EXAMPLES OF FEIGENBAUM JULIA SETS WITH SMALL HAUSDORFF DIMENSION

ARTUR AVILA AND MIKHAIL LYUBICH

To Bodil Branner on her 60th birthday

Abstract. We give examples of infinitely renormalizable quadratic polynomials $F_c : z \mapsto z^2 + c$ with stationary combinatorics whose Julia sets have Hausdorff dimension arbitrary close to 1. The combinatorics of the renormalization involved is close to the Chebyshev one. The argument is based upon a new tool, a "Recursive Quadratic Estimate" for the Poincaré series of an infinitely renormalizable map.

1. Introduction

One of the most remarkable objects in complex dynamics are the fixed points of the Douady-Hubbard renormalization operator. Such objects have a distinguished place in the dictionary between rational maps and Kleinian groups (see [Mc2]). Existence of the renormalization fixed points established in the works of Sullivan [S2] and McMullen [Mc2] (under certain assumptions) implies many beautiful features (self-similarity, universality, hairyness,...) of Feigenbaum Julia sets (see §2.2 for the definition). However, even with this thorough information, some basic questions concerning measure and dimension of these Julia sets have remained unsettled.

One of the key questions (asked, for instance, in [Mc2]) regarding the geometry of Feigenbaum Julia sets has been the following: *Is the Hausdorff dimension of a Feigenbaum Julia set always equal to* 2? In [AL] we supply a fairly large class of Feigenbaum Julia sets with $\mathrm{HD}(J) < 2$, thus giving a negative answer to the above question. In this paper we show that in fact the dimension of a Feigenbaum Julia set can be arbitrarily close to 1:

Theorem 1.1. *There exists a sequence of Feigenbaum quadratic polynomial $F_p : z \mapsto z^2 + c_p$ with $c_p \in \mathbb{R}$, $c_p \to -2$, such that $\mathrm{HD}(J(F_p)) \to 1$ as $p \to \infty$.*

Hausdorff dimension is closely related to another geometric characteristic of the Julia set, the *critical exponent* δ_{cr} (see §2.3). In fact, for a Feigenbaum map F_c,

$$\mathrm{HD}(J(F_c)) = \delta_{\mathrm{cr}}(J(F_c)),$$

provided $\mathrm{meas}(J(F_c)) = 0$ [AL],[1] and the same is true for the associated renormalization fixed point f_c. This allows us to reduce Theorem 1.1 to the following two results.

Theorem 1.2. *Let f_p be the fixed point of the renormalization operator of period p with combinatorics closest to the Chebyshev one. Then $\delta_{\mathrm{cr}}(f_p) \to 1$ as $p \to \infty$.*

The proof of this theorem is based upon a "Recursive Quadratic Estimate" for the Poincaré series which provides a new efficient tool for getting bounds on the critical exponent.

Theorem 1.3. *For large p,* $\mathrm{area}\, J(f_p) = 0$.

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[1]It follows from Bishop's work [B] that for any F_c, $\mathrm{HD}(J(F_c)) \le \delta_{\mathrm{cr}}(J(F_c))$, provided $\mathrm{meas}(J(F_c)) = 0$.

Remarks. 1. The class of Feigenbaum maps with $\mathrm{HD}(J(f)) < 2$ supplied in [AL] is qualitatively the same as the class treated in Yarrington's thesis [Y] (see also §9 of [AL]) for which $\mathrm{area}(J) = 0$ (which in turn, is qualitatively the same as the class of infinitely renormalizable maps for which *a priori* bounds were established in [L2]). Though Theorem 1.3 is not formally covered by [AL, Y], it is proved by a similar method, which becomes more direct in our situation. Similarly, to prove Theorem 1.2 we adjust the method of [AL] to the Chebyshev combinatorics, which makes it (in this combinatorial case) simpler and more powerful.

2. There is a plenty of quadratic polynomials whose Julia sets have Hausdorff dimension two [Sh]. However, it is still unknown whether there exist Feigenbaum Julia sets with this property.

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2. Basic concepts

2.1. **Notations.** $\mathbb{D}_r(z) \equiv \{w \in \mathbb{C}, |w - z| < r\}$, $\mathbb{D}_r \equiv \mathbb{D}_r(0)$. A domain is a connected open subset of $\mathbb{C}$. A topological disk is a simply connected domain. $U \Subset V$ means that U is *compactly contained* in V.

Notation $a \asymp b$ means that $C^{-1} < a/b < C$ with a constant $C > 0$ independent of particular a and b under consideration; $a \approx b$ means that a is close to b.

We usually denote the p-fold iterate of a map f by f^p, but occasionally use a more forceful notation $f^{\circ p}$.

Let $\omega(x) \equiv \omega_f(x) = \cap_{m \geq 0} \overline{\{f^k(x),\, k \geq m\}}$ denote the *ω-limit set* of x.

For a quadratic-like map $f : U \to V$ (see below) with the critical point at 0, let $\mathcal{O}(f) \equiv \overline{\{f^k(0),\, k > 0\}}$ denote its *postcritical set*.

2.2. **Quadratic-like maps and renormalization.** A *quadratic-like map* is a holomorphic double covering map $f : U \to V$ where $U, V \subset \mathbb{C}$ are topological disks and $U \Subset V$. Such a map has a unique critical point which we will assume to be 0. Let $K(f) \equiv \cap_{k=0}^{\infty} f^{-k}(U)$ denote the *filled Julia set* of f and let $J(f) \equiv \partial K(f)$ denote its *Julia set*.

Two quadratic-like germs f and g are said to be *hybrid equivalent* if there exists a quasiconformal map $h : \mathbb{C} \to \mathbb{C}$ satisfying $h(f(x)) = g(h(x))$ for x near $J(f)$ such that $\overline{\partial} h | J(f) = 0$. Any quadratic-like map $f : U \to V$ with connected Julia set is hybrid equivalent to a unique quadratic polynomial $F : z \mapsto z^2 + c$ called the *straightening* of f [DH]. Moreover, the dilatation $\mathrm{Dil}(h)$ of the (appropriately chosen) conjugacy h depends only on $\mathrm{mod}(V \setminus U)$, and $\mathrm{Dil}(h) \to 1$ as $\mathrm{mod}(V \setminus U) \to \infty$.

The Julia set $J(f)$ of a quadratic-like map is either connected or Cantor. If $J(f)$ is connected, there exists a unique repelling or parabolic fixed point $\beta = \beta(f) \in J(f)$ such that $J(f) \setminus \{\beta(f)\}$ is connected. The other fixed point is denoted by $\alpha = \alpha(f)$. We will only consider quadratic-like maps with connected Julia set.

A quadratic-like map which is considered only up to choice of domains is called a *quadratic-like germ*. More precisely, one says that two quadratic-like maps with connected Julia sets represent the same germ if they have a common Julia set and coincide in a neighborhood of it. We shall consider quadratic-like germs up to affine conjugacy.

A quadratic-like map $f : U \to V$ is called *renormalizable* with period $p > 1$ if there exist topological disks $U' \Subset V'$ containing the critical point such that

(1) $g \equiv f^p : U' \to V'$ is a quadratic-like map with connected Julia set $J(f')$ called a *pre-renormalization* of f;

(2) For every $1 \leq k \leq p-1$, either $f^k(J(g)) \cap J(g) = \emptyset$ or $f^k(J(g)) \cap J(g) = \{\beta(g)\}$.

The renormalization operator R is defined on the space of germs by letting $Rf = g$. The minimal $p = p(f) > 1$ for which f is renormalizable is called the *renormalization period* of f. In what follows, the operator R will always correspond the this minimal period.

A quadratic-like map $f : U \to V$ is said to be a *renormalization fixed point* if f is renormalizable and $Rf = f$. In other words, $f^p(x) = \lambda f(\lambda^{-1}x)$ near $J(g)$ for some $\lambda \in \mathbb{D} \smallsetminus \{0\}$, where p is the renormalization period of f and g is a pre-renormalization of f.

2.3. **Poincaré series.** Let $f : U \to V$ be a quadratic-like map.

Sullivan's Poincaré series [S1] is defined as follows:

$$\Xi_\delta(z) = \sum_{k=0}^{\infty} \sum_{f^k(w)=z} |Df^k(w)|^{-\delta}, \quad z \in V \smallsetminus \mathcal{O}(f), \quad \delta > 0.$$

It follows from the Koebe Distortion Theorem that $\Xi_\delta(z) \leq C(z, z')^\delta \Xi_\delta(z')$ for any $z, z' \in V \smallsetminus \mathcal{O}(f)$. In particular, Ξ_δ is finite or infinite independently of z.

The function $\delta \mapsto \Xi_\delta$ is obviously convex. By definition, the *critical exponent*, $\delta_{\mathrm{cr}}(f) \in [0, \infty]$, is the unique value of δ that separates convergent Ξ_δ from divergent ones. The critical exponent is well defined on the level of germs.

It is easy to see that Ξ_2 is always finite (area argument) and, since $J(f)$ is assumed to be connected, $\Xi_1 = \infty$ (length argument), see §2.9 of [AL]. Thus we actually have $\delta_{\mathrm{cr}}(f) \in [1, 2]$.[2]

2.3.1. *Poincaré series for subfamilies of orbits.* An *orbit of length* $k \geq 0$ is a sequence $(x_0, ..., x_k)$, where $x_k \in V$ and $f(x_i) = x_{i+1}$ for $0 \leq i < k$. An orbit of zero length is called *trivial.*

Given a family $\mathcal{F}$ of orbits $(x_0, ..., x_k)$, we define a function $\mathbb{C} \to [0, \infty]$

$$\Xi_\delta(\mathcal{F})(z) = \sum_{k=0}^{\infty} \sum_{(x_0,\ldots,x_k=z)\in\mathcal{F}} |Df^k(x_0)|^{-\delta}$$

(to keep notation shorter, we do not explicitly mention f). Let $\Xi_\delta^{[j]}$ denote the truncation of Ξ_δ at level j,

$$\Xi_\delta^{[j]}(\mathcal{F})(z) = \sum_{k=0}^{j} \sum_{(x_0,\ldots,x_k=z)\in\mathcal{F}} |Df^k(x_0)|^{-\delta},$$

with convention that $\Xi^{[j]} = 0$ for $j < 0$. Note that $\Xi^{[0]}(\mathcal{F})$ is equal to 1 or 0 depending on whether $\mathcal{F}$ contains the trivial orbit or not.

2.3.2. *Arrow notation.* Let us introduce a convenient notation for certain families of orbits. By $D \leftarrow E$, we will understand the family of orbits $(x_0, ..., x_k)$ with $x_0 \in E$ and $x_k \in D$. The family of orbits $(x_0, ..., x_k)$ with $x_0 \in E$, $x_k \in D$ and $x_1, ..., x_{k-1} \in F$ will be denoted $D \underset{F}{\leftarrow} E$. A "plus sign" over the arrow will indicate that only non-trivial orbits are considered. The juxtaposition of arrows will denote composition in the natural way. For instance,

$$D \overset{+}{\underset{F}{\leftarrow}} D \underset{F}{\leftarrow} E,$$

denotes the family of orbits $(x_0, ..., x_k)$, with $x_0 \in E$, $x_k \in D$, such that $x_i \in D$ for some $0 \leq i < k$, and $x_1, ..., x_{i-1}, x_{i+1}, ..., x_{k-1} \in F$.

[2] In fact, $\delta_{\mathrm{cr}} > 1$, unless $J(f)$ is a real analytic curve.

3. Quadratic recursive estimate

We will now introduce a version of the Quadratic Recursive Estimate which is sufficient for purposes of this paper (see [AL] for a finer version). We shall restrict ourselves to the case of renormalization fixed points. The argument is based on a combinatorial breakdown of orbits which exploits the scaling self-similarity of the dynamics.

Let $f : U \to V$ be a fixed point of renormalization of period p, $f^p(x) = \lambda f(\lambda^{-1}x)$ near 0. Let $U' = \lambda U$, $V' = \lambda V$, and let $gx \equiv f^p : U' \to V'$. Let $A = V \smallsetminus U$, $A' = V' \smallsetminus U'$. We assume that $V' \subset U$, g is the first return from U' to V', and that $\mathcal{O}(f)$ does not intersect $\bar{A}'$.

Lemma 3.1. *Let $s_j(\delta) = \sup_{z\in A'} \Xi_\delta^{[j]}(A' \leftarrow U)(z)$. Then*

$$s_{j+1}(\delta) \le P_\delta(s_j(\delta)),$$

where $x \mapsto P_\delta(x)$ is a quadratic polynomial with positive coefficients which can be expressed explicitly in terms of the Poincaré series $\Xi_\delta(\mathcal{F})$ over families $\mathcal{F}$ of orbits that do not accumulate on 0.

If P_δ has a positive fixed point s then

$$\sup_{z\in A'} \Xi_\delta(A' \leftarrow U)(z) = \lim s_j \le s,$$

so that $\delta_{\rm cr}(f) \le \delta$.

Proof. In what follows, the sup is always taken over z, the terminal point of the orbit in question. We will also omit the truncation parameter (j or $j+1$) in the notation.

We can decompose $A \leftarrow U$ into two groups: $A \xleftarrow[U\smallsetminus V']{} U \smallsetminus V'$ and $A \xleftarrow[U\smallsetminus V']{} A' \leftarrow U$. This gives the inequality

$$\sup \Xi_\delta(A \leftarrow U) \le \sup \Xi_\delta(A \xleftarrow[U\smallsetminus V']{} U \smallsetminus V') + \sup \Xi_\delta(A \xleftarrow[U\smallsetminus V']{} A') \sup \Xi_\delta(A' \leftarrow U). \tag{3.1}$$

In turn, we can decompose $A' \xleftarrow{+} U$ into two groups:

(1) $A' \xleftarrow[U\smallsetminus A']{} U \smallsetminus A'$, which can be further decomposed into

$$A' \xleftarrow[U\smallsetminus V']{} U \smallsetminus V', \quad A' \xleftarrow[U\smallsetminus A']{} U', \quad \text{and} \quad A' \xleftarrow[U\smallsetminus A']{} U' \xleftarrow[U\smallsetminus V']{} U' \smallsetminus V';$$

(2) $A' \xleftarrow[U\smallsetminus A']{+} A' \leftarrow U$, which can be further decomposed into

$$A' \xleftarrow[U\smallsetminus V']{+} A' \leftarrow U \quad \text{and} \quad A' \xleftarrow[U\smallsetminus A']{} U' \xleftarrow[U\smallsetminus V']{} A' \leftarrow U.$$

This gives the following inequality

$$\begin{aligned} \sup \Xi_\delta(A' \leftarrow U) \le 1 &+ \sup \Xi_\delta(A' \xleftarrow[U\smallsetminus V']{} U \smallsetminus V') \\ &+ \sup \Xi_\delta(A' \xleftarrow[U\smallsetminus A']{} U') \left(1 + \sup \Xi_\delta(U' \xleftarrow[U\smallsetminus V']{} U \smallsetminus V')\right) \\ &+ \sup \Xi_\delta(A' \xleftarrow[U\smallsetminus V']{+} A') \sup \Xi_\delta(A' \leftarrow U) \\ &+ \sup \Xi_\delta(A' \xleftarrow[U\smallsetminus A']{} U') \sup \Xi_\delta(U' \xleftarrow[U\smallsetminus V']{} A') \sup \Xi_\delta(A' \leftarrow U), \end{aligned} \tag{3.2}$$

where the first term, 1, accounts for the trivial orbits.

Notice that since $x \mapsto f^p x$ is the first return map from U' to V', if $(x_0, ..., x_k)$ belongs to $A \leftarrow U$ then $(\lambda x_0, ..., f^{kp}(\lambda x_0))$ belongs to $A' \xleftarrow[U\smallsetminus A']{} U'$. This correspondence is readily seen to be a bijection between $A \leftarrow U$ and $A' \xleftarrow[U \setminus A']{} U'$ preserving the weights of the Poincaré series. Hence

$$\Xi_\delta(A \leftarrow U)(x) = \Xi_\delta(A' \xleftarrow[U \setminus A']{} U')(\lambda x)$$

and

$$\sup \Xi_\delta(A' \xleftarrow[U \setminus A']{} U') = \sup \Xi_\delta(A \leftarrow U). \tag{3.3}$$

Plugging (3.1) into (3.3), and then plugging the resulting expression for $\sup \Xi_\delta(A' \xleftarrow[U \setminus A']{} U')$ into the 2nd and 4th lines of (3.2), we obtain

$$\sup \Xi_\delta(A' \leftarrow U) \leq \alpha + \beta \sup \Xi_\delta(A' \leftarrow U) + \gamma \sup \Xi_\delta(A' \leftarrow U)^2,$$

where

$$\alpha = 1 + \sup \Xi_\delta(A' \xleftarrow[U \setminus V']{} U \setminus V') + \sup \Xi_\delta(A \xleftarrow[U \setminus V']{} U \setminus V') \left(1 + \sup \Xi_\delta(U' \xleftarrow[U \setminus V']{} U \setminus V')\right), \tag{3.4}$$

$$\begin{aligned} \beta = {} & \sup \Xi_\delta(A' \xleftarrow[U \setminus V']{+} A') + \sup \Xi_\delta(A \xleftarrow[U \setminus V']{} A') \left(1 + \sup \Xi_\delta(U' \xleftarrow[U \setminus V']{} U \setminus V')\right) \\ & + \sup \Xi_\delta(A \xleftarrow[U \setminus V']{} U \setminus V') \sup \Xi_\delta(U' \xleftarrow[U \setminus V']{} A'), \end{aligned} \tag{3.5}$$

and

$$\gamma = \sup \Xi_\delta(A \xleftarrow[U \setminus V']{} A') \sup \Xi_\delta(U' \xleftarrow[U \setminus V']{} A'). \tag{3.6}$$

This is the desired quadratic recurrence estimate for $\sup \Xi_\delta(A' \leftarrow U)$. The above three formulas give an explicit expression of the coefficients α, β and γ of P_δ in terms of Poincaré series over families of orbits that do not accumulate on 0.

For the last statement, notice that $s_j \leq P^j(s_{-1}) = P^j(0) \leq s$ for all j. Thus, for every $z \in A'$ we have $\Xi_\delta(z) \leq \sup \Xi_\delta(A' \leftarrow U) = \lim_{j \to \infty} s_j \leq s$, which shows that $\delta_{\mathrm{cr}}(f) \leq \delta$. $\square$

4. Renormalization with combinatorics closest to Chebyshev

In this section we will show that the critical exponent of maps with combinatorics "close to Chebyshev" can be arbitrarily close to 1. Our bounds on the critical exponent will be based on direct estimates of the coefficients of the quadratic recursive polynomial corresponding to a nearly Chebyshev map.

4.1. **Basic properties.** Let $Ч(x) = 2 - x^2$ be the Chebyshev polynomial. Let f_p *be the fixed point of the renormalization operator of period* p*, with (real) combinatorics closest to Chebyshev*: f_p is combinatorially characterized among fixed points of renormalization of period p by being (up to affine conjugacy) a real-symmetric quadratic-like germ such that $f_p(0) > 0$ and $f_p^i(0) < 0$, $1 < i < p$. The existence of f_p is a particular case of a result of Sullivan [MS].

We normalize f_p so that its orientation preserving fixed point is -2. Let $-1 < \lambda_p < 0$ be the scaling factor of f_p. Then we have near zero:

$$g_p := f_p^{\circ p}(x) = \lambda_p f_p(\lambda_p^{-1} x).$$

Notice that $[-2, 2] \subset J(f_p)$ and $f_p : [-2, 2] \to [-2, 2]$ is a unimodal map. Let $\alpha_p > 0$ stand for the orientation reversing fixed point of f_p.

A basic fact is that all of the f_p belong to some fixed *Epstein class*, that is, there exists $\epsilon > 0$ such that $f_p : [-2, 2] \to [-2, 2]$ extends to a real-symmetric double covering onto the slit plane $\mathbb{C} \setminus (\mathbb{R} \setminus (-2 - \epsilon, 2 + \epsilon))$. (The natural topology in such an Epstein class makes it a compact space.)

This is a consequence of the real *a priori* bounds, see [MS]. This yields a number of nice properties of the maps f_p. The ones that are relevant for us are summarized in the following lemma:

Lemma 4.1. *Let $p \geq 3$, $T' = (-\alpha_p, \alpha_p)$, $\mathbb{V}' = \{z : |z| < \alpha_p\}$, and let $\mathbb{U}'$ be the component of $f_p^{-p}(\mathbb{V}')$ containing 0. Let $\mathbb{U} = \lambda^{-1}\mathbb{U}'$ and $\mathbb{V} = \lambda^{-1}\mathbb{V}'$. Then*

1. *f_p extends to a double covering onto the slit plane $\mathbb{C} \smallsetminus (\mathbb{R} \smallsetminus T)$;*
2. *$f_p \to Ч$ uniformly in $[-2, 2]$ (in particular $f_p(0) \to 2$ and $\alpha_p \to 1$);*
3. *the maps $g_p : \mathbb{U}' \to \mathbb{V}'$ and $f_p : \mathbb{U} \to \mathbb{V}$ are quadratic-like for p sufficiently large;*
4. *$\mathrm{mod}(\mathbb{V} \smallsetminus \overline{\mathbb{U}}) = \mathrm{mod}(\mathbb{V}' \smallsetminus \overline{\mathbb{U}'}) \to \infty$;*
5. *$\lambda_p \to 0$.*

Proof. Let $S_k \subset [-2, 2]$ be the component of $(f_p|[-2, 2])^{-(p-k)}(T')$ containing $f_p^k(0)$, $k = 0, 1, \ldots, p$. Since the intervals $[-2, -\alpha_p]$ and $[\alpha_p, 2]$ are monotonically mapped by f_p onto $[-2, \alpha_p]$, the maps $f_p : S_k \to S_{k+1}$ are diffeomorphisms for $k = 1, 2, \ldots, p - 1$. This implies the first assertion by rescaling.

The second assertion follows from the compactness of the Epstein class and the first assertion.

Moreover, $|S_1|^{-1/p} \approx \mathrm{dist}(S_1, f_p(0))^{-1/p} \approx 4$, where "4" is the multiplier of the orientation preserving fixed point -2 of Ч. Since f_p belongs to the Epstein class, the component of $f_p^{-(p-1)}(\mathbb{V}')$ containing $f_p(0)$ is contained in the round disk with diameter S_1. Hence $(\mathrm{diam}\, \mathbb{U}')^{-1/p} \approx 2$, which implies assertions (3) and (4) for g_p. The corresponding assertions for f_p are obtained by rescaling.

Since

$$\lambda_p = \frac{\mathrm{diam}\, J(g_p)}{\mathrm{diam}\, J(f_p)} \leq \frac{1}{4}\,\mathrm{diam}\, \mathbb{U}',$$

assertion (5) follows, too. □

4.2. **Estimates for the coefficients.** We will now use the information provided by Lemma 4.1 to give direct estimates on the coefficients of the quadratic recursive estimate.

The following lemma gives control of expansion along the orbits that stay away from 0:

Lemma 4.2. *For every $x \in \mathbb{C} \smallsetminus \{-2, 2\}$, there exists $K = K(x)$ with the following properties:*

1. *If $Ч^m(y) = x$, $m \geq 1$, then $|D\,Ч^m(y)| \geq K2^m$;*
2. *For any $\epsilon > 0$ and $p \geq p_0(\epsilon)$, if $x \in \frac{1}{2}\mathbb{V}$ and $f_p^m(y) = x$, $m \geq 1$, with $f_p^k(y) \notin \mathbb{D}_\epsilon$, $0 \leq k \leq m - 1$, then $|Df_p^m(y)| \geq K(2 - \epsilon)^m$.*

Moreover, K depends only on the distance from x to $\{-2, 2\}$ and goes to infinity as x goes to infinity.

Proof. Consider the map $T : \mathbb{C} \smallsetminus \mathbb{D} \to \mathbb{C}$, $T(z) = -(z + z^{-1})$ semi-conjugating $z \mapsto z^2$ to Ч; $T(z^2) = Ч(T(z))$. If $x = T(x')$, $y = T(y')$ and $Ч^m(y) = x$ with $m \geq 1$, then $DЧ^m(y) = DT(x')DT(y)^{-1}2^m x'y'^{-1}$. Since $|y'| = |x'|^{1/2^m} \leq \sqrt{|x'|}$, we have:

$$|DЧ^m(y)| \geq \frac{|DT(x')|}{|DT(y')|}|x'|^{1/2}2^m. \tag{4.1}$$

Since $|DT(y')| \leq 2$ for all $y' \in \mathbb{C} \smallsetminus \mathbb{D}$ and $|DT(x')|$ is bounded away from zero for x outside a neighborhood of $\{-2, 2\}$, (4.1) implies (1).

Since the dynamics of Ч outside a neighborhood of 0 is hyperbolic and hence Hölder stable, the second statement follows easily from Lemma 4.1. □

For $0 < \rho < 1$, let $V' = V'_\rho = \mathbb{D}_\rho$ and $U' = U'_{\rho,p} = f_p^{-p}(V')|0$. It follows from Lemma 4.1 that for $p > p_0(\rho)$, the map $g_p = f_p^{\circ p} : U' \to V'$ is a quadratic-like pre-renormalization of $f_p : U \to V$, where $U = U_{\rho,p} = \lambda_p^{-1}U'$ and $V = V_{\rho,p} = \lambda_p^{-1}V'$. *In what follows, ρ and p will be usually suppressed in the notation.*

The following two lemmas give control of expansion along the orbits that originate near 0.

Lemma 4.3. *For every $0 < \rho \le 1/10$, $0 < \kappa \le 1/10$, and $p > p_0(\kappa, \rho)$, we have*

$$|f(y) - 2| \le |y|^{2-\kappa} \tag{4.2}$$

for any $y \in \mathbb{D}_{e^{-\kappa^{-2}}} \smallsetminus U'$.

Proof. Since the map $f^{p-1} : [f(0), 2] \to [f^p(0), -2]$ has bounded distortion,

$$|f(0) - 2| \asymp |Df^{p-1}(f(0))|^{-1}.$$

Let $W = W(p, \rho)$ be the connected component of $f^{-(p-1)}(V')$ containing $f(0)$. Similarly, since the map $f^{p-1} : W \to \mathbb{D}_\rho$ has bounded distortion,

$$\mathrm{dist}(f(0), \partial W) \asymp \rho |Df^{p-1}(f(0))|^{-1}.$$

Hence for some $\eta > 0$,

$$\mathrm{dist}(f(0), \partial W) \ge \eta\rho |f(0) - 2|.$$

It follows that for $y \notin U'$ we have: $2|y|^2 \ge \eta\rho|f(0) - 2|$. On the other hand, since $|f(0) - 2| \to 0$ as $p \to \infty$, we have: $\eta\rho > |f(0) - 2|^{\kappa/4}$ for $p > p_0(\kappa, \rho)$. Hence $2|y|^2 \ge |f(0) - 2|^{1+\kappa/4}$. It implies by an elementary calculation that

$$|y|^{2-\kappa} \ge 2|y|^2 + (2|y|^2)^{(1+\kappa/4)^{-1}} \ge |f(y) - f(0)| + |f(0) - 2| \ge |f(y) - 2|,$$

provided $0 < \kappa \le 1/10$ and $|y| < e^{-\kappa^{-2}}$ □

Lemma 4.4. *For every $\epsilon > 0$, $0 < \rho < \rho_0(\epsilon)$, and for any period $p \ge p_0(\epsilon, \rho)$, the following property holds. Assume that $y \in A'$ and let $m \ge 2$ be the minimal moment such that $|f^m(y) + 2| > 1/10$. Then*

$$|Df^m(y)| \ge (2 - \epsilon)^m.$$

Proof. A simple consideration of the local dynamics near -2 shows that

$$|Df^{m-1}(f(y))| \asymp |f(y) - 2|^{-1}. \tag{4.3}$$

Hence $m \le K - \log|f_p(y) - 2|/\log\eta_p$, where $\eta_p = |Df(-2)|$. Since $\eta_p \to 4$ as $p \to \infty$, we have

$$(2 - \epsilon)^m \le (2 - \epsilon)^{K - \frac{\log|f(y)-2|}{\log\eta_p}} \le 2^K |f(y) - 2|^{\frac{-1+\kappa}{2}}$$

for $0 < \kappa < \kappa(\epsilon)$ and $p > p_0(\epsilon)$.

Set $\rho = e^{-\kappa^{-2}}$. By Lemma 4.3, if $p > p_0(\rho)$ then $y \in V' \smallsetminus U'$ implies (4.2). On the other hand, (4.3) and (4.2) yields:

$$|Df^m(y)| = |Df(y)|\,|Df^{m-1}(f(y))| > K^{-1}|y|\,|f(y) - 2|^{-1} \ge K^{-1}|f(y) - 2|^{\frac{1}{2-\kappa} - 1}.$$

Thus, we just have to check

$$K^{-1}|f(y) - 2|^{\frac{1}{2-\kappa} - 1} \ge 2^K |f(y) - 2|^{\frac{-1+\kappa}{2}},$$

that is,

$$|f(y) - 2|^{\frac{\kappa(1-\kappa)}{4-2\kappa}} \le \frac{1}{K2^K},$$

which follows from (4.2) and $|y| < \rho = e^{-\kappa^{-2}}$, provided κ is small enough. □

Note that we have obtained the same lower bound $(\log 2 - \epsilon)$ for the Lyapunov exponents of orbits that stay away from 0 and for those that originate quite near 0. It is because the multiplier of the postcritical fixed point -2 is big $(2^2 - \epsilon)$.

Lemma 4.5. *For every $\epsilon > 0$, $0 < \rho < \rho_0(\epsilon)$, and $p > p_0(\epsilon, \rho)$, we have*

(1) *If* $(x_0,...,x_k) \in (A \underset{U\setminus V'}{\longleftarrow} U \setminus U')$ *then* $|Df^k(x_0)| \geq K(2-\epsilon)^k$, *where* $K = K(p,\rho) \to \infty$ *as* $p \to \infty$*;*
(2) *If* $(x_0,...,x_k) \in (V' \underset{U\setminus V'}{\longleftarrow} U \setminus U')$ *then* $|Df^k(x_0)| \geq K(2-\epsilon)^k$ *for some absolute* K*.*

Proof. Let us deal with the first statement. Notice that since $\mathrm{mod}(\mathbb{V} \setminus \overline{\mathbb{U}})$ is large when p is large, for fixed ρ we also have $\lim_{p\to\infty} \mathrm{mod}(V \setminus \overline{U}) = \infty$. Since $\mathrm{mod}(U \setminus J(f)) \geq \mathrm{mod}(V \setminus \overline{U})$, we see that the distance $M(p,\rho)$ between ∂U and 0 satisfies $\lim_{p\to\infty} M(p,\rho) = \infty$. Since $x_k \in A$, we have $|x_k| \geq M(p,\rho)$. If $x_0 \notin V'$ then Lemma 4.2 shows that $|Df^k(x_0)| \geq K(p,\rho)(2-\epsilon)^k$, where $\lim_{p\to\infty} K(p,\rho) = \infty$. If $x_0 \in A'$, we let $2 \leq k_0 \leq k$ be minimal with $|f^{k_0}(x_0) + 2| > 1/10$. Then by Lemma 4.2, $|Df^{k-k_0}(f^{k_0}(x_0))| \geq K(p,\rho)(2-\epsilon)^{k-k_0}$, where $\lim_{p\to\infty} K(p,\rho) = \infty$, and by Lemma 4.4, $|Df^{k_0}(x_0)| \geq (2-\epsilon)^{k_0}$, so $|Df^k(x_0)| \geq K(p,\rho)(2-\epsilon)^k$, and the first statement follows.

The second statement is analogous. □

Lemma 4.6. *Let* $\delta > 1$*. Then*

$$\lim_{p\to\infty} \sup \Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U \setminus U') = 0, \quad 0 < \rho < \rho_0(\delta),$$

$$\lim_{\rho\to 0} \limsup_{p\to\infty} \sup \Xi_\delta(V' \overset{+}{\underset{U\setminus V'}{\longleftarrow}} A') = 0,$$

$$\limsup_{p\to\infty} \sup \Xi_\delta(V' \underset{U\setminus V'}{\longleftarrow} U \setminus V') \leq K \equiv K(\delta), \quad 0 < \rho < \rho_0(\delta).$$

Proof. By the first statement of Lemma 4.5, for every $x \in A$ we have

$$\Xi_\delta(A \underset{U\setminus V'}{\longleftarrow} U \setminus U')(x) \equiv \Xi_\delta(\mathcal{F})(x) \leq \sum_{k\geq 1} \sum_{(x_0,\ldots,x_k=x)\in\mathcal{F}} |Df^k(x_0)|^{-\delta}$$

$$\leq \sum_{k\geq 1} 2^k K^{-\delta}(2-\epsilon)^{-\delta k} = K^{-\delta} \sum_{k\geq 1} \left(\frac{2}{(2-\epsilon)^\delta}\right)^k,$$

where $K = K(p,\rho)$, $\epsilon = \epsilon(p,\rho)$ satisfy $\lim_{p\to\infty} K(p,\rho) = \infty$, $\lim_{\rho\to 0}\lim_{p\to\infty} \epsilon(p,\rho) = 0$. The first estimate follows.

Let $m = m(p,\rho)$ be the minimal return time from A' to V'. Then $\lim_{\rho\to 0} \liminf_{p\to\infty} m(p,\rho) = \infty$. By the second statement of Lemma 4.5, for every $x \in V'$, we have

$$\Xi_\delta(V' \overset{+}{\underset{U\setminus V'}{\longleftarrow}} A')(x) \equiv \Xi_\delta(\mathcal{F})(x) \leq \sum_{k\geq m} \sum_{(x_0,\ldots,x_k=x)\in\mathcal{F}} |Df^k(x_0)|^{-\delta}$$

$$\leq \sum_{k\geq m} 2^k K^{-\delta}(2-\epsilon)^{-\delta k} = K^{-\delta} \sum_{k\geq m} \left(\frac{2}{(2-\epsilon)^\delta}\right)^k,$$

where K is an absolute constant and $\epsilon = \epsilon(p,\rho)$ satisfies $\lim_{\rho\to 0}\lim_{p\to\infty} \epsilon(p,\rho) = 0$. This gives the second estimate.

By the second statement of Lemma 4.5, for every $x \in V'$ we have

$$\Xi_\delta(V' \underset{U\setminus V'}{\longleftarrow} U \setminus V')(x) \equiv \Xi_\delta(\mathcal{F})(x) \leq \sum_{k\geq 1} \sum_{(x_0,\ldots,x_k=x)\in\mathcal{F}} |Df^k(x_0)|^{-\delta}$$

$$\leq \sum_{k\geq 1} 2^k K^{-\delta}(2-\epsilon)^{-\delta k} = K^{-\delta} \sum_{k\geq 1} \left(\frac{2}{(2-\epsilon)^\delta}\right)^k,$$

where K is an absolute constant and $\epsilon = \epsilon(p,\rho)$ satisfies $\lim_{\rho\to 0}\lim_{p\to\infty} \epsilon(p,\rho) = 0$. This gives the last estimate. □

Proof of Theorem 1.2. Since any quadratic-like map with connected Julia set satisfies $\delta_{\rm cr} \geq 1$, we only have to show that for every $\delta > 1$ and for every p sufficiently large, $\delta_{\rm cr}(f) \leq \delta$, where $f = f_{p,\rho}$ is some quadratic-like representative of f_p.

Fix some $\delta > 1$. By Lemma 4.6, we can choose $\rho > 0$ so that

$$\limsup_{p\to\infty} \sup \Xi_\delta(V' \xleftarrow[U\setminus V']{+} A') \leq \frac{1}{4}.$$

Let P_δ be the quadratic polynomial defined in Lemma 3.1. Notice obvious inequalities

$$\max\left\{\Xi_\delta(A' \xleftarrow[U\setminus V']{} U\setminus V'),\ \Xi_\delta(U' \xleftarrow[U\setminus V']{} U\setminus V')\right\} \leq \Xi_\delta(V' \xleftarrow[U\setminus V']{} U\setminus V'),$$

$$\max\left\{\Xi_\delta(A \xleftarrow[U\setminus V']{} U\setminus V'),\ \Xi_\delta(A \xleftarrow[U\setminus V']{} A')\right\} \leq \Xi_\delta(A \xleftarrow[U\setminus V']{} U\setminus U'),$$

$$\max\left\{\Xi_\delta(A' \xleftarrow[U\setminus V']{+} A'),\ \Xi_\delta(U' \xleftarrow[U\setminus V']{} A')\right\} \leq \Xi_\delta(V' \xleftarrow[U\setminus V']{+} A').$$

By Lemma 4.6, when p grows, the constant coefficient (3.4) of P_δ stays bounded, the linear coefficient (3.5) becomes smaller than $1/3$, and the quadratic term (3.6) goes to 0. In particular, for p large P_δ takes $[0, 2P_\delta(0)]$ into itself, and hence it has a fixed point. By Lemma 3.1, $\delta_{\rm cr}(f) \leq \delta$ as desired. $\square$

Corollary 4.7. *Let $F_p : z \mapsto z^2 + c_p$ be the straightening of f_p. Then*

$$\lim_{p\to\infty} \delta_{\rm cr}(F_p) = 1. \tag{4.4}$$

Proof. By Lemma 4.1, the germ f_p has a quadratic-like representative $f_p : U_p \to V_p$ with a big modulus: $\mathrm{mod}(V_p \setminus U_p) \to \infty$ as $p \to \infty$. Hence for p large, there is a quasiconformal conjugacy between f_p and F_p with a small dilatation. This easily implies (see for instance Lemma 3.15 of [AL]) that f_p and F_p have close critical exponents. $\square$

5. Lebesgue measure of the Julia set

Below $f = f_p : U \to V$ will be the fixed point of nearly Chebyshev renormalization of period p, and $f' = f^p : U' \to V'$ will be its pre-renormalization as constructed in Lemma 4.1. Thus, $f'(z) = \lambda f(\lambda^{-1}z)$, where $\lambda = \lambda_p \in (-1, 0)$ is the scaling factor of f. We let as above $A = V \setminus U$, $A' = V' \setminus U'$. Furthermore, we let $U^k = \lambda^k U$, $V^k = \lambda^k V$, and $A^k = V^k \setminus U^k$.

We will need the following combinatorial lemma.

Lemma 5.1. *Let*

$$u^k = \sup \Xi_\delta(V^k \xleftarrow[U\setminus V^k]{+} A^k),$$

$$v^k_j = \sup \Xi^{[j]}_\delta(A^k \xleftarrow[U\setminus V^{k+1}]{+} A^k),$$

$$v^k = \lim_{j\to\infty} v^k_j = \sup \Xi_\delta(A^k \xleftarrow[U\setminus V^{k+1}]{+} A^k).$$

Then

$$u^{k+1} \leq \sup \Xi_\delta(V' \xleftarrow[U\setminus V']{+} A') + u^k(1+v^k) \sup \Xi_\delta(A \xleftarrow[U\setminus V']{} A')(1+\sup \Xi_\delta(V' \xleftarrow[U\setminus V']{} U\setminus V')), \tag{5.1}$$

$$v^k_{j+1} \leq (1+v^k_j)u^k(1+\sup \Xi_\delta(A \xleftarrow[U\setminus V']{} U\setminus V')). \tag{5.2}$$

Proof. Let $B^k = U \setminus (A^k \cup V^{k+1})$. Let us prove the first estimate.

We can decompose $V^{k+1} \xleftarrow[U \setminus V^{k+1}]{+} A^{k+1}$ into two groups:

$$V^{k+1} \xleftarrow[B^k]{+} A^{k+1},$$

which takes into account the orbits that do not land at the annulus A^k, and

$$V^{k+1} \xleftarrow[B^k]{} A^k \xleftarrow[U \setminus V^{k+1}]{} A^k \xleftarrow[B^k]{} A^{k+1},$$

which accounts for the orbits landing at A^k and marks the first and the last landings. Thus

$$\begin{aligned} u^{k+1} &= \sup \Xi_\delta(V^{k+1} \xleftarrow[U \setminus V^{k+1}]{+} A^{k+1}) \le \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{+} A^{k+1}) \\ &+ \left(\sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} A^k) \cdot \sup \Xi_\delta(A^k \xleftarrow[U \setminus V^{k+1}]{} A^k) \cdot \sup \Xi_\delta(A^k \xleftarrow[B^k]{} A^{k+1}) \right). \end{aligned} \tag{5.3}$$

Notice that

$$\Xi_\delta(V^{k+1} \xleftarrow[B^k]{+} A^{k+1})(x) = \Xi_\delta(V' \xleftarrow[U \setminus V']{+} A')(\lambda^{-k}(x)), \tag{5.4}$$

$$\Xi_\delta(A^k \xleftarrow[B^k]{} A^{k+1})(x) = \Xi_\delta(A \xleftarrow[U \setminus V']{} A')(\lambda^{-k} x), \tag{5.5}$$

and

$$\sup \Xi_\delta(A^k \xleftarrow[U \setminus V^{k+1}]{} A^k) = 1 + \sup \Xi_\delta(A^k \xleftarrow[U \setminus V^{k+1}]{+} A^k) = 1 + v^k, \tag{5.6}$$

the 1 accounting for trivial orbits. Plugging (5.4) – (5.6) into (5.3) we get

$$u^{k+1} \le \sup \Xi_\delta(V' \xleftarrow[U \setminus V']{+} A') + (1 + v^k) \sup \Xi_\delta(A \xleftarrow[U \setminus V']{} A') \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} A^k). \tag{5.7}$$

We can decompose $V^{k+1} \xleftarrow[B^k]{} A^k$ into two groups,

$$V^{k+1} \xleftarrow[U \setminus V^k]{} A^k, \quad \text{and} \quad V^{k+1} \xleftarrow[B^k]{} U^k \setminus V^{k+1} \xleftarrow[U \setminus V^k]{} A^k.$$

Thus

$$\begin{aligned} \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} A^k) &\le \sup \Xi_\delta(V^{k+1} \xleftarrow[U \setminus V^k]{} A^k) \\ &+ \sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} U^k \setminus V^{k+1}) \sup \Xi_\delta(U^k \setminus V^{k+1} \xleftarrow[U \setminus V^k]{} A^k). \end{aligned} \tag{5.8}$$

Notice that

$$\Xi_\delta(V^{k+1} \xleftarrow[B^k]{} U^k \setminus V^{k+1})(x) = \Xi_\delta(V' \xleftarrow[U \setminus V']{} U \setminus V')(\lambda^{-k} x), \tag{5.9}$$

$$\begin{aligned} \max \left\{ \sup \Xi_\delta(V^{k+1} \xleftarrow[U \setminus V^k]{} A^k), \sup \Xi_\delta(U^k \setminus V^{k+1} \xleftarrow[U \setminus V^k]{} A^k), \sup \Xi_\delta(A^k \xleftarrow[U \setminus V^k]{+} A^k) \right\} \\ = \sup \Xi_\delta(V^k \xleftarrow[U \setminus V^k]{+} A^k) = u^k. \end{aligned} \tag{5.10}$$

Plugging (5.9) and (5.10) into (5.8), and plugging the resulting expression for $\sup \Xi_\delta(V^{k+1} \xleftarrow[B^k]{} A^k)$ into (5.7) gives (5.1).

Let us prove the second estimate. We will omit the truncation parameter (j or $j + 1$).

We can rewrite $A^k \xleftarrow[U\setminus V^{k+1}]{+} A^k$ as $A^k \xleftarrow[B^k]{+} A^k \xleftarrow[U\setminus V^{k+1}]{} A^k$. Thus

$$\sup \Xi_\delta(A^k \xleftarrow[U\setminus V^{k+1}]{+} A^k) \le \sup \Xi_\delta(A^k \xleftarrow[B^k]{+} A^k) \sup \Xi_\delta(A^k \xleftarrow[U\setminus V^{k+1}]{} A^k). \tag{5.11}$$

Plugging (5.6) into (5.11) we get

$$v^k \le (1+v^k) \sup \Xi_\delta(A^k \xleftarrow[B^k]{+} A^k). \tag{5.12}$$

We can split $A^k \xleftarrow[B^k]{+} A^k$ into two groups: $A^k \xleftarrow[U\setminus V^k]{+} A^k$ and $A^k \xleftarrow[B^k]{} U^k \setminus V^{k+1} \xleftarrow[U\setminus V^k]{} A^k$. Thus

$$\begin{aligned}\sup \Xi_\delta(A^k \xleftarrow[B^k]{+} A^k) \le &\sup \Xi_\delta(A^k \xleftarrow[U\setminus V^k]{+} A^k) \\ &+ \sup \Xi_\delta(A^k \xleftarrow[B^k]{} U^k \setminus V^{k+1}) \sup \Xi_\delta(U^k \setminus V^{k+1} \xleftarrow[U\setminus V^k]{} A^k).\end{aligned} \tag{5.13}$$

Notice that

$$\Xi_\delta(A^k \xleftarrow[B^k]{} U^k \setminus V^{k+1})(x) = \Xi_\delta(A \xleftarrow[U\setminus V']{} U \setminus V')(\lambda^{-k}x), \tag{5.14}$$

Plugging (5.14) and (5.10) into (5.13) we get

$$\sup \Xi_\delta(A^k \xleftarrow[B^k]{+} A^k) \le u^k + u^k \sup \Xi_\delta(A \xleftarrow[U\setminus V']{} U \setminus V'). \tag{5.15}$$

Plugging (5.15) into (5.12) gives (5.2). $\square$

Proof of Theorem 1.3. By Lemma 4.6, there exists $K \equiv K(2) > 0$ such that if one takes ρ sufficiently small, then for all p sufficiently large we have

$$\sup \Xi_2(V' \xleftarrow[U\setminus V']{+} A') < \frac{1}{100}, \tag{5.16}$$

$$\sup \Xi_2(A \xleftarrow[U\setminus V']{} A') < \frac{1}{5K+5}, \tag{5.17}$$

$$\sup \Xi_2(V' \xleftarrow[U\setminus V']{} U \setminus V') < 2K, \tag{5.18}$$

$$\sup \Xi_2(A \xleftarrow[U\setminus V']{} U \setminus V') < \frac{1}{100}. \tag{5.19}$$

Let us show by induction that for every $k \ge 0$ we have

$$u^k \le \frac{1}{10}, \tag{5.20}$$

where u^k is as in Lemma 5.1. Notice that $u^0 = 0$, so (5.20) holds for $k = 0$. Assuming that (5.20) holds for some k, notice that (5.2) and (5.19) imply

$$v^k_{j+1} \le \frac{1}{5}(1 + v^k_j),$$

for $j \ge -1$. Since $v^k_{-1} = 0$, this implies by induction that $v^k_j \le \frac{1}{4}$ for every $j \ge -1$, so $v^k = \lim_{j\to\infty} v^k_j \le \frac{1}{4}$. By (5.1) and (5.16) – (5.18), we have

$$u^{k+1} \le \frac{1}{100} + \frac{1}{10}\frac{5}{4}\frac{1}{5K+5}(2K+1) \le \frac{1}{10}.$$

By induction, (5.20) holds for all $k \ge 0$.

Let

$$X_k = \cup_{r\geq 1} f^{-r} V^k.$$

Then

$$\operatorname{area}(X_k \cap A^k) = \int_{A^k} 1_{X^k} dx = \int_{V^k} \Xi_2(V^k \xleftarrow[U\smallsetminus V^k]{+} A^k) dx \leq \int_{V^k} u^k dx \leq \frac{1}{10} \operatorname{area}(V^k).$$

Notice that $X^k \cap V^k = U^k \cup (X^k \cap A^k)$. Thus,

$$\frac{\operatorname{area}(X^k \cap V^k)}{\operatorname{area} V^k} \leq \frac{1}{10} + \frac{\operatorname{area} U^k}{\operatorname{area} V^k} \leq \frac{1}{5}, \tag{5.21}$$

where we have used that $\operatorname{area} U^k \leq \frac{1}{10} \operatorname{area} V^k$, which holds since $\operatorname{mod}(V^k \smallsetminus \overline{U^k}) = \operatorname{mod} A$ is big for large p (by Lemma 4.1).

The conclusion of the argument is standard. Let

$$X = \{x \in J(f),\, 0 \in \omega(x)\}.$$

Notice that X is fully invariant: $X = f^{-1}(X) = f(X)$. By [L1], for almost every $x \in J(f)$, $\omega(x) \subset \omega(0)$. Since $\omega(0)$ is a minimal set containing 0, we conclude that $\operatorname{area} X = \operatorname{area} J(f)$. Let us show that $\operatorname{area} X = 0$.

Assume that this is not the case. By the Lebesgue Density Points Theorem, there exists a density point $x \in X$. Let $r_k \geq 0$ be minimal such that $f^{r_k}(x) \in V^k$. We may assume that x is not a preimage of 0, so that $r_k \to \infty$. Let W^k be the connected component of $f^{-r_k}(V^k)$ containing x. Then $f^{r_k} : W^k \to V^k$ admits a univalent extension onto $\mathbb{V}^k \equiv \lambda^k \mathbb{V}$, and since $\operatorname{mod}(\mathbb{V}^k \smallsetminus \overline{V^k})$ is big, it has distortion bounded by 2. It also follows that W^k contains a round disk of radius $\frac{1}{10} \operatorname{diam}(W^k)$. Since $r_k \to \infty$ and $W^k \subset f^{-r_k}(V)$, $\limsup W^k \subset K(f)$. Since $K(f)$ has empty interior, we conclude that $\operatorname{diam}(W^k) \to 0$. Notice that

$$\frac{\operatorname{area}(V^k \smallsetminus X)}{\operatorname{area} V^k} \leq 10 \frac{\operatorname{area}(W^k \smallsetminus X^k)}{\operatorname{area} W^k} \leq 1000 \frac{\operatorname{area}(\mathbb{D}_{\operatorname{diam}(W^k)}(x) \smallsetminus X)}{\operatorname{area}(\mathbb{D}_{\operatorname{diam}(W^k)}(x)},$$

and since x is a density point of X, we have

$$\frac{\operatorname{area}(V^k \smallsetminus X)}{\operatorname{area}(V^k)} \to 0. \tag{5.22}$$

Obviously, $X \subset X_k$, so (5.22) and (5.21) give the desired contradiction. □

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Laboratoire de Probabilités et Modèles aléatoires, Université Pierre et Marie Curie–Boite courrier 188, 75252–Paris Cedex 05, France

E-mail address: artur@ccr.jussieu.fr

Mathematics Department and IMS, SUNY Stony Brook, Stony Brook, NY 11794, USA

E-mail address: mlyubich@math.sunysb.edu

Department of Mathematics, University of Toronto, Ontario, Canada M5S 3G3

E-mail address: misha@math.toronto.edu