CORRECTIONS TO ‘SUR LES A_∞-CATÉGORIES’

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ABSTRACT. These are corrections to K. Lefèvre’s Ph. D. thesis ‘Sur les A_∞-catégories’.

1. The errors

B. Keller and K. Lefèvre thank I. Moerdijk for pointing out [?] the following errors in [?]:

1) In the proof of point c) of Lemme 1.3.2.3 on page 37, line 5, it is claimed that the comultiplication on $\mathsf{Gr}\, C$ vanishes. This is wrong, in general.
2) In the proof of Lemme 1.3.2.7, on pages 43 and 44, the morphism j will not respect the filtration defined at the bottom of page 43, in general. Therefore, the end of the proof is incorrect.

The following error was found on August 6, 2005:

> In Theorem 1.3.1.1 on page 31, it is erroneously claimed that the cofibrant dg algebras are exactly the dg algebras isomorphic to almost free dg algebras. This statement is erroneously attributed to V. Hinich.

A counterexample is obtained by taking the free algebra $k\langle x\rangle$ in one indeterminate of degree 1 and endowing it with the differential d such that $d(x) = x^2$ (note that this is the cobar construction on a non cocomplete coalgebra). The underlying complex of this dg algebra is contractible (the differential is an isomorphism in odd degrees and vanishes in even degrees). If the algebra was cofibrant (and thus cofibrant and fibrant), then its identity morphism would be homotopic to the zero morphism. But a homotopy between two morphisms f and g is an f-g-derivation H of degree -1. It is determined by its value at the generator x. For degree reasons, we have $H(x) = 0$.

2. How to correct error 1)

Error 1) is corrected as follows: We have to show that the morphism of complexes

$$\mathsf{Gr}\, C \to \mathsf{Gr}(B\Omega C)$$

is a quasi-isomorphism. Let us fix $n \geq 0$. We have to show that

$$\mathsf{Gr}_n\, C \to \mathsf{Gr}_n(B\Omega C)$$

is a quasi-isomorphism. We endow $B\Omega C$ with the additional decreasing filtration F'_l, $l \in \mathbf{N}$, whose term F'_l is generated by all mth tensor powers of C where $m \geq l$. We endow $\mathsf{Gr}(B\Omega C)$ with the induced filtration, still denoted by F'_l. Then $F'_l\, \mathsf{Gr}_n(B\Omega C)$ vanishes for all $l \geq n$, because C is admissible and the subquotient

$$\mathsf{Gr}'_l\, \mathsf{Gr}_n(B\Omega C)$$

is the complex which is shown to be contractible in the last part of the proof on page 37.

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3. How to correct error 2)

We correct error 2) by more carefully choosing the splitting of the sequence

$$0 \to K \to A \to \Omega D \to 0$$

at the top of page 43. Indeed, we already know from Lemme 1.3.2.6 that the cobar constructions preserves cofibrations. In particular, since every object in Cogc is cofibrant, the dg algebra ΩD is cofibrant. So the surjective morphism $p : A \to \Omega D$ admits a section which is a morphism of *differential* graded algebras (and not only of graded algebras). We choose such a section to obtain a decomposition

$$A \xrightarrow{\sim} K \oplus \Omega D$$

of complexes. Then the differential is compatible with this decomposition, *i.e.* given by a matrix

$$\begin{bmatrix} d_K & 0 \\ 0 & d_{\Omega D} \end{bmatrix}$$

so that the contribution d' of the original proof vanishes. Now the terms (I) and (II) of the lower half of page 43 become quasi-isomorphic thanks to the Künneth theorem and the last, erroneous, part of the proof can simply be omitted.

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