0.1. **The Reconstruction Theorem of Kontsevich–Soibelman.** Let $\{e_1, \ldots, e_m\}$ be a basis for $N \cong \mathbb{Z}^m$, and let

$$\delta(n) : N \to \mathbb{Z}$$

be the linear function defined by $\delta(n) = \lambda_1 + \ldots + \lambda_m$ for for $n = \lambda_1 e_1 + \ldots + \lambda_m e_m$.

Theorem 0.1. *Consistent scattering diagrams (or $\hat{\mathbf{g}}$-complexes), up to equivalence are in bijective correspondence with elements of the group*

$$\hat{G} = \lim_{\longleftarrow} G_{\leq k}$$

Proof. One direction is immediate: given any consistent $\hat{\mathbf{g}}$-complex, we define an element of $\hat{G}$ given by the path-ordered product along any path between arbitrary points in $M_{\mathbb{R}}^{+}$ and $M_{\mathbb{R}}^{-}$. Conversely, we will first show that from any element

$$g \in G_{\leq k}$$

we can construct a consistent $\mathbf{g}$-complex, up to equivalence. We construct the set of walls from $G_{\leq k}$ as follows. Let

$$P := \{n \in N^+ \mid \delta(n) \leq k\}$$

Note that P is finite subset of N^+. The subdivisions of $n^\perp$, for each $n \in P$, into polyhedral cones in $M_{\mathbb{R}}$ is in bijective correspondence with partitions of the set P, as defined in [B, Example 2.5]. Since P is finite, it admits finitely many partitions, and hence we obtain finitely many codimension one cones (walls), say $\{d_i\}_{i \in I}$, for a finite index set I, with $d_i \subset n^\perp$, for some $n \in P$. Now, we assign to each such wall, a wall-crossing automorphism.

Note that for any point $x \in M_{\mathbb{R}}$, there is a unique decomposition of g with respect to x, given by

$$g = g_+^x \cdot g_0^x \cdot g_-^x$$

where $g_0^x \in G_0^x$, and $g_\pm^x \in G_\pm^x$. Moreover, there are natural projection maps

$$\begin{aligned} \pi_*^x : G &\longrightarrow G_*^x \\ g &\longmapsto g_*^x \end{aligned}$$

for each $* \in \{0, +, -\}$. We define the wall-crossing automorphism associated to each wall d_i, for $i \in I$, denoted by $\Phi(d_i)$, by

$$\Phi(d_i) := g_0^{x_i}$$

where $x_i \in M_{\mathbb{R}}$ is a generic point of d_i, and $g_0^{x_i} = \pi_0^{x_i}(g)$. Note that, since we have a finite set of walls, the decomposition $\mathbf{g} = \mathbf{g}_+^x \oplus \mathbf{g}_0^x \oplus \mathbf{g}_-^x$ of the Lie algebra varies discretely with x to order k, for a general point $x \in n^\perp$ for any $n \in P$. Therefore, the image of g_0^x under the projection map $G \mapsto G_{\leq k}$ is constant for $x \in d$, along each wall $d \in \{d_i\}_{i \in I}$.

It remains to show that the $\mathbf{g}$-complex we defined, by describing the set of walls along with wall-crossing automorphisms on each wall, is consistent. For this, it suffices to show that for any generic path γ connecting two arbitrary points $\Theta_+ \in M_{\mathbb{R}}^+$ and $\Theta_- \in M_{\mathbb{R}}^-$, the path ordered product is the same. We will show this indeed is the case, by showing that the path ordered product for any such path equals $g \in G_{\leq k}$.

Assume that γ traces out N walls d_i, with generic points x_i such that $\gamma(t_i) = x_i$. Recall that we had defined for each wall, the wall crossing automorphism by considering the decomposition of g for x_i, that is, $g = g_+^{x_i} \cdot g_0^{x_i} \cdot g_-^{x_i}$ and setting $\Phi(d_i) = g_0^{x_i}$. Hence, the path ordered product along γ by definition, up to a sign convention depending on the orientation of γ, is

$$\Phi_{\mathcal{D}} = g_0^{x_N} \dots g_0^{x_1}$$

We need to show $g_0^{x_N} \dots g_0^{x_1} = g$. We proceed similarly as in [GHKK, Thm. 1.17].

We first show inductively that the factorization of the element g, given by the generic point x_j of the wall d_j, for any $j \in \{1, \dots, N\}$, can be written as

$$g = (g_+)(g_0^{x_j})(g_0^{x_{j-1}} \dots g_0^{x_1})$$

for some $g_+ \in G_+^{x_j}$. For $j = 1$, writing the factorization of g for x_1, we obtain $g = g_+^{x_1} g_0^{x_1} g_-^{x_1}$. From [GHKK, pg 26] or [B, Lemma 3.1], we have $g_-^{x_1} \in G_+^{\Theta_+}$, where $\Theta_+ = \gamma(0)$. Hence $g_-^{x_1} \in G_-^{x_1} \cap G_+^{\Theta_+} = 1$ (the last equality holds since d_1 is the first wall traced by γ). If the claim is true for $j - 1$, then we have

$$g = (g')g_0^{x_{j-1}}(g_0^{x_{j-2}} \dots g_0^{x_1}) \tag{0.1}$$

for $g' \in G_+^{x_{j-1}}$. Now, writing the decomposition of g' for x_j, we get $g' = g_+^{x_j} \cdot g_0^{x_j} \cdot g_-^{x_j} \in G_+^{x_{j-1}}$. Note that $g_-^{x_j} = 1$: this again follows by [B, Lemma 3.1], since $G_+^{x_{j-1}} \cap G_-^{x_j} = 1$ (as there are no walls between d_{j-1} and d_j). Hence,

$$g' = g_+^{x_j} \cdot g_0^{x_j} \tag{0.2}$$

Now, from equations (0.1) and (0.2) we obtain

$$g = (g_+^{x_j})g_0^{x_j}(g_0^{x_{j-1}} g_0^{x_{j-2}} \dots g_0^{x_1})$$

for $g_+^{x_j} \in G_+^{x_j}$. Hence, the claim follows. Note that since, d_N is the last wall, we have $g_+^{x_N} \in G_+^{x_N} \cap G_-^{\Theta_-} = 1$, where $\Theta_- = \gamma(1)$. Therefore, we obtain

$$\Phi_{\mathcal{D}} = g_0^{x_N}(g_0^{x_{N-1}} g_0^{x_{j-2}} \dots g_0^{x_1}) = g.$$

This shows the consistency of the **g**-complex we defined. The bijective correspondence between equivalence classes of consistent $\hat{\mathbf{g}}$-complexes and elements of $\hat{G}$ follows by taking the limit as $k \to \infty$. $\square$

References

[GHKK] Gross, Mark, Paul Hacking, Sean Keel, and Maxim Kontsevich. "Canonical bases for cluster algebras." Journal of the American Mathematical Society 31, no. 2 (2018): 497-608.

[B] Bridgeland, Tom. "Scattering diagrams, Hall algebras and stability conditions." arXiv preprint arXiv:1603.00416 (2016).