Stability of Abhyankar valuations

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Let K be a field equipped with a Krull valuation $|.|$ (throughout the whole paper we will use the *multiplicative notation*) and let L be a finite extension of K. Let $|.|_1, \ldots, |.|_n$ be the valuations on L extending $|.|$, and for every i, let e_i and f_i be the ramification and inertia indexes of the valued field extension $(K, |.|) \hookrightarrow (L, |.|_i)$. One always has $\sum e_i f_i \leqslant [L : F]$, and the extension L of the valued field $(K, |.|)$ is said to be *defectless* if the equality holds. We say that $(K, |.|)$ is *stable* if every finite extension of it is defectless.

Remark. The product $e_i f_i$ can also be intepreted as the degree of the *graded* residue extension induced by $(K, |.|) \hookrightarrow (L, |.|_i)$.

Examples. Any algebraically closed field is stable; any complete, discretely valued field is stable; the function field of an irreducible smooth algebraic curve, endowed with the discrete valuation associated to a closed point of the curve, is stable; any valued field whose residue characteristic is zero is stable

Let us now give a more involved example. Let G be an ordered abelian group containing $|K^*|$ and let $r = (r_1, \ldots, r_n)$ be a n-uple of elements of G. We denote by $\eta_{K,r}$ the valuation on $K(T_1, \ldots, T_n)$ that sends any polynomial $\sum a_I T^I$ to $\max |a_I| \cdot r^I$ (with $T = (T_1, \ldots, T_n)$).

Theorem. *If the valued field $(K, |.|)$ is stable, so is $(K(T_1, \ldots, T_n), \eta_{K,r})$ for every $r = (r_1, \ldots, r_n)$ as above.*

It has been given several proofs by Gruson, Temkin, Ohm, Kuhlmann, Teissier (see [3], [9], [7], [6], [8]; to the author's knowledge, the first proof working in full generality was that of Kuhlmann, the preceeding proofs requiring some extra-assumptions on K and/or on the r_i's). In what follows we will present a new proof of this theorem, which is based upon model-theoretic tools; it is part of a joint work (which is still at its very beginning) with Ehud Hrushovski and François Loeser.

Step 1. By induction, it is sufficient to prove the theorem for $n = 1$. One then reduces straightforwardly to the case where K is algebraically closed; note that this step requires to understand what happens when K is replaced with one of its finite extensions, and this is here that our stability assumption is used.

Step 2. Now we fix a finite extension F of $K(T)$, and an element r in an ordered group G containing $|K^*|$. We want to prove that F is a defectless extension of $(K(T), \eta_r)$. For that purpose, let us consider a non-trivially valued, algebraically closed extension L of K such that $r \in |L^*|$. Let E_0 be a finite dimensional K-vector subspace of F. Let $\langle . \rangle$ be any extension of $\eta_{L,r}$ to F_L. Using (part of) the seminal work [4] of Haskell, Hrushovski and Macpherson on the elimination of imaginaries in the theory ACVF, together with some further results by Hrushovski and Loeser in[5], one gets the following:

1) the restriction of $\langle . \rangle$ to $L \otimes_K E_0$ is a norm which is definable with parameters in $K \cup \{r\}$;

2) as a consequence of 1), there exists a basis $e_1, \ldots, e_d$ of E_0 over K and elements $s_1, \ldots, s_d$ of $|K^*| \cdot r^{\mathbf{Q}}$ such that $\langle \sum a_i e_i \rangle = \max |a_i| s_i$ for every d-uple $(a_i) \in L^d$.

The formula given in 2) immediately implies that the *graded* reduction $\widetilde{L \otimes_K E_0}^{\mathrm{gr}}$ is equal to $\widetilde{L}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{E_0}^{\mathrm{gr}}$. As a consequence, $\widetilde{F_L}^{\mathrm{gr}}$ is nothing but the graded fraction field of $\widetilde{L}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{F}^{\mathrm{gr}}$. As $\widetilde{L(T)}^{\mathrm{gr}}$ is itself equal by a direct computation to the graded fraction field of $\widetilde{L}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K(T)}^{\mathrm{gr}}$, we eventually get

$$\widetilde{F_L}^{\mathrm{gr}} = \widetilde{L(T)}^{\mathrm{gr}} \otimes_{\widetilde{K(T)}^{\mathrm{gr}}} \widetilde{F}^{\mathrm{gr}}.$$

In particular $[\widetilde{F_L}^{\mathrm{gr}} : \widetilde{L(T)}^{\mathrm{gr}}] = [\widetilde{F}^{\mathrm{gr}} : \widetilde{K(T)}^{\mathrm{gr}}]$ (note that here, all graded reductions involved should be understood with respect to $\langle . \rangle$ and its restrictions to the various fields). The author has proved in [1] (a mistake in the latter is corrected in [2]), using also the aformentioned work by Haskell, Hrushovski and Macpherson, that the restriction induces a *bijection* between the set of extensions of $\eta_{L,r}$ to F_L and the set of extensions of $\eta_{K,r}$ to F. It thus follows from the above that F is a defectless extension of $(K(T), \eta_{K,r})$ if and only if F_L is a defectless extension of $(L(T), \eta_{L,r})$. Hence by replacing K with a suitable valued extension, we may and do ssume that $r \in |K^*| \neq \{1\}$ (and that K is still algebraically closed).

Step 3. Let $b \in |K^*|$. Let us choose $\lambda \in K$ such that $|\lambda| = b$ and let τ be the image of T/λ in the residue field k of $(K(T), \eta_{K,b})$; note that $k = \widetilde{K}(\tau)$. Let b^- and b^+ be elements of an ordered group containing $|K^*|$ which are infinitely close to b (with respect to $|K^*|$), with $b^- < b < b^+$. The valuation η_{K,b^-} (resp. η_{K,b^+}) is the composition of $\eta_{K,b}$ and of the discrete valuation $\langle . \rangle_0$ (resp. $\langle . \rangle_\infty$) of k that corresponds to $\tau = 0$ (resp. $\tau = \infty$), and the extensions of η_{K,b^-} (resp. η_{K,b^+}) to F are compositions of extensions of $\eta_{K,b}$ and of extensions of $\langle . \rangle_0$ (resp. $\langle . \rangle_\infty$). Since $(k, \langle . \rangle_0)$ and $(k, \langle . \rangle_\infty)$ are stable, we see that the following are equivalent:

i) F is a defectless extension of $(K(T), \eta_{K,b^-})$;
ii) F is a defectless extension of $(K(T), \eta_{K,b})$;
iii) F is a defectless extension of $(K(T), \eta_{K,b^+})$.

In the same spirit, let ε be an element of an ordered group containing $|K^*|$ which is infinitely close to zero with respect to $|K^*|$. The valuation $\eta_{K,\varepsilon}$ is the composition of the discrete valuation $\langle . \rangle'_0$ of $K(T)$ corresponding to the closed point $T = 0$ and of the valuation of K. Since both $(K, |.|)$ and $(K(T), \langle . \rangle'_0)$ are stable, $(K(T), \eta_{K,\varepsilon})$ is stable; in particular, F is a defectless extension of $(K(T), \eta_{K,\varepsilon})$.

Step 4. We will now use the theory of "stable completions" introduced by Hrushovski and Loeser in [5] as kind of a model-theoretic avatar of Berkovich spaces. Let X be an irreducible, smooth, projective curve over K whose function field is isomorphic to F, and such that $K(T) \hookrightarrow F$ is induced by a finite map $f \colon X \to \mathbf{P}^1_K$; the latter induces a map $\hat{f} \colon \hat{X} \to \hat{\mathbf{P}}^1_K$, where the "hat" denotes the stable completion. Let M be the class of algebraically closed valued extension of K. For every $L \in \mathsf{M}$, and any $s \in |L^*|$, the valuation $\eta_{L,s}$ appears as a point of $\hat{\mathbf{P}}^1(L)$, whose pre-images on $\hat{X}(L)$ correspond to the extensions of $\eta_{L,s}$ to F_L. We denote by Δ_L the set of $s \in |L^*|$ such that the extension F_L of $(L(T), \eta_s)$ is defectless.

A fundamental result by Hrushovski and Loeser asserts that $\hat{X}$ is definable (this is specific to the one-dimensional case). This leads, together with the "o-minimality of the value group", to the following fact: there exist finitely many disjoint non-empty intervals $I_1 < I_2 < \ldots < I_m$ of $|K^*|$ with endpoints in $|K| \cup \{+\infty\}$ (and with at least one element of $|K^*|$ lying between I_j and I_{j+1} for every $j < n$) such that for every $L \in \mathsf{M}$ one has $\Delta_L = \coprod I_{j,L}$, where we denote by $I_{j,L}$ the interval of $|L^*|$ with the same definition as that of I_j.

Step 5. Let $L \in \mathsf{M}$ such that there exist ε as in step 3 in the value group $|L^*|$. By step 3, the extension F of $(K(T), \eta_{K,\varepsilon})$ is defectless. By step 2, this implies that the extension F_L of $(L, \eta_{L,\varepsilon})$ is defectless. Therefore $\varepsilon \in \Delta_L$. This implies that $m \geqslant 1$ and that the lower bound of I_1 is equal to zero: indeed, if it were an element $c \in |K^*|$, then $\Delta_L = \coprod I_{j,L}$ would be contained in $[c; +\infty[_L$, contradicting the fact that $\varepsilon < c$.

The interval I_1 is thus of the form $]0; b[$, $[0; b]$ or $]0; +\infty[$. We will exclude $]0; b[$ and $]0; b]$. This will show that $\Delta_K = |K^*|$ and will end the proof.

Step 6. *Assume that I_1 is equal to $]0; b[$ or $]0; b]$ with $b \in |K^*|$.* Choose $L \in \mathsf{M}$ such that there exists elements b^- and b^+ as in step 3 in the value group $|L^*|$. Since $b^- \in]0; b[_L \subset I_{1,L}$, the extension F_L of $(L(T), \eta_{L,b^-})$ is defectless. By step 2, this implies that F is a defectless extension of $(K(T), \eta_{K,b^-})$ as well. By step 3, F is then a defectless extension of $(K(T), \eta_{K,b})$; therefore $b \in \Delta_K$ and $I_1 =]0; b]$. This implies the existence of $c > b$ such that $I_j \subset]c; +\infty[$ for every $j \geqslant 2$.

Since F is a defectless extension of $(K(T), \eta_{K,b})$, by using again step 3, we see that F is a defectless extension of $(K(T), \eta_{K,b^+})$. By step 2, the extension F_L of $(L(T), \eta_{L,b^+})$ is defectless. Hence $b^+ \in \Delta_L$, but the latter consists of $]0; b]_L$ and of elements of $|L^*|$ which belong to $I_{j,L}$ for $j \geqslant 2$, hence are greater than c; since $b < b^+ < c$, we get a contradiction.

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Differential-Henselian Fields

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(joint work with Matthias Aschenbrenner, Joris van der Hoeven)

After some twenty years of study we established in Spring 2014 some decisive results about the model theory of the valued differential field $\mathbb{T}$ of transseries, similar to what Tarski achieved for the field of real numbers in the 1940s, and Ax, Kochen, Ersov, and Macintyre in the 1960s and 1970s for henselian valued fields like $\mathbb{C}((t))$ and the p-adic fields. We finished the program outlined in [1], and hope to make the results available soon. Some of our work deals with general valued differential fields with continuous derivation. Below we discuss the notion of differential-henselianity in this setting.

Let K be a *valued differential field*: a field K with a valuation $v : K^\times \to \Gamma$ whose residue field $\mathbf{k} := \mathcal{O}/\mathfrak{m}$ has characteristic zero, and also equipped with a derivation $\partial : K \to K$. Here $\Gamma = v(K^\times)$ is the value group, $\mathcal{O} = \mathcal{O}_K$ is the valuation ring of v, and $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Also $C = C_K := \{f \in K : \partial(f) = 0\}$ is the constant field of the differential field K. We often write f' for $\partial(f)$.

We focus on the case that ∂ is continuous (for the valuation topology). If the derivation is *small* in the sense that $\partial(\mathfrak{m}) \subseteq \mathfrak{m}$, then ∂ is continuous. As a partial converse, if ∂ is continuous, then some multiple $a\partial$ with $a \in K^\times$ is small. *From now on we assume that our derivation ∂ on K is small.* This has the effect that also $\partial(\mathcal{O}) \subseteq \mathcal{O}$, and so ∂ induces a derivation on the residue field; we view $\mathbf{k}$ below as equipped with this induced derivation. Examples of such K include:

(i) $\mathbf{k}(t)$ and $\mathbf{k}((t))$ with the t-adic valuation and $\partial = t\frac{d}{dt}$, with $\mathbf{k}$ any field of characteristic zero;
(ii) Hardy fields, with $\mathcal{O} = \{$germs in K of bounded functions$\}$; see [3];
(iii) $\mathbb{T}$, the valued differential field of transseries; see [5, 1];
(iv) $\mathbf{k}((t^\Gamma))$ where $\mathbf{k}$ is any differential field of characteristic zero, Γ any ordered abelian group, and $\partial(\sum_\gamma a_\gamma t^\gamma) := \sum_\gamma a'_\gamma t^\gamma$; see [9].

We say that K has *few constants* if v is trivial on C, that is, $v(C^\times) = \{0\}$, and that K has *many constants* if $v(C^\times) = \Gamma$. In (i), (ii), (iii) we have few constants, and in (iv) we have $C = C_{\mathbf{k}}((t^\Gamma))$: many constants. If K has many constants, then K is *monotone* in the sense of [4], that is, $v(f) \leq v(f')$ for all $f \in K$. But the examples in (i) are also monotone.

Asymptotic K are defined by another interaction of valuation and derivation:

$$0 < v(f) \leq v(g) \implies v(f') \leq v(g').$$