CLUSTER TILTING FOR ONE-DIMENSIONAL HYPERSURFACE SINGULARITIES

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Abstract. In this article we study Cohen-Macaulay modules over one-dimensional hypersurface singularities and the relationship with representation theory of associative algebras using methods of cluster tilting theory. We give a criterion for existence of cluster tilting objects and their complete description by homological method using higher almost split sequences and results from birational geometry. We obtain a large class of 2-CY tilted algebras which are finite dimensional symmetric and satisfies $\tau^2 = \mathrm{id}$. In particular, we compute 2-CY tilted algebras for simple/minimally elliptic curve singularities.

Introduction

Motivated by the Fomin-Zelevinsky theory of cluster algebras [FZ1, FZ2, FZ3], a tilting theory in cluster categories was initiated in [BMRRT]. For a finite dimensional hereditary algebra H over a field k, the associated cluster category $\mathcal{C}_H$ is the orbit category $\mathcal{D}^b(H)/F$, where $\mathcal{D}^b(H)$ is the bounded derived category of finite dimensional H-modules and the functor $F\colon \mathcal{D}^b(H) \to \mathcal{D}^b(H)$ is $\tau^{-1}[1] = S^{-1}[2]$. Here τ denotes the translation associated with almost split sequences/triangles and S the Serre functor on $\mathcal{D}^b(H)$ [BK]. (See [CCS] for an independent definition of the cluster category when H is of Dynkin type A_n).

An object T in a cluster category $\mathcal{C}_H$ was defined to be a (cluster) tilting object if $\mathrm{Ext}^1_{\mathcal{C}_H}(T,T) = 0$, and if $\mathrm{Ext}^1_{\mathcal{C}_H}(X, X \coprod T) = 0$, then X is in $\operatorname{add} T$. The corresponding endomorphism algebras, called cluster tilted algebras, were investigated in [BMR1] and subsequent papers. A useful additional property of a cluster tilting object was that even the weaker condition $\mathrm{Ext}^1_{\mathcal{C}_H}(X,T) = 0$ implies that X is in $\operatorname{add} T$, called Ext-configuration in [BMRRT]. Such a property also appears naturally in the work of the second author on a higher theory of almost split sequences in module categories [I1, I2] and the corresponding modules were called maximal 1-orthogonal. For a category $\operatorname{mod}\Lambda$ of finite dimensional modules over a preprojective algebra of Dynkin type Λ over an algebraically closed field k, the property corresponding to the above definition of cluster tilting object in a cluster category was called maximal rigid [GLSc]. Also in this setting it was shown that being maximal 1-orthogonal was a consequence of being maximal rigid. The same result holds for the stable category $\underline{\operatorname{mod}}\Lambda$.

The categories $\mathcal{C}_H$ and $\underline{\operatorname{mod}}\Lambda$ are both triangulated categories [Ke, H], with finite dimensional homomorphism spaces, and they have Calabi-Yau dimension 2 (2-CY for short) (see [BMRRT, Ke];[AR, 3.1,1.2][C][Ke, 8.5]). The last fact means that there is a Serre functor $S = \Sigma^2$, where Σ is the shift functor in the triangulated category.

For an arbitrary 2-CY triangulated category $\mathcal{C}$ with finite dimensional homomorphism spaces over a field k, a cluster tilting object T in $\mathcal{C}$ was defined to be an object satisfying the stronger property discussed above, corresponding to the property of being maximal 1-orthogonal/Ext-configuration [KR]. The corresponding class of algebras, containing the cluster tilted ones, have been called 2-CY tilted. With this concept many results have been generalised from cluster categories, and from the stable categories $\underline{\operatorname{mod}}\Lambda$, to this more general setting in [KR], which moreover contains several results which are new also in the first two cases.

One of the important applications of classical tilting theory has been the construction of derived equivalences: Given a tilting bundle T on a smooth projective variety X, the total right derived functor of $\mathrm{Hom}(T, \ )$ is an equivalence from the bounded derived category of coherent

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sheaves on X to the bounded derived category of finite dimensional modules over the endomorphism algebra of T. Analogously, cluster tilting theory allows one to establish equivalences between very large factor categories appearing in the local situation of Cohen-Macaulay modules and categories of modules over finite-dimensional algebras. Namely, if $\underline{\mathrm{CM}}(R)$ is the stable category of maximal Cohen-Macaulay modules over an isolated hypersurface singularity, then $\underline{\mathrm{CM}}(R)$ is 2-CY. If it contains a cluster tilting object T, then the functor $\mathrm{Hom}(T, \ )$ induces an equivalence between the quotient of $\underline{\mathrm{CM}}(R)$ by the ideal of morphisms factoring through τT and the category of finite-dimensional modules over the endomorphism algebra $B = \mathrm{End}(T)$. It is then not hard to see that B is symmetric and the indecomposable nonprojective B-modules are τ-periodic of τ-period at most 2. In this article, we study examples of this setup arising from finite, tame and wild CM-type isolated hypersurface singularities R. The endomorphism algebras of the cluster tilting objects in the tame case occur in lists in [BS, Er, Sk]. We also obtain a large class of symmetric finite dimensional algebras where the stable AR-quiver consists only of tubes of rank one or two. Examples of selfinjective algebras whose stable AR-quiver consists only of tubes of rank one or three were known previously [AR].

In the process we investigate the relationship between cluster tilting and maximal rigid objects. It is of interest to know if the first property implies the second one in general. In this paper we provide interesting examples where this is not the case. The setting we deal with are the simple isolated hypersurface singularities R in dimension one over an algebraically closed field k, with the stable category $\underline{\mathrm{CM}}(R)$ of maximal Cohen-Macaulay R-modules being our 2-CY category. These singularities are indexed by the Dynkin diagrams, and in the cases D_n for odd n and E_7 we give examples of maximal rigid objects which are not cluster tilting.

We also investigate the other Dynkin diagrams, and it is interesting to notice that there are also cases with no nonzero rigid objects (A_n, n even, E_6, E_8), and cases where the maximal rigid objects coincide with the cluster tilting objects (A_n, n odd and D_n, n even). In the last case we see that both loops and 2-cycles can occur for the associated 2-CY tilted algebras, whereas this never happens for the cases $\mathcal{C}$ and $\underline{\mathrm{mod}}\Lambda$ [BMRRT, BMR2, GLSc]. The results are also valid for any odd dimensional simple hypersurface singularity, since the stable categories of Cohen-Macaulay modules are all triangle equivalent (see [Kn, So]).

We shall construct a large class of one-dimensional hypersurface singularities R having a cluster tilting object including examples coming from simple singularities and minimally elliptic singularities. We classify all rigid object in $\mathrm{CM}(R)$ for these R, in particular, we give a bijection between cluster tilting objects in $\mathrm{CM}(R)$ and elements in a symmetric group. Our method is based on a higher theory of almost split sequences [I1, I2], and a crucial role is played by the endomorphism algebras $\mathrm{End}_R(T)$ (called 'three dimensional Auslander algebras') of cluster tilting objects T in $\mathrm{CM}(R)$. These algebras have global dimension three, and have 2-CY tilted algebras as stable factors. The functor $\mathrm{Hom}_R(T, \ ) : \mathrm{CM}(R) \to \mathrm{mod}\,\mathrm{End}_R(T)$ sends cluster tilting objects in $\mathrm{CM}(R)$ to tilting modules over $\mathrm{End}_R(T)$. By comparing cluster mutations in $\mathrm{CM}(R)$ and tilting mutation over $\mathrm{End}_R(T)$, we can apply results on tilting mutation due to Riedtmann-Schofield [RS] to get information on cluster tilting objects in $\mathrm{CM}(R)$.

We focus on the interplay between cluster tilting theory and birational geometry. In [V1, V2], Van den Bergh established a relationship between crepant resolutions of singularities and certain algebras called non-commutative crepant resolutions via derived equivalence. It is known that three dimensional Auslander algebras of cluster tilting objects of three dimensional normal Gorenstein singularities are 3-CY in the sense that the bounded derived category of finite length modules is 3-CY, and form a class of non-commutative crepant resolution [I2, IR]. Thus we have a connection between cluster tilting theory and birational geometry. We translate Katz's criterion [Kat] for three dimensional cA_n-singularities on existence of crepant resolutions to a criterion for one-dimensional hypersurface singularities on existence of cluster tilting objects. Consequently the class of hypersurface singularities, which are shown to have cluster tilting objects by using higher almost split sequences, are exactly the class having cluster tilting objects. However we do not know whether the number of cluster tilting objects has a meaning in birational geometry.

In section 2 we investigate maximal rigid objects and cluster tilting objects in $\underline{\mathrm{CM}}(R)$ for simple one-dimensional hypersurface singularities. We decide whether extension spaces are zero or not by using covering techniques. In section 3 we point out that we could also use the computer program `Singular` [GP] to accomplish the same thing. In section 4 we construct a cluster tilting objects for a large class of isolated hypersurface singularities, where the associated 2-CY tilted algebras can be of finite, tame or wild representation type. We also give a formula for the number of cluster tilting and indecomposable rigid objects. In section 5 we establish a connection between existence of cluster tilting objects and existence of small resolutions. In section 6 we give a geometric approach to the results in section 4. Section 7 is devoted to computing some concrete examples of 2-CY tilted algebras. In section 8 we generalize results from section 2 to 2-CY triangulated categories with only a finite number of indecomposable objects.

We refer to [Y] as a general reference for representation theory of Cohen-Macaulay rings, and [AGV] for classification of singularities.

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1. Main results

Let $(R,\mathfrak{m})$ be a local complete d-dimensional commutative noetherian Gorenstein isolated singularity and $R/\mathfrak{m} = k \subset R$, where k is an algebraically closed field. We denote by $\mathrm{CM}(R)$ the category of maximal Cohen-Macaulay modules over R. Then $\mathrm{CM}(R)$ is a Frobenius category, and so the stable category $\underline{\mathrm{CM}}(R)$ is a triangulated category with shift functor $\Sigma = \Omega^{-1}$ [H].

We collect some fundamental results.

- We have AR-duality
$$\underline{\mathrm{Hom}}(X,Y) \simeq D\,\mathrm{Ext}^1(Y,\tau X)$$
with $\tau \simeq \Omega^{2-d}$ [Au]. In particular, $\underline{\mathrm{CM}}(R)$ is $(d-1)$-CY.
- If R is a hypersurface singularity, then $\Sigma^2 = \mathrm{id}$ [Ei].
- (Knörrer periodicity)
$$\underline{\mathrm{CM}}(k[[x_0,\cdots,x_d,y,z]]/(f+yz)) \simeq \underline{\mathrm{CM}}(k[[x_0,\cdots,x_d]]/(f))$$
for any $f \in k[[x_0,\cdots,x_d]]$ [Kn] ([So] in characteristic two).

Consequently, if d is odd, then $\tau = \Omega$ and $\underline{\mathrm{CM}}(R)$ is 2-CY. If d is even, then $\tau = \mathrm{id}$ and $\underline{\mathrm{CM}}(R)$ is 1-CY, hence any non-free CM R-module M satisfies $\mathrm{Ext}^1_R(M,M) \neq 0$.

Definition 1.1. *Let $\mathcal{C} = \mathrm{CM}(R)$ or $\underline{\mathrm{CM}}(R)$. We call an object $M \in \mathcal{C}$*

- rigid *if* $\mathrm{Ext}^1_R(M,M) = 0$*,*
- maximal rigid *if it is rigid and any rigid $N \in \mathcal{C}$ satisfying $M \in \mathrm{add}\,N$ satisfies $N \in \mathrm{add}\,M$,*
- cluster tilting *if* $\mathrm{add}\,M = \{X \in \mathcal{C} \mid \mathrm{Ext}^1_R(M,X) = 0\} = \{X \in \mathcal{C} \mid \mathrm{Ext}^1_R(X,M) = 0\}$*.*

Cluster tilting objects are maximal rigid, thought the converse does not necessarily hold. If $\mathcal{C}$ is 2-CY, then $M \in \mathcal{C}$ is cluster tilting if and only if $\mathrm{add}\,M = \{X \in \mathcal{C} \mid \mathrm{Ext}^1_R(M,X) = 0\}$.

Let $M \in \mathcal{C}$ be a basic cluster tilting object and X an indecomposable summand of $M = X \oplus N$. Then there exist triangles/short exact sequences (called *exchange sequences*)

$$X \xrightarrow{g_1} N_1 \xrightarrow{f_1} Y \quad \text{and} \quad Y \xrightarrow{g_0} N_0 \xrightarrow{f_0} X$$

such that $N_i \in \mathrm{add}\,N$, f_i is a minimal right $(\mathrm{add}\,N)$-approximation, and g_i is a minimal left $(\mathrm{add}\,N)$-approximation. Then $Y \oplus N$ is a basic cluster tilting object again called *(cluster) mutation* of M [BMRRT]. It is known that there are no more basic cluster tilting objects containing N [IY].

Let $R = k[[x, y, x_2, \cdots, x_d]]/(f)$ be a simple hypersurface singularity so that in characteristic zero f is one of the following polynomials,

$$\begin{array}{lll}(A_n) & x^2 + y^{n+1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (n \geq 1)\\ (D_n) & x^2y + y^{n-1} + z_2^2 + z_3^2 + \cdots + z_d^2 & (n \geq 4)\\ (E_6) & x^3 + y^4 + z_2^2 + z_3^2 + \cdots + z_d^2 & \\ (E_7) & x^3 + xy^3 + z_2^2 + z_3^2 + \cdots + z_d^2 & \\ (E_8) & x^3 + y^5 + z_2^2 + z_3^2 + \cdots + z_d^2 & \end{array}$$

Then R is of finite Cohen-Macaulay representation type [Ar, GK, Kn].

We shall show the following result in section 2 using additive functions on the AR quiver. We shall explain another proof using `Singular` in section 3.

Theorem 1.2. *Let k be an algebraically closed field of characteristic zero and R a simple hypersurface singularity of dimension $d \geq 1$.*

(1) Assume that d is even. Then $\underline{\mathrm{CM}}(R)$ does not have non-zero rigid objects.

(2) Assume that d is odd. Then the number of indecomposable rigid objects, cluster tilting objects, maximal rigid objects, and indecomposable summands of maximal rigid objects in $\underline{\mathrm{CM}}(R)$ are as follows:

f	*indec. rigid*	*cluster tilting*	*max. rigid*	*summands of max. rigid*
(A_n) n : *odd*	2	2	2	1
(A_n) n : *even*	0	0	1	0
(D_n) n : *odd*	2	0	2	1
(D_n) n : *even*	6	6	6	2
(E_6)	0	0	1	0
(E_7)	2	0	2	1
(E_8)	0	0	1	0

We also consider a *minimally elliptic* curve singularity $T_{p,q}(\lambda)$ $(p \leq q)$. Assume for simplicity that our base field k is algebraically closed of characteristic zero. Then these singularities are given by the equations

$$x^p + y^q + \lambda x^2 y^2 = 0,$$

where $\frac{1}{p}+\frac{1}{q} \leq \frac{1}{2}$ and certain values of $\lambda \in k$ have to be excluded. They are tame Cohen-Macaulay representation type [D, Kah, DG]. We divide into two cases.

(i) Assume $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. This case occurs if and only if $(p, q) = (3, 6)$ or $(4, 4)$, and $T_{p,q}(\lambda)$ is called *simply elliptic*. The corresponding coordinate rings can be written in the form

$$T_{3,6}(\lambda) = k[[x, y]]/(y(y - x^2)(y - \lambda x^2))$$

and

$$T_{4,4}(\lambda) = k[[x, y]]/(xy(x - y)(x - \lambda y)),$$

where in both cases $\lambda \in k \setminus \{0, 1\}$.

(ii) Assume $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. Then $T_{p,q}(\lambda)$ does not depend on the continuous parameter λ, and is called a *cusp* singularity. In this case the corresponding coordinate rings can be written in the form

$$T_{p,q} = k[[x, y]]/((x^{p-2} - y^2)(x^2 - y^{q-2})).$$

We shall show the following result in section 6 by applying a result in birational geometry.

Theorem 1.3. *Let k be an algebraically closed field of characteristic zero and R a minimally elliptic curve singularity $T_{p,q}(\lambda)$.*

(a) *R has a cluster tilting object if and only if $p = 3$ and q is even or if both p and q are even.*

(b) *The number of indecomposable rigid objects, cluster tilting objects, and indecomposable summands of cluster tilting objects in* $\underline{\mathrm{CM}}(R)$ *are as follows:*

p, q	*indec. rigid*	*cluster tilting*	*summands of cluster tilting*
$p = 3,\ q : even$	6	6	2
$p, q : even$	14	24	3

We also prove the following general theorem, which includes both Theorem 1.2 (except the assertion on maximal rigid objects) and Theorem 1.3. The 'if' part in (a) and the assertion (b) are proved in section 4 by a purely homological method. The proof of (a), including another proof of the 'if' part, is given in section 6 by applying Katz's criterion in birational geometry.

Theorem 1.4. *Let* $R = k[[x, y]]/(f)$ $(f \in (x, y))$ *be a one-dimensional reduced hypersurface singularity.*

(a) R *has a cluster tilting object if and only if* f *is a product* $f = f_1 \cdots f_n$ *with* $f_i \notin (x, y)^2$.
(b) *The number of indecomposable rigid objects, cluster tilting objects, and indecomposable summands of cluster tilting objects in* $\underline{\mathrm{CM}}(R)$ *are as follows:*

indec. rigid	*cluster tilting*	*summands of cluster tilting*
$2^n - 2$	$n!$	$n - 1$

The following result gives a bridge between cluster tilting theory and birational geometry. The terminologies are explained in section 5.

Theorem 1.5. *Let* $(R, \mathfrak{m})$ *be a three dimensional isolated* cA_n *singularity defined by the equation* $g(x, y)+zt$ *and* R' *a one dimensional singularity defined by* $g(x, y)$*. Then the following conditions are equivalent.*

(a) $\operatorname{Spec} R$ *has a small resolution.*
(b) $\operatorname{Spec} R$ *has a crepant resolution.*
(c) $(R, \mathfrak{m})$ *has a non-commutative crepant resolution.*
(d) $\underline{\mathrm{CM}}(R)$ *has a cluster tilting object.*
(e) $\underline{\mathrm{CM}}(R')$ *has a cluster tilting object.*
(f) *The number of irreducible power series in the prime decomposition of* $g(x, y)$ *is* $n + 1$.

We end this section by giving an application to finite dimensional algebras. A *2-CY tilted algebra* is an endomorphism ring $\operatorname{End}_{\mathcal{C}}(M)$ of a cluster tilting object T in a 2-CY triangulated category $\mathcal{C}$. In section 7, we shall show the following result and compute 2-CY tilted algebras associated with minimally elliptic curve singularities.

Theorem 1.6. *Let* $(R, \mathfrak{m})$ *be an isolated hypersurface singularity and* Γ *a 2-CY tilted algebra coming from* $\underline{\mathrm{CM}}(R)$*. Then we have the following.*

(a) Γ *is a symmetric algebra.*
(b) *All components in the stable AR-quiver of* Γ *are tubes of rank 1 or 2.*

For example, put

$$R = k[[x, y]]/((x - \lambda_1 y) \cdots (x - \lambda_n y)) \quad \text{and} \quad M = \bigoplus_{i=1}^{n} k[[x, y]]/((x - \lambda_1 y) \cdots (x - \lambda_i y))$$

for distinct elements $\lambda_i \in k$. Then M is a cluster tilting object in $\mathrm{CM}(R)$ by Theorem 4.1, so $\Gamma = \underline{\mathrm{End}}_R(M)$ satisfies the conditions in Theorem 1.6. Since $\mathrm{CM}\, R$ has wild Cohen-Macaulay representation type if $n > 4$ [DG, Th. 3], we should get a family of examples of finite dimensional symmetric k-algebras whose stable AR-quiver consists only of tubes of rank 1 or 2, and are of wild representation type.

2. Simple hypersurface singularities

Let R be a one-dimensional simple hypersurface singularity. In this case the AR-quivers are known for $\mathrm{CM}(R)$ [DW], and so also for $\underline{\mathrm{CM}}(R)$. We use the notation from [Y].

In order to locate the indecomposable rigid modules M, that is, the modules M with $\mathrm{Ext}^1(M,M) = 0$, the following lemmas are useful, where part (a) of the first one is proved in [HKR], and the second one is a direct consequence of [KR] (generalizing [BMR1]).

Lemma 2.1. (a) *Let $\mathcal{C}$ be an abelian or triangulated k-category with finite dimensional homomorphism spaces. Let $A \xrightarrow{\binom{f_1}{f_2}} B_1 \oplus B_2 \xrightarrow{(g_1,g_2)} C$ be a short exact sequence or a triangle, where A is indecomposable, B_1 and B_2 nonzero, and (g_1, g_2) has no nonzero indecomposable summand which is an isomorphism. Then* $\mathrm{Hom}(A,C) \neq 0$.

(b) *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an almost split sequence in $\mathrm{CM}(R)$, where R is an isolated hypersurface singularity, and B has at least two indecomposable nonprojective summands in a decomposition of B into a direct sum of indecomposable modules. Then* $\mathrm{Ext}^1(C,C) \neq 0$.

Proof. (a) See [HKR].

(b) Using (a) together with the above AR-formula and $\tau^2 = \mathrm{id}$, we obtain $D\,\mathrm{Ext}^1(C,C) \simeq \underline{\mathrm{Hom}}(\tau^{-1}C,C) = \underline{\mathrm{Hom}}(\tau C,C) \simeq \underline{\mathrm{Hom}}(A,C) \neq 0$, where $D = \mathrm{Hom}_k(\ ,k)$. □

Lemma 2.2. *Let T be a cluster tilting object in the* Hom*-finite connected 2-*CY *category $\mathcal{C}$, and* $\Gamma = \mathrm{End}_{\mathcal{C}}(T)$.

(a) *The functor $G = \mathrm{Hom}_{\mathcal{C}}(T,\ )\colon \mathcal{C} \to \mathrm{mod}\,\Gamma$ induces an equivalence of categories $\overline{G}\colon \mathcal{C}/\mathrm{add}(\tau T) \to \mathrm{mod}\,\Gamma$.*

(b) *The AR-quiver for Γ is as a translation quiver obtained from the AR-quiver for $\mathcal{C}$ by removing the vertices corresponding to the indecomposable summands of τT.*

(c) *Assume $\tau^2 = \mathrm{id}$. Then we have the following.*

  (i) *Γ is a symmetric algebra.*

  (ii) *The indecomposable nonprojective Γ-modules have τ-period one or two.*

  (iii) *If $\mathcal{C}$ has an infinite number of nonisomorphic indecomposable objects, then all components in the stable AR-quiver of Γ are tubes of rank one or two.*

(d) *If $\mathcal{C}$ has only a finite number n of nonisomorphic indecomposable objects, and T has t nonisomorphic indecomposable summands, then there are $n-t$ nonisomorphic indecomposable Γ-modules.*

Proof. For (a) and (b) see [BMR1][KR]. Since $\mathcal{C}$ is 2-CY, we have $\tau = \Sigma$, and a functorial isomorphism

$$D\,\mathrm{Hom}_{\mathcal{C}}(T,T) \simeq \mathrm{Hom}_{\mathcal{C}}(T,\Sigma^2 T) = \mathrm{Hom}_{\mathcal{C}}(T,\tau^2 T) \simeq \mathrm{Hom}_{\mathcal{C}}(T,T).$$

This shows that Γ is symmetric. Let C be an indecomposable nonprojective Γ-module. Viewing C as an object in $\mathcal{C}$ we have $\tau_{\mathcal{C}}^2 C \simeq C$, and τC is not a projective Γ-module since C is not removed. Hence we have $\tau_{\Gamma}^2 C \simeq C$. If $\mathcal{C}$ has an infinite number of nonisomorphic indecomposable objects, then Γ is of infinite type. Then each component of the AR-quiver is infinite, and hence is a tube of rank one or two. Finally, (d) is a direct consequence of (a). □

We also use that in our cases we have a covering functor $\Pi\colon k(\mathbb{Z}Q) \to \underline{\mathrm{CM}}(R)$, where Q is the appropriate Dynkin quiver and $k(\mathbb{Z}Q)$ is the mesh category of the translation quiver $\mathbb{Z}Q$ [Rie][Am], (see also [I1, Section 4.4] for another explanation using functorial method).

For the one-dimensional simple hypersurface singularities we have the cases A_n (n even or odd), D_n (n odd or even), E_6, E_7 and E_8. We now investigate them case by case.

Proposition 2.3. *In the case A_n (with n even) there are no indecomposable rigid objects.*

Proof. We have the stable AR-quiver

$$I_1 \rightleftarrows I_2 \rightleftarrows \cdots \rightleftarrows I_{n/2}$$

Here, and later, a dotted line between two indecomposable modules means that they are connected via τ.

Since $\tau I_j \simeq I_j$ for each j, $\operatorname{Ext}^1(I_j, I_j) \neq 0$ for $j = 1, \cdots, n/2$. Hence no I_j is rigid. □

Proposition 2.4. *In the case A_n (with n odd) the maximal rigid objects coincide with the cluster tilting objects. There are two indecomposable ones, and the corresponding 2-CY tilted algebras are $k[x]/(x^{\frac{(n+1)}{2}})$*

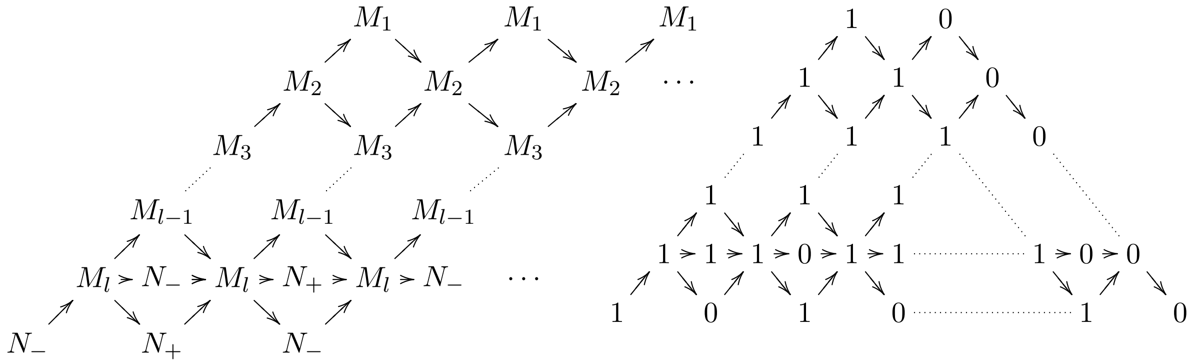
Proof. We have the stable AR-quiver

$$\begin{tikzcd}
 & & & & N_- \arrow[dd, dash, dashed] \arrow[dl, shift left] \\
M_1 \arrow[r, shift left] & M_2 \arrow[l, shift left] \arrow[r, shift left] & \cdots \arrow[l, shift left] \arrow[r, shift left] & M_{(n-1)/2} \arrow[l, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & \\
 & & & & N_+ \arrow[ul, shift left]
\end{tikzcd}$$

Since $\tau M_i \simeq M_i$ for $i = 1, \cdots, n - 1/2$, we have

$$\operatorname{Ext}^1(M_i, M_i) \simeq \underline{\operatorname{Hom}}(M_i, \tau M_i) \simeq \underline{\operatorname{Hom}}(M_i, M_i) \neq 0.$$

So only the indecomposable objects N_- and N_+ could be rigid. We use covering techniques and additive functions to compute the support of $\underline{\operatorname{Hom}}(N_-, \ )$. For simplicity, we write $l = (n-1)/2$

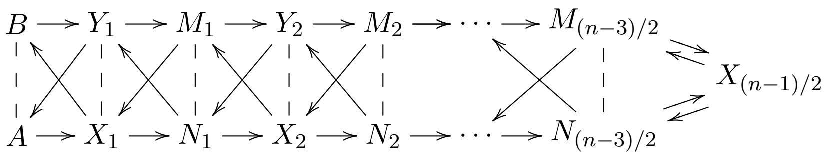


We see that $\underline{\operatorname{Hom}}(N_-, N_+) = 0$, so $\operatorname{Ext}^1(N_+, N_+) = \operatorname{Ext}^1(N_+, \tau N_-) = 0$, and $\operatorname{Ext}^1(N_-, N_-) = 0$. Since $\operatorname{Ext}^1(N_+, N_-) \neq 0$, we see that N_+ and N_- are exactly the maximal rigid objects. Further $\underline{\operatorname{Hom}}(N_-, M_i) \neq 0$ for all i, so $\operatorname{Ext}^1(N_+, M_i) \neq 0$ and $\operatorname{Ext}^1(N_-, M_i) \neq 0$ for all i. This shows that N_+ and N_- are also cluster tilting objects.

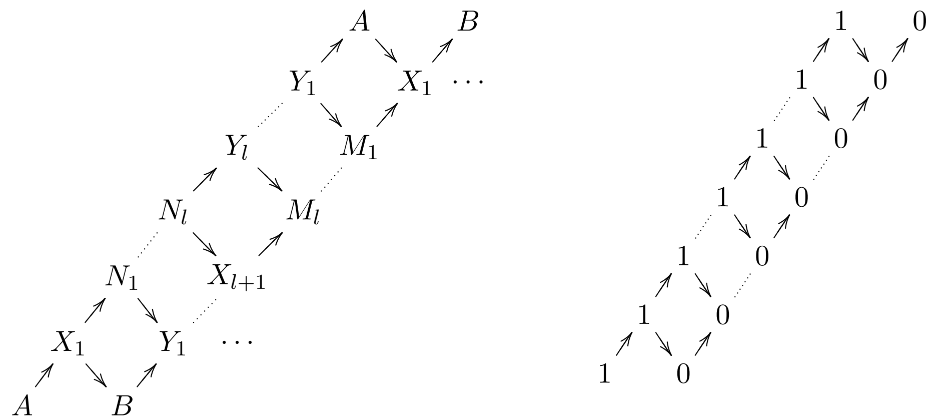
The description of the cluster tilted algebras follows directly from the above picture. □

Proposition 2.5. *In the case D_n with n odd we have two maximal rigid objects, which both are indecomposable, and neither one is cluster tilting.*

Proof. We have the AR-quiver



Using Lemma 2.1, the only candidates for being indecomposable rigid are A and B. We compute the support of $\underline{\operatorname{Hom}}(A, \ )$

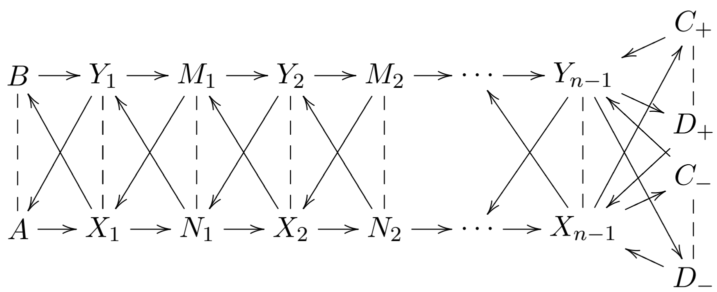


where $B = \tau A$ and $l = (n-3)/2$. We see that $\underline{\mathrm{Hom}}(A, B) = 0$, so that $\mathrm{Ext}^1(A, A) = 0$. Then A is clearly maximal rigid. Since $\underline{\mathrm{Hom}}(A, M_1) = 0$, we have $\mathrm{Ext}^1(A, N_1) = 0$, so A is not cluster tilting. Alternatively, we could use that we see that $\underline{\mathrm{End}}(A)^{\mathrm{op}} \simeq k[x]/(x^2)$, which has two indecomposable modules, whereas $\underline{\mathrm{CM}}(R)$ has $2n-3$ indecomposable objects. If A was cluster tilting, $\underline{\mathrm{End}}(A)^{\mathrm{op}}$ would have had $2n-3-1 = 2n-4$ indecomposable modules, by Lemma 2.2. $\square$

Proposition 2.6. *In the case D_{2n} with n a positive integer we have that the maximal rigid objects coincide with the cluster tilting ones. There are 6 of them, and each is a direct sum of two nonisomorphic indecomposable objects.*

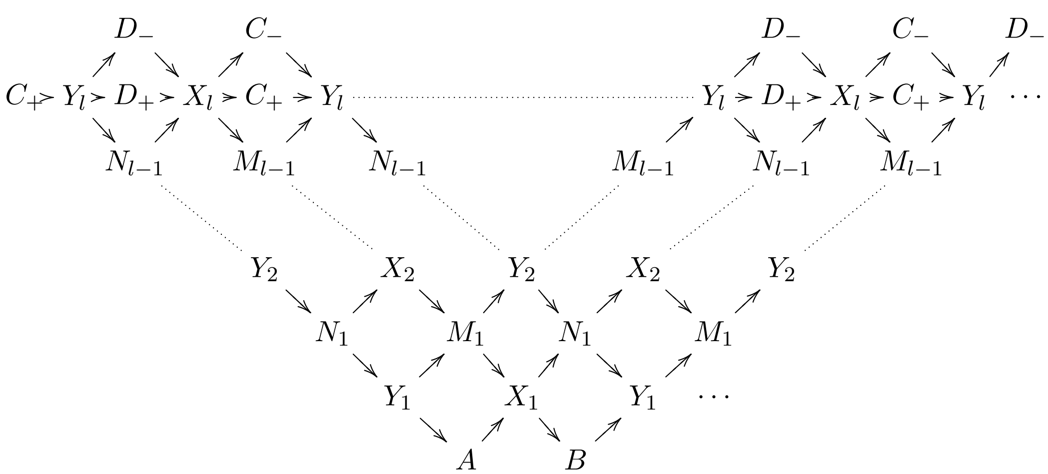
The corresponding 2-CY-tilted algebras are given by the quiver with relations $\cdot \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \cdot$ $\alpha\beta\alpha = 0 = \beta\alpha\beta$ in the case D_4, and by $\gamma \circlearrowright \cdot \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \cdot$ with $\gamma^{n-1} = \beta\alpha$, $\gamma\beta = 0 = \alpha\gamma$ and $\cdot \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \cdot$ with $(\alpha\beta)^{n-1}\alpha = 0 = (\beta\alpha)^{n-1}\beta$ for $2n > 4$.

Proof. We have the AR-quiver

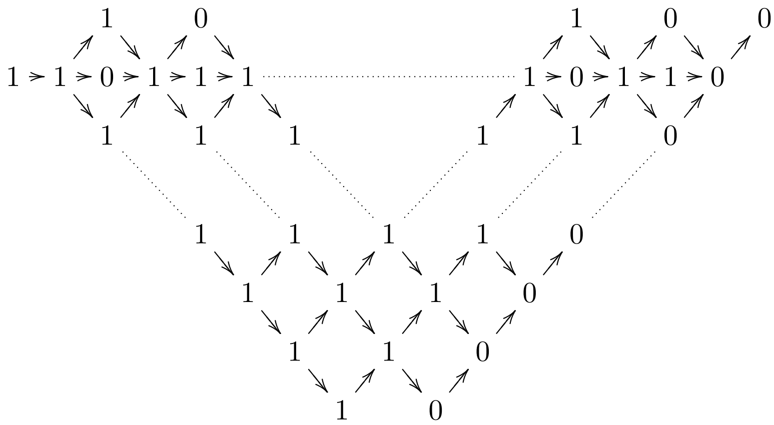


By Lemma 2.1, the only possible indecomposable rigid objects are: A, B, C_+, C_-, D_+, D_-.

We compute the support of $\underline{\mathrm{Hom}}(C_+, \ )$:

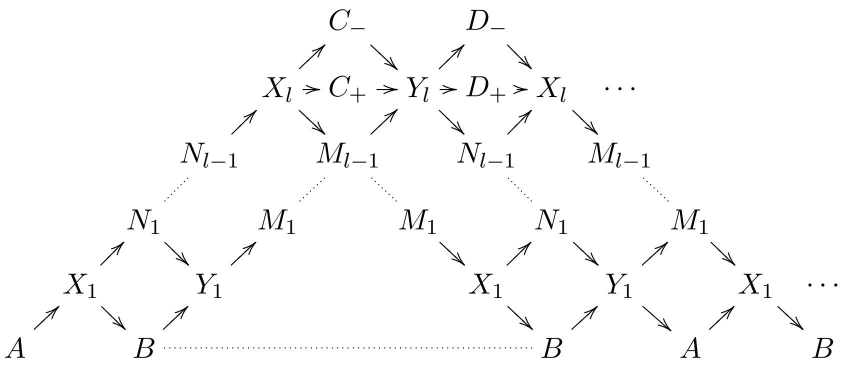


where $l = n - 1$

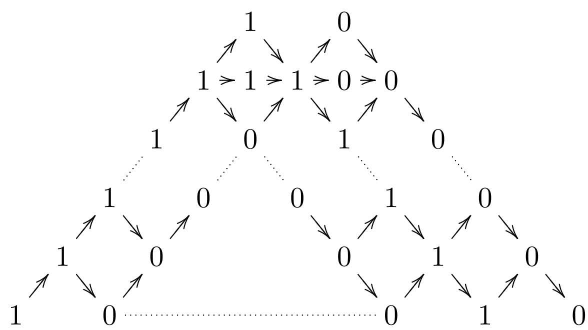


We see that $\underline{\mathrm{Hom}}(C_+, D_+) = 0$, so $\mathrm{Ext}^1(C_+, C_+) = 0 = \mathrm{Ext}^1(D_+, D_+)$. Further, $\underline{\mathrm{Hom}}(C_+, C_-) = 0$, so $\mathrm{Ext}^1(C_+, D_-) = 0$. By symmetry $\mathrm{Ext}^1(D_-, D_-) = 0 = \mathrm{Ext}^1(C_-, C_-)$ and $\mathrm{Ext}^1(D_+, C_-) = 0$. Also $\mathrm{Ext}^1(C_+, A) = 0$, $\mathrm{Ext}^1(C_+, B) \neq 0$, so $\mathrm{Ext}^1(D_+, B) = 0$, $\mathrm{Ext}^1(D_+, A) \neq 0$. Further $\mathrm{Ext}^1(C_+, X) \neq 0$ for $X \neq A, D_-, C_+$.

We now compute the support of $\underline{\mathrm{Hom}}(A,\ )$



where $l = n - 1$ and we have an odd number of columns and rows.



We see that $\underline{\mathrm{Hom}}(A, B) = 0$, so $\mathrm{Ext}^1(A, A) = 0$, hence also $\mathrm{Ext}^1(B, B) = 0$. Since $\underline{\mathrm{Hom}}(A, D_-) = 0$, we have $\mathrm{Ext}^1(A, C_-) = 0$, hence $\mathrm{Ext}^1(B, D_-) = 0$. Since $\underline{\mathrm{Hom}}(A, C_-) \neq 0$, we have $\mathrm{Ext}^1(A, D_-) \neq 0$, so $\mathrm{Ext}^1(B, D_+) \neq 0$.

It follows that $C_+ \coprod D_-$, $C_- \coprod D_+$, $C_+ \coprod A$, $D_+ \coprod B$, $A \coprod C_-$ and $B \coprod D_-$ are maximal rigid.

These are also cluster tilting: We have $\underline{\mathrm{Hom}}(A, X_i) \neq 0$, $\underline{\mathrm{Hom}}(A, N_i) \neq 0$, so $\mathrm{Ext}^1(B, X_i) \neq 0$, $\mathrm{Ext}^1(B, N_i) \neq 0$. Similarly, $\mathrm{Ext}^1(A, Y_i) \neq 0$, $\mathrm{Ext}^1(A, M_i) \neq 0$. Also $\underline{\mathrm{Hom}}(C_+, Y_i) \neq 0$, $\underline{\mathrm{Hom}}(C_+, N_i) \neq 0$, so $\mathrm{Ext}^1(D_+, Y_i) \neq 0$, $\mathrm{Ext}^1(D_+, N_i) \neq 0$. Hence $\mathrm{Ext}^1(C_+, X_i) \neq 0$, $\mathrm{Ext}^1(C_+, M_i) \neq 0$. So $\mathrm{Ext}^1(D_-, Y_i) \neq 0$, $\mathrm{Ext}^1(D_-, N_i) \neq 0$, $\mathrm{Ext}^1(C_-, X_i) \neq 0$, $\mathrm{Ext}^1(C_-, M_i) \neq 0$. We see that each indecomposable rigid object can be extended to a cluster tilting object in exactly two ways, which we would know from a general result in [IY].

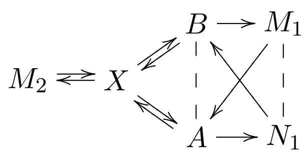
The exchange graph is as follows:

$$\begin{tikzcd}[column sep=small]
 & \{C_+, D_-\} \arrow[dl, dash] \arrow[dr, dash] & \\
\{B, D_-\} \arrow[d, dash] & & \{A, C_+\} \arrow[d, dash] \\
\{B, D_+\} \arrow[dr, dash] & & \{A, C_-\} \arrow[dl, dash] \\
 & \{C_-, D_+\} &
\end{tikzcd}$$

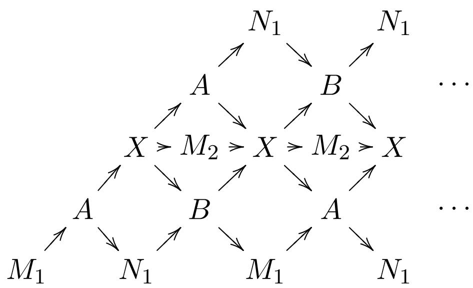
Considering the above pictures, we get the desired description of the corresponding 2-CY tilted algebras in terms of quivers with relations. $\square$

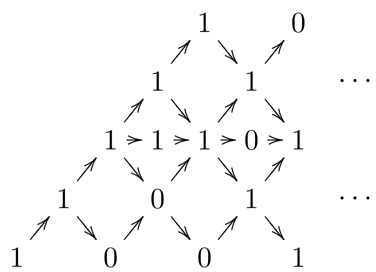
Proposition 2.7. *In the case E_6 there are no indecomposable rigid objects.*

Proof. We have the AR-quiver



The only candidates for indecomposable rigid objects according to Lemma 2.1 are M_1 and N_1. We compute the support of $\underline{\mathrm{Hom}}(M_1, \ )$.

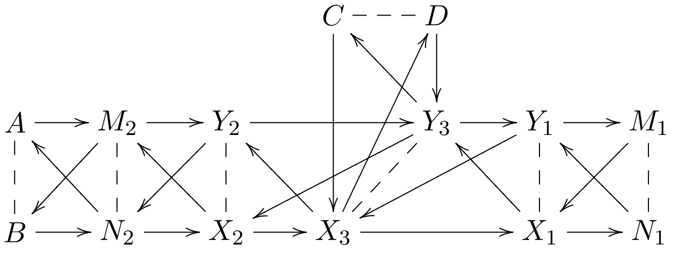




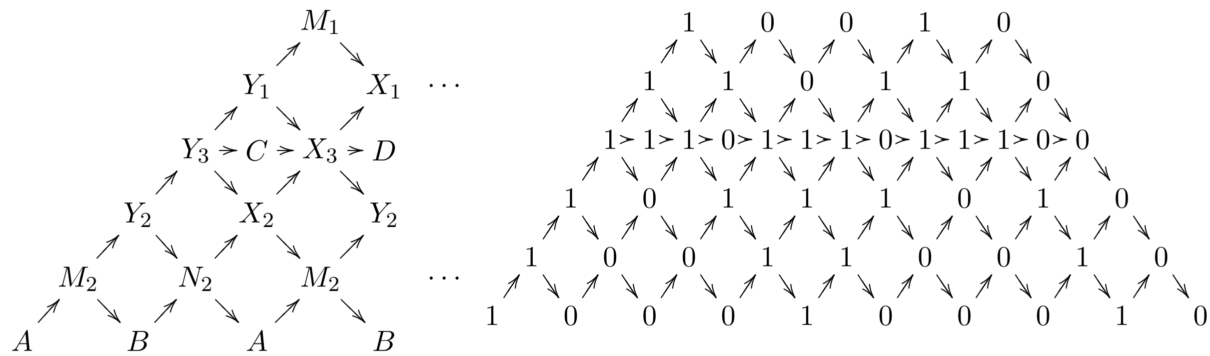
We see that $\underline{\mathrm{Hom}}(M_1, N_1) \neq 0$, so that $\mathrm{Ext}^1(M_1, M_1) \neq 0$ and $\mathrm{Ext}^1(N_1, N_1) \neq 0$. $\square$

Proposition 2.8. *In the case E_7 there are two maximal rigid objects, which both are indecomposable, and neither of them is cluster tilting.*

Proof. We have the AR-quiver

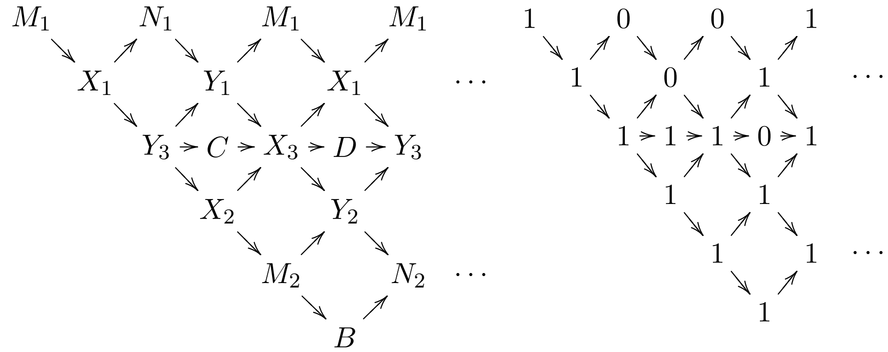


Using Lemma 2.1, we see that the only candidates for indecomposable rigid objects are A, B, M_1, N_1, C and D. We first compute the support of $\underline{\mathrm{Hom}}(A, \ )$.



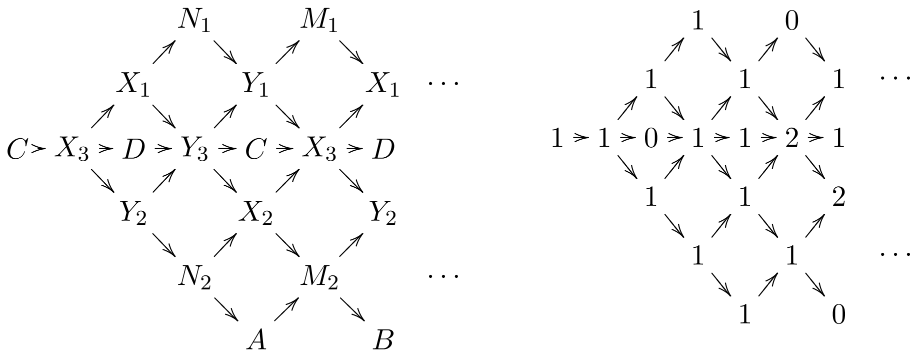
We see that $\mathrm{Ext}^1(A, A) = 0$, and so also $\mathrm{Ext}^1(B, B) = 0$, so A and B are rigid.

Next we compute the support of $\underline{\mathrm{Hom}}(M_1, \ )$.



We see that $\mathrm{Ext}^1(M_1, M_1) \neq 0$ and $\mathrm{Ext}^1(N_1, N_1) \neq 0$, so that M_1 and N_1 are not rigid.

Then we compute the support of $\underline{\mathrm{Hom}}(C, \ )$.

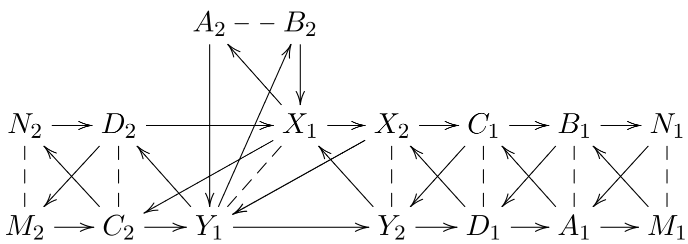


We see that $\mathrm{Ext}^1(C, C) \neq 0$ and $\mathrm{Ext}^1(D, D) \neq 0$, so that C and D are not rigid. Hence A and B are the rigid indecomposable objects, and they are maximal rigid.

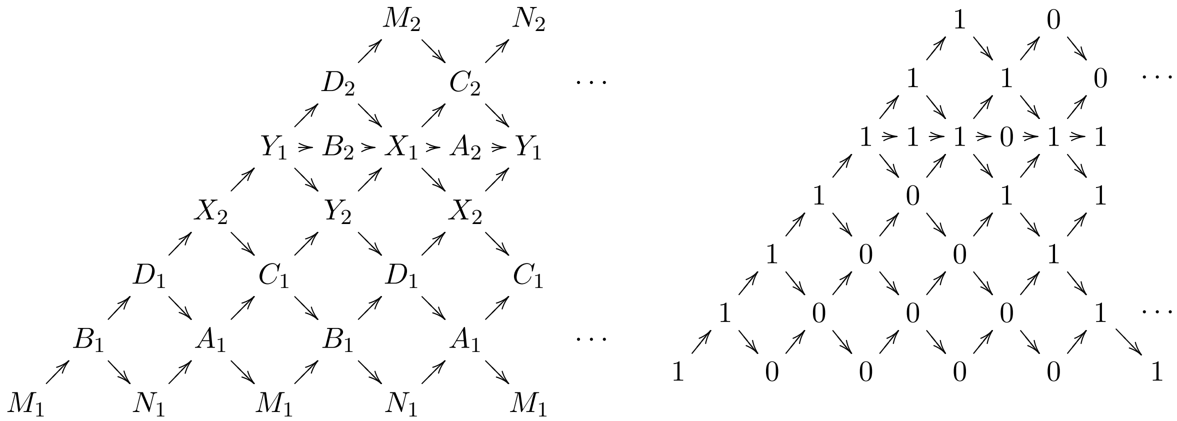
Since $\mathrm{Ext}^1(A, C) = 0$, we see that A and hence B is not cluster tilting. $\square$

Proposition 2.9. *In the case E_8 there are no indecomposable rigid objects.*

Proof. We have the AR-quiver

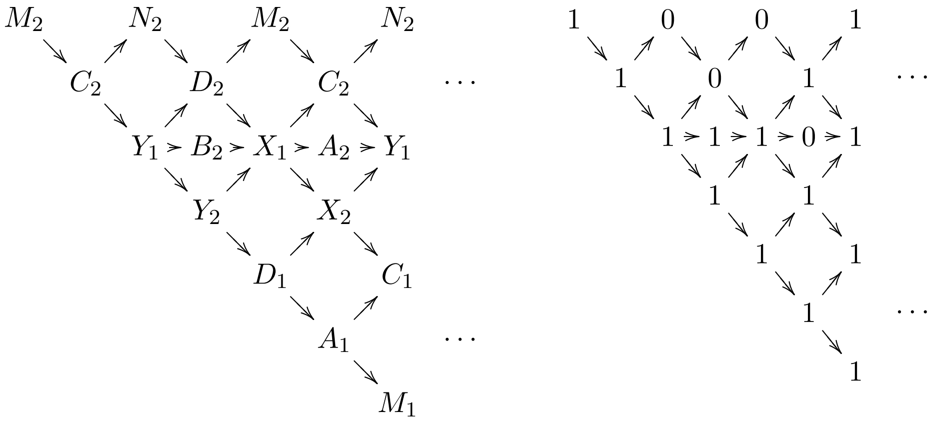


The only candidates for indecomposable rigid objects are M_1, N_1, M_2, N_2, A_2 and B_2, by Lemma 2.1. We first compute the support of $\underline{\mathrm{Hom}}(M_1, \ )$:



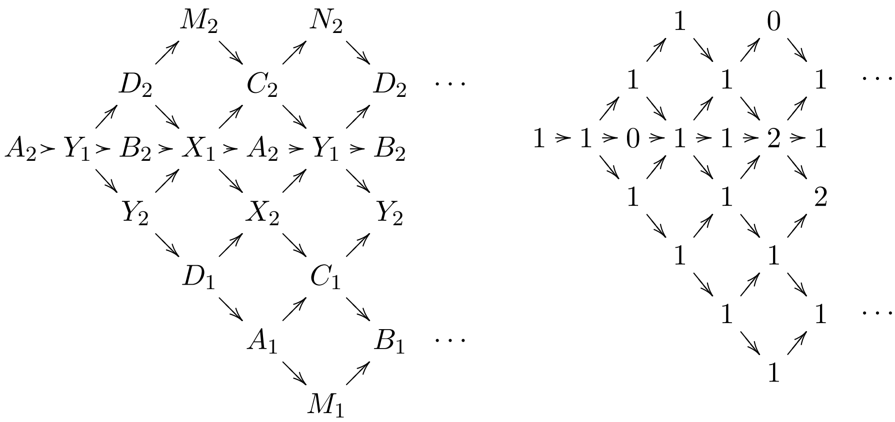
We see that $\mathrm{Ext}^1(M_1, M_1) \neq 0$, and hence $\mathrm{Ext}^1(N_1, N_1) \neq 0$.

Next we compute the support of $\underline{\mathrm{Hom}}(M_2, \ )$:



We see that $\mathrm{Ext}^1(M_2, M_2) \neq 0$, and hence $\mathrm{Ext}^1(N_2, N_2) \neq 0$.

Finally we compute the support of $\underline{\mathrm{Hom}}(A_2, \ )$:



It follows that $\mathrm{Ext}^1(A_2, A_2) \neq 0$, and similarly $\mathrm{Ext}^1(B_2, B_2) \neq 0$. Hence there are no indecomposable rigid objects. $\square$

3. Computation with `Singular`

An alternative way to carry out computations of Ext^1–spaces in the stable category of maximal Cohen-Macaulay modules is to use the computer algebra system `Singular`, see [GP]. Let

$$R = k[x_1, x_2, \dots, x_n]_{\langle x_1, x_2, \dots, x_n \rangle}/I$$

be a Cohen-Macaulay local ring which is an isolated singularity, and M and N two maximal Cohen-Macaulay modules. Denote by $\widehat{R}$ the completion of R. Since all the spaces $\mathrm{Ext}^i_R(M, N)$ $(i \geq 1)$ are finite-dimensional over k and the functor $\mathrm{mod}\, R \to \mathrm{mod}\, \widehat{R}$ is exact, maps the maximal Cohen-Macaulay modules to maximal Cohen-Macaulay modules and the finite length modules to finite length modules, we can conclude that

$$\dim_k(\mathrm{Ext}^i_R(M,N)) = \dim_k(\mathrm{Ext}^i_{\widehat{R}}(\widehat{M},\widehat{N})).$$

As an illustration we show how to do this for the case E_7.

Proposition 3.1. *In the case E_7 there are two maximal rigid objects, which both are indecomposable and neither of them is cluster tilting.*

By [Y] the AR-quiver of $\underline{\mathrm{CM}}(R)$ has the form

$$\begin{tikzcd}
 & & & C \arrow[r, dash, dashed] \arrow[dd] & D \arrow[d] & & \\
A \arrow[r] \arrow[d, dash, dashed] & M_2 \arrow[r] \arrow[dl] \arrow[d, dash, dashed] & Y_2 \arrow[rr] \arrow[dl] \arrow[d, dash, dashed] & & Y_3 \arrow[r] \arrow[ul] \arrow[dll] & Y_1 \arrow[r] \arrow[dll] \arrow[d, dash, dashed] & M_1 \arrow[dl] \arrow[d, dash, dashed] \\
B \arrow[r] & N_2 \arrow[r] \arrow[ul] & X_2 \arrow[r] \arrow[ul] & X_3 \arrow[rr] \arrow[ul] \arrow[uur] & & X_1 \arrow[r] \arrow[ul] & N_1 \arrow[ul]
\end{tikzcd}$$

By Lemma 2.1 only the modules A,B,C,D,M_1,N_1 can be rigid. Since $N=\tau(M), B=\tau(A)$, $N_1=\tau(M_1)$, the pairs of modules (A,B), (C,D) and (M_1,N_1) are rigid or not rigid simultaneously. By [Y] we have the following presentations:

$$R \xrightarrow{x^2+y^3} R \xrightarrow{x} R \longrightarrow A \longrightarrow 0,$$

$$R \xrightarrow{\begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}} R \xrightarrow{x\begin{pmatrix} x & y \\ y^2 & -x \end{pmatrix}} C \longrightarrow 0,$$

$$R \xrightarrow{\begin{pmatrix} x^2 & y \\ xy^2 & -x \end{pmatrix}} R \xrightarrow{\begin{pmatrix} x & y \\ xy^2 & -x^2 \end{pmatrix}} M_1 \longrightarrow 0,$$

so we can use the computer algebra system `Singular` in order to compute the Ext^1–spaces between these modules.

> `Singular` (call the program ``Singular'')
> `LIB ``homolog.lib'';` (call the library of homological algebra)
> `ring S = 0,(x,y),ds;` (defines the ring $S=\mathbb{Q}[x,y]_{\langle x,y\rangle}$)
> `ideal I = x3 + xy3;` (defines the ideal x^3+xy^3 in S)
> `qring R = std(I);` (defines the ring $\mathbb{Q}[x,y]_{\langle x,y\rangle}/I$)
> `module A = [x];`
> `module C = [x2, xy2], [xy, -x2];`
> `module M1 = [x2, xy2], [y, -x2];` (define modules A,C,M_1)
> `list l = Ext(1,A,A,1);`
`// dimension of` Ext^1`:  -1` (Output: $\mathrm{Ext}^1_R(A,A)=0$)
> `list l = Ext(1,C,C,1);`
`// ** redefining l **`
`// dimension of` Ext^1`:  0` (the Krull dimension of $\mathrm{Ext}^1_R(C,C)$ is 0)
`// vdim of` Ext^1`:  2` ($\dim_k(\mathrm{Ext}^1_R(C,C))=2$)
> `list l = Ext(1,M1,M1,1);`
`// ** redefining l **`
`// dimension of` Ext^1`:  0`
`// vdim of` Ext^1`:  10`
> `list l = Ext(1,A,C,1);`
`// ** redefining l **`
`// dimension of` Ext^1`:  -1`

This computation shows that the modules A and B are rigid, C,D,M_1 and N_1 are not rigid and since $\mathrm{Ext}^1_R(A,C)=0$, there are no cluster tilting objects in the stable category $\underline{\mathrm{CM}}(R)$.

4. One-dimensional hypersurface singularities

We shall construct a large class of one-dimensional hypersurface singularities having a cluster tilting object, then classify all cluster tilting objects. Our method is based on higher theory of almost split sequences and Auslander algebras studied in [I1, I2]. We also use a relationship between cluster tilting objects in $\mathrm{CM}(R)$ and tilting modules over the endomorphism algebra of a cluster tilting object [I2]. Then we shall compare cluster mutation with tilting mutation by using results due to Riedtmann-Schofield [RS].

In this section, we usually consider tilting objects in $\mathrm{CM}(R)$ instead of $\underline{\mathrm{CM}}(R)$.

Let k be an infinite field, $S := k[[x,y]]$ and $\mathfrak{m} := (x,y)$. We fix $f \in \mathfrak{m}$ and write $f = f_1 \cdots f_n$ for irreducible formal power series $f_i \in \mathfrak{m}$ $(1 \le i \le n)$. Put

$$S_i := S/(f_1 \cdots f_i) \ \text{ and } \ R := S_n = S/(f).$$

We assume that R is reduced, so we have $(f_i) \neq (f_j)$ for any $i \neq j$.

Our main results in this section are the following.

Theorem 4.1. (a) $\bigoplus_{i=1}^n S_i$ *is a rigid object in* $\mathrm{CM}(R)$.

(b) $\bigoplus_{i=1}^n S_i$ *is a cluster tilting object in* $\mathrm{CM}(R)$ *if the following condition (A) is satisfied.*
(A) $f_i \notin \mathfrak{m}^2$ *for any* $1 \le i \le n$.

Let $\mathfrak{S}_n$ be the symmetric group of degree n. For $w \in \mathfrak{S}_n$ and $I \subseteq \{1, \cdots, n\}$, we put

$$S_i^w := S/(f_{w(1)} \cdots f_{w(i)}), \ M_w := \bigoplus_{i=1}^n S_i^w \ \text{ and } \ S_I := S/(\prod_{i \in I} f_i).$$

Theorem 4.2. *Assume that (A) is satisfied.*

(a) *There are exactly* $n!$ *cluster tilting objects* M_w $(w \in \mathfrak{S}_n)$ *and exactly* $2^n - 1$ *indecomposable rigid objects* S_I $(\emptyset \neq I \subseteq \{1, \cdots, n\})$ *in* $\mathrm{CM}(R)$.

(b) *For any* $w \in \mathfrak{S}_n$, *there are exactly* $n!$ *Cohen-Macaulay tilting* $\mathrm{End}_R(M_w)$*-modules* $\mathrm{Hom}_R(M_w, M_{w'})$ $(w' \in \mathfrak{S}_n)$ *of projective dimension at most one. Moreover, all algebras* $\mathrm{End}_R(M_w)$ $(w \in \mathfrak{S}_n)$ *are derived equivalent.*

It is interesting to compare with results in [IR], where two-dimensional (2-Calabi-Yau) algebras Γ are treated and a bijection between elements in an affine Weyl group and tilting Γ-modules of projective dimension at most one is given. Here the algebra is one-dimensional, and Weyl groups appear.

Here we consider three examples.

(a) Let R be a curve singularity of type A_{2n-1} or D_{2n+2}, so

$$R = S/((x-y^n)(x+y^n)) \ \text{ or } \ R = S/(y(x-y^n)(x+y^n)).$$

By our theorems, there are exactly 2 or 6 cluster tilting objects and exactly 3 or 7 indecomposable rigid objects in $\mathrm{CM}(R)$, which fits with our computations in section 1.

(b) Let R be a curve singularity of type $T_{3,2q+2}(\lambda)$ or $T_{2p+2,2q+2}(\lambda)$, so

$$R = S/((x-y^2)(x-y^q)(x+y^q)) \qquad (R = S/(y(y-x^2)(y-\lambda x^2)) \text{ for } q = 2),$$
$$R = S/((x^p-y)(x^p+y)(x-y^q)(x+y^q)) \qquad (R = S/(xy(x-y)(x-\lambda y)) \text{ for } p = q = 1).$$

By our theorems, there are exactly 6 or 24 cluster tilting objects and exactly 7 or 15 indecomposable rigid objects in $\mathrm{CM}(R)$.

(c) Let $\lambda_i \in k$ $(1 \le i \le n)$ be mutually distinct elements in k. Put

$$R := S/((x-\lambda_1 y) \cdots (x-\lambda_n y)).$$

By our theorems, there are exactly $n!$ cluster tilting objects and exactly $2^n - 1$ indecomposable rigid objects in $\mathrm{CM}(R)$.

First of all, Theorem 4.1(a) follows immediately from the following observation.

Proposition 4.3. *For* $g_1, g_2 \in \mathfrak{m}$ *and* $g_3 \in S$, *put* $R := S/(g_1 g_2 g_3)$. *If* g_1 *and* g_2 *have no common factor, then* $\mathrm{Ext}^1_R(S/(g_1 g_3), S/(g_1)) = 0 = \mathrm{Ext}^1_R(S/(g_1), S/(g_1 g_3))$.

Proof. We have a projective resolution

$$R \xrightarrow{g_2} R \xrightarrow{g_1 g_3} R \to S/(g_1 g_3) \to 0.$$

Applying $\mathrm{Hom}_R(\ , S/(g_1))$, we have a complex

$$S/(g_1) \xrightarrow{g_1 g_3 = 0} S/(g_1) \xrightarrow{g_2} S/(g_1).$$

This is exact since g_1 and g_2 have no common factor. Thus we have the former equation, and the other one can be proved similarly. □

Our plan of proof of Theorem 4.1(b) is the following.

(i) First we shall prove Theorem 4.1 under the following stronger assumption:
(B) $\mathfrak{m} = (f_1, f_2) = \cdots = (f_{n-1}, f_n)$.

(ii) Then we shall prove the general statement of Theorem 4.1.

We need the following general result in [I1, I2].

Proposition 4.4. *Let R be a complete local Gorenstein ring of dimension at most three and M a rigid Cohen-Macaulay R-module which is a generator. Then the following conditions are equivalent.*

(a) *M is a cluster tilting object in* $\mathrm{CM}(R)$.

(b) $\mathrm{gl.dim}\,\mathrm{End}_R(M) \leq 3$.

(c) *For any $X \in \mathrm{CM}(R)$, there exists an exact sequence $0 \to M_1 \to M_0 \to X \to 0$ with $M_i \in \mathrm{add}\, M$.*

(d) *For any indecomposable direct summand X of M, there exists an exact sequence $0 \to M_2 \to M_1 \to M_0 \xrightarrow{a} X$ with $M_i \in \mathrm{add}\, M$ and a is a right almost split map in $\mathrm{add}\, M$.*

Proof. (a)⇔(b) Apply [I2, Th. 5.1(3)] for $d = m = 1$ and $n = 2$ there.

(a)⇔(c) See [I1, Prop. 2.2.2].

(a)⇒(d) See [I1, Th. 3.3.1].

(d)⇒(b) For any simple $\mathrm{End}_R(M)$-module S, there exists an indecomposable direct summand X of M such that S is the top of the projective $\mathrm{Hom}_R(M, X)$. Then the sequence in (d) gives a projective resolution $0 \to \mathrm{Hom}_R(M, M_2) \to \mathrm{Hom}_R(M, M_1) \to \mathrm{Hom}_R(M, M_0) \to \mathrm{Hom}_R(M, X) \to S \to 0$. Thus we have $\mathrm{pd}\, S \leq 3$ and $\mathrm{gl.dim}\,\mathrm{End}_R(M) \leq 3$. □

The sequence in (d) is called a *2-almost split sequence* when X is non-projective. There is a close relationship between 2-almost split sequences and exchange sequences [IY].

We shall construct exact sequences satisfying the above condition (d) in Lemma 4.5 and Lemma 4.6 below.

We use the equality

$$\mathrm{Hom}_R(S_i, S_j) \simeq \begin{cases} (f_{i+1} \cdots f_j)/(f_1 \cdots f_j) & i < j \\ S/(f_1 \cdots f_j) & i \geq j. \end{cases}$$

Lemma 4.5. (a) *We have exchange sequences*

$$0 \to S_i \xrightarrow{(f_{i+1}\ -1)} S_{i+1} \oplus S_{i-1} \xrightarrow{\binom{1}{f_{i+1}}} S/(f_1 \cdots f_{i-1} f_{i+1}) \to 0,$$

$$0 \to S/(f_1 \cdots f_{i-1} f_{i+1}) \xrightarrow{(f_i\ 1)} S_{i+1} \oplus S_{i-1} \xrightarrow{\binom{-1}{f_i}} S_i \to 0.$$

(b) *Under the assumption (B), we have a 2-almost split sequence*

$$0 \to S_i \xrightarrow{(f_{i+1}\ -1)} S_{i+1} \oplus S_{i-1} \xrightarrow{\begin{pmatrix} f_i & 1 \\ f_i f_{i+1} & f_{i+1} \end{pmatrix}} S_{i+1} \oplus S_{i-1} \xrightarrow{\binom{-1}{f_i}} S_i \to 0$$

in $\mathrm{add} \bigoplus_{i=1}^n S_i$ *for any* $1 \leq i < n$.

Proof. (a) Consider the map $a := \binom{-1}{f_i} : S_{i+1} \oplus S_{i-1} \to S_i$. Any morphism from S_j to S_i factors through $1 : S_{i+1} \to S_i$ (resp. $f_i : S_{i-1} \to S_i$) if $j > i$ (resp. $j < i$). Thus a is a minimal right $(\mathrm{add} \bigoplus_{j \neq i} S_j)$-approximation.

It is easily checked that $\mathrm{Ker}\, a = \{s \in S_{i+1} \mid \overline{s} \in f_i S_i\} = (f_i)/(f_1 \cdots f_{i+1}) \simeq S/(f_1 \cdots f_{i-1} f_{i+1})$, where we denote by $\overline{s}$ the image of s via the natural surjection $S_{i+1} \to S_i$.

Consider the surjective map $b := \binom{1}{f_{i+1}} : S_{i+1} \oplus S_{i-1} \to S/(f_1 \cdots f_{i-1} f_{i+1})$. It is easily checked that $\operatorname{Ker} b = \{s \in S_{i+1} \mid \overline{s} \in (f_{i+1})/(f_1 \cdots f_{i-1} f_{i+1})\} = (f_{i+1})/(f_1 \cdots f_{i+1}) \simeq S_i$, where we denote by $\overline{s}$ the image of s via the natural surjection $S_{i+1} \to S/(f_1 \cdots f_{i-1} f_{i+1})$.

(b) This sequence is exact by (a). Any non-isomorphic endomorphism of S_i is multiplication with an element in $\mathfrak{m}$, which is equal to (f_i, f_{i+1}) by (B). Since f_{i+1} (resp. f_i) $: S_i \to S_i$ factors through $1 : S_{i+1} \to S_i$ (resp. $f_i : S_{i-1} \to S_i$), we have that a is a right almost split map. $\square$

Now we choose $f_{n+1} \in \mathfrak{m}$ such that $\mathfrak{m} = (f_n, f_{n+1})$, and f_{n+1} and $f_1 \cdots f_n$ has no common factor.

Lemma 4.6. *Under the assumption (B), we have an exact sequence*

$$0 \to S_{n-1} \xrightarrow{(f_n\ -f_{n+1})} S_n \oplus S_{n-1} \xrightarrow{\binom{f_{n+1}}{f_n}} S_n$$

with a right almost split map $\binom{f_{n+1}}{f_n}$ *in* $\operatorname{add} \bigoplus_{i=1}^n S_i$.

Proof. Consider the map $a := \binom{f_{n+1}}{f_n} : S_n \oplus S_{n-1} \to S_n$. Any morphism from S_j $(j < n)$ to S_n factors through $f_n : S_{n-1} \to S_n$.

Any non-isomorphic endomorphism of S_n is multiplication with an element in $\mathfrak{m} = (f_{n+1}, f_n)$. Since $f_n : S_n \to S_n$ factors through $f_n : S_{n-1} \to S_n$, we have that a is a right almost split map.

It is easily checked that $\operatorname{Ker} a = \{s \in S_{n-1} \mid f_n s \in f_{n+1} S_n\} = (f_{n+1}, f_1 \cdots f_{n-1})/(f_1 \cdots f_{n-1})$, which is isomorphic to S_{n-1} by the choice of f_{n+1}. $\square$

Thus we finished the proof of Theorem 4.1 under the stronger assumption (B).

To show the general statement of Theorem 4.1, we need some preliminary observations.

Lemma 4.7. *Let R and R' be complete local Gorenstein rings with $\dim R = \dim R'$ and M a rigid object in $\mathrm{CM}(R)$ which is a generator. Assume that there exists a surjection $R' \to R$, and we regard $\mathrm{CM}(R)$ as a full subcategory of $\mathrm{CM}(R')$. If $R' \oplus M$ is a cluster tilting object in $\mathrm{CM}(R')$, then M is a cluster tilting object in $\mathrm{CM}(R)$.*

Proof. We use the equivalence (a)⇔(c) in Proposition 4.4. For any $X \in \mathrm{CM}(R)$, there exists an exact sequence $0 \to N_1 \to N_0 \xrightarrow{f} X \to 0$ with $N_i \in \operatorname{add}(R' \oplus M)$ and a minimal right $\operatorname{add}(R' \oplus M)$-approximation f of X. Since f is right minimal, we have $N_1 \in \operatorname{add} M$. Since M is a generator of R, we have that $N_0 \in \operatorname{add} M$. Thus M satisfies condition (c) in Proposition 4.4. $\square$

Next let us consider cluster mutation in $\mathrm{CM}(R)$. We use the notation introduced at the beginning of this section.

Lemma 4.8. *For $w \in \mathfrak{S}_n$, we assume that M_w is a cluster tilting object in $\mathrm{CM}(R)$. Then, for $1 \le i < n$ and $s_i = (i\ i+1)$, we have exchange sequences*

$$0 \to S_i^w \to S_{i+1}^w \oplus S_{i-1}^w \to S_i^{ws_i} \to 0 \quad \textit{and} \quad 0 \to S_i^{ws_i} \to S_{i+1}^w \oplus S_{i-1}^w \to S_i^w \to 0.$$

Proof. Without loss of generality, we can assume $w = 1$. Then the assertion follows from Lemma 4.5(a). $\square$

Immediately, we have the following.

Proposition 4.9. *Assume that M_w is a cluster tilting object in $\mathrm{CM}(R)$ for some $w \in \mathfrak{S}_n$.*

(a) *The mutations of M_w are M_{ws_i} $(1 \le i < n)$.*

(b) *$M_{w'}$ is a cluster tilting object in $\mathrm{CM}(R)$ for any $w' \in \mathfrak{S}_n$.*

Proof. (a) This follows from Lemma 4.8.

(b) This follows from (a) since $\mathfrak{S}_n$ is generated by s_i $(1 \le i < n)$. $\square$

Now we shall prove Theorem 4.1. Since k is an infinite field, we can take irreducible formal power series $g_i \in \mathfrak{m}$ $(1 \le i \le n)$ such that $h_{2i-1} := f_i$ and $h_{2i} := g_i$ satisfy the following conditions:

- $(h_i) \neq (h_j)$ for any $i \neq j$.

- $\mathfrak{m} = (h_1, h_2) = (h_2, h_3) = \cdots = (h_{2i-1}, h_{2i})$.

Put $R' := S/(h_1 \cdots h_{2n-1})$. This is reduced by the first condition.

Since we have already proved Theorem 4.1 under the assumption (B), we have that $\bigoplus_{i=1}^{2n-1} S/(h_1 \cdots h_i)$ is a cluster tilting object in $\mathrm{CM}(R')$. By Proposition 4.9, $\bigoplus_{i=1}^{2n-1} S/(h_{w(1)} \cdots h_{w(i)})$ is a cluster tilting object in $\mathrm{CM}(R')$ for any $w \in \mathfrak{S}_{2n-1}$. In particular,

$$(\bigoplus_{i=1}^{n} S/(f_1 \cdots f_i)) \oplus (\bigoplus_{i=1}^{n-1} S/(f_1 \cdots f_n g_1 \cdots g_i))$$

is a cluster tilting object in $\mathrm{CM}(R')$. Moreover we have surjections

$$R' \to \cdots \to S/(f_1 \cdots f_n g_1 g_2) \to S/(f_1 \cdots f_n g_1) \to R.$$

Using Lemma 4.7 repeatedly, we have that $\bigoplus_{i=1}^{n}(S/(f_1 \cdots f_i))$ is a cluster tilting object in $\mathrm{CM}(R)$. Thus we have proved Theorem 4.1. $\Box$

In the rest we shall show Theorem 4.2. We recall results on tilting mutation due to Riedtmann-Schofield [RS]. For simplicity, a tilting module means a tilting module of projective dimension at most one.

Let Γ be a module-finite algebra over a complete local ring with n simple modules. Their results remain valid in this setting. Recall that, for basic tilting Γ-modules T and U, we write

$$T \leq U$$

if $\mathrm{Ext}^1_\Gamma(T, U) = 0$. Then $\leq$ gives a partial order. On the other hand, we call a Γ-module T *almost complete tilting* if $\mathrm{pd}_\Gamma T \leq 1$, $\mathrm{Ext}^1_\Gamma(T, T) = 0$ and T has exactly $(n-1)$ non-isomorphic indecomposable direct summands.

Proposition 4.10. (a) *Any almost complete tilting* Γ*-module has at most two complements.*
(b) *T and U are neighbors in the partial order if and only if there exists an almost complete tilting Γ-module which is a common direct summand of T and U.*
(c) *Assume $T \leq U$. Then there exists a sequence $T = T_0 < T_1 < T_2 < \cdots < U$ satisfying the following conditions.*
  (i) *T_i and T_{i+1} are neighbors.*
  (ii) *Either $T_i = U$ for some i or the sequence is infinite.*

If the conditions in (b) above are satisfied, we call T a *(tilting) mutation* of U.

We also need the following easy observation on Cohen-Macaulay tilting modules.

Lemma 4.11. *Let Γ be a module-finite algebra over a complete local Gorenstein ring R such that $\Gamma \in \mathrm{CM}(R)$, and T and U tilting Γ-modules. Assume $U \in \mathrm{CM}(\Gamma)$.*
(a) *If $T \leq U$, then $T \in \mathrm{CM}(\Gamma)$.*
(b) *Let P be a projective Γ-module such that $\mathrm{Hom}_R(P, R)$ is a projective Γ^{op}-module. Then $P \in \mathrm{add}\, U$.*

Proof. (a) Recall that $T \leq U$ holds if and only if there exists an exact sequence $0 \to T \to U_0 \to U_1 \to 0$ with $U_i \in \mathrm{add}\, U$. Thus the assertion holds.

(b) There exists an exact sequence $0 \to P \to U_0 \to U_1 \to 0$ with $U_i \in \mathrm{add}\, U$, which must split since $\mathrm{Ext}^1_R(U, P) = 0$. $\Box$

Finally, let us recall the following relation between cluster tilting and tilting (see [I2, Th. 5.3.2] for (a), and (b) is clear).

Proposition 4.12. *Let R be a complete local Gorenstein ring and M, N and N' cluster tilting objects in* $\mathrm{CM}(R)$.
(a) $\mathrm{Hom}_R(M, N)$ *is a tilting* $\mathrm{End}_R(M)$*-module of projective dimension at most one.*
(b) *If N' is a mutation of N, then* $\mathrm{Hom}_R(M, N')$ *is a mutation of* $\mathrm{Hom}_R(M, N)$.

Now we shall prove Theorem 4.2. Fix $w \in \mathfrak{S}_n$ and put $\Gamma := \mathrm{End}_R(M_w)$. The functor $\mathrm{Hom}_R(M_w, \ ) : \mathrm{CM}(R) \to \mathrm{CM}(\Gamma)$ is fully faithful since M_w is a generator of R. By Theorem 4.1, M_w is a cluster tilting object in $\mathrm{CM}(R)$. By Proposition 4.12(a), $\mathrm{Hom}_R(M_w, M_{w'})$ ($w' \in \mathfrak{S}_n$) is a Cohen-Macaulay tilting Γ-module.

(b) Take any Cohen-Macaulay tilting Γ-module U. Since $P := \mathrm{Hom}_R(M_w, R)$ is a projective Γ-module such that $\mathrm{Hom}_R(P, R) = M_w = \mathrm{Hom}_R(R, M_w)$ is a projective Γ^{op}-module, we have $P \in \mathrm{add}\, U$ by Lemma 4.11(b). In particular, each Cohen-Macaulay tilting Γ-module has at most $(n-1)$ mutations which are Cohen-Macaulay by Proposition 4.10(a)(b). Conversely, any Cohen-Macaulay tilting Γ-module of the form $\mathrm{Hom}_R(M_w, M_{w'})$ ($w' \in \mathfrak{S}_n$) has precisely $(n-1)$ mutations $\mathrm{Hom}_R(M_w, M_{w's_i})$ ($1 \le i < n$) which are Cohen-Macaulay by Proposition 4.9 and Proposition 4.12(b).

Now we shall show that U is isomorphic to $\mathrm{Hom}_R(M_w, M_{w'})$ for some w'. Since $\Gamma \le U$, there exists a sequence

$$\Gamma = T_0 < T_1 < T_2 < \cdots < U$$

satisfying the conditions in Proposition 4.10(c). By Lemma 4.11(a), each T_i is Cohen-Macaulay. Since Γ has the form $\Gamma = \mathrm{Hom}_R(M_w, M_w)$, the above argument implies that each T_i has the form $\mathrm{Hom}_R(M_w, M_{w'})$ for some $w' \in \mathfrak{S}_n$. Since $\mathfrak{S}_n$ is a finite group, the above sequence must be finite. Thus $U = T_i$ holds for some i, hence the proof is completed.

(a) Let U be a cluster tilting object in $\mathrm{CM}(R)$. Again by Proposition 4.12(a), $\mathrm{Hom}_R(M_w, U)$ is a Cohen-Macaulay tilting Γ-module. By part (b) which we already proved, $\mathrm{Hom}_R(M_w, U)$ is isomorphic to $\mathrm{Hom}_R(M_w, M_{w'})$ for some $w' \in \mathfrak{S}_n$. Thus U is isomorphic to $M_{w'}$, and the former assertion is proved.

For the latter assertion, we only have to show that any rigid object in $\mathrm{CM}(R)$ is a direct summand of some cluster tilting object in $\mathrm{CM}(R)$. This is valid by the following general result in [BIRS, Th. 1.9]. $\square$

Proposition 4.13. *Let $\mathcal{C}$ be a 2-CY Frobenius category with a cluster tilting object. Then any rigid object in $\mathcal{C}$ is a direct summand of some cluster tilting object in $\mathcal{C}$.*

We end this section with the following application to dimension three.

Now let $S'' := k[[x, y, u, v]]$, $f_i \in \mathfrak{m} = (x, y)$ ($1 \le i \le n$) and $R'' := S''/(f_1 \cdots f_n + uv)$. For $w \in \mathfrak{S}_n$ and $I \subseteq \{1, \cdots, n\}$, we put

$$U_i^w := (u, f_{w(1)} \cdots f_{w(i)}) \subset R'', \ M_w := \bigoplus_{i=1}^{n} U_i^w \ \text{ and } \ U_I := (u, \prod_{i \in I} f_i) \subset S''.$$

We have the following result.

Corollary 4.14. *Under the assumption (A), we have the following.*

(a) *There are exactly $n!$ indecomposable rigid objects M_w ($w \in \mathfrak{S}_n$) and exactly $2^n - 1$ indecomposable rigid objects U_I ($\emptyset \ne I \subset \{1, \cdots, n\}$) in* $\mathrm{CM}(R'')$.

(b) *There are non-commutative crepant resolutions* $\mathrm{End}_{R''}(M_w)$ *($w \in \mathfrak{S}_n$) of R'', which are derived equivalent.*

Proof. (a) This follows from Knörrer periodicity $\underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R'')$.

(b) Any cluster tilting object gives a non-commutative crepant resolution. See 5.4 below. $\square$

For example,

$$k[[x, y, u, v]]/((x - \lambda_1 y) \cdots (x - \lambda_n y) + uv)$$

has a non-commutative crepant resolution for distinct elements $\lambda_1, \cdots, \lambda_n \in k$.

5. Link with birational geometry

There is another approach to the investigation of cluster tilting objects for maximal Cohen-Macaulay modules, using birational geometry. More specifically there is a close connection between resolutions of three dimensional Gorenstein singularities and cluster-tilting theory, provided by the so-called non-commutative crepant resolutions of Van den Bergh. This provides at the same time alternative proofs for geometric results, using cluster tilting objects. The aim of this section is to establish this link with small resolutions. We give relevant criteria for having small resolutions, and apply them to give an alternative approach to most of the results in the previous sections.

Let $(R, \mathfrak{m})$ be a complete normal Gorenstein algebra of Krull dimension 3 over an algebraically closed field k, and let $X = \operatorname{Spec}(R)$. A resolution of singularities $Y \xrightarrow{\pi} X$ is called

- crepant, if $\omega_Y \cong \pi^*\omega_X \cong \mathcal{O}_Y$.
- small, if the relative dimension of the exceptional locus of π is smaller than one.

A small resolution is automatically crepant, but the converse is in general not true. However, both types of resolutions coincide for certain important classes of three-dimensional singularities.

A *cDV (compound Du Val)* singularity is a three dimensional singularity given by the equation

$$f(x, y, z) + tg(x, y, z, t) = 0,$$

where $f(x, y, z)$ defines a simple surface singularity and $g(x, y, z, t)$ is arbitrary.

It is called cA_n (respectively, cD_n, cE_n) if the intersection of $f + tg$ with a generic hyperplane in k^4 is an A_n (respectively, D_n, E_n) surface singularity. By definition, a *generic* hyperplane means that the coefficients defining this hyperplane belong to a Zariski (dense) open subset of k^4. Note that any cDV singularity is terminal.

Theorem 5.1. [Re, Cor. 1.12, Th. 1.14] *Let X be a Gorenstein threefold singularity.*

(a) *If X has a small resolution, then it is cDV.*
(b) *If X is an isolated cDV singularity, then any crepant resolution of X is small.*

There is a close connection with the non-commutative crepant resolutions of Van den Bergh defined as follows.

Definition 5.2. [V2, Def. 4.1] *Let $(R, \mathfrak{m})$ be a normal Gorenstein domain of Krull dimension three. An R-module M gives rise to a non-commutative crepant resolution if*

(i) *M is reflexive,*
(ii) *$A = \operatorname{End}_R(M)$ is Cohen-Macaulay as an R–module,*
(iii) $\operatorname{gl.dim}(A) = 3$.

The following result establishes a useful connection.

Theorem 5.3. [V1, Cor. 3.2.11][V2, Th. 6.6.3] *Let $(R, \mathfrak{m})$ be an isolated cDV singularity of Krull dimension three. Then there exists a crepant resolution of $X = \operatorname{Spec} R$ if and only if there exists a non-commutative one in the sense of Definition 5.2.*

The existence of a non-commutative crepant resolution turns out to be equivalent to the existence of a cluster tilting object in the triangulated category $\underline{\mathrm{CM}}(R)$.

Theorem 5.4. [I2, Th. 5.2.1][IR, Th. 8.9] *Let $(R, \mathfrak{m})$ be a normal Gorenstein isolated singularity of Krull dimension three. Then the existence of a non-commutative crepant resolution is equivalent to the existence of a cluster tilting object in the stable category of maximal Cohen-Macaulay modules $\underline{\mathrm{CM}}(R)$.*

Proof. For convenience of the reader, we give an outline of the proof (see also Proposition 4.4).

Let us first assume that M is a cluster tilting object in $\underline{\mathrm{CM}}(R)$. Then M is automatically reflexive. From the exact sequence

$$0 \longrightarrow \Omega(M) \longrightarrow F \longrightarrow M \longrightarrow 0$$

we obtain

$$0 \longrightarrow \operatorname{End}_R(M) \longrightarrow \operatorname{Hom}_R(F, M) \longrightarrow \operatorname{Hom}_R(\Omega(M), M) \longrightarrow \operatorname{Ext}^1_R(M, M) \longrightarrow 0. \tag{1}$$

Since M is rigid, $\operatorname{Ext}^1_R(M, M) = 0$. Moreover, $\operatorname{depth}(\operatorname{Hom}_R(F, M)) = \operatorname{depth}(M) = 3$ and $\operatorname{depth}(\operatorname{Hom}_R(\Omega(M), M) \geq 2$, and hence $\operatorname{depth}(\operatorname{End}_R(M)) = 3$ and $A = \operatorname{End}(M)$ is maximal Cohen-Macaulay over R.

For the difficult part of this implication, claiming that $\operatorname{gl.dim}(A) = 3$, we refer to [I1, Th. 3.6.2].

For the other direction, let M be a module giving rise to a non-commutative crepant resolution. Then by [IR, Th. 8.9] there exists another module M' giving rise to a non-commutative crepant resolution, which is maximal Cohen-Macaulay and contains R as a direct summand.

By the assumption, $\operatorname{depth}(\operatorname{End}_R(M')) = 3$ and we can apply [IR, Lem. 8.5] to the exact sequence(1) to deduce that $\operatorname{Ext}^1_R(M', M') = 0$, so that M' is rigid. The difficult part saying that M' is cluster tilting is proven in [I2, Th. 5.2.1]. $\square$

We now summarize the results of this section.

Theorem 5.5. *Let $(R, \mathfrak{m})$ be a three dimensional isolated* cDV *singularity. Then the following are equivalent.*

(a) $\operatorname{Spec} R$ *has a small resolution.*
(b) $\operatorname{Spec} R$ *has a crepant resolution.*
(c) $(R, \mathfrak{m})$ *has a non-commutative crepant resolution.*
(d) $\underline{\mathrm{CM}}(R)$ *has a cluster tilting object.*

We have an efficient criterion for existence of a small resolution of a cA_n singularity.

Theorem 5.6. [Kat, Th. 1.1] *Let $X = \operatorname{Spec}(R)$ be an isolated cA_n singularity.*

(a) *Let $Y \longrightarrow X$ be a small resolution. Then the exceptional curve in Y is a chain of n projective lines and X has the form $g(x, y) + uv$, where the curve singularity $g(x, y)$ has $n + 1$ distinct branches at the origin.*
(b) *If X has the form $g(x, y) + uv$, where the curve singularity $g(x, y)$ has $n + 1$ distinct branches at the origin, then X has a small resolution.*

Using the criterion of Katz together with Knörrer periodicity, we get additional equivalent conditions in a special case.

Theorem 5.7. *Let $(R, \mathfrak{m})$ be a three dimensional isolated cA_n singularity defined by the equation $g(x, y) + zt$. Then the following conditions are equivalent in addition to (a)-(d) in Theorem 5.5.*

(e) *Let R' be a one dimensional singularity defined by $g(x, y)$. Then $\underline{\mathrm{CM}}(R')$ has a cluster tilting object.*
(f) *The number of irreducible power series in the prime decomposition of $g(x, y)$ is $n + 1$.*

Proof. (a)$\Leftrightarrow$(f) This follows from Theorem 5.6.

(d)$\Leftrightarrow$(e) By the Knörrer correspondence there is an equivalence of triangulated categories between the stable categories $\underline{\mathrm{CM}}(R) \cong \underline{\mathrm{CM}}(R')$. For, the equivalence of these stable categories given in [Kn], [So] is induced by an exact functor taking projectives to projectives. $\square$

Theorem 5.8. *Assume that the equivalent conditions in Theorem 5.7 are satisfied. Then the following numbers are equal.*

(a) *One plus the number of irreducible components of the exceptional curve of a small resolution of* $\operatorname{Spec} R$.
(b) *The number of irreducible power series in the prime decomposition of $g(z, t)$.*
(c) *The number of simple modules of non-commutative crepant resolutions of $(R, \mathfrak{m})$.*
(d) *One plus the number of non-isomorphic indecomposable summands of cluster tilting objects in $\underline{\mathrm{CM}}(R)$.*

Proof. (a) and (b) are equal by Theorem 5.6.

(a) and (c) are equal by [V1, Th. 3.5.6].

(c) and (d) are equal by [IR, Cor. 8.8]. $\square$

6. Application to curve singularities

In this section we apply results in the previous section to some curve singularities to investigate whether they have cluster tilting object or not. In addition to simple singularities, we study some other nice singularities. In what follows we refer to [AGV] as a general reference for classification of singularities.

To apply results in previous sections to minimally elliptic singularities, we also consider a three-dimensional hypersurface singularity

$$T_{p,q,2,2}(\lambda) = k[[x,y,u,v]]/(x^p + y^q + \lambda x^2y^2 + uv).$$

To apply Theorem 5.7 to a curve singularity, we have to know that the corresponding three dimensional singularity is cA_n. It is given by the following result, where we denote by $\operatorname{ord}(g)$ the degree of the lowest term of a power series g.

Proposition 6.1. *We have the following properties of three-dimensional hypersurface singularities:*

(a) A_n *(*$n \geq 1$*) is a* cA_1*–singularity,*
(b) D_n *(*$n \geq 4$*) and* E_n *(*$n = 6,7,8$*) are* cA_2*–singularities,*
(c) $T_{3,q,2,2}(\lambda)$ *(*$q \geq 6$*) is a* cA_2*–singularity,*
(d) $T_{p,q,2,2}(\lambda)$ *(*$p \geq q \geq 4$*) is a* cA_3*–singularity,*
(e) $k[[x,y,z,t]]/(x^2+y^2+g(z,t))$ *(*$g \in k[[z,t]]$*) is a* cA_m*–singularity if* $m = \operatorname{ord}(g) - 1 \geq 1$.

We shall give a detailed proof at the end of this section. In view of Theorem 5.7 and Proposition 6.1, we have the following main result in this section.

Theorem 6.2. (a) *A simple three dimensional singularity satisfies the equivalent conditions in Theorem 5.8 if and only if it is of type* A_n *(*n *is odd) or* D_n *(*n *is even).*
(b) *A* $T_{p,q,2,2}(\lambda)$*–singularity satisfies the equivalent conditions in Theorem 5.8 if and only if* $p = 3$ *and* q *is even or if both* p *and* q *are even.*
(c) *A singularity* $k[[x,y,u,v]]/(uv + f_1 \cdots f_n)$ *with irreducible and mutually prime* $f_i \in (x,y) \subset k[[x,y]]$ *(*$1 \leq i \leq n$*) satisfies the equivalent conditions in Theorem 5.8 if and only if* $f_i \notin (x,y)^2$ *for any* i*.*

Proof. Each singularity is cA_m by Proposition 6.1, and defined by an equation of the form $g(x,y) + uv$. By Theorem 5.8, we only have to check whether the number of irreducible power series factors of $g(x,y)$ is $m+1$ or not.

(a) For an A_n–singularity, we have $m = 1$ and $g(x,y) = x^2 + y^{n+1}$. So g has two factors if and only if n is odd.

For a D_n–singularity, we have $m = 2$ and $g(x,y) = (x^2 + y^{n-2})y$. So g has three factors if and only if n is even.

For an E_n–singularity, we have $m = 2$ and $g(x,y) = x^3 + y^4$, $x(x^2 + y^3)$ or $x^3 + y^5$. In each case, g does not have three factors.

(b) First we consider the simply elliptic case. We have $m = 2$ and $g(x,y) = y(y - x^2)(y - \lambda x^2)$ for $(p,q) = (3,6)$, and $m = 3$ and $g(x,y) = xy(x-y)(x-\lambda y)$ for $(p,q) = (4,4)$. In both cases, g has $m+1$ factors.

Now we consider the cusp case. We have $m = 2$ for $p = 3$ and $m = 3$ for $p > 3$, and $g(x,y) = (x^{p-2} - y^2)(x^2 - y^{q-2})$. So g has $m+1$ factors if and only if $p = 3$ and q is even or if both p and q are even.

(c) We have $m = \sum_{i=1}^{n} \operatorname{ord}(f_i) - 1$ and $g = f_1 \cdots f_n$. So g has $m+1$ factors if and only if $\operatorname{ord}(f_i) = 1$ for any i. □

Immediately we have the following conclusion.

Corollary 6.3. (a) *A simple curve singularity has a cluster tilting object if and only if it is of type* A_n *(*n *is odd) or* D_n *(*n *is even). The number of non-isomorphic indecomposable summands of cluster tilting objects is* 1 *for type* A_n *(*n *is odd) and* 2 *for type* D_n *(*n *is even).*
(b) *A* $T_{p,q}(\lambda)$*-singularity has a cluster tilting object if and only if* $p = 3$ *and* q *is even or if both* p *and* q *are even. The number of non-isomorphic indecomposable summands of cluster tilting objects is* 2 *if* $p = 3$ *and* q *is even, and* 3 *if both* p *and* p *are even.*
(c) *A singularity* $R = k[[x,y]]/(f_1 \cdots f_n)$ *with irreducible and mutually prime* $f_i \in (x,y) \subset k[[x,y]]$ *(*$1 \leq i \leq n$*) has a cluster tilting object if and only if* $f_i \notin (x,y)^2$ *for any* i*. In this case, the number of non-isomorphic indecomposable summands of cluster tilting objects in* $\underline{\mathrm{CM}}R$ *is* n*.*

Summarizing with Theorem 4.2, we have completed the proof of Theorem 1.4.

In the rest of this section, we shall prove Proposition 6.1.

Let k be an algebraically closed field of characteristic zero, $R = k[[x_1, x_2, \dots, x_n]]$ the local ring of formal power series and $\mathfrak{m}$ its maximal ideal. We shall need the following standard notions.

Definition 6.4. *For $f \in \mathfrak{m}^2$ we denote by $J(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$ its Jacobi ideal. The Milnor number $\mu(f)$ is defined as*

$$\mu(f) := \dim_k(R/J(f)).$$

The following lemma is standard (see for example [AGV, GLSh]):

Lemma 6.5. *A hypersurface singularity $f = 0$ is isolated if and only if $\mu(f) < \infty$.*

Definition 6.6 ([AGV])**.** *Two hypersurface singularities $f = 0$ and $g = 0$ are called* right *equivalent ($f \overset{r}{\sim} g$) if there exists an algebra automorphism $\varphi \in Aut(R)$ such that $g = \varphi(f)$.*

Note, that $f \overset{r}{\sim} g$ implies an isomorphism of k–algebras

$$R/(f) \cong R/(g).$$

The following lemma is straightforward, see for example [GLSh, Lem. 2.10].

Lemma 6.7. *Assume $f \overset{r}{\sim} g$, then $\mu(f) = \mu(g)$.*

In what follows, we shall need the next standard result on classification of singularities, see for example [GLSh, Cor. 2.24].

Theorem 6.8. *Let $f \in \mathfrak{m}^2$ be an isolated singularity with Milnor number μ. Then*

$$f \overset{r}{\sim} f + g$$

*for any $g \in \mathfrak{m}^{\mu+2}$. One also says that f is $(\mu + 1)$ –*determined *meaning that the equivalence class of f is determined by its $(\mu + 1)$ –jet.*

We shall need the following easy lemma.

Lemma 6.9. *Let $f = x^2 + y^2 + p(x, y, z)$, where*

$$p(x, y, z) = z^n + p_1(x, y)z^{n-1} + \cdots + p_n(x, y)$$

is a homogeneous form of degree $n \geq 3$. Then

$$f \overset{r}{\sim} x^2 + y^2 + z^n.$$

Proof. Write $p(x, y, z) = z^n + xu + yv$ for some homogeneous forms u and v of degree $n - 1$. Then

$$x^2 + y^2 + z^n + xu + yv = (x + u/2)^2 + (y + v/2)^2 + z^n - (u^2 + v^2)/4.$$

After a change of variables $x \mapsto x + u/2$, $y \mapsto y + v/2$ and $z \mapsto z$ we reduce f to the form

$$f = x^2 + y^2 + z^n + h,$$

where $h \in \mathfrak{m}^{2(n-1)} \subset \mathfrak{m}^{n+1}$. Note that $\mu(x^2 + y^2 + z^n) = n - 1$, hence by Theorem 6.8 we have

$$f \overset{r}{\sim} x^2 + y^2 + z^n.$$

□

Now we are ready to give a proof of Proposition 6.1. We only have to show the assertion (e) since other cases are special cases of this. We denote by H the hyperplane in a four-dimensional space defined by the equation $t = \alpha x + \beta y + \gamma z$, $\alpha, \beta, \gamma \in k$. We put

$$g(z, t) = a_0 z^{m+1} + a_1 z^m t + \cdots + a_{m+1} t^{m+1} + (\text{higher terms}).$$

Then the intersection of H with the singularity defined by the equation $x^2 + y^2 + g(z, t)$ is given by the equation $f = h + (\text{higher terms})$, where

$$h = x^2 + y^2 + a_0 z^{m+1} + a_1 z^m(\alpha x + \beta y + \gamma z) + \cdots + a_{m+1}(\alpha x + \beta y + \gamma z)^{m+1}.$$

Now we consider the case $m = 1$. We have $h \overset{r}{\sim} x^2 + y^2 + z^2$ since any quadratic form can be diagonalized using linear transformations. By Lemma 6.7, we have $\mu(h) = \mu(x^2 + y^2 + z^2) = 1$. Hence $f \overset{r}{\sim} h \overset{r}{\sim} x^2 + y^2 + z^2$ by Theorem 6.8.

Next we consider the case $m \geq 2$. Assume $\alpha \in k$ satisfies $a_0 + a_1\alpha + \cdots + a_{m+1}\alpha^{m+1} \neq 0$. By Lemma 6.9, we have $h \overset{r}{\sim} x^2+y^2+z^{m+1}$. By Lemma 6.7, we have $\mu(h) = \mu(x^2+y^2+z^{m+1}) = m$. Hence $f \overset{r}{\sim} h \overset{r}{\sim} x^2 + y^2 + z^{m+1}$ by Theorem 6.8.

Consequently, $x^2 + y^2 + g(z, t)$ is cA_m. □

7. Examples of 2-CY tilted algebras

Since the 2-CY tilted algebras coming from maximal Cohen-Macaulay modules over hypersurfaces have some nice properties, it is of interest to have more explicit information about such algebras. This section is devoted to some such computations for algebras coming from minimally elliptic singularities. We obtain algebras appearing in classification lists for some classes of tame self-injective algebras [Er, BS].

We start with giving some general properties which are direct consequences of Lemma 2.2.

Theorem 7.1. *Let $(R, \mathfrak{m})$ be an isolated hypersurface singularity and Γ a 2-CY tilted algebra coming from $\underline{\mathrm{CM}}(R)$. Then we have the following.*

(a) *Γ is a symmetric algebra.*
(b) *All components in the stable AR-quiver of Γ are tubes of rank 1 or 2.*

We now start with our computations of 2-CY tilted algebras coming from minimally elliptic singularities. We first introduce and investigate two classes of algebras, and then show that they are isomorphic to 2-CY tilted algebras coming from minimally elliptic singularities.

For a quiver Q with finitely many vertices and arrows we define the radical completion $\widehat{kQ}$ of the path algebra kQ by the formula

$$\widehat{kQ} = \varprojlim kQ/\operatorname{rad}^n(kQ).$$

The reason we deal with completion is the following: Let Q be a finite quiver, J the ideal of $\widehat{kQ}$ generated by the arrows and $I \subseteq J^2$ a complete ideal such that $\Lambda = \widehat{kQ}/I$ is finite-dimensional.

Lemma 7.2. *The ideal I is generated in $\widehat{kQ}$ by a minimal system of relations, that is, a set of elements $\rho_1, \cdots, \rho_n$ of I whose images form a k-basis of $I/IJ + JI$.*

The lemma is shown by a standard argument (cf [BMR3, Section 3]). Its analogue for the non complete path algebra is not always true. For example, for the algebra $\Lambda = B_{2,2}(\lambda)$ defined below, the elements $\rho_1, \cdots, \rho_n$ listed as generators for I form a minimal system of relations. So they generate I in $\widehat{kQ}$. They also yield a k-basis of $I'/I'J + JI' \xrightarrow{\sim} I/IJ + JI$, where $I' = I \cap kQ$ and $J' = J \cap kQ$. But they do not generate the ideal I' of kQ since, as one can show, the quotient $kQ/\langle\rho_1, \cdots, \rho_n\rangle$ is infinite-dimensional.

On the other hand, the ideal I' is generated by the preimage $\rho_1, \cdots, \rho_n$ of a basis of $I'/I'J' + J'I'$ if the quotient $kQ/\langle\rho_1, \cdots, \rho_n\rangle$ is finite-dimensional, since then the ideal $\langle\rho_1, \cdots, \rho_n\rangle$ contains a power of J'. This happens for example for the algebra $A_2(\lambda)$ as defined below, cf. also [Sk, 5.9] and [BS, Th. 1].

We know that for all vertices i, j of Q, we have

$$\dim_k e_i(I/IJ + JI)e_j = \dim_k \operatorname{Ext}^2_\Lambda(S_i, S_j)$$

where S_i and S_j denote the simple Λ-modules corresponding to the vertices i and j. When Λ is 2-CY tilted, then

$$\dim \operatorname{Ext}^1_\Lambda(S_j, S_i) \geq \dim \operatorname{Ext}^2_\Lambda(S_i, S_j)$$

(see [BMR3, KR]). Thus the number of arrows in Q is an upper bound on the number of elements in a minimal system of relations.

Definition 7.3. *(1) For $q \geq 2$ and $\lambda \in k^*$ we write $A_q(\lambda) = \widehat{kQ}/I$, where*

$$Q = \begin{tikzcd} \cdot \arrow[loop left, "\varphi"] \arrow[r, shift left, "\alpha"] & \cdot \arrow[l, shift left, "\beta"] \arrow[loop right, "\psi"] \end{tikzcd}$$

and

$$I = \langle \psi\alpha - \alpha\varphi, \beta\psi - \varphi\beta, \varphi^2 - \beta\alpha, \psi^q - \lambda\alpha\beta \rangle.$$

If $q = 2$, then we additionally assume $\lambda \neq 1$. (It can be shown that for $q \geq 3$ we have $A_q(\lambda) \cong A_q(1)$, so we drop the parameter λ in this case.)

(2)For $p, q \geq 1$ and $\lambda \in k^*$ we write $B_{p,q}(\lambda) = \widehat{kQ}/I$, where

$$Q = \begin{tikzcd} \cdot \arrow[loop left, "\varphi"] \arrow[r, shift left, "\alpha"] & \cdot \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & \cdot \arrow[l, shift left, "\delta"] \arrow[loop right, "\psi"] \end{tikzcd}$$

and

$$I = \langle \beta\alpha - \varphi^p, \gamma\delta - \lambda\psi^q, \alpha\varphi - \delta\gamma\alpha, \varphi\beta - \beta\delta\gamma, \delta\psi - \alpha\beta\delta, \psi\gamma - \gamma\alpha\beta \rangle.$$

For $p = q = 1$ we additionally assume $\lambda \neq 1$.

When $p = q = 1$, the generators φ and ψ can be excluded and $B_{1,1}(\lambda)$ is given by the completion of the path algebra of the quiver

$$Q = \begin{tikzcd} \cdot \arrow[r, shift left, "\alpha"] & \cdot \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & \cdot \arrow[l, shift left, "\delta"] \end{tikzcd}$$

modulo the relations

$$I = \langle \alpha\beta\alpha - \delta\gamma\alpha, \alpha\beta\delta - \lambda\delta\gamma\delta, \gamma\alpha\beta - \lambda\gamma\delta\gamma, \beta\delta\gamma - \beta\alpha\beta \rangle.$$

For $(p, q) \neq (1, 1)$ we have $B_{p,q}(\lambda) \cong B_{p,q}(1)$. In particular, for $p = 1$ and $q \geq 2$ the algebra is isomorphic to $\widehat{kQ}/I$, where

$$Q = \begin{tikzcd} \cdot \arrow[r, shift left, "\alpha"] & \cdot \arrow[l, shift left, "\beta"] \arrow[r, shift left, "\gamma"] & \cdot \arrow[l, shift left, "\delta"] \arrow[loop right, "\psi"] \end{tikzcd}$$

and

$$I = \langle \gamma\delta - \psi^q, \alpha\beta\alpha - \delta\gamma\alpha, \beta\alpha\beta - \beta\delta\gamma, \delta\psi - \alpha\beta\delta, \psi\gamma - \gamma\alpha\beta \rangle.$$

It turns out that the algebras $A_q(\lambda)$ and $B_{p,q}(\lambda)$ are finite dimensional. In order to show this it suffices to check that all oriented cycles in $\widehat{kQ}/I$ are nilpotent.

Lemma 7.4. *In the algebra $A_q(\lambda)$ the following zero relations hold:*

$$\alpha\beta\alpha = 0, \beta\alpha\beta = 0, \alpha\varphi^2 = \psi^2\alpha = 0, \varphi^2\beta = \beta\psi^2 = 0, \varphi^4 = 0, \psi^{q+2} = 0.$$

Proof. We have to consider separately the cases $q = 2$ and $q \geq 3$.

Let $q = 2$, then we assumed $\lambda \neq 1$. We have

$$\alpha\beta\alpha = \alpha\varphi^2 = \psi^2\alpha = \lambda^{-1}\alpha\beta\alpha,$$

hence $\alpha\beta\alpha = 0$. In a similar way we obtain $\beta\alpha\beta = 0$. Then $\alpha\varphi^2 = \alpha\beta\alpha = 0$, $\varphi^2 = \beta\alpha\beta\alpha = 0$ and the remaining zero relations follow analogously.

Let $q \geq 3$. Then

$$\psi^q\alpha = \alpha\beta\alpha = \alpha\varphi^2 = \psi^2\alpha,$$

so $(1 - \psi^{q-2})\psi^2\alpha = 0$ and hence

$$\psi^2\alpha = \alpha\beta\alpha = 0$$

in $\widehat{kQ}/I$. The remaining zero relations follow similarly. □

Lemma 7.5. *We have the following relations in $B_{p,q}(\lambda)$:*

$$\varphi^{p+2} = 0, \psi^{q+2} = 0, \gamma\alpha\varphi = \psi\gamma\alpha = 0, \varphi\beta\delta = \beta\delta\psi = 0.$$

Moreover, $\alpha\beta \cdot \delta\gamma = \delta\gamma \cdot \alpha\beta$. For $q \geq p \geq 2$ we have

$$(\alpha\beta)^2 = (\delta\gamma)^2 = 0,$$

for $q > p = 1$ we have

$$(\alpha\beta)^3 = 0, (\delta\gamma)^2 = 0, (\alpha\beta)^2 \cdot (\delta\gamma) = 0$$

and for $p = q = 1$

$$(\alpha\beta)^3 = (\gamma\delta)^3 = 0, (\alpha\beta)^2 = \alpha\beta \cdot \delta\gamma = \lambda(\delta\gamma)^2.$$

The proof is completely parallel to the proof of the previous lemma and is therefore skipped. □

The main result of this section is the following

Theorem 7.6. *(a) Let R be a $T_{3,2q+2}(\lambda)$–singularity, where $q \geq 2$ and $\lambda \in k^*$. Then in the triangulated category $\underline{\mathrm{CM}}(R)$ there exists a cluster tilting object with the corresponding 2-CY-tilted algebra isomorphic to $A_q(\lambda)$.*

(b)For $R = T_{2p+2,2q+2}(\lambda)$ the category $\underline{\mathrm{CM}}(R)$ has a cluster tilting object with endomorphism algebra isomorphic to $B_{p,q}(\lambda)$.

Proof. (a) We consider first the case of $T_{3,2q+2}(\lambda)$.

The coordinate ring of $T_{3,6}(\lambda)$ is isomorphic to

$$R = k[[x, y]]/(y(y - x^2)(y - \lambda x^2)),$$

where $\lambda \neq 0, 1$. Consider Cohen-Macaulay modules M and N given by the two-periodic free resolutions

$$\begin{cases} M = (R \xrightarrow{y-x^2} R \xrightarrow{y(y-\lambda x^2)} R), \\ N = (R \xrightarrow{y(y-x^2)} R \xrightarrow{y-\lambda x^2} R). \end{cases}$$

It is cluster tilting by Theorem 4.1 or Corollary 6.3. In order to compute the endomorphism algebra $\underline{\mathrm{End}}(M \coprod N)$, note that

$$\underline{\mathrm{End}}(M) \cong k[\varphi]/\langle\varphi^4\rangle,$$

where $\varphi = (x, x)$ is an endomorphism of M viewed as a two-periodic map of a free resolution. In $\underline{\mathrm{End}}(M)$ we have $(y, y) = (x, x)^2 = \varphi^2$. Similarly,

$$\underline{\mathrm{End}}(N) \cong k[\psi]/\langle\psi^4\rangle,\ \psi = (x, x),\ (y, y) = \lambda(x, x)^2 = \lambda\psi^2,$$

and

$$\underline{\mathrm{Hom}}(M, N) = k^2 = \langle(1, y), (x, xy)\rangle, \quad \underline{\mathrm{Hom}}(N, M) = k^2 = \langle(y, 1), (xy, x)\rangle.$$

The isomorphism $A_2(\lambda) \longrightarrow \underline{\mathrm{End}}(M \coprod N)$ is given by

$$\varphi \mapsto (x, x), \psi \mapsto (x, x), \alpha \mapsto (1, y), \beta \mapsto (y, 1).$$

Assume now $q \geq 3$ and $R = T_{3,2q+2}$. By [AGV] we may write

$$R = k[[x, y]]/((x - y^2)(x^2 - y^{2q})).$$

Consider the Cohen-Macaulay module $M \coprod N$, where

$$\begin{cases} M = (R \xrightarrow{x-y^2} R \xrightarrow{x^2-y^{2q}} R), \\ N = (R \xrightarrow{(x-y^2)(x+y^q)} R \xrightarrow{x-y^q} R). \end{cases}$$

Again, by a straightforward calculation

$$\underline{\mathrm{End}}(M) \cong k[\varphi]/\langle\varphi^4\rangle,\ \varphi = (y, y), \quad \underline{\mathrm{End}}(N) \cong k[\psi]/\langle\psi^{q+2}\rangle,\ \psi = (y, y)$$

and

$$\underline{\mathrm{Hom}}(M, N) = k^2 = \langle(1, x + y^q), (y, y(x + y^q))\rangle,$$
$$\underline{\mathrm{Hom}}(M, N) = k^2 = \langle(x + y^q, 1), (y(x + y^q), y)\rangle.$$

If $q \geq 4$ then $\underline{\mathrm{End}}(M \coprod N)$ is isomorphic to $\widehat{kQ}/I$, where

$$Q = \begin{tikzcd} \cdot \arrow[loop left, "\varphi"] \arrow[r, shift left, "\alpha"] & \cdot \arrow[l, shift left, "\beta"] \arrow[loop right, "\psi"] \end{tikzcd}$$

and the relations are

$$\beta\alpha = \varphi^2, \alpha\beta = 2\psi^q, \alpha\varphi = \psi\alpha, \varphi\beta = \beta\psi$$

for

$$\varphi = (y, y), \psi = (y, y), \alpha = (1, x + y^q), \beta = (x + y^q, 1).$$

By rescaling all generators $\alpha \mapsto 2^a\alpha, \beta \mapsto 2^b\beta, \varphi \mapsto 2^f\varphi, \psi \mapsto 2^g\psi$ for properly chosen $a, b, f, g \in \mathbb{Q}$ one can easily show $\underline{\mathrm{End}}(M \coprod N) \cong A_q$.

The case $q = 3$ has to be considered separately, since this time the relations are

$$\beta\alpha = \varphi^2 + \varphi^3, \alpha\beta = 2\psi^q, \alpha\varphi = \psi\alpha, \varphi\beta = \beta\psi.$$

We claim that there exist invertible power series $u(t), v(t), w(t), z(t) \in k[[t]]$ such that the new generators

$$\varphi' = u(\varphi)\varphi, \psi' = v(\psi)\psi, \alpha' = \alpha w(\varphi) = w(\psi)\alpha, \beta' = \beta z(\psi) = z(\varphi)\beta$$

satisfy precisely the relations of the algebra A_3. This is fulfilled provided we have the following equations in $k[[t]]$:

$$\begin{cases} zw = u^2(1 + tu) \\ zw = 2v^3 \\ uw = vw \\ uz = vz. \end{cases}$$

This system is equivalent to

$$u(t) = (2 - t)^{-1} = \frac{1}{2}(1 + \frac{t}{2} + (\frac{t}{2})^2 + \dots)$$

and hence the statement is proven.

The case of $T_{2p+2,2q+2}(\lambda)$ is essentially similar. For $p = q = 1$ we have

$$R = k[[x, y]]/(xy(x - y)(x - \lambda y)).$$

Take

$$\begin{cases} M = (R \xrightarrow{x-y} R \xrightarrow{xy(x-\lambda y)} R), \\ N = (R \xrightarrow{x(x-y)} R \xrightarrow{y(x-\lambda y)} R), \\ K = (R \xrightarrow{xy(x-y)} R \xrightarrow{x-\lambda y} R). \end{cases}$$

By Theorem 4.1 or Corollary 6.3, $M \coprod N \coprod K$ is cluster tilting. Moreover, $B_{1,1}(\lambda) \simeq \underline{\mathrm{End}}(M \coprod N \coprod K)$.

Let now

$$R = k[[x, y]]/((x^p - y)(x^p + y)(y^q - x)(y^q + x)),$$

where $(p, q) \neq (1, 1)$ and

$$\begin{cases} M = (R \xrightarrow{x^p-y} R \xrightarrow{(y^q+x)(y^q-x)(x^p+y)} R), \\ N = (R \xrightarrow{(x^p-y)(x^p+y)} R \xrightarrow{(y^q-x)(y^q+x)} R), \\ K = (R \xrightarrow{(x^p-y)(x^p+y)(y^q+x)} R \xrightarrow{y^q-x} R). \end{cases}$$

By Theorem 4.1 or Corollary 6.3, $M \coprod N \coprod K$ is cluster tilting, and by a similar case-by-case analysis it can be verified that $\underline{\mathrm{End}}(M \coprod N \coprod K) \cong B_{p,q}$. □

We have seen that the algebras $A_q(\lambda)$ and $B_{p,q}(\lambda)$ are symmetric, and the indecomposable nonprojective modules have τ-period at most 2, hence Ω-period dividing 4 since $\tau = \Omega^2$ in this case. A direct computation shows that the Cartan matrix is nonsingular. Note that these algebras appear in Erdmann's list of algebras of *quaternion type* [Er], see also [Sk], that is, in addition to the above properties, the algebras are tame. Note that for the corresponding algebras, more relations are given in Erdmann's list. This has to do with the fact that we are working with the completion, as discussed earlier. In our case all relations correspond to different arrows in the quiver. The simply elliptic ones also appear in Białkowski-Skowroński's list of weakly symmetric tubular algebras with a nonsingular Cartan matrix.

This provides a link between some stable categories of maximal Cohen-Macaulay modules over isolated hypersurface singularities, and some classes of finite dimensional algebras, obtained via cluster tilting theory.

Previously a link between maximal Cohen-Macaulay modules and finite dimensional algebras was given with the canonical algebras of Ringel, via the categories $\mathrm{coh}\mathbb{X}$ of coherent sheaves on weighted projective lines in the sense of Geigle-Lenzing [GL]. Here the category of vector bundles is equivalent to the category of graded maximal Cohen-Macaulay modules with degree zero maps, over some isolated singularity. And the canonical algebras are obtained as endomorphism algebras of certain tilting objects in $\mathrm{coh}\mathbb{X}$ which are vector bundles.

Note that it is known from work of Dieterich [D], Kahn [Kah], Drozd and Greuel [DG] that minimally elliptic curve singularities have tame Cohen-Macaulay representation type. Vice versa, any Cohen-Macaulay tame reduced hypersurface curve singularity is isomorphic to one of the $T_{p,q}(\lambda)$, see [DG]. Moreover, simply elliptic singularities are tame of polynomial growth and cusp singularities are tame of exponential growth. Furthermore, the Auslander-Reiten quiver of the corresponding stable categories of maximal Cohen-Macaulay modules consists of tubes of rank one or two, see [Kah, Th. 3.1] and [DGK, Cor. 7.2].

It should follows from the tameness of $\mathrm{CM}(T_{3,p}(\lambda))$ and $\mathrm{CM}(T_{p,q}(\lambda))$ that the associated 2-CY tilted algebras are tame.

We point out that in the wild case we can obtain symmetric 2-CY tilted algebras where the stable AR-quiver consists of tubes of rank one and two, and most of them should be wild. It was previously known that there are examples of wild selfinjective algebras whose AR-quivers consist of tubes of rank one or three [AR].

8. Appendix: 2-CY triangulated categories of finite type

In this section, we consider more general situation than in section 2. Let k be an algebraically closed field and $\mathcal{C}$ a k-linear connected 2-Calabi-Yau triangulated category with only finitely many indecomposable objects. We show that it follows from the shape of the AR quiver of $\mathcal{C}$ whether cluster tilting objects (resp. non-zero rigid objects) exist in $\mathcal{C}$ or not. Let us start with giving the possible shapes of the AR quiver of $\mathcal{C}$.

Proposition 8.1. *The AR quiver of $\mathcal{C}$ is $\mathbf{Z}\Delta/H$ for a Dynkin diagram Δ and a weakly admissible subgroup H of $\mathrm{Aut}(\mathbf{Z}\Delta)$ which contains $F \in \mathrm{Aut}(\mathbf{Z}\Delta)$ defined by the list below. Moreover, H is generated by a single element $h \in \mathrm{Aut}(\mathbf{Z}\Delta)$ in the list below.*

Δ	$\mathrm{Aut}(\mathbf{Z}\Delta)$	F	h
(A_n) $n: odd$	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(\frac{n+3}{2},1)$	$(k,1)$ $(k\vert\frac{n+3}{2},\ \frac{n+3}{2k}$ *is odd*$)$
(A_n) $n: even$	$\mathbf{Z}$	$n+3$	k $(k\vert n+3)$
(D_n) $n: odd$	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(n,1)$	$(k,1)$ $(k\vert n)$
(D_4)	$\mathbf{Z}\times S_3$	$(4,0)$	(k,σ) $(k\vert 4,\ \sigma^{\frac{4}{k}}=1)$
(D_n) $n: even,\ n>4$	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(n,0)$	$(k,0)$ $(k\vert n)$ *or* $(k,1)$ $(k\vert n,\ \frac{n}{k}$ *is even*$)$
(E_6)	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(7,1)$	$(1,1)$ *or* $(7,1)$
(E_7)	$\mathbf{Z}$	10	$1,2,5$ *or* 10
(E_8)	$\mathbf{Z}$	16	$1,2,4,8$ *or* 16

Proof By [XZ] (see also [Am, 4.0.4]), the AR quiver of $\mathcal{C}$ is $\mathbf{Z}\Delta/H$ for a Dynkin diagram Δ and a weakly admissible subgroup H of $\mathrm{Aut}(\mathbf{Z}\Delta)$. Since $\mathcal{C}$ is 2-Calabi-Yau, H contains F. By [Am, 2.2.1], H is generated by a single element h. By the condition $F \in \langle h\rangle$, we have the above list. $\square$

Theorem 8.2. *(1) $\mathcal{C}$ has a cluster tilting object if and only if the AR quiver of $\mathcal{C}$ is $\mathbf{Z}\Delta/\langle h\rangle$ for a Dynkin diagram Δ and $h \in \mathrm{Aut}(\mathbf{Z}\Delta)$ in the list below.*

Δ	$\mathrm{Aut}(\mathbf{Z}\Delta)$	h
(A_n) n : *odd*	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(\frac{n+3}{6},1)$ $(3\|n)$ *or* $(\frac{n+3}{2},1)$
(A_n) n : *even*	$\mathbf{Z}$	$\frac{n+3}{3}$ $(3\|n)$ *or* $n+3$
(D_n) n : *odd*	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(k,1)$ $(k\|n)$
(D_4)	$\mathbf{Z}\times S_3$	(k,σ) $(k\|4,\ \sigma^{\frac{4}{k}}=1,\ (k,\sigma)\neq(1,1))$
(D_n) n : *even*, $n>4$	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(k,\overline{k})$ $(k\|n)$
(E_6)	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(7,1)$
(E_7)	$\mathbf{Z}$	10
(E_8)	$\mathbf{Z}$	8 *or* 16

(2) $\mathcal{C}$ does not have a non-zero rigid object if and only if the AR quiver of $\mathcal{C}$ is $\mathbf{Z}\Delta/\langle h\rangle$ for a Dynkin diagram Δ and $h \in \mathrm{Aut}(\mathbf{Z}\Delta)$ in the list below.

Δ	$\mathrm{Aut}(\mathbf{Z}\Delta)$	h
(A_n) n : *odd*	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$-$
(A_n) n : *even*	$\mathbf{Z}$	1
(D_n) n : *odd*	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$-$
(D_4)	$\mathbf{Z}\times S_3$	$(1,1)$
(D_n) n : *even*, $n>4$	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(1,0)$
(E_6)	$\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$	$(1,1)$
(E_7)	$\mathbf{Z}$	1
(E_8)	$\mathbf{Z}$	1 *or* 2

Proof Our method is based on the computation of additive functions in section 2.

(1) Assume that h is on the list. Then one can check that $\mathcal{C}$ has a cluster tilting object. For example, consider the (D_n) case here. Fix a vertex $x \in \mathbf{Z}\Delta$ corresponding to an end point of Δ which is adjacent to the branch vertex of Δ. Then the subset $\{(1,1)^l x \mid l \in \mathbf{Z}\}$ of $\mathbf{Z}\Delta$ is stable under the action of h, and gives a cluster tilting object of $\mathcal{C}$.

Conversely, assume that $\mathcal{C}$ has a cluster tilting object. Then one can check that h is on the list. For example, consider the (A_n) case with even n here. By [I1], cluster tilting objects correspond to dissections of a regular $(n+3)$-polygon into triangles by non-crossing diagonals. The action of h shows that it is invariant under the rotation of $\frac{2k\pi}{n+3}$-radian. Since the center of the regular $(n+3)$-polygon is contained in some triangle or its edge, we have $\frac{2k\pi}{n+3} = 2\pi,\ \frac{4\pi}{3},\ \pi$ or $\frac{2\pi}{3}$. Since $k|n+3$ and n is even, we have $k = n+3$ or $\frac{n+3}{3}$.

(2) If h is on the list above, then one can easily check that $\mathcal{C}$ does not have a non-zero rigid objects. Conversely, if h is not on the list, then one can easily check that at least one indecomposable object which corresponds to an end point of Δ is rigid. $\square$

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