BULK UNIVERSALITY AND CLOCK SPACING OF ZEROS FOR ERGODIC JACOBI MATRICES WITH A.C. SPECTRUM

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Abstract. By combining some ideas of Lubinsky with some soft analysis, we prove that universality and clock behavior of zeros for OPRL in the a.c. spectral region is implied by convergence of $\frac{1}{n}K_n(x,x)$ for the diagonal CD kernel and boundedness of the analog associated to second kind polynomials. We then show that these hypotheses are always valid for ergodic Jacobi matrices with a.c. spectrum and prove that the limit of $\frac{1}{n}K_n(x,x)$ is $\rho_\infty(x)/w(x)$ where ρ_∞ is the density of zeros and w is the a.c. weight of the spectral measure.

1. Introduction

Given a finite measure, $d\mu$, of compact and not finite support on $\mathbb{R}$, one defines the orthonormal polynomials, $p_n(x)$ (or $p_n(x,d\mu)$ if the μ-dependence is important), by applying Gram–Schmidt to $1, x, x^2, \dots$. Thus, p_n is a polynomial of degree exactly n with leading positive coefficient so that

$$\int p_n(x)p_m(x)\,d\mu(x) = \delta_{nm} \tag{1.1}$$

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See [43, 15, 41] for background on these OPRL (orthogonal polynomials on the real line).

Associated to μ is a family of Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$, $a_n > 0$, b_n real, determined by the recursion relation ($p_{-1}(x) \equiv 0$)

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x) \tag{1.2}$$

The $\{p_n(x)\}_{n=0}^\infty$ are an orthonormal basis of $L^2(\mathbb{R}, d\mu)$ (since $\operatorname{supp}(d\mu)$ is compact) and (1.2) says that multiplication by x is given in this basis by the tridiagonal Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \cdots \\ a_1 & b_2 & a_2 & \cdots \\ 0 & a_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.3}$$

If we restrict (as we normally will) to μ normalized by $\mu(\mathbb{R}) = 1$, then μ can be recovered from J as the spectral measure for the vector $(1, 0, 0, \dots)^t$. Favard's theorem says there is a one-one correspondence between sets of bounded Jacobi parameters, that is,

$$\sup_n |a_n| = \alpha_+ < \infty \qquad \sup_n |b_n| = \beta < \infty \tag{1.4}$$

and probability measures with compact and not finite support under this $\mu \to J \to \mu$ correspondence.

We will use this to justify spectral theory notation for things like $\operatorname{supp}(d\mu)$ which we will denote $\sigma(d\mu)$ since it is the spectrum of J, $\sigma(J)$. We will use $\sigma_{\text{ess}}(d\mu)$ for the essential spectrum, and if

$$d\mu(x) = w(x)\,dx + d\mu_{\text{s}}(x) \tag{1.5}$$

where $d\mu_{\text{s}}$ is Lebesgue singular, then we define

$$\Sigma_{\text{ac}}(d\mu) = \{x \mid w(x) > 0\} \tag{1.6}$$

determined up to sets of Lebesgue measure 0, so $\Sigma_{\text{ac}} \neq \emptyset$ means $d\mu$ has a nonvanishing a.c. part.

We will also suppose

$$\inf_n a_n = \alpha_- > 0 \tag{1.7}$$

which is no loss since it is known [11] that if the inf is 0, then $\Sigma_{\text{ac}} = \emptyset$, and we will only be interested in cases where $\Sigma_{\text{ac}} \neq \emptyset$.

One of our concerns in this paper is the zeros of $p_n(x, d\mu)$. These are not only of intrinsic interest; they enter in Gaussian quadrature and also as the eigenvalues of $J_{n;F}$, the upper left $n \times n$ corner of J, and so, relevant to statistics of eigenvalues in large boxes, a subject on which there is an enormous amount of discussion in both the mathematics and the physics literature.

These zeros are all simple and lie in $\mathbb{R}$. $d\nu_n$ is the normalized counting measure for the zeros, that is,

$$\nu_n(S) = \frac{1}{n}\,\#(\text{zeros of } p_n \text{ in } S) \tag{1.8}$$

In many cases, $d\nu_n$ converges to a weak limit, $d\nu_\infty$, called the density of zeros or density of states (DOS). If this weak limit exists, we say that the DOS exists. It often happens that $d\nu_\infty$ is $d\rho_{\mathfrak{e}}$, the equilibrium measure for $\mathfrak{e} = \sigma_{\text{ess}}(d\mu)$. This is true, for example, if $\rho_{\mathfrak{e}}$ is equivalent to $dx \restriction \mathfrak{e}$ and $\Sigma_{\text{ac}} = \mathfrak{e}$, a theorem of Widom [49] and Van Assche [48] (see also Stahl–Totik [42] and Simon [37]). If $d\nu_\infty$ has an a.c. part, we use $\rho_\infty(x)$ for $d\nu_\infty/dx$ and we use $\rho_{\mathfrak{e}}(x)$ for $d\rho_{\mathfrak{e}}/dx$. More properly, $d\nu_\infty$ is the "density of states measure" (so $\int_{-\infty}^x d\nu_\infty$ is the "integrated density of states") and $\rho_\infty(x)$ the "density of states."

We are especially interested in the fine structure of the zeros near some point $x_0 \in \sigma(d\mu)$. We define $x_j^{(n)}(x_0)$ by

$$x_{-2}^{(n)}(x_0) < x_{-1}^{(n)}(x_0) < x_0 \le x_0^{(n)}(x_0) < x_1^{(n)}(x_0) < \dots \tag{1.9}$$

requiring these to be all of the zeros near x_0. It is known that if x_0 is not isolated from $\sigma(d\mu)$ on either side, that is, for all $\delta > 0$,

$$(x_0 - \delta, x_0) \cap \sigma(d\mu) \neq \emptyset \neq (x_0, x_0 + \delta) \cap \sigma(d\mu) \tag{1.10}$$

then for each fixed j,

$$\lim_{n\to\infty} x_j^{(n)}(x_0) = x_0 \tag{1.11}$$

We are interested in clock behavior named after the spacing of numerals on a clock—meaning equal spacing of the zeros nearby to x_0:

Definition. We say that there is *quasi-clock behavior* at $x_0 \in \sigma(d\mu)$ if and only if for each fixed $j \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{x_{j+1}^{(n)}(x_0) - x_j^{(n)}(x_0)}{x_1^{(n)}(x_0) - x_0^{(n)}(x_0)} = 1 \tag{1.12}$$

We say there is *strong clock behavior* at x_0 if and only if the DOS exists and for each fixed $j \in \mathbb{Z}$,

$$\lim_{n\to\infty} n(x_{j+1}^{(n)}(x_0) - x_j(x_0)) = \frac{1}{\rho_\infty(x_0)} \tag{1.13}$$

Obviously, strong clock behavior implies quasi-clock behavior. Thus far, the only cases where it is proven there is quasi-clock behavior, one has strong clock behavior but, as we will explain in Section 7, we think there are examples where one has quasi-clock behavior at x_0 but not strong clock behavior. Before this paper, all examples known with strong clock behavior have $\rho_\infty = \rho_{\mathfrak{e}}$, but we will find several examples where there is strong clock behavior with $\rho_\infty \neq \rho_{\mathfrak{e}}$ in Section 7. In that section, we will say more about:

Conjecture. For any μ, quasi-clock behavior holds at a.e. $x_0 \in \Sigma_{\text{ac}}(d\mu)$.

In this paper, one of our main goals is to prove this result for ergodic Jacobi matrices. A major role will be played by the CD (for Christoffel–Darboux) kernel, defined for $x, y \in \mathbb{C}$ by

$$K_n(x,y) = \sum_{j=0}^{n} \overline{p_j(x)}\, p_j(y) \tag{1.14}$$

the integral kernel for the orthogonal projection onto polynomials of degree at most n in $L^2(\mathbb{R}, d\mu)$; see Simon [38] for a review of some important aspects of the properties and uses of this kernel. We will repeatedly make use of the CD formula,

$$K_n(x,y) = \frac{a_{n+1}[\overline{p_{n+1}(x)}\, p_n(y) - \overline{p_n(x)}\, p_{n+1}(y)]}{\bar{x} - y}, \tag{1.15}$$

the Schwarz inequality,

$$|K_n(x,y)|^2 \leq K_n(x,x) K_n(y,y) \tag{1.16}$$

and the reproducing property,

$$\int K_n(x,y) K_n(y,z)\, d\mu(y) = K_n(x,z). \tag{1.17}$$

It is a theorem (see Simon [40]) that if the DOS exists, then

$$\frac{1}{n+1}\, K_n(x,x)\, d\mu(x) \xrightarrow{\text{weak}} d\nu_\infty(x) \tag{1.18}$$

and, in general, $\frac{1}{n+1} K_n(x,x)\, d\mu(x)$ has the same weak limit points as $d\nu_n$. This suggests that a.c. parts converge pointwise, that is, one hopes that for a.e. $x_0 \in \Sigma_{\text{ac}}$,

$$\frac{1}{n+1}\, K_n(x_0,x_0) \to \frac{\rho_\infty(x_0)}{w(x_0)} \tag{1.19}$$

This has been proven for regular (in the sense of Stahl–Totik [42]; see also Simon [37]) measures with a local Szegő condition in a series of papers of which the seminal ones are Máté–Nevai-Totik [30] and Totik [45]. We will prove it for ergodic Jacobi matrices.

We say *bulk universality* holds at $x_0 \in \operatorname{supp}(d\mu)$ if and only if uniformly for a, b in compact subsets of $\mathbb{R}$, we have

$$\frac{K_n(x_0 + \frac{a}{n}, x_0 + \frac{b}{n})}{K_n(x_0, x_0)} \to \frac{\sin(\pi\rho(x_0)(b-a))}{\pi\rho(x_0)(b-a)} \tag{1.20}$$

We use the term "bulk" here because (1.20) fails at edges of the spectrum; see Lubinsky [24]. We also note that when (1.20) holds, typically (and in all cases below) for z, w complex, one has

$$\frac{K_n(x_0 + \frac{z}{n}, x_0 + \frac{w}{n})}{K_n(x_0, x_0)} \to \frac{\sin(\rho(x_0)(w-\bar{z}))}{\rho(x_0)(w-\bar{z})} \tag{1.21}$$

Freud [15] proved bulk universality for measures on $[-1, 1]$ with $d\mu_{\rm s} = 0$ and strong conditions on $w(x)$. Because of related results (but with variable weights) in random matrix theory, this result was re-examined and proven in multiple interval support cases with analytic weights by Kuijlaars–Vanlessen [21]. A significant breakthrough was made by Lubinsky [25], whose contributions we return to shortly.

It is a basic result of Freud [15], rediscovered by Levin (in [23]), that

Theorem 1.1 (Freud–Levin Theorem)**.** *Bulk universality at x_0 implies strong clock behavior at x_0.*

Remarks. 1. The proof (see [15, 23, 38]) relies on the CD formula, (1.15), which implies that if y_0 is a zero of p_n, then the other zeros of p_n are the points y solving $K_n(y, y_0) = 0$ and the fact that the zeros of $\sin(\pi\rho(x_0)(b-a))$ are at $b - a = j/\rho(x_0)$ with $j \in \mathbb{Z}$.

2. Szegő [43] proved strong clock behavior for Jacobi polynomials and Erdős–Turán [12] for a more general class of measures on $[-1, 1]$. Simon [34, 35, 36, 22] has a series on the subject. The paper with Last [22] was one motivation for Levin–Lubinsky [23].

3. Lubinsky (private communication) has emphasized to us that this part of [23] is due to Levin alone—hence our name for the result.

It is also useful to define

$$\rho_n = \frac{1}{n}\, w(x_0) K_n(x_0, x_0) \tag{1.22}$$

so (1.19) is equivalent to

$$\rho_n \to \rho_\infty(x_0) \tag{1.23}$$

We say *weak bulk universality* holds at x_0 if and only if, uniformly for a, b on compact subsets of $\mathbb{R}$, we have

$$\frac{K_n(x_0 + \frac{a}{n\rho_n}, x_0 + \frac{b}{n\rho_n})}{K_n(x_0, x_0)} \to \frac{\sin(\pi(b-a))}{\pi(b-a)} \tag{1.24}$$

the form in which universality is often written, especially in the random matrix literature. Notice that

$$\text{weak universality} + (1.23) \Rightarrow \text{universality} \tag{1.25}$$

Notice also that (1.24) could hold in case where ρ_n does not converge as $n \to \infty$. The same proof that verifies Theorem 1.1 implies

Theorem 1.2 (Weak Freud–Levin Theorem)**.** *Weak bulk universality at x_0 implies quasi-clock behavior at x_0.*

With this background in place, we can turn to describing the main results of this paper: five theorems, proven one per section in Sections 2–6.

The first theorem is an abstraction, extension, and simplification of Lubinsky's second approach to universality [26]. In [25], Lubinsky found a beautiful way of going from control of the diagonal CD kernel to the off-diagonal (i.e., to universality). It depended on the ability to control limits not only of $\frac{1}{n}K_n(x_0,x_0)$ but also $\frac{1}{n}K_n(x_0+\frac{a}{n},x_0+\frac{a}{n})$—what we call the Lubinsky wiggle. We will especially care about the *Lubinsky wiggle condition*:

$$\lim_{n\to\infty} \frac{K_n(x_0+\frac{a}{n},x_0+\frac{a}{n})}{K_n(x_0,x_0)} = 1 \tag{1.26}$$

uniformly for $a \in [-A,A]$ for each A. In addition to this, in [25], Lubinsky needed a simple but clever inequality and, most significantly, a comparison model example where one knows universality holds. For $[-1,1]$, he took Legendre polynomials (i.e., $d\mu = \frac{1}{2}\chi_{[-1,1]}(x)\,dx$). In extending this to more general sets, one uses approximation by finite gap sets as pioneered by Totik [46]. Simon [39] then used Jacobi matrices in isospectral tori for a comparison model on these finite gap sets, while Totik [47] used polynomials mappings and the results for $[-1,1]$.

For ergodic Jacobi matrices where $\sigma(d\mu)$ is often a Cantor set, it is hard to find comparison models, so we will rely on a second approach developed by Lubinsky [26] that seems to be able to handle any situation that his first approach can and which does not rely on a comparison model. Our first theorem, proven in Section 2, is a variant of this approach. We need a preliminary definition:

Definition. Let $d\mu$ be given by (1.5). A point x_0 is called a *Lebesgue point* of $d\mu$ if and only if $w(x_0) > 0$ and

$$\lim_{\delta\downarrow 0}\,(2\delta)^{-1}\int_{x_0-\delta}^{x_0+\delta} |w(x)-w(x_0)|\,dx = 0 \tag{1.27}$$

$$\lim_{\delta\downarrow 0}\,(2\delta)^{-1}\mu_{\rm s}(x_0-\delta,x_0+\delta) = 0 \tag{1.28}$$

Standard maximal function methods (see Rudin [32]) show Lebesgue a.e. $x_0 \in \Sigma_{\text{ac}}(d\mu)$ is a Lebesgue point.

Theorem 1. *Let x_0 be a Lebesgue point of μ. Suppose that*

(1) *The Lubinsky wiggle condition* (1.26) *holds uniformly for $a \in [-A, A]$ and any $A < \infty$.*
(2) *We have*

$$\liminf_{n\to\infty} \frac{1}{n+1}\, K_n(x_0, x_0) > 0 \tag{1.29}$$

(3) *For any ε, there is $C_\varepsilon > 0$ so that for any $R < \infty$, there is an N so that for all $n > N$ and all $z \in \mathbb{C}$ with $|z| < R$, we have*

$$\frac{1}{n+1}\, K_n\biggl(x_0 + \frac{z}{n}\,, x_0 + \frac{z}{n}\biggr) \le C_\varepsilon \exp(\varepsilon |z|^2) \tag{1.30}$$

Then weak bulk universality, and so, quasi-clock behavior, holds at x_0.

Remarks. 1. If one replaces (1.30) by

$$C \exp(A|z|) \tag{1.31}$$

then the result can be proven by following Lubinsky's argument in [26]. He does not assume (1.31) directly but rather hypotheses that he shows imply it (but which is invalid in case $\operatorname{supp}(d\mu)$ is a Cantor set).

2. Because our Theorem 3 below is so general, we doubt there are examples where (1.30) holds but (1.31) does not, but we feel our more general abstract result is clarifying.

3. The strategy we follow is Lubinsky's, but the tactics differ and, we feel, are more elementary and illuminating.

In [26], the only examples where Lubinsky can verify his wiggle condition are the situations where Totik [47] proves universality using Lubinsky's first method. To go beyond that, we need the following, proven in Section 3:

Theorem 2. *Let $\Sigma \subset \Sigma_{\text{ac}}$. Suppose for a.e. $x_0 \in \Sigma$, we have that condition* (3) *of Theorem 1 holds and that*
(4) $\lim_{n\to\infty} \frac{1}{n+1} K_n(x_0, x_0)$ *exists and is strictly positive.*
Then condition (1) *of Theorem 1 holds for a.e. $x_0 \in \Sigma$.*

Of course, (4) implies condition (2). So we obtain:

Corollary 1.3. *If* (3) *and* (4) *hold for a.e. $x_0 \in \Sigma$, then for a.e. $x_0 \in \Sigma$, we have weak universality and quasi-clock behavior.*

By (1.25), we see

Corollary 1.4. *If* (3) *and* (4) *hold for a.e.* $x_0 \in \Sigma$*, and if the DOS exists and the limit in* (4) *is* $\rho_\infty(x)/w(x)$*, then for a.e.* $x \in \Sigma$*, we have universality and strong clock behavior.*

Next, we need to examine when (1.30) holds. We will not only obtain a bound of the type (1.31) but one that does not need to vary N with R and is universal in z. We will use transfer matrix techniques and notation.

Given Jacobi parameters, $\{a_n, b_n\}_{n=1}^\infty$, we define

$$A_j(z) = \begin{pmatrix} \frac{z-b_j}{a_j} & -\frac{1}{a_j} \\ a_j & 0 \end{pmatrix} \tag{1.32}$$

so that (1.2) is equivalent to

$$\begin{pmatrix} p_n(x) \\ a_n p_{n-1}(x) \end{pmatrix} = A_n(x) \begin{pmatrix} p_{n-1}(x) \\ a_{n-1} p_{n-2}(x) \end{pmatrix} \tag{1.33}$$

We normalize, placing a_n on the lower component, so that

$$\det(A_j(z)) = 1 \tag{1.34}$$

The transfer matrix is then defined by

$$T_n(z) = A_n(z) \dots A_1(z) \tag{1.35}$$

so

$$\begin{pmatrix} p_n(x) \\ a_n p_{n-1}(x) \end{pmatrix} = T_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{1.36}$$

If $\tilde p_n$ are the OPRL associated to the once stripped Jacobi parameters $\{a_{n+1}, b_{n+1}\}_{n=1}^\infty$, and

$$q_n(x) = -a_1^{-1} \tilde p_{n-1}(x) \tag{1.37}$$

with $q_0 = 0$, then

$$T_n(z) = \begin{pmatrix} p_n(z) & q_n(z) \\ a_n p_{n-1}(z) & a_n q_{n-1}(z) \end{pmatrix} \tag{1.38}$$

Here is how we will establish (1.30)/(1.31):

Theorem 3. *Fix* $x_0 \in \mathbb{R}$*. Suppose that*

$$\sup_n \frac{1}{n+1} \sum_{j=0}^n \|T_j(x_0)\|^2 \le C < \infty \tag{1.39}$$

Then for all $z \in \mathbb{C}$ *and all* n*,*

$$\frac{1}{n+1} \sum_{j=0}^n \left\| T_j\left(x_0 + \frac{z}{n+1} \right) \right\|^2 \le C \exp(2C\alpha_-^{-1}|z|) \tag{1.40}$$

Moreover, if

$$\sup_n \|T_n(x_0)\|^2 = C < \infty \tag{1.41}$$

then for all $z \in \mathbb{C}$ *and* n*,*

$$\left\| T_n\left(x_0 + \frac{z}{n+1}\right)\right\| \leq C^{1/2} \exp(C\alpha_-^{-1}|z|) \tag{1.42}$$

Remarks. 1. Our proof is an abstraction of ideas of Avila–Krikorian [5] who only treated the ergodic case.

2. α_- is given by (1.7).

3. There is a conjecture, called the Schrödinger conjecture (see [29]), that says (1.41) holds for a.e. $x_0 \in \Sigma_{\text{ac}}(d\mu)$.

Our last two theorems below are special to the ergodic situation. Let Ω be a compact metric space, $d\eta$ a probability measure on Ω, and $S\colon \Omega \to \Omega$ an ergodic invertible map of Ω to itself. Let A, B be continuous real-valued functions on Ω with $\inf_\omega A(\omega) > 0$. Let

$$\alpha_+ = \|A\|_\infty \qquad \beta = \|B\|_\infty \qquad \alpha_- = \|A^{-1}\|_\infty^{-1} \tag{1.43}$$

For each $\omega \in \Omega$, J_ω is the Jacobi matrix with

$$a_n(\omega) = A(S^{n-1}\omega) \qquad b_n(\omega) = B(S^{n-1}\omega) \tag{1.44}$$

(1.43) is consistent with (1.4) and (1.7). Usually one only takes Ω, a measure space, and A, B bounded measurable functions, but by replacing Ω by $([\alpha_-, \alpha_+] \times [-\beta, \beta])^\infty \equiv \widetilde{\Omega}$ and mapping $\Omega \to \widetilde{\Omega}$ by $\omega \mapsto (A(S^n\omega), B(S^n\omega))_{n=-\infty}^\infty$, we get a compact space model equivalent to the original measure model. We use $d\mu_\omega$ for the spectral measure of J_ω and $p_n(x,\omega)$ for $p_n(x, d\mu_\omega)$.

The canonical example of the setup with a.c. spectrum is the almost Mathieu equation. α is a fixed irrational, λ a nonzero real, $\Omega = \partial\mathbb{D}$, the unit circle $\{e^{i\theta} \mid \theta \in [0, 2\pi)\}$

$$a_n \equiv 1 \qquad b_n = 2\lambda\cos(\pi\alpha n + \theta)$$

(so $S(e^{i\theta}) = e^{i\theta}e^{i\pi\alpha}$, $d\eta(\theta) = d\theta/2\pi$). If $0 \neq |\lambda| < 1$, it is known (see [1, 2, 4, 16]) that the spectrum is purely a.c. and is a Cantor set. It is also known [16] that if $|\lambda| \geq 1$, there is no a.c. spectrum.

We will prove the following in Section 5:

Theorem 4. *Let* $\{J_\omega\}_{\omega\in n}$ *be an ergodic family with* Σ_{ac}*, the common essential support of the a.c. spectrum of* J_ω*, of positive Lebesgue measure. Then for a.e. pairs* $(x,\omega) \in \Sigma_{\text{ac}} \times \Omega$*,*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^n |p_j(x,w)|^2 \text{ exists} \tag{1.45}$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} |q_j(x,w)|^2 \textit{ exists}$$

In Section 6, we will prove

Theorem 5. *For a.e.* (x,ω) *in* $\Sigma_{\rm ac}\times\Omega$*, the limit in* (1.45) *is* $\rho_\infty(x)/w_\omega(x)$ *where* ρ_∞ *is the density of the a.c. part of the DOS.*

Note. This is, of course, an analog of the celebrated results of Máté–Nevai–Totik [30] (for $[-1,1]$) and Totik [45] (for general sets $\mathfrak{e}$ containing open intervals) for regular measures obeying a local Szegő condition.

Theorems 3–5 show the applicability of Theorem 2, and so lead to

Corollary 1.5. *For any ergodic Jacobi matrix, we have universality and strong clock behavior for a.e.* ω *and a.e.* $x_0\in\Sigma_{\rm ac}$*.*

In particular, the almost Mathieu equation has strong clock behavior for the zeros.

Remark. It is possible to show that for the almost Mathieu equation there is universality for a.e. $x_0\in\Sigma_{\rm ac}$ and *every* ω. Our current approach to this uses that the Schrödinger conjecture is true for the almost Mathieu operator, a recently announced result [3].

For $n=1,2,3,4,5$, Theorem n is proven in Section $n+1$. Section 7 has some further remarks.

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2. Lubinsky's Second Approach

In this section, we will prove Theorem 1. We begin with two overall visions relevant to the proof. First, the so-called "sinc kernel" [27], $\sin\pi z/\pi z$ enters as the Fourier transform of a suitable multiple of the characteristic function of $[-\pi,\pi]$.

Second, the ultimate goal of quasi-clock spacing is that on a $1/n\rho_n$ scale, zeros are a unit distance apart, so on this scale

$$\#\text{ of zeros in } [0,n]\sim n \tag{2.1}$$

Lubinsky's realization is that the Lubinsky wiggle condition and Markov–Stieltjes inequalities (see below) imply the difference of the two sides of (2.1) is bounded by 1. This is close enough that, together with some complex variable magic, one gets unit spacing.

The complex variable magic is encapsulated in the following result whose proof we defer until the end of the section:

Theorem 2.1. *Let f be an entire function with the following properties:*

(a)
$$f(0) = 1 \tag{2.2}$$

(b)
$$\sup_{x\in\mathbb{R}} |f(x)| < \infty \tag{2.3}$$

(c)
$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx \le 1 \tag{2.4}$$

(d) *f is real on $\mathbb{R}$.*

(e) *All the zeros of f lie on $\mathbb{R}$ and if these zeros are labelled by*
$$\dots \le z_{-2} \le z_{-1} < 0 < z_1 \le z_2 \le \dots \tag{2.5}$$
with $z_0 \equiv 0$, then
$$|z_j - z_k| \ge |j-k| - 1 \tag{2.6}$$

(f) *For each $\varepsilon > 0$, there is C_ε with*
$$|f(z)| \le C_\varepsilon e^{\varepsilon |z|^2} \tag{2.7}$$

Then
$$f(z) = \frac{\sin(\pi z)}{\pi z} \tag{2.8}$$

Remarks. 1. (2.6) allows f a priori to have double zeros but not triple or higher zeros.

2. It is easy to see there are examples where (2.7) holds for some but not all of ε and where (2.8) is false, so (2.7) is sharp.

Proof of Theorem 2 given Theorem 2.1. (This part of the argument is essentially in Lubinsky [26].) Fix $a \in \mathbb{R}$ and let
$$f_n(z) = \frac{K_n(x_0 + \frac{a}{n\rho_n}, x_0 + \frac{a+z}{n\rho_n})}{K_n(x_0, x_0)} \tag{2.9}$$

By (1.29), (1.30), and (1.16), the f_n are uniformly bounded on each disk $\{z \mid |z| < R\}$, so by Montel's theorem, we have compactness that shows it suffices to prove that any limit point $f(z)$ has the form (2.8). We will show that this putative limit point obeys conditions (a)–(f) of Theorem 2.1.

By the Lubinsky wiggle condition (1.26), (a) holds. By Schwarz inequality, (1.11) and the wiggle condition,

$$\sup_{x\in\mathbb{R}} |f(x)| = 1 \tag{2.10}$$

which is stronger than (b).

By (1.17),

$$\int_{|y-x_0-\frac{a}{n\rho_n}|\leq\frac{R}{n\rho_n}} |K_n(x,y)|^2 w(y)\, dy \leq K_n(x,x) \tag{2.11}$$

for each $R<\infty$. Changing variables and using the Lebesgue point condition leads to

$$\int_{-R}^{R} |f(y)|^2\, dy \leq 1 \tag{2.12}$$

which yields (2.4) (see Lubinsky [26] for more details). In this, one uses (1.29) and (1.30) to see that

$$0 < \inf \rho_n < \sup \rho_n < \infty. \tag{2.13}$$

That f is real on $\mathbb{R}$ is immediate; the reality of zeros follows from Hurwitz's theorem and the fact (see, e.g., [38]) that $p_{n+1}(x) - cp_n(x)$ has only real zeros for c real.

The Markov–Stieltjes inequalities (see [28, 15, 38]) assert that if $x_1, x_2, \dots$ are successive zeros of $p_n(x) - cp_{n-1}(x)$ for some c, then for $j \geq k+2$,

$$\mu([x_j, x_k]) \geq \sum_{\ell=k+1}^{j-1} \frac{1}{K_n(x_\ell, x_\ell)} \tag{2.14}$$

Using the fact that the z_j (including z_0) are, by Hurwitz's theorem, limits of x_j's scaled by $n\rho_n$ and the Lubinsky wiggle condition to control limits of $n\rho_n/K_n(x_\ell, x_\ell)$, one finds (see Lubinsky [26] for more details) that (2.6) holds. Here one uses that x_0 is a Lebesgue point to be sure that

$$\frac{1}{x_k - x_j} \int_{x_j}^{x_k} d\mu(y) \to w(x_0) \tag{2.15}$$

Finally, (1.30) implies (2.7). Thus, (2.8) holds. □

The following will reduce the proof of Theorem 2.1 to using conditions (a)–(e) to improving the bound (2.7).

Proposition 2.2. (a) *Fix $a>0$. If f is measurable, real-valued and supported on $[-a,a]$ with*

$$\int_{-a}^{a} f(x)^2\, dx \leq 2a \quad \textit{and} \quad \int_{-a}^{a} f(x)\, dx = 2a \tag{2.16}$$

then

$$f(x) = \chi_{[-a,a]}(x) \quad a.e. \tag{2.17}$$

(b) *If f is real-valued and continuous on $\mathbb{R}$ and $\widehat{f}$ is supported on $[-\pi,\pi]$ with*

$$\int_{-\infty}^{\infty} f(x)^2\,dx \le 1 \quad \text{and} \quad f(0) = 1 \tag{2.18}$$

then

$$f(x) = \frac{\sin(\pi x)}{\pi x} \tag{2.19}$$

(c) *If f is an entire function, real on $\mathbb{R}$ with* (2.18)*, and for all $\delta > 0$, there is C_δ with*

$$|f(z)| \le C_\delta \exp((\pi+\delta)|\mathrm{Im}\, z|) \tag{2.20}$$

then (2.8) *holds.*

Proof. (a) Essentially this follows from equality in the Schwarz inequality. More precisely, (2.16) implies

$$\int_{-a}^{a} |f(x) - \chi_{[-a,a]}(x)|^2\,dx \le 0 \tag{2.21}$$

(b) Apply part (a) to $(2\pi)^{1/2}\widehat{f}(k)$ with $a=\pi$.

(c) By the Paley–Wiener theorem, (2.20) implies that $\widehat{f}$ is supported on $[-\pi,\pi]$. □

Thus, we are reduced to going from (2.7) to (2.20).

By $f(0) = 1$, the reality of the zeros and (2.7), we have, by the Hadamard factorization theorem (see Titchmarsh [44, Sect. 8.24]) that

$$f(z) = e^{Az} \prod_{j\ne 0} \left(1 - \frac{z}{z_j}\right) e^{z/z_j} \tag{2.22}$$

with A real. For $x \in \mathbb{R}$, define $z_j(x)$ to be a renumbering of the z_j, so

$$\dots \le z_{-1}(x) < x \le z_0(x) \le z_1(x) \le \dots \tag{2.23}$$

By $|z_j - z_k| \ge |k-j| - 1$, we see that

$$z_{n+1}(x) - x \ge n \qquad x - z_{-(n+1)}(x) \ge n \tag{2.24}$$

In particular, $(x - 1.1, x + 1.1)$ can contain at most $z_0(x), z_{\pm 1}(x), z_{\pm 2}(x)$. Removing the open intervals of size $2/10$ about each of the five points $|z_\ell(x) - x|$ ($\ell = 0, \pm 1, \pm 2$) from $[0,1]$ leaves at least one $\delta > 0$, that is, we can pick $\delta(x)$ in $[0,1]$ so for all j,

$$|z_j(n) - (x \pm \delta)| \ge \tfrac{1}{10} \tag{2.25}$$

Moreover, by (2.24), for $n = 1, 2, \dots$,

$$|z_{\pm(n+2)}(x) - (x \pm \delta)| \geq n \tag{2.26}$$

Since

$$\frac{|1 - (x+iy)/z_j|^2}{|(1 - (x+\delta/z_j)(1-x-\delta)/z_j)|} \leq 1 + \frac{(y^2+\delta^2)}{|z_j - (x+\delta)||z_j - (x-\delta)|} \tag{2.27}$$

we conclude from (2.22) that

$$\begin{aligned} \frac{|f(x+iy)|^2}{|f(x-\delta)||f(x+\delta)|} &\leq \left[1 + \frac{y^2+1}{(\frac{1}{100})}\right]^5 \prod_{n=1}^{\infty}\left(1 + \frac{1+y^2}{n^2}\right)^2 \\ &\leq C(1+y^{10})\left(\frac{\sinh \pi\sqrt{y^2+1}}{\pi\sqrt{y^2+1}}\right)^2 \end{aligned} \tag{2.28}$$

Thus, for any ε, there is a C_ε with

$$|f(x+iy)| \leq C_\varepsilon \exp((\pi+\varepsilon)|y|) \tag{2.29}$$

for every $x + iy \in \mathbb{C}$, which is (2.20). This concludes the proof of Theorem 2.1.

Remark. It is possible to show, using the Phragmén–Lindelöf principle [44], that if one assumes, instead of (2.7), the stronger $|f(z)| \leq Ce^{|z|^\delta}$, then it is possible to weaken (2.6) to

$$|z_j| \geq |j| - 1 \tag{2.30}$$

for if (2.30) holds, then (2.22) implies that

$$|f(iy)| \leq C(1+|y|)e^{\pi|y|} \tag{2.31}$$

Applying Phragmén–Lindelöf to $(1 - iz)^{-1} f(z) e^{i\pi z}$ on the sectors $\arg z \in [0, \pi/2]$ and $[\pi/2, \pi]$ proves that

$$|f(x+iy)| \leq C(1+|z|)e^{\pi|y|} \tag{2.32}$$

3. Doing the Lubinsky Wiggle

Our goal in this section is to prove Theorem 2.

Proof of Theorem 2. By Egorov's theorem (see Rudin [32, p. 73]), for every ε, there exists a compact set $\mathcal{L} \subset \Sigma$ with $|\Sigma \setminus \mathcal{L}| < \varepsilon$ (with $|\cdot|$ = Lebesgue measure) so that on $\mathcal{L}$, $\frac{1}{n+1} K_n(x,x) \equiv \tilde{q}_n(x)$ converges uniformly to a limit we will call $\tilde{q}(x)$. If we prove that (1.26) holds for a.e. $x_0 \in \mathcal{L}$, then by taking a sequence of ε's going to 0, we get that (1.26) holds for a.e. $x_0 \in \Sigma$

By Lebesgue's theorem on differentiability of integrals of L^1-functions (see Rudin [32, Thm 7.7]) applied to the characteristic function of $\mathcal{L}$, for a.e. $x_0 \in \mathcal{L}$,

$$\lim_{\delta\downarrow 0} (2\delta)^{-1} |(x_0-\delta, x_0+\delta)\cap \mathcal{L}| = 1 \tag{3.1}$$

We will prove that (1.26) holds for all x_0 with (3.1) and with condition (4).

$\frac{1}{n+1} K_n(x+\frac{a}{n}+\frac{\bar z}{n}, x+\frac{a}{n}+\frac{z}{n})$ is analytic in z, so by a Cauchy estimate and a real,

$$\begin{aligned}\left|\frac{d}{da}\,\tilde q_n\left(x+\frac{a}{n}\right)\right| &\le \sup_{|z|\le 1} \frac{1}{n+1}\left|K_n\left(x+\frac{a}{n}+\frac{\bar z}{n}, x+\frac{a}{n}+\frac{z}{n}\right)\right| \\ &= \sup_{|z|\le 1}\left|\tilde q_n\left(x+\frac{a}{n}+\frac{z}{n}\right)\right| \end{aligned}\tag{3.2}$$

By a Schwarz inequality, for $x,y\in\mathbb{C}$,

$$\frac{1}{n+1}\,|K_n(x,y)| \le (\tilde q_n(x)\tilde q_n(y))^{1/2} \tag{3.3}$$

Thus, using the assumed (1.30), for any x_0 for which (1.30) holds and any $A<\infty$, there are N_0 and C so for $n\ge N_0$,

$$\left|\tilde q_n\left(x_0+\frac{a}{n}\right) - \tilde q_n\left(x_0+\frac{b}{n}\right)\right| \le C|a-b| \tag{3.4}$$

for all a,b with $|a|\le A$, $|b|\le A$.

Since each $\tilde q_n$ is continuous and the convergence is uniform on $\mathcal{L}$, $\tilde q$ is continuous on $\mathcal{L}$. Thus, we have for each $A<\infty$,

$$\sup\left\{\left|\tilde q\left(x_0+\frac{a}{n}\right)-\tilde q(x_0)\right| \;\middle|\; |a|<A,\ x_0+\frac{a}{n}\in\mathcal{L}\right\}\to 0 \tag{3.5}$$

as $n\to\infty$. By the uniform convergence,

$$\sup\left\{\left|\tilde q_n\left(x_0+\frac{a}{n}\right)-\tilde q_n(x_0)\right| \;\middle|\; |a|<A,\ x_0+\frac{a}{n}\in\mathcal{L}\right\}\to 0 \tag{3.6}$$

We next use the fact that (3.1) holds. It implies that

$$\sup_{|b|\le A} n \operatorname{dist}\left(x_0+\frac{b}{n}, \mathcal{L}\right)\to 0 \tag{3.7}$$

or equivalently, for any ε, there is an N_1 so for $n\ge N_1$ and $|b|<A$, there exists $|a|<A$ (a will be n-dependent) so that $|a-b|<\varepsilon$ and

$x_0 + \frac{a}{n} \in \mathcal{L}$. We have that

$$\left|\tilde q_n\left(x_0+\frac{b}{n}\right)-\tilde q_n(x_0)\right| \le \left|\tilde q_n\left(x_0+\frac{b}{n}\right)-\tilde q_n\left(x_0+\frac{a}{n}\right)\right|+\left|\tilde q_n\left(x_0+\frac{a}{n}\right)-\tilde q_n(x_0)\right| \tag{3.8}$$

where $|b-a|<\varepsilon$ and $x_0+\frac{a}{n}\in\mathcal{L}$. By (3.4), if $n \ge \max(N_0, N_1)$, the first term is bounded by $C\varepsilon$, and by (3.7), the second term goes to zero, that is,

$$\sup_{|b|<A}\left|\tilde q_n\left(x_0+\frac{b}{n}\right)-\tilde q_n(x_0)\right| \to 0 \tag{3.9}$$

Since $\tilde q_n(x_0) \to \tilde q(x_0) \neq 0$, we have

$$\sup_{|b|<A}\left|\frac{\tilde q_n(x_0+\frac{b}{n})}{\tilde q_n(x_0)}-1\right| \to 0 \tag{3.10}$$

as $n\to\infty$, which is (1.26). □

4. Exponential Bounds for Perturbed Transfer Matrices

In this section, our goal is to prove Theorem 3. As noted in the introduction, our approach is an extension of a theorem of Avila–Krikorian [5, Lemma 3.1] exploiting that one can avoid using cocycles and so go beyond the apparent limitation to ergodic situations. The argument here is related to but somewhat different from variation of parameters techniques (see, e.g., Jitomirskaya–Last [17] and Killip–Kiselev–Last [19]) and should have wide applicability.

Proof of Theorem 3. Fix n and define for $j=1,2,\dots,n$,

$$\tilde A_j = A_j\left(x_0+\frac{z}{n+1}\right) \tag{4.1}$$

$$A_j = A_j(x_0) \tag{4.2}$$

$$T_j = A_j \dots A_1 \qquad \tilde T_j = \tilde A_j \dots \tilde A_1 \tag{4.3}$$

(Note that $\tilde A_j$ and $\tilde T_j$ are both j- and n-dependent.)

Note that, by (1.32),

$$\tilde A_j - A_j = a_j^{-1}\begin{pmatrix} \frac{z}{n+1} & 0 \\ 0 & 0\end{pmatrix} \tag{4.4}$$

so that

$$\|\tilde A_j - A_j\| \le \alpha_-^{-1}\,\frac{|z|}{n+1} \tag{4.5}$$

Write

$$T_j^{-1}\tilde T_j = (T_j^{-1}\tilde A_j T_{j-1})(T_{j-1}^{-1}\tilde A_{j-1}T_{j-2})\dots(T_1^{-1}\tilde A_1 T_0) \tag{4.6}$$

$$= (1 + B_j)(1 + B_{j-1}) \dots (1 + B_1) \tag{4.7}$$

where

$$B_k = T_k^{-1}(\tilde{A}_k - A_k)T_{k-1} \tag{4.8}$$

Here we used

$$A_k T_{k-1} = T_k \tag{4.9}$$

Since T_k has determinant 1 (see (1.34)), we have

$$\|T_k^{-1}\| = \|T_k\| \tag{4.10}$$

So, by (4.5),

$$\|B_k\| \leq \|T_k\| \, \|T_{k-1}\| \alpha_-^{-1} \frac{|z|}{n+1} \tag{4.11}$$

Thus, since

$$\|1 + B_j\| \leq 1 + \|B_j\| \leq \exp(\|B_j\|) \tag{4.12}$$

(4.7) implies that

$$\|\tilde{T}_j\| \leq \|T_j\| \exp\biggl(\alpha_-^{-1}|z| \biggl[\frac{1}{n+1} \sum_{k=1}^{j} \|T_k\| \, \|T_{k-1}\|\biggr]\biggr) \tag{4.13}$$

By the Schwarz inequality, for $j = 1, 2, \dots, n$,

$$\begin{aligned} \frac{1}{n+1} \sum_{k=1}^{j} \|T_k\| \, \|T_{k-1}\| &\leq \frac{1}{n+1} \sum_{k=0}^{j} \|T_k\|^2 \\ &\leq \frac{1}{n+1} \sum_{k=0}^{n} \|T_k\|^2 \end{aligned} \tag{4.14}$$

Using (1.39) and (4.13), we find

$$\|\tilde{T}_j\| \leq \|T_j\| \exp(C\alpha_-^{-1}|z|) \tag{4.15}$$

This clearly holds for $j = 0$ also. Squaring and summing,

$$\frac{1}{n+1} \sum_{j=0}^{n} \|\tilde{T}_j\|^2 \leq \biggl(\frac{1}{n+1} \sum_{j=0}^{n} \|T_j\|^2\biggr) \exp(2C\alpha_-^{-1}|z|) \tag{4.16}$$

which is (1.40).

Note that (1.41) implies (1.39) so that (1.42) is just (4.15). □

We note that the argument above can also be used for more general perturbative bounds. For example, suppose that

$$C_1 \equiv \sup_n \|T_n(x_0)\| < \infty \tag{4.17}$$

for a given set of Jacobi parameters. Let $a_n' = a_n + \delta a_n$ and $b_n' = b_n + \delta b_n$ with

$$C_2 \equiv \sum_{n=1}^{\infty} |\delta a_n| + |\delta b_n| < \infty \tag{4.18}$$

and

$$\alpha_-' = \inf a_n' > 0 \tag{4.19}$$

Defining $\tilde{A}_n, \tilde{T}_n$ at energy x_0 but with $\{a_n', b_n'\}_{n=1}^{\infty}$ Jacobi parameters, one gets

$$\|\tilde{A}_k - A_k\| \leq C_3[\alpha_-^{-1} + (\alpha_-')^{-1}](|\delta a_k| + |\delta b_k|) \tag{4.20}$$

for some universal constant C_3. Thus

$$\|B_k\| \leq C_3 C_1^2 [\alpha_-^{-1} + (\alpha_n')^{-1}](|\delta a_k| + |\delta b_k|) \tag{4.21}$$

and

$$\|\tilde{T}_n\| \leq C_1 \exp(C_1^2 C_2 C_3 [\alpha_-^{-1} + (\alpha_-')^{-1}]) \tag{4.22}$$

providing another proof of a standard ℓ^1 perturbation result.

5. Ergodic Jacobi Matrices and Cesàro Summability

In this section, our goal is to prove Theorem 4. We fix an ergodic Jacobi matrix setup. We will need to use special solutions found by Deift–Simon in 1983:

Theorem 5.1 (Deift–Simon [10]). *For any Jacobi matrix with $\Sigma_{\rm ac}(d\mu_\omega)$ (which is a.e. ω-independent) of positive measure, for a.e. pairs $(x,\omega) \in \Sigma_{\rm ac} \times \Omega$ (a.e. with respect to $dx \otimes d\eta(\omega)$), there exist sequences $\{u_n^\pm(x,\omega)\}_{n=-\infty}^{\infty}$ so that*

$$T_n(x,\omega) \begin{pmatrix} u_1^\pm(x,\omega) \\ a_0 u_0^\pm(x,\omega) \end{pmatrix} = \begin{pmatrix} u_{n+1}^\pm(x,\omega) \\ a_n u_n^\pm(x,\omega) \end{pmatrix} \tag{5.1}$$

with the following properties:

$$\text{(i)} \qquad u_n^-(x,\omega) = \overline{u_n^+(x,\omega)} \tag{5.2}$$

$$\text{(ii)} \qquad a_n(u_{n+1}^+ u_n^- - u_{n+1}^- u_n^+) = -2i \tag{5.3}$$

$$\text{(iii)} \qquad |u_n^+(x,\omega)| = |u_0^+(x, S^n\omega)| \tag{5.4}$$

$$\text{(iv)} \qquad \int |u_n^+(x,\omega)|^2 \, d\eta(\omega) < \infty \tag{5.5}$$

$$\text{(v)} \qquad u_0^\pm \text{ is real} \tag{5.6}$$

Of course, by (5.4), the integral in (5.5) is n-independent. For later purposes (see Section 6), we will need an explicit formula for this integral. In fact, we will need explicit formulae for u_0, u_{-1} in terms of the m-function.

One defines for $\operatorname{Im} z > 0$, $\tilde{u}_n^+(z,\omega)$ to solve (i.e., (5.1))

$$a_n\tilde{u}_{n+1}^+ + (b_n - z)\tilde{u}_n^+ + a_{n-1}\tilde{u}_{n-1}^+ = 0 \tag{5.7}$$

with $\sum_{n=1}^\infty |\tilde{u}_n^+|^2 < \infty$. This determines $\tilde{u}_n^+$ up to a constant, and so,

$$m(z,\omega) = -\frac{\tilde{u}_1^+(z,\omega)}{a_0\tilde{u}_0^+(z,\omega)} \tag{5.8}$$

is normalization-independent and obeys, by (5.7),

$$m(z,\omega) = \frac{1}{-z + b_1 - a_1^2 m(z, S\omega)} \tag{5.9}$$

[*Note*: We have suppressed the ω-dependence of a_n, b_n.]

As usual with solutions of (5.9),

$$m(z,\omega) = \int \frac{d\mu_\omega^+(x)}{x - z} \tag{5.10}$$

where $d\mu_\omega^+$ is the measure associated to the half-line Jacobi matrix, J_ω.

For a.e. $x \in \Sigma_{\text{ac}}$ and a.e. ω, $m(x + i0, \omega)$ exists and has

$$\operatorname{Im} m(x + i0, \omega) > 0 \qquad (\text{a.e. } x \in \Sigma_{\text{ac}}) \tag{5.11}$$

We normalize the solution u^+ obeying Theorem 5.1 by defining:

$$u_0^+(x,\omega) = \frac{1}{a_0[\operatorname{Im} m(x + i0, \omega)]^{1/2}} \tag{5.12}$$

$$u_1^+(x,\omega) = -\frac{m(x + i0, \omega)}{[\operatorname{Im} m(x + i0, \omega)]^{1/2}} \tag{5.13}$$

(We have listed all the formulae because [10] only consider the case $a_n \equiv 1$.) u_n^+ are then determined by the difference equation and u_n^- by (5.2).

Of course, we have

$$p_n = \frac{u_{n+1}^+ - u_{n+1}^-}{u_1^+ - u_1^-} \tag{5.14}$$

since both sides obey the same difference equations with $p_{-1} = 0$ (since $u_0^+ = u_0^-$) and $p_0 = 1$.

By (5.14), to prove Theorem 4 we need to show

$$\frac{1}{n}\sum_{j=0}^{n-1}(u_{j+1}^+ - u_{j+1}^-)^2 \tag{5.15}$$

exists. This follows from the existence of

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} |u_j^+|^2 \tag{5.16}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (u_j^+)^2 \tag{5.17}$$

From (5.4) and the ergodic theorem (plus (5.5)), the a.e. ω existence of the limit in (5.16) is immediate. In cases like the almost Mathieu equation with Diophantine frequencies where u_n^+ is almost periodic, one also gets the existence of the limit in (5.17) directly, but there are examples, like the almost Mathieu equation with frequencies whose dual has singular continuous spectrum, where the phase of u_n^+ is not almost periodic. So this argument does not work in general. In fact, we will eventually prove that for a.e. (x,ω) in $\Sigma_{\text{ac}} \times \Omega$ (see Theorem 6.3)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (u_j^+)^2 = 0 \tag{5.18}$$

It would be interesting to have a direct proof of this (for the periodic case, see [41]) rather than the indirect path we will take.

Define the 2×2 matrix

$$U_n(x,\omega) = \frac{1}{(-2i)^{1/2}} \begin{pmatrix} u_{n+1}^+(x,\omega) & u_{n+1}^-(x,\omega) \\ a_n u_n^+(x,\omega) & a_n u_n^-(x,\omega) \end{pmatrix} \tag{5.19}$$

(where we fix once and for all a choice of $\sqrt{-2i}$). By (5.3),

$$\det(U_n(x,\omega)) = 1 \tag{5.20}$$

and, by (5.1),

$$T_n(x,\omega) U_0(x,\omega) = U_n(x,\omega) \tag{5.21}$$

or

$$T_n(x,\omega) = U_n(x,\omega) U_0(x,\omega)^{-1} \tag{5.22}$$

For now, we fix $x \in \Sigma_{\text{ac}}$ with

$$E([a_0(\omega)^2 \operatorname{Im} m(x+i0,\omega)]^{-1}) < \infty \tag{5.23}$$

(known Lebesgue a.e. by Kotani theory; see [33, 10]), so U_n can be defined and is in L^2. We are heading towards a proof of

Theorem 5.2. *For any fixed matrix, Q, a.e. ω, as matrices*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} T_j(x,\omega)^t Q T_j(x,\omega) \tag{5.24}$$

exists.

Proof of Theorem 4 given Theorem 5.2. Pick $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then the $1,1$ matrix element of $T_j(x,\omega)^t Q T_j(x,\omega)$ is $p_j(x,\omega)^2$, so (1.45) holds. Similarly, the $2,2$ matrix element is $q_j(x,\omega)^2$. □

(5.22) plus (5.5) will imply critical a priori bounds on $\|T_n(x,\,\cdot\,)\|_{L^1(d\eta)}$. It will be convenient to use the Hilbert–Schmidt norm on these 2×2 matrices.

Lemma 5.3. *We have*

$$\sup_n \int \|T_n(x,\omega)\|\,d\eta(\omega) < \infty \tag{5.25}$$

Proof. Since $\det(U_n)=1$,

$$\|U_n(x,\omega)^{-1}\| = \|U_n(x,\omega)\| \tag{5.26}$$

Thus, by (5.22),

$$\|T_n(x,\omega)\| \le \|U_n(x,\omega)\|\,\|U_0(x,\omega)\| \tag{5.27}$$

By the Schwarz inequality,

$$\begin{aligned}\sup_n \int \|T_n(x,\omega)\|\,d\eta(\omega) &\le \sup_n \int \|U_n(x,\omega)\|^2\,d\eta(\omega)\\ &= \int \|U_0(x,\omega)\|^2\,d\eta(\omega)\\ &< \infty\end{aligned}$$

by (5.5) and the fact that since (5.4) holds and we use Hilbert–Schmidt norms,

$$\|U_j(x,\omega)\| = \|U_0(x,S^j\omega)\| \tag{5.28}$$

□

Let $A_j(\omega)$ be the matrix (1.32) with $a_j=a_j(\omega)$, $b_j=b_j(\omega)$ and let

$$A(\omega)\equiv A_1(\omega) \tag{5.29}$$

so

$$A_j(\omega) = A(S^{j-1}\omega) \tag{5.30}$$

and the transfer matrix for J_ω is

$$T_n(\omega) = A(S^{n-1}\omega)\dots A(\omega) \tag{5.31}$$

Now form the suspension

$$\widehat{\Omega} = \Omega\times \mathbb{SL}(2,\mathbb{C}) \tag{5.32}$$

and define $\widehat{S}\colon \widehat{\Omega}\to\widehat{\Omega}$ by

$$\widehat{S}(\omega,C) = (S\omega, A(\omega)C) \tag{5.33}$$

so

$$\widehat{S}^n(\omega,C) = (S^n\omega, T_n(\omega)C) \tag{5.34}$$

Theorem 5.4. *There exists an $\widehat{S}$-invariant probability measure, $d\nu$, on $\widehat{\Omega}$ whose projection onto Ω is $d\eta$ and with*

$$\int \|C\| \, d\nu(\omega, C) < \infty \tag{5.35}$$

Proof. Pick any probability measure μ_0 on $\mathbb{SL}(2,\mathbb{C})$ with $\int \|C\|^k \, d\mu_0(C) < \infty$ for all k. For example, one could take $d\mu_0(C) = Ne^{-\|C\|^2} d\,\mathrm{Haar}(C)$ where N is a normalization constant. Let $\widehat{S}_*$ be induced on measures on $\widehat{\Omega}$ by $[\widehat{S}_*(\nu)](f) = \nu(f \circ \widehat{S})$. Let

$$\nu_n = \widehat{S}_*^n(\eta \otimes \mu_0) \tag{5.36}$$

Then the invariance of η under S_* implies the projection of ν_n is η and

$$\begin{aligned} \int \|C\| \, d\nu_n &= \int \|T_n(\omega)C\| \, d\eta \otimes d\mu_0 \\ &\leq \left(\int \|T_n(\omega)\| \, d\eta \right) \left(\int \|C\| \, d\mu_0 \right) \end{aligned} \tag{5.37}$$

which, by (5.25), is uniformly bounded in n.

Let $\tilde{\nu}_n$ be the Cesàro averages of ν_n, that is,

$$\tilde{\nu}_n = \frac{1}{n} \sum_{j=0}^{n-1} \nu_j \tag{5.38}$$

So, by (5.37),

$$\sup_n \int \|C\| \, d\tilde{\nu}_n < \infty \tag{5.39}$$

so $\{\tilde{\nu}_n\}$ are tight, that is,

$$\lim_{K \to \infty} \sup_n \tilde{\nu}_n \{C \mid \|C\| \geq K\} \to 0$$

which implies that $\tilde{\nu}_n$ has a weak limit point in probability measures on $\widetilde{\Omega}$. This weak limit point is invariant and, by (5.39), it obeys (5.35). $\square$

Lemma 5.5. *Let $L < \infty$. Let*

$$\widehat{\Omega}_L = \{(\omega, C) \mid \|U_0(\omega)\| < L, \, \|C\| < L\} \tag{5.40}$$

Then for any ε, there is a K so that for a.e. $(\omega, C) \in \widehat{\Omega}_L$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\substack{j \in B(K,\omega,C) \\ 0 \leq j \leq n-1}} \|T_j(\omega)C\|^2 \leq \varepsilon \tag{5.41}$$

where

$$B(K, \omega, C) = \{j \mid \|T_j(\omega)C\| \geq K\} \tag{5.42}$$

Proof. Since $U_0(\omega) \in L^2(d\eta)$, we have

$$\lim_{s\to\infty} \int_{\|U_0(\omega)\|\geq s} \|U_0(\omega)\|^2 \, d\eta(\omega) = 0 \tag{5.43}$$

so for any $\delta > 0$, there exists $s(\delta)$ so that the integral is less than δ.

Let $\widetilde{B}(\widetilde{K}, \omega)$ be defined by

$$\widetilde{B}(\widetilde{K}, \omega) = \{j \mid \|U_j(\omega)\| \geq \widetilde{K}\} \tag{5.44}$$

By the Birkhoff ergodic theorem and (5.28) for a.e. ω,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{j\in\widetilde{B}(\widetilde{K},\omega) \\ 0\leq j\leq n-1}} \|U_j(\omega)\|^2 = \int_{\|U_0(\omega)\|\geq\widetilde{K}} \|U_0(\omega)\|^2 \, d\eta \leq \delta \tag{5.45}$$

if $\widetilde{K} \geq s(\delta)$.

Given ε and L, let $\delta = \varepsilon/L^2$ and $K \geq L^2 s(\delta)$. Since

$$\|T_j(\omega)C\| \leq \|U_j(\omega)\| L^2 \tag{5.46}$$

if $(\omega, C) \subset \Omega_L$,

$$B(K, \omega, C) \subset \widetilde{B}\biggl(\frac{K}{L^2}, \omega\biggr)$$

So, by (5.45) and (5.46),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{j\in B(K,\omega,C) \\ 0\leq j\leq n-1}} \|T_j(\omega)C\|^2 \leq L^2\delta = \varepsilon \tag{5.47}$$

which is (5.40). □

Proof of Theorem 5.2. Without loss, suppose $\|Q\| \leq 1$. Define on $\widehat{\Omega}$

$$f_n(\omega, C) = \frac{1}{n} \sum_{j=0}^{n-1} C^t T_j(x,\omega)^t Q T_j(x,\omega) C \tag{5.48}$$

If we prove that this has a pointwise limit for ν a.e. (ω, C), we are done: since η is the projection of ν, for η a.e. ω, there are some C for which (5.48) has a limit. But C is invertible, so $(C^t)^{-1} f_n C^{-1}$ has a limit, that is, (5.24) does.

Notice that if

$$h(\omega, C) = C^t Q C \tag{5.49}$$

then $f_n(\omega, C)$ is a Cesàro average of $h(\widehat{S}^j(\omega, C))$, so we can almost use the ergodic theorem except we only know a priori that $\int \|h(\omega, C)\|^{1/2} \, d\nu < \infty$, not $\int \|h(\omega, C)\| \, d\nu < \infty$, so we need to use Lemma 5.5.

Fix L and consider $(\omega, C) \in \widehat{\Omega}_L$. Let

$$h_K(\omega, C) = \begin{cases} C^t Q C & \text{if } \|C\| \leq K \\ 0 & \text{if } \|C\| > K \end{cases} \tag{5.50}$$

Then, since $\|Q\| \leq 1$,

$$\|h_K(\widehat{S}^j(\omega, C)) - h(\widehat{S}^j(\omega, C))\| \leq \begin{cases} 0 & \text{if } j \notin B(K, \omega, C) \\ \|T_j(\omega)C\|^2 & \text{if } j \in B(K, \omega, C) \end{cases} \tag{5.51}$$

It follows that if

$$f_n^{(K)}(\omega, C) = \frac{1}{n} \sum_{j=0}^{n-1} h_K(\widehat{S}^j(\omega, C)) \tag{5.52}$$

then

$$\|f_n^{(K)}(\omega, C) - f_n(\omega, C)\| \leq \text{sum on left side of (5.41)}$$

So, by Lemma 5.5,

$$\limsup_{n\to\infty} \|f_n^{(K)}(\omega, C) - f_n(\omega, C)\| \leq \varepsilon \tag{5.53}$$

if

$$K \geq K(\varepsilon, L) \tag{5.54}$$

given by the lemma.

For any finite K, h_K is bounded, so the Birkhoff ergodic theorem and the invariance of ν imply, for a.e. (ω, C), $\lim f_n^{(K)}(\omega, C)$ exists. Thus (5.53) and (5.54) imply that $\lim f_n^{(K)}(\omega, C)$ forms a Cauchy sequence as $K \to \infty$ (among, say, integer values), and that its limit is also $\lim f_n(\omega, C)$, for a.e. $(\omega, C) \in \widehat{\Omega}_L$.

Since L is arbitrary and $\nu(\widehat{\Omega} \setminus \widehat{\Omega}_L) \to 0$ on account of $\int \|U_0(\omega)\|^2 \, d\nu < \infty$, we see that f_n has a limit for a.e. ω, C. $\square$

6. Equality of the Local and Microlocal DOS

Our main goal in this section is to prove Theorem 5. We know from Theorem 4 that for a.e. $\omega \in \Omega$ and $x_0 \in \Sigma_{\text{ac}}$, we have that

$$\frac{1}{n+1} K_n(x_0, x_0) \to k_\omega(x_0) \tag{6.1}$$

some positive function. By Theorems 1 and 2, this implies that the spacing of zeros at a.e. Lebesgue point is

$$x_{j+1}^{(n)}(x_0) - x_j^{(n)}(x_0) \sim \frac{1}{n w_\omega(x_0) k_\omega(x_0)} \tag{6.2}$$

Thus, for fixed K large, in an interval $(x_0 - \frac{K}{n}, x_0 + \frac{K}{n})$, the number of zeros is $2Kw(x_0)k(x_0)$. On the other hand, if $\rho_\infty(x_0)$ is the density of states, for a.e. x_0 in the a.c. part of the support of $d\nu_\infty$, the number of zeros in $(x_0 - \delta, x_0 + \delta)$ is approximately $2\delta n\rho(x_0)$. If δ were K/n, this would tell us that

$$w_\omega(x_0)k_\omega(x_0) = \rho_\infty(x_0) \tag{6.3}$$

which is precisely (1.23).

Of course, ρ_∞ is defined by first taking $n \to \infty$ and then $\delta \downarrow 0$, so we cannot set $\delta = K/n$, but (6.3) is an equality of a local density of zeros obtained by taking intervals with $O(n)$ zeros as $n \to \infty$ and a microlocal individual spacing as in (6.2).

So define

$$\rho_L(x_0, \omega) = w_\omega(x_0)k_\omega(x_0) \tag{6.4}$$

the microlocal DOS. Notice that we have indicated an ω-dependence of ρ_L because, at this point, we have not proven ω-independence. ω-independence often comes from the ergodic theorem—we determined the existence of $k_\omega(x_0)$ using the ergodic theorem, but unlike for ρ_∞, the underlying measure was only invariant, not ergodic, and indeed, k_ω, the object we controlled is *not* ω-independent.

Of course, once we prove $\rho_L = \rho_\infty$, ρ_L will be proven ω-independent, but we will, in fact, go the other way: we first prove that ρ_L is ω-independent, use that to show that if u is the Deift–Simon wave function, then the average of u^2 (not $|u|^2$) is zero, and use that to prove that $\rho_L = \rho_\infty$.

Theorem 6.1. *Suppose that J_ω is a family of ergodic Jacobi matrices. Let $\rho_L(x, \omega)$ be given by* (6.1)/(6.4) *for $x \in \Sigma_{\text{ac}}$, $\omega \in \Omega$. Then for a.e. $x \in \Sigma_{\text{ac}}$, $\rho_L(x, \omega)$ is a.e. ω-independent.*

Proof. Since $\rho_L(x, \omega)$ is jointly measurable for $(x, \omega) \in \Sigma_{\text{ac}} \times \Omega$, $\rho_L(x, \cdot\,)$ is measurable for a.e. x. Since S is ergodic, it suffices to prove that $\rho_L(x, S\omega) = \rho_L(x, \omega)$ for a.e. (x, ω).

Let $p_n(x, \omega)$ be the OPs for J_ω. Then the zeros of $p_{n-1}(x, S\omega)$ and $p_n(x, \omega)$ interlace. It follows for any interval $[x_0 - \frac{A}{n}, x_0 + \frac{A}{n}] = I_{n,A}(x_0)$,

$$|\#\text{ of zeros of } p_n(x, \omega) \text{ in } I_{n,A}(x_0) - \#\text{ of zeros of } p_{n-1}(x, S\omega) \text{ in } I_{n,A}(x_0)| \leq 2 \tag{6.5}$$

If $\rho_L(x_0, S\omega) \neq \rho_L(x_0, \omega)$ and $A = k\rho_L(x_0, \omega)^{-1}$ with k large, it is easy to get a contradiction between (6.5) and (6.2). Thus, $\rho_L(x, \omega) = \rho_L(x, S\omega)$ as claimed. ∎

Next, we need a connection between ρ_L and u. Recall (see (5.14))

$$p_n(x,\omega) = \frac{\operatorname{Im} u_{n+1}^+(x,\omega)}{\operatorname{Im} u_1^+(x,\omega)} \tag{6.6}$$

and, by (5.13),

$$\operatorname{Im} u_1^+(x,\omega) = -[\operatorname{Im} m(x+i0,\omega)]^{1/2} \tag{6.7}$$

and, by (5.10), for a.e. $x \in \Sigma_{\text{ac}}$,

$$\operatorname{Im} m(x+i0,\omega) = \pi w_\omega(x) \tag{6.8}$$

Thus, if we define

$$\operatorname{Av}_\omega(f_j(\omega)) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n f_j(\omega) \tag{6.9}$$

then

$$\rho_L(x,\omega) = \frac{1}{\pi} \operatorname{Av}_\omega([\operatorname{Im} u_j^+(x,\omega)]^2) \tag{6.10}$$

Note that $\operatorname{Im} u_j^+(x,\omega)$ is not $\operatorname{Im} u_0^+(x,S^j\omega)$, so we cannot write (6.10) as an integral. In fact, the ω-independence of the right side of (6.10) (because of ω-independence of the left side) will have important consequences.

To see where we are heading, we note the following result of Kotani [20]; see Damanik [9, Thm. 5]:

Theorem 6.2 (Kotani [20])**.** *For a.e. $x \in \Sigma_{\text{ac}}$,*

$$\rho_\infty(x) = \frac{1}{2\pi} \int |u_0^+(x,\omega)|^2 \, d\eta(x) \tag{6.11}$$

Remarks. 1. [20, 9] treat $a_n \equiv 1$, but it is easy to accommodate general a_n.

2. Kotani's theorem is not stated in this form but rather as (see eqn. (22) in Damanik [9])

$$\pi\rho_\infty(x) = \int \operatorname{Im} G_\omega(0,0;x+i0)\, d\eta(\omega) \tag{6.12}$$

where G_ω is the whole-line Green's function. Because G_ω is reflectionless, G_ω is pure imaginary and

$$\operatorname{Im}(G_\omega(0,0;x+i0)) = [2a_0^2 \operatorname{Im} m(x+i0,\omega)]^{-1} \tag{6.13}$$

$$= \tfrac{1}{2}\, |u_0^+(x,\omega)|^2 \tag{6.14}$$

by (5.12).

Thus, the key to proving $\rho_L = \rho_\infty$ will be to show that

$$\mathrm{Av}_\omega([\operatorname{Im} u_j^+(x,\omega)]^2) = \mathrm{Av}_\omega([\operatorname{Re} u_j^+(x,\omega)]^2) \tag{6.15}$$

Note that (6.10) includes that the $\mathrm{Av}_\omega([\operatorname{Im} u_j^+]^2)$ exists and, by the ergodic theorem, $\mathrm{Av}_\omega(|u_j^+|^2)$ exists, so we know for a.e. $(x,\omega) \in \Sigma_{\mathrm{ac}} \times \Omega$ that $\mathrm{Av}_\omega([\operatorname{Re} u_j^+(x,\omega)]^2)$ exists. We are heading towards:

Theorem 6.3. *Suppose $x \in \Sigma_{\mathrm{ac}}$ is such that $\rho_L(x,\omega)$ exists for a.e. ω and is ω-independent, and that*

$$\nu_\infty((-\infty, x]) \neq \tfrac{1}{2} \tag{6.16}$$

Then for a.e. ω,

$$\mathrm{Av}_\omega((u_j^+(x,\omega))^2) = 0 \tag{6.17}$$

Proof of Theorem 5 given Theorem 6.3. (6.16) fails at at most a single x in Σ_{ac}, so (6.17) holds for a.e. $(x,\omega) \in \Sigma_{\mathrm{ac}} \times \Omega$. Its real part implies (6.15), and so for a.e. (x,ω),

$$\mathrm{Av}_\omega([\operatorname{Im} u_j^+(x,\omega)]^2) = \tfrac{1}{2}\,\mathrm{Av}_\omega(|u_j^+(x,\omega)|^2) \tag{6.18}$$

$$= \tfrac{1}{2} \int |u_0^+(x,\omega)|^2 \, d\eta(x) \tag{6.19}$$

by the ergodic theorem. By (6.10), (6.11), and the definition (6.1)/(6.4) of ρ_L, we see that the limit in (1.45) is $\rho_\infty(x)/w_\omega(x)$. □

Proof of Theorem 6.3. Fix $x \in \Sigma_{\mathrm{ac}}$ (at each stage, we work up to sets of Lebesgue measure 0). Define $\varphi(\omega) \in (0, 2\pi)$ by

$$\operatorname{Arg}(-m(x+i0,\omega)) = -\varphi(\omega) \tag{6.20}$$

Then $\varphi(\omega) \in (0,\pi)$ by $\operatorname{Im} m > 0$. Let (φ and s_n also depend on x)

$$s_n(\omega) = \sum_{j=1}^n \varphi(S^{j-1}\omega) \tag{6.21}$$

Then by (5.8) and (5.4),

$$u_n^+(x,\omega) = e^{-is_n(\omega)} u_0^+(x, S^n\omega) \tag{6.22}$$

and

$$u_{n+j}^+(x,\omega) = e^{-is_n(\omega)} u_j^+(x, S^n\omega) \tag{6.23}$$

It follows that for each fixed n,

$$\mathrm{Av}_\omega((\operatorname{Im} u_j^+((x, S^n\omega))^2) = \mathrm{Av}_\omega((\operatorname{Im} e^{is_n(\omega)} u_j^+(x,\omega))^2) \tag{6.24}$$

If s, x, y are real,

$$\begin{aligned}(\operatorname{Im}(e^{is}(x+iy)))^2 &= (x\sin s + y\cos s)^2 \\ &= y^2 + (\sin^2 s)(x^2 - y^2) + xy(\sin 2s)\end{aligned} \tag{6.25}$$

and thus,

$$\begin{aligned}\text{LHS of (6.24)} = \text{Av}_\omega([\text{Im}(u_j^+(x,\omega))]^2) &+ \sin^2 s_n(\omega) R(\omega) \\ &+ \tfrac{1}{2}\sin(2s_n(\omega))I(\omega)\end{aligned} \tag{6.26}$$

where

$$R(\omega) = \text{Av}_\omega(\text{Re}((u_j^+(x,\omega))^2)) \tag{6.27}$$

$$I(\omega) = \text{Av}_\omega(\text{Im}((u_j^+(x,\omega))^2)) \tag{6.28}$$

(all such averages having been previously shown to exist).

We know that for a.e. (x,ω), for $n=0,1,2,\dots$, LHS of (6.24) exists and is n-independent (and equal to $\rho_L(x,\omega)$). For such (x,ω), (6.26) implies that for all n,

$$\sin s_n(\omega)[\sin s_n(\omega)R(\omega) + \cos s_n(\omega) I(\omega)] = 0 \tag{6.29}$$

We want to consider two cases:

Case 1. For a positive measure set of ω,

$$s_2(\omega)=\pi \qquad s_4(\omega)=2\pi \qquad s_6(\omega)=3\pi \qquad \dots \tag{6.30}$$

Case 2. For a.e. ω, there is an $n(\omega)$ so

$$s_{2j}(\omega) = j\pi \quad (j=1,\dots,n-1) \qquad s_{2n}(\omega)\neq n\pi \tag{6.31}$$

In Case 1, for such ω, we have $\frac{s_n(\omega)}{n\pi}\to\frac{1}{2}$. It follows by standard Sturm oscillation theory (see, e.g., [18]) that $\frac{s_n(\omega)}{n\pi}\to\nu_\infty((-\infty,x])$ for almost every ω. Thus, the hypothesis (6.16) eliminates Case 1.

For Case 2, suppose first that n is odd, so $s_{2(n-1)}(\omega)$ is a multiple of 2π and (6.21) for $2n-1$ and $2n$ imply

$$\sin(\varphi_{2n-1})[\sin(\varphi_{2n-1})R + \cos(\varphi_{2n-1})I] = 0 \tag{6.32}$$

$$\sin(\varphi_{2n-1}+\varphi_{2n})[\sin(\varphi_{2n-1}+\varphi_{2n})R + \cos(\varphi_{2n-1}+\varphi_{2n})I] = 0 \tag{6.33}$$

Since $\varphi_{2n-1}\in(0,\pi)$, $\sin(\varphi_{2n-1})\neq 0$ and since $\varphi_{2n-1}+\varphi_{2n}\in(0,2\pi)\setminus\{\pi\}$, (for if it equals π, then $s_{2n}=n\pi$!), $\sin(\varphi_{2n-1}+\varphi_{2n})\neq 0$.

The determinant of equations (6.32)/(6.33) is

$$-\sin(\varphi_{2n-1})\sin(\varphi_{2n-1}+\varphi_{2n})\sin(\varphi_{2n})\neq 0 \tag{6.34}$$

since

$$\sin(A)\cos(B) - \sin(B)\cos(A) = \sin(A-B) \tag{6.35}$$

Here $\neq 0$ in (6.34) comes from $\varphi_{2n}\in(0,\pi)$, so $\sin(\varphi_{2n})\neq 0$.

The nonzero determinant means that (6.32)/(6.33) $\Rightarrow I=R=0$, that is, $\text{Av}_\omega((u_j^+)^2)=0$ for a.e. ω. If n is even, $s_{2(n-1)}(\omega)$ is an odd multiple of π and all equations pick up minus signs, so the argument is unchanged. □

7. Assorted Remarks

1. We have proven for general ergodic Jacobi matrices that for a.e. $(x,\omega)\in\Sigma_{\text{ac}}\times\Omega$,

$$\frac{1}{n+1}\,K_n(x,x;\omega)\to\frac{\rho_\infty(x)}{w_\omega(x)} \tag{7.1}$$

Here ρ_∞ is the Radon–Nikodym derivative of the a.c. part of $d\rho_\infty$. Based on [30, 45], where results of this type are proven for regular measures, one expects

$$\rho_\infty(x)=\rho_{\mathfrak{e}}(x) \tag{7.2}$$

Here $\mathfrak{e}$ is the essential spectrum of J_ω and $\rho_{\mathfrak{e}}$ its equilibrium measure. In [37], it is proven (see Thm. 1.15 there)

Theorem 7.1. *If* Σ_{ac} *is not empty, then* (7.2) *holds if and only if, for* $\rho_{\mathfrak{e}}$ *a.e.* x*,*

$$\gamma(x)=0 \tag{7.3}$$

In particular, for examples where (7.3) fails on a set of positive Lebesgue measure in $\mathfrak{e}$ (e.g., [6, 7, 13, 14]), (7.2) may not hold. On the other hand, for examples like the almost Mathieu equation where it is known that (7.3) holds on all of $\mathfrak{e}$ (see [8]), (7.2) holds. The moral is that (7.2) holds some, but not all, of the time for ergodic Jacobi matrices.

2. Here is an interesting example that provides a deterministic problem where one has strong clock behavior but with a density of zeros, ρ_∞, which is not $\rho_{\mathfrak{e}}$. Let $d\mu$ be a measure on $[-2,2]$ of the form (N is a normalization constant)

$$d\mu(x)=N^{-1}\biggl[\chi_{[-1,1]}(x)\,dx+\sum_{n=1}^\infty e^{-n^2}\delta_{x_n}\biggr] \tag{7.4}$$

where $\{x_n\}$ is a dense subset of $[-2,2]\backslash(-1,1)$. Then, as in Example 5.8 of [37], ρ_∞ exists and is the equilibrium measure for $[-1,1]$ (not $\mathfrak{e}=[-2,2]$). Moreover, the method of [25] shows that for $x\in(-1,1)$,

$$\frac{1}{n+1}\,K_n(x,x)\to\frac{\rho_\infty(x)}{N^{-1}} \tag{7.5}$$

Using either the method of this paper (i.e., of [26]) or the method of [25], one proves universality with ρ_∞.

3. Example 5.8 of [37] provides a measure with $\sigma_{\text{ess}}(\mu)=[-2,2]$ but $\Sigma_{\text{ac}}=[-2,0]$ and where ν_n has multiple weak limits, including the equilibrium measures for $[-2,0]$ and for $[-2,2]$. By general principles [42], the set of limits is connected, so uncountable. One would like to

prove that quasi-clock behavior nevertheless holds for the a.c. spectrum of this model as this will provide a key test for the conjecture that quasi-clock behavior always holds on $\Sigma_{\rm ac}$.

4. What has sometimes been called the Schrödinger conjecture (see [29]) says that for any Jacobi matrix and a.e. $x \in \Sigma_{\rm ac}(\mu)$, we have a solution, u_n, with

$$0 < \inf_n |u_n| \leq \sup_n |u_n| < \infty \tag{7.6}$$

and $u_{-1} = 0$. Invariance of $\Sigma_{\rm ac}$ under rank one perturbations then proves that for a.e. $x \in \Sigma_{\rm ac}(\mu)$, the transfer matrix is bounded. Thus, Theorem 3 in the strong form would always be applicable.

5. While (6.16) is harmless since it only eliminates at most one x, one can ask if (6.17) holds even if (6.16) fails. Using periodic problems, it is easy to construct ergodic cases where $\arg u_n^+ = -\pi n/2$, so (6.29) provides no information on $I(\omega)$. Nevertheless, in these cases, one can show $R(\omega) = I(\omega) = 0$. We have not been able to find an example where for a set of positive measure ω's, $s_{2n}(\omega) = n\pi$, $s_{2n+1}(\omega) = n\pi + \varphi$ with φ some fixed point in $(0,\pi) \setminus \{\frac{\pi}{2}\}$. In that case, it might happen that $R(\omega) \neq 0$, $I(\omega) \neq 0$. So it remains open if we need to exclude the x with (6.16).

6. While we could use soft methods in Section 3, at one point in our research we used an explicit formula for the derivative of $\frac{1}{n} K_n(x_0 + \frac{a}{n}, x_0 + \frac{a}{n})$ as a function of a that may be useful in other contexts, so we want to mention it. We start with a variation of parameters formula (discussed, e.g., in [17, 19]) that, in terms of the second kind polynomials of (1.38),

$$p_n(x) - p_n(x_0) = (x - x_0) \sum_{m=0}^{n-1} (p_n(x_0) q_m(x_0) - p_m(x_0) q_n(x_0)) p_m(x) \tag{7.7}$$

which implies

$$p_n'(x_0) = \sum_{m=0}^{n-1} (p_n(x_0) q_m(x_0) - p_m(x_0) q_n(x_0)) p_m(x_0) \tag{7.8}$$

Since

$$\frac{d}{da} \frac{1}{n} K_n\left(x_0 + \frac{a}{n}, x_0 + \frac{a}{n}\right)\bigg|_{a=0} = \frac{1}{n^2} \sum_{j=0}^{n} 2p_j'(x_0) p_j(x_0) \tag{7.9}$$

this leads to

$$\frac{d}{da}\frac{1}{n}K_n\left(x_0+\frac{a}{n},x_0+\frac{a}{n}\right)\Big|_{a=0}$$
$$=\frac{2}{n^2}\sum_{j=0}^{n}\left[p_j(x_0)^2\left(\sum_{k=0}^{j}p_k(x_0)q_k(x_0)\right)-q_j(x_0)p_j(x_0)\left(\sum_{k=0}^{j}p_k(x_0)^2\right)\right] \tag{7.10}$$

As noted in [38], if $\frac{1}{n}\sum_{j=0}^{n}p_j(x_0)^2$ and $\frac{1}{n}\sum_{j=0}^{n}p_j(x_0)q_j(x_0)$ have limits and $\sup_n[\frac{1}{n}\sum_{j=0}^{n}q_j(x_0)^2]<\infty$, then the right side of (7.10) goes to 0.

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