

INFINITESIMAL PERTURBATIONS OF RATIONAL MAPS

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ABSTRACT. We analyze the infinitesimal effect of holomorphic perturbations of the dynamics of a structurally stable rational map on a neighborhood of its Julia set. This implies some restrictions on the behavior of critical points.

1. INTRODUCTION

A major open problem in one-dimensional dynamics is to show that hyperbolic rational maps are dense among all rational maps (Fatou conjecture). After the work of Mañé-Sad-Sullivan [MSS], this problem reduces to show that all structurally stable rational maps are hyperbolic. In other words, one should show that non-hyperbolic rational maps can be perturbed to a map with different dynamics (in the topological sense).

An obvious difficulty to perturb rational maps is that there are not enough perturbations, since the space of rational maps of a given degree is finite dimensional. It is then hard to show that any of them has non-trivial effects on the dynamics.

A natural approach is then to perturb rational maps inside a broader class of dynamical systems, and later realize the perturbation as a rational one. One such class consists of quasiregular maps (the quasiconformal analogous to rational maps). The space of quasiregular perturbations of a given rational map is infinite dimensional, and it is indeed much simpler to change the dynamics of a rational map through a quasiregular perturbation. Of course one is then left with the problem of realizing the perturbation as a rational one: this problem was attacked by Thurston in some simple situations, and is the basic problem in [R], where it is solved under certain geometric assumptions on the recurrence of critical points. Another application of quasiregular perturbations, due to Shishikura [Sh], is a sharp bound on the number of non-repelling periodic orbits of rational maps of a given degree.

(One should note the resemblance to a usual approach to rigidity of rational maps: to show that two maps are affinely conjugate, one first looks for qc conjugacies.)

Another approach to analyzing perturbations of rational maps is to observe that the Julia set of a structurally stable rational map R depends holomorphically on a neighborhood of R . Linearizing this holomorphic dependence for the case of a infinitesimal perturbation $R + \lambda v$ we get a representation $v = \alpha \circ R - DR\alpha$ for some α which is a qc vector field on $J(R)$. Looking for non-trivial perturbations is then reduced to finding some v which can not satisfy such an equation. A successful application of those ideas is for instance Mañé's proof of instability of Herman rings [Ma]. For other applications, the fact that the space of such v is finite dimensional can be a major obstacle, and this is the problem we try to address here.

Our aim in this paper is to present a technique of infinitesimal analysis of certain quasiregular perturbations. We consider a rational map R and a quasiregular perturbation $R + \lambda v$ of R which is holomorphic on a neighborhood of the Julia set $J(R)$. This is enough to increase a lot our freedom if we assume $J(R) \neq \overline{\mathbb{C}}$. We then show that if R is structurally stable then the “Julia set” of $R + \lambda v$ depends holomorphically on λ for small λ . Linearizing this holomorphic dependence we get the representation $v = \alpha \circ R - DR\alpha$ on $J(R)$. Thus we increased the space of “test functions” against which to check the possibility of the above representation from a finite dimensional space to a much bigger space.

We present a direct application of this procedure by considering the holomorphic dependence of iterates of critical points c_λ of $R + \lambda v$. The fact that the speed of motion of those iterates must be bounded reflects on the fact that certain sequences depending on v are bounded, and this imposes restrictions on the dynamics of R .

As a corollary of this analysis, we are able to show that a rational map such that $J(R) \subset \mathbb{C}$ (the case $J(R) \neq \overline{\mathbb{C}}$ reduces to this one by a Moebius change of coordinates), with one critical point satisfying a summability condition and an additional technical condition (which can be interpreted as “the space of holomorphic vector fields in a neighborhood of $J(R)$ is big enough”) is not structurally stable. More precisely we have:

Theorem 1.1 (see Corollary 2.10). *Let R be a rational map such that $J(R) \subset \mathbb{C}$ and such that there exists a critical point $c \in J(R)$ with*

$$(1.1) \quad \sum_{k=0}^{\infty} \frac{1}{|DR^k(R(c))|} < \infty.$$

If there exists a vector field v which is holomorphic on a neighborhood of the closure of the orbit of c and such that

$$(1.2) \quad \sum_{k=0}^{\infty} \frac{v(R^k(c))}{DR^k(R(c))} \neq 0$$

then R is not structurally stable.

(Notice that the summability condition (1.1) implies (1.2) for many *continuous* vector fields v .)

This recovers a result of Levin and Makienko ([Le], [M]), which is based on a different approach (analysis of a Ruelle transfer operator). Indeed this work was motivated as an attempt to give a geometric interpretation to those results, in the hope that both points of view could be eventually merged.

We finally discuss the meaning of this technique in the better understood case of polynomial maps.

Although we do not develop it here, infinitesimal analysis is also an important technique in real one-dimensional dynamics. Indeed this path was thoroughly developed in the case of unimodal maps in [ALM], and the results of that work could shed more light to the setting we consider here.

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1.1. Notation and definitions. We let \mathbb{C} denote the complex plane and $\overline{\mathbb{C}}$ the Riemann Sphere. We let $\mathbb{D}_\varepsilon \subset \mathbb{C}$ denote the disk with radius ε around 0. The identity map is denoted id .

We assume the reader is familiar with the theory of quasiconformal maps. The Beltrami differential of a qc map h is $\mu = \overline{\partial}h/\partial h$. The dilatation of h is $\text{Dil}(h) = \text{ess-sup}(1 + |\mu|)/(1 - |\mu|)$. h is said to be normalized if it fixes 0, 1 and ∞ . We will also say that $h : X \rightarrow \overline{\mathbb{C}}$ is a qc map on X if it extends to a (global) qc map.

A quasiconformal vector field α of $\overline{\mathbb{C}}$ is a continuous vector field with locally integrable distributional derivatives $\overline{\partial}\alpha$ and $\partial\alpha$ in L^1 and $\overline{\partial}\alpha \in L^\infty$. If α is only defined on a set $X \subset \overline{\mathbb{C}}$ then we say α is qc on X if it has a qc extension.

Let U be a complex manifold with a base point $\lambda_0 \in U$. Let $X \subset \overline{\mathbb{C}}$ be any set. A holomorphic motion of X over U (based on λ_0) is a family $H_\lambda : X \rightarrow \overline{\mathbb{C}}$, $\lambda \in U$, of injective maps H_λ such that for each $x \in X$, $\lambda \mapsto H_\lambda(x)$ is holomorphic and $H_{\lambda_0} = \text{id}$. In this case each H_λ is qc on X and the continuous extension $H_\lambda : \overline{X} \rightarrow \mathbb{C}$ is also a holomorphic motion.

In this paper we will usually let $U \subset \mathbb{C}$ be a domain containing 0 and consider holomorphic motions based on 0. We observe that the Beltrami differential $\overline{\partial}H_\lambda/\partial H_\lambda$ of a holomorphic motion depends holomorphically on λ , and

$$\alpha = \left. \frac{dH_\lambda}{d\lambda} \right|_{\lambda=0}$$

is a qc vector field on X (see [ALM]).

2. RATIONAL MAPS

Let us quickly recall the basic theory of rational maps (see [MS]). Let Rat_d be the space of rational maps of degree d . Let $R \in \text{Rat}_d$. We define as usual the Fatou set of R , $F(R)$, as the biggest open set where $\{R^n\}$ is a normal family. The Julia set of R is then defined as $J(R) = \overline{\mathbb{C}} \setminus F(R)$. The Julia set is the closure of repelling periodic orbits of R , and there are at most finitely many periodic orbits in $F(R)$. It is easy to see that R takes connected components of $F(R)$ to connected components of $F(R)$. Moreover, according to Sullivan, every connected component $U \subset F(R)$ is either periodic ($R^k(U) = U$ for some $k > 0$) or is eventually mapped to some periodic component, and there is only a finite number of such periodic components.

We say that R_0 is structurally stable if there exists a neighborhood $U \subset \text{Rat}_d$ of R_0 such that for every $R \in U$, there exists a homeomorphism $h_R : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ such that $h_R \circ R_0 \circ h_R^{-1} = R$. In this case the h_R can be chosen to form a holomorphic motion over U based on R_0 (see [MS], §7). In particular, $\lim_{R \rightarrow R_0} \text{Dil } h_R = 1$ and $\lim_{R \rightarrow R_0} h_R = \text{id}$ uniformly on \mathbb{C} .

It is known that if R is structurally stable then every periodic component U of $F(R)$ (of period k) must be attracting, that is, there exists $w \in U$ such that $R^k(w) = w$, $|DR^k(w)| < 1$ and for every $z \in U$, $f^{kn}(z) \rightarrow w$.

Let $\text{Rat}_d^\infty \subset \text{Rat}_d$ be the space of all rational maps with degree d such that $\infty \in F(R)$. Let $\text{Rat}_d^a \subset \text{Rat}_d$ be the space of maps for which all periodic components of $F(R)$ are attracting and Rat_d^s be the space of structurally stable maps R .

2.1. Quasiregular maps. A map $\tilde{R} : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is called a quasiregular map if it is the composition of a rational map R with a qc map h , $\tilde{R} = R \circ h$. One defines Beltrami differentials and dilatation of quasiregular maps in the natural way.

Let us now consider a map \tilde{R} which is a composition of a qc map h and a rational map R , $\tilde{R} = h \circ R$. Let \tilde{h} be a qc map with the same Beltrami differential of \tilde{R} . Then $\tilde{R} \circ \tilde{h}^{-1}$ is a rational map. So \tilde{R} can also be seen as a composition of a rational map and a qc map $\tilde{R} = (\tilde{R} \circ \tilde{h}^{-1}) \circ \tilde{h}$, and is a quasiregular map. It follows:

Lemma 2.1. *The composition of quasiregular maps is quasiregular, furthermore $\text{Dil}(\tilde{R}_1 \circ \tilde{R}_2) \leq \text{Dil}(\tilde{R}_1) \text{Dil}(\tilde{R}_2)$.*

Given a quasiregular map \tilde{R} , let us consider the support K of its Beltrami differential $\bar{\partial}\tilde{R}/\partial\tilde{R}$. Let us say that \tilde{R} is tame if there exists $n > 0$ such that for $m \geq n$, $\tilde{R}^m(K) \cap K = \emptyset$. Notice that in this case, for all $m > 0$, $\text{Dil}(\tilde{R}^m) \leq \text{Dil}(\tilde{R})^n$.

Lemma 2.2. *Let \tilde{R} be a tame quasiregular map. Then there exists a normalized qc map h such that $\text{Dil}(h) \leq \sup \text{Dil}(\tilde{R}^n)$ and $h \circ \tilde{R} \circ h^{-1}$ is a rational map.*

Proof. Let μ_n be the Beltrami differential of \tilde{R}^n , so that $\mu_{n+1} = \tilde{R}^*(\mu_n)$. Since \tilde{R} is tame, it follows that for every x , the sequence $\mu_n(x)$ is eventually constant. Let μ be the pointwise limit of μ_n and let h be a normalized quasiconformal map with Beltrami differential μ . We have $\tilde{R}^*(\mu) = \mu$, so that $h \circ \tilde{R} \circ h^{-1}$ is a rational map. \square

Remark 2.1. The conclusion of the above Lemma also holds if the tame assumption is replaced by the weaker condition $\sup \text{Dil}(\tilde{R}^n) < \infty$.

2.2. Quasiregular perturbations. An admissible perturbation of a rational map R is a family R_λ , $\lambda \in \mathbb{D}_\varepsilon$, of quasiregular maps such that

- $R_0 = R$,
- $\lim_{\lambda \rightarrow 0} R_\lambda(z) = R(z)$ uniformly on $\bar{\mathbb{C}}$,
- $\lim_{\lambda \rightarrow 0} \text{Dil}(R_\lambda) = 1$,
- there is a neighborhood V_1 of $J(R)$ such that $(\lambda, z) \mapsto R_\lambda(z)$ is holomorphic on $\mathbb{D}_\varepsilon \times V_1$,
- there is a neighborhood V_2 of the attracting periodic orbits of R and of the critical points of R on $F(R)$ such that $R_\lambda = R$.

Lemma 2.3. *Let R_λ be an admissible family through $R \in \text{Rat}_d^q$. Then there exists $n > 0$, $\delta > 0$ and a family of qc maps h_λ , $\lambda \in \mathbb{D}_\delta$ such that $h_\lambda^{-1} \circ R_\lambda \circ h_\lambda \in \text{Rat}_d$, $\lim_{\lambda \rightarrow 0} h_\lambda = \text{id}$ and $\text{Dil}(h_\lambda) \leq \text{Dil}(R_\lambda)^n$.*

Proof. Let V_1 and V_2 be as in the definition of admissible family and let $K = \bar{\mathbb{C}} \setminus V_1 \setminus V_2$, so that for any λ , $\bar{\partial}R_\lambda = 0$ out of K . Let $V \subset V_2$ be a neighborhood of the attracting cycles of R with $R(\bar{V}) \subset V$. Since $R \in \text{Rat}_d^q$, there exists $n > 0$ such that $R^n(K) \subset V$. Taking δ small we may assume that $R_\lambda^n(K) \subset V$ for $|\lambda| < \delta$. By Lemma 2.2, there exists a family h_λ of normalized qc maps such that $h_\lambda^{-1} \circ R_\lambda \circ h_\lambda \in \text{Rat}_d$, and $\text{Dil} h_\lambda \leq (\text{Dil} R_\lambda)^n$. In particular, $\lim_{\lambda \rightarrow 0} \text{Dil} h_\lambda = 1$ and since the h_λ are normalized, $\lim_{\lambda \rightarrow 0} h_\lambda = \text{id}$ in the uniform topology. \square

Remark 2.2. The h_λ we constructed actually satisfies $\bar{\partial}h_\lambda = 0$ on $J(R)$. Indeed, it is clear that $h_\lambda(J(R)) \subset V_1$ for λ small. In particular, R_λ^n is holomorphic on $h_\lambda(J(R))$ for all n . It is easy to check that the proof of Lemma 2.2 gives $\bar{\partial}h_\lambda = 0$ on $J(R)$.

Lemma 2.4. *Let R_λ be an admissible family through $R \in \text{Rat}_d^s$. Then there exists a family of qc maps H_λ , $\lambda \in \mathbb{D}_\delta$ such that $H_\lambda^{-1} \circ R_\lambda \circ H_\lambda = R$, $\lim_{\lambda \rightarrow 0} H_\lambda = \text{id}$ and $\lim_{\lambda \rightarrow 0} \text{Dil}(H_\lambda) = 1$.*

Proof. The previous lemma gives us a family $\tilde{R}_\lambda = h_\lambda^{-1} \circ R_\lambda \circ h_\lambda \in \text{Rat}_d$ which is continuous on $\lambda = 0$ (since $\lim_{\lambda \rightarrow 0} h_\lambda = \text{id}$). Since R is structurally stable, there exists a family of quasiconformal maps \tilde{h}_λ such that $\tilde{h}_\lambda^{-1} \circ \tilde{R}_\lambda \circ \tilde{h}_\lambda = R$, and moreover $\lim \tilde{h}_\lambda = \text{id}$ and $\lim \text{Dil}(\tilde{h}_\lambda) = 1$. Setting $H_\lambda = h_\lambda \circ \tilde{h}_\lambda$ we get the result. \square

Theorem 2.5. *Let R_λ be an admissible family through $R \in \text{Rat}_d^s$, and let H_λ be a family of qc maps over \mathbb{D}_δ as above such that $H_\lambda \circ R \circ H_\lambda^{-1} = R_\lambda$. Then there exists $\varepsilon > 0$ and a holomorphic motion h_λ of $J(R)$ over \mathbb{D}_ε such that $h_\lambda(J(R)) = H_\lambda(J(R))$ and $h_\lambda \circ R \circ h_\lambda^{-1} = R_\lambda$.*

Proof. Let W be a neighborhood of $J(R)$ where R_λ is holomorphic and such that all periodic orbits of R in W in fact belong to $J(R)$. Let $J(R) \subset W' \subset \overline{W'} \subset W$ be a smaller neighborhood. Taking $\varepsilon < \delta$ small, we may assume that $J(R) \subset H_\lambda^{-1}(W') \subset \overline{H_\lambda^{-1}(W')} \subset W$ for all $\lambda \in \mathbb{D}_\varepsilon$.

It is easy to see that p is a periodic orbit of R on $H_\lambda^{-1}(W')$ (of period n and multiplier $\rho = DR^n(p)$) if and only if $H_\lambda(p) \in W'$ is a periodic orbit of R_λ (with period n and multiplier ρ_λ) and $\ln(|\rho|)/K \leq \ln(|\rho_\lambda|) \leq K \ln(\rho)$ with K only depending on $\text{Dil}(H_\lambda)$, and in particular, since the former is uniformly bounded, does not depend on λ . So $H_\lambda(J(R))$ is the closure of the set of repelling periodic orbits of R_λ contained in $\overline{W'}$.

Let $P_n(\lambda)$ be the set of repelling periodic orbits of period less than n of R_λ on $H_\lambda(J(R))$. The above discussion shows that $H_\lambda(P_n(0)) = P_n(\lambda)$. Fixing λ_0 , if $p_{\lambda_0} \in P_n(\lambda_0)$ and p_λ , λ near λ_0 is its holomorphic continuation (as a repelling periodic orbit), we have $p_\lambda \in P_n(\lambda)$. Since the $P_n(\lambda)$ are finite sets with the same cardinality and \mathbb{D}_δ is simply connected, it follows that there exists a unique holomorphic motion $h_\lambda^n : P_n(0) \rightarrow P_n(\lambda)$, $\lambda \in \mathbb{D}_\delta$. Passing to the limit $n \rightarrow \infty$ and using the extension theorem for holomorphic motions of [MSS], we obtain the desired holomorphic motion of $J(R)$. \square

Remark 2.3. It is easy to see that if $c \in J(R)$ is a critical point of R then $h_\lambda(c)$ is a critical point of R_λ .

2.3. Infinitesimal perturbations. Let now $R \in \text{Rat}_d^\infty$. Let us consider the space of vector fields v which are holomorphic on a neighborhood of $J(R)$. This space can be interpreted as a tangent space to certain quasiregular perturbations due to the following obvious lemma.

Lemma 2.6. *Let $R \in \text{Rat}_d^\infty$ and let v be a holomorphic vector field on a neighborhood of $J(R)$. Then there exists an admissible family R_λ through R and a bounded neighborhood V of $J(R)$ such that $R_\lambda = R + \lambda v$ on a neighborhood of $J(R)$.*

Proof. Let \tilde{v} be a C^∞ vector field coinciding with v on a neighborhood of $J(R)$ and vanishing on a neighborhood of the union of attracting periodic orbits of R , critical points on $F(R)$, $R^{-1}(\infty)$ and ∞ . Then it is easy to check that $R_\lambda = R + \lambda \tilde{v}$ is quasiregular for λ small and has the required properties. \square

Corollary 2.7. *Let $R \in \text{Rat}_d^s \cap \text{Rat}_d^\infty$ and let v be a holomorphic vector field on a bounded neighborhood V of $J(R)$. Then there exists $\delta > 0$ and a holomorphic motion h_λ , $\lambda \in \mathbb{D}_\delta$ of $J(R)$ such that $h_\lambda(J(R)) \subset V$ and $h_\lambda \circ R \circ h_\lambda^{-1} = R + \lambda v$. Moreover if $c \in J(R)$ is a critical point then $h_\lambda(c)$ is a critical point of $R + \lambda v$.*

Theorem 2.8. *Let $R \in \text{Rat}_d^s \cap \text{Rat}_d^\infty$ and let v be a holomorphic vector field on a neighborhood of $J(R)$. Then there exists a unique qc vector field α such that $v = \alpha \circ R - DR\alpha$ on $J(R)$.*

Proof. Just take

$$\alpha = \left. \frac{d}{d\lambda} h_\lambda \right|_{\lambda=0}$$

and linearize $h_\lambda \circ R \circ h_\lambda^{-1} = R + \lambda v$. \square

Corollary 2.9. *Let $R \in \text{Rat}_d^s \cap \text{Rat}_d^\infty$ and let v be a holomorphic vector field on a neighborhood of $J(R)$. Then for any $c \in J(R)$ critical point of R , the sequence*

$$\left. \frac{d}{d\lambda} (R + \lambda v)^n(c) \right|_{\lambda=0} = DR^{n-1}(R(c)) \sum_{k=0}^{n-1} \frac{v(R^k(c))}{DR^k(R(c))}$$

is uniformly bounded.

Proof. Let α be as in the previous Theorem. Then it is easy to see by induction that

$$\alpha(R^n(c)) = \left. \frac{d}{d\lambda} (R + \lambda v)^n(c) \right|_{\lambda=0} = DR^{n-1}(R(c)) \sum_{k=0}^{n-1} \frac{v(R^k(c))}{DR^k(R(c))}.$$

(Since R is structurally stable, there are no critical relations so $DR^k(R(c))$ is non-zero for $k \geq 0$.)

Since α is a qc vector field, it must be uniformly bounded on $J(R)$, so on $R^n(c)$.

(Another reasoning is that the above formulas give the speed of motion of the n -th iterate of the critical point c after perturbation by v . Since those iterates depend holomorphically over a definite neighborhood of 0, so Cauchy estimates give the desired bound. This conclusion can be obtained without the use of holomorphic motions.) \square

2.4. Measures. Let us say that a critical point $c \in J(R)$ of R is summable if

$$\sum_{k=0}^{\infty} \frac{1}{|DR^k(R(c))|} < \infty.$$

In this case, for a continuous map $v : \overline{\mathbb{C}} \rightarrow \mathbb{C}$ we let

$$\mu_{R,c}(v) = \sum_{k=0}^{\infty} \frac{v(R^k(c))}{DR^k(R(c))},$$

so that $\mu_{R,c}$ is a complex measure. It is clear that $\mu_{R,c}$ does not vanish over continuous functions.

Let us say that $\mu_{R,c}$ is non-trivial if it does not vanish over the space of holomorphic functions on a neighborhood of $\overline{\text{orb}_R(c)}$. It should be noted that if $\overline{\text{orb}_R(c)}$ does not disconnect the plane (for instance, if it is a Cantor set) then $\mu_{R,c}$ is non-trivial by the Mergelyan Approximation Theorem.

With those definitions, Theorem 1.1 can be restated as follows:

Corollary 2.10. *Let $R \in \text{Rat}_d^\infty$ and assume that c is a summable critical point such that $\mu_{R,c}$ is non-trivial. Then R is not structurally stable.*

Proof. Assume by contradiction that R is structurally stable, and let v be a holomorphic vector field on a neighborhood of $\overline{\text{orb}_R(c)}$ such that $\mu_{R,c}(v) \neq 0$. By the Runge Approximation Theorem, there exists a meromorphic vector field \tilde{v} on $\overline{\mathbb{C}}$ which is uniformly close to v on $\overline{\text{orb}_R(c)}$. Since $J(R) \neq \overline{\mathbb{C}}$, it has empty interior so we may assume that the poles of \tilde{v} do not intersect $J(R)$, so \tilde{v} is holomorphic in a neighborhood of $J(R)$.

Then

$$\mu_{R,c}(\tilde{v}) = \lim_{\lambda \rightarrow 0} \frac{1}{DR^{n-1}(R(c))} \frac{d}{d\lambda} (R + \lambda v)^n(c) \Big|_{\lambda=0}.$$

Since c is a summable critical point, $|DR^n(R(c))| \rightarrow \infty$, so by Corollary 2.9, $\mu_{R,c}(\tilde{v}) = 0$. This is a contradiction since $\mu_{R,c}(\tilde{v})$ approximates $\mu_{R,c}(v)$. \square

Remark 2.4. When $\omega(c)$ is a non-minimal set (this is often the case if $\overline{\text{orb}_R(c)}$ is not a Cantor set) it is usually possible to tackle successfully the problem of structural stability using other techniques.

Remark 2.5. Peter Makienko has informed me that more general conditions for non-triviality can be obtained using the theory of universal algebras. For instance, if $J(R)$ is the boundary of one of the Fatou components (this is the case for polynomials), $\mu_{R,c}$ is always non-trivial.

Remark 2.6. Michael Benedicks pointed to me that a complex atomic measure can indeed vanish over, say, complex polynomials (for a discussion on this problem see [BSZ]).

APPENDIX A. APPENDIX: INFINITESIMAL PERTURBATIONS OF POLYNOMIAL MAPS AND HYBRID CLASSES

The results we obtained in the setting of rational maps have clear analogues for polynomial maps. Instead of presenting those straightforward applications, we will use this Appendix to present a link to Douady-Hubbard's Theory of polynomial-like maps.

A polynomial-like map of degree d is a holomorphic degree d ramified covering map $f : U \rightarrow U'$, where U and U' are Jordan disks and $\overline{U} \subset U'$. Such maps were introduced by [DH]. We assume the reader is familiar with this theory. The filled-in Julia set of f , $K(f)$ is the set of non-escaping points under f .

A polynomial of degree d naturally restricts to a polynomial-like map: just take $U' = \mathbb{D}_r$ for r big enough. This allows us to consider polynomial-like perturbations of a polynomial map. A holomorphic perturbation of a polynomial on a neighborhood of the filled-in Julia set $K(P)$ of a polynomial P (with, say, $K(P)$ connected) will usually give rise to a polynomial like map. Such perturbations are more restrictive than the ones we consider here. Nevertheless they have interesting geometric interpretations. The reason is that the space of polynomial-like maps with connected filled-in Julia set fibers over the space of polynomials with connected Julia set (modulo conformal equivalence). Each fiber is called a hybrid class¹ and the polynomial (or rather its conformal class) in each hybrid class is called the straightening of the class.

¹Two polynomial-like maps f and g with connected Julia set are hybrid equivalent if there is a quasiconformal conjugacy h between f and g on a neighborhood of their filled-in Julia sets satisfying $\bar{\partial}h|_{K(f)} = 0$.

In this case, a polynomial-like perturbation of a polynomial map may be projected back to the space of polynomials with the straightening². This approach, while more geometric, gives slightly weaker results, since $J(P) = \partial K(P)$, so the space of holomorphic perturbations on $J(P)$ (which we considered before) is usually bigger.

Let us elaborate a little bit more the case of quadratic polynomial maps, that is, we will consider polynomials of the form $p_c = z^2 + c$ (see Remark A.2 for higher degree). To keep things non-trivial, we will consider only parameters c in the Mandelbrot set \mathcal{M} , the set of all c such that the critical orbit of p_c is non-escaping.

According to Lyubich in [L], §4, the space \mathcal{C} of germs³ of normalized quadratic-like maps with a connected Julia set has a complex analytic structure. The tangent space to a germ $f \in \mathcal{C}$ is precisely the space of germs of vector fields which are holomorphic on a neighborhood of $K(f)$ and whose derivative vanish to order 2 at 0. This space is laminated by hybrid classes \mathcal{H}_c , $c \in \mathcal{M}$, which are codimension-one analytic submanifolds. The quadratic family $c \mapsto z^2 + c$ is a transverse to this lamination, see Theorem 4.11 of [L]. Finally, the filled-in Julia set of $f \in \mathcal{C}$ depends holomorphically along \mathcal{H}_c . Since the critical orbit of p_c is contained in $K(p_c)$ we have as before:

Lemma A.1. *Let $c \in \mathcal{M}$ and let v be a holomorphic vector field on a neighborhood of $K(p_c)$. If v is tangent to the hybrid class of c , say, along an analytic path $f_\lambda \in \mathcal{H}_c$,*

$$f_0 = p_c, \quad \left. \frac{d}{d\lambda} f_\lambda \right|_{\lambda=0} = v,$$

then $v = \alpha \circ p_c - \alpha Dp_c$ for some qc vector field α , and in particular, the sequence

$$\left. \frac{d}{d\lambda} (f_\lambda)^n(0) \right|_{\lambda=0} = Dp_c^{n-1}(c) \sum_{k=0}^{n-1} \frac{v(p_c^k(c))}{Dp_c^k(p_c(c))}$$

is uniformly bounded.

Remark A.1. In Section 4.4 of [L], it is actually shown that a tangent vector field v to the hybrid class of a polynomial-like map f admits a representation $v = \alpha \circ f - \alpha Df$ satisfying $\bar{\partial}\alpha|_{K(f)} = 0$. Moreover, those two conditions characterize the tangent space to the hybrid class of f .

This allows us to characterize the tangent space to the hybrid class of polynomials satisfying the summability condition:

Lemma A.2. *Assume p_c has a summable critical point. Then $\mu_{p_c,0}$ does not vanish over the $T_{p_c}\mathcal{C}$.*

Proof. According to Przytycki (see [R]), under the summability assumption one has that $K(p_c)$ is a full compact set with empty interior. By the Mergelyan Approximation Theorem, any continuous function over $K(p_c)$ can be approximated by a polynomial in $T_{p_c}\mathcal{C}$. The result follows since $\mu_{p_c,0}$ does not vanish over continuous functions. \square

²Although the straightening is not in general continuous at polynomial-like maps, it is continuous at polynomials, which is enough for our purposes. This continuity is analogous to the one of Lemma 2.3, and can be proved in a similar way (see also Remark 2.2).

³Two quadratic-like maps f and g with connected filled-in Julia set give rise to the same germ if $K(f) = K(g)$ and $f|_{K(f)} = g$.

Theorem A.3. *If p_c has a summable critical point then the tangent space to \mathcal{H}_c is the kernel of $\mu_{p_c,0}$.*

Proof. Indeed, $\mu_{p_c,0}$ vanishes over the tangent space to \mathcal{H}_c by Lemma A.1, but does not vanish completely by Lemma A.2. The result follows since \mathcal{H}_c is codimension-one. \square

Let us mention a curious application of transversality of the quadratic family:

Corollary A.4. *Assume the critical point of p_c is summable. Then*

$$(A.1) \quad \mu_{p_c,c}(1) = \sum_{k=0}^{\infty} \frac{1}{Dp_c^k(c)} \neq 0.$$

Proof. Indeed, the tangent to the quadratic family is the set of constant vector fields. Transversality implies that the tangent vector field 1 does not belong to the tangent space of \mathcal{H}_c . \square

(This same result seems to have been independently obtained by Levin using other techniques.)

In [R], the same result is obtained with a stronger condition than summability (and under those conditions, an interpretation for $\mu_{p_c,c}(1)$ is given in terms of ratios of similarity between Julia and Mandelbrot sets).

Remark A.2. The results described here for quadratic-like maps, in particular Theorem A.3 and Corollary A.4 can be also obtained in the context of unicritical polynomial-like maps of degree d (hybrid conjugate to $z^d + c$). In this case there is still a lamination of the space \mathcal{C}_d of germs with connected Julia set by hybrid classes, and the generalized Mandelbrot set \mathcal{M}_d is a global transversal (although a given hybrid class might intersect \mathcal{M}_d in more than one point). In fact for our purposes it is only necessary to generalize (in a straightforward way) Sections 4.4 and 4.5 of [L] which deal with the description of the tangent space of \mathcal{C}_d .

Problem 1. *Let $c \in \mathcal{M}_d$ be such that p_c is non-hyperbolic. Is it true that*

$$(A.2) \quad Dp_c^{n-1}(c) \sum_{k=0}^{n-1} \frac{1}{Dp_c^k(c)}$$

is unbounded?

The previous Corollary shows that (A.2) is unbounded at least in the summable case. The work of [ALM] indicates that this should hold for real quadratic maps whose postcritical set is a non-minimal set. By Corollary 2.9, an affirmative answer in the general case would imply that the Fatou conjecture is valid in this setting.

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