analytic dimension $d - \delta$. □

11.2. The axiomatic setting

11.2.1. — We use the general categorical setting and the notation introduced in 2.2, especially $\mathfrak{T}, \mathfrak{F}$, and $\mathfrak{L}$ (Definition 2.2.1, 2.2.5, 2.2.7), but we assume that $\mathfrak{F} = \mathfrak{T}$ or $\mathfrak{F} = \mathfrak{Coh}$ (Examples 2.2.8 and 2.2.9; see also Convention 2.2.12).

We fix a property P as in 2.3. We assume that P satisfies conditions $(\mathsf{H}_{\mathrm{reg}})$, (F) and (O) of 2.3.15 (see Examples 2.3.17 and 2.3.18 for properties of interest that satisfy those conditions). If X is an object of $\mathfrak{T}$ and if D is an object of $\mathfrak{F}_X$, it makes sense to say that D satisfies P at a given point of X, or that D satisfies P (see Remark 2.3.8 and Lemma-Definition 2.4.1). Let $\varphi\colon Y \to X$ be a morphism between k-analytic spaces or between schemes belonging to $\mathfrak{T}$. We shall say that an object D of $\mathfrak{F}_Y$ satisfies P fiberwise at a point y of Y if $D_{Y_{\varphi(y)}}$ satisfies P at y, and that it satisfies P fiberwise if satisfies P fiberwise at every point of Y.

11.2.2. Lemma. — *Let $Y \to X$ be a morphism of k-affinoid spaces, let y be a point of Y and let x be its image in X. Let $\mathfrak{p}$ be a prime ideal of $\mathscr{O}_{X,x}$, and let Z be an integral closed analytic subspace of X inducing the quotient map $\mathscr{O}_{X,x} \to \mathscr{O}_{X,x}/\mathfrak{p}$. Let D be an object of $\mathfrak{F}_Y$. Assume that there exists a non-empty Zariski open subset Z' of Z such that $D_{Y\times_X Z'}$ satisfies P. Then D_y satisfies P fiberwise over $\mathfrak{p}$ with respect to* $\operatorname{Spec} \mathscr{O}_{Y,y} \to \operatorname{Spec} \mathscr{O}_{X,x}$.

Proof. — Set $T = Y \times_X Z$. By definition of Z, the local ring $\mathscr{O}_{Z,x}$ is the domain $\mathscr{O}_{X,x}/\mathfrak{p}$, and $\mathscr{O}_{T,y}$ is equal to $\mathscr{O}_{Y,y}/\mathfrak{p}\mathscr{O}_{Y,y}$; hence the fiber of $\operatorname{Spec} \mathscr{O}_{Y,y} \to \operatorname{Spec} \mathscr{O}_{X,x}$ over $\mathfrak{p}$ is equal to the generic fiber of $\operatorname{Spec} \mathscr{O}_{T,y} \to \operatorname{Spec} \mathscr{O}_{Z,x}$. We denote by ξ the generic point of Z^{al}; note that the generic point of $\operatorname{Spec} \mathscr{O}_{Z,x}$ lies over ξ (by flatness).

By assumption, $D_{T\times_Z Z'} = D_{Y\times_X Z'}$ satisfies P. Since P satisfies $(\mathsf{H}_{\mathrm{reg}})$ and the map $T \to T^{\mathrm{al}}$ is flat as a morphism of locally ringed spaces, $D^{\mathrm{al}}_{(T\times_Z Z')^{\mathrm{al}}}$ satisfies P as well; as $(T\times_Z Z')^{\mathrm{al}}$ contains by definition of Z' the generic fiber of $T^{\mathrm{al}} \to Z^{\mathrm{al}}$, the object $D^{\mathrm{al}}_{T^{\mathrm{al}}}$ satisfies P at every point of T^{al} lying above ξ. Since P satisfies $(\mathsf{H}_{\mathrm{reg}})$ and the map $\operatorname{Spec} \mathscr{O}_{T,y} \to \operatorname{Spec} \mathscr{O}_{T^{\mathrm{al}},y^{\mathrm{al}}}$ is regular, it follows that $D_{T,y}$ satisfies P at every point of $\operatorname{Spec} \mathscr{O}_{T,y}$ lying above ξ. In particular, $D_{T,y}$ satisfies P at every point of the generic fiber of $\operatorname{Spec} \mathscr{O}_{T,y} \to \operatorname{Spec} \mathscr{O}_{Z,x}$; but at such a point, the validity of P is equivalent to *fiberwise* validity with respect to $\operatorname{Spec} \mathscr{O}_{T,y} \to \operatorname{Spec} \mathscr{O}_{Z,x}$. The generic fiber of the map $\operatorname{Spec} \mathscr{O}_{T,y} \to \operatorname{Spec} \mathscr{O}_{Z,x}$ is equal to the fiber of $\operatorname{Spec} \mathscr{O}_{Y,y} \to \operatorname{Spec} \mathscr{O}_{X,x}$ over $\mathfrak{p}$, so D_y satisfies P fiberwise above $\mathfrak{p}$ with respect to $\operatorname{Spec} \mathscr{O}_{Y,y} \to \operatorname{Spec} \mathscr{O}_{X,x}$. □

11.2.3. Lemma. — *Let $Y \to X$ be a morphism of k-analytic spaces and let r be a k-free polyradius. Let us denote by $\mathfrak{s}$ both Shilov sections $X \to X_r$ and $Y \to Y_r$*

(1.2.16). Let y be a point of Y, let x be its image in X, and let y' be any pre-image of y in Y_r lying above $\mathfrak{s}(x)$; e.g., $y' = \mathfrak{s}(y)$. Let D be an object of $\mathfrak{F}_Y$.

(1) *The following are equivalent:*
   (i) *D_{Y_r} satisfies P at y'.*
   (ii) *D satisfies P at y.*
(2) *The following are equivalent:*
   (iii) *D_{Y_r} satisfies P fiberwise at y'.*
   (iv) *D satisfies P fiberwise at y.*

Proof. — Since k_r is analytically separable over k (Example 2.6.4), (i) $\iff$ (ii) follows from Proposition 2.6.6 and the fact that P satisfies $(\mathsf{H}_{\text{reg}})$.

Let us prove that (iii) $\iff$ (iv). The fiber $(Y_r)_{\mathfrak{s}(x)}$ is equal to $Y_x \times_{\mathscr{H}(x)} \mathscr{H}(\mathfrak{s}(x))$. By definition of the Shilov section, $\mathfrak{s}(x)$ is the point $\eta_{\mathscr{H}(x),r}$ of the fiber $\mathscr{M}(\mathscr{H}(x)_r)$; this implies that $\mathscr{H}(\mathfrak{s}(x))$ is an analytically separable extension of $\mathscr{H}(x)$ (Example 2.6.4, applied to the field $\mathscr{H}(x)$). Therefore it follows again from Proposition 2.6.6 and the fact that P satisfies $(\mathsf{H}_{\text{reg}})$ that D_{Y_x} satisfies P at y if and only if $D_{(Y_r)_{\mathfrak{s}(x)}}$ satisfies P at y'. Otherwise said, (iii) $\iff$ (iv). □

11.2.4. New technical conditions. — Let us introduce now some technical conditions that make sense for P. Each of these involves a local morphism $A \to B$ between local noetherian rings, and "fiberwise" will be understood with respect to $\operatorname{Spec} B \to \operatorname{Spec} A$; the maximal ideal of A will be denoted by $\mathfrak{m}_A$.

Condition $(\mathsf{T}_{\text{weak}})$*, when $\mathfrak{F} = \mathfrak{T}$.* — For every *flat* morphism $A \to B$ of $\mathfrak{L}$, the following implications hold:

- If B satisfies P, then A satisfies P.
- If A satisfies P, then B satisfies P if it satisfies P fiberwise.

Condition $(\mathsf{T}_{\text{strong}})$*, when $\mathfrak{F} = \mathfrak{T}$.* — For every *flat* morphism $A \to B$ of $\mathfrak{L}$, the following implications hold:

- If B satisfies P, then A satisfies P.
- If A satisfies P, then B satisfies P if it satisfies P fiberwise at the closed point of $\operatorname{Spec} B$; i.e., if $B/\mathfrak{m}_A B$ satisfies P.

Condition $(\mathsf{T}'_{\text{weak}})$*, when $\mathfrak{F} = \mathfrak{Coh}$.* — For every morphism $A \to B$ of $\mathfrak{L}$, every finitely generated A-module M and every non-zero and A-*flat* finitely generated B-module N the following implications hold:

- If $N \otimes_A M$ satisfies P, then M satisfies P.
- If M satisfies P, then $N \otimes_A M$ satisfies P if N satisfies P fiberwise.

Condition $(\mathsf{T}'_{\text{strong}})$*, when $\mathfrak{F} = \mathfrak{Coh}$.* — For every morphism $A \to B$ of $\mathfrak{L}$, every finitely generated A-module M and every non-zero and A-*flat* finitely generated B-module N the following implications hold:

- If $N \otimes_A M$ satisfies P, then M satisfies P.
- If M satisfies P, then $N \otimes_A M$ satisfies P if N satisfies P fiberwise at the closed point of Spec B; i.e., if the $B/\mathfrak{m}_A B$-module $N/\mathfrak{m}_A N$ satisfies P.

11.2.5. Example. — We assume that $\mathfrak{F} = \mathfrak{T}$. The following properties satisfy ($\mathsf{T}_{\text{strong}}$): being regular ([**EGA IV**$_2$], Prop. 6.5.1); being CI ([**Avr75**]); being Gorenstein ([**Mat86**], Thm. 23.4). The property of being R_m for some specified m satisfies (T_{weak})([**EGA IV**$_2$], Prop. 6.5.3).

11.2.6. Example. — We assume that $\mathfrak{F} = \mathfrak{Coh}$. The property of being CM satisfies ($\mathsf{T}'_{\text{strong}}$) by [**EGA IV**$_2$], Cor. 6.3.5. The property of being S_m for some specified m satisfies ($\mathsf{T}'_{\text{weak}}$)by [**EGA IV**$_2$], Prop. 6.4.1.

11.2.7. Remark. — All properties mentioned in Example 11.2.5 and 11.2.6 also satisfy (H_{reg}), (F), and (O); see Examples 2.3.17 and 2.3.18 for precise references.

11.3. The main theorem

11.3.1. Theorem. — *Let $\mathfrak{F}$ and P be as in 11.2.1. Let $Y \to X$ be a morphism of k-analytic spaces. Let y be a point of Y and let x be its image in X; in what follows, "fiberwise" will always be relative to the morphism $Y \to X$.*

(1) *Assume that $\mathfrak{F} = \mathfrak{T}$ and that P satisfies (T_{weak}).*
   (1a) *If Y satisfies P at y and is X-flat at y, then X satisfies P at x.*
   (1b) *If Y satisfies P fiberwise everywhere and is X-flat at y, and X satisfies P at x, then Y satisfies P at y.*
   (1c) *If Y is X-flat at y and satisfies P fiberwise at y, the space X satisfies P at x, and P satisfies ($\mathsf{T}_{\text{strong}}$), then Y satisfies P at y.*

(2) *Assume now that $\mathfrak{F} = \mathfrak{Coh}$, let $\mathscr{E}$ be a coherent sheaf on X, and let $\mathscr{F}$ be a coherent sheaf on Y. Assume that P satisfies ($\mathsf{T}'_{\text{weak}}$).*
   (2a) *If $\mathscr{F} \boxtimes \mathscr{E}$ satisfies P at y and is X-flat at y then $\mathscr{E}$ satisfies P at x.*
   (2b) *If $\mathscr{F}$ is X-flat at y and satisfies P fiberwise everywhere, and $\mathscr{E}$ satisfies P at x, then $\mathscr{F} \boxtimes \mathscr{E}$ satisfies P at y.*
   (2c) *If $\mathscr{F}$ is X-flat at y and satisfies P fiberwise at y, the coherent sheaf $\mathscr{E}$ satisfies P at x, and P satisfies ($\mathsf{T}'_{\text{strong}}$), then $\mathscr{F} \boxtimes \mathscr{E}$ satisfies P at y.*

Proof. — We are going to prove (2). The proof of (1) is *mutatis mutandis* exactly the same (one simply has to replace everywhere $\mathscr{E}$ with X or $\mathscr{O}_X$, replace $\mathscr{F} \boxtimes \mathscr{E}$ and $\mathscr{F}$ with Y and $\mathscr{O}_Y$, and properties (T_{weak}) and ($\mathsf{T}_{\text{strong}}$) with ($\mathsf{T}'_{\text{weak}}$) and ($\mathsf{T}'_{\text{strong}}$), respectively).

We can assume that Y and X are k-affinoid. Assertion (2a) then follows from the first of the two properties in (T_{weak}). We now focus on assertions (2b) and (2c).

11.3.1.1. *First reductions.* — We assume that one of the following conditions is satisfied:

(A) $\mathscr{E}$ satisfies P at x, the coherent sheaf $\mathscr{F}$ is X-flat at y, and $\mathscr{F}$ satisfies P fiberwise;

(B) $\mathscr{E}$ satisfies P at x, the coherent sheaf $\mathscr{F}$ is X-flat at y and satisfies P fiberwise at y, and P satisfies $(\mathsf{T}_{\text{strong}})$.

Our goal is to show that $\mathscr{F} \boxtimes \mathscr{E}$ satisfies P at y. Since $\mathrm{Flat}(\mathscr{F}/X)$ is (Zariski)-open by Theorem 10.3.2 and P satisfies (O) by assumption, we can shrink Y so that (B) can be replaced with:

(B′) $\mathscr{E}$ satisfies P at x, the coherent sheaf $\mathscr{F}$ is X-flat everywhere and satisfies P fiberwise *at every point of* Y_x, and P satisfies $(\mathsf{T}_{\text{strong}})$.

11.3.1.2. *Beginning of the proof under assumption* (A). — We assume that (A) holds, and we want to prove that $\mathscr{F} \boxtimes \mathscr{E}$ satisfies P at y; we argue by induction on $\dim_x X$.

Assume that $\dim_x X = 0$. This means that x is an isolated rigid point of X (1.4.7); note that $\mathscr{O}_{X,x}$ is then artinian. As P satisfies $(\mathsf{T}_{\text{weak}})$ and $\mathscr{E}$ satisfies P at x, it suffices to prove that $\mathscr{F}_y/\mathfrak{m}_x\mathscr{F}_y$ satisfies P. But since x is rigid, $\mathscr{F}_y/\mathfrak{m}_x\mathscr{F}_y = \mathscr{F}_{Y_x,y}$, which satisfies P by assumption (i); hence we are done.

Assume now that $\dim_x X > 0$, and that the assertion holds for strictly smaller local dimensions. As P satisfies $(\mathsf{T}_{\text{weak}})$ and $\mathscr{E}$ satisfies P at x, it suffices to prove that $\mathscr{F}_{Y,y}$ satisfies P fiberwise with respect to $\mathrm{Spec}\,\mathscr{O}_{Y,y} \to \mathrm{Spec}\,\mathscr{O}_{X,x}$.

Let us fix a prime ideal $\mathfrak{p}$ of $\mathscr{O}_{X,x}$, and prove that $\mathscr{F}_y$ satisfies P fiberwise above $\mathfrak{p}$ with respect to $\mathrm{Spec}\,\mathscr{O}_{Y,y} \to \mathrm{Spec}\,\mathscr{O}_{X,x}$. One can shrink X (and accordingly Y) so that $\mathfrak{p}$ is induced by an integral closed analytic subspace Z of X (2.1.5). By Lemma 11.2.2, it suffices to exhibit a non-empty Zariski-open subset Z' of Z satisfying the following:

$(\star)$ The coherent sheaf $\mathscr{F}_{Y \times_X Z'}$ satisfies P.

Since Z is integral, Lemma 2.4.8 ensures that the subset Z' of Z consisting of points at which Z is normal and $\mathscr{O}_Z$ satisfies P is Zariski-open and non-empty. We shall prove that Z' satisfies $(\star)$. We argue by contradiction, so *we assume from now on that Z' does* not *satisfy* $(\star)$.

11.3.1.3. — Let D be the subset of $Y \times_X Z$ consisting of points at which at which $\mathscr{F}$ does not satisfy P. Since P satisfies (O), D is a Zariski-closed subset of $Y \times_X Z$. Since we assume that Z' does not satisfy $(\star)$, the intersection $D \cap (Y \times_X Z')$ is non-empty. In particular, there exists an irreducible component D_0 of D whose intersection D_0' with $Y \times_X Z'$ is non-empty, hence is a dense Zariski-open subset of D_0. Let D_0'' be the intersection of D_0' with the set of points of D_0, that do not lie on any other component of D; this is a dense Zariski-open subset of D_0 which is Zariski-open in D as well. Let d be the dimension of D_0, and let δ be the infimum of the relative dimension of the morphism $D_0'' \to Z'$ (seen as a function from D_0'' to $\mathbf{Z}_{\geqslant 0}$). The set of points of D_0''

at which the relative dimension of $D_0'' \to Z'$ is precisely equal to δ is a non-empty Zariski-open subset of D_0'' ([**Duc07b**], Thm. 4.9). We can therefore find an affinoid domain Y_0 of $Y \times_X Z'$ enjoying the following properties:

- $\varnothing \neq Y_0 \cap D \subset D_0''$ (in particular, $Y_0 \cap D$ is purely of dimension d);
- $Y_0 \cap D \to Z'$ is purely of relative dimension δ.

By Theorem 11.1.5, there exists a non-empty affinoid domain D_1 of $Y_0 \cap D$ (endowed, say, with its reduced structure), an affinoid domain V of Z', and a Zariski-closed subset S of V of pure dimension $d-\delta$ which contains the image of D_1. By construction, $D_1 \subset D \cap (Y_0 \times_{Z'} V)$ and $\dim D_1 = d$. Moreover, we can shrink D_1 so that $D_1 = T \cap D$ for some affinoid domain T of $Y_0 \times_{Z'} V$ (Remark 1.3.13). By replacing V with any of its connected components that intersect the image of D_1 and T by the pre-image of this component in $Y_0 \times_{Z'} V$, we get a triple (T, V, S) satisfying the following:

- V is a connected affinoid domain of Z';
- S is a purely $(d-\delta)$-dimensional Zariski-closed subset of V;
- T is an affinoid domain of $Y \times_X V$ such that $T \cap D$ is non-empty and purely d-dimensional, $T \cap D \to V$ is purely of relative dimension δ, and the image of $T \cap D$ in V is contained in S.

Note that under these assumptions S is non-empty (it contains the image of $T \cap D$), hence V is non-empty. And since Z is irreducible, it is purely of dimension $\dim Z$, whence the equality $\dim V = \dim Z$. Moreover, V is contained Z'; the latter is normal, and $\mathscr{O}_Z$ satisfies P on Z'. As a consequence V is normal (hence integral, because it is non-empty and connected) and $\mathscr{O}_V$ satisfies P.

11.3.1.4. *Proof of the inequality* $d-\delta < \dim Z$. — The next step consists in proving that $d-\delta < \dim Z$, which will be crucial for applying the induction hypothesis. Note that since Z contains the non-empty $(d-\delta)$-dimensional Zariski-closed subset S of V, we have $d-\delta \leqslant \dim Z$. We are going to show by contradiction that the strict inequality holds; so we suppose that $d-\delta = \dim Z$ $(= \dim V)$. Choose t in $T \cap D$ such that $d_k(t) = d$, and let z be the image of t in V. We then have $d_k(z) = d-\delta = \dim V$ by 1.4.14.

Let r be a k-free polyradius such that $|k_r^\times| \neq \{1\}$ and such that T_r and V_r are strictly k_r-affinoid. Let $\mathfrak{s}$ denote both Shilov sections $X \to X_r$ and $Y \to Y_r$, and set $\mathscr{F}_r = \mathscr{F}_{Y_r}$. The field k_r is analytically separable over k. This implies, as V is normal as an affinoid domain of Z', that V_r is normal. This also implies, since P satisfies $(\mathsf{H}_{\mathrm{reg}})$, that $D_r \cap T_r$ is precisely the set of points of T_r at which $\mathscr{F}_r$ does not satisfy P. Moreover, it follows from 1.4.8 that

$$d_{k_r}(\mathfrak{s}(z)) = d_k(z) = \dim_k V = \dim_{k_r} V_r,$$

and from Lemma 11.2.3 that $\mathscr{F}_r$ satisfies P fiberwise at every point lying over $\mathfrak{s}(z)$ (because we are currently working under assumption (A); i.e., $\mathscr{F}$ satisfies P fiberwise).

Since V_r and T_r are strict over the non-trivially valued field k_r, the fiber of $T_r \cap D_r$ above $\mathfrak{s}(z)$ (which is non-empty because it contains $\mathfrak{s}(t)$) has an $\mathscr{H}(\mathfrak{s}(z))$-rigid point s. By Lemma 10.3.6, there exists a quasi-étale morphism $V' \to V_r$, and a pre-image s' of s in $T' := V' \times_{V_r} T_r$ such that $T' \to V'$ is inner at s'; we can shrink V' so that it is k_r-affinoid. Let z' be the image of s' on V'. Let us write down some consequences of the quasi-étaleness of $V' \to V_r$.

(i) The space V' is normal by Proposition 5.5.5, and

$$d_{k_r}(z') = d_{k_r}(\mathfrak{s}(z)) = \dim V_r = \dim V';$$

this implies that $\mathscr{O}_{V',z'}$ is artinian (Corollary 3.2.9), hence a field by normality (reducedness would be sufficient). Since $T' \to V'$ is inner at s', Theorem 6.3.3 then ensures that $\mathscr{O}_{T'_{z'},s'}$ is flat over $\mathscr{O}_{T',s'}$.

(ii) The mophism Spec $\mathscr{O}_{T'_{z'},s'} \to$ Spec $\mathscr{O}_{T_z,s}$ is regular (Theorem 5.5.3).

(iii) The morphism Spec $\mathscr{O}_{T',s'} \to$ Spec $\mathscr{O}_{T,s}$ is flat (Corollary 5.3.2; it is in fact even regular by Theorem 5.5.3).

The property P satisfies ($\mathsf{H}_{\mathrm{reg}}$), and the coherent sheaf $\mathscr{F}_r$ satisfies P fiberwise at s. This implies the following, by using successively (ii), (i), and (iii):

- $\mathscr{F}_{T'}$ satisfies P fiberwise at s';
- $\mathscr{F}_{T'}$ satisfies P at s';
- $\mathscr{F}_r$ satisfies P at s.

But s lies on D_r, which is the set of points of T_r at which $\mathscr{F}_r$ does not satisfy P, a contradiction. Hence our assumption that $\dim Z = d-\delta$ was wrong, so $d-\delta < \dim Z$, as announced.

11.3.1.5. *End of the proof under assumption* (A). — We choose a point τ in the non-empty space $T \cap D$, and we denote by σ the image of τ in S. Since $\mathscr{O}_V$ satisfies P, the module $\mathscr{O}_{V,\sigma}$ satisfies P. We shall now prove that for every prime ideal $\mathfrak{q}$ of $\mathscr{O}_{V,\sigma}$, the module $\mathscr{F}_{T,\tau}$ satisfies P fiberwise above $\mathfrak{q}$; since P satisfies ($\mathsf{T}_{\mathrm{weak}}$), this will imply that $\mathscr{F}$ satisfies P at τ, hence contradict the fact that τ lies on D – and end the proof when (A) holds.

We fix $\mathfrak{q}$ as above, and we shrink V so that the quotient map $\mathscr{O}_{V,\sigma} \to \mathscr{O}_{V,\sigma}/\mathfrak{q}$ is induced by an integral closed analytic subspace W of V (2.1.5). By Lemma 11.2.2, it suffices to prove the existence of a non-empty Zariski-open subset W' of W such that $\mathscr{F}_{T\times_V W'}$ satisfies P.

Let us first assume that $W = V$. By definition of D, the coherent sheaf $\mathscr{F}_T$ satisfies P at any point of T outside D, and in particular above the Zariski-open subset $V\backslash S$. One has $\dim S = d-\delta$, and $d-\delta < \dim Z = \dim V$ (11.3.1.4). Therefore $S \subsetneq W$ and we may take $W' = W\backslash S$.

Assume now that $W \neq V$. Since V is irreducible one has then

$$\dim W < \dim V = \dim Z \leqslant \dim_x X$$

(the latter inequality comes from the fact that Z contains x by definition). Since W^{al} is integral, it follows from Lemma 2.4.8 that there exists a non-empty Zariski open subset W' of W such that the coherent sheaf $\mathscr{O}_{W'}$ satisfies P everywhere. Now $\mathscr{F}_{T\times_V W'}$ satisfies P fiberwise (because so does $\mathscr{F}$) and $\dim W' = \dim W < \dim_x X$. By the induction hypothesis, $\mathscr{F}_{T\times_V W'}$ satisfies P, and we are done.

11.3.1.6. *Proof under assumption* (B′). — We assume that (B′) holds, and we shall prove that $\mathscr{F}\boxtimes\mathscr{E}$ satisfies P at every point of Y_x (note that the original point y does not play any role in (B′)).

Let r be a k-free polyradius such that $|k^\times| \neq \{1\}$ and such that Y_r and X_r are strictly k_r-affinoid; let $\mathfrak{s}$ be the Shilov section of $X_r \to X$. Due to Lemma 11.2.3, we can replace k with k_r, the spaces Y and X with Y_r and X_r respectively, the point x with $\mathfrak{s}(x)$, and the sheaves $\mathscr{F}$ and $\mathscr{E}$ with $\mathscr{F}_{Y_r}$ and $\mathscr{E}_{X_r}$ respectively, to reduce to the case where $|k^\times| \neq \{1\}$ and Y and X are strict.

The set of points of Y at which $\mathscr{F}\boxtimes\mathscr{E}$ satisfies P is a Zariski-open subset of Y because P satisfies O. It is therefore sufficient to prove that the coherent sheaf $\mathscr{F}\boxtimes\mathscr{E}$ satisfies P at every *rigid* point of the strictly $\mathscr{H}(x)$-analytic space Y_x. So, let ω be any $\mathscr{H}(x)$-rigid point of Y_x. Lemma 10.3.6 ensures that there exists a strictly k-analytic space X', a quasi-étale morphism $X' \to X$ and a point ω' on $Y' := Y\times_X X'$ lying above y and such that $Y' \to X'$ is inner at ω'; we can assume that X' is affinoid. Since P satisfies ($\mathsf{H}_{\mathrm{reg}}$), it follows from from Proposition 5.5.4 that we can, by pulling-back all the data to X', reduce to the case where $Y \to X$ is inner at ω.

Since P satisfies ($\mathsf{T}'_{\mathrm{strong}}$) and $\mathscr{E}$ satisfies P at x, it suffices to prove that the quotient $\mathscr{F}_{Y,\omega}/\mathfrak{m}_x\mathscr{F}_{Y,\omega}$ satisfies P. We can shrink X so that $\mathfrak{m}_x$ is induced by a closed analytic subspace Z of X. One has $\mathscr{O}_{Z,x} = \mathscr{O}_{X,x}/\mathfrak{m}_x$; as a consequence, $\mathscr{O}_{Z,x}$ is a field. One also has $\mathscr{O}_{Y,\omega}/\mathfrak{m}_x\mathscr{O}_{Y,\omega} = \mathscr{O}_{Y\times_X Z,\omega}$. The morphism $Y\times_X Z \to Z$ is boundaryless at ω, and $\mathscr{O}_{Z,x}$ is a field. It follows from Theorem 6.3.3 that $\mathscr{O}_{Y_x,\omega}$ is flat over $\mathscr{O}_{Y\times_X Z,\omega}$. The module $\mathscr{F}_{Y_x,\omega}$ satisfies P by assumption (B′), and P satisfies ($\mathsf{T}_{\mathrm{strong}}$). Therefore $\mathscr{F}_{Y\times_X Z,\omega} = \mathscr{F}_{Y,\omega}/\mathfrak{m}_x\mathscr{F}_{Y,\omega}$ satisfies P. □

11.3.2. Remark. — In the proof of the theorem, we needed twice (in 11.3.1.4 and 11.3.1.6) to "spread out" a property from the generic fiber, which we achieved by using Theorem 6.3.3. But the latter only holds at an *inner* point. Therefore we first needed to reduce to the inner case, by applying Lemma 10.3.6, which requires to start from a *rigid* point of the relevant fiber. This need of enough rigid points was the reason why we had twice to reduce to the strict case (by extending scalars to some k_r).

Theorem 11.3.1 above deals with a general property satisfying some axioms, because we wanted to give a unified proof, and to emphasize the assumptions we actually need. But of course, the properties of interest are those mentioned in Examples 11.2.5

and 11.2.6. For that reason, we are now going to write (a particular case of) this theorem with *explicit* properties involved. Note that assertion (1a) below is part of Lemma 4.5.2; assertion (3) is a simple application of (1) and (2).

11.3.3. Theorem **(Concrete version of Theorem 11.3.1)**

Let $Y \to X$ be a morphism of k-analytic spaces. Let $\mathscr{E}$ be a coherent sheaf on X, and let $\mathscr{F}$ be a coherent sheaf on Y. Let y be a point of Y and let x be its image in X; in what follows, "fiberwise" will always be relative to the morphism $Y \to X$. Let m be a non-negative integer.

(1) *Properties of the ambient spaces.*
   - (1a) *If Y is regular (resp. R_m, resp. CI, resp. Gorenstein) at y and Y is X-flat at y, then X is regular (resp. R_m, resp. CI, resp. Gorenstein) at x.*
   - (1b) *If Y is X-flat at y and is fiberwise R_m everywhere, and X is R_m at x, then Y is R_m at y.*
   - (1c) *If Y is X-flat at y and is fiberwise regular (resp. Gorenstein, resp. CI) at y, and X is regular (resp. Gorenstein, resp. CI) at x, then Y is regular (resp. Gorenstein, resp. CI) at y.*

(2) *Properties of coherent sheaves.*
   - (2a) *If $\mathscr{F} \boxtimes \mathscr{E}$ is CM (resp. S_m) at y and $\mathscr{F}$ is X-flat at y, then $\mathscr{E}$ is CM (resp. S_m) at x.*
   - (2b) *If $\mathscr{F}$ is X-flat at y and is fiberwise S_m everywhere, and $\mathscr{E}$ is S_m at x, then $\mathscr{F} \boxtimes \mathscr{E}$ is S_m at y.*
   - (2c) *If $\mathscr{F}$ is X-flat at y and fiberwise CM at y, and $\mathscr{E}$ is CM at x, then $\mathscr{F} \boxtimes \mathscr{E}$ is CM at y.*

(3) *Reducedness and normality.*
   - (3a) *If Y is reduced (resp. normal) at y and Y is X-flat at y, then X is reduced (resp. normal) at x.*
   - (3b) *If Y is X-flat at y and is fiberwise reduced (resp. fiberwise normal) everywhere, and X is reduced (resp. normal) at x, then Y is reduced (resp. normal) at y.*

APPENDIX A

GRADED COMMUTATIVE ALGEBRA

The purpose of this appendix is to introduce *graded* commutative algebra, after Temkin [**Tem04**]. Most classical notions of classical commutative algebra have graded counterparts, and the usual theorems often remain *mutatis mutandis* true in the graded context, with similar proofs; one only has essentially to add the word "graded" or "homogeneous" at suitable places. We shall therefore give almost no proofs. The justifications are left to the reader, who can also fruitfully read [**Tem04**].

A.1. Basic definitions

A.1.1. Definition. — In this memoir, a *graded ring* will always be an $\mathbf{R}_+^\times$*-graded ring*; i.e., a ring A equipped with a decomposition $A = \bigoplus_{r\in\mathbf{R}_+^\times} A^r$ as an abelian group, satisfying the condition $A^r \cdot A^s \subset A^{rs}$ for all r, s (the notation relative to the graduation is then *multiplicative*).

A.1.2. — Let A be a graded ring. It follows from the definition that $1 \in A^1$, and that A^1 is a usual ring. For any $r > 0$, the summand A^r is called the set of *homogeneous elements of degree* r; any homogeneous nonzero element has thus a well-defined degree, but 0 is homogeneous of degree r for all positive r.

Note that any ring can be considered as a trivially graded ring; i.e., a graded ring in which any element is homogeneous of degree 1.

A.1.3. — If A is a graded ring and if Γ is a subgroup of $\mathbf{R}_+^\times$, we shall denote by A^Γ the graded subring $\bigoplus_{\gamma\in\Gamma} A^\gamma$ of A.

A.1.4. — A morphism of graded rings $f\colon A \to B$ is a usual ring homomorphism f from A to B such that $f(A^r) \subset B^r$ for every $r > 0$.

A.1.5. — An ideal I of a graded ring A is called *homogeneous* if $I = \bigoplus_r I_r$ (as an abelian group) or, what amounts to the same, if I admits a set of generators consisting of homogeneous elements. If I is such an ideal, the quotient A/I inherits a graduation such that $A \to A/I$ is a morphism of graded rings.

A.1.6. — A *graded domain* is a graded ring whose underlying commutative ring is a domain. A graded ring A is a graded domain if and only if $A \neq \{0\}$ and the product of two *homogeneous* nonzero elements of A is always nonzero.

A.1.7. — A *graded field* is a graded ring in which every *homogeneous* nonzero element is invertible. If K is a graded field, K^1 is a field (whose characteristic will be also called the characteristic of K), and the set of degrees of homogeneous nonzero elements of K is a subgroup of $\mathbf{R}_+^\times$; we shall denote it by $\mathfrak{D}(K)$.

The reader should be aware that the commutative ring underlying a graded field is *not* a field in general. For example, let K be a field and let r be an element of $\mathbf{R}^\times \backslash \{1\}$. Let L be the graded ring whose underlying commutative ring is $K[T, T^{-1}]$ and whose graduation is such that $L^{r^i} = KT^i$ for every i (and $L^s = \{0\}$ if $s \notin r^{\mathbf{Z}}$). Then any homogeneous nonzero element of L is invertible, hence L is a graded field; but the ring L is not a field.

A.1.8. — Let I be a homogeneous ideal of a graded ring A. We shall say that I is *prime* if A/I is a graded domain; this is the case if and only if $I \neq A$ and $ab \in I \Rightarrow a \in I$ or $b \in I$ for every pair (a, b) of homogeneous elements of A. We shall say that I is *maximal* if A/I is a graded field; this is the case if and only if $I \neq A$ and I is maximal among all proper homogeneous ideals of A; note that every maximal homogeneous ideal is prime. By Zorn's lemma any proper homogeneous ideal of A is contained in a maximal homogeneous ideal; in particular if $A \neq \{0\}$ it contains a maximal homogeneous ideal.

A.1.9. — If A is a graded ring and if S is a multiplicative set of homogeneous elements of A, there is a well-defined *graded localization* $S^{-1}A$. For $r > 0$, every element of $(S^{-1}A)^r$ can be written as a fraction $\frac{a}{b}$ with $a \in A^{rs}$ and $b \in S^s$ for some $s > 0$; moreover two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if there exists $e \in S$ such that $e(ad - bc) = 0$. The localization $S^{-1}A$ comes with a natural morphism of graded rings $A \to S^{-1}A$, which sends any homogeneous element a to $\frac{a}{1}$.

For instance, any graded domain A has a *graded field of fractions* Frac A, given by the above construction for S consisting of all non-zero homogeneous elements of A.

A.1.10. — The above construction of a graded field whose underlying ring is not a field (A.1.7) can be generalized as follows. Let us start from any graded field K, let $r = (r_1, \ldots, r_n)$ be a family of positive real numbers and let $T = (T_1, \ldots, T_n)$ be a family of indeterminates. We denote by $K[T_1/r_1, \ldots, T_n/r_n]$, or by $K[T/r]$ for short, the graded domain whose underlying domain is $K[T]$, and whose graduation is such that $K[T/r]^s$ is equal to $\bigoplus_{I\in \mathbf{Z}^n_{\geqslant 0}} K^{sr^{-I}}T^I$ for every positive s (in other words, this is the only graduation extending that of K and such that every T_i is homogeneous of degree r_i); if $r_i = 1$ for some i we shall write T_i instead of $T_i/1$.

We denote by $K(T/r)$ the graded field of fractions of the graded domain $K[T/r]$. If $r_1, \ldots, r_n$ are linearly independent as elements of the $\mathbf{Q}$-vector space $\mathbf{R}^\times_+/\mathfrak{D}(K)^{\mathbf{Q}}$, the commutative ring underlying $K(T/r)$ is $K[T, T^{-1}]$. Thus if $n = 1$, if K is trivially graded and if $r \neq 1$ we recover the aforementioned example.

A.2. Graded linear algebra

We fix a graded ring A.

A.2.1. Definition. — A *graded A-module* is a (usual) A-module M equipped with a decomposition $M = \bigoplus_{r>0} M^r$ as an abelian group, such that $A^r M^s \subset M^{rs}$ for all r, s.

A.2.2. — Let M and N be two graded A-module and let $f\colon N \to M$ be an A-linear map. For r a positive real number, the map $f\colon M \to N$ is called *homogeneous of degree r* if $f(M^s) \subset N^{rs}$ for every $s > 0$. The map f is called *homogeneous* (without a mention of the degree) if it is homogeneous of degree r for some $r > 0$.

We denote by $A\text{-}\mathsf{Mod}_{\mathrm{g}}$ the category whose objects are graded A-modules and whose arrows are homogeneous A-linear maps of degree 1.

A.2.3. — Let M, N and P be graded A-modules. An A-bilinear map $b\colon M\times N \to P$ is called *homogeneous* if for every $r > 0$, every $m \in M^r$ and every $n \in N^r$, the two maps $b(m, \cdot)\colon N \to P$ and $b(\cdot, n)\colon M \to P$ are homogeneous of degree r.

A.2.4. — A submodule N of a graded A-module M is called a *graded* submodule of M if $N = \bigoplus_{r>0} N \cap M^r$ as an abelian group or, what amounts to the same, if N can be generated by a set of homogeneous elements. If N is a graded submodule of A, it inherits a natural graduation with $N^r = N \cap M^r$ for every $r > 0$, and the inclusion $N \hookrightarrow M$ is homogeneous of degree 1. The quotient M/N also inherits a natural graduation for which $(M/N)^r = M^r/N^r$ for every $r > 0$ (as an abelian group); it makes M/N a graded A-module, and the quotient map $M \to M/N$ is then homogeneous of degree 1. Note that we may view A as a graded A-module; its graded submodules are precisely its homogeneous ideals.

If $f\colon N \to M$ is a homogeneous A-linear map bewteen two graded A-modules, its image is a graded submodule of M, and its kernel is a graded submodule of N. Therefore f is injective if and only if $(f(n) = 0) \Rightarrow (n = 0)$ for every *homogeneous* element n of N.

If M is a graded A-module, for every $r > 0$ we define $M(r)$ as the graded A-module whose underlying A-module is M and whose graduation is such that $M(r)^s = M^{rs}$ for every $s > 0$.

A.2.5. — If $(M_i)_{i\in I}$ is any family of graded A-modules, the usual direct sum $\bigoplus_i M_i$ inherits a graduation, with $(\bigoplus_i M_i)^r = \bigoplus M_i^r$ for all r; the graded module $\bigoplus_i M_i$ is the disjoint sum of the M_i's in the category $A\text{-}\mathsf{Mod}_{\mathrm{g}}$. Combining this construction with that of quotients, we see that $A\text{-}\mathsf{Mod}_{\mathrm{g}}$ admits arbitrary colimits.

A.2.6. — Let M be a graded A-module and let $(m_i)_{i\in I}$ be a family of homogeneous elements of M, say $m_i \in M^{r_i}$ for every i. This family gives rise to a homogeneous A-linear map of degree 1 from $\bigoplus_i A(r_i^{-1})$ to M, that sends any homogeneous element (a_i) to $\sum a_i m_i$. The family (m_i) is said to be free (resp. generating, resp. a basis) if this map is injective (resp. surjective, resp. bijective).

A.2.7. — Let K be a graded field. Graded K-modules will be rather called graded K-vector spaces. If E is such a space, it has a basis; moreover, all bases of E have the same cardinality, which is called the *dimension* of E. If E' is any graded subspace of E, there exists a graded subspace E'' of E such that $E = E' \oplus E''$.

A.2.8. Definition. — Let A be a graded ring, and let M and N be two graded A-modules. The covariant functor from $A\text{-}\mathsf{Mod}_{\mathrm{g}}$ that sends P to the set of homogeneous A-bilinear maps from $M \times N$ to P is representable. The object that represents it is called the *graded tensor product* of M and N over A.

A.2.9. — If M and N are two graded A-modules, the A-module underlying their graded tensor product over A is the usual tensor product $M \otimes_A N$, and its summand of homogeneous elements of given degree r is generated *as an abelian group* by all elements of the form $m \otimes n$ with $m \in M^s$ and $n \in N^{r^{-1}s}$ for some s.

Hence we shall also denote by $M \otimes_A N$ the graded tensor product of M and N over A.

A.2.10. — For every graded A-module M, the functor $M\otimes$ from $A\text{-}\mathsf{Mod}_{\mathrm{g}}$ to itself commutes with colimits.

A.2.11. — A graded A-module M is called *flat* if $M\otimes\colon A\text{-}\mathsf{Mod}_{\mathrm{g}} \to A\text{-}\mathsf{Mod}_{\mathrm{g}}$ preserves injections. Any graded vector space over a graded field K is flat over K.

Let M be a graded A-module. Since $M\otimes$ commutes with colimits, M is flat if and only if $M\otimes$ preserves injections $N \hookrightarrow N'$ in $A\text{-}\mathsf{Mod}_{\mathrm{g}}$ such that N' is generated by N and a *finite* family $(e_1, \dots, e_m)$ of homogeneous elements. By induction on m, this is the case if and only if $M\otimes$ preserves injections as above with $m = 1$. Let $N \hookrightarrow N'$ be such an injection. The quotient N'/N can then be generated by a single homogeneous element, hence is isomorphic to $(A/I)(r)$ for some homogeneous ideal

I of A and some positive real number r. By the same piece of diagram-chasing as in the classical case, $M \otimes_A N \to M \otimes_A N'$ is injective as soon as $M \otimes_A I \to M$ is so. Using again commutation with colimits, we eventually see that M is flat if and only if $M \otimes_A J \to M$ is injective for every finitely generated homogeneous ideal J of A.

A.3. Graded algebras and graded extensions

A.3.1. Definition. — A *graded A-algebra* is a graded ring B endowed with a morphism of graded rings from $A \to B$; a morphism of graded A-algebras is a morphism of graded rings commuting with the structure maps from A.

A.3.2. — Any graded A-algebra inherits a structure of a graded A-module. In particular if B and C are two graded A-algebras, the graded tensor product $B \otimes_A C$ makes sense. Since B and C are usual A-algebras, $B \otimes_A C$ admits a natural structure of a ring; the latter together with the graduation of $B \otimes_A C$ makes $B \otimes_A C$ a graded ring, which is the amalgamated sum of B and C along A in the category of graded rings.

A.3.3. — Let K be a graded field and let L be a graded K-algebra. If L is nonzero then the structure morphism $K \to L$ is injective. We shall call L a *graded extension* of K if L is a graded field; let us assume from now on that this is the case.

Let $(x_i)_{i \in I}$ be a family of homogeneous elements of L, say $x_i \in L^{r_i}$ for every i. Evaluating polynomials at (x_i) yields a morphism of graded K-algebras from $K[T/r]$ to L (here $T = (T_i)_{i \in I}$, and we extend straightforwardly the definition of $K[T/r]$ given in A.1.10 to the case of an arbitrary family of indeterminates). The elements x_i are said to be *algebraically independent* over K if this morphism is injective. A maximal family of homogeneous elements of L that are algebraically independent over K is called a transcendance basis of L over K; there exist transcendance bases of L over K, all of which have the same cardinality; the latter is called the *transcendance degree* of L over K.

If x is a homogeneous element of L of degree $r > 0$, the singleton family $\{x\}$ is algebraically independent over K if $P(x) \neq 0$ for all nonzero homogeneous elements P of $K[T/r]$ (where T is now a single indeterminate). If this is the case we say that x is *transcendental* over K. The element x is said to be *algebraic* over K if it is not transcendental over K. If x is algebraic over K, the ideal I generated by the homogeneous elements $P \in K[T/r]$ such that $P(x) = 0$ is principal (like any homogeneous ideal of $K[T/r]$); if $x \neq 0$, the *minimal polynomial* of x is the unitary homogeneous generator of I (the condition $x \neq 0$ ensures that r is well-defined; if $x = 0$, its minimal polynomial should certainly be T, but it could be seen as belonging to $K[T/r]$ for any r, without canonical choice).

A homogeneous element x of L is algebraic over K if and only if the graded K-algebra generated by x is finite-dimensional (and hence a graded field). As a consequence, the homogeneous elements of L that are algebraic over K are the homogeneous elements of a graded subfield of L, which is called the *algebraic closure* of K inside L.

A graded extension L of K is said to be algebraic if all its homogeneous elements are algebraic over K or, otherwise said, if its transcendance degree is zero.

A.3.4. — Let $K \hookrightarrow L$ be a graded extension of graded fields. Let r be a positive real number, let x be a non-zero element of L^r algebraic over K, and let $P \in K[T/r]$ be its minimal polynomial. If n denotes the monomial degree of P, the constant coefficient of P (which is non-zero) has degree r^n, hence $r^n \in \mathfrak{D}(K)$. Morevoer every non-zero coefficient is of degree r^i for some i; as a consequence, x is algebraic over $K^{r^{\mathbf{Z}}}$.

Hence if L is algebraic over K, then $\mathfrak{D}(L)/\mathfrak{D}(K)$ is torsion and L^Γ is algebraic over F^Γ for every subgroup Γ of $\mathbf{R}_+^\times$; in particular, L^1 is algebraic over K^1. Conversely, assume that L^1 is algebraic over F^1 and $\mathfrak{D}(L)/\mathfrak{D}(K)$ is torsion. Let x be a homogeneous element of L. Since $\mathfrak{D}(L)/\mathfrak{D}(K)$ is torsion, there exists a non-zero homogeneous element a of K and $n \in \mathbf{Z}_{\geqslant 0}$ such that $ax^n \in L^1$; hence ax^n is algebraic over K^1, and x is algebraic over K.

A.3.5. — Let $K \hookrightarrow L$ be a graded extension of graded fields. Let (r_i) be a system of representatives of the quotient $\mathfrak{D}(L)/\mathfrak{D}(K)$. For every i, let x_i be a non-zero element of L^{r_i}; let (y_j) be a basis of the K^1-vector space L^1. The family $(x_i y_j)_{i,j}$ is then a basis of the graded K-vector space L (the verification is straightforward and left to the reader). In particular,

(a) $$[L : K] = [L^1 : K^1] \cdot [\mathfrak{D}(L) : \mathfrak{D}(K)]$$

(this is an equality of cardinal numbers, possibly infinite). For every subgroup Γ of $\mathbf{R}_+^\times$ we thus have

(b) $$[L^\Gamma : K^\Gamma] = [L^1 : K^1] \cdot [\mathfrak{D}(L^\Gamma) : \mathfrak{D}(K^\Gamma)] \leqslant [L^1 : K^1] \cdot [\mathfrak{D}(L) : \mathfrak{D}(K)] = [L : K].$$

Let (s_ℓ) be a family of positive real numbers lifting a basis of the $\mathbf{Q}$-vector space $\mathfrak{D}(L)^{\mathbf{Q}}/\mathfrak{D}(K)^{\mathbf{Q}}$. For every ℓ, let z_ℓ be a non-zero element of L^{s_ℓ}; let (t_λ) be a transcendence basis of L^1 over K^1. The family $(t_\lambda) \coprod (z_\ell)$ is then a transcendance basis of L over K. Indeed, a straightforward computation (left to the reader) show that it consists of algebraically independent elements. Denote by F the graded subfield of L generated by K and the family $(t_\lambda) \coprod (z_\ell)$. By construction, L^1 is algebraic over F^1 and $\mathfrak{D}(L)/\mathfrak{D}(F)$ is torsion, whence our claim. In particular,

(c) $$\operatorname{tr.\,deg}(L/K) = \operatorname{tr.\,deg}(L/K) + \dim_{\mathbf{Q}} D(L)^{\mathbf{Q}}/\mathfrak{D}(K)^{\mathbf{Q}}$$

(this is an equality of cardinal numbers, possibly infinite).

A.3.6. — The classical theory of algebraic extensions admits a graded counterpart, for which a general reference is the first section of [**Duc13**]. Let us simply recall here some basic definitions and properties. The proofs (most of which are outlined in [**Duc13**]) essentially consist of a "graded transcription" of Bourbaki's approach; see [**Bou81**], Chapt. V and especially §7 about separable extensions, §9 about normal extensions (they are called "quasi-galoisiennes" by Bourbaki), and §10 about Galois extensions.

We fix an algebraic graded extension $K \hookrightarrow L$. We denote by $\mathrm{Gal}(L/K)$ the Galois group of L over K; i.e., the group of K-automorphisms of the graded field L. If $[L : K]$ is finite, then $\mathrm{Gal}(L/K)$ is finite of cardinality $\leqslant [L : K]$; in general, it has a natural topology making it a profinite group (by considering the system of all finite graded subextensions of L stable under $\mathrm{Gal}(L/K)$).

We say that the graded extension $K \hookrightarrow L$ is *radicial* if for every homogeneous element a of L there exist $n \geqslant 0$ such that $a^{p^n} \in L_{\mathrm{sep}}$, where p is the characteristic exponent of L; i.e., p is equal to the characteristic of L if the latter is positive, and to 1 otherwise. If L is radicial over K, we have $\mathrm{Gal}(L/K) = \{\mathrm{Id}_L\}$ (because $a \mapsto a^p$ is an endomorphism of the graded ring L, hence is injective since L is a graded field).

Let r be a positive real number. An element x of L^r is called *separable* over K if $x = 0$ or if its minimal polynomial P satisfies the condition $P'(x) \neq 0$. There exists a graded subfield L_{sep} of L such that L^r_{sep} is for every $r > 0$ the set of separable elements of L^r. The graded subfield L_{sep} is called the *separable closure* of K inside L, and L is called a separable extension of K if $L_{\mathrm{sep}} = L$ (this is automatically the case if char. $L = 0$). The graded extension $L_{\mathrm{sep}} \hookrightarrow L$ is *radicial.*

The graded extension $K \hookrightarrow L$ is called *normal* if for every $r > 0$ and every element x of $L^r \backslash \{0\}$, the minimal polynomial P of x splits in L; i.e., P can be written $\prod_{1 \leqslant i \leqslant n}(T - x_i)$ with $x_i \in L^r$ for all i.

The graded extension $K \hookrightarrow L$ is called *Galois* if it is both separable and normal. This is the case if and only if the fixed graded field of $\mathrm{Gal}(L/K)$ is equal to K. If moreover $[L : K]$ is finite, L is Galois over K if and only if the cardinality of $\mathrm{Gal}(L/K)$ is equal to $[L : K]$.

The graded extension $K \hookrightarrow L$ is normal if and only if the fixed graded field of $\mathrm{Gal}(L/K)$ is radicial over K.

In fact, for the graded extension $K \hookrightarrow L$ to be normal (resp. Galois), it suffices that there exists a subgroup G of $\mathrm{Gal}(L/K)$ whose fixed graded field is radicial over K (resp. is equal to K); and if this is the case, G is dense in $\mathrm{Gal}(L/K)$.

Let Γ be a subgroup of $\mathbf{R}_+^\times$. The graded subfield L^Γ of K^Γ is stable under $\mathrm{Gal}(L/K)$, whence we get a restriction morphism $\mathrm{Gal}(L/K) \to \mathrm{Gal}(L^\Gamma/K^\Gamma)$. Let H denote the image of this map (this is a closed subgroup of $\mathrm{Gal}(L^\Gamma/K^\Gamma)$), and let F denote the fixed graded field of $\mathrm{Gal}(L/K)$. The fixed graded field of H is equal to $F \cap K^\Gamma = F^\Gamma$. Since F^Γ is radical over K^Γ (resp. equal to K^Γ) if F is radicial over K

(resp. equal to K), we see that if L is normal (resp. Galois) over K then L^Γ is normal (resp. Galois) over K^Γ and $\mathrm{Gal}(L^\Gamma/K^\Gamma) = G$.

A.4. Graded valuations

A.4.1. Definition. — Let K be a graded field and let Γ be an ordered abelian group with multiplicative notation. A Γ-valued *graded valuation* on K is a map $|\cdot|$ defined on the set of homogeneous elements of K with values in $\Gamma \cup \{0\}$ which satisfies the following conditions:

1. $|1| = 1, |0| = 0$, and $|ab| = |a| \cdot |b|$ for every pair (a, b) of homogeneous elements of K;
2. for every pair (a, b) of homogeneous elements of K *of the same degree* we have $|a + b| \leqslant \max(|a|, |b|)$.

A.4.2. — If we do not need to focus on the group Γ, or if the latter is clear from the context, we shall simply talk about a *graded valuation* on K; if K is a field (viewed as a trivially graded field), a graded valuation on K is nothing but a classical Krull valuation on K. Two graded valuations

$$|\cdot| : \coprod K^r \to \Gamma_0 \text{ and } |\cdot|' : \coprod K^r \to \Gamma'_0$$

are called *equivalent* if there exist an ordered abelian group Γ'', two increasing embeddings $i \colon \Gamma'' \hookrightarrow \Gamma$ and $j \colon \Gamma'' \hookrightarrow \Gamma'$, and a Γ''-valuation $|\cdot|''$ on K such that $|\cdot| = i \circ |\cdot|''$ and $|\cdot|' = j \circ |\cdot|''$.

A.4.3. — For any graded valuation $|\cdot|$ on K, the set $\{|a|\}_{a \in \bigcup_r K^r \setminus \{0\}}$ is an ordered group which is called the *value group* of $|\cdot|$. And the set

$$\bigoplus_r \{\lambda \in K^r,\ |\lambda| \leqslant 1\}$$

is a graded subring of K which is called the *graded ring of* $|\cdot|$. It is a local graded ring; i.e., it has a unique maximal homogeneous ideal, namely $\bigoplus_r \{\lambda \in K^r,\ |\lambda| < 1\}$. The residue graded field of this local graded ring is called the *residue graded field* of $|\cdot|$. Two graded valuations on K are equivalent if and only if they have the same graded ring.

A.4.4. — A graded subring A of K is a graded valuation ring of K (i.e., the graded ring of some graded valuation on K) if and only if for every non-zero homogeneous element λ of K, one has $\lambda \in A$ or $\lambda^{-1} \in A$; or, what amounts to the same, if and only if A is a graded local subring of K which is maximal for the domination relation (the latter is defined similarly as for usual local rings). Hence by Zorn's lemma, every graded local subring of K is dominated by a graded valuation ring of K. It follows that any graded valuation on a graded subfield of K extends to K (with possibly larger value group).

A.4.5. — Let $|\cdot|$ be a graded valuation on K, let A be its graded ring and let k be its graded residue field. If $|\cdot|'$ is a graded valuation on k, the pre-image of the graded ring of $|\cdot|'$ inside A is a graded valuation ring of K. The corresponding graded valuation is called the *composition* of $|\cdot|$ and $|\cdot|'$ and has the same residue graded field as $|\cdot|'$.

A.4.6. — Let $S = (S_i)_{1\leqslant i\leqslant n}$ be a family of indeterminates and let $s = (s_i)_{1\leqslant i\leqslant n}$ be a family positive real numbers. If $|\cdot|$ is any graded valuation on K, we shall denote by $|\cdot|_{\mathrm{Gauß}}$ the valuation on $K(S/s)$ that sends any homogeneous element $\sum_{I\in\mathbf{Z}^n_{\geqslant 0}} a_I S^I$ to $\max_I |a|_I$. It has the same value group as $|\cdot|$ and is characterized by the following properties:

- $|\cdot|_{\mathrm{Gauß}}$ is an extension of K to $K(S/s)$ such that $|S_i|_{\mathrm{Gauß}} = 1$ for every i.
- the images of the S_i's in the residue graded field of $|\cdot|_{\mathrm{Gauß}}$ are algebraically independent over the residue graded field of $|\cdot|$.

A.4.7. Graded reduction. — Let A be a ring equipped with a sub-multiplicative semi-norm $\|\cdot\|$; i.e., $\|\cdot\|$ is a map from A to $\mathbf{R}_+$ such that $\|0\| = 0$, $\|1\| \leqslant 1$ and

$$\|-a\| = \|a\|,\ \|a+b\| \leqslant \max(\|a\|, \|b\|),\ \text{ and }\ \|ab\| \leqslant \|a\|\cdot\|b\|$$

for every $(a, b) \in A^2$; one then has $\|1\| = 1$ unless $\|\cdot\| = 0$. We shall denote by $\widetilde{A}$ the residue *graded* ring of A in the sense of Temkin [**Tem04**], which is by definition equal to

$$\bigoplus_{r>0} \{x \in A, \|x\| \leqslant r\}/\{x \in A, \|x\| < r\}.$$

Note that $\widetilde{A}^1$ is the usual residue ring of A. If a is any element of A and if r is a positive real number such that $\|a\| \leqslant r$, we shall denote by $\widetilde{a}^r$ the image of a in $\widetilde{A}^r$. If $\|a\| \neq 0$ we shall write $\widetilde{a}$ instead of $\widetilde{a}^{\|a\|}$; if $\|a\| = 0$ we set $\widetilde{a} = 0$.

A.4.8. — Let k be a field equipped with a valuation $|\cdot| : k \to \mathbf{R}_+$. The previous construction provides a graded residue ring $\widetilde{k}$, which is easily seen to be a graded field. The field $\widetilde{k}^1$ is the residue field of the valuation $|\cdot|$ in the classical sense, and the group of degrees $\mathfrak{D}(\widetilde{k})$ is equal to $|k^\times|$. Hence $\widetilde{k}$ encodes information on both the residue field and the value group of $|\cdot|$ (for other manifestations of this phenomenon, see A.4.11 below).

Note that if $|\cdot|$ is the *trivial* valuation (i.e., $|x| = 1$ for all $x \in k^\times$) then $\widetilde{k} = \widetilde{k}^1$; if not, it does not seem that $\widetilde{k}$ can be interestingly interpreted as a residue graded field in the sense of A.4.3.

A.4.9. Example. — Let p be a prime number. There is an isomorphism of graded algebras over $\mathbf{F}_p = \widetilde{\mathbf{Q}_p}^1$ from $\mathbf{F}_p(T/|p|)$ to $\widetilde{\mathbf{Q}_p}$, which sends T to $\widetilde{p}$ (see A.1.10 for the meaning of the notation $\mathbf{F}_p(T/|p|)$).

A.4.10. Example. — Let k be a field endowed with a valuation $|\cdot| : k \to \mathbf{R}_+$. Let $(r_1, \dots, r_n)$ be a family of positive real numbers, and let $T = (T_1, \dots, T_n)$ be a family of indeterminates. The formula $\sum a_I T^I \mapsto \max |a|_I \cdot r^I$ defines a real-valued valuation on $k(T)$. It follows from its very definition that there exists a (unique) $\widetilde{k}$-isomorphism of graded fields

$$\widetilde{k}\left(\frac{\tau}{r}\right) \simeq \widetilde{k(T)}$$

sending τ_i to $\widetilde{T}_i$ for every i, where $\tau = (\tau_1, \dots, \tau_n)$ is a family of indeterminates.

A.4.11. Graded interpretation of classical invariants. — Let $k \hookrightarrow L$ be an isometric extension of real-valued fields. Classical valuation theory assigns four invariants to such an extension, which are (possibly infinite) cardinal numbers:

- The ramification index e, which is the cardinality of $|L^\times| / |k^\times|$.
- The inertia index f, which is the dimension of the $\widetilde{k}^1$-vector space $\widetilde{L}^1$.
- The rational rank r, which is the dimension of the $\mathbf{Q}$-vector space $|L^\times|^{\mathbf{Q}} / |k^\times|^{\mathbf{Q}}$.
- The residue transcendence degree d, which is the transcendence degree of $\widetilde{L}^1$ over $\widetilde{k}^1$.

The product ef and the sum $r + d$ admit natural interpretations in terms of graded reduction:

(1) The product ef is the dimension of the $\widetilde{k}$-graded vector space $\widetilde{L}$, by equality (a) of A.3.5.

(2) The sum $r + d$ is the transcendence degree of the graded extension $\widetilde{k} \hookrightarrow \widetilde{L}$ by equality (c) of A.3.5. Note that it is always bounded by the usual transcendence degree of L over k (this is the so-called *Abhyankar inequality*, cf. [**Bou85**], Chapitre VI, §10, n° 3, Cor. 1).

A.4.12. Graded reduction of algebraic extensions. — Let $k \hookrightarrow L$ be an algebraic isometric extension of real-valued (non-graded) fields, and let Γ be a subgroup of $\mathbf{R}_+^\times$. The graded field $\widetilde{L}$ is algebraic over $\widetilde{k}$, as we see by reducing to the case where L is finite over k, in which case $\widetilde{L}$ is finite over $\widetilde{k}$ by A.4.11 (1). This implies that $\widetilde{L}^\Gamma$ is algebraic over $\widetilde{k}^\Gamma$ (A.3.4).

Assume first that L is radicial over k. In this case, $\widetilde{L}$ is radicial over $\widetilde{k}$ (which implies that $\widetilde{L}^\Gamma$ is radicial over $\widetilde{k}^\Gamma$). Indeed, if char. $k = 0$ then $L = k$ and $\widetilde{L} = \widetilde{k}$. And if char. $k = p > 0$ then for every element a of L, there exists $n \in \mathbf{Z}_{\geqslant 0}$ such that $a^{p^n} \in k$; hence for every homogeneous element α of $\widetilde{L}$, there exists $n \in \mathbf{Z}_{\geqslant 0}$ such that $\alpha^{p^n} \in \widetilde{k}$.

Assume now that L is henselian (e.g., L is complete) and Galois over k. In this case, the valuation of L is preserved by the Galois action, and $\mathrm{Gal}(L/k)$ acts therefore in a natural way on $\widetilde{L}$. By Prop. 2.11 of [**Duc13**], the graded field $\widetilde{L}$ is normal over $\widetilde{k}$ and the natural map $\mathrm{Gal}(L/k) \to \mathrm{Gal}(\widetilde{L}/\widetilde{k})$ is surjective.

By A.3.6, this implies that the graded field $\widetilde{L}^{\Gamma}$ is normal over $\widetilde{k}^{\Gamma}$ and the natural map $\mathrm{Gal}(L/k) \to \mathrm{Gal}(\widetilde{L}^{\Gamma}/\widetilde{k}^{\Gamma})$ is surjective.

INDEX OF NOTATION

General conventions. From the beginning of chapter 3 till the end of the memoir (except the appendix), k denotes an analytic field without any specific assumption (Convention before the introduction of Chapter 3). In Chapter 3 and parts of Chapters 7, 8 and 9, Γ denotes a subgroup of $\mathbf{R}_+^\times$ such that $\Gamma \cdot |k^\times| \neq \{1\}$; i.e., $\Gamma \neq \{1\}$ whenever k is trivially valued.

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