Hilbert schemes of points on singular surfaces: combinatorics, geometry, and representation theory

Balázs Szendrői, University of Oxford

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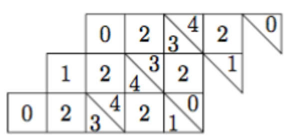
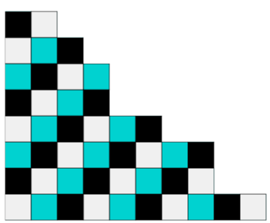




Hilbert schemes of points: affine case

A: a finitely generated commutative unital $\mathbb{C}$-algebra.

$X = \mathrm{Spec}(A)$: the algebraic variety over $\mathbb{C}$ whose ring of functions is A.

The **n-th Hilbert scheme of points of X** parametrizes the set of codimension n ideals $I \lhd A$:

$$\mathrm{Hilb}^n(X) = \{I \lhd A \colon \dim_{\mathbb{C}} A/I = n\}.$$

Grothendieck: this set carries the structure of a quasiprojective algebraic scheme over $\mathbb{C}$.

Hilbert schemes of points: affine case, geometry

Geometric interpretation For $I \in \mathrm{Hilb}^n(X)$, we get a surjection

$$A \twoheadrightarrow A/I$$

defining a subscheme (subvariety)

$$Z = \mathrm{Spec}(A/I) \subset X = \mathrm{Spec}\, A$$

of finite length n. So we can think of $\mathrm{Hilb}^n(X)$ as parametrizing **finite length subschemes** of the geometric space $X = \mathrm{Spec}\, A$.

Construction Choosing $P_1, \ldots, P_n$ distinct points in X, we can let $Z = \cup P_i$ and

$$I = I_Z = \{f \in A \colon f(P_i) = 0\} \lhd A.$$

Then

$$I_Z \in \mathrm{Hilb}^n(X).$$

This construction however does not give all codimension n ideals.

Hilbert schemes of points: affine case, example

Example $A = \mathbb{C}[x, y]$, corresponding to the affine plane $X = \mathrm{Spec}(A) = \mathbb{C}^2$.

- $\langle 1 \rangle \in \mathrm{Hilb}^0(X)$ corresponds to the empty subscheme.
- $\langle x, y \rangle \in \mathrm{Hilb}^1(X)$ corresponds to the origin in $\mathbb{C}^2$.
- $\langle x - \alpha, y - \beta \rangle \in \mathrm{Hilb}^1(X)$ for $\alpha, \beta \in \mathbb{C}$ corresponds to $P = (\alpha, \beta) \in \mathbb{C}^2$. Indeed
$$\mathrm{Hilb}^1(X) \cong X.$$
- $\langle x^2 - 1, y \rangle = \langle x + 1, y \rangle \cap \langle x - 1, y \rangle \in \mathrm{Hilb}^2(X)$ corresponds to the pair of points $Z = (1, 0) \cup (-1, 0)$ in $\mathbb{C}^2$.
- $\langle x^2, y \rangle \in \mathrm{Hilb}^2(X)$ gives a **length-two fat subscheme supported at the origin**;
$$A/I = \mathbb{C}[x, y]/\langle x^2, y \rangle = \mathbb{C}[x]/\langle x^2 \rangle$$
is an Artinian ring with nilpotent elements.
- $\langle x^2, xy, y^2 \rangle \in \mathrm{Hilb}^3(X)$ gives a length-three fat subscheme at the origin.

Hilbert schemes of points: global case

Let X be a general quasiprojective algebraic variety. We can then define

$$\mathrm{Hilb}^n(X) = \{Z \subset X \text{ a subscheme of length } n\}.$$

Once again, a collection of n distinct points of X gives $Z = \cup P_i \in \mathrm{Hilb}^n(X)$. The Hilbert scheme parametrizes, in a geometric way, collisions between points of X.

Indeed, a subscheme $Z \subset X$ of length n has **support** $\mathrm{Supp}(Z) \subset X$, a set of unordered points in X together with multiplicities summing to n. This gives rise to the **Hilbert–Chow morphism**

$$\phi_{\mathrm{HC}} \colon \mathrm{Hilb}^n(X) \to S^n(X)$$

to the n-th **symmetric product** of X

$$S^n(X) = \overbrace{X \times \ldots \times X}^{n} / \mathfrak{S}_n$$

where $\mathfrak{S}_n$ is the symmetric group.

Geometry and topology of Hilbert schemes of points

The Hilbert scheme has its own geometry over $\mathbb{C}$, and hence topology. Its topology is a combination of

- the global topology of the space X, and
- the local topology of Hilbert schemes of local $\mathbb{C}$-algebras $\mathcal{O}_{X,x}$.

For this talk, one object of interest is the generating function

$$\boxed{Z_X(q) = 1 + \sum_{n\geq 1} \chi_{\text{top}}(\text{Hilb}^n(X))\, q^n}$$

We are also interested in geometric questions such as

- when is $\text{Hilb}^n(X)$ nonsingular;
- when is $\text{Hilb}^n(X)$ irreducible?

Smooth curves

Let first $X = C$ be a smooth connected algebraic curve over $\mathbb{C}$. Then the Hilbert–Chow morphism is an isomorphism:

$$\phi_{\mathrm{HC}} \colon \mathrm{Hilb}^n(C) \cong S^n(C).$$

Slogan: "in one dimension, there is only one way for points to collide".

This in particular shows that $\mathrm{Hilb}^n(C)$ is irreducible and nonsingular.

Theorem (MacDonald)

$$Z_C(q) = (1-q)^{-\chi_{\mathrm{top}}(C)}.$$

Example If $C = \mathbb{A}^1$, then $\mathrm{Hilb}^n(C) = \mathbb{A}^n$ ("Newton's theorem on symmetric functions"), and so

$$Z_{\mathbb{A}^1}(q) = 1 + q + q^2 + \ldots = (1-q)^{-1}.$$

Smooth surfaces

Let now X be a smooth algebraic surface over $\mathbb{C}$.

Theorem (Fogarty) The algebraic variety $\mathrm{Hilb}^n(X)$ is irreducible and nonsingular. The Hilbert–Chow morphism

$$\phi_{\mathrm{HC}} \colon \mathrm{Hilb}^n(X) \to S^n(X)$$

is a resolution of singularities of the symmetric product.

Theorem (Göttsche)

$$\boxed{Z_X(q) = E(q)^{\chi_{\mathrm{top}}(X)}}$$

where

$$E(q) = \prod_m (1-q^m)^{-1}$$

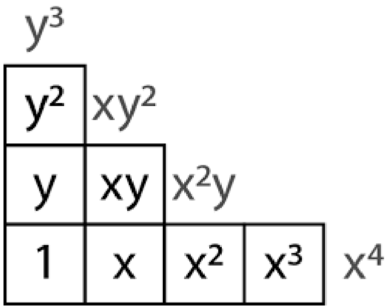
is essentially the Dedekind eta function.

Remark In particular, up to a power of q, this is a **modular function** of q.

Smooth surfaces: an example

Example, continued Return to $X = \mathbb{C}^2$, the affine plane, corresponding to the ring $A = \mathbb{C}[x, y]$. Special ideals: **monomial ideals** attached to **partitions**.

Example Let $\lambda = (4, 2, 1)$, a partition of 7.



We get the monomial ideal

$$I_\lambda = \langle x^4, x^2y, xy^2, y^3 \rangle \in \mathrm{Hilb}^7(\mathbb{C}^2).$$

Smooth surfaces: an example

Example, continued Return to $X = \mathbb{C}^2$, the affine plane, corresponding to the ring $A = \mathbb{C}[x, y]$. Special ideals: **monomial ideals** attached to **partitions**.

Using the technique of torus localization, we obtain

$$\begin{aligned} \chi_{\text{top}}(\text{Hilb}^n(\mathbb{C}^2)) &= \#\{\text{monomial ideals of colength n}\} \\ &= \#\{\lambda \text{ a partition of } n\} \\ &= p(n) \end{aligned}$$

and so

$$\boxed{Z_{\mathbb{C}^2}(q) = 1 + \sum_{n \geq 1} p(n) q^n = \prod_m (1 - q^m)^{-1}}$$

as stated by Göttsche's formula!

Singular curves

Next, let $X = C$ be a **singular** algebraic curve over $\mathbb{C}$ with a finite number of **planar** singularities $P_i \in C$.

The corresponding Hilbert schemes $\mathrm{Hilb}^n(C)$ are of course singular (already for $n = 1$!) but known to be irreducible.

Theorem (conjectured by Oblomkov and Shende, proved by Maulik)

$$\boxed{Z_C(q) = (1-q)^{-\chi(C)} \prod_{j=1}^{k} Z^{(P_i,C)}(q)}$$

Here each $Z^{(P_i,C)}(q)$ is a highly nontrivial local term that can be expressed in terms of the HOMFLY polynomial of the embedded link of the singularity $P_i \in C$.

Singular surfaces

In joint work with Gyenge and Némethi, followed by further work with Craw, Gammelgaard and Gyenge, we explored the case of **singular algebraic surfaces**.

As in the curve case, one is only likely to get results for restricted classes of singularities. We study the simplest possible class: **rational double points**.

There are many equivalent characterisations of surface rational double points. The most useful for us will be the following.

Definition A surface rational double point $P \in X$ is a quotient singularity locally analytically of the form

$$P = [(0,0)] \in X = \mathbb{C}^2/\Gamma$$

for a finite matrix group

$$\Gamma < \mathrm{SL}(2,\mathbb{C}).$$

Classification of surface rational double points

Definition A surface rational double point $P \in X$ is a quotient singularity locally of the form $P = [(0,0)] \in X = \mathbb{C}^2/\Gamma$ for a finite group $\Gamma < \mathrm{SL}(2,\mathbb{C})$.

We have

$$A = \mathbb{C}[X] = \mathbb{C}[x,y]^{\Gamma},$$

the ring of invariants.

Such groups/singularities correspond to finite subgroups of the rotation group $\mathrm{SO}(3)$, and so come in three families.

- Abelian groups $\Gamma = C_{r+1}$, called type A_r.
- (Binary) dihedral groups, called type D_r.
- Exceptional groups (tetrahedral, octahedral, icosahedral), called types E_6, E_7, E_8.

Via the **McKay correspondence**, these subgroups of $\mathrm{SL}(2,\mathbb{C})$ can be related to simply laced (finite and affine) Dynkin diagrams, hence their names.

Singular and equivariant Hilbert schemes

Our main interest is in the spaces $\mathrm{Hilb}^n(X)$ for $X = \mathbb{C}^2/\Gamma$ with coordinate ring $A = \mathbb{C}[x,y]^\Gamma$. These are singular spaces for $n \geq 1$.

Given the action of the group Γ on $\mathbb{C}^2$, one can define **equivariant Hilbert schemes** also, for any finite-dimensional representation $\rho \in \mathrm{Rep}(\Gamma)$ of Γ:

$$\mathrm{Hilb}^\rho(\mathbb{C}^2) = \{I \lhd \mathbb{C}[x,y]\ \Gamma\text{-invariant}\colon \mathbb{C}[x,y]/I \simeq_\Gamma \rho\}.$$

Their topological Euler characteristics can be collected into a master generating function

$$\boxed{Z_{\mathbb{C}^2,\Gamma}(q_0,\ldots,q_r) = \sum_{m_0,\ldots,m_r=0}^{\infty} \chi_{\mathrm{top}}\left(\mathrm{Hilb}^{m_0\rho_0+\ldots+m_r\rho_r}(\mathbb{C}^2)\right) q_0^{m_0}\cdot\ldots\cdot q_r^{m_r}}$$

where $\mathrm{Irrep}(\Gamma) = \{\rho_0, \rho_1, \ldots, \rho_r\}$.

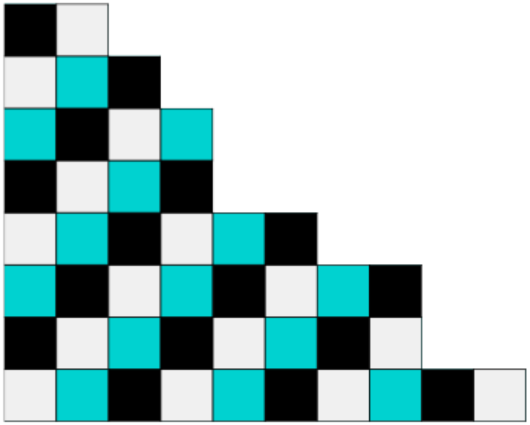
This function $Z_{\mathbb{C}^2,\Gamma}(q_0,\ldots,q_r)$ turns out to be closely related to the function $Z_X(q)$ attached to the singular surface $X = \mathbb{C}^2/\Gamma$.

The abelian case

The case of an abelian group Let Γ be the group of type A_r

$$\Gamma = \left\{ \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} : \omega^{r+1} = 1 \right\} < \mathrm{SL}(2, \mathbb{C}).$$

Monomial ideals in $\mathbb{C}[x, y]$ are Γ-equivariant, and correspond to partitions that are **coloured** by $r + 1$ colours, in the following way (here $r = 2$ so $\Gamma \cong C_3$):



The abelian case: coloured box counting

We apply torus localization again. We get a coloured version of the partition counting problem:

$$\chi_{\mathrm{top}}\left(\mathrm{Hilb}^{\sum m_i\rho_i}(\mathbb{C}^2)\right) \;=\; \#\{\lambda \text{ a coloured partition with } m_i \text{ boxes of colour } i\}$$

and so

$$\boxed{Z_{\mathbb{C}^2,\Gamma}(q_0,\ldots,q_r) = 1 + \sum_{\lambda}\prod_j q_j^{\mathrm{col}_j(\lambda)}}$$

is the **coloured generating function of partitions** (for diagonal colouring).

Example For type A_1, $\Gamma \cong C_2$ and we get the generating function of partitions in the checkerboard colouring

$$\boxed{Z_{\mathbb{C}^2,C_2}(q_0,q_1) = E(q_0q_1)^2 \cdot \sum_{m=-\infty}^{\infty} q_0^{m^2} q_1^{m^2+m}}$$

The equivariant generating function in general

For abelian $\Gamma < \mathrm{SL}(2,\mathbb{C})$, the generating function of diagonally coloured partitions can be determined purely combinatorially, and one gets a similar formula to the A_1 case.

However, the answer has a Lie-theoretic flavour, and generalises to all types in the following way.

Theorem (essentially due to Nakajima) In all types, the equivariant generating function has the following expression, with $q = \prod_i q_i^{\delta_i}$:

$$\boxed{Z_{\mathbb{C}^2,\Gamma}(q_0,\ldots,q_r) = E(q)^{r+1} \sum_{\mathbf{m}\in\mathbb{Z}^r} q^{\frac{1}{2}\mathbf{m}^t C \mathbf{m}} \prod_{i=1}^{r} q_i^{m_i}}$$

Here r, C, δ_i are the rank, Cartan matrix and Dynkin indices corresponding to the type of the group Γ.

The singular generating function

Our main interest was not in the equivariant function, but the function

$$Z_X(q) = 1 + \sum_{n \geq 1} \chi_{\mathrm{top}}(\mathrm{Hilb}^n(X)) q^n$$

attached to the singular geometry $X = \mathbb{C}^2/\Gamma$.

Theorem (Gyenge–Némethi–Sz., 2015) Let Γ be of type A_r or D_r. Then, with $q = \prod_i q_i^{\delta_i}$ and $\xi = \exp(\frac{2\pi i}{1+h})$, we have

$$\boxed{Z_X(q) = Z_{\mathbb{C}^2,\Gamma}(q_0, q_1, \ldots, q_r)|_{q_1=q_2=\ldots=q_r=\xi}}$$

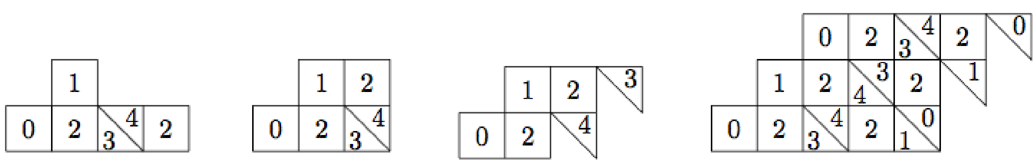
where h is the Coxeter number of the Lie algebra of the corresponding type.

The Theorem implies in particular that the function $Z_X(q)$ is modular.

We conjectured that the result also holds in type E.

Some aspects of the proof

- For type A_r, the argument is purely combinatorial and only involves coloured partitions.
- Coloured partitions have a Lie-theoretic meaning as **elements of a crystal basis** (of a certain representation of the affine Lie algebra)
- For type D_r, the argument has two parts:
  1. the combinatorics of the crystal basis in type D_r, and
  2. the study of the geometry of stratifications of Hilbert schemes indexed by crystal basis elements.



The McKay quiver of Γ

Return to our finite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, with $\mathrm{Irrep}(\Gamma) = \{\rho_0, \rho_1, \ldots, \rho_r\}$. Let V be the canonical 2-dim rep of Γ.

The **McKay graph** of Γ has

- vertex set $\{0, 1, \ldots, r\}$;
- $\dim \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes V)$ edges from i to j.

McKay (1980): this graph is an extended Dynkin diagram of ADE type.

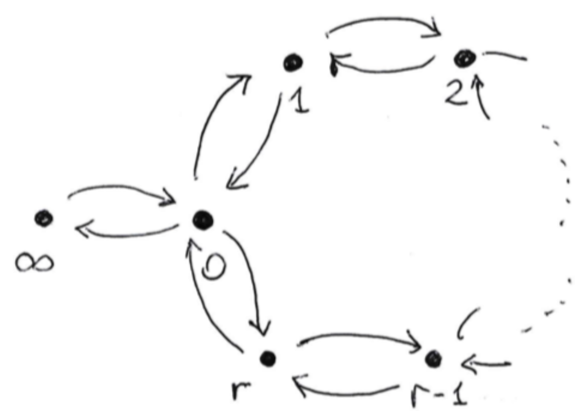
McKay quiver: turn the McKay graph into a quiver (oriented graph) by introducing a pair of opposite arrows for each edge. Extend by an additional vertex labelled ∞, with a pair of arrows to and from vertex 0. Call Q the resulting quiver on the vertex set $V(Q) = \{0, 1, \ldots, r, \infty\}$, with edge set $E(Q)$.

The McKay quiver of abelian Γ

Return to $\Gamma \cong \mu_{r+1}$ of type A_r

$$\Gamma = \left\{ \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} : \omega^{r+1} = 1 \right\} < \mathrm{SL}(2, \mathbb{C}).$$

The (extended) McKay quiver Q looks as follows:



Representations of the McKay quiver

We want to study representation (quiver) varieties of the extended McKay quiver Q. These depend on two parameters:

- the **dimension vector** $d \in \mathbb{N}^{r+1}$, attaching to each vertex i a non-negative integer d_i (with vertex ∞ always carrying dimension 1);
- the **stability parameter** $\theta \in \mathbb{Q}^{r+1}$.

Given this data, we fix a set of vector spaces $\{V_i : I \in V(Q)\}$ of dimension d_i attached to each vertex, with V_∞ of dimension 1, and we consider the collection of all linear maps $\{\varphi_{ij} \colon V_i \to V_j \colon (ij) \in E(Q)\}$, subject two conditions:

- they should satisfy the **preprojective relations**;
- they should be **semistable** with respect to the parameter θ (King).

Let $U_\theta(d)$ denote the space of all linear maps satisfying these two conditions. This is a locally closed subvariety of an affine space.

Quiver varieties

The space $U_\theta(d)$ carries an action of the group $G = \prod_{i=0}^{r} \mathrm{GL}(V_i)$. Orbits of this group parametrise **isomorphism classes** of representations of Q, which are θ-semistable with dimension vector d and satisfy the relations.

Define the **(Nakajima) quiver variety**

$$\mathfrak{M}_\theta(d) = U_\theta(d) /\!\!/_\theta\, G,$$

the Geometric Invariant Theory (GIT) quotient of $U_\theta(d)$ by the group G.

Example 1 (Kronheimer–Nakajima) Choose $d_1 = \{\dim \rho_i\}$. Then for generic stability condition θ, the GIT quotient $\mathfrak{M}_\theta(d_1)$ is independent of θ, and is isomorphic to the minimal resolution Y of the surface singularity $X = \mathbb{C}^2/\Gamma$.

Example 2 (folklore) Let $d_n = \{n \cdot \dim \rho_i\}$ for some natural number n. Choose the stability condition $\theta = 0$. Then the GIT quotient $\mathfrak{M}_0(d_n)$ is affine (general fact), and is isomorphic to the n-th symmetric product $S^n(X)$. In particular, $\mathfrak{M}_0(d_1) \cong X$.

Generic and special stability parameters

We continue to work with this setup: fix $d_n = \{n \cdot \dim \rho_i\}$, and study the space $\mathfrak{M}_\theta(d_n)$ as the stability parameter $\theta \in \mathbb{Q}^{r+1}$ varies.

By general principles of variation of GIT (Thaddeus, Dolgachev-Hu), we expect a wall-and-chamber structure, with stability parameters in open chambers giving nice GIT quotients $\mathfrak{M}_\theta(d_n)$, while the quotient $\mathfrak{M}_{\theta_0}(d_n)$ becomes more singular for parameters θ_0 lying in walls.

The general setup will also induce morphisms

$$\mathfrak{M}_\theta(d_n) \to \mathfrak{M}_{\theta_0}(d_n)$$

relating different quiver varieties.

Example (continued) With $d_1 = \{\dim \rho_i\}$ as above, moving from a generic stability condition θ to $\theta = 0$ gives a morphism $\mathfrak{M}_\theta(d_1) \to \mathfrak{M}_0(d_1)$ which can be identified with the minimal resolution $Y \to X = \mathbb{C}^2/\Gamma$.

A distinguished chamber in stability space

Theorem (Varagnolo–Vasserot, Kuznetsov) Fix $n \geq 1$. There exists a distinguished open chamber $C^+ \subset \mathbb{Q}^{r+1}$ inside stability space, so that for $\theta \in C^+$,

$$\mathfrak{M}_\theta(d_n) \cong \mathrm{Hilb}^{n \cdot \rho_{\mathrm{reg}}}(\mathbb{C}^2)$$

where on the right we have the Γ-equivariant Hilbert scheme of $\mathbb{C}^2$ corresponding to $n \cdot \rho_{\mathrm{reg}} \in \mathrm{Rep}(\Gamma)$, with $\rho_{\mathrm{reg}} \in \mathrm{Rep}(\Gamma)$ is the regular representation. The morphism to the stability space at zero stability can be identified with

$$\mathrm{Hilb}^{n \cdot \rho_{\mathrm{reg}}}(\mathbb{C}^2) \to S^n(\mathbb{C}^2/\Gamma)$$

which is a minimal resolution of singularities.

Example (continued again) For $n = 1$, we this fits with a theorem of Kapranov and Vasserot, the isomorphism

$$\mathrm{Hilb}^{\rho_{\mathrm{reg}}}(\mathbb{C}^2) \cong Y$$

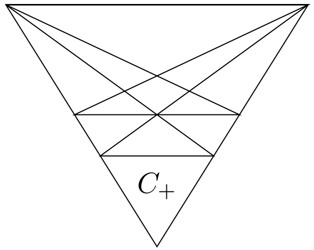
between the minimal resolution Y of X and the so-called Γ-Hilbert scheme.

The wall-and-chamber structure of stability space

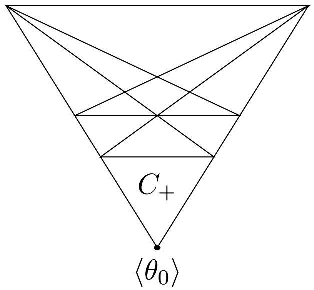
In a recent paper, Bellamy and Craw understood the structure of the entire stability space, at least as far as generic open chambers are concerned.

Theorem (Bellamy–Craw, 2018) The closed cone $\bar{C}^+ \subset \mathbb{Q}^{r+1}$ can be identified with the nef cone (closed ample cone) of the variety $\mathrm{Hilb}^{n\cdot\rho_{\mathrm{reg}}}(\mathbb{C}^2)$. There is a larger cone $N \subset \mathbb{Q}^{r+1}$, with a finite (combinatorially described) wall-and-chamber structure, open chambers of which correspond to ample cones of birational models of $\mathrm{Hilb}^{n\cdot\rho_{\mathrm{reg}}}(\mathbb{C}^2)$.

Example Let $\Gamma \cong \mu_3$, corresponding to Dynkin type A_2, and $n = 3$.



A distinguished corner of stability space



Theorem (Craw–Gammelgaard–Gyenge–Sz., 2019) For a distinguished ray $\langle\theta_0\rangle \in \partial\bar{C}_+$, we have an isomorphism

$$\mathfrak{M}_{\theta_0}(d_n) \cong \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$$

between a quiver variety and the Hilbert scheme of points of the surface singularity.

A distinguished corner of stability space (continued)

Theorem (continued) The resulting chain of morphisms

$$\mathfrak{M}_\theta(d_n) \to \mathfrak{M}_{\theta_0}(d_n) \to \mathfrak{M}_0(d_n)$$

can be identified with the chain

$$\mathrm{Hilb}^{n\cdot\rho_{\mathrm{reg}}}(\mathbb{C}^2) \to \mathrm{Hilb}^n(\mathbb{C}^2/\Gamma) \to S^n(\mathbb{C}^2/\Gamma)$$

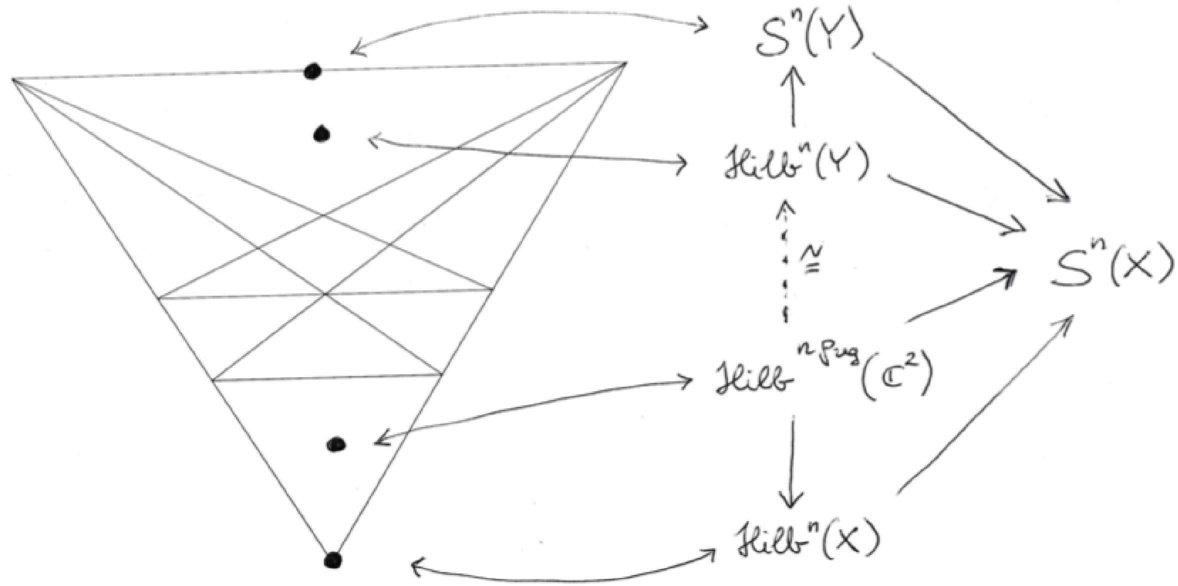
which includes the Hilbert–Chow morphism of the singular variety $X = \mathbb{C}^2/\Gamma$.

Corollary The Hilbert scheme $\mathrm{Hilb}^n(X)$ of the surface singularity $X = \mathbb{C}^2/\Gamma$ is an irreducible, normal quasiprojective variety with a unique symplectic (Calabi–Yau) resolution.

This is about as nice as one could hope for! Irreducibility was known before (Xudong Zheng, 2017). Conjecturally this property **characterises** surface rational double points among all varieties of dimension at least 2.

Many spaces in one diagram

We get the following diagram of GIT-induced morphisms, including the Hilbert–Chow morphisms of both the singularity $X = \mathbb{C}^2/\Gamma$ and its minimal resolution Y.



Back to Euler characteristics

As opposed to the combinatorial story, which only applies to type A and type D singularities, the quiver story is completely general. We have identified the resolution of singularities

$$\mathrm{Hilb}^{n\cdot\rho_{\mathrm{reg}}}(\mathbb{C}^2) \to \mathrm{Hilb}^n(X)$$

with a map

$$\mathfrak{M}_\theta(d_n) \to \mathfrak{M}_{\theta_0}(d_n)$$

between quiver varieties.

This suggests that the conjecture of Gyenge–Némethi–Sz. about the generating function of Euler characteristics of $\mathrm{Hilb}^n(X)$ could be approached this way.

Specialising stability parameters in geometry

Nakajima, 2009: the fibres of the map $\mathfrak{M}_\theta(d_n) \to \mathfrak{M}_{\theta_0}(d_n)$ between quiver varieties are themselves (Lagrangian subvarieties in) quiver varieties associated with **finite ADE quivers**.

This looks like it gives an approach to the conjecture. However, computing the Euler characteristics of fibres directly is still hard! Nevertheless...

Theorem (Nakajima, 2020) For Γ of **arbitrary type**, with $q = \prod_i q_i^{\delta_i}$ and $\xi = \exp(\frac{2\pi i}{1+h})$, the generating function of the Hilbert scheme of points of the surface singularity $X = \mathbb{C}^2/\Gamma$ is related to the equivariant generating function by the formula

$$\boxed{Z_X(q) = Z_{\mathbb{C}^2,\Gamma}(q_0, q_1, \ldots, q_r)|_{q_1=q_2=\ldots=q_r=\xi}}$$

where h is the Coxeter number of the Lie algebra of the corresponding type. In other words, the conjecture of Gyenge–Némethi–Sz. from 2015 holds.

How does he do it?

Specialising stability parameters in representation theory

Nakajima, 2009:

- the collection of spaces $\{\mathfrak{M}_\theta(d) \colon d \in \mathbb{N}^{r+1}\}$, for generic stability parameter and **all** dimension vectors, give rise to a **representation** of the **affine Lie algebra** $\widehat{\mathfrak{g}}$ attached to the McKay quiver as Dynkin diagram;
- going from a generic stability parameter to a degenerate one corresponds to **branching of representations** with respect to subalgebras of $\widehat{\mathfrak{g}}$;
- specifically, going from a generic parameter θ to our special ray θ_0 corresponds to considering representations of $\widehat{\mathfrak{g}}$ as representations of the finite-dimensional Lie algebra $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$.

This gives the following interpretation of the GyNSz conjecure: the generating function of Euler characteristics of our spaces $\mathrm{Hilb}^n(\mathbb{C}^2/\Gamma)$ is given by the graded **quantum dimension**, taken at a specific root of unity, of the basic representation of the affine Lie algebra, restricted to $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$.

Quantum dimensions of standard modules

It turns out that in computing this quantum dimension, a lot of cancellations happen, and the GyNSz conjecture is reduced to the following statement.

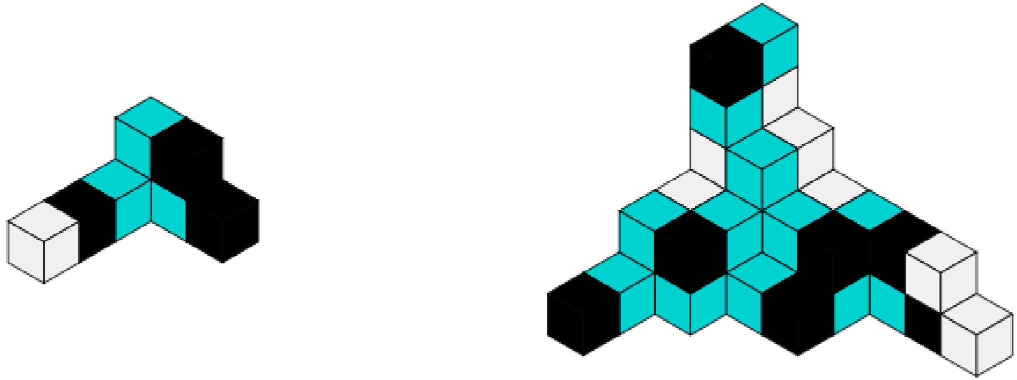
Theorem (Nakajima, 2020) The quantum dimention of an arbitrary so-called standard module of $U_q(L\mathfrak{g})$ of type ADE at the root of unity $\xi = \exp(\frac{2\pi i}{1+h})$ is equal to 1.

While this statement fits into more general conjectures in representation theory, it appears that this was a new result Nakajima needed to prove for E_7, E_8. For E_8, his proof relies on his own earlier computations of characters, done on a supercomputer, as well as further miraculous cancellations such as

$$(-4) + 18 + (-23) + 10 = 1.$$

Further directions

- Other walls in the space of stability parameters - some interesting geometry and combinatorics - work in progress by Gyenge, Sz. and others
- This is the rank 1 story - how about higher rank? Work in progress by Gammelgaard
- How much of the picture exists for a finite subgroup $G < \mathrm{SL}(3, \mathbb{C})$? Some really interesting combinatorics, ideas from Donaldson–Thomas theory... for another time
- Can we really understand why this simple substitution works? Nakajima's proof still relies on some mysterious cancellations...



Thank you!