TROPICAL FUNCTIONS ON A SKELETON

by

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Abstract. — We prove a general finiteness statement for the ordered abelian group of tropical functions on skeleta in Berkovich analytifications of algebraic varieties. Our approach consists in working in the framework of stable completions of algebraic varieties, a model-theoretic version of Berkovich analytifications, for which we prove a similar result, of which the former one is a consequence.

Contents

1. Introduction

1.1. The general context: skeleta in Berkovich geometry. — Let F be a complete non-archimedean field. Among the several frameworks available for doing analytic geometry over F (Tate, Raynaud, Berkovich, Huber...), Berkovich's is the one that encapsulates in the most natural way the deep links between non-archimedean and tropical (or polyhedral) geometry.

Indeed, every Berkovich space X over F contains plenty of natural "tropical" subspaces, which are called *skeleta*. Roughly speaking, a skeleton of X is a subset S of X on which the sheaf of functions of the form $\log|f|$ with f a section of $\mathscr{O}_X^\times$ induces a piecewise linear structure; i.e., using such functions one can equip S with a piecewise linear atlas, whose charts are modelled on (rational) polyhedra and whose transition maps are piecewise affine (with rational linear part).

This definition is rather abstract, but there are plenty of concrete examples of skeleta. The prototype of such objects is the "standard skeleton" S_n of $(\mathbf{G}_{\mathrm{m}}^n)^{\mathrm{an}}$, that consists of all Gauss norms with arbitrary real parameters; the family $(\log|T_1|, \ldots, \log|T_n|)$ induces a piecewise-linear isomorphism $S_n \simeq \mathbf{R}^n$.

Now if X is an arbitrary analytic space and if $\varphi_1, \ldots, \varphi_m$ are quasi-finite maps from X to $(\mathbf{G}_{\mathrm{m}}^n)^{\mathrm{an}}$, then $\bigcup_j \varphi_j^{-1}(S_n)$ is a skeleton by [**Duc12**], Theorem 5.1 (it consists only of points whose Zariski-closure is n-dimensional, so it is empty if $\dim X < n$), and $\varphi_j^{-1}(S_n) \to S_n$ is a piecewise immersion for all j; of course, every piecewise-linear subspace of $\bigcup_j \varphi_j^{-1}(S_n)$ is still a skeleton.

Skeleta were introduced by Berkovich in his seminal work [**Ber99**] on the homotopy type of analytic spaces, where he proved that any compact analytic space with a polystable formal model admits a deformation retraction to a skeleton (isomorphic to the dual complex of the special fiber), and used it to show that quasi-smooth analytic spaces are locally contractible; they play a key role in the theory of real integration on Berkovich spaces [**CLD**]. Let us mention that all skeleta encountered in these works are at least locally of the form described above; i.e., piecewise-linear subspaces of finite unions $\bigcup \varphi_j^{-1}(S_n)$ for quasi-finite maps $\varphi_j \colon X \to (\mathbf{G}_{\mathrm{m}}^n)^{\mathrm{an}}$.

1.2. Our main result. — If S is a skeleton of an analytic space X and if f is a regular invertible functions defined on a neighborhood of S, then $\log|f|$ is a piecewise-linear function on S, and our purpose is to understand what are the piecewise linear functions on S that can arise this way in the *algebraic* situation.

Let us make precise what we mean. Let X be an *algebraic* variety over F, say irreducible of dimension n; let us call *log-rational* any real-valued function of the form $\log|f|$ for f a non-zero rational function on X, viewed as defined over U^{an} for U the maximal open subset of X on which f is well-defined and invertible. Let $\varphi_1, \ldots, \varphi_m$ be (algebraic) quasi-finite maps from X to $\mathbf{G}_{\mathrm{m}}^n$ (the corresponding analytic maps will also be denoted $\varphi_1, \ldots, \varphi_m$). Let S be a subset of the skeleton $\bigcup \varphi_j^{-1}(S_n)$ defined by a Boolean combination of inequalities between log-rational functions. Our main theorem is the following finiteness result.

***Main Theorem* (Berkovich setting).** — *Let X be an irreducible algebraic variety over F of dimension n and assume F is algebraically closed. Let S be as above. Then there exists finitely many non-zero rational functions $f_1, \dots, f_\ell$ on X such that the following holds.*

(1) *The functions* $\log|f_1|, \dots, \log|f_\ell|$ *identify S with a piecewise-linear subset of $\mathbf{R}^\ell$ (i.e., a subset defined by a Boolean combination of inequalities between $\mathbb{Q}$-affine functions).*

(2) *The group of restrictions of log-rational functions to S is stable under* min *and* max *and is generated under addition, substraction,* min *and* max *by the (restrictions of the) functions* $\log|f_i|$ *and the constants* $\log|a|$ *for $a \in F^\times$.*

Let us mention that statement (1) is implicitly established in [**Duc12**] (see op. cit., proof of Theorem 5.1); what is really new here is statement (2). And let us insist on the assumption that F is algebraically closed: for a general F the theorem does not hold, as shown by a counter-example due to Michael Temkin (Remark 7.6).

1.3. About our proof. — In fact, we do not work directly with Berkovich spaces but with the model-theoretic avatar of this geometry, namely the theory of *stable completions* of algebraic varieties which was introduced by two of the authors in [**HL16**]. Thus, what we actually prove is Theorem 7.2 which is a version of the result above in this model-theoretic framework – the final transfer to Berkovich spaces being straightforward.

Let us give some explanations. Let X be an algebraic variety over a valued field F. We denote by $\widehat{X}$ the stable completion of X. The standard skeleton S_n of $(\mathbf{G}_\mathrm{m}^n)^\mathrm{an}$ has a natural counterpart $\widehat{S}_n$ (1) on $\widehat{\mathbf{G}_\mathrm{m}^n}$, and $\bigcup \varphi_j^{-1}(\widehat{S}_n)$ makes sense as a subset of $\widehat{X}$; moreover, the inequalities between log-regular functions that cut out S inside $\bigcup \varphi_j^{-1}(S_n)$ also make sense here, and cut out a subset $\widehat{S}$ of $\bigcup \varphi_j^{-1}(\widehat{S}_n)$. By Corollary 4.5, this subset is F-definably homeomorphic to an F-definable subset of Γ^N for some N. It follows moreover from its construction that $\widehat{S}$ is contained in the subset $X^\#$ of $\widehat{X}$ consisting of strongly stably dominated types (or, otherwise said, of Abhyankar valuations), and even in its subset $X^\#_\mathrm{gen}$ of Zariski-generic points. And now Theorem 7.2 tells the following.

1. This is not in accordance with the standard usage of the $\widehat{\cdot}$ functor, since S_n or S are not definable subsets of $\mathbf{G}_\mathrm{m}^n$. But we use this notation specifically for S_n and subsets S of S_n to indicate that we are now working in the model-theoretic framework.

***Main Theorem* (Model-theoretic setting). —** *Let F be an algebraically closed field endowed with a non-trivial valuation* $\mathrm{val} : F \to \Gamma \cup \{\infty\}$*. Let X be an irreducible algebraic variety over F. Let Υ be an iso-definable subset of $X^{\#}_{\mathrm{gen}}$ which is Γ-internal, that is, F-definably isomorphic to an F-definable subset of Γ^N for some N.*

There exists finitely many non-zero rational functions $f_1, \ldots, f_\ell$ on X such that the following holds.

(1) *The functions* $\mathrm{val}(f_1), \ldots, \mathrm{val}(f_\ell)$ *identify topologically Υ with an F-definable subset of Γ^ℓ.*

(2) *The group of restrictions of val-rational functions to Υ is stable under* min *and* max *and generated under addition, substraction,* min *and* max *by the (restrictions of the) functions* $\mathrm{val}(f_i)$ *and the constants* $\mathrm{val}(a)$ *for $a \in F^\times$ (as the terminology suggests, a val-rational function is a Γ-valued function of the form* $\mathrm{val}(f)$ *with f rational, defined on the stable completion of the invertibility locus of f).*

Let us start with a remark. The Γ-internal subsets we are really interested in for application to Berkovich theory seem to be of a very specific form (they are definable subsets of $\bigcup \varphi_j^{-1}(\widehat{S}_n)$ for some family (φ_j) of quasi-finite maps from X to $\mathbf{G}_{\mathrm{m}}^n$) and our main theorem deals at first sight with fare more general Γ-internal subsets. But this is somehow delusive; indeed, we show (Theorem 4.8) that every Γ-internal subset of $X^{\#}_{\mathrm{gen}}$ is contained in some finite union $\bigcup \varphi_j^{-1}(\widehat{S}_n)$ as above.

We are now going to describe roughly the main steps of the proof of our main theorem.

Step 1. — This first step has nothing to do with valued fields and concerns general divisible abelian ordered groups. Basically, one proves the following. Let D be an M-definable closed subset of Γ^n for some divisible ordered group M contained in a model Γ of DOAG, let $g_1, \ldots, g_m$ be $\mathbb{Q}$-affine M-definable functions on Γ^n, and let f be any continuous *and Lipschitz* M-definable map from D to Γ, such that for every x in D there is some index i with $f(x) = g_i(x)$. Then under these assumptions, f lies in the set of functions from D to Γ generated under addition, substraction, min and max by the g_i, the coordinate functions and M: this is Theorem 3.13. Here the Lipschitz condition refers to a Lipschitz constant in $\mathbb{Z}_{\geqslant 0}$, so that it is a void condition when M has no non-trivial convex subgroup and D is definably compact, but meaningful in general.

Step 2. — We start with proving a finiteness result in the spirit of our theorem under a weaker notion of generation. More precisely, we show (Theorem 5.7) the existence of $f_1, \ldots, f_\ell$ as in our statement such that (1) holds and such that the following weak version of (2) holds, with H denoting the group of Γ-valued functions on Υ generated by the $\mathrm{val}(f_i)$ and the constants $\mathrm{val}(a)$ for $a \in F^\times$: *for every non-zero rational function g on X there exist finitely many elements $h_1, \ldots, h_r$ of H such that Υ is covered by its definable subsets* $\{\mathrm{val}(g) = \mathrm{val}(h_i)\}$ *for* $i = 1, \ldots, r$.

The key point for this step is the purely valuation-theoretic fact that an Abhyankar extension of a defectless valued field is still defectless. It has been given several proofs in the literature, some of which are purely algebraic, some of which are more geometric. For the sake of completeness and for consistency with the general viewpoint of this paper, we give a new one in Appendix A, (Theorem A.1) which is model-theoretic and based upon [**HL16**]. It follows already from Theorem 5.7 that skeleta are endowed with a canonical piecewise $\mathbb{Z}$-affine structure. In particular this implies the existence of canonical volumes for skeleta as we spell out in Section 8.

Step 3. — One strengthens the statement of Step 2 by showing (Proposition 6.10) that the f_i can even be chosen so that all functions $(\mathrm{val}(g))|_\Upsilon$ as above are Lipschitz, when seen as functions on $\mathrm{val}(f)(\Upsilon) \subseteq \Gamma^m$. This is done by using an interpretation of the Lipschitz property in terms of coarsenings and refinements of valuations.

Step 4. — One proves that the set of functions on Υ of the form $\mathrm{val}(g)$ is stable under min and max. This follows from orthogonality between the residue field and the value group sorts in ACVF, see Lemma 7.1.

Step 5. — By the very choice of the f_i, every function $\mathrm{val}(g)|_\Upsilon$ gives rise *via* the embedding $\mathrm{val}(f)|_\Upsilon$ to a definable function on $\mathrm{val}(f)(\Upsilon)$ that belongs piecewise to the group generated by $\mathrm{val}(F^\times)$ and the coordinate functions $x_1, \ldots, x_\ell$ (Step 2) and is moreover Lipschitz (Step 3); it is thus (Step 1) equal to $t(x_1, \ldots, x_\ell, a)$ where t is a term in $\{+, -, \min, \max\}$ and a a tuple of elements of $\mathrm{val}(F^\times)$. Then $\mathrm{val}(g)|_\Upsilon = t(\mathrm{val}(f_1)|_\Upsilon, \ldots, \mathrm{val}(f_\ell)|_\Upsilon, a)$ and we are done.

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2. Preliminaries

2.1. Stably dominated types. — The aim of this section is to review some of the material from [**HL16**] that we will use in this paper. The reader is refered to [**HL16**] or to the surveys [**Duc13**] or [**Duc16**] for more detailed information. In this paper, we shall work in the framework of [**HL16**], namely the theory ACVF of algebraically closed valued fields K with nontrivial valuation in the geometric language $\mathcal{L}_{\mathcal{G}}$ of [**HHM06**]. We recall that this language is an extension of the classical three-sorted language with sorts VF, Γ and RES for the valued field, value group and residue field sorts, and additional symbols val and res for the valuation and residue maps, obtained by adding new sorts S_m and T_m, $m \geqslant 1$, corresponding respectively to lattices in K^m and to the elements of the reduction of such lattices modulo the maximal ideal of the valuation ring. By the main result of [**HHM06**] ACVF has elimination of imaginaries in $\mathcal{L}_{\mathcal{G}}$.

Recall that in a theory T admitting elimination of imaginaries in a given language $\mathcal{L}$, for $M \models T$ and $A \subseteq M$, a type $p(\overline{x})$ in $S_{\overline{x}}(M)$ is said to be A-definable if for every $\mathcal{L}$-formula $\varphi(\overline{x}, \overline{y})$ there exists an $\mathcal{L}_A$-formula $d_p\varphi(\overline{y})$ such that for every $\bar{b}$ in M, $\varphi(\overline{x}, \bar{b}) \in p$ if and only if $M \models d_p\varphi(\bar{b})$. If $p \in S_{\overline{x}}(M)$ is definable via $d_p\varphi$, then the same scheme gives rise to a unique type $p_{|N}$ for any elementary extension N of M. There is a general notion of stable domination for A-definable types: stably dominated types are in some sense "controlled by their stable part". In the case of ACVF, there is concrete characterisation of A-definable stably dominated types as those which are orthogonal to Γ, meaning that for every elementary extension N of M, if $\overline{a} \models p_{|N}$, one has $\Gamma(N) = \Gamma(N\overline{a})$.

Let X be an A-definable set in ACVF, with A an $\mathcal{L}_{\mathcal{G}}$-structure. A basic result in [**HL16**] states that there exists a strict A-pro-definable set $\widehat{X}$ such that for any $C \supseteq A$, $\widehat{X}(C)$ is equal to the set of C-definable stably dominated types on X ([**HL16**, Theorem 3.1]). Here by pro-definable we mean a pro-object in the category of definable sets and strict refers to the fact that the transition morphisms can be chosen to be surjective. Morphisms in the category of pro-definable sets are called definable morphisms.

In fact $\widehat{X}$ can be endowed with a topology that makes it a pro-definable space in the sense of [**HL16**, Section 3.3]. In this setting there is a model theoretic version of compactness, namely definable compactness: a pro-definable space X is said to be definably compact if every definable type on X has a limit in X. In an o-minimal structure M, this notion is equivalent to the usual one, namely a definable subset $X \subseteq M^n$ is definably compact if and only if it is closed and bounded.

2.2. Γ-internal sets. — Let us fix a valued field k and a quasi-projective variety X over k. We denote by Γ the value group of k. The structure induced is that of an ordered abelian group in the language of ordered groups, in particular it is o-minimal. We extend Γ to $\Gamma_\infty = \Gamma \cup \{\infty\}$ with ∞ larger than any element of Γ. A pro-definable set is called iso-definable if it is pro-definably isomorphic to a definable set. A Γ-internal subset Z of $\widehat{X}$, or more generally of $\widehat{X \times \Gamma_\infty^m}$, is an iso-definable subset such that there exists a surjective definable morphism $D \to Z$ with D a definable subset of some Γ_∞^r.

By [**HL16**, Theorem 6.2.8], if Z is a k-iso-definable and Γ-internal subset of $\widehat{X}$, there exists some finite k-definable set w and a continuous injective definable morphism $f : Z \hookrightarrow \Gamma_\infty^w$. In particular if Z is definably compact such an f is a homeomorphism onto its image.

2.3. The Zariski-generic case. — Assume that k is algebraically closed. Then the definable injection $Z \hookrightarrow \Gamma_\infty^w$ alluded to above can be obtained by using (locally) valuations of regular functions. Thus if X is irreducible and Z only consists of Zariski-dense points, we can find a dense open subset U of X and invertible functions $g_1, \dots, g_w$ on U such that the functions $\mathrm{val}(g_i)$ induce a definable bijection between Z and a k-definable subset of Γ^w (without ∞). Moreover, by shrinking U and adding some extra invertible functions to the g_i, we can assume that g induces a closed immersion $U \hookrightarrow \mathbf{G}_{\mathrm{m}}^w$; then the functions $\mathrm{val}(g_i)$ induce a (definably) *proper* map $\widehat{U} \to \Gamma^w$ and thus a definable *homeomorphism* between Z and its image.

2.4. Retractions to skeleta. — Since multiplication does not belong to the structure on the value group sort Γ, we have to consider generalized intervals, which are obtained by concatenating a finite number of (oriented) closed intervals in Γ_∞. Such a generalized interval I has an origin o_I and an end point e_I.

We may now define strong deformation retractions. Fix a valued field k and a quasi-projective variety X over k. A strong deformation retraction of $\widehat{X}$ onto $\Upsilon \subseteq \widehat{X}$ is continuous definable morphism

$$H : I \times \widehat{X} \longrightarrow \widehat{X}$$

such that

- The restriction of H to $\{o_I\} \times \widehat{X}$ is the identity on $\widehat{X}$.
- The restriction of H to $I \times \Upsilon$ is the identity on $I \times \Upsilon$.
- The image of the restriction H_{e_I} of H to $\{e_I\} \times \widehat{X}$ is contained in Υ.
- For every $(t, a) \in I \times \widehat{X}$, $H_{e_I}(H(t, a)) = H_{e_I}(a)$.

A special case of the main result of [**HL16**] states the following:

2.5. Theorem. — *Let X be a quasi-projective variety over a valued field k. Then there is a strong deformation retraction*

$$H : I \times \widehat{X} \longrightarrow \widehat{X}$$

onto a Γ-internal subset $\Upsilon \subseteq \widehat{X}$ and a definable injection $\Upsilon \to \Gamma_\infty^w$ for some finite definable set w, which is an homeomorphism onto its image and such that for each irreducible component W of X, $\Upsilon \cap \widehat{W}$ is everywhere of o-minimal dimension $\dim(W)$.

We shall call such a Γ-internal set Υ a *retraction skeleton* of $\widehat{X}$. Note that this is what is called a skeleton in [**HL16**], but we have decided to change the terminology to avoid conflict with the literature.

2.6. Remark. — When X is smooth and irreducible, there exists a deformation retraction as above with Υ consisting only of Zariski-generic points: this follows from the proof of Theorem 11.1.1 in [**HL16**], see also Chapter 12 of [**HL16**]; so if k is a model of ACVF then Υ can be topologically and k-definably identified with a subset of some Γ^m by using valuations of non-zero rational functions (2.3).

Note that the smoothness assumption cannot be dropped for the above: if X is a cubic nodal curve, any retraction skeleton Υ of $\widehat{X}$ contains the nodal point (and any definable topological embedding from Υ into some Γ_∞^w will send the nodal point to a w-uple with at least one infinite coordinate).

2.7. Strongly stably dominated types. — In fact all retraction skeleta of $\widehat{X}$ are contained in the subspace $X^\# \subseteq \widehat{X}$ of strongly stably dominated types on X. The study of the space $X^\#$ is the subject of Chapter 8 of [**HL16**]. Loosely speaking the notion of strongly stably dominated corresponds to a strong form of the Abhyankar property for valuations namely that the transcendence degrees of the extension and of the residue field extension coincide. An important property of $X^\#$ is that it has a natural structure of (strict) ind-definable subset of $\widehat{X}$. Furthermore, by [**HL16**, Theorem 8.4.2], $X^\#$ is exactly the union of all the retraction skeleta of $\widehat{X}$.

It seems plausible that very general Γ-internal subsets of $\widehat{X}$ can be rather pathological, but those contained in $X^\#$ should be reasonable. We shall see below that this is indeed the case at least for Γ-internal subsets of $X^\#$ that consist of Zariski-generic points (when X is irreducible). When X is irreducible, we will denote by $X^\#_{\rm gen}$ the subset of $X^\#$ consisting of Zariski-generic points.

2.8. Connection with Berkovich spaces. — Let k be a valued field with $\mathrm{val}(k) \subseteq \mathbb{R}_\infty$, which we assume to be complete. Let X be a separated and reduced algebraic variety of finite type over k. Denote by $X^{\rm an}$ its analytification in the sense of Berkovich. Chapter 14 of [**HL16**] is devoted to a detailed study of how one can deduce statements about X^{an} from similar statements about $\widehat{X}$. This comes from the fact that, if one denotes by $k^{\max}$ a maximally complete algebraically closed extension of k with value group $\mathbb{R}$ and residue field the algebraic closure of the residue field of k, there is a canonical and functorial map $\pi : \widehat{X}(k^{\max}) \to X^{\mathrm{an}}$ which is continuous, surjective, and closed. When $k = k^{\max}$, π is actually an homeomorphism. Furthermore, any definable morphism $g : \widehat{X} \to \Gamma_\infty$ induces a unique map $\tilde{g} : X^{\mathrm{an}} \to \mathbb{R}_\infty$ which is continuous if g is, and any strong deformation retraction $H : I \times \widehat{X} \to \widehat{X}$ induced canonically a strong deformation retraction $\tilde{H} : I(\mathbb{R}_\infty) \times X^{\mathrm{an}} \to X^{\mathrm{an}}$ compatible with π for any $t \in I(\mathbb{R}_\infty)$. Thus, if one defines a retraction skeleton Σ in X^{an} as the image under π of the $k^{\max}$-points of a retraction skeleton in $\widehat{X}$, we obtain that when X is quasi-projective there exists a strong deformation retraction of X^{an} onto a retraction skeleton Σ. Furthermore, the fact that retraction skeleta in $\widehat{X}$ are contained in $X^{\#}$ implies that any point of Σ, as a type over $(k, \mathbb{R})$, extends to a unique stably dominated type; this type is strongly stably dominated and, restricted to $(k, \mathbb{R})$, it determines an Abhyankar extension of the valued field k, cf. Theorem 14.2.1 in [**HL16**].

3. Finite generation and Lipschitz functions in DOAG

In this section, we work in the theory of divisible ordered abelian groups which is denoted by DOAG, and by definable we mean definable with parameters. We shall usually denote by Γ a model of DOAG. We start with the definition of w-combination and w-generation.

3.1. Definition. — Let X and Y be definable topological spaces and g, $f_1, \dots, f_n$ be definable continuous functions from X to Y. We say g is a *w-combination* of $f_1, \dots, f_n$ if for every $x \in X$, there is some $i \in \{1, \dots, n\}$ such that $f_i(x) = g(x)$. Notationally, we use $[g = f_i]$ to denote the set $\{x \in X : g(x) = f_i(x)\}$. Hence, g is a w-combination by $f_1, \dots, f_n$ iff $X = \bigcup_{i=1}^n [g = f_i]$.

In contrast, there is a stronger notion of combination that is very specific to DOAG.

3.2. Definition. — Let X be a definable topological space and let g and f_i, $i \in I$, be definable continuous functions $X \to \Gamma$. We say that g is an *ℓ-combination* of the f_i if g lies in the $(\min, \max)$-lattice generated by $(f_i)_{i \in I}$. More explicitly, there are $f_1, \dots, f_n$ in $(f_i)_{i \in I}$ such that g is a function obtained by $f_1, \dots, f_n$ and finitely many operations of $\min, \max$.

We shall also use the following variants of w and ℓ-combination.

3.3. Definition. — Let X be a definable topological space and let g and f_i, $i \in I$, be definable continuous functions $X \to \Gamma$ be definable continuous functions for $i \in I$. We say that g is a $(w,+)$-*combination* of the f_i if there exist $h_1, \dots, h_n$ in the abelian group generated by the functions f_i, $i \in I$ such that g is a w-combination of the h_i. We say that g is an $(\ell,+)$-*combination* of the f_i if g can be described by a formula involving only $+, -$, min and max and finitely many f_i.

We say that a given set of functions containing the f_i and stable under w-combination is w-generated by the f_i if it consists precisely of the set of all w-combinations of the f_i. We define $(w,+)$, ℓ and $(\ell,+)$-generation in an analogous way.

3.4. Example. — Let $X = \Gamma^n$ and $m_k : X \to \Gamma$ be the definable function which to $(x_1, \dots, x_n)$ assigns the k-th smallest x_i. Clearly, m_k is a w-combination of the coordinate functions $x_1, \dots, x_n$. On the other hand, it is not hard to see that

$$m_k(x) = \min_{U \subseteq \{1,\dots,n\}, |U|=k} \max_{i \in U} x_i$$

Hence the $m_k(x)$ are even ℓ-combinations of $x_1, \dots, x_n$.

However, the two notions of combinations do not agree in general.

3.5. Example. — Let I be the interval $[0, \infty) \subseteq \mathbb{Q}$. Let $D = I \times \{1,2\} \subseteq \mathbb{Q}^2$ and $f_1 = 0$, $f_2 = x_1$. Consider g that is equal to f_i on $I \times \{i\}$ for $i = 1, 2$. Clearly g is a w-combination of f_1 and f_2. However, we claim that g is not an $(\ell,+)$-combination of coordinate functions. Indeed, if it were, then it would extend to a continuous $\mathbb{Q}$-definable function g' on $\mathbb{Q}^2$. Let Γ be a model of DOAG containing $\mathbb{Q}$ and in which there is some $c > n$ for all $n \in \mathbb{N}$. Since $\operatorname{tp}(1,c) = \operatorname{tp}(\alpha,c)$ for any $1 > \alpha > 0$ and $g(1,c) = c$, so $g'(\alpha,c) = c$. However $g'(0,c) = g(0,c) = 0$, in contradiction with the continuity of g'. For a connected version of this example, replace D by $D' = D \bigcup \{0\} \times [1,2]$ and set $g = 0$ on $\{0\} \times [1,2]$.

This example suggests that interaction of the ambient space and the topology of D plays a role in distinguishing the two notions of combinations. To proceed towards a topological characterisation for such properties, we need the following.

3.6. Definition. — Let T be an o-minimal expansion of DOAG and $\Gamma \models T$ with $D \subseteq \Gamma^n$ definable. We say that D is convex if for any u and v in D, $\frac{u+v}{2} \in D$.

3.7. Remark. — When T is an o-minimal expansion of the theory of real closed fields RCF, this is equivalent to the usual definition of convexity for definable sets. For $u, v \in D$, let $L \subseteq [0,1]$ be $\{\alpha : \alpha u + (1-\alpha)v \in D\}$. By our notion of convexity, L contains $\mathbb{Z}[1/2] \cap [0,1]$. By o-minimality, L must be $[0,1]$ with at most finitely many points in $(0,1)$ removed. But removing any point from $(0,1)$ would lead to a violation of convexity.

Note further that for D convex, working inside the smallest affine subspace containing D, one has $\mathrm{cl}(\mathrm{int}(D)) = \mathrm{cl}(D)$.

Lastly, recall that for any definable subset D of some Γ^n, a function $f : D \to \Gamma$ is called $\mathbb{Q}$-*affine* if $f = \sum_{i=1}^n m_i x_i + c$ where $m_i \in \mathbb{Q}$ and $c \in \Gamma$. Such functions are the most basic definable continuous functions on D. We say f is $\mathbb{Z}$-*affine* if the m_i are all in $\mathbb{Z}$.

3.8. Proposition. — *Let Γ be a divisible ordered abelian group and let $f_1, \ldots, f_m$ be $\mathbb{Q}$-affine functions on Γ^n. Let $D \subseteq \Gamma^n$ be definable and $g : D \to \Gamma$ be a continuous definable function. Assume that g is a w-combination of $f_1, \ldots, f_m$. Then the following are equivalent:*

1. *g is an ℓ-combination of $f_1, \ldots, f_m$.*
2. *g extends to a continuous definable function $g' : \Gamma^n \to \Gamma$ that is a w-combination of $f_1, \ldots, f_m$.*
3. *g extends to a continuous definable function $g' : D' \to \Gamma$ on some convex definable set D' containing D that is a w-combination by $f_1, \ldots, f_m$.*
4. *For any $x, y \in D$, there is $i \in \{1, \ldots, m\}$ such that $f_i(x) \leqslant g(x)$ and $g(y) \leqslant f_i(y)$.*
5. *For some collection S of subsets of $\{1, \ldots, m\}$, $g = \min_{X \in S} \max_{i \in X} f_i$.*

Proof. — The implications (5) $\implies$ (1) $\implies$ (2) $\implies$ (3) are clear.

For (3) $\implies$ (4), by working in an elementary extension, we may assume that Γ is a model of the theory of real closed fields RCF. By Remark 3.7 and after replacing D by the convex set D' in (3), we may assume the line segment $[x, y]$ connecting x, y is in D. Replace g by g' given by (3) as well. Let $I_j \subseteq [x, y]$ be $\{z : g(z) = f_i(z)\}$. By continuity of g and o-minimality, we know that the sets I_j are finite unions of closed intervals and $\bigcup_{j=1}^m I_j = [x, y]$. Consider the canonical parameterization $h : [0,1] \to [x, y], \alpha \mapsto \alpha y + (1-\alpha)x$, and let $f'_i = f_i \circ h$, $g' = g \circ h$ and $I'_j = h^{-1}(I_j)$. Since the functions f_i are $\mathbb{Q}$-affine, the functions f'_i are of the form $a_i x + b_i$ for some $a_i, b_i \in \Gamma$. Let k be the j such that a_j is the greatest amongst all the j such that $I'_j \neq \varnothing$. If there are multiple such j, pick any. By induction, for a to the right of I'_k, we have $g'(a) \leqslant f'_k(a)$. Similarly, for a to the left of I'_k, we have $f'_k(a) \leqslant g'(a)$. In particular we have $f'_k(0) = f_k(x) \leqslant g'(0) = g(x)$ and $g'(1) = g(y) \leqslant f_k(y) = f'_k(1)$.

For (4) $\implies$ (5), consider S to be the collection of subsets $X \subseteq \{1, \dots, m\}$ such that $g \leqslant \max_{i \in X} f_i$ on the entire D. Set $f := \min_{X \in S} \max_{i \in X} f_i$. We claim that $g = f$. Clearly $g \leqslant f$, so it suffices to show that $g \geqslant f$. For each $W \notin S$, there is some y_W such that $g(y_W) > f_i(y_W)$ for every $i \in W$. By (4), for each $x \in D$, there is i^x_W such that $f_{i^x_W}(x) \leqslant g(x)$ and $f_{i^x_W}(y_W) \geqslant g(y_W)$. Note that $i^x_W \notin W$. Let $X = \{i^x_W : W \notin S\}$. We have that $X \in S$ because otherwise, $i^x_X \in X$. For this x, we have that $\max_{i \in X} f_i(x) \geqslant g(x)$ and $f_i(x) \leqslant g(x)$ for any $i \in X$, hence $f(x) \leqslant \max_{i \in X} f_i(x) = g(x)$. □

3.9. Corollary. — *Let $D \subseteq \Gamma^n$ be a definable convex set. The set of definable continuous functions from D to Γ is $(\ell, +)$-generated by the constants and all rational multiples of coordinate functions.*

Proof. — By quantifier elimination, we can find $\mathbb{Q}$-affine functions $f_1, \dots, f_n$ such that g is a w-combination of $f_1, \dots, f_n$. By Proposition 3.8, we have that g is in fact an ℓ-combination of $f_1, \dots, f_n$. □

Proposition 3.8 suggests that the agreement of w-combination and ℓ-combination is related to the existence of continuous extensions to an ambient convex space. This motivates the following definition.

3.10. Definition. — For a tuple $x \in \Gamma^n$, define $|x| = \max_{i=1}^n |x_i|$. Let $D \subseteq \Gamma^n$ and $f : D \to \Gamma$ a definable function. We say f is *Lipschitz* if there is some $M \in \mathbb{N}$ such that $|f(x) - f(y)| \leqslant M|x - y|$.

Note that Lipschitz functions are automatically continuous and clearly the class of Lipschitz functions depends on the embedding of D in Γ^n. Our purpose is now to investigate Lipschitz definable functions on closed definable sets; a first step will consist in reducing to the definably compact case, by using the two following lemmas.

3.11. Lemma. — *Let Γ be a model of* DOAG*, let D be a subset of Γ^n definable over some set A of parameters, and let $f \colon D \to \Gamma$ be a Lipschitz A-definable map. Let (f_i) be a finite family of $\mathbb{Q}$-affine A-definable functions such that f is a w-combination of the $f_i|_D$. Then f admits a unique continuous extension $\overline{f}$ to $\overline{D}$, the set $\overline{D}$ and the function $\overline{f}$ are A-definable, and $\overline{f}$ is Lipschitz and w-generated by the $f_i|_{\overline{D}}$.*

Proof. — The uniqueness of $\overline{f}$ is clear, as well as the A-definability of $\overline{D}$ and $\overline{f}$ if the latter exists, as one sees by using the definition of the closure and of the limit (with ε and δ...). The same reasoning also show that the set of points of ∂D at which f admits a limit is A-definable. Moreover if $\overline{f}$ exist it inherits obviously the Lipschitz property of f, and it is also w-generated by the (restrictions of) the f_i: indeed, the susbet of $\overline{D}$ consisting of points x such that there is some i with $f(x) = f_i(x)$ is closed and contains D, thus is the whole of $\overline{D}$.

It thus remain to show the existence of $\overline{f}$, and this can be done after enlarging the model Γ. We can thus suppose that it is equal to the additive group of some real closed field. Let x be a point of $\overline{D}\backslash D$. There exists a half-line L-emanating from x such that $(x, y) \subseteq D$ for some y; taking y close enough to x we can assume that $f = f_j$ on (x, y) for some j. Then the limit of f at x along the direction of L exists and is equal to $f_j(x)$. The Lipschitz property then ensures that this limit does not depend on L, let us denote it by $\overline{f}(x)$. Since D is defined by affine inequalities, there is a positive $\gamma \in \Gamma$ such that for every y in Γ^n with $\|x - y\| < \gamma$ (say for the Euclidean norm) then either $(x, y) \subseteq D$ or $(x, y) \cap D = \varnothing$. Thus if y is a point of D with $\|x - y\| < \gamma$ then $|f(y) - \overline{f}(x)| \leqslant N\|x - y\|$ where N is an upper bound for the slopes of the f_i. So $f(y)$ tends to $\overline{f}(x)$ when the point y of D tends to x. □

3.12. Lemma. — *Let M be either $\{0\}$ or a model of* DOAG*, let Γ be a model of* DOAG *containing M, and let ρ be an element of Γ with $\rho > M$. Let $Z \subseteq \Gamma^n$ be an M-definable subset. Let $x_1, \ldots, x_n : Z \to \Gamma$ denote the coordinate functions of Z and le $h : Z \to \Gamma$ be an M definable function.*

Assume that there exists a term t in $\{+, -, \max, \min\}$ and $\gamma = (\gamma_1, \ldots, \gamma_l)$ in Γ^ℓ such that $h|_{Z_\rho} = t(x_1, \ldots, x_n, \gamma)|_{Z_\rho}$, where $Z_\rho = Z \cap [-\rho, \rho]^n$.

Then there is a term t' in $\{+, -, \max, \min\}$ and an element β of M such that $h = t'(x_1, \ldots, x_n, \beta)$.

Proof. — Assume first that M is a model of DOAG. By our assumption, there exists a term t in $\{+, -, \max, \min\}$ and a tuple $\gamma = (\gamma_1, \ldots, \gamma_l) \in \Gamma^\ell$ such that $h|_{Z_\rho} = t(x_1, \ldots, x_n, \gamma)|_{Z_\rho}$. By model-completeness of DOAG, the γ_i can be chosen in $M \oplus \mathbb{Q} \cdot \rho$. Thus there is $m > 0$ such that for each i, there exist integers k_i and $\beta_i \in M$) with $\gamma_i = \frac{k_i}{m}\rho + \beta_i$. Let ν denote ρ/m. We have

$$h|_{Z_{m\nu}} = t(x_1, \ldots, x_n, (k_i\nu + \beta_i))|_{Z_{m\nu}}.$$

Viewing the above as a first-order formula with constants in the model M and a variable for ν, using o-minimality and model-completeness of M, we have some $\nu_0 \in M_{>0}$ such that for any $\nu' > \nu_0$, the following holds in M:

$$h|_{Z_{m\nu'}} = t(x_1, \ldots, x_n, (k_i\nu' + \beta_i))|_{Z_{m\nu'}}.$$

Take $\nu(x) = \max\{|x_1|, \ldots, |x_n|, 2\nu_0\}$ and

$$t'(f_1, \ldots, f_n, \beta) = t(x_1, \ldots, x_n, (k_i\nu(x) + \beta_i)).$$

We then have

$$h = t'(x_1, \ldots, x_n, \beta)$$

by construction, which ends the proof when $M \neq \{0\}$.

If $M = \{0\}$, set $\Gamma' = \Gamma \oplus \delta$ where δ is positive and infinitesimal with respect to Γ, set $M' = \mathbb{Q} \cdot \delta$ and let us denote by Z' and h' the objects deduced from Z and h by base-change to Γ'. Applying the above yields a term θ in $\{+, -, \max, \min\}$ and an element β of $\mathbb{Q} \cdot \delta$ such that $h' = \theta(x_1, \ldots, x_n, \beta)$. By reducing modulo the convex subgroup $\mathbb{Q} \cdot \delta$ of Γ' we see that $h = \theta(x_1, \ldots, x_n, 0)$. □

3.13. Theorem. — *Let $M \models$ DOAG or $M = \{0\}$ and let Γ be a model of DOAG containing M. Let $D \subseteq \Gamma^m$ be an M-definable set. Let $g : D \to \Gamma$ be a Lipschitz definable function over M. Let $f_1, \ldots, f_n$ be $\mathbb{Q}$-affine functions over M such that g is a w-combination of $f_1, \ldots, f_n$. Then g is an $(\ell, +)$-combination of the f_i, the constant M-valued functions and the coordinate functions.*

Proof. — By Lemma 3.11 we can assume that D is closed. We may then enlarge the model Γ and assume that it contains some ρ with $\rho > M$. Let D_ρ denote the intersection of D with $[-\rho, \rho]^m$. This is a definably compact subset of Γ^m which is definable over $M_\rho := M \oplus \mathbb{Q} \cdot \rho$. If we prove that $g|_{D_\rho}$ is an $(\ell, +)$-combination of the f_i, Lemma 3.12 above will allow us to conclude that g is an $(\ell, +)$-combination of the f_i, the coordinate functions and some constant functions with values in M. We thus may and do assume that D is definably compact. By considering a submodel of M over which everything is defined, we reduce to the case where M is of finite height r, i.e. has exactly r non-trivial convex subgroups, and we proceed by induction on r. The case $r = 0$ is obvious since the definably compact set D is then either empty or equal to $\{0\}$. Assume now that $r > 0$ and that the result holds true for smaller values of the height.

Let M_0 be the smallest non-trivial convex subgroup of M, and $\overline{M} = M/M_0$ be the quotient. We are first going to explain why we can assume that $g(D(M)) \subseteq M_0^m$; this is tautological if $M_0 = M$, so we assume (just for this reduction step) that $M_0 \neq M$. In this case $\overline{M}$ is a model of DOAG of height $r-1$, and the natural map carrying M to $\overline{M}$ induces a map that carries $D(M)$ to a definably compact definable subset $\overline{D}$ of $\overline{M}^m$ (see [**CHY**, Theorem 4.1.1] for example). Furthermore, since g is Lipschitz, it descends to a definable function $\overline{g} : \overline{D}(\overline{M}) \to \Gamma_{\overline{M}}$, which is Lipshitz as well and is a $(w, +)$-combinations of the $\overline{f_i}$. By the induction hypothesis, we then know that $\overline{g}$ is of the form $\tau(\overline{f_1}, \ldots, \overline{f_n})$, where τ is a term involving constants, projections and $+, -, \min, \max$ only. Replacing g by $g - \tau(f_1, \ldots, f_n)$, we may assume that $g(D) \subseteq M_0^m$, as announced.

Our next step consists in showing the existence of a nice cell decomposition of D (the inclusion $g(D(M)) \subseteq M_0^m$ will not be used here).

3.14. Claim. — *The set D can be divided into finitely many definable closed cells D_i such that g is $\mathbb{Q}$-affine on each D_i and for $i \neq j$, either there exists a $\mathbb{Z}$-affine function H and a hyperplane $H_a = H^{-1}(a)$ such that $D_i \cap D_j = D_i \cap H_a = D_j \cap H_a$, $H - a \geqslant 0$ on D_i and $H - a \leqslant 0$ on D_j.*

Proof of Claim. — We shall use the notion of special linear decompositions from [**Ele18**]. In fact we will need only to consider bounded linear cells in Γ^n. They are defined by induction on n. In Γ^0 the origin is a cell. If C is a bounded linear cell in Γ^{n-1}, f and g are $\mathbb{Q}$-affine functions on Γ^{n-1}, with $f < g$ on C, the relative interval $(f < g)_C = \{(x', y) \in C \times \Gamma; f(x') < y < g(x')\}$ and the graph $\Gamma(f)_C = \{(x', y) \in C \times \Gamma; f(x') = y\}$ are bounded linear cells in Γ^n. Denote by $\pi : \Gamma^n \to \Gamma^{n-1}$ the projection to the $n-1$ first coordinates. A special linear decomposition of a bounded definable set $Y \subseteq \Gamma^n$ is defined recursively in [**Ele18**] as follows. When $n = 1$ any cell decomposition is special. A linear decomposition C of Y, $n > 1$, is special if:

(1) $\pi(C)$ is a special linear decomposition of $\pi(Y)$.
(2) For every pair of cells $\Gamma(f)_S$ and $\Gamma(g)_T$ in C with S in the closure of T, $f|_S < g|_S$ or $f|_S > g|_S$ $f|_S = g|_S$.
(3) For every pair of cells cells $(f, g)_T$ and X in C, where $X = \Gamma(h)_S$, $(h, k)_S$ or $(k, h)_S$, there is no $c \in cl(S) \cap cl(T)$ such that $f(c) < h(c) < g(c)$.

Take a linear decomposition of D and g. By [**Ele18**, Fact 2.2] there exists a special linear decomposition $\mathcal{D}$ refining it. Let $\mathcal{D}'$ be the set of all $C \in \mathcal{D}$ such that, for any $C' \neq C$, C is not contained in the closure of C'. For each C_i in $\mathcal{D}$, we denote its closure by D_i. It follows from [**Ele18**, Fact 2.3] that the cells D_i cover D.

Let us prove the claim when $D_i \cap D_j \neq \varnothing$. We shall prove this by induction on the dimension n of the ambient space. For $n = 1$ the claim trivially holds. Let D_i and D_j be given. One proves easily by induction on n that $D_i \cap D_j$ is the closure of a single cell $C \in \mathcal{D}$. We shall separate into the following two cases.

Case 1: Assume $C = (f, g)_S$ for some cell S in Γ^{n-1} with f, g both $\mathbb{Q}$-affine functions. In this case, $\pi(C_i) \neq \pi(C_j)$ and $cl(\pi(C)) \subseteq \pi(D_i) \cap \pi(D_j)$. So by induction hypothesis, there is a hyperplane H in Γ^{n-1} given by a $\mathbb{Z}$-affine equation such that $\pi(D_i) \cap \pi(D_j) = H \cap \pi(D_i) = H \cap \pi(D_j)$, and $\pi(D_i)$ and $\pi(D_j)$ lie in different sides of H. Consider the vertical hyperplane above H (the hyperplane defined by the same equation in Γ^n), call it H_a. We claim that H_a works. Clearly $D_i \cap D_j \subseteq D_i \cap H_a$ and $D_i \cap D_j \subseteq D_j \cap H_a$. Since $C = (f, g)_S$, it must be the case that $C_i = (h, k)_T$ for some affine functions h and k and T a cell. Using condition (3) in the definition the special linear decomposition, it must be the case that for any $c \in S$, $h(c) = f(c)$ and $k(c) = g(c)$. This shows that $H_a \cap D_i = D_i \cap D_j$. The same proof works for D_j.

Case 2: Assume $C = \Gamma(f)_S$ for for some cell S in Γ^{n-1} and some affine function f. There are two possibilities: either $\pi(C_i) = \pi(C_j)$ or $\pi(C_i)$ and $\pi(C_j)$ are disjoint. If $\pi(C_i) = \pi(C_j)$, then we may assume that C_i is above C_j. Then the average of the lower bound (or the defining function when C_i is a graph of a function) of C_i and the upper bound of C_j (or the defining function of C_j when it is a graph of a function) gives a hyperplane that satisfies the property.

In the other case, $\pi(C_i)$ and $\pi(C_j)$ are disjoint. By the inductive hypothesis, there is a hyperplane H' given by an equation $g(x') = 0$ in Γ^{n-1}, with g a $\mathbb{Q}$-affine function, $x' = (x_1, \ldots, x_{n-1})$ satisfying the claim about the intersection for the closures of $\pi(C_i)$ and $\pi(C_j)$. We deal with the case when $C_i = (f_i, g_i)_{\pi(C_i)}$ and $C_j = (f_j, g_j)_{\pi(C_j)}$, the other cases being completely similar. Since our linear decomposition is special, we have that $f|_{\pi(C)} = f_i|_{\pi(C)}$ or $g_i|_{\pi(C)}$. Without loss of generality we may assume the latter case happens, then it must be the case that $f|_{\pi(C)} = f_j|_{\pi(C)}$ by the case assumption and the fact our decomposition is special. Moreover, we may assume that $g|_{\pi(C_i)} > 0$ and $g|_{\pi(C_j)} < 0$.

Consider the hyperplane H_N with equation $Ng(x') - (x_n - f(x')) = 0$. We will prove that for N a large enough integer, H_N lies strictly above the graph of g_i on C_i and strictly below the graph of f_j on C_j. This implies that $H_N \cap D_i$ and $H_N \cap D_j$ are both equal to the closure of C, as desired. It is enough to prove that if X_i is the closure of the graph of g_i on C_i, there is a natural number M such that the inequality $x_n - f(x') \leqslant Mg(x')$ holds on X_i, and the analogous statement for the closure Y_j of the graph of f_j on C_j. Let us prove this statement for X_i, the proof for Y_j being exactly the same.

Consider all 1-dimensional faces of X_i that intersect the hyperplane $g(x') = 0$. Let F be such a face (a line segment). The hypotheses force the point p_F of intersection of F with $\mathrm{g}(x') = 0$ to lie in the closure of C. Thus both $g(x')$ and $x_n - f(x')$ vanish at the point p_F. The restriction of $g(x')$ and $x_n - f(x')$ to the line segment F are linear functions on F vanishing at one endpoint of F, thus there exists a positive integer M_F such that $x_n - f(x') \leqslant M_F g(x')$ on F. In fact this inequality holds on the whole half-line L_F containing F with origin p_F. Take $M = \max(M_F)$. Now consider R a RCF-expansion of Γ. Let Y be the convex hull of the half-lines L_F in that expansion. We have $x_n - f(x') \leqslant Mg(x')$ on Y. But Y contains $X_i(R)$, since if P is a convex definably compact polyhedron of R^n, and if F is a face of P of any dimension, then the convex hull of all half-lines directed by 1-faces intersecting F contains P, hence the result.

It remains to prove the claim when $D_i \cap D_j = \varnothing$. We use induction on the dimension n. For $n = 1$ the statement is clear. If $\pi(D_i) \cap \pi(D_j) = \varnothing$ then by the induction hypothesis there exists an hyperplane H'_a in Γ^{n-1} satisfying the claim for $\pi(D_i) \cap \pi(D_j)$ and $\pi^{-1}(H'_a)$ will do the job. If $\pi(D_i) = \pi(D_j)$, we proceed by averaging as in case 2 above. If $\pi(D_i) \neq \pi(D_j)$ and $\pi(D_i) \cap \pi(D_j) \neq \varnothing$, we choose an hyperplane H'_a in Γ^{n-1} with equation $g(x') = 0$ satisfying the claim for $\pi(D_i) \cap \pi(D_j)$. We may assume $g \geqslant 0$ on D_i and $g \leqslant 0$ on D_j. Denote by $D'_i = D_i \cap \pi^{-1}(H'_a)$ and $D'_j = D_j \cap \pi^{-1}(H'_a)$. We have $D'_i \cap D'_j = \varnothing$. By the induction hypothesis there is some $\mathbb{Z}$-affine function f on Γ^n such that $f < 0$ on D'_i and $f > 0$ on D'_i. By an argument similar to the one used in Case 2 above on checks that for N any large enough integer, the affine function $Ng' - f$ is strictly positive on D_i and strictly negative on D_j. □

Now we would like to use our cell decomposition and our separation hyperplanes to build affine functions that will appear in the $(\ell, +)$-combination we are seeking for describing g. For this purpose, the inclusion $g(D(M)) \subseteq M_0^m$ will be crucial.

3.15. Claim. — *For any two distinct cells D' and D'' contained in D as in Claim 3.14, there is a function $f_{D',D''}$ in the group generated by $f_1, \ldots, f_n$, the constant functions and the coordinate functions such that*

$$(*) \qquad g|_{D'} \leqslant f_{D',D''}|_{D'} \text{ and } f_{D',D''}|_{D''} \leqslant g|_{D''}.$$

Proof of Claim. — Let H and a be given for D' and D'' as in Claim 3.14. Replacing H by $H - a$, without loss of generality, we may assume that $H|_{D'} \leqslant 0$ and $H|_{D''} \geqslant 0$. If $D' \cap D'' = \varnothing$, using definable compactness of D' and D'', we get there exists $a \in M_0$ such that $H|_{D'} < -a < 0 < a < H|_{D''}$. Moreover, by our assumption that $g(D(M))) \subseteq M_0$, there is $b \in M_0$ such that $-b < g(D) < b$. For $k \in \mathbb{N}$, consider the function $h_k = -kH + b$. On D', $h_k(D') > ka$, so $h_k - g(D) > ka - b$. On D'', $-h_k > ka - b$, so $g - h_k(D'') > ka - 2b$. Since M_0 has height 1, for k large enough condition $(*)$ is satisfied.

If $D' \cap D'' \neq \varnothing$, take $c \in D' \cap D''$ and let G be the $\mathbb{Q}$-affine function such that $g = G$ on D'. Replacing g by $g - G$, we may assume that $g = 0$ on D'. Translating our entire set by c, we may assume that c is the origin. Thus $g(0) = 0$, g is $\mathbb{Q}$-affine on D'', hence g is actually the restriction of a $\mathbb{Q}$-linear function on D''. Without loss of generality, we may further assume that for H given in Claim 3.14, $H|_{D''} \leqslant 0$ and $H|_{D'} \geqslant 0$. So on D', for any $m > 0$, $mH \geqslant 0 = g$. It remains to find a $m > 0$ such that $mH \leqslant g$ on D''. For any $b \in D''$, if $H(b) = 0$, then $b \in D'' \cap H_0 = D' \cap H_0$, hence $g(b) = 0$. By the fact that g is $\mathbb{Q}$-linear and H is $\mathbb{Z}$-linear, on a definably compact neighborhood N of $D'' \cap H_0$ in D'', there is $m \in \mathbb{N}$ such that $|g(x)| \leqslant m|H(x)|$. Since $H \leqslant 0$ on D'' by assumption, we have that $g(x) \geqslant mH(x)$ for $x \in N$. Working in an elementary extension, we may assume that we are working in the theory of real closed fields, thus D'' can be covered by line segments passing through the origin. For each line segment L under the standard parameterization of

L, we know that $g(x) - mH(x)|_L$ is a function of the form $x \mapsto cx$ for some c, by linearity of g and H on D''. Note further that c is greater or equal to zero on some interval containing 0, witnessed by N, hence it is greater or equal to 0 throughout the segment. Thus we have proven that $g(x) - mH(x) \geqslant 0$ on D''. □

We now conclude the proof of Theorem 3.13. Note that g is a w-combination of the f_i; so it is a fortiori a w-combination of the set of functions obtained by adding all the functions $f_{D',D''}$ to the functions f_i. Take x and y in D. If they belong to the same cell D', then $g(x) = f_i(x)$ and $g(y) = f_i(y)$ for some i and condition (4) in Proposition 3.8 is satisfied. If they belong to two distinct cells D' and D'', then $g(x) \leqslant f_{D',D''}(x)$ and $f_{D',D''}(y) \leqslant g(y)$ by Claim 3.15. Thus, by the implication (4) $\Longrightarrow$ (1) in Proposition 3.8, we obtain that g is an ℓ-combination of the functions f_i and $f_{D',D''}$, which concludes the proof. □

The proof of Claim 3.15 actually yields the following convenient way to check if a given function is Lipschitz on a definably compact set.

3.16. Corollary. — *Let D' and D'' be two definably compact convex sets such that $D' \cap D'' = D' \cap H_a = D'' \cap H_a \neq \varnothing$ with H_a an hyperplane defined by a $\mathbb{Z}$-affine function. Assume further that g a continuous function that is affine on D' and D'' respectively, then g is Lipschitz on $D' \cup D''$.*

3.17. Remark. — Note that one can have definably compact versions of Example 3.5 by replacing $[0, \infty)$ with $[0, c]$ for some $c > n \cdot 1$ for all $n \in \mathbb{N}$. However, the function g there is not Lipschitz because $|(0, c) - (1, c)| = 1$ and $|g(0, c) - g(1, c)| = c$.

3.18. Remark. — In the case of homogeneous linear equations, with no parameters, equivalence of ℓ-combination and w-combination goes back to work of Beynon [**Bey75**], see also §5.2 of [**Gla99**] for related results. In 2011, as a student, Daniel Lowengrub rediscovered and partially generalized Beynon's results. He also gave Example 3.5 showing that they do not hold over non-archimedean parameters. Here we fully generalized them, after replacing continuity by a Lipschitz condition. Our proofs in this section make use of his ideas.

4. Complements about Γ-internal sets

4.1. Preimages of the standard Γ-internal subset of $\widehat{\mathbf{G}_{\mathrm{m}}^n}$. — Let k be a model of ACVF and let X be an n-dimensional k-scheme of finite type. Let us denote by $X^{\#}_{\mathrm{gen}}$ the subset of $X^{\#}$ consisting of points lying over the generic point of an n-dimensional irreducible component of X (so if X is irreducible, $X^{\#}_{\mathrm{gen}}$ has the same meaning as above). Let $f \colon X \to \mathbf{G}_{\mathrm{m}}^n$ be a k-morphism, and let $\widehat{S}_n$ be the image of the definable topological embedding from Γ^n into $\widehat{\mathbf{G}_\mathrm{m}^n}$ that sends a n-tuple γ to the generic point r_γ of the closed n-ball with valuative radius γ and centered at the origin.

This set $\widehat{S}_n$ is the archetypal example of a Γ-internal subset. The purpose of what follows is to show that its preimage Υ under f is Γ-internal and purely n-dimensional, and more generally that a finite union of such preimages (for various f) is Γ-internal and purely n-dimensional. This is a model-theoretic version of a result that is known in the Berkovich setting, see [**Duc12**], Theorem 5.1.

For this, we shall need an alternative description of Υ which is of independent interest and is the analogue of [**Duc12**], Theorem 3.4 (1). If x is a point of $\widehat{X}$, the *tropical* dimension of f at x is the infimum of $\dim \mathrm{val}(f)(\widehat{V}) = \dim \mathrm{val}(f)(V)$ for V a (v+g)-open definable subset of X with $x \in \widehat{V}$.

4.2. Lemma. — *Let V be a definable subset of X such that* $\mathrm{val}(f)(V)$ *is of dimension n. Then $\widehat{V} \cap \Upsilon \neq \varnothing$.*

Proof. — Since $\mathrm{val}(f)(V)$ is n-dimensional, it contains a product of intervals $\prod_{1\leqslant i\leqslant n}[a_i, b_i]$ with $a_i < b_i$. Let K be the field sort of the monster model and let L be a model of ACVF containing K and with group $\Gamma \oplus \mathbb{Q}^n$ with the lexicographic ordering. For every i, set $c_i = (a_i + b_i)/2$ and $d_i = c_i + (0, \ldots, 1, 0, \ldots, 0)$ where the 1 lies at the i-th position of the $\mathbb{Q}^n$ factor. Set also $\gamma = (c_1, \ldots, c_n)$ and $\delta = (d_1, \ldots, d_n)$. By construction, there exists some $v \in V(L)$ such that $\mathrm{val}(f(v)) = \delta$.

By covering X with finitely many affine charts, and by considering in each of them a system of inequalities defining V as well as a finite set of generators of the ring of regular functions, we can write V as a union $\bigcup V_\lambda$ of definable subsets with $\widehat{V_\lambda}$ definably compact, each of the V_λ being defined by a positive combination of non-strict inequalities involving scalars in Γ. But since Γ is cofinal in $\Gamma \oplus \mathbb{Z}^n$, we still have $V(L) = \bigcup_\lambda V_\lambda(L)$. In particular, there exists a definable subset W of V with $\widehat{W}$ definably compact and such that $v \in W(L)$.

By construction, the type of $f(v)$ over K is Abhyankar with pure group parameters $d_1, \ldots, d_n$. Under the coarsening of the valuation of L corresponding to the convex subgroup $\mathbb{Q}^n$ of $\mathrm{val}(L)$, this type gives rise to the stably dominated type r_γ. Applying this coarsening to v gives rise to a stably dominated type w on X over K which lies on $\widehat{W}$ since the latter is definably compact, and whose image on $\widehat{\mathbf{G}_\mathrm{m}^n}$ is r_γ. Thus $w \in \widehat{V} \cap \Upsilon$. □

4.3. Proposition. — *With the above notation, Υ is exactly the set of points of $\widehat{X}$ at which the tropical dimension of f is equal to n.*

Proof. — Let $(f_1, \ldots, f_n)$ be the family of invertible functions defining f. Let $x \in \widehat{X}$. A point x of $\widehat{X}$ belongs to Υ if and only if $f_1, \ldots, f_n$ is an Abhyankar basis at x.

Now let $x \in \widehat{X} \backslash \Upsilon$. Then $f_1, \ldots, f_n$ is *not* an Abhyankar basis at x. Therefore there exists a polynomial $\sum a_I T^I$ such that

$$\operatorname{val}\left(\sum a_I f^I(x)\right) > \min_I \operatorname{val}(a_I) + \operatorname{val}(f^I(x)).$$

Let V be the subset of X defined by the inequality

$$\operatorname{val}\left(\sum a_I f^I\right) > \min_I \operatorname{val}(a_I) + \operatorname{val}(f^I).$$

It is a definable (v+g)-open subset and $x \in \widehat{V}$. Moreover by the very definition of V, for every $y \in V$ there exists two distinct multi-indices I and J with $\operatorname{val}(a_I) + \operatorname{val}(f^I(y)) = \operatorname{val}(a_J) + \operatorname{val}(f^J(y))$, which shows that $\operatorname{val}(f)(V)$ is contained in a finite union of $(n-1)$-dimensional subspaces of Γ^n. As a consequence, the tropical dimension of f at x is at most $n-1$.

Conversely, let $x \in \Upsilon$ and let V a (v+g)-open definable subset of X with $x \in \widehat{V}$. Since $f(x) \in \widehat{S}_n$, the residue transcendence degree at x is at least n, which shows that x is Zariski-generic in some irreducible component Y of X (equipped with its reduced structure). There is a dense open subset Z of Y such that f induces a finite flat map from Z to a dense open subscheme of $\mathbf{G}_{\mathrm{m}}^n$; then the induced map $\widehat{Z} \to \widehat{\mathbf{G}_{\mathrm{m}}^n}$ is open by [**HL16**, Corollary 9.7.2], and since $x \in \widehat{Z}$, it follows that f is open around x. In particular, $f(\widehat{V})$ contains a neighborhood Ω of $f(x)$. Since $x \in \Upsilon$, the image $f(x)$ is equal to r_γ for some $\gamma \in \Gamma^n$. The intersection $\Omega \cap \widehat{S}_n$ then contains $\{r_\delta\}_{\delta \in B}$ for B some product of n open intervals containing γ, so $\operatorname{val}(f)(\widehat{V})$ contains B, and is in particular n-dimensional. The tropical dimension of f at x is thus equal to n. □

4.4. Theorem. — *The preimage $\Upsilon := f^{-1}(\widehat{S}_n)$ is a purely n-dimensional Γ-internal subset of $\widehat{X}$ contained in $X^{\#}_{\mathrm{gen}}$.*

Proof. — Set $\Upsilon = f^{-1}(\widehat{S}_n)$. Let x be a point of Υ. Then $f(x) = r_\gamma$ for some γ, so that the residue field of x has transcendence degree at least n. As X is of dimenion n, this transcendence degree is exactly n, and x belongs to $X^{\#}$ and is Zariski-generic in some n-dimensional irreducible component of X. Thus by arguing componentwise, we may and do assume that X is integral and that f is dominant (otherwise $\Upsilon = \varnothing$ and we are done). Let F be the function field of X; through the map $f = (f_1, \ldots, f_n)$ one can see F as a finite extension of $k(T_1, \ldots, T_n)$. It follows from Theorem 2.8 of [**Duc12**] (which itself rests on the definability of $\widehat{C}$ for C a curve) that there are finitely many non-zero elements $g_1, \ldots, g_m$ of F that separate universally (i.e., after any ground field extension) the extensions to F of Gauss valuations on $k(T_1, \ldots, T_n)$. Set

$h = (f_1, \dots, f_n, g_1, \dots, g_m)$; up to removing a proper Zariski-closed subset from X (which will not alterate Υ whose points are Zariski-generic), we can assume that h is a regular map from X to $\mathbf{G}_{\mathrm{m}}^{n+m}$, and even (by adding some extra functions if needed), that it is a closed immersion; what will be needed here is only topological properness of $\mathrm{val}(h)\colon \widehat{X} \to \Gamma^{n+m}$. By choice of the g_i, the restriction of $\mathrm{val}(h)$ to Υ is injective. Hence in order to conclude, it suffices to prove that $\mathrm{val}(h)(\Upsilon)$ is a k-definable subset of Γ^{n+m}.

For that purpose, first note that $P := \mathrm{val}(h)(\widehat{X}) = \mathrm{val}(h)(X)$ is a k-definable subset of Γ^{n+m}; more precisely, it is closed and purely n-dimensional (this is well-known, see for instance [**Duc12**], Theorem 1.2, statement c2). Let π be the projection from Γ^{n+m} to Γ^n, and let Q be the set of $p \in P$ such that $\pi(R)$ is n-dimensional for every definable neighborhood R of p. Then Q is a k-definable subset of P, and we are going to prove that $\mathrm{val}(h)(\Upsilon) = Q$, which will end the proof.

Let $x \in \Upsilon$ and let R be a definable open neighborhood of $\mathrm{val}(h(x))$ in P. Then $R = \mathrm{val}(h)((\mathrm{val}(h))^{-1}(R))$ so that $\pi(R) = \mathrm{val}(f)(\mathrm{val}(h)^{-1}(R))$. Since $x \in \Upsilon$, Prop. 4.3 ensures that f is of tropical dimension n at x, so $\pi(R)$ is n-dimensional and $\mathrm{val}(h(x)) \in Q$.

Conversely, let p be a point of Q. Assume that it does not belong to $\mathrm{val}(h)(\Upsilon)$. By properness of $\mathrm{val}(h)$ there exists some definable neighborhood R of p in P such that $(\mathrm{val}(h))^{-1}(R)$ does not intersect Υ. It follows then from Lemma 4.2 that $\mathrm{val}(f)((\mathrm{val}(h))^{-1}(R))$ is of dimension $\leqslant n-1$; thus $\pi(R)$ is of dimension $n-1$ and p does not belong to Q, which leads to a contradiction. □

4.5. Corollary. — *Let X be an n-dimensional k-scheme of finite type and let $\varphi_1, \dots, \varphi_m$ be morphisms from X to $\mathbf{G}_{\mathrm{m}}^n$. The finite union $\bigcup \varphi_j^{-1}(\widehat{S}_n)$ is a purely n-dimensional Γ-internal subset of $\widehat{X}$ contained in $X^{\#}_{\mathrm{gen}}$.*

Proof. — By Theorem 4.4 this holds for each $\varphi_j^{-1}(\widehat{S}_n)$; moreover if j and ℓ are two indices then $\varphi_j^{-1}(\widehat{S}_n) \cap \varphi_\ell^{-1}(\widehat{S}_n)$ is definable in both $\varphi_j^{-1}(\widehat{S}_n)$ and $\varphi_\ell^{-1}(\widehat{S}_n)$ by [**HL16**], Lemma 8.2.9, so Υ is Γ-internal. □

Our purpose is now to prove that conversely, if X is irreducible, every Γ-internal subset of $X^{\#}_{\mathrm{gen}}$ is contained in some finite union $\bigcup_j \varphi_j^{-1}(\widehat{S}_n)$ as above (Theorem 4.8); this is an instance of the general principle according to which Γ-internal subsets of $X^{\#}$ are expected to be reasonable (while general Γ-internal subsets of $\widehat{X}$ can likely be rather pathological). Originally we used this result through its corollary 4.9 for proving Theorem 7.2, but we finally do not need it anymore. Nonetheless, we have chosen to keep it in this paper, because it seems to us of independent interest, and shows that the main objects considered in this work are more tractable than one could think at first sight.

4.6. Proposition. — *Let X be an integral n-dimensional k-scheme of finite type and let Υ be a purely n-dimensional Γ-internal subset of $X^{\#}_{\mathrm{gen}}$. There exists a dense open subset U of X and finitely many morphisms $\varphi_1, \dots, \varphi_m$ from U to $\mathbf{G}_{\mathrm{m}}^n$ such that $\Upsilon \subseteq \bigcup_j \varphi_j^{-1}(\widehat{S}_n)$.*

Proof. — Up to shrinking X we might assume that there exist finitely many invertible functions $f_1, \dots, f_r$ on X such that $\mathrm{val}(f)$ induces a definable homeomorphism between Υ and a definable susbet of Γ^r (2.3). For every subset I of $\{1, \dots, r\}$ of cardinality n, let f_I be the map from X to $\mathbf{G}_{\mathrm{m}}^n$ given by the f_i with $i \in I$. Since Υ is of pure dimension n, for every $x \in \Upsilon$ there is at least one subset I of $\{1, \dots, r\}$ of cardinality n such that the tropical dimension of f_I at x is n. By Proposition 4.3, this means that

$$\Upsilon \subseteq \bigcup_{I \subseteq \{1,\dots,r\}, |I|=n} f_I^{-1}(\widehat{S}_n). \qquad \square$$

4.7. Proposition. — *Let X be an n-dimensional integral scheme of finite type over k, and let $\Upsilon \subseteq X^{\#}_{\mathrm{gen}}$ be a Γ-internal subset defined over k. Then Υ is contained in some purely n-dimensional Γ-internal subset of $X^{\#}_{\mathrm{gen}}$ defined over k.*

Proof. — By shrinking X we can assume that it is quasi-projective. In view of Proposition 4.6 and Corollary 4.5, a finite union of purely n-dimensional Γ-internal subsets of $X^{\#}_{\mathrm{gen}}$ is still a purely dimensional Γ-internal subset of $X^{\#}_{\mathrm{gen}}$. This allows to cut Υ into finitely many k-definable pieces and to argue piecewise. We thus can assume that Υ is purely d-dimensional for some d, and we argue by descending induction on d, so we assume that our proposition holds if the Γ-internal subset involved is equidimensional of dimension $> d$.

Let α be a k-definable embedding from Υ into some Γ^m given by finitely many non-zero rational functions. By [**HL16**, Theorem 11.1.1], there exists a pro-definable deformation retraction $h : I \times \widehat{X} \to \widehat{X}$ preserving α with a Γ-internal purely n-dimensional image Υ_{targ} contained in $X^{\#}$. Let $\Upsilon_s = \{p \in \Upsilon : h(t,p) = p \text{ for any } t\}$. By its very definition, Υ_s is contained in the set Υ' of Zariski-dense points of Υ_{targ}, which is a purely n-dimensional Γ-internal subset of $X^{\#}_{\mathrm{gen}}$. It therefore suffices to prove the proposition for the open complement of Υ_s in Υ, which is still purely d-dimensional. Otherwise said, we can assume that $\Upsilon_s = \varnothing$.

Let $\Upsilon'' = h(I, \Upsilon)$. We claim that it is iso-definable, and thus Γ-internal. By [**HL16**, Lemma 2.2.8], Υ'' is strict pro-definable. Since $\Upsilon \subseteq X^{\#}$, the set Υ'' is contained in $X^{\#}$ as well by [**HL16**, Theorem 11.1.1] and the latter is strict ind-definable. Hence by compactness, we see that $h(I, \Upsilon)$ is a strict pro-definable subset of a definable set, thus is iso-definable. Note also that the homotopy built in [**HL16**] is Zariski-generalizing, so $\Upsilon'' \subseteq X^{\#}_{\mathrm{gen}}$.

Since $\Upsilon_s = \varnothing$, for every $p \in \Upsilon$ there are some a_p, b_p in I with $a_p < b_p$ such that $h|_{[a_p,b_p]} : [a_p, b_p] \to \widehat{X}$ is injective. Since Υ'' is Γ-internal, the induced function $h : I \times \Upsilon \to \Upsilon''$ is a definable function in the o-minimal sense. Let $x = h(p,t)$ be a point of Υ'', with t and p defined over k. We claim that $\dim_p \Upsilon'' = d + 1$. Indeed, since $\dim_p \Upsilon = d$, there exists a point q in U that specializes to p (when viewed as a type over k) and such that $\alpha(q)$ is d-dimensional over k (i.e., its coordinates generate a group of rational rank d over $\Gamma(k)$). Now up to replacing t if necessary by an endpoint of an interval containing t on which $h(p,\cdot)$ is constant, we may assume that there exists a non-singleton segment $J \subseteq I$ having t as one of its endpoints such that $h(p,\cdot)|_J$ is injective. If K is some subinterval of J containing t defined over $k(q)$ and on which $h(q,\cdot)$ is constant then since h is continuous and thus is compatible with specialisation, both endpoints of K have to specialize to t. Thus there exists a non-singleton segment K contained in J and defined over $k(q)$, having one endpoint τ that specializes to t, on which $h(q,\cdot)$ is injective. Now let us choose an element τ' of K that specializes to t and such that $k(\tau')$ is of dimension 1 over $k(q)$. By construction $h(q,\tau')$ is a point of Υ'' that specializes to $h(p,t)$ and that is $(d+1)$-dimensional over k, whence our claim.

It follows that Υ'' is of pure dimension $d+1$, and it contains Υ. By induction Υ'' is contained in some purely n-dimensional Γ-internal subset of $X^{\#}_{\mathrm{gen}}$, and we are done. □

Combining Proposition 4.6 and Proposition 4.7 we get the following structure theorem for Γ-internal subsets of $X^{\#}_{\mathrm{gen}}$.

4.8. Theorem. — *Let X be an n-dimensional integral scheme of finite type over k, and let $\Upsilon \subseteq X^{\#}_{\mathrm{gen}}$ be a Γ-internal subset defined over k. There exists a dense open subset U of X and finitely many morphisms $\varphi_1, \ldots, \varphi_m$ from U to $\mathbf{G}_{\mathrm{m}}^n$ such that $\Upsilon \subseteq \bigcup_j \varphi_j^{-1}(\widehat{S}_n)$.* □

This theorem has an interesting consequence concerning the closure $\overline{\Upsilon}$ of Υ, or at least its subset $\overline{\Upsilon}_{\mathrm{gen}}$ consisting of Zariski-generic points (let us mention that the general structure of the closure of an arbitrary Γ-internal subset is poorly understood).

4.9. Corollary. — *Let X be an n-dimensional integral scheme of finite type over k, and let $\Upsilon \subseteq X^{\#}_{\mathrm{gen}}$ be a Γ-internal subset. The set $\overline{\Upsilon}_{\mathrm{gen}}$ is contained in $X^{\#}$ and is Γ-internal.*

5. A first finiteness result

The aim in this section is to prove a finiteness result, Theorem 5.7, which is weaker than our main theorem but will be needed in its proof.

5.1. Notation. — Throughout this section we fix a valued field k, an n-dimensional integral k-scheme of finite type X, and a Γ-internal subset Υ of $X^{\#}_{\mathrm{gen}}$. Every non-zero k-rational function $f \in k(X)$ gives rise to a k-definable map $\mathrm{val}(f) \colon \Upsilon \to \Gamma$. The set of all such maps is denoted by $\mathbb{S}_k(\Upsilon)$, or simply by $\mathbb{S}(\Upsilon)$ if the ground field k is clearly understood from the context. Elements of $\mathbb{S}(\Upsilon)$ will be called *regular functions* from $\Upsilon \to \Gamma$. By a *constant* function on Υ we shall always mean a *k-definable* constant function; i.e., an element of $\mathrm{val}(k) \otimes \mathbb{Q}$.

Assume that $\mathrm{val}(k)$ is divisible, in which case $\mathbb{S}(\Upsilon)$ contains the constant functions. We shall then say for short that $\mathbb{S}(\Upsilon)$ is finitely $(w,+)$-generated *up to constant functions* if there exist a finite subset E of $\mathbb{S}(\Upsilon)$ such that $\mathbb{S}(\Upsilon)$ is $(w,+)$ -generated by E and the constant functions.

5.2. Remark. — For a subset E of $\mathbb{S}(\Upsilon)$ to w-generate $\mathbb{S}(\Upsilon)$, it suffices by compactness that for every $p \in \Gamma$ and every $f \in \mathbb{S}(\Upsilon)$ there exists $g \in E$ such that $f(p) = g(p)$.

5.3. Lemma. — *Assume that* $\mathrm{val}(k)$ *is divisible. The following are equivalent.*

(i) *There exists a regular embedding $\alpha \colon \Upsilon \hookrightarrow \Gamma^n$ such that for every $\beta \in \mathbb{S}(\Upsilon)$, the map $\beta \circ \alpha^{-1} \colon \alpha(\Upsilon) \to \Gamma$ is a piecewise affine function (with integral linear coefficients and arbitrary constant terms).*

(ii) *There exists a regular embedding $\alpha \colon \Upsilon \hookrightarrow \Gamma^n$ such that for every $p \in \Upsilon$ and every algebraically closed field F containing k, if $c \models p|_F$, then the value group of $F(c)$ is generated by $\alpha(c)$ over $\Gamma(F)$.*

(iii) *The set $\mathbb{S}(\Upsilon)$ is finitely $(w,+)$-generated up to constant functions.*

Proof. — If there exists α as in (i), then α satisfies (ii) as well. Assume that α satisfies (ii), and let G be the subgroup generated by all components of α. Let $f \in \mathbb{S}(\Upsilon)$ and let F be the algebraic closure of k (in the monster model). For every $g \in G$ and $\alpha \in F$, let $\Upsilon_{g,\alpha}$ be the subset of Υ consisting of all p such that $\mathrm{val}(f(p)) = \mathrm{val}(\alpha(p)) + \mathrm{val}(g(p))$; this is a definable subset of Υ, and Υ is covered by the sets $\Upsilon_{g,\alpha}$. It follows that it is covered by finitely many of them, whence (iii). Now assume that (iii) is fulfilled; then if one takes for α the map given by any finite system of generators of $\mathbb{S}(\Upsilon)$ modulo the constant functions, (i) is fulfilled (note that (iii) and the existence of a regular embedding of Υ forces α to be itself an embedding). □

Our purpose is now to show that if $\mathrm{val}(k)$ is divisible and k is defectless, $\mathbb{S}(\Upsilon)$ is finitely $(w,+)$-generated up to constant functions. (Recall that a valued field F is called defectless or stable if every finite extension of F is defectless; to avoid any risk of confusion with the model-theoretic notion of stability use the terminology defectless instead of stable.) The core of the proof is the following proposition about valued field extensions.

5.4. Proposition. — *Let $F \hookrightarrow K \hookrightarrow L$ be finitely generated extensions of valued fields, with $K = F(a)$ and $L = K(b)$. We make the following assumptions:*

(1) *F is defectless;*

(2) *K is Abhyankar over F;*

(3) $\mathrm{res}(K) = \mathrm{res}(F)$*;*

(4) $\mathrm{val}(L) = \mathrm{val}(K)$*;*

(5) *L is finite over K.*

Then there exists a quantifier-free formula $\varphi(x, y)$ in the language of valued fields with parameters in F such that $L \models \varphi(a, b)$, and such that whenever $L' = F(a', b')$ is a valued field extension with $L' \models \varphi(a', b')$ and the residue field of $K' := F(a')$ is a regular exension of $\mathrm{res}(F)$*, then* $\mathrm{val}(L') = \mathrm{val}(K')$*.*

Proof. — Since F is defectless, K is defectless as well (this was proved by Kuhlmann in [**Kuh10**], but for the reader's convenience we give a new proof of this fact in Appendix A with model-theoretic tools based upon [**HL16**], see Theorem A.1). Therefore L^{h} is a defectless finite extension of K^{h}; let d denote its degree. By assumption one has $\mathrm{val}(L^{\mathrm{h}}) = \mathrm{val}(K^{\mathrm{h}})$, so that $\mathrm{res}(L^{\mathrm{h}})$ is of degree d over $\mathrm{res}(F^{\mathrm{h}})$. Otherwise said, $\mathrm{res}(L)$ is of degree d over $\mathrm{res}(K)$.

Now let $c_1, \dots, c_r$ be elements of $\mathrm{res}(L)$ that generate it over $\mathrm{res}(F)$; for every i, let P_i be a polynomial in i variables with coefficients in $\mathrm{res}(F)$ such that $P_i[c_1, \dots, c_{i-1}, T]$ is the minimal polynomial of c_i over $\mathrm{res}(F)[c_1, \dots, c_{i-1}]$. Choose a monic lift Q_i of P_i with coefficients in the ring of integers of F, and an element R_i of $F(X)[Y]$ such that $F(a)[b]$ is a lift of c_i. Let $\Phi(x, y)$ be the formula

$$\mathrm{val}(R_i(x, y)) = 0 \text{ and } \mathrm{val}[Q_i(R_1(x, y), \dots, R_i(x, y))] > 0 \text{ for all } i.$$

Now L^{h} is a compositum of L and K^{h}, so it is generated by b over K^{h}. Hence there exists a sub-tuple b' of b of size d such that b is contained in the K^{h}-vector space generared by b'. As K^{h} is the definable closure of K, the latter property can be rephrased as $\Psi(a, b)$ for some formula Ψ in the language of valued fields, with parameters in F.

Now let $L' := F(a', b')$ be a valued extension of F, and set $K' = F(a')$. Assume that $\mathrm{res}(K')$ is a regular extension of F, and that

$$L' \models \Phi(a', b') \text{ and } \Psi(a', b').$$

Then $\Psi(a', b')$ ensures that $(L')^{\mathrm{h}}$ is at most d-dimensional over $(K')^{\mathrm{h}}$, while $\Phi(a', b')$ ensures that $\mathrm{res}(L')$ contains a field isomorphic to $\mathrm{res}(F)(c_1, \dots, c_r) =$

$\mathrm{res}(L)$. Since $\mathrm{res}(K')$ is regular over $\mathrm{res}(K) = \mathrm{res}(F)$, the residue field $\mathrm{res}(L')$ contains a field isomorphic to $\mathrm{res}(L) \otimes_{\mathrm{res}(F)} \mathrm{res}(K')$, which is of degree d over $\mathrm{res}(K')$. As a consequence,

$$[L' : K'] = [\mathrm{res}(L') : \mathrm{res}(K')] = d \text{ and } \mathrm{val}(L') = \mathrm{val}(K'). \qquad \square$$

5.5. Generic types of closed balls. — In practice, the above proposition will be applied for a realizing the generic type of a ball over F. Let us collect here some basic facts about such types. If γ is an element of Γ, we denote by r_γ the type of the closed ball of (valuative) radius γ, which belongs to $\widehat{\mathbb{A}^1}$ and even to $\mathbb{A}^{1\#}$. More generally if $\gamma = (\gamma_1, \ldots, \gamma_n)$ we shall denote by r_γ the type $r_{\gamma_1} \otimes \ldots \otimes r_{\gamma_n}$, which is the generic type of the n-dimensional ball of polyradius $(\gamma_1, \ldots, \gamma_n)$ and belongs to $\mathbb{A}^{n\#}$.

Now let F be a valued field, let K be a valued extension of F and let $a_1, \ldots, a_r, a_{r+1}, \ldots, a_n$ be elements of $L^\times$. Assume the following:

(1) the group elements $\mathrm{val}(a_1), \ldots, \mathrm{val}(a_r)$ are $\mathbb{Z}$-linearly independent over $\mathrm{val}(F)$;

(2) one has $\mathrm{val}(a_i) = 0$ for $i = r+1, \ldots, n$ and the residue classes of the a_i for $i = r+1, \ldots, n$ are algebraically independent over $\mathrm{res}(F)$.

Set $\gamma_i = \mathrm{val}(a_i)$ for $i = 1, \ldots, n$. Then under these assumptions one has $a \models r_\gamma|_{F(\gamma)}$.

Conversely, assume that $a \models r_\gamma|_{F(\gamma)}$. Then $\mathrm{val}(a_i) = 0$ for $i = r+1, \ldots, n$, the residue classes of the a_i for $i = r+1, \ldots, n$ are algebraically independent over $\mathrm{res}(F)$ and $\mathrm{res}(F(a_{r+1}, \ldots, a_n)$ is generated by the residue classes of the a_i, so is purely transcendental of degree $n - r$ over $\mathrm{res}(F)$. In particular, this is a regular extension of $\mathrm{res}(F)$. Now the $\mathrm{val}(a_i)$ for $i = 1, \ldots, r$ are $\mathbb{Z}$-linearly independent, the group $\mathrm{val}(F(a_1, \ldots, a_n))$ is generated over the group $\mathrm{val}(F(a_{r+1}, \ldots a_n)) = \mathrm{val}(F)$ by the $\mathrm{val}(a_i)$ for $i = 1, \ldots, r$, so it is free of rank r modulo $\mathrm{val}(F)$; and the residue field of $F(a_1, \ldots, a_n)$ is equal to that of $F(a_{r+1}, \ldots, a_n)$, so it is purely transcendental of degree $n - r$ over $\mathrm{res}(F)$; in particular, this is a regular extension of the latter.

5.6. Lemma. — *Let F be a valued field and let p be a strongly dominated (global) type with canonical parameter of definition $\gamma \in \Gamma^n$. Let $b \models p|_{F(\gamma)}$ and set $K = F(b)$. Then:*

1. γ is definable over $F(b)$;

2. $F(b)$ is an Abhyankar extension of F.

Proof. — Let us start with (1). Let Φ be an automorphism of the monster model fixing $F(b)$ pointwise. One has to show that Φ fixes γ, or p – this amounts to the same. Set $\delta = \Phi(\gamma)$ and $q = \Phi(p)$. Let A be a Φ-invariant subset of Γ containing γ. Since p is orthogonal to Γ, the restriction $p|_{F(\gamma)}$

implies a complete type r over $F(A)$, which coincides necessarily with the type of b over $F(A)$. Thus p contains the type of b over $F(A)$, and so does $\Phi(p)$ since both b and A are Φ-invariant. So p and $\Phi(p)$ are two global generically stable $F(A)$-definable types that coincide over $F(A)$; it follows that they are equal.

Now we prove (2). By replacing γ by a suitable subtuple if necessary, we may and do assume that $\gamma = (\gamma_1, \ldots, \gamma_n)$ where the γ_i are $\mathbb{Z}$-linearly independent over $\mathrm{val}(F)$. Now choose $c = (c_1, \ldots, c_n)$ realizing r_γ over $F(b)$. Then by stable domination, b realizes p over $F(\gamma, c)$ and in particular over $F(c)$. The type p is strongly stably dominated, and it is definable over $F(c)$ by construction. So $\mathrm{res}(F(b))$ is of transcendence degree $\dim X$ over $\mathrm{res}(F(c))$, and $F(b, c)$ is thus Abhyankar over $F(c)$, hence over F since $F(c)$ is Abhyankar over F. Then $F(b)$ is Abhyankar over F. □

5.7. Theorem. — *Let k be a defectless valued field with divisible value group. Let X be an n-dimensional integral k-scheme of finite type, and let Υ be a Γ-internal subset of $X^{\#}_{\mathrm{gen}} \subseteq \widehat{X}$. Then $\mathbb{S}(\Upsilon)$ is finitely $(w, +)$-generated up to constant functions.*

Proof. — We shall prove the following: for every $p \in \Upsilon$, there exists a k-definable subset W of Υ containing p and finitely many functions $a_1, \ldots, a_n$ in $k(X)^\times$ such that for every $x \in W$ and every $f \in \mathbb{S}(\Upsilon)$, the element $\mathrm{val}(f(x))$ of Γ belongs to the group generated by $\mathrm{val}(k)$ and the $\mathrm{val}(a_i(x))$. This will allow us to conclude. Indeed, assume that this statement has been proved. Then by compactness there is a finite cover $\mathscr{W}$ of Υ with finitely many sets W as above. Hence $\mathbb{S}(\Upsilon)$ is $(\ell, +)$-generated by the a_i up to constant functions.

So, let $p \in \Upsilon$. By Lemma 5.6, one can realize p over F by a tuple (b, a) such that the following are satisfied:

(1) b is algebraic over $F(a)$;

(2) $a = (a_1, \ldots, a_r, \ldots, a_n)$ where all the a_i are invertible, the elements $\mathrm{val}(a_1), \ldots, \mathrm{val}(a_r)$ are $\mathbb{Z}$-linearly independent over $\mathrm{val}(F)$ and generate $\mathrm{val}(F(a, b))$ modulo $\mathrm{val}(F)$, $\mathrm{val}(a_i) = 0$ for all i between $r+1$ and n, and the residue classes of $a_{r+1}, \ldots, a_n$ are algebraically independent over F; otherwise said, if one sets $\gamma_i = \mathrm{val}(a_i)$ then a realizes the type r_γ over F;

(3) p is definable over $F(\gamma_1, \ldots, \gamma_r)$.

(Note that (2) can be achieved since $\mathrm{val}(F(b, a))/\mathrm{val}(F)$ is torsion-free, which comes from our assumption that $\mathrm{val}(F)$ is divisible.)

Since p is Zariski-generic, a and b can be interpreted as a tuple of rational functions on X, giving rise to a map π from a dense open subset of X to $\mathbb{A}^n_F$. In particular, π induced a map (which we still denote by π) from Υ to $\widehat{\mathbb{A}^n_F}$, and condition (2) above can be rephrased by telling that $\pi(p) = r_\gamma$

$\pi(p)|_{F(\gamma)} = r_\gamma|_{F(\gamma)}$; as both $\pi(p)$ and r_γ are generically stable types defined over $F(\gamma)$, it follows that $\pi(p) = r_\gamma$.

Moreover, the tower $F(a_{r+1}, \ldots, a_n) \subseteq F(a) \subseteq F(a, b)$ fulfills the conditions of Proposition 5.4; hence the latter provides a formula $\varphi(y, x_1, \ldots, x_r)$ with coefficients in $F(a_{r+1}, \ldots, a_n)$, which we can see as the evaluation at $(a_{r+1}, \ldots, a_n)$ of a formula $\psi(y, x_1, \ldots, x_n)$.

Now let W be the subset of Υ defined as the set of types q satisfying the following conditions, with $\delta_i := \mathrm{val}(a_i(q))$

(a) $\pi(q) = r_\delta$;

(b) $\delta_i = 0$ for $r + 1 \leqslant i \leqslant r$;

(c) $\psi(b(q), a_1(q), \ldots, a_n(q))$ holds.

Then W is an F-definable subset of Υ – as far as condition (a) is concerned this is by Lemma 8.2.9 in [**HL16**], and it contains p. Now let q be a point of W. Set $b' = b(q)$ and $a' = a(q)$, and $\gamma'_i = \mathrm{val}(a'_i)$ for all i. Condition (a) and (b) ensure that $F(a')$ has a residue field which is regular over $\mathrm{res}(F(a'_{r+1}, \ldots, a'_n))$. Indeed, up to applying an invertible monomial transformation to $(a'_1, \ldots, a'_r)$ and renormalizing, we can assume that there is some s such that $\mathrm{val}(a'_1), \ldots, \mathrm{val}(a'_s)$ are free modulo $\mathrm{val}(F)$ and that $\mathrm{val}(a'_t) = 0$ for $s+1 \leqslant t \leqslant r$, in which case the result is obvious since the residue field we consider is then purely transcendental of degree $r-s$ over that of $F(a'_{r+1}, \ldots, a'_n)$.

Using the fact that $F(a'_{r+1}, \ldots, a_n) \simeq F(a_{r+1}, \ldots, a_n)$ as valued extensions of F (with a'_i corresponding to a_i) and the definition of ψ, we see that $\mathrm{val}(F(b', a')) = \mathrm{val}(F(a'))$. Otherwise said, the value group of q is generated by the $a_i(q)$ and $\mathrm{val}(F)$, which ends the proof. □

Our purpose is now to show how the results of section 5 extend quite straightforwardly, at least on affine charts, when Υ is not assumed to consist only of Zariski-generic points.

5.8. A more general setting. — We still denote by k a defectless valued field with divisible value group. Let X be an *affine* k-scheme of finite type, and let Υ be a Γ-internal subset of $X^{\#} \subseteq \widehat{X}$. Let $X_1, \ldots, X_m$ be the irreducible Zariski-closed subsets of X whose generic point supports an element of Υ; for each i, set

$$X'_i = X_i \backslash \bigcup_{j, X_j \subsetneq X_i} X_j$$

and $\Upsilon_i = \Upsilon \cap \widehat{X'_i}$. By construction, $\Upsilon = \coprod \Upsilon_i$ and for all i, Υ_i consists only in Zariski-generic points in in $\widehat{X'_i}$. We denote by $\mathbb{S}(\Upsilon)$ the set of k-definable functions of the form $\mathrm{val}(f)$ with f a regular function on X.

5.9. Proposition. — *There exists a finite set E of regular functions on X such that for every $f \in \mathbb{S}(\Upsilon)$, there exists a finite covering $(D_a)_a$ of Υ by closed definable subsets and, for each a, an element λ of k, a finite family $(e_1, \ldots, e_\ell)$ of elements of E, and a finite family $(\varepsilon_1, \ldots, \varepsilon_\ell)$ of elements of $\{-1, 1\}$ such that:*

- ◇ $\varepsilon_j = 1$ *if* e_j *vanishes on* D_a *;*
- ◇ $f = \mathrm{val}(\lambda e_1^{\varepsilon_1} \ldots e_\ell^{\varepsilon_\ell})$ *identically on* D_a.

Proof. — For all i, we can apply Theorem 5.7 to the integral scheme X_i' and the Γ-internal set Υ_i; let E_i be the finite set of rational functions on X_i' provided by this theorem. Write $E_i = \{g_{ij}/h_{ij}\}_j$ where g_{ij} and h_{ij} are non-zero regular functions on the integral affine scheme X_i. For all (i, j), let g_{ij}' and h_{ij}' denote lifts of g_{ij} and h_{ij} to the ring $\mathscr{O}_X(X)$. We then might take for E the set of all g_{ij}' and h_{ij}'. □

6. Specialisations and Lipschitz embeddings

As before, Υ is a Γ-internal subset of $X^{\#}_{\mathrm{gen}}$ for X a separated integral scheme of finite type over K. The goal of this section is to show the existence of regular embeddings of Υ in some Γ^n such that $\mathbb{S}(\Upsilon)$ becomes exactly the set of Lipschitz definable functions under certain assumptions. We begin with some definitions.

6.1. Definition. — Let $\alpha : \Upsilon \to \Gamma^n$ be a definable morphism and set $W = \alpha(\Upsilon)$.

1. We say α is *regular* if α is given by $\mathrm{val}(f)$ for a tuple of regular functions f. item If α is an embedding, then we say $\alpha^{-1} : W \to \Upsilon$ is an *integral parameterization* if for any rational function f defined on Υ, $\mathrm{val}(f) \circ \alpha^{-1}$ is piecewise $\mathbb{Z}$-affine. We will also call α integral in this case.
2. If α is an embedding, then we say $\alpha^{-1} : W \to \Upsilon$ is *Lipschitz* if for any non zero rational function f on X, $\mathrm{val}(f) \circ \alpha^{-1}$ is a Lipschitz function. We will also say α is Lipschitz.
3. We say α is a *good embedding* if it is integral and Lipschitz.

It is immediate from the definition that if $\alpha : \Upsilon \to \Gamma^n$ is a regular embedding that is Lipschitz and f is another regular function on X, then $(\alpha, f) : \Upsilon \to \Gamma^{n+1}$ is also Lipschitz. Before the discussion of good embeddings, we need to recall some basic facts about $\mathrm{ACV^2F}$ and specialisations, that will provide an important criterion for the existence of good embeddings.

6.2. $\mathrm{ACV^2F}$**-specialisations.** — We consider a triple (K_2, K_1, K_0) of fields with surjective, non-injective places $r_{ij} : K_i \to K_j$ for $i > j$, with $r_{20} = r_{10} \circ r_{21}$, such structures are also called $\mathrm{V^2F}$. The places r_{21} and r_{20} give rise to two valuations on K_2, which we denote by val_{21} and val_{20} respectively. We denote by Γ_{ij} and RES_{ij} the corresponding value groups and residue fields. We consider (K_2, K_1, K_0) as a substructure of a model of the theory $\mathrm{ACV^2F}$ introduced in [**HL16**, Chapter 9.3]. We will use K_{210} to denote the structure (K_2, K_1, K_0). It is clearly an expansion of (K_2, val_{21}) via an expansion of the residue field and an expansion of (K_2, val_{20}) by a convex subgroup in the value group. We will focus on the latter expansion.

Let X be an affine integral scheme of finite type over K_2, we will use X_{20} when we view X as a definable set in an ambient model of ACVF extending (K_2, val_{20}) and X_{21} is defined analogously. There is a natural map $s : X_{20}^{\#} \to X_{21}^{\#}$ which can be described as follows. By [**HL16**, Lemma 9.3.8], we have that p generates a complete p_{210} in $\mathrm{ACV^2F}$. By [**HL16**, Lemma 9.3.10], we see that for stably dominated types $p \in \widehat{X_{20}}$, the corresponding p_{210} as an $\mathrm{ACV^2F}$-type is stably dominated. Let $\dim(p)$ denote the dimension of the Zariski closure of p. Let $L \models \mathrm{ACV^2F}$ extending K_{210} and $c \models p|L$. Since p corresponds to an Abhyankar point in the space of valuations, we see that the residual transcendence degree of $\mathrm{tp}_{\,21}(c/L)$ is still $\dim(p)$, so $\mathrm{tp}_{\,21}(c/L)$ extends to a type $s(p)$ in $X_{21}^{\#}$. (Note that here we work in the restricted language where the only valuation is val_{21}.)

6.3. Lemma. — *Let $Y \subseteq X_{20}^{\#}$ be an $\mathrm{ACV^2F}_{K_{210}}$-definable set, then $s|_Y$ is a definable function.*

Proof. — By the way s is defined, it is a pro-definable function by considering the φ-definitions. Hence it is definable by compactness. □

We need one last lemma before stating our criterion with respect to specialisations.

6.4. Lemma. — *Let $(K_2, K_1, K_0) \models \mathrm{ACV^2F}$ and Y be a definable set of imaginaries in $\mathrm{ACVF}_{K_{21}}$. If Y is Γ_{20}-internal as a definable set in $\mathrm{ACV^2F}_{K_{210}}$, then Y is Γ-internal in K_{21}.*

Proof. — By the classification of imaginaries in ACVF [**HHM06**, Theorem 1.01], if Y is not Γ-internal in K_{21}, there is an $\mathrm{ACVF}_{K_{21}}$-definable map (possibly after expanding the language by some constants) that is generically surjective onto the residue field. By assumption, Y is Γ_{20}-internal as an $\mathrm{ACV^2F}_{K_{210}}$ set. This yields a generically surjective map $\Gamma_{20} \to \mathrm{RES}_{21}$. Composing with the dominant place $\mathrm{RES}_{21} \to \mathrm{RES}_{20}$, we obtain an $\mathrm{ACV^2F}$-definable map $\Gamma_{20} \to \mathrm{RES}_{20}$ that is generically surjective. By [**HL16**, Lemma 9.3.1(4)], one checks immediately that the two sorts Γ_{20} and RES_{20} are orthogonal in $\mathrm{ACV}^2\mathrm{F}$, hence a contradiction. □

6.5. Specialisable maps and Lipschitz condition. — Now we introduce the notion of specialisations of maps. Let (K, v) be a valued field, we denote by $\rho(K)$ the set of convex subgroups of $\Gamma(K)$. Clearly, if K is of transcendence degree m over the prime field, then $|\rho(K)| \leqslant m+1$. For each $\Delta \in \rho(K)$, we have a valuation $\mathrm{val}_{21} : K \to \Gamma(K)/\Delta$ given by quotienting out by Δ, which gives rise to a $\mathrm{V}^2\mathrm{F}$ structure we shall denote by $K[\Delta]$. Each choice of Δ specifies an expansion of ACVF_K to $\mathrm{ACV}^2\mathrm{F}_K$ by interpreting the convex subgroup to be the convex hull of Δ. Moreover, by varying Δ one exhausts all the possible expansions of ACVF_K to $\mathrm{ACV}^2\mathrm{F}_K$. Let X be an integral separated scheme of finite type over K as before. We write X_Δ to denote X as a definable set in $\mathrm{ACVF}_{K[\Delta]}$. We use s_Δ to denote the map s defined in Section 6.2 when we expand ACVF_K to $\mathrm{ACV}^2\mathrm{F}_{K[\Delta]}$. Note that the $\mathrm{ACV}^2\mathrm{F}_{K[\Delta]}$ describe all the possible expansions of $\mathrm{ACV}^2\mathrm{F}$ with constants for K. Let $\Upsilon \subseteq X^\#$ be a Γ-internal set definable over K. We use Υ_Δ to denote $s_\Delta(\Upsilon)$. Similarly, if $\alpha : \Upsilon \to \Gamma^n$ is some regular embedding. We use $\alpha_\Delta : \Upsilon_\Delta \to \Gamma_{21}^n$ to denote the corresponding map.

6.6. Definition. — Let $\alpha : \Upsilon \hookrightarrow \Gamma^n$ be a regular embedding and let K be a field over which α is defined. We say α is *specialisable* if for every convex subgroup Δ of $\Gamma(K)$ the map α_Δ is still an embedding.

6.7. Remark. — If $\alpha\colon \Upsilon \hookrightarrow \Gamma^n$ is a regular embedding and $\beta\colon \Upsilon \to \Gamma^m$ is any regular map, then $(\alpha, \beta)\colon \Upsilon \to \Gamma^{n+m}$ is a regular embedding.

6.8. Remark. — Assume $\alpha : \Upsilon \to \Gamma^n$ is specialisable and defined over K, and $\Upsilon' \subseteq \Upsilon$ is definable but not necessarily over K. If α is specialisable, so is $\alpha|_{\Upsilon'}$. Indeed, let $L \supseteq K$ be such that Υ' is defined over L, any expansion of ACVF_L to $\mathrm{ACV}^2\mathrm{F}_L$ by some Δ' gives an expansion of ACVF_K to $\mathrm{ACV}^2\mathrm{F}_K$ by some Δ. As α is specialisable, α_Δ is an embedding for any Δ, thus $\alpha|_{\Upsilon'}$ is specialisable.

6.9. Theorem. — *Let X be an affine integral scheme of finite type over a valued field K and let $\Upsilon \subseteq X^\#$ be a Γ-internal subset. Let K_0 be a finitely generated field over which all the above is defined. Then there exists a K_0^{alg}-definable regular embedding of Υ that is specialisable.*

Proof. — For each $\Delta \in \rho(K_0)$, by Lemma 6.4, we have that $\Upsilon_\Delta \subseteq X_\Delta^\#$ is Γ-internal in $\mathrm{ACVF}_{K_0[\Delta]}$.

Consider X as embedded in some affine space. We use H_d to denote the image in $\mathcal{O}_X(X)$ of the set of polynomials of degree at most d. By [**HL16**, Corollary 6.2.5], for each Δ, there is an integer $d(\Delta)$ such that for any $d \geqslant d(\Delta)$

there exist finitely many $h_i \in H_d$ providing a regular embedding of Υ_Δ in Γ^s for some s. Moreover, the functions h_i are defined over $K_0[\Delta]^{alg}$ and by [**HL16**, Lemma 6.2.2], we may assume that they form a basis in the sense that any $h \in H_d$ is w-generated by the h_i. Call such an embedding α_d. Thus if one takes d to be greater or equal to all the $d(\Delta)$, then we see that α_d is specialisable. □

6.10. Proposition. — *Let X be an affine integral scheme of finite type over a valued field K and let $\Upsilon \subseteq X^{\#}$ be a Γ-internal subset. If $\alpha : \Upsilon \hookrightarrow \Gamma^n$ is a specialisable embedding, then the image of $\mathbb{S}(\Upsilon)$ is contained in the group of Lipschitz functions. In other words, all the log-rational functions are Lipschitz and α is Lipschitz.*

Proof. — We let $W = \alpha(\Upsilon)$ and use p_w to denote $\alpha^{-1}(w)$ for $w \in W$. Assume there is some $f \in K(X)$ such that $w \mapsto p_w(f)$ is not Lipschitz. Going to an elementary extention, we may assume there is $w_1, w_2 \in W$ such that $|p_{w_2}(f) - p_{w_1}(f)| > n|w_1 - w_2|$ for all $n \in \mathbb{N}$. Take C to be the convex subgroup generated by $|w_1 - w_2|$. Consider L to be the same field with the valuation given by quotienting out by C. By our assumption on specialisability, we have that α_L is an embedding. However, we have $\overline{w_1} = \overline{w_2}$, while $\overline{p_{w_1}(f)} = p_{w_1}(f) + C \neq p_{w_2}(f) + C = \overline{p_{w_2}(f)}$, a contradiction. □

6.11. Corollary. — *Let X be an affine integral scheme of finite type over an algebraically closed valued field K and let $\Upsilon \subseteq X^{\#}$ be a Γ-internal set. Then there exists a good embedding $\Upsilon \hookrightarrow \Gamma^n$.*

Proof. — By Theorem 6.9 and Proposition 6.10 there exists a Lipschitz embedding $\alpha : \Upsilon \hookrightarrow \Gamma^n$. On the other hand, by Theorem 5.7, $\mathbb{S}(\Upsilon)$ is is finitely $(w, +)$-generated up to constants, hence by Lemma 5.3 there exists an integral embedding $\alpha' : \Upsilon \hookrightarrow \Gamma^{n'}$. Thus the concatenation α and α' provides a good embedding $\Upsilon \hookrightarrow \Gamma^{n+n'}$. □

7. The main theorem

In this section, we prove the theorem stated in Section 1.3 and we transfer it into the Berkovich setting.

7.1. Lemma. — *Let k be a valued field with infinite residue field, let X be a geometrically integral k-scheme and let $\Upsilon \subseteq X^{\#}_{\text{gen}}$ be a k-definable Γ-internal subset defined over k. The group $\mathbb{S}(\Upsilon)$ is stable under* min *and* max.

Proof. — It is enough to prove stability under min. Let p be a point of Υ. If there exists a scalar a of valuation zero such that $\mathrm{val}(f(p) + ag(p)) > \min(\mathrm{val}(f)(p), \mathrm{val}(g(p)))$ then $\mathrm{res}(a)$ is a well-defined element of the residue field which we call $\theta(p)$; otherwise we set (say) $\theta(p) = 0$. Then θ is a k-definable map from the Γ-internal set Υ to the residue field. By orthogonality between the value group and the residue sorts, θ has finite image. Since k has infinite residue field, there exists an element $a \in \mathscr{O}_k^\times$ whose residue class does not belong to the image of θ. Then $f + ag \neq 0$ and $\mathrm{val}(f(p) + ag(p)) = \min(\mathrm{val}(f(p)), \mathrm{val}(g(p)))$ for all $p \in \Upsilon$. □

In the situation of the lemma above, it thus makes sense to speak about an $(\ell, +)$-generating system of elements of $\mathbb{S}(\Upsilon)$. As for $(w, +)$-generation, we shall say for short that $\mathbb{S}(\Upsilon)$ is finitely $(\ell, +)$-generated up to the constant functions if there exists a finite subset E of $\mathbb{S}(\Upsilon)$ such that E and the k-definable constant functions (i.e., the constant functions taking values in $\mathbb{Q} \otimes \mathrm{val}(k^\times)$) $(\ell, +)$-generate $\mathbb{S}(\Upsilon)$.

7.2. Theorem. — *Let k be an algebraically closed valued field. Let X be an integral scheme of finite type over k and let $\Upsilon \subseteq X^\#_{\mathrm{gen}}$ be a Γ-internal subset defined over k. The group $\mathbb{S}(\Upsilon)$ is stable under* min *and* max *and is finitely $(\ell, +)$-generated up to constant functions.*

Proof. — By Theorem 6.9, there is a k-definable good embedding $\alpha : \Upsilon \to \Gamma^n$ for some n. By Theorem 5.7, $\mathbb{S}(\Upsilon)$ is $(w, +)$-finitely generated up to constant functions. Let $f_1, \dots, f_m$ be finitely many k-rational functions whose valuations $(w, +)$-generate $\mathbb{S}(\Upsilon)$ up to constant functions, adjoining the $\mathrm{val}(f_i)$ as new coordinates of α, we may furthermore assume that $\mathbb{S}(\Upsilon)$ is $(w, +)$-generated by the components of α and the constant functions. By possibly enlarging once again α and replacing X with a suitable dense Zariski-open subset we can also assume that $\alpha = \mathrm{val}(f)$ for some closed immersion $f \colon X \to \mathbf{G}_{\mathrm{m}}^n$; in particular, α is definably proper and induces a definable homeomorphism $\Upsilon \simeq \alpha(\Upsilon)$.

Let f in $\mathbb{S}(\Upsilon)$. Since α is a specialisable embedding whose coordinates $(w, +)$-generate $\mathbb{S}(\Upsilon)$ up to the constant functions, the composition $f \circ \alpha^{-1}$ viewed as a Γ-valued function on $\alpha(\Upsilon)$ is piecewise $\mathbb{Z}$-affine and Lipschitz. In view of Theorem 3.13, this implies that $f \circ \alpha^{-1}$ is an ℓ-combination of finitely many $\mathbb{Z}$-affine functions, so that f itself is an $(\ell, +)$-combination of the components of α and of constant functions. □

7.3. Remark. — Assume that k is algebraically closed and let $(f_1, \dots, f_n)$ be a family of rational functions on X such that $\mathbb{S}(\Upsilon)$ is $(w, +)$-generated (resp. $(\ell, +)$-generated) by the $\mathrm{val}(f_i)$ and the constant (k-definable) functions. Then for every algebraically closed extension L of k, the $\mathrm{val}(f_i)$ and the L-definable constant functions $(w, +)$-generate (resp. $(\ell, +)$-generate) $\mathbb{S}_L(\Upsilon)$ (work with a bounded family of rational functions and use compactness).

Our purpose is now to state and prove the Berkovich avatar of our main theorem. We fix a non-archimedean complete field F. For all n, we denote by $S_{F,n}$ the closed subset $\{\eta_r\}_{r\in(\mathbf{R}_+^\times)^n}$ of $\mathbf{G}_{\mathrm{m},F}^{n,\mathrm{an}}$, where η_r is the semi-norm $\sum a_I T \mapsto \max|a_I|r^I$.

In [**Duc12**], 4.6 a general notion of a skeleton is defined for an F-analytic space; the subset $S_{n,F}$ of $\mathbf{G}_{\mathrm{m},F}^{n,\mathrm{an}}$ is the archetypal example of such an object.

But this notion is however slightly too analytic for our purposes here: indeed, if X is an algebraic variety over F then X^{an} might have plenty of skeleta in the sense of [**Duc12**] that cannot be handled by our methods, since they would not correspond to any Γ-internal subset of $\widehat{X}$, by lack of algebraic definability. For instance, assume that F is algebraically closed and non-trivially valued, and let f be any non-zero analytic function of $\mathbf{A}_F^{1,\mathrm{an}}$ with countably many zeroes. Let U be the non-vanishing locus of f, and let Σ be the preimage of $S_{1,F}$ under $f\colon U \to \mathbf{G}_{\mathrm{m}}$. Then Σ is a skeleton in the sense of [**Duc12**], but topologically this is only a locally finite graph, with countably many branch points. We thus shall need to focus on “algebraic” skeleta.

7.4. Theorem. — *Let us assume that F is algebraically closed. Let X be an integral F-scheme of finite type, and let n be its dimension. Let $\varphi_1,\ldots,\varphi_r$ be maps $U_i \to \mathbf{G}_{\mathrm{m},F}^n$ where the U_i are dense open subsets of X, and let $S \subseteq X^{\mathrm{an}}$ be a subset of $\bigcup_i \varphi_i^{-1}(S_n)$ defined by a Boolean combination of norm inequalities between non-zero rational functions.*

There exist finitely many non-zero rational functions $f_1,\ldots,f_m$ on X such that the following hold.

(1) *The functions $\log|f_1|,\ldots,\log|f_m|$ identify S with a piecewise-linear subset of $\mathbf{R}^m$ (i.e., a subset defined by a Boolean combination of inequalities between $\mathbb{Q}$-affine functions).*

(2) *The group of real-valued functions on S of the form $\log|g|$ for g a non-zero rational function on X is stable under* min *and* max *and is $(\ell,+)$-generated by the $\log|f_i|$ and the constant functions of the form $\log|\lambda|$ with $\lambda\in F^\times$.*

Proof. — The subset $\widehat{S}(L)$ of $\widehat{X}$ given by the same definition as S mutatis mutandis is a Γ-internal set containd in $X^{\#}_{\mathrm{gen}}$ to which we can thus apply Theorem7.2. The theorem above then follows by noticing that if L denotes a non-archimedean maximally complete extension of F with value group the whole of $\mathbf{R}_+^\times$, then S is naturally homeomorphic to $\widehat{S}(L)$. □

7.5. Remark. — Note that by Proposition 4.6 the condition that S is a subset of some $\bigcup_i \varphi_i^{-1}(S_n)$ holds as soon as S is the image of $\Upsilon(L)$ under the projection $\widehat{X}(L) \to X^{\mathrm{an}}$ with Υ some F-definable Γ-internal subset of $X^{\#}_{\mathrm{gen}}$ and L as in the above proof.

7.6. Remark. — We insist that we require that the ground field be algebraically closed. Indeed, our theorems (for stable completions as well as for Berkovich spaces) definitely do not hold over an arbitrary non-Archimedean field, *even in a weaker version with* $(w,+)$*-generation instead of* $(\ell,+)$*-generation*, as witnessed by a counter-example that was communicated to the authors by Michael Temkin. For the reader's convenience we will first detail the original counter-example which is written in the Berkovich's language, and then a model-theoretic variant thereof in the Hrushovski-Loeser's language.

7.6.1. *The Berkovich version.* — Let F be a non-archimedean field and let $\mathbb{F}$ be the completion of an algebraic closure of F ; assume that the residue field of F is of positive characteristic p and that F admits an immediate extension L of degree p, say $L := F(\alpha)$ with $\alpha \in \mathbb{F}$. By general valuation theory, the distance r between α and F is not achieved.

For every $s \geqslant r$ let ξ_s be the image on $\mathbf{P}^{1,\mathrm{an}}_F$ of the Shilov point of the closed $\mathbb{F}$-disc with center α and radius s. If $s > r$ there exists β_s in F with $|\alpha - \beta_s| \leqslant s$, and ξ_s is the Shilov point of the closed F-disc with center β_s and radius s; but as far as ξ_r is concerned, it is the Shilov point of an affinoid domain V of $\mathbb{P}^{1,\mathrm{an}}_F$ without rational point.

Let v be a rigid point of V. It corresponds to an element ω of $\mathbb{F}$ algebraic over F and whose distance to F is equal to r and not achieved. Therefore the extension $F(\omega)$ has defect over F, which forces its degree to be divisible by p. Otherwise said, $[\mathscr{H}(v) : F]$ is divisible by p.

In particular if f is any non-zero element of $F(T)$, the divisor of $f|_V$ has degree divisible by p, so that there exists some $s > r$ such that the slope of $\log|f|$ on (ξ_r, ξ_s) is divisible by p.

Now assume that there exists a finite set E of non-zero rational functions such that on the skeleton $[\xi_r, \infty)$, every function of the form $\log|g|$ with g in $F(T)^\times$ belongs piecewise to the group generated by the $\log|h|$ for $h \in E \cup F^\times$. Then there would exist some $s > r$ such that for every g as above, all slopes of $(\log|g|)|_{[\xi_r,\xi_s]}$ are divisible by p. Taking $g = T - \beta_s$ yields to a contradiction.

7.6.2. *The model-theoretic version.* — Let F be a perfect valued field of positive residue characteristic p such that there exists an irreducible separable polynomial $P \in F[T]$ with the following property: *the smallest closed ball containing all roots of* P *has no* F*-rational points, but any bigger* F*-definable closed ball has one* F*-point* (it is not difficult to exhibit such pairs (F, P); the easiest case is that of pure characteristic p, where one can take any perfect field F with an height 1 valuation having an element s with $\mathrm{val}(s) < 0$ such that $T^p - T - s$ has no root in F, and take $P =: T^p - T - s$; for instance, the perfect closure of $\mathbf{F}_p(s)$ equipped with the $(1/s)$-adic valuation will do the job).

Let b be the smallest closed ball containing the roots of P, and let B be a bigger F-definable closed ball. Let I be the interval between their generic points; this is a Γ-internal subset of $\widehat{\mathbf{P}^1}$ contained in $\mathbf{P}^{1,\#}_{\text{gen}}$. This interval is naturally parameterized by the interval $[V, v]$ where V is the valuative radius of B and v is that of b, and we will identify them. In particular a linear function from I to Γ has a well-defined slope. We will be interested in the germ of functions on I towards the endpoint v. The number of roots in b of every irreducible polynomial of $F[T]$ is divisible by p, for otherwise averaging the roots would produce an F-rational point in b. Hence every polynomial, and indeed every rational function in $F(T)$, has slope divisible by p on some interval (i, v) inside I. If the group of functions $\text{val}(f)|I$ were finitely $(w, +)$-generated m up to constants, there would be a fixed $i < 0$ (defined over F) such that all val-rational functions have slope divisible by p on $[i, v]$. Now pick an F-rational point a in the closed ball containing b of valuative radius $(i + v)/2$; then $T - a$ has slope one on $((i + v)/2, v]$, contradiction.

8. Applications to (motivic) volumes of skeleta

It follows from Theorem 5.7 on finite $(w, +)$-generation that skeleta are endowed with a canonical piecewise $\mathbb{Z}$-affine structure. In this section we explain how this implies the existence of canonical volumes for skeleta.

8.1. Some Grothendieck semirings of Γ. — We shall consider various Grothendieck semirings of Γ analogous to those introduced in §9 of [**HK06**] (see also [**HK08**] for a detailed study of the rich structure of such semirings). Let Γ be a non-trivial divisible ordered abelian group and let A be a fixed subgroup of Γ. We work in the theory DOAG_A of (non-trivial) divisible ordered Abelian groups with distinguished constants for elements of the subgroup A. Fix a non negative integer N. One defines a category $\Gamma(N)$ as follows (since there is no risk of confusion we omit the A from the notation). An object of $\Gamma(N)$ is a finite disjoint union of subsets of Γ^N defined by linear equalities and inequalities with $\mathbb{Z}$-coefficients and constants in A. A morphism f between to two objects X and Y of $\Gamma(N)$ is a bijection such that there exists a finite partition $X = \bigcup_{1 \leqslant i \leqslant r} X_i$ with X_i in $\Gamma(N)$, matrices $M_i \in \text{GL}_N(\mathbb{Z})$ and constants $a_i \in A^N$, such that for $x \in X_i$, $f(x) = M_i x + a_i$. We denote by $K_+(\Gamma(N))$ the Grothendieck semigroup of this category, that is the free abelian semigroup generated by isomorphism classes of objects of $\Gamma(N)$ modulo the cut and past relation. The inclusion map $\Gamma^N \to \Gamma^{N+1}$ given by $x \mapsto (x, 0)$ induces an inclusion functor $\Gamma(N) \to \Gamma(N + 1)$ and we denote by $\Gamma(\infty)$ the colimit of the categories $\Gamma(N)$. We may identify the Grothendieck semigroup $K_+(\Gamma(\infty))$ of

$\Gamma(\infty)$ with the colimit of the semigroups $K_+(\Gamma(N))$. It is endowed with a natural structure of a semiring. We may also consider the full subcategory $\Gamma^{\mathrm{bdd}}(N)$ of $\Gamma(N)$ consisting of bounded sets, that is definable subsets of $[-\gamma, \gamma]^N$ for some non negative $\gamma \in \Gamma$ (which can be chosen in A), and the corresponding full subcategory $\Gamma^{\mathrm{bdd}}(\infty)$ of $\Gamma(\infty)$ and its Grothendieck semiring. The above categories admit natural filtrations $F^{\cdot}$ by dimension, with F^n the subcategory generated by objects of o-minimal dimension $\leqslant n$ and we will also consider the induced filtration on Grothendieck rings.

8.2. Volumes. — Let R be a real closed field. Fix integers $0 \leqslant n \leqslant N$. Let W be a bounded piecewise $\mathbb{Z}$-linear definable subset of R^N of o-minimal dimension n. We denote by $\mathrm{vol}_n(W)$ its n-dimensional volume which can defined in the following way. After decompositing into simplices, it is enough to define the volume of a simplex spanned by $n+1$-points, which one can do via the classical formula over $\mathbb{R}$, chosing the normalization such that, for any family $(e_1, \dots, e_n)$ of n vectors in R^N with integer coordinates which can be completed to a basis of the abelian group $\mathbb{Z}^N$, the volume of the simplex with vertices the origin and the endpoints of $e_1, \dots, e_n$ is $\frac{1}{n!}$. When $R = \mathbb{R}$, vol_n is well-defined thanks to the existence of the Lebesgue measure. In general, the well-definedness of vol_n follows from the case of $\mathbb{R}$ since after increasing R one can assume it is an an elementary extension of $\mathbb{R}$.

Thus, for any embedding $\beta : A \to R$ with R a real closed field and any integer n, vol_n induces a morphism $\mathrm{vol}_{n,\beta} : F^n K_+^{\mathrm{bdd}}(\Gamma(N))/F^{n-1} K_+^{\mathrm{bdd}}(\Gamma(N)) \to R$ which stabilizes to a morphism $\mathrm{vol}_{n,\beta} : F^n K_+^{\mathrm{bdd}}(\Gamma(\infty))/F^{n-1} K_+^{\mathrm{bdd}}(\Gamma(\infty)) \to R$.

8.3. Motivic volumes of skeleta. — Let us assume that we are in the setting of Theorem 5.7, that is, k is a defectless valued field with divisible value group, X is an n-dimensional integral k-scheme of finite type and Υ is a Γ-internal subset of $X^{\#}_{\mathrm{gen}} \subseteq \widehat{X}$. Then, by Theorem 5.7, $\mathbb{S}(\Upsilon)$ is finitely $(w,+)$-generated up to constant functions. Let $\alpha : \Upsilon \to \Gamma^N$ be a definable embedding of the form $(\mathrm{val}(f_1), \cdots, \mathrm{val}(f_N))$ where the functions $\mathrm{val}(f_i)$ are $(w,+)$-generating $\mathbb{S}(\Upsilon)$ up to constant functions. We take for A the group $\mathrm{val}(k^\times)$.

8.4. Proposition. — *The class of $\alpha(\Upsilon)$ in $K_+(\Gamma(\infty))$ does not depend on α.*

Proof. — Consider $\alpha' : \Upsilon \to \Gamma^{N'}$ another definable embedding of the form $(\mathrm{val}(f'_1), \cdots, \mathrm{val}(f'_{N'}))$ with the functions $\mathrm{val}(f'_i)$ $(w,+)$-generating $\mathbb{S}(\Upsilon)$ up to constant functions. After adding zeroes we may assume $N = N'$. Since the functions $\mathrm{val}(f_i)$ are $(w,+)$-generating $\mathbb{S}(\Upsilon)$ up to constant functions, there exists a finite partition of Υ into definable pieces Υ_j such that on each Υ_j we may write $(\mathrm{val}(f_i')) = M_j((\mathrm{val}(f_i))) + a_j$ with M_j a matrix with coefficients in $\mathbb{Z}$ and $a_j \in \Gamma^N$. Exchanging α and α' we get that the matrix M_j lies in $\mathrm{GL}_N(\mathbb{Z})$. □

Thus, to any Γ-internal subset Υ of $X^{\#}_{\mathrm{gen}} \subseteq \widehat{X}$, we may assign a well defined motivic volume $\mathrm{MV}(\Upsilon)$ in the ring $K_+(\Gamma(\infty))$, namely the class of $\alpha(\Upsilon)$ for any embedding α as above.

If Υ is contained in a definably compact set, $\alpha(\Upsilon)$ is bounded, thus $\mathrm{MV}(\Upsilon)$ lies in $F^n K_+^{\mathrm{bdd}}(\Gamma(\infty))$ and we can consider its n-dimensional volume $\mathrm{vol}_{n,\beta}(\mathrm{MV}(\Upsilon))$ in R for any embedding $\beta : \Gamma \to R$ with R a real closed field. Similarly, any definable subset of Υ of o-minimal dimension $m \leqslant n$ contained in a definably compact set has an m-dimensional volume in R.

8.5. Berkovich variants. — These constructions admit direct variants in the Berkovich setting which are transfered from the previous section 8.3 similarly as in the proof of Theorem 7.4.

Fix an algebraically closed non-archimedean complete field F with value group A. Let X be an integral F-scheme of finite type and of dimension n. Let $S \subseteq X^{\mathrm{an}}$ be an algebraic skeleton as in the statement of Theorem 7.4. Then one can assign similarly as above a well defined class $\mathrm{MV}(S)$ to S in in $K_+(\mathbb{R}(\infty))$. Furthermore, if S is relatively compact, since $A \subseteq \mathbb{R}$, one can consider its n-dimensional volume $\mathrm{vol}_n(\mathrm{MV}(S))$ in $\mathbb{R}$, or more generally its m-dimensional volume if S of dimension $\leqslant m$.

8.6. Remark. — Note that all the invariants defined above (motivic and actual volumes) are invariant under birational automorphisms and Galois actions.

Appendix A. Abhyankar valuations are defectless: a model-theoretic proof

Let K be a field equipped with a Krull valuation v and let L be a finite extension of K. Let $v_1, \ldots, v_n$ be the valuations on L extending v, and for every i, let e_i and f_i be the ramification and inertia indexes of the valued field extension $(K, v) \hookrightarrow (L, v_i)$. One always has $\sum e_i f_i \leqslant [L : F]$, and the extension L of the valued field (K, v) is said to be *defectless* if the equality holds. We shall say that (K, v) is *defectless* if every finite extension of it is defectless (such a field is also often called *stable* in the litterature, but we think that defectless is a better terminology, if only because stable has a totally different meaning in model theory).

We shall use here the notion of the *graded* residue field of a valued field in the sense of Temkin, see [**Tem04**] (we will freely apply the basic facts about graded commutative algebra which are proved therein). A more model-theoretic approach of the latter was introduced independently by the second author and Kazhdan in [**HK06**] with the notation $\mathrm{RV}(\cdot)$ which we have decided to adopt here. The key point making this notion relevant for the study of defect is the following obvious remark: the product $e_i f_i$ can also be interpreted as the degree of the graded residue extension $\mathrm{RV}(K, v) \hookrightarrow \mathrm{RV}(L, v_i)$.

Examples. Any algebraically closed valued field is defectless; any complete discretely valued field is defectless ; the function field of an irreducible smooth algebraic curve, endowed with the discrete valuation associated to a closed point of the curve, is defectless; any valued field whose residue characteristic is zero is defectless.

The purpose of this appendix is to give a new proof of the following well-known theorem.

A.1. Theorem. — *Let (K, v) be a defectless valued field, and let G be an abelian ordered group containing $v(K^\times)$. Let $g = (g_1, \ldots, g_n)$ be a finite family of elements of G. Endow $K(T) = K(T_1, \ldots, T_n)$ with the "Gauss extension v_g of v with parameter g", i.e.*

$$v_g(\sum a_I T^I) = \min_I v(a_I) + Ig.$$

The valued field $(K(T), v_g)$ is still defectless.

This result has been given several proofs by Gruson, Temkin, Ohm, Kuhlmann, Teissier (see [**Gru68**], [**Tem10**], [**Ohm89**], [**Kuh10**], [**Tei14**]). To our knowledge, the first proof working in full generality was that of Kuhlmann, the preceeding proofs requiring some additional assumptions on K and/or on the g_i.

Proof. — It is rather long, with several steps.

A.1.1. *First easy reduction.* — By a straightforward induction argument, we reduce to the case where $n = 1$, and we write now T instead of T_1 and g instead of g_1.

A.1.2. *Reduction to the case where K is algebraically closed.* — We choose an arbitrary extension w of v to an algebraic closure $\overline{K}$ of K, and we endow the field $\overline{K}(T)$ with the Gauss valuation w_g. We assume that $(\overline{K}(T), w_g)$ is defectless, and we want to prove that $(K(T), v_g)$ is defectless too; this is the step in which our defectlessness assumption on K will be used. So, let F be a finite extension of $K(T)$, and let us prove that it is defectless.

We begin with a general remark which we will use several times. Let K' be a finite extension of K. For every extension v' of v on K' there is a unique extension of v_g on $K'(T)$ whose restriction to K' coincides with v', namely the Gauss valuation v'_g (indeed, for such an extension the $\mathrm{RV}(T_i)$ will be algebraically independent over $\mathrm{RV}(K')$, so this extension is necessarily a Gauss extension of v'). Then it follows by a direct explicit computation that

$$\mathrm{RV}(K'(T)) = \mathrm{RV}(K(T)) \otimes_{\mathrm{RV}(K)} \mathrm{RV}(K'),$$

(where K' is endowed with v' and $K'(T)$ with v'_g) which implies that $K'(T)$ is a defectless extension of $K(T)$.

Let us first handle the case where F is separable over $K(T)$. Let K' be the separable closure of K in F. By the remark above, $K'(T)$ is a defectless extension of $K(T)$, and it is therefore sufficient to prove that F is a defectless extension of $K'(T)$, thus we can assume that $K' = K$. The tensor product $L := \overline{K} \otimes_K F$ is then a field, and L is a defectless extension of $\overline{K}(T)$ since $\overline{K}(T)$ is defectless by assumption. Let $w_1, \ldots, w_d$ be the extensions of w_g to L; for every i, let L_i be the graded field $\mathrm{RV}(L, w_i)$. We have by assumption

$$[F : K(T)] = [L : \overline{K}(T)] = \sum_i [\mathrm{RV}(L_i) : \mathrm{RV}(\overline{K}(T))].$$

Now each $\mathrm{RV}(L_i)$ is a finite extension of $\mathrm{RV}(\overline{K}(T))$, so it is defined over $\mathrm{RV}(E(T))$ for E a suitable finite extension of K contained in $\overline{K}$, which can be chosen to work for all i. Let us set

$$E_i = \mathrm{RV}(F \otimes_K E, w_i|_{F\otimes_K E}).$$

By construction, E_i ontains a graded subfield of degree $[\mathrm{RV}(L_i) : \mathrm{RV}(\overline{K}(T))]$ over $\mathrm{RV}(E(T))$, so that we have

$$[F\otimes_K E : E(T)] = [F : K(T)] = \sum_i [\mathrm{RV}(L_i) : \mathrm{RV}(\overline{K}(T))] \leqslant \sum_i [E_i : \mathrm{RV}(E(T))].$$

Then

$$[F \otimes_K E : E(T)] = \sum_i [E_i : \mathrm{RV}(E(T))]$$

and $F \otimes_K E$ is a defectless extension of $E(T)$. Moreover, $E(T)$ is a defectless extension of $K(T)$ by the remark at the beginning of the proof. Therefore $F \otimes_K E$ is a defectless extension of $K(T)$ as well, which in turn forces F to be defectless over $K(T)$. We thus are done when F is separable over $K(T)$.

Now let us handle the general case. In order to prove that F is defectless over $K(T)$ we may enlarge F, and so we can assume that it is normal over $K(T)$. Let F_0 be the subfield of F consisting of Galois-invariant elements. This is a purely inseparable extension of $K(T)$, and F is separable (and even Galois) over F_0. Since F_0 is a finite extension of $K(T)$, it is contained in $K_0(T^{1/p^m})$ for some integer m and some purely inseparable finite extension K_0 of K (indeed, if $f \in K(T)$ then for every ℓ the p^ℓ-th root f^{1/p^ℓ} is contained in the radicial extension generated by T^{1/p^ℓ} and the p^ℓ-th roots of the coefficients of f).

It is now sufficient to prove that $F \otimes_{F_0} K_0(T^{1/p^m})$ (which is a field since F and $K_0(T^{1/p^m})$ are respectively separable and purely inseparable over F_0) is defectless over $K(T)$. But $F \otimes_{F_0} K_0(T^{1/p^m})$ is separable over $K_0(T^{1/p^m})$, so it is defectless over $K_0(T^{1/p^m})$ by the above; and $K_0(T^{1/p^m})$ is defectless over $K(T)$ by direct computation, resting on the fact that K_0 is defectless over K, which ends this first step.

We thus may and do assume from now on that K is algebraically closed.

A.1.3. *Reduction to the case of a rational radius.* — Let F be a finite extension of $K(T)$. We want to prove that F is a defectless over the valued field $(K(T), v_g)$ and our purpose now is to reduce to the case where g belongs to $v(K^\times)$.

Let us fix a non-trivially valued, algebraically closed extension L of K whose value group is contained in G and contains g; let v_L denote the valuation of L. We are going to prove that F_L is defectless over $(L(T), v_{L,g})$ if and only if F is defectless over $K(T)$, which will allow to replace (K, v) with (L, v_L) and thus assume that $g \in v(K^\times)$.

Let w be any extension of $v_{L,g}$ to $F_L := F \otimes_{K(T)} L(T)$; in what follows, $L(T)$ and its subfields are understood as endowed with (the restriction of) w. Let E be a finite dimensional K-vector subspace of F. Using (part of) the work [**HHM06**] by Haskell, Hrushovski and Macpherson on the elimination of imaginaries in the theory ACVF, together with some further results in [**HL16**], one gets the following:

(1) the restriction of w to $L \otimes_K E$ is a norm which is definable with parameters in $K \cup \{g\}$;

(2) as a consequence of (1), there exists a basis $e_1, \ldots, e_d$ of E over K and elements $h_1, \ldots, h_d$ of $v(K^\times) \oplus \mathbb{Q}g$ such that $w(\sum a_i e_i) = \min v(a_i) + h_i$ for every d-uple $(a_i) \in L^d$.

The formula given in (2) immediately implies that the *graded* reduction $\mathrm{RV}(L \otimes_K E)$ is equal to $\mathrm{RV}(L) \otimes_{\mathrm{RV}(K)} \mathrm{RV}(E)$. By a limit argument it follows that $\mathrm{RV}(F_L)$ is nothing but the graded fraction field of $\mathrm{RV}(L) \otimes_{\mathrm{RV}(K)} \mathrm{RV}(F)$. As $\mathrm{RV}(L(T))$ is itself equal by a direct computation to the graded fraction field of $\mathrm{RV}(L) \otimes_{\mathrm{RV}(K)} \mathrm{RV}(K(T))$, we eventually get

$$\mathrm{RV}(F_L) = \mathrm{RV}(L(T)) \otimes_{\mathrm{RV}(K(T))} \mathrm{RV}(F).$$

In particular we have the equality

$$(*) \quad [\mathrm{RV}(F_L) : \mathrm{RV}(L(T))] = [\mathrm{RV}(F) : \mathrm{RV}(K(T)].$$

This holds for all extensions w of $v_{L,g}$ to F_L (we remind that w is implicitly involved in the above equality). Let E, resp. E_L, be the set of extensions of v_g to F, resp. of $v_{L,g}$, to F_L. There is a natural restriction map from E_L to E.

Now since K is algebraically closed, it follows from Theorem 2.8 (2) of [**Duc12**] that there exists a finite subset S of F that separates universally (i.e., this still holds after an arbitrary ground field extension) the extensions of Gauss valuations of $K(T)$ to F. In particular, S separates the extensions of v_g to F as well as the extensions of $v_{L,g}$ to F_L; it follows that $E_L \to E$ is injective. We claim that $E_L \to E$ is surjective. Indeed, to see this, we may enlarge F and assume it is Galois over $K(T)$. Now let $\omega \in E$ and let w be an arbitrary element of E_L. The restriction $w|_F$ belongs to E, so is equal to $\omega \circ \varphi$ for some $\varphi \in \mathrm{Gal}(F/K(T))$. Then $w \circ \varphi^{-1}$ is a preimage of ω in E_L.

Therefore $E_L \to E$ is bijective. In view of $(*)$ above, this implies that F is a defectless extension of $(K(T), v_g)$ if and only if F_L is a defectless extension of $(L(T), v_{L,g})$, as announced.

Hence we may and do assume from now on that $g \in v(K^\times) \neq \{0\}$ *(and that* K *is still algebraically closed).*

A.1.4. *Some specialisations.* — Let $h \in v(K^\times)$. Let us choose $\lambda \in K$ such that $v(\lambda) = h$ and let τ be the image of T/λ in the residue field k of $(K(T), v_h)$; note that $k = \mathsf{res}(K)(\tau)$, and that τ is transcendental over $\mathsf{res}(K)$. Let h^- and h^+ be elements of an abelian ordered group containing $v(K^\times)$ which are infinitely close to b (with respect to $v(K^\times)$), with $h^- < h < h^+$. The valuation v_{h^-}, resp. v_{h^+} is the composition of v_h and of the discrete valuation u_∞, resp. u_0, of k that corresponds to $\tau = \infty$, resp. $\tau = 0$, and the extensions of v_{h^-}, resp. v_{h^+}, to F are compositions of extensions of v_h and of extensions of u_∞, resp. u_0. Since (k, u_0) and (k, u_∞) are defectless, we see that the following are equivalent :

(i) F is a defectless extension of $(K(T), v_{h^-})$;
(ii) F is a defectless extension of $(K(T), v_h)$;
(iii) F is a defectless extension of $(K(T), v_{h^+})$.

In the same spirit, let θ be an element of an abelian ordered group containing $v(K^\times)$ and larger than any element of $v(K^\times)$. The valuation v_θ is the composition of the discrete valuation ω of $K(T)$ corresponding to the closed point $T = 0$ and of the valuation of K. Since both (K, v) and $(K(T), \omega)$ are defectless, $(K(T), v_\theta)$ is defectless; in particular, F is a defectless extension of $(K(T), v_\theta)$.

A.1.5. *The core of the proof.* — Let X be an irreducible, smooth, projective curve over K whose function field is isomorphic to F, and such that $K(T) \hookrightarrow F$ is induced by a finite map $f\colon X \to \mathbf{P}^1_K$; the latter induces a map $\widehat{f}\colon \widehat{X} \to \widehat{\mathbf{P}^1_K}$. It follows from Riemann-Roch that there exists a line bundle $\mathscr{L}$ on X such that the quotients s/t for s and t running through the set of non-zero global sections of $\mathscr{L}$ generates $K(X)^\times$ universally (see [**HL16**], 7.1; this is the key input for the proof therein that $\widehat{X}$ is definable, and not merely pro-definable).

We keep the notation $h, \lambda, etc.$ from above. Let x be a preimage of v_h in $\widehat{X}(K)$, and let β be a branch emanating from x above the branch pointing from v_h toward the point at infinity of $\mathbf{P}^1(K)$. The branch β corresponds to some valuation u_β on the residue field $\mathsf{res}(K(x))$ lying over u_∞. The ramification index of u_β over u_∞ is then the largest integer $N > 0$ (necessarily bounded above by $[F : K(T)]$) for which there exists a function $f \in K(X)^\times$ "whose slope along β is equal to $\pm\frac{1}{N}$"; that is, it satisfies the equality $\mathrm{val}(f) = \pm\frac{1}{N}\mathrm{val}(T)$ on a small enough section of β.

Let N be a non-zero integer with prime decomposition $N = \prod p_i^{n_i}$. There exists $f \in K(X)^\times$ with slope $\pm 1/N$ on a section of β if and only if there exists for every i two non-zero sections s and t of $\mathscr{L}$ such that the denominator of the slope of s/t along β is a multiple of $p_i^{n_i}$ (note that the slope of such s/t has a denominator bounded by $[F : K(T)]$ and that it is uniformly bounded since $\mathrm{H}^0(X, \mathscr{L})$ is finite-dimensional).

It follows immediately that the set of points $h \in v(K^\times)$ such that condition (i) of A.1.4 is fulfilled is K-definable, and even universally K-definable; i.e., there exis a K-definable subset D of Γ such that for every model (L, w) of ACVF containing K and every $h \in w(L^\times)$, the extension F_L of $(L(T), w_{h^-})$ is defectless if and only if $h \in D(L)$; this also means by equivalence (i) $\iff$ (iii) in A.1.4 that the extension F_L of $(L(T), w_h)$ is defectless if and only if $h \in D(L)$.

By o-minimality, we can write D as a finite disjoint union of intervals (each of them both open and closed in D) $I_1 \coprod \ldots \coprod I_m$ with endpoints in $v(K^\times) \cup \{-\infty, +\infty\}$.

A.1.6. *Study of I_m.* — Let (L, w) be a model of ACVF containing K and such that $w(L^\times)$ contains an element θ larger than every element of $w(K^\times)$. By A.1.4, the extension F of $(K(T), v_\theta)$ is defectless. By A.1.3, this implies that the extension F_L of (L, w_θ) is defectless. Therefore $\theta \in D(L)$. This implies that $m \geqslant 1$ and that the upper bound of I_m is equal to $+\infty$.

The interval I_m is thus of the form $[b, -\infty)$, $(b, +\infty)$ or $(-\infty, +\infty)$. We will exclude $[b, -\infty)$ and $(b, +\infty)$, which will show that $D = \Gamma$ and in particular that $g \in D(K)$, meaning that F is a defectless extension of $(K(T), v_g)$.

A.1.7. *End of the proof.* — Assume that I_m is equal to $[b, -\infty)$ or $(b, +\infty)$ with $b \in v(K^\times)$. Choose a model (L, w) of ACVF containing K and such that there exist elements b^- and b^+ as in A.1.4 in $w(L^\times)$. We then have $b^+ \in I_m(L)$, which means that the extension F_L of $(L(T), v_{b^+})$ is defectless. By A.1.3 this implies that F is a defectless extension of $(K(T), v_{b^+})$ as well. By A.1.4 F is then a defectless extension of $(K(T), v_b)$; therefore $b \in D(K)$ and $I_m = [b, +\infty)$. This implies the existence of $c < b$ such that $I_j \subseteq (-\infty, c]$ for every $j < m$.

Since F is a defectless extension of $(K(T), v_b)$, by using again A.1.4 , we see that F is a defectless extension of $(K(T), v_{b^-})$. By A.1.3, the extension F_L

of $(L(T), w_{b^-})$ is defectless. Hence $b^- \in D(L)$, but there is a contradiction since $D \cap (c, b) = \varnothing$. □

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