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Graded quiver varieties and derived categories

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To the memory of Dieter Happel

Abstract. Inspired by recent work of Hernandez–Leclerc and Leclerc–Plamondon we investigate the link between Nakajima's graded affine quiver varieties associated with an acyclic connected quiver Q and the derived category of Q. As Leclerc–Plamondon have shown, the points of these varieties can be interpreted as representations of a category, which we call the (singular) Nakajima category $\mathcal{S}$. We determine the quiver of $\mathcal{S}$ and the number of minimal relations between any two given vertices. We construct a δ-functor Φ taking each finite-dimensional representation of $\mathcal{S}$ to an object of the derived category of Q. We show that the functor Φ establishes a bijection between the strata of the graded affine quiver varieties and the isomorphism classes of objects in the image of Φ. If the underlying graph of Q is an ADE Dynkin diagram, the image is the whole derived category; otherwise, it is the category of 'line bundles over the non-commutative curve given by Q'. We show that the degeneration order between strata corresponds to Jensen–Su–Zimmermann's degeneration order on objects of the derived category. Moreover, if Q is an ADE Dynkin quiver, the singular category $\mathcal{S}$ is weakly Gorenstein of dimension 1 and its derived category of singularities is equivalent to the derived category of Q.

1. Introduction

Let Q be a Dynkin quiver, i.e. a quiver whose underlying graph is an ADE Dynkin diagram Δ. The (affine) graded quiver varieties associated with Q were introduced by Nakajima in [35]. In type A, they generalize Ginzburg–Vasserot's graded nilpotent orbit closures [13]. They have been of great importance in

(1) Nakajima's geometric study [35] of the finite-dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ associated with Δ,

(2) his related study of cluster algebras in [34, 36], cf. also the survey [30].

Let us elaborate on the second point: In [36], Nakajima showed how to use categories of perverse sheaves on graded quiver varieties in order to investigate the cluster algebra $\mathcal{A}_Q$ associated with Q by Fomin–Zelevinsky [8]. He did so not only for Dynkin quivers but more generally for arbitrary bipartite quivers (where each vertex either has only incoming or only outgoing arrows). He showed that the dual Grothendieck ring associated with these categories (almost) yields a monoidal categorification of $\mathcal{A}_Q$ in the sense of Hernandez–Leclerc [16], who had constructed monoidal categorifications in types A_n and D_4 (they extend their results to all linearly oriented quivers of type A or D in their recent article [17]). Qin [37, 38] has generalized Nakajima's construction of graded quiver varieties to all acyclic quivers Q and Kimura–Qin [28] have used these varieties to extend Nakajima's results on cluster algebras to this generality.

In Section 9 of their remarkable study [18] of deformed Grothendieck rings of quantum affine algebras, Hernandez–Leclerc proved that the graded quiver varieties associated with certain special weights are isomorphic to varieties of representations of Q in such a way that Nakajima's stratification corresponds to the natural stratification by orbits. This description was extended by Leclerc–Plamondon [31], who showed that the quiver varieties in a much larger class are isomorphic to varieties of representations of the repetitive algebra [14, 19] associated with Q, where Nakajima's stratification again corresponds to the natural one by orbits. Let us call LP-varieties the graded quiver varieties covered by Leclerc–Plamondon's construction. Via Happel's equivalence [14] between the stable category of the repetitive algebra of Q and the derived category of Q, Leclerc–Plamondon's isomorphism yields a map from a given LP-variety to the set of isomorphism classes of the derived category of Q and, as shown in [31], the fibers of this map are precisely the Nakajima strata. In this article, we extend this last result in two directions simultaneously:

(1) from LP-varieties to all graded quiver varieties,

(2) from Dynkin quivers to arbitrary acyclic quivers (using Qin's definition [37,38] of graded quiver varieties).

Along the way, we obtain information on graded affine quiver varieties as well as on their desingularization by Nakajima's smooth (quasi-projective) graded quiver varieties. Among other results,

- we determine the quiver of the singular Nakajima category $\mathcal{S}$, whose representations form the (affine) graded quiver varieties,
- we determine the number of minimal relations between the vertices of the quiver of $\mathcal{S}$; remarkably, there are *no* relations if Q is a connected non-Dynkin quiver,
- we construct the stratifying functor Φ from the category of finite-dimensional $\mathcal{S}$-modules to the derived category of Q and use it to describe the strata and their closures in terms of the derived category,
- we describe the fibers of Nakajima's desingularization map using Φ in the spirit of theorems by Lusztig [32], Savage–Tingley [43] and Shipman [44],
- we extend Happel's equivalence [14] by showing that, for a Dynkin quiver Q, the singular category $\mathcal{S}$ is weakly Gorenstein and that its derived category of singularities is equivalent to the derived category of Q,
- we vastly generalize the preceding point by showing that for any configuration C of vertices of $\mathcal{S}$ satisfying a certain natural condition, the associated quotient $\mathcal{S}_C$ of $\mathcal{S}$ is weakly Gorenstein with associated derived category of singularities equivalent to the derived category of Q.

We refer to Section 2 for a more detailed description of our main results. In the companion paper [26], we show how to use Nakajima's desingularization map to generalize recent results by Cerulli–Feigin–Reineke [4, 5] on quiver Grassmannians.

Let us emphasize that throughout, we use framed quiver varieties. As shown by Crawley–Boevey [6], from the point of view of the geometry of the individual quiver varieties, the framing may be neglected. However, it is essential in the applications to quantum affine algebras and cluster algebras alluded to above as well as in the homological approach we use. We hope to come back to the relation of this approach with that of Frenkel–Khovanov–Schiffmann [9] in future work.

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2. Notation and main results

2.1. Repetition quivers and Happel's theorem. Let Q be a quiver. Thus, Q is an oriented graph given by a set of vertices Q_0, a set of arrows Q_1 and two maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ taking an arrow to its source vertex respectively its target vertex. We assume that Q is finite (both Q_0 and Q_1 are finite) and acyclic (there are no oriented cycles in Q).

The *repetition quiver* $\mathbb{Z}Q$, cf. [41], has the set of vertices $\mathbb{Z}Q_0$ formed by all pairs (i, p), where i belongs to Q_0 and p is an integer. For each arrow $\alpha : i \to j$, it has the arrows $(\alpha, p) : (i, p) \to (j, p)$ and $\sigma(\alpha, p) : (j, p-1) \to (i, p)$, where p runs through the integers. If β is an arbitrary arrow of $\mathbb{Z}Q$, we put $\sigma(\beta) = \sigma(\alpha, p)$ if $\beta = (\alpha, p)$ and $\sigma(\beta) = (\alpha, p-1)$ if $\beta = \sigma(\alpha, p)$. We denote by $\tau : \mathbb{Z}Q \to \mathbb{Z}Q$ the automorphism of $\mathbb{Z}Q$ given by the left translation by one unit: we have $\tau(i, p) = (i, p-1)$ and $\tau(\beta) = \sigma^2(\beta)$ for all $i \in Q_0$, $p \in \mathbb{Z}$, and for all arrows β of $\mathbb{Z}Q$. For example, when Q is the quiver $1 \to 2 \to 3$, the repetition quiver has the form given in Figure 1.

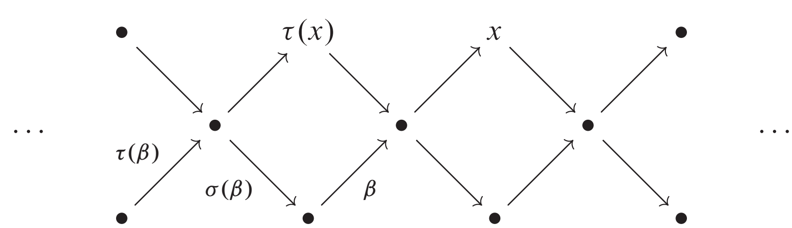


Figure 1. The repetition quiver $\mathbb{Z}Q$ for Q of type A_3.

Let k be a field. Following [11, 40], we define the *mesh category* $k(\mathbb{Z}Q)$ to be the k-category whose objects are the vertices of $\mathbb{Z}Q$ and whose morphism space from a to b

is the space of all k-linear combinations of paths from a to b modulo the subspace spanned by all elements ur_xv, where u and v are paths and

$$r_x = \sum_{\beta : y \to x} \sigma(\beta)\beta : \quad \begin{array}{ccccc} & & y_1 & & \\ & \overset{\sigma(\beta_1)}{\nearrow} & & \overset{\beta_1}{\searrow} & \\ \tau(x) & & \vdots & & x \\ & \underset{\sigma(\beta_s)}{\searrow} & & \underset{\beta_s}{\nearrow} & \\ & & y_s & & \end{array}$$

is the *mesh relator* associated with a vertex x of $\mathbb{Z}Q$. Here the sum runs over all arrows $\beta : y \to x$ of $\mathbb{Z}Q$. For example, in the mesh category $k(\mathbb{Z}\vec{A}_2)$ associated with the quiver $Q = \vec{A}_2 : 1 \to 2$, the composition of any two consecutive arrows vanishes. The computation of the morphism spaces in $k(\mathbb{Z}Q)$ is easy using additive functions, cf. [11, Section 6.5].

Let kQ be the path algebra of Q. It is a finite-dimensional, hereditary k-algebra. For each vertex i of Q, we write e_i for the associated idempotent of kQ (the 'lazy path at i') and $P_i = e_i kQ$ for the indecomposable projective kQ-module whose head is the simple module S_i concentrated at the vertex i. Let $\operatorname{mod} kQ$ be the category of all k-finite-dimensional right kQ-modules. Let $\mathcal{D}_Q$ be the bounded derived category $\mathcal{D}^b(\operatorname{mod} kQ)$. It is a Krull–Schmidt category [14] and a triangulated category. We write Σ for its shift (= suspension) functor. Let $\operatorname{ind}(\mathcal{D}_Q)$ be a full subcategory of $\mathcal{D}_Q$ whose objects form a set of representatives of the isomorphism classes of indecomposable objects of $\mathcal{D}_Q$. The following theorem is [14, Proposition 4.6] and [15, Theorem 5.6].

Theorem 2.2 (Happel, 1987). *There is a canonical fully faithful functor*

$$H : k(\mathbb{Z}Q) \to \operatorname{ind}(\mathcal{D}_Q)$$

taking each vertex $(i, 0)$ *to the indecomposable projective module* P_i, $i \in Q_0$. *It is an equivalence iff* Q *is a Dynkin quiver (i.e. its underlying graph is a disjoint union of ADE Dynkin diagrams).*

The dichotomy between Dynkin quivers and non-Dynkin quivers which appears in this theorem is responsible for the distinction between these two cases which we have to introduce in many of our proofs. Let $\nu : \mathcal{D}_Q \to \mathcal{D}_Q$ be the autoequivalence given by the derived tensor product with the k-dual of kQ considered as a bimodule. We have an isomorphism, bifunctorial in $L, M \in \mathcal{D}_Q$,

$$D \operatorname{Hom}_{\mathcal{D}_Q}(L, M) = \operatorname{Hom}_{\mathcal{D}_Q}(M, \nu L), \tag{2.2.1}$$

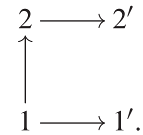
where D denotes the duality over k. This means that ν is the *Serre functor of* $\mathcal{D}_Q$. As shown in [14], via the embedding H, the autoequivalence τ of the mesh category corresponds to the *Auslander–Reiten translation* $\tau_{\mathcal{D}_Q} = \Sigma^{-1}\nu$, which we will also denote by τ. For Dynkin quivers, the combinatorial descriptions of ν (equivalently: Σ) and of the image of $\operatorname{mod} kQ$ in $\mathcal{D}_Q$ are given in [11, Section 6.5].

For later use, we record the following isomorphism, which follows from Serre duality (2.2.1): For $L, M \in \mathcal{D}_Q$ and $p \in \mathbb{Z}$, we have

$$D \operatorname{Ext}^p_{\mathcal{D}_Q}(L, M) = \operatorname{Hom}_{\mathcal{D}_Q}(M, \Sigma^{-(p-1)}\tau L), \tag{2.2.2}$$

where, as usual, we write $\operatorname{Ext}^p_{\mathcal{D}_Q}(L, M)$ for $\operatorname{Hom}_{\mathcal{D}_Q}(L, \Sigma^p M)$.

2.3. Graded affine quiver varieties. Let Q be a finite acyclic quiver as in Section 2.1 and let k be the field of complex numbers. The *framed quiver* $\widetilde{Q}$ is obtained from Q by adding, for each vertex i, a new vertex i' and a new arrow $i \to i'$. For example, if Q is the quiver $1 \to 2$, the framed quiver is



Let $\mathbb{Z}\widetilde{Q}$ be the repetition quiver of $\widetilde{Q}$. We refer to the vertices (i', p), $i \in Q_0$, $p \in \mathbb{Z}$, as the *frozen vertices* of $\mathbb{Z}\widetilde{Q}$ and mark them by squares as in the examples in Figure 2 associated with quivers whose underlying graphs are A_2 respectively D_4. For a vertex $x = (i, p)$, we put $\sigma(x) = (i', p-1)$ and for a vertex (i', p), we put $\sigma(i', p) = (i, p-1)$.

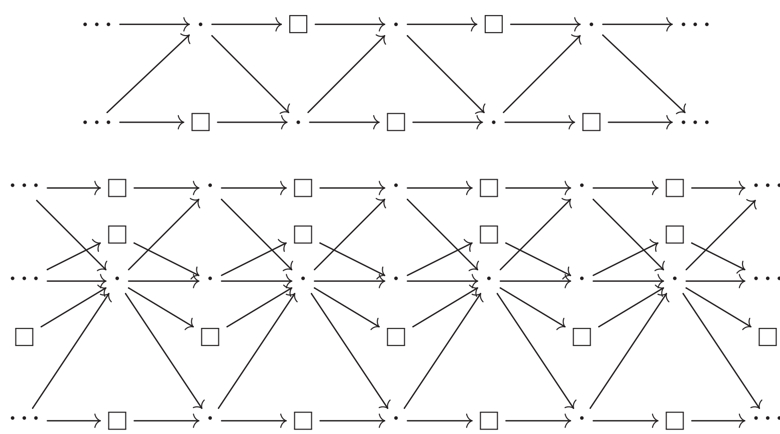


Figure 2. The quivers $\mathbb{Z}\widetilde{Q}$ associated with A_2 and D_4.

The *regular (or smooth) Nakajima category* $\mathcal{R}$ is the mesh category $k(\mathbb{Z}\widetilde{Q})$, where we take into account the presence of frozen vertices by *only imposing* the mesh relations r_x associated with *non-frozen vertices* x. The *singular Nakajima category* $\mathcal{S}$ is the full subcategory of $\mathcal{R}$ whose objects are the frozen vertices. In the main body of this article, we will work more generally with the quotient $\mathcal{S}_C$ of $\mathcal{S}$ associated with a configuration of vertices C, cf. Section 3.3. This generality will in particular ensure that our results do contain those of [31] as special cases. For simplicity, in this description of the main results, we restrict ourselves to the case where $\mathcal{S}_C = \mathcal{S}$. We write $\mathcal{R}_0$ and $\mathcal{S}_0$ for the sets of objects of the categories $\mathcal{R}$ and $\mathcal{S}$. An $\mathcal{S}$*-module* is a k-linear functor $M : \mathcal{S}^{\mathrm{op}} \to \operatorname{Mod} k$, where $\operatorname{Mod} k$ is the category of k-vector spaces (cf. Section 3.1) Let $w : \mathcal{S}_0 \to \mathbb{N}$ be a *dimension vector*, i.e. a function with finite support. The *affine graded quiver variety* $\mathcal{M}_0(w)$ is the variety of $\mathcal{S}$-modules M such that $M(u) = k^{w(u)}$ for each vertex u in $\mathcal{S}_0$. Notice that such a module is given by the images of the morphisms of $\mathcal{S}$ and that these have to satisfy all the relations that hold in $\mathcal{S}$. This shows that the set $\mathcal{M}_0(w)$ canonically identifies with a Zariski closed subset of the finite-dimensional affine space

$$\prod_{u_1, u_2} \operatorname{Hom}_k(\operatorname{Hom}_{\mathcal{S}}(u_1, u_2), k^{w(u_2) \times w(u_1)}),$$

where the product ranges over all objects u_1, u_2 of $\mathcal{S}$. Thus, the set $\mathcal{M}_0(w)$ becomes indeed canonically an affine variety. By [31, Theorem 2.4], based on [29,32], this definition of $\mathcal{M}_0(w)$ is equivalent to Nakajima's original definition in [35] when Q is a Dynkin quiver. The proof of [31] also shows that when Q is bipartite (each vertex is a source or a sink), our definition of $\mathcal{M}_0(w)$ agrees with Nakajima's in [36] and when Q is an arbitrary acyclic quiver with Kimura–Qin's in [28].

Neither the original definition of $\mathcal{M}_0(w)$ nor the above variant are very explicit. However, we can make the above definition more explicit by describing the category $\mathcal{S}$ by its quiver $Q_{\mathcal{S}}$ with an admissible set of relations, cf. [12, Chapter 8] and [1, Section II.3]. Since the objects of $\mathcal{S}$ are pairwise non-isomorphic, we can identify the set of vertices of $Q_{\mathcal{S}}$ with $\mathcal{S}_0$ and then the number of arrows from $\sigma(y)$ to $\sigma(x)$ in $Q_{\mathcal{S}}$ equals

$$\dim \operatorname{Ext}^1_{\mathcal{S}}(S_{\sigma(x)}, S_{\sigma(y)}),$$

where $S_{\sigma(x)}$ is the simple module associated with $\sigma(x)$. Moreover, the number of relations from $\sigma(y)$ to $\sigma(x)$ equals

$$\dim \operatorname{Ext}^2_{\mathcal{S}}(S_{\sigma(x)}, S_{\sigma(y)}).$$

Theorem 2.4 (Corollary 3.10). *For each integer $p \geq 1$ and all vertices x, y of $\mathbb{Z}Q$, we have a canonical isomorphism*

$$\operatorname{Ext}^p_{\mathcal{S}}(S_{\sigma(x)}, S_{\sigma(y)}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}_Q}(H(x), \Sigma^p H(y)),$$

where H is Happel's embedding (Theorem 2.2). These spaces vanish if no connected component of Q is a Dynkin quiver and $p \geq 2$.

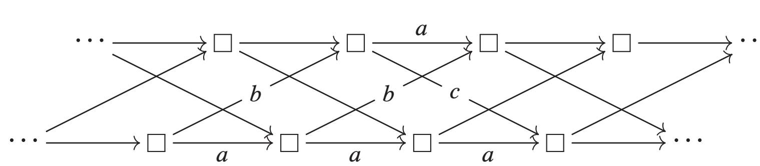
Thanks to the theorem and to formula (2.2.2), we find that the number of arrows, respectively minimal relations, from $\sigma(x)$ to $\sigma(y)$ equals

$$\dim \operatorname{Hom}_{\mathcal{D}_Q}(H(x), \tau H(y)) \text{ respectively } \dim \operatorname{Hom}_{\mathcal{D}_Q}(H(x), \Sigma^{-1}\tau H(y)).$$

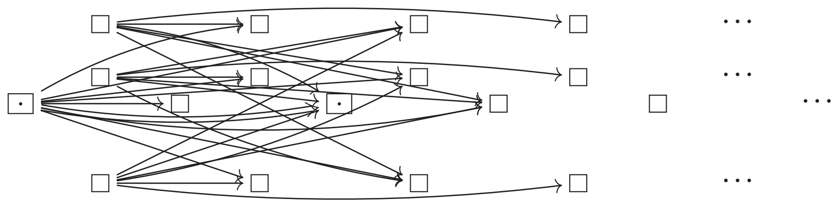
It is not hard to see that this last dimension vanishes if no connected component of Q is a Dynkin quiver. Thus, we obtain the following corollary.

Corollary 2.5. *If Q is connected and not a Dynkin quiver, then for each dimension vector w, the graded affine quiver variety $\mathcal{M}_0(w)$ is isomorphic to an affine space.*

Let us consider two examples of Dynkin quivers: For the quiver $Q : 1 \to 2$, we find that $Q_{\mathcal{S}}$ is the quiver



and that $\mathcal{S}$ is isomorphic to the path category of $Q_{\mathcal{S}}$ modulo the ideal generated by all relations of the form $ab - ba$, $ac - ca$ and $a^3 - cb$ (we denote all horizontal arrows by a, all rising arrows by b and all descending arrows by c). This example is deceptively simple. The great complexity of the category $\mathcal{S}$ becomes visible when we look at the quiver of $\mathcal{S}$ in the case of D_4. In the following drawing, we only depict the arrows which start at the leftmost vertex on each row.



Thus, the complete quiver is obtained from the one displayed by adding all translates of the indicated arrows. Notice that there is a double arrow between the two vertices marked by a dotted box. This implies that the variety of representations with dimension vector (d_1, d_2) of the Kronecker quiver $1 \rightrightarrows 2$ is isomorphic to the graded affine quiver variety $\mathcal{M}_0(w)$ of type D_4 with dimension vector w such that $w(\sigma(x)) = d_1$, $w(\sigma(\tau^{-2}(x)) = d_2$ for $x = (0, 1)$ and $w(y) = 0$ for all other frozen vertices y. The stratification of this variety given by the orbits of the base change group $\mathrm{GL}_{d_1} \times \mathrm{GL}_{d_2}$ has infinitely many strata already for $(d_1, d_2) = (1, 1)$. On the other hand, the Nakajima stratification, which we will recall in the next section, always has finitely many strata.

2.6. Stratification. We keep the assumptions of Section 2.3. In particular, Q is an acyclic quiver and $\mathcal{R}$ and $\mathcal{S}$ are the associated regular and singular Nakajima categories. Let $v : \mathcal{R}_0 \setminus \mathcal{S}_0 \to \mathbb{N}$ and $w : \mathcal{S}_0 \to \mathbb{N}$ be dimension vectors. Let $\widetilde{\mathcal{M}}(v, w)$ be the set of $\mathcal{R}$-modules M such that

$$M(x) = k^{v(x)}, \quad M(\sigma(x)) = k^{w(\sigma(x))} \quad \text{for all } x \in \mathbb{Z}Q_0$$

and that M is *stable*, i.e. we have $\mathrm{Hom}_{\mathcal{R}}(S_x, M) = 0$ for each simple module S_x associated with a non-frozen vertex $x \in \mathbb{Z}Q_0$. Equivalently, M does not contain any non-zero submodule supported only on non-frozen vertices. Let G_v be the product of the groups $\mathrm{GL}(k^{v(x)})$, where x runs through the non-frozen vertices. By base change in the spaces $k^{v(x)}$, the group G_v acts freely on the set $\widetilde{\mathcal{M}}(v, w)$. The *graded quiver variety* $\mathcal{M}(v, w)$ is the quotient $\widetilde{\mathcal{M}}(v, w)/G_v$. For this definition and the following facts, we refer to Nakajima's work [35, 36] for the case where Q is Dynkin or bipartite and to Qin [37, 38] and Kimura–Qin [28] for the extension to the case of an arbitrary acyclic quiver Q. The set $\mathcal{M}(v, w)$ canonically becomes a smooth quasi-projective variety and the projection map

$$\pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w)$$

taking an $\mathcal{R}$-module M to its restriction $M|_{\mathcal{S}}$ is a proper map. Moreover, when v varies, the graded affine quiver variety $\mathcal{M}_0(w)$ is stratified by the images of the non-empty ones among the open subsets $\mathcal{M}^{\mathrm{reg}}(v, w) \subset \mathcal{M}(v, w)$ formed by the classes of the modules $M \in \mathcal{M}(v, w)$ which, in addition, are *co-stable*, i.e. we have $\mathrm{Hom}_{\mathcal{R}}(M, S_x) = 0$ for each non-frozen vertex x (by [37, Proposition 4.1.3.8], this is equivalent to Nakajima's original description). The morphism π induces an isomorphism of each $\mathcal{M}^{\mathrm{reg}}(v, w)$ onto its image in $\mathcal{M}_0(w)$.

Recall that a *δ-functor* from an abelian to a triangulated category is (roughly) an additive functor transforming short exact sequences into triangles, cf. e.g. [25]. If no connected component of Q is a Dynkin quiver, let $\mathcal{V}$ denote the additive subcategory of $\mathcal{D}_Q$ whose indecomposable objects are the sums of objects in the image of Happel's embedding. The category $\mathcal{V}$ becomes exact when endowed with all the sequences giving rise to triangles in $\mathcal{D}_Q$.

Theorem 2.7 (Sections 4.1 and 4.12). *There is a canonical δ-functor*

$$\Phi : \operatorname{mod} \mathcal{S} \to \mathcal{D}_Q$$

taking the simple module $S_{\sigma(x)}$ associated with $x \in \mathbb{Z}Q_0$ to $H(x)$ (cf. Theorem 2.2*) and such that two modules M_1, M_2 belonging to $\mathcal{M}_0(w)$ lie in the same stratum if and only if $\Phi(M_1)$ is isomorphic to $\Phi(M_2)$ in the derived category $\mathcal{D}_Q$. Moreover, if no connected component of Q is a Dynkin quiver, then Φ arises from an exact functor* $\operatorname{mod} \mathcal{S} \to \mathcal{V}$.

The theorem is inspired by results obtained for Dynkin quivers and particular choices of w by Hernandez–Leclerc [18] and by Leclerc–Plamondon [31]. It suggests that the varieties $\mathcal{M}_0(w)$ should be related to the moduli stack of objects of $\mathcal{D}_Q$ introduced and studied by Toën–Vaquié [45]. The following theorem further underlines the geometric relevance of the derived category.

Theorem 2.8 (Section 4.18). *Under the bijection between strata of $\mathcal{M}_0(w)$ and isomorphism classes in its image under Φ, the degeneration order among strata corresponds to the degeneration order of Jensen–Su–Zimmermann* [22] *among isomorphism classes in the derived category $\mathcal{D}_Q$.*

Note that for Dynkin quivers Q, the degeneration order on strata of LP-varieties coincides with the degeneration order on orbits in the representation spaces of the repetitive algebra and also with the Hom-order on isomorphism classes of representations of the repetitive algebra, cf. [31, Remark 3.15].

Now consider the projection π as a morphism $\coprod_v \mathcal{M}(v, w) \to \mathcal{M}_0(w)$. The following theorem is a consequence of Nakajima's slice theorem (see [35, Section 3.3] and [28, Section 2.4]):

Theorem 2.9 (Section 4.19). *For each module $M \in \mathcal{M}_0(w)$, the fiber $\pi^{-1}(\{M\})$ is homeomorphic to the Grassmannian of $\mathcal{D}_Q$-submodules of the right $\mathcal{D}_Q$-module*

$$D \operatorname{Hom}_{\mathcal{D}_Q}(\Phi(M), ?) : \mathcal{D}_Q^{\mathrm{op}} \to \operatorname{mod} k.$$

Notice that each fiber contains a distinguished point: the zero submodule. It corresponds to the pre-image of M under the isomorphism induced by π from a suitable $\mathcal{M}^{\mathrm{reg}}(v, w)$ onto the unique stratum containing M.

2.10. Description of Φ via Kan extensions. Recall that a k-category is a category whose morphism spaces carry k-vector space structures such that the composition is bilinear. For a k-category $\mathcal{C}$, let $\operatorname{Mod}(\mathcal{C})$ denote the category of all *right $\mathcal{C}$-modules*, i.e. all k-linear functors $M : \mathcal{C}^{\mathrm{op}} \to \operatorname{Mod} k$, cf. Section 3.1.

The inclusion $\mathcal{S} \to \mathcal{R}$ yields the restriction functor $\operatorname{res} : \operatorname{Mod}(\mathcal{R}) \to \operatorname{Mod}(\mathcal{S})$. This functor admits a left adjoint K_L and a right adjoint K_R, the *left* and the *right Kan extension*, cf. [33]:

$$\begin{array}{c} \operatorname{Mod}(\mathcal{R}) \\ K_L \Big\uparrow \; \Big\downarrow \mathrm{res} \; \Big\uparrow K_R \\ \operatorname{Mod}(\mathcal{S}). \end{array}$$

As we will see in Section 4.3, they have simple and concrete descriptions. Both Kan extensions are fully faithful (and so res is a localization of abelian categories in the sense of [10]). They are linked by a canonical morphism

$$\text{can} : K_L \to K_R. \tag{2.10.1}$$

By definition, the *intermediate Kan extension* K_{LR} is its image, so that we have canonical morphisms

$$K_L \twoheadrightarrow K_{LR} \rightarrowtail K_R. \tag{2.10.2}$$

The functor K_{LR} restricted to certain subcategories plays an important role in [4]. For special vectors w, the following proposition follows from [31, Section 3.3].

Proposition 2.11 (Section 4.9). *Let $w : \mathcal{S}_0 \to \mathbb{N}$ be a dimension vector. Further, let $M \in \mathcal{M}_0(w)$. Then the module $K_{LR}(M)$ is both stable and co-stable and thus yields a point $\widetilde{M}$ in $\mathcal{M}^{\text{reg}}(v, w)$ for a suitable v. The unique stratum containing M is $\pi(\mathcal{M}^{\text{reg}}(v, w))$ and $\widetilde{M}$ is the unique pre-image of M under $\pi : \mathcal{M}^{\text{reg}}(v, w) \to \mathcal{M}_0(w)$.*

It is not hard to check that K_{LR} is in fact an equivalence from $\text{Mod}(\mathcal{S})$ onto the full subcategory of $\text{Mod}(\mathcal{R})$ whose objects are the modules which are both stable and co-stable. The geometric meaning of the functor taking a stable $\mathcal{R}$-module L to $K_{LR}(\text{res}(L))$ is given by the following proposition, which is essentially implicit in Nakajima's work [35].

Proposition 2.12 (Section 4.10). *If L is a stable $\mathcal{R}$-module belonging to $\mathcal{M}(v, w)$ and $K_{LR}(\text{res}\, L)$ is of dimension vector (v^0, w), then the unique closed G_v-orbit in the closure of $G_v L$ in the affine variety $\text{rep}(\mathcal{R}^{\text{op}}, v, w)$ of representations of $\mathcal{R}^{\text{op}}$ of dimension vector (v, w) is that of $K_{LR}(\text{res}\, L) \oplus S$, where S denotes the semi-simple $k(\mathbb{Z}Q)$-module of dimension vector $v - v^0$.*

By applying the above proposition to $L = K_{LR}(M)$ for an $\mathcal{S}$-module M (notice that $\text{res}(K_{LR}(M))$ identifies with M), we see in particular that the G_v-orbit of $K_{LR}(M)$ is closed in the variety $\text{rep}(\mathcal{R}^{\text{op}}, v, w)$.

For each $\mathcal{S}$-module M, the morphisms $K_L(M) \to K_{LR}(M) \to K_R(M)$ become invertible when restricted to $\mathcal{S}$. Thus, the modules $CK(M)$ and $KK(M)$ defined by

$$KK(M) = \ker(K_L(M) \to K_{LR}(M)),$$
$$CK(M) = \text{cok}(K_{LR}(M) \to K_R(M))$$

vanish on $\mathcal{S}$. Now we have an obvious isomorphism $\mathcal{R}/\langle \mathcal{S} \rangle \xrightarrow{\sim} k(\mathbb{Z}Q)$, where $\langle \mathcal{S} \rangle$ is the ideal generated by the identical morphisms of $\mathcal{S}$. Therefore, we may view $CK(M)$ and $KK(M)$ as $k(\mathbb{Z}Q)$-modules. The following proposition shows in particular that these modules are injective respectively projective and that KK and CK determine Φ.

Proposition 2.13 (Section 5.20). *For $M \in \text{mod}(\mathcal{S})$, we have functorial isomorphisms of $k(\mathbb{Z}Q)$-modules*

$$KK(M) = \text{Hom}_{\mathcal{D}_Q}(H(?), \tau\Phi(M)) \quad \textit{and} \quad CK(M) = D\,\text{Hom}_{\mathcal{D}_Q}(\Phi(M), H(?)),$$

where H is Happel's embedding (Theorem 2.2).

2.14. Gorenstein homological algebra. Let us assume that Q is connected. The construction of the stratifying functor Φ of Theorem 2.7 is given in Section 4.1. The proof of its exactness properties is quite different depending on whether Q is a Dynkin quiver or not: If Q is not Dynkin, we give a direct argument in Section 4.22. In the Dynkin case, we use Gorenstein homological algebra (cf. Section 5): Let us assume that Q is a Dynkin quiver and $\mathcal{S}$ the associated singular Nakajima category. Recall that an $\mathcal{S}$-module M is *finitely presented* if there is a projective presentation

$$P_1 \to P_0 \to M \to 0$$

with finitely generated projective modules P_0 and P_1. An $\mathcal{S}$-module M is *right-bounded* if, for all $p \gg 0$, the space $M(i, p)$ vanishes for all $i \in Q_0$; it is *pointwise finite-dimensional* if all the spaces $M(u)$, $u \in \mathcal{S}_0$, are finite-dimensional.

Proposition 2.15 (Sections 5.6 and 5.9). *The following hold.*

(a) *The category $\mathcal{S}$ is coherent, i.e. its category of finitely presented modules is abelian.*

(b) *The category $\mathcal{S}$ is weakly Gorenstein of dimension* 1 *in the sense that, for all $p > 1$, we have*

$$\mathrm{Ext}^p_{\mathcal{S}}(M, P) = 0$$

for each right-bounded pointwise finite-dimensional module M and each finitely generated projective module P.

An $\mathcal{S}$-module M is *Gorenstein-projective* if $\mathrm{Ext}^p_{\mathcal{S}}(M, P) = 0$ for all $p > 1$ and all finitely generated projective $\mathcal{S}$-modules P. Let $\mathrm{gpr}(\mathcal{S})$ (resp. $\mathrm{gin}(\mathcal{S})$) be the category of finitely presented Gorenstein-projective (resp. Gorenstein-injective) $\mathcal{S}$-modules. For each finitely generated $\mathcal{S}$-module M, let ΩM be the kernel of a surjective morphism $P \to M$, where P is finitely generated projective.

Theorem 2.16 (Sections 5.12 and 5.16). *The category* $\mathrm{gpr}(\mathcal{S})$ *is a Frobenius category. There is a canonical equivalence from its stable category* $\underline{\mathrm{gpr}}(\mathcal{S})$ *to* $\mathcal{D}_Q$ *sending* $\Omega S_{\sigma(x)}$ *to* $H(x)$ *(Theorem* 2.2*) for each* $x \in \mathbb{Z}Q_0$.

Now the functor Φ is obtained as the composition

$$\mathrm{mod}(\mathcal{S}) \xrightarrow{\Omega} \underline{\mathrm{gpr}}(\mathcal{S}) \xrightarrow{\sim} \mathcal{D}_Q,$$

which shows in particular that it is a δ-functor.

Let $\mathrm{proj}(\mathcal{R})$ denote the category of the finitely generated projective $\mathcal{R}$-modules. The following theorem allows us to view the regular category $\mathcal{R}$ as an Auslander category for the Gorenstein projective $\mathcal{S}$-modules.

Theorem 2.17 (Section 5.21). *The restriction functor induces equivalences*

$$\mathrm{proj}(\mathcal{R}) \to \mathrm{gpr}(\mathcal{S}) \quad \textit{and} \quad \mathrm{inj}(\mathcal{R}) \to \mathrm{gin}(\mathcal{S}).$$

It yields isomorphisms from the quiver $\mathbb{Z}\widetilde{Q}$ onto the Auslander–Reiten quivers of $\mathrm{gpr}(\mathcal{S})$ *and* $\mathrm{gin}(\mathcal{S})$ *so that the frozen vertices correspond to the projective-injective vertices.*

In particular, we obtain that $\mathrm{proj}(\mathcal{R})$ admits a natural structure of standard (in the sense of [42, Section 2.3, p. 63]) Frobenius category whose projectives are the finite direct sums of indecomposable projectives associated with the frozen vertices of $\mathbb{Z}\widetilde{Q}$ and where each mesh ending in a non-frozen vertex yields an Auslander–Reiten conflation. More generally, in Section 5, we will prove the above results (except coherence) for the quotients $\mathcal{R} \to \mathcal{R}_C$ and $\mathcal{S} \to \mathcal{S}_C$ associated with suitable configurations C, cf. Section 3.3. In the philosophy of [21], the Frobenius category $\mathcal{E} = \mathrm{proj}(\mathcal{R}_C)$ 'admits a resolution', namely itself, and so one expects its category of projectives $\mathrm{proj}(\mathcal{S}_C)$ to be Gorenstein and the category itself to be equivalent to the category Gorenstein-projective modules over $\mathrm{proj}(\mathcal{S}_C)$. Technically, the categories we consider do not quite fit into the framework of [21] but the philosophy of that paper is compatible with our findings.

3. Homological properties of the Nakajima categories

3.1. Notations and recollections. Let k be a field and let $\mathrm{Mod}\, k$ be the category of k-vector spaces. Recall that a *k-category* is a category whose morphism spaces are endowed with k-vector space structures such that the composition is bilinear. Let $\mathcal{C}$ be a k-category. We denote by $\mathrm{Mod}(\mathcal{C})$ the category of *right $\mathcal{C}$-modules*, i.e. k-linear functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}(k)$. For each object x of $\mathcal{C}$, we have the *free module*

$$x^\wedge = x^\wedge_{\mathcal{C}} = \mathcal{C}(?, x) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\, k$$

and the *cofree module*

$$x^\vee = x^\vee_{\mathcal{C}} = D(\mathcal{C}(x, ?)) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}\, k.$$

Here, we write $\mathcal{C}(u, v)$ for the space of morphisms $\mathrm{Hom}_{\mathcal{C}}(u, v)$ and D for the duality over the ground field k. For each object x of $\mathcal{C}$ and each $\mathcal{C}$-module M, we have canonical isomorphisms

$$\text{(3.1.1)} \qquad \mathrm{Hom}(x^\wedge, M) = M(x) \quad \text{and} \quad \mathrm{Hom}(M, x^\vee) = D(M(x)).$$

In particular, the module $x^\wedge$ is projective and $x^\vee$ is injective. A module is *finitely generated* if it is a quotient of a finite direct sum of modules $x^\wedge$; it is *finitely cogenerated* if it is a submodule of a finite direct sum of modules $x^\vee$. If x is an object of $\mathcal{C}$ whose endomorphism algebra is local, then the free module $x^\wedge$ admits a unique *simple quotient* S_x, which is also the unique simple submodule of $x^\vee$. By Kaplansky's theorem [23], if the endomorphism ring of each object x of $\mathcal{C}$ is local, each projective module over $\mathcal{C}$ is a direct sum of free modules $x^\wedge$.

Let us make the following assumptions on $\mathcal{C}$.

Assumption 3.2. (a) The morphism spaces of $\mathcal{C}$ are finite-dimensional.

(b) The category $\mathcal{C}$ is *directed*, i.e. the endomorphism algebra of each object is k and $\mathcal{C}$ is endowed with an order relation such that $\mathcal{C}(x, y) \neq 0$ implies $x \leq y$.

A $\mathcal{C}$-module M is *pointwise finite-dimensional* if $M(x)$ is finite-dimensional for each object x of $\mathcal{C}$. It is *right bounded* if there is a finite set of objects E such that each object x with $M(x) \neq 0$ is less than or equal to an object of E. For example, the modules $x^\wedge$ are pointwise finite-dimensional and right bounded. Let M be a pointwise finite-dimensional, right bounded module. Then M admits a projective cover $P \to M$ by a (usually infinite) coproduct P of free modules $x^\wedge$ and the multiplicity of $x^\wedge$ in P is finite and equals the dimension of $\operatorname{Hom}(M, S_x)$. Moreover, the kernel of $P \to M$ is again right bounded and pointwise finite-dimensional. Thus, the module M admits a minimal projective resolution

$$\cdots \to P_1 \to P_0 \to M \to 0,$$

where each object P_i is a coproduct of free modules $x^\wedge$ and the multiplicity of $x^\wedge$ in P_i equals the dimension of $\operatorname{Ext}^i(M, S_x)$.

3.3. Resolutions for the simple $\mathcal{R}_C$-modules. Let Q be a connected acyclic quiver and let C be a subset of the set of vertices of the repetition quiver $\mathbb{Z}Q$. Let $\mathcal{R}_C$ be the quotient of $\mathcal{R}$ by the ideal generated by the identities of the frozen vertices not belonging to $\sigma^{-1}(C)$ and let $\mathcal{S}_C$ be the full subcategory of $\mathcal{R}_C$ formed by the vertices in $\sigma^{-1}(C)$. We make the following assumption on C.

Assumption 3.4. For each non-frozen vertex x of $\mathbb{Z}\widetilde{Q}$, the sequences

(3.4.1)
$$0 \to \mathcal{R}_C(?, x) \to \bigoplus_{x \to y} \mathcal{R}_C(?, y),$$
$$0 \to \mathcal{R}_C(x, ?) \to \bigoplus_{y \to x} \mathcal{R}_C(y, ?)$$

are exact, where the sums range over all arrows of $\mathbb{Z}\widetilde{Q}$ whose source (respectively, target) is x.

The assumption holds, for example, if C is the set of all vertices of $\mathbb{Z}Q$. It also holds in the following situation: Assume that $\mathcal{E}$ is a Hom-finite exact Krull–Schmidt category which is standard (in the sense of [42, Section 2.3, p. 63]) and whose Auslander–Reiten quiver is the full subquiver of $\mathbb{Z}\widetilde{Q}$ formed by the non-frozen vertices and the vertices $\sigma^{-1}(c)$, $c \in C$, where the latter correspond to the projective indecomposables of $\mathcal{E}$. Then the sequences (3.4.1) are associated with Auslander–Reiten conflations of $\mathcal{E}$ and hence are exact. For example, one can take $\mathcal{E}$ to be the category of finite-dimensional modules over the repetitive algebra of an iterated tilted algebra B of Dynkin type. The case where B itself is the path algebra of a Dynkin quiver was considered by Leclerc–Plamondon [31].

In fact, as we will see in Theorem 5.23, when the assumption holds, the given set C always comes from the choice of a Hom-finite exact Krull–Schmidt category which is moreover standard and whose stable Auslander–Reiten quiver is $\mathbb{Z}Q$.

Another sufficient condition for the assumption to hold is due to Iyama: According to [20, Section 7.4 (2)], the assumption holds if for each vertex x of $\mathbb{Z}Q$, there is a vertex c in C such that there is a non-zero morphism from x to c in the mesh-category $\mathcal{R}_C$. Notice that for this it is sufficient that the following condition (R) holds:

(R) for each vertex x of $\mathbb{Z}Q$, there is a vertex c in C such that the space of morphisms $k(\mathbb{Z}Q)(x, c)$ in the mesh category of $\mathbb{Z}Q$ does not vanish.

This is the first condition which Riedtmann imposed on the 'combinatorial configurations' in her sense, cf. [41, Definition 2.3] and [3].

Returning to the general setup we have the following lemma.

Lemma 3.5. *For each (non-frozen) vertex x of $\mathbb{Z}Q$, we have (co-)resolutions of simple $\mathcal{R}_C$-modules:*

(a) $0 \to \tau(x)^\wedge \to \sigma(x)^\wedge \to S_{\sigma(x)} \to 0$,

(b) $0 \to S_{\sigma(x)} \to \sigma(x)^\vee \to x^\vee \to 0$,

(c) $0 \to \tau(x)^\wedge \to \bigoplus_{y \to x} y^\wedge \to x^\wedge \to S_x \to 0$,

(d) $0 \to S_x \to x^\vee \to \bigoplus_{x \to y} y^\vee \to \tau^{-1}(x)^\vee \to 0$,

where in (c) *and* (d), *the sum ranges over all arrows $y \to x$ with target x in the quiver $\mathbb{Z}\widetilde{Q}$, respectively all arrows $x \to y$ with source x.*

The proof is an exercise. By the isomorphisms (3.1.1), we immediately obtain the following corollary.

Corollary 3.6. *For each $\mathcal{R}_C$-module M and each vertex x of $\mathbb{Z}Q_0$, we have canonical isomorphisms in the derived category of vector spaces, where the first term on the right is always in degree* 0:

(a) $\mathrm{RHom}(M, S_{\sigma(x)}) = (DM(\sigma(x)) \to DM(x))$,

(b) $\mathrm{RHom}(S_{\sigma(x)}, M) = (M(\sigma(x)) \to M(\tau(x)))$,

(c) $\mathrm{RHom}(M, S_x) = (DM(x) \to \bigoplus_{x \to y} DM(y) \to DM(\tau^{-1}(x)))$,

(d) $\mathrm{RHom}(S_x, M) = (M(x) \to \bigoplus_{y \to x} M(y) \to M(\tau(x)))$.

(e) *In particular, we have a canonical isomorphism*

$$D\,\mathrm{RHom}(S_x, M) = \mathrm{RHom}(M, \Sigma^2 S_{\tau(x)}), \tag{3.6.1}$$

where Σ denotes the suspension functor.

Let $\langle C \rangle$ denote the ideal of $\mathcal{R}_C$ generated by the identities of the vertices in $\sigma^{-1}(C)$. By Happel's theorem (2.2), we have a fully faithful embedding

$$H : \mathcal{R}_C/\langle C \rangle \to \mathcal{D}_Q,$$

which is an equivalence if and only if Q is a Dynkin quiver. If Q is Dynkin, let Σ be the unique bijection of the vertices of $\mathbb{Z}Q$ such that

$$H(\Sigma x) = \Sigma H(x).$$

If Q is arbitrary acyclic, for each non-frozen vertex $x \in \mathbb{Z}Q_0$, let

$$x^\wedge_{\mathcal{D}} = (\mathcal{R}_C/\langle C \rangle)(?, x) = \mathcal{D}_Q(?, x),$$
$$x^\vee_{\mathcal{D}} = D(\mathcal{R}_C/\langle C \rangle)(x, ?)) = D\mathcal{D}_Q(x, ?),$$

where, for simplicity, we omit the Happel functor H from the notations. Moreover, put

$$P_C(x) = \bigoplus_{\sigma(y) \in C} \mathcal{D}_Q(y, x) \otimes \sigma(y)^\wedge, \tag{3.6.2}$$
$$I_C(x) = \prod_{\sigma^{-1}(y) \in C} D\mathcal{D}_Q(x, y) \otimes \sigma^{-1}(y)^\vee.$$

Theorem 3.7. *The following hold.*

(a) *Suppose that Q is a Dynkin quiver. For each non-frozen vertex $x \in \mathbb{Z}Q_0$, we have a resolution of $\mathcal{R}_C$-modules*

$$0 \to (\Sigma^{-1}x)^\wedge \to P_C(x) \to x^\wedge \to x^\wedge_{\mathcal{D}} \to 0 \tag{3.7.1}$$

and a coresolution

$$0 \to x^\vee_{\mathcal{D}} \to x^\vee \to I_C(x) \to (\Sigma x)^\vee \to 0. \tag{3.7.2}$$

(b) *Suppose that Q is not a Dynkin quiver. For each non-frozen vertex $x \in \mathbb{Z}Q_0$, we have a resolution of $\mathcal{R}_C$-modules*

$$0 \to P_C(x) \to x^\wedge \to x^\wedge_{\mathcal{D}} \to 0$$

and a coresolution

$$0 \to x^\vee_{\mathcal{D}} \to x^\vee \to I_C(x) \to 0.$$

Proof. Note that the category $\mathcal{R}_C$ satisfies Assumption 3.2. Thus, to check the claims, it suffices to compute the extensions between the simple modules S_u, where u is any vertex of $\mathbb{Z}\widetilde{Q}$, and $x^\wedge_{\mathcal{D}}$ respectively $x^\vee_{\mathcal{D}}$. For this, we use Lemma 3.5. Let y be a non-frozen vertex. We have

$$\mathrm{RHom}(x^\wedge_{\mathcal{D}}, S_{\sigma(y)}) = \mathrm{RHom}(x^\wedge_{\mathcal{D}}, \sigma(y)^\vee \to y^\vee) = (0 \to D\mathcal{D}_Q(y,x)).$$

This yields the term $P_C(x)$ in the resolution (3.7.1). Similarly, we find

$$\begin{aligned}\mathrm{RHom}(x^\wedge_{\mathcal{D}}, S_y) &= \mathrm{RHom}(x^\wedge, (y^\vee \to \bigoplus_{y\to z} z^\vee \to \tau^{-1}(y)^\vee)) \\ &= (D\mathcal{D}_Q(y,x) \to \prod_{y\to z} D\mathcal{D}_Q(z,x) \to D\mathcal{D}_Q(\tau^{-1}(y),x)).\end{aligned}$$

This complex is also obtained by applying $\mathrm{Hom}(x^\wedge, ?)$ to the complex of $\mathcal{D}_Q$-modules

$$y^\vee_{\mathcal{D}} \to \prod_{y\to z} z^\vee_{\mathcal{D}} \to \tau^{-1}(y)^\vee_{\mathcal{D}}$$

which is associated with the Auslander–Reiten triangle

$$y \to \bigoplus_{y\to z} z \to \tau^{-1}(y) \to \Sigma y$$

of $\mathcal{D}_Q$. Thus, we have an exact sequence of $\mathcal{D}_Q$-modules

$$0 \to S^{\mathcal{D}}_y \to y^\vee_{\mathcal{D}} \to \prod_{y\to z} z^\vee_{\mathcal{D}} \to \tau^{-1}(y)^\vee_{\mathcal{D}} \to S^{\mathcal{D}}_{\Sigma y} \to 0,$$

where $S^{\mathcal{D}}_y$ is the simple $\mathcal{D}_Q$-module associated with the vertex y. It follows that the homology of $\mathrm{RHom}(x^\wedge_{\mathcal{D}}, S_y)$ is given by $S^{\mathcal{D}}_y(x)$ in degree 0 and $S^{\mathcal{D}}_{\Sigma y}(x) = S^{\mathcal{D}}_y(\Sigma^{-1}(x))$ in degree 2. This yields the projective resolution in (a). In the non-Dynkin case, no object $H(y)$, $y \in \mathbb{Z}Q_0$, is isomorphic to $\Sigma^{-1}H(x)$. Thus, the homology of $\mathrm{RHom}(x^\wedge_{\mathcal{D}}, S_y)$ in degree 2 vanishes and we find the projective resolution in (b). A similar argument yields the injective co-resolutions in (a) and (b). □

3.8. Resolutions for the simple $\mathcal{S}_C$-modules. We keep the notations and assumptions of Section 3.3. Notice that for each frozen vertex $\sigma(x)$, $x \in \mathbb{Z}Q_0$, the restriction of the free $\mathcal{R}_C$-module $\sigma(x)^\wedge_{\mathcal{R}_C}$ to $\mathcal{S}_C$ is the free $\mathcal{S}_C$-module $\sigma(x)^\wedge_{\mathcal{S}_C}$ and similarly for the co-free modules associated with the frozen vertices. In particular, the restrictions to $\mathcal{S}_C$ of the modules $P_C(x)$ and $I_C(x)$ defined in (3.6.2) are still projective respectively injective. By abuse of notation, we denote the restricted modules by the same symbols $P_C(x)$ and $I_C(x)$.

Theorem 3.9. *Suppose that Q is connected. Let x be a vertex of $\mathbb{Z}Q$.*

(a) *If Q is a Dynkin quiver, the simple $\mathcal{S}_C$-module $S_{\sigma^{-1}(x)}$ admits a minimal projective resolution of the form*

$$\cdots \to P_C(\Sigma^{-2}x) \to P_C(\Sigma^{-1}x) \to P_C(x) \to \sigma^{-1}(x)^\wedge \to S_{\sigma^{-1}(x)} \to 0$$

and the simple $\mathcal{S}_C$-module $S_{\sigma(x)}$ admits a minimal injective resolution of the form

$$0 \to S_{\sigma(x)} \to \sigma(x)^\vee \to I_C(x) \to I_C(\Sigma x) \to I_C(\Sigma^2 x) \to \cdots.$$

(b) *If Q is not a Dynkin quiver, the simple $\mathcal{S}_C$-module $S_{\sigma^{-1}(x)}$ admits a minimal projective resolution of the form*

$$0 \to P_C(x) \to \sigma^{-1}(x)^\wedge \to S_{\sigma^{-1}(x)} \to 0$$

and the simple $\mathcal{S}_C$-module $S_{\sigma(x)}$ admits a minimal injective resolution of the form

$$0 \to S_{\sigma(x)} \to \sigma(x)^\vee \to I_C(x) \to 0.$$

Proof. Part (a) of Lemma 3.5 yields an exact sequence of $\mathcal{R}_C$-modules

$$0 \to x^\wedge \to \sigma^{-1}(x)^\wedge \to S_{\sigma^{-1}(x)} \to 0.$$

We restrict it to $\mathcal{S}_C$ and now have to construct a minimal resolution for $\mathrm{res}(x^\wedge)$, where res denotes the restriction functor. If Q is not a Dynkin quiver, part (b) of Theorem 3.7 yields the exact sequence of $\mathcal{R}_C$-modules

$$0 \to P_C(x) \to x^\wedge \to x^\wedge_{\mathcal{D}} \to 0.$$

Since the restriction of $x^\wedge_{\mathcal{D}}$ to $\mathcal{S}_C$ vanishes, we find that $\mathrm{res}(x^\wedge)$ is isomorphic to the restriction of $P_C(x)$ to $\mathcal{S}_C$, which yields the projective resolution in (b). If Q is a Dynkin quiver, part (a) of Theorem 3.7 yields the exact sequence of $\mathcal{R}_C$-modules

$$0 \to (\Sigma^{-1}x)^\wedge \to P_C(x) \to x^\wedge \to x^\wedge_{\mathcal{D}} \to 0.$$

Since the restriction $\mathrm{res}(x^\wedge_{\mathcal{D}})$ vanishes, we obtain a short exact sequence

$$0 \to \mathrm{res}((\Sigma^{-1}x)^\wedge) \to P_C(x) \to \mathrm{res}(x^\wedge) \to 0$$

and more generally an exact sequence

$$0 \to \mathrm{res}((\Sigma^{-(p+1)}x)^\wedge) \to P_C(\Sigma^{-p}x) \to \mathrm{res}((\Sigma^{-p}x)^\wedge) \to 0$$

for each $p \geq 0$. We obtain the desired resolution by splicing these sequences together. The construction of the injective co-resolutions is analogous. □

Corollary 3.10. *Let x and y be vertices of $\mathbb{Z}Q$. For each $p \geq 1$, we have an isomorphism*

$$\mathrm{Ext}^p_{\mathcal{S}_C}(S_{\sigma(x)}, S_{\sigma(y)}) = \mathcal{D}_Q(H(x), \Sigma^p H(y)).$$

If Q is not a Dynkin quiver, these spaces vanish for all $p \geq 2$.

Proof. We use the minimal injective co-resolutions in Theorem 3.9. Suppose that Q is a Dynkin quiver and $p \geq 1$. We have

$$\begin{aligned}\mathrm{Ext}^p_{\mathcal{S}_C}(S_{\sigma(x)}, S_{\sigma(y)}) &= \mathrm{Hom}(S_{\sigma(x)}, I_C(\Sigma^{p-1} y)) \\ &= \mathrm{Hom}(S_{\sigma(x)}, \prod D\mathcal{D}_Q(\Sigma^{p-1} y, z) \otimes \sigma^{-1}(z)^{\vee}).\end{aligned}$$

Now the space $\mathrm{Hom}(S_{\sigma(x)}, \sigma^{-1}(z)^{\vee})$ vanishes unless $\sigma(x) = \sigma^{-1}(z)$, i.e. $z = \sigma^2(x) = \tau(x)$. Hence we find

$$\begin{aligned}\mathrm{Ext}^p_{\mathcal{S}_C}(S_{\sigma(x)}, S_{\sigma(y)}) &= D\mathcal{D}_Q(\Sigma^{p-1} y, \tau(x)) \\ &= D\mathcal{D}_Q(\Sigma^{p-1} y, \Sigma^{-1}\nu x) \\ &= \mathcal{D}_Q(x, \Sigma^p y).\end{aligned}$$

□

4. The stratifying functor Φ

4.1. Construction of Φ. Let Q be a connected acyclic quiver and let C be a subset of the set of frozen vertices of the repetition quiver $\mathbb{Z}\widetilde{Q}$ which satisfies Assumption 3.4. Notice that $\mathrm{Mod}(\mathcal{R}_C)$ is a subcategory of $\mathrm{Mod}(\mathcal{R})$ and similarly $\mathrm{Mod}(\mathcal{S}_C)$ a subcategory of $\mathrm{Mod}(\mathcal{S})$. Let $\mathrm{res}^C : \mathrm{Mod}(\mathcal{R}_C) \to \mathrm{Mod}(\mathcal{S}_C)$ be the restriction functor. Clearly, it is just the restriction of the functor $\mathrm{res} : \mathrm{Mod}(\mathcal{R}) \to \mathrm{Mod}(\mathcal{S})$ to the subcategories under consideration. The left and right adjoints K_L and K_R of res take the subcategory $\mathrm{Mod}(\mathcal{S}_C)$ of $\mathrm{Mod}(\mathcal{S})$ to $\mathrm{Mod}(\mathcal{R}_C)$ and thus induce left and right adjoints K_L^C and K_R^C of res^C so that we have

$$\begin{array}{c}\mathrm{Mod}(\mathcal{R}_C)\\ K_L^C \uparrow \quad \downarrow \mathrm{res}^C \quad \uparrow K_R^C\\ \mathrm{Mod}(\mathcal{S}_C).\end{array}$$

The functor res^C is still a localization of abelian categories in the sense of [10]. *In the sequel, we will omit the exponents C in the notation for the functors K_L^C and K_R^C and simply write K_L and K_R*. We have the canonical morphism

(4.1.1) $$\mathrm{can} : K_L \to K_R.$$

(which is just the restriction to $\mathrm{Mod}(\mathcal{S}_C)$ of the canonical morphism between the non-restricted functors). By definition, the *intermediate Kan extension* K_{LR} is its image, the *functor KK* its kernel and the *functor CK* its cokernel so that we have the following diagram of functors from $\mathrm{Mod}(\mathcal{S}_C)$ to $\mathrm{Mod}(\mathcal{R}_C)$:

$$0 \longrightarrow KK \longrightarrow K_L \longrightarrow K_R \longrightarrow CK \longrightarrow 0,\qquad K_L \twoheadrightarrow K_{LR} \rightarrowtail K_R.$$

The *kernel* $\mathcal{N}$ of the functor res is the abelian subcategory formed by the modules which vanish on the frozen vertices. Clearly, the category $\mathcal{N}$ is isomorphic to the category of modules over the quotient $\mathcal{R}_C/\langle \mathcal{S}_C \rangle$ of $\mathcal{R}_C$ by the ideal generated by the identities of the objects of $\mathcal{S}_C$. Notice that this quotient is isomorphic to the mesh category $k(\mathbb{Z}Q)$. We will identify

$$\mathcal{N} = \operatorname{Mod}(\mathcal{R}_C/\langle \mathcal{S}_C \rangle) = \operatorname{Mod}(k(\mathbb{Z}Q)).$$

Composition with the functor res sends the morphisms

$$K_L \to K_{LR} \to K_R$$

to isomorphisms. Thus, the images of KK and CK lie in the subcategory

$$\mathcal{N} = \operatorname{Mod}(k(\mathbb{Z}Q)).$$

In Theorem 4.8, we will see that for each finite-dimensional $\mathcal{S}_C$-module M, the $k(\mathbb{Z}Q)$-module $CK(M)$ is a finitely cogenerated injective module. Thus, there is an object $\Phi(M)$ in the derived category $\mathcal{D}_Q$ such that

$$CK(M)(x) = D\operatorname{Hom}(\Phi(M), H(x))$$

functorially in $x \in \mathbb{Z}Q$. Clearly, the map $M \mapsto \Phi(M)$ underlies a *k-linear functor*

$$\Phi : \operatorname{mod}(\mathcal{S}_C) \to \mathcal{D}_Q.$$

We call Φ the *stratifying functor* because of Theorem 2.7. Our construction does not make it clear which exactness properties the stratifying functor has but we will show the following theorem. Recall from Section 2.6 that if no connected component of Q is a Dynkin quiver, then $\mathcal{V}$ denotes the additive subcategory of $\mathcal{D}_Q$ whose indecomposable objects are the sums of objects in the image of Happel's embedding. The category $\mathcal{V}$ becomes exact when endowed with all the sequences giving rise to triangles in $\mathcal{D}_Q$.

Theorem 4.2. *The following hold.*

(a) *If no connected component of Q is a Dynkin quiver, then the functor Φ is exact as a functor* $\operatorname{mod}(\mathcal{S}_C) \to \mathcal{V}$.

(b) *If Q is a Dynkin quiver, then Φ underlies a δ-functor* $\operatorname{mod}(\mathcal{S}_C) \to \mathcal{D}_Q$.

We will prove part (a) in Section 4.22. Part (b) will follow from the alternative construction of Φ via Gorenstein homological algebra in Section 5.20.

4.3. Computation of the right Kan extension. Let us show how to compute, in principle, the right Kan extension of a finite-dimensional $\mathcal{S}$-module M. Let $L = K_R(M)$. This module is still pointwise finite-dimensional and left bounded (since it is a submodule of a finitely cogenerated injective $\mathcal{R}$-module). Thus, the module $L = K_R(M)$ satisfies

$$L(u) = 0 \tag{4.3.1}$$

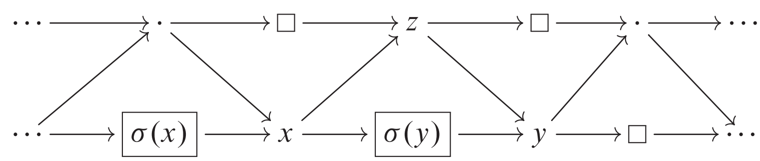
for all vertices u sufficiently far to the left in $\mathbb{Z}\widetilde{Q}$. Moreover, for all vertices x of $\mathbb{Z}Q$, we have

$$L(\sigma(x)) = M(\sigma(x)) \tag{4.3.2}$$

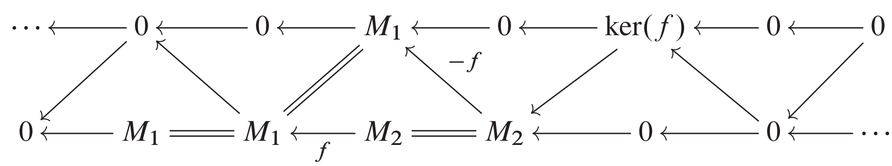
and the sequence

$$0 \to L(x) \to \bigoplus_{y \to x} L(y) \to L(\tau(x)) \tag{4.3.3}$$

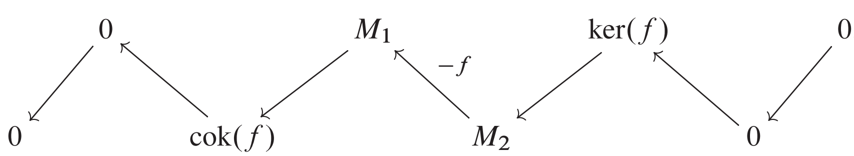
is exact because $\operatorname{Hom}(S_x, L) = \operatorname{Ext}^1(S_x, L) = 0$. Thus, we can compute $L(x)$ once we know $L(\tau(x))$ and $L(y)$ for all vertices y of $\mathbb{Z}\widetilde{Q}$ which are sources of arrows $y \to x$ of $\mathbb{Z}\widetilde{Q}$. Using (4.3.1), (4.3.2) and (4.3.3) we can compute $K_R(L)$ by 'knitting from left to right' as in the following example: We consider the quiver Q of type A_2 and the vertices x and y of $\mathbb{Z}\widetilde{Q}$ given by



Let M be the $\mathcal{S}$-module M such that $M(u) = 0$ for all elements u distinct from $\sigma(x)$ and $\sigma(y)$, $M(\sigma(x)) = M_1$, $M(\sigma(y)) = M_2$ and the only non-trivial value of M on an arrow is given by a linear map $f : M_2 \to M_1$. If we apply the above algorithm, we find that $K_R(M)$ is the following representation of $\mathcal{R}$:



If $P \to M$ is an epimorphism with projective P, then so is $K_L(P) \to K_L(M)$ and so $K_{LR}(M)$ is the image of the induced map $K_L(P) \to K_R(M)$. This shows that $K_{LR}(M) \subset K_R(M)$ is the submodule generated by the spaces $M(\sigma(v))$, where v runs through the vertices of $\mathbb{Z}Q$. By taking the quotient modulo this submodule, we obtain the $k(\mathbb{Z}Q)$-module $CK(M)$, which in our example is given by



We see that it is the injective $k(\mathbb{Z}Q)$-module $I_x^{d_x} \oplus I_z^{d_z} \oplus I_y^{d_y}$, where $d_x = \dim \operatorname{cok}(f)$, $d_z = \operatorname{rk}(f)$, $d_y = \dim \ker(f)$.

4.4. The representability theorem. In this subsection we shall prove Theorem 4.8.

Lemma 4.5. *An $\mathcal{R}_C$-module M belongs to the image of K_R, respectively K_L, if and only if, for each $N \in \mathcal{N}$, we have*

$$\operatorname{Hom}(N, M) = 0 \quad \textit{and} \quad \operatorname{Ext}^1(N, M) = 0,$$

respectively

$$\operatorname{Hom}(M, N) = 0 \quad \textit{and} \quad \operatorname{Ext}^1(M, N) = 0.$$

Proof. This is a general characterization of the image of the adjoint of a localization functor, cf. [10, Lemme 1, p. 370]. □

Lemma 4.6. *Let M be an $\mathcal{S}_C$-module and let $N \in \mathcal{N}$. We have canonical isomorphisms*

$$\operatorname{Hom}(N, CK(M)) \xrightarrow{\sim} \operatorname{Ext}^1(N, K_{LR}M)$$

and

$$\operatorname{Hom}(KK(M), N) \xrightarrow{\sim} \operatorname{Ext}^1(K_{LR}M, N).$$

Proof. The first isomorphism is obtained by applying the functor $\operatorname{Hom}(N, ?)$ to the exact sequence

$$0 \to K_{LR}(M) \to K_R(M) \to CK(M) \to 0$$

and using Lemma 4.5. Dually, one obtains the second isomorphism. □

We will see in Theorem 4.8 below that if M is a finite-dimensional $\mathcal{S}_C$-module, then $KK(M)$ is projective and $CK(M)$ is injective. The main step in the proof is the following lemma.

Lemma 4.7. *Let M be a finite-dimensional $\mathcal{S}_C$-module and let*

(4.7.1) $$0 \to N' \to N \to N'' \to 0$$

be an exact sequence in $\mathcal{N}$.

(a) *If N' is right bounded and pointwise finite-dimensional, then the left exact functor* $\operatorname{Hom}(KK(M), ?)$ *transforms* (4.7.1) *into an exact sequence.*

(b) *If N'' is left bounded and pointwise finite-dimensional, then the left exact functor* $\operatorname{Hom}(?, CK(M))$ *transforms* (4.7.1) *into an exact sequence.*

Proof. We prove (a), the proof of (b) being dual. By Lemma 4.6, on the subcategory $\mathcal{N}$, the functor $\operatorname{Hom}(KK(M), ?)$ is isomorphic to $\operatorname{Ext}^1(K_{LR}(M), ?)$. We have an exact sequence

$$E^0(K_{LR}(M), N'') \to E^1(K_{LR}(M), N') \to E^1(K_{LR}(M), N) \to E^1(K_{LR}, N'') \\ \to E^2(K_{LR}(M), N'),$$

where we abbreviate $\operatorname{Hom}(\cdot,\cdot)$ by $E^0(\cdot,\cdot)$ and $\operatorname{Ext}^p(\cdot,\cdot)$ by $E^p(\cdot,\cdot)$, $p \geq 1$. Here the group $\operatorname{Hom}(K_{LR}(M), N'')$ vanishes since $K_{LR}(M)$ is a quotient of $K_L(M)$. The claim will follow once we show that $\operatorname{Ext}^2(K_{LR}(M), N')$ vanishes. Since N' is right bounded and pointwise finite-dimensional, it is the inverse limit of a countable system N'_p, $p \in \mathbb{N}$, of finite-dimensional $\mathcal{R}_C$-modules. We have an exact sequence

$$0 \to \lim{}^1 \mathrm{E}^1(K_{LR}(M), N'_p) \to \mathrm{E}^2(K_{LR}(M), N') \to \lim \mathrm{E}^2(K_{LR}(M), N'_p) \to 0,$$

where $\lim^1$ is the first right derived functor of the inverse limit functor lim, cf. [46, Application 3.5.10]. Since $K_{LR}(M)$ is finite-dimensional, it admits a finite resolution by finitely generated projective $\mathcal{R}_C$-modules. Now $\operatorname{Hom}(P, N'_p)$ is finite-dimensional for each finitely generated projective P and each N'_p. So the spaces $\operatorname{Ext}^1(K_{LR}(M), N'_p)$ are all finite-dimensional and the $\lim^1$ term in the above sequence vanishes by the Mittag-Leffler Lemma. It remains to be shown that each $\operatorname{Ext}^2(K_{LR}(M), N'_p)$ vanishes. For this, since N'_p is of finite length, it suffices to check that $\operatorname{Ext}^2(K_{LR}(M), S_x)$ vanishes for each vertex x of $\mathbb{Z}Q$. Indeed, by Lemma 3.6, we have

$$\mathrm{Ext}^2(K_{LR}(M), S_x) = D\,\mathrm{Hom}(S_{\tau^{-1}(x)}, K_{LR}(M))$$

and this last space vanishes because $K_{LR}(M)$ is a submodule of $K_R(M)$. □

Theorem 4.8. *If M is a finite-dimensional $\mathcal{S}_C$-module, then*

- *$KK(M)$ is a finitely generated projective $\mathcal{R}_C/\langle \mathcal{S}_C\rangle$-module,*
- *$CK(M)$ is a finitely cogenerated injective $\mathcal{R}_C/\langle \mathcal{S}_C\rangle$-module.*

Proof. Since M is a finite-dimensional $\mathcal{S}_C$-module, it is finitely generated. Thus the module $K_L(M)$ is also finitely generated and hence right bounded and pointwise finite-dimensional. These properties are inherited by the submodule $KK(M)$ of $K_L(M)$. Moreover, this submodule is supported on the vertices of $\mathbb{Z}Q$. Thus, there is a surjection $P \to KK(M)$ where P is a direct sum of projective $k(\mathbb{Z}Q)$-modules $u_i^{\wedge}$, $i \in I$, such that the family of the vertices u_i is right bounded and each vertex x of $\mathbb{Z}Q$ occurs at most finitely many times among the u_i, $i \in I$. It follows that P is right bounded and pointwise finite-dimensional and so is the kernel M' of $P \to M$. Thus, by Lemma 4.7, the functor $\mathrm{Hom}(KK(M), ?)$ takes the sequence

$$0 \to M' \to P \to KK(M) \to 0$$

to an exact sequence. Thus, the morphism $P \to KK(M)$ splits and $KK(M)$ is a direct factor of P. Let us show that P is finitely generated. First we notice that $K_{LR}(M)$ is finite-dimensional since it is pointwise finite-dimensional and both right and left bounded. Therefore, for each vertex x of $\mathbb{Z}Q$, by Corollary 3.6, the space

$$\mathrm{Hom}(KK(M), S_x) = \mathrm{Ext}^1(K_{LR}(M), S_x)$$

is finite-dimensional and vanishes for all but finitely many vertices x. Hence $KK(M)$ must be a finite sum of projective $k(\mathbb{Z}Q)$-modules $x^{\wedge}$. The proof of the second assertion is dual. □

4.9. Description of the strata. Our goal in this subsection is to prove the description of the strata of $\mathcal{M}_0(w)$ given in Proposition 2.11. We use the notations and assumptions of Section 4.1. We define an arbitrary (not necessarily finite-dimensional) $\mathcal{R}_C$-module U to be *stable* if we have $\mathrm{Hom}(N, U) = 0$ for each module N in the kernel $\mathcal{N}$ of the restriction functor res. If U is finite-dimensional, it is stable if and only if $\mathrm{Hom}(S_x, U) = 0$ for each vertex x of $\mathbb{Z}Q$. Dually, U is *co-stable* if $\mathrm{Hom}(U, N) = 0$ for each N in $\mathcal{N}$. Clearly submodules of stable modules are stable and quotient modules of co-stable modules are co-stable. Now let us fix an $\mathcal{S}_C$-module M. By the adjunctions between K_L, res and K_R, for each $\mathcal{S}_C$-module M, the module $K_R(M)$ is stable and $K_L(M)$ is co-stable. Since $K_{LR}(M)$ is a submodule of $K_R(M)$ and a quotient of $K_L(M)$, it is both stable and co-stable. This yields the first statement of Proposition 2.11. If we apply the functor res to the canonical morphism

$$\mathrm{can} : K_L(M) \to K_R(M),$$

we obtain the composition of the adjunction morphisms

$$\mathrm{res}\, K_L(M) \xleftarrow{\sim} M \xrightarrow{\sim} \mathrm{res}\, K_R(M)$$

and in particular the restriction res(can) is invertible. Since the restriction functor is exact, it also makes the canonical morphisms

$$K_L(M) \twoheadrightarrow K_{LR}(M) \rightarrowtail K_R(M)$$

invertible and so $\mathrm{res}(K_{LR}(M))$ is canonically isomorphic to M. Now let us assume that M is finite-dimensional of dimension vector w and belongs to $\mathcal{M}_0(w)$. Then $K_{LR}(M)$ is finite-dimensional: Indeed, $K_L(M)$ is right bounded, $K_R(M)$ is left bounded and both are pointwise finite-dimensional. Let (v, w) be the dimension vector of $K_{LR}(M)$. Since $\mathrm{res}(K_{LR}(M))$ is isomorphic to M, the second component of the dimension vector of $K_{LR}(M)$ is indeed the dimension vector w of M. This also shows that the image under π of the point $\widetilde{M}$ of $\mathcal{M}^{\mathrm{reg}}(v, w)$ corresponding to $K_{LR}(M)$ equals M, which therefore does belong to the stratum $\pi(\mathcal{M}^{\mathrm{reg}}(v, w))$. Now the other assertions of Proposition 2.11 are immediate from the facts recalled in Section 2.6.

4.10. Intermediate extensions and closed orbits. Our goal in this subsection is to prove Proposition 2.12. We use the notations and assumptions of Section 4.1 above. If M is a finite-dimensional $\mathcal{R}$-module of dimension vector (v, w), the *G_v-orbit of M* is the orbit corresponding to M in the affine variety $\mathrm{rep}(\mathcal{R}^{\mathrm{op}}, v, w)$ of representations of $\mathcal{R}^{\mathrm{op}}$ with dimension vector (v, w). By abuse of language, we say that a G_v-stable subset of $\mathrm{rep}(\mathcal{R}^{\mathrm{op}}, v, w)$ *contains* a module if it contains the orbit corresponding to the module.

Lemma 4.11. *Let*

$$0 \to L \to M \to N \to 0$$

be an exact sequence of finite-dimensional $\mathcal{R}_C$-modules. If $\mathrm{res}(L) = 0$ (resp. $\mathrm{res}(N) = 0$), then the closure of the G_v-orbit of M contains $L_{\mathrm{ss}} \oplus N$ (resp. $L \oplus N_{\mathrm{ss}}$), where L_{ss} is the semi-simple module with the same dimension vector as L.

Proof. For each vertex x of $\mathbb{Z}Q$, we choose an isomorphism $M_x = L_x \oplus N_x$ which provides a splitting of the sequence

$$0 \to L_x \to M_x \to N_x \to 0.$$

For an invertible scalar t, let $g(t)$ be the element of G_v which acts by $t \oplus \mathbf{1}$ on $L_x \oplus N_x$. When t tends to zero, the representation $g(t).x$, where x is a point in the orbit of M, tends to $L \oplus N$. Since L is a nilpotent representation of $k(\mathbb{Z}Q)$, its G_v-orbit contains L_{ss} in its closure. Thus, the G_v orbit of M contains $L_{\mathrm{ss}} \oplus N$ in its closure. The proof of the second statement is analogous. □

Let us now prove Proposition 2.12. Since L is stable, we have an exact sequence

$$0 \to K_{LR}(\mathrm{res}\, L) \to L \to N \to 0,$$

where $\mathrm{res}(N) = 0$. By the lemma, the closure of the G_v-orbit of L contains $K_{LR}(\mathrm{res}\, L) \oplus N_{\mathrm{ss}}$. Now let U be a module in the unique closed G_v-orbit in the closure of the G_v-orbit of L (the existence of this unique orbit is guaranteed by geometric invariant theory). Let us show that the orbit of U contains $K_{LR}(\mathrm{res}\, L) \oplus N_{\mathrm{ss}}$. We have $\mathrm{res}(U) = \mathrm{res}(L)$ since the restriction to $\mathcal{S}_C$ of a module is constant on the closure of its G_v-orbit. Thus, the morphisms

$$K_L(\mathrm{res}(M)) \to K_R(\mathrm{res}(M)) \quad \text{and} \quad K_L(\mathrm{res}(U)) \to K_R(\mathrm{res}(U))$$

are equal and so $K_L(\mathrm{res}(M)) \to K_R(\mathrm{res}(M))$ factors through the adjunction morphism

$$\varepsilon : U \to K_R(\mathrm{res}\, U).$$

Therefore, the module $K_{LR}(\mathrm{res}\, M)$ is contained in $\mathrm{im}(\varepsilon) \subset K_R(\mathrm{res}\, U)$. Let i denote the inclusion $K_{LR}(\mathrm{res}\, M) \subset \mathrm{im}(\varepsilon)$. Now by the lemma, the closure of the orbit of U contains $\mathrm{im}(\varepsilon) \oplus (\ker(\varepsilon))_{\mathrm{ss}}$ and the closure of the orbit of $\mathrm{im}(\varepsilon)$ contains $K_{LR}(\mathrm{res}\, M) \oplus (\mathrm{cok}(i))_{\mathrm{ss}}$. Thus the orbit of U, which equals its closure, contains the object

$$K_{LR}(\mathrm{res}\, M) \oplus (\mathrm{cok}(i))_{\mathrm{ss}} \oplus (\ker(\varepsilon))_{\mathrm{ss}}.$$

This shows that U is isomorphic to $K_{LR}(\mathrm{res}\, M) \oplus N_{\mathrm{ss}}$, as claimed.

4.12. Characterization of the strata. Our goal in this subsection is to prove the characterization of the strata of $\mathcal{M}_0(w)$ given in Theorem 2.7. We use the notations and assumptions of Section 4.1. We need the following lemmas. For a vector $v : \mathbb{Z}Q_0 \to \mathbb{Z}$, we define $C_q v : \mathbb{Z}Q_0 \to \mathbb{Z}$ by

$$(C_q v)(x) = v(x) - \Big(\sum_{y \to x} v(y)\Big) + v(\tau(x)), \quad x \in \mathbb{Z}Q_0,$$

where the sum ranges over all arrows $y \to x$ of $\mathbb{Z}Q$. The index q reminds us that C_q is a 'quantum Cartan matrix', cf. [36, Section 3.1]. Notice that the linear map $v \mapsto C_q v$ is injective on the space of finitely-supported vectors.

Lemma 4.13. *Let U be a finite-dimensional $\mathcal{R}_C$-module of dimension vector (v, w). If U is stable and co-stable, the vector*

$$x \mapsto \dim \mathrm{Ext}^1(S_x, U), \quad x \in \mathbb{Z}Q_0,$$

equals $w\sigma - C_q v$ (where $w\sigma$ is the composition $w \circ \sigma$).

Proof. By part (d) of Corollary 3.6, the space $\mathrm{Ext}^1(S_x, U)$ is the homology in degree 1 of the complex

$$0 \to U(x) \to \bigoplus_{y \to x} U(y) \to U(\tau(x)) \to 0,$$

where the sum ranges over all arrows $y \to x$ of $\mathcal{R}_C$. Since U is stable and co-stable, the homologies in degree 0 and 2 of this complex vanish. Thus, the dimension of the homology in degree 1 equals

$$-\dim U(x) + \Big(\sum_{y \to x} \dim U(y)\Big) - \dim U(\tau(x)) = -(C_q v)(x) + w(\sigma(x)). \qquad \square$$

Lemma 4.14. *Let M be an $\mathcal{S}_C$-module. Let (v, w) be the dimension vector of $K_{LR}(M)$. Then, for each vertex x of $\mathbb{Z}Q$, the multiplicity of the indecomposable $H(x)$ in $\Phi(M)$ equals*

$$\dim \mathrm{Ext}^1(S_x, K_{LR}(M)) = (w\sigma - C_q v)(x).$$

Proof. Recall from Section 4.1 that $CK(M)$ isomorphic to the finitely cogenerated injective module $D\,\mathrm{Hom}(\Phi(M), H(?))$. The multiplicity of an indecomposable $H(x)$ in $\Phi(M)$

equals the multiplicity of the injective indecomposable

$$D\,\mathrm{Hom}(H(x), H(?)) = Dk(\mathbb{Z}Q)(x, ?)$$

in $CK(M)$. This multiplicity equals the multiplicity of the simple S_x in the socle of $CK(M)$, that is to say the dimension of $\mathrm{Hom}(S_x, CK(M))$. Now by Lemma 4.6, we have the isomorphism

$$\mathrm{Hom}(S_x, CK(M)) = \mathrm{Ext}^1(S_x, K_{LR}(M)).$$

By Lemma 4.13, the dimension of the right hand side equals $(w\sigma - C_q v)(x)$. □

Theorem 2.7 is now an easy consequence: Let w be a dimension vector for $\mathcal{S}_C$ (i.e. w vanishes on the vertices not belonging to C). Proposition 2.11 shows that two $\mathcal{S}_C$-modules M_1 and M_2 belong to the same stratum of $\mathcal{M}_0(w)$ iff the $\mathcal{R}_C$-modules $K_{LR}(M_1)$ and $K_{LR}(M_2)$ have the same dimension vector (v, w) and in this case, the objects $\Phi(M_1)$ and $\Phi(M_2)$ are isomorphic, by Lemma 4.14. Conversely, if $\Phi(M_1)$ and $\Phi(M_2)$ are isomorphic, then by the same lemma, we have

$$w\sigma - C_q v_1 = w\sigma - C_q v_2,$$

where (w, v_i) is the dimension vector of $K_{LR}(M_i)$, $i = 1, 2$. Since C_q is injective on the space of dimension vectors, we find that $K_{LR}(M_1)$ and $K_{LR}(M_2)$ have the same dimension vector, which implies that M_1 and M_2 lie in the same stratum, by Proposition 2.11.

4.15. Resolution of the intermediate extension. For future reference, we record the following lemma:

Lemma 4.16. *Let M be a finite-dimensional $\mathcal{S}_C$-module.*

(a) *The $\mathcal{R}_C$-module $K_{LR}(M)$ has a minimal injective resolution with finitely cogenerated terms*

$$0 \to K_{LR}(M) \to I^0 \to I^1 \to 0,$$

where I^0 is the direct sum of the modules $\sigma(x)^\vee$ with multiplicity equal to the dimension of $\mathrm{Hom}(S_{\sigma(x)}, M)$*, $x \in \mathbb{Z}Q_0$, and I^1 contains the summand $x^\vee$ with multiplicity equal to the multiplicity of $H(x)$ as a direct factor of $\Phi(M)$, $x \in \mathbb{Z}Q_0$.*

(b) *The $\mathcal{R}_C$-module $K_{LR}(M)$ has a minimal projective resolution with finitely generated terms*

$$0 \to P_1 \to P_0 \to K_{LR}(M) \to 0,$$

where P_0 is the direct sum of the modules $\sigma(x)^\wedge$ with multiplicity equal to the dimension of $\mathrm{Hom}(M, S_{\sigma(x)})$*, $x \in \mathbb{Z}Q_0$, and P_1 contains the summand $x^\wedge$ with multiplicity equal to the multiplicity of $H(\tau^{-1}(x))$ as a direct factor of $\Phi(x)$.*

Remark 4.17. One can show that if the projective $\mathcal{S}_C$-modules coincide with the injective ones, then I^1 contains no direct factor $\sigma(x)^\vee$ and P_1 no direct factor $\sigma(x)^\wedge$.

Proof. (a) The module $K_{LR}(M)$ is finite-dimensional (cf. Section 4.9) and thus admits an injective resolution with finitely cogenerated terms I^p and the multiplicity of the indecomposable injective $u^\vee$ associated with a vertex u of $\mathbb{Z}\widetilde{Q}$ equals $\dim \mathrm{Ext}^p_{\mathcal{R}_C}(S_u, K_{LR}(M))$. Let x

be a vertex of $\mathbb{Z}Q$. We have $\operatorname{Hom}(S_x, K_{LR}(M)) = 0$ since $K_{LR}(M)$ is stable. Since we have $\operatorname{Hom}(S_{\sigma(x)}, CK(M)) = 0$, we find

$$\operatorname{Hom}(S_{\sigma(x)}, K_{LR}(M)) = \operatorname{Hom}(S_{\sigma(x)}, K_R(M)) = \operatorname{Hom}_{\mathcal{S}_C}(S_{\sigma(x)}, M).$$

The multiplicity of $x^\vee$ in I^1 equals $\operatorname{Ext}^1_{\mathcal{R}_C}(S_x, K_{LR}(M))$ and this equals the multiplicity of $H(x)$ in $\Phi(M)$ by Lemma 4.14. The proof of (b) is similar but uses the duality isomorphism (3.6.1) in addition. □

4.18. On the degeneration order. Our goal in this subsection is to prove Theorem 2.8. We may and will assume that Q is connected. Let (v, w) and (v', w) be dimension vectors of $\mathcal{R}_C$ associated with non-empty subsets $\mathcal{M}^{\mathrm{reg}}(v, w)$ and $\mathcal{M}^{\mathrm{reg}}(v', w)$ of the corresponding smooth quiver varieties. Let M and M' be $\mathcal{S}_C$-modules belonging to the corresponding strata. Let us assume that $\Phi(M) \leq \Phi(M')$ in the degeneration order of [22] and show that the stratum $\pi(\mathcal{M}^{\mathrm{reg}}(v', w))$ is contained in the closure of $\pi(\mathcal{M}^{\mathrm{reg}}(v, w))$. Recall from [37, Corollary 4.1.3.14] that this is the case iff we have $v'(x) \leq v(x)$ for all vertices x of $\mathbb{Z}Q_0$. Now by Proposition 2.11, if we denote by $\underline{\dim}\, U$ the dimension vector of a module U, we have

$$(v, w) = \underline{\dim}\, K_{LR}(M)$$

and

$$(v', w) = \underline{\dim}\, K_{LR}(M').$$

So we need to show the inequality

$$\underline{\dim}\, K_{LR}(M') \leq \underline{\dim}\, K_{LR}(M).$$

Indeed, by definition [22], the relation $\Phi(M) \leq \Phi(M')$ means that there is an object Z of $\mathcal{D}_Q$ and a triangle

$$\Phi(M') \to \Phi(M) \oplus Z \to Z \to \Sigma\Phi(M'). \tag{4.18.1}$$

If Q is a Dynkin quiver, then thanks to the triangle equivalence of Theorem 5.18, we can find a finite-dimensional $\mathcal{S}_C$-module U such that $\Phi(U)$ is isomorphic to Z. If Q is not a Dynkin quiver, then a priori, the object Z may not belong to the image of Φ but we claim that we can always replace it with an object in the image. For this, let $H^i_{\mathcal{A}}$ denote the homology functors for the heart $\mathcal{A}$ of $\mathcal{D}_Q$ whose indecomposable objects are the indecomposable regular kQ-modules and all the objects $\tau^p P$, where $p \in \mathbb{Z}$ and P is an indecomposable projective kQ-module. Then $\Phi(M)$ and $\Phi(M')$ lie in $\mathcal{A}$ but Z may have non-vanishing homologies in several degrees. However, if we apply $H^{-1}_{\mathcal{A}}$ to the triangle (4.18.1), we find the exact sequence

$$0 \to H^{-1}_{\mathcal{A}}(Z) \to H^{-1}_{\mathcal{A}}(Z) \to \Phi(M') \to \Phi(M) \oplus H^0_{\mathcal{A}}(Z).$$

Since $H^{-1}_{\mathcal{A}}(Z)$ is finite-dimensional, the second morphism of the sequence must be invertible and so the map $\Phi(M') \to \Phi(M) \oplus H^0_{\mathcal{A}}(Z)$ is injective. Thus, the following sequence is left exact

$$0 \to \Phi(M') \to \Phi(M) \oplus H^0_{\mathcal{A}}(Z) \to H^0_{\mathcal{A}}(Z) \to 0. \tag{4.18.2}$$

It is also right exact because $H^1_{\mathcal{A}}(\Phi(M'))$ vanishes. The category $\mathcal{A}$ contains the subcategory of all regular kQ-modules as a torsion subcategory and the corresponding category of torsion-free objects is the category $\mathcal{V}$ formed by the direct sums of the objects $\tau^p(P)$, where $p \in \mathbb{Z}$ and P is an indecomposable projective kQ-module. Since $\Phi(M')$ is torsion-free, the map

$$\Phi(M) \oplus H^0_{\mathcal{A}}(Z) \to H^0_{\mathcal{A}}(Z)$$

induces an isomorphism in the torsion parts. Thus, if we apply the functor $A \to A_{tf}$ (which takes an object to its torsion-free quotient) to the exact sequence (4.18.2), we obtain an exact sequence

$$0 \to \Phi(M') \to \Phi(M) \oplus H^0_{\mathcal{A}}(Z)_{tf} \to H^0_{\mathcal{A}}(Z)_{tf} \to 0.$$

So after replacing Z with $H^0_{\mathcal{A}}(Z)_{tf}$, we have a short exact sequence of objects in $\mathcal{V}$:

$$0 \to \Phi(M') \to \Phi(M) \oplus Z \to Z \to 0. \tag{4.18.3}$$

By the first part of Theorem 2.7, each object of $\mathcal{V}$ is isomorphic to the image under Φ of a finite-dimensional semi-simple $\mathcal{S}_C$-module. So we can find a finite-dimensional semi-simple $\mathcal{S}_C$-module U such that $\Phi(U)$ is isomorphic to Z.

From now on, Q may be Dynkin or non-Dynkin. By the surjectivity at the level of extensions stated in Lemma 5.15, respectively in Corollary 4.25, we can lift the triangle (4.18.1), respectively the short exact sequence (4.18.3), to a short exact sequence of $\mathcal{S}_C$-modules

$$0 \to M' \to E \to U \to 0.$$

Since K_L is right exact and K_R is left exact, the image

$$0 \to K_{LR}(M') \to K_{LR}(E) \to K_{LR}(U) \to 0$$

of this sequence is exact at the terms $K_{LR}(M')$ and $K_{LR}(U)$ but cannot be expected to be exact at $K_{LR}(E)$. Thus, we find the inequality

$$\underline{\dim}(K_{LR}(M')) + \underline{\dim}(K_{LR}(U)) \le \underline{\dim}(K_{LR}(E)).$$

Now we also know that we have

$$\begin{aligned} \underline{\dim}\, E &= \underline{\dim}\, M' + \underline{\dim}\, U \\ &= \underline{\dim}\, M + \underline{\dim}\, U \\ &= \underline{\dim}(M \oplus U) \end{aligned}$$

and that $\Phi(E)$ is isomorphic to $\Phi(M) \oplus Z = \Phi(M) \oplus \Phi(U) = \Phi(M \oplus U)$. By Lemma 4.14 and the injectivity of the map $v \mapsto C_q v$, we conclude that we have

$$\underline{\dim}(K_{LR}(E)) = \underline{\dim}(K_{LR}(M \oplus U)).$$

Thus, we obtain the inequality

$$\underline{\dim}\, K_{LR}(M) \ge \underline{\dim}\, K_{LR}(M')$$

as claimed.

Conversely, suppose that we have $v'(x) \le v(x)$ for all vertices $x \in \mathbb{Z}Q_0$. Then the class of $\Phi(M')$ in the split Grothendieck group $K_0^{\text{split}}(\mathcal{D}_Q)$ is obtained from that of $\Phi(M)$ by adding a positive integer linear combination of elements of the form

$$[U] - [E] + [\tau^{-1}(U)] \tag{4.18.4}$$

associated with Auslander–Reiten triangles

$$U \to E \to \tau^{-1}(U) \to \Sigma U \tag{4.18.5}$$

for indecomposables U of $\mathcal{D}_Q$. By the transitivity of the degeneration relation, we may assume that the class of $\Phi(M')$ is obtained from that of $\Phi(M)$ by adding a single element (4.18.4). This means that we have decompositions

$$\Phi(M) \xrightarrow{\sim} V \oplus E \quad \text{and} \quad \Phi(M') \xrightarrow{\sim} V \oplus U \oplus \tau^{-1}(U)$$

for some V in $\mathcal{D}_Q$. Now if we add a split triangle over the identity of $V \oplus \tau^{-1}(U)$ to the triangle (4.18.5), we obtain a triangle

$$V \oplus \tau^{-1}(U) \oplus U \to V \oplus \tau^{-1}(U) \oplus E \to \tau^{-1}(U) \to \Sigma(V \oplus \tau^{-1}(U) \oplus U),$$

and this is of the form

$$\Phi(M') \to \Phi(M) \oplus Z \to Z \to \Phi(M').$$

Thus, we have $\Phi(M) \le \Phi(M')$ as claimed.

4.19. Description of the fibers. Our goal in this subsection is to prove Theorem 2.9. We first determine which fibres are non-empty. Let w be a dimension vector of $\mathcal{S}_C$ (i.e. a dimension vector of $\mathcal{S}$ whose support is contained in C). Let M be a point of $\mathcal{M}_0(w)$. Let $L_0 = K_{LR}(M)$ and $(v_0, w) = \underline{\dim}\, L_0$. Recall from Section 4.1 that

$$CK(M) = K_R(M)/K_{LR}(M)$$

is an injective module over $k(\mathbb{Z}Q)$ isomorphic to $D\mathcal{D}_Q(\Phi(M), ?)$. For a dimension vector u of $k(\mathbb{Z}Q)$, let $\mathrm{Gr}_u(CK(M))$ denote the quiver Grassmannian of $k(\mathbb{Z}Q)$-submodules $N \subset CK(M)$ such that $\underline{\dim}\, N = u$.

Lemma 4.20. *Let v be a dimension vector of the mesh category $k(\mathbb{Z}Q)$. The fiber of $\pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w)$ over M is non-empty iff the quiver Grassmannian $\mathrm{Gr}_{v-v_0}(CK(M))$ is non-empty.*

Proof. Suppose the fiber is non-empty. Then there is a stable $\mathcal{R}_C$-module L with the properties that $\underline{\dim}\, L = (v, w)$ and $\mathrm{res}(L) = M$. The adjunction morphisms yield a commutative diagram

$$\begin{array}{ccccc}
K_L(M) & & \xrightarrow{\text{can}} & & K_R(M) \\
\| & \searrow\!\!\!\!\to & & \nearrow & \| \\
\| & & K_{LR}(M) & & \| \\
\| & & \downarrow & & \| \\
K_L(\mathrm{res}\, L) & \xrightarrow[\varepsilon L]{} & L & \xrightarrow[\eta L]{} & K_R(\mathrm{res}\, L).
\end{array}$$

Here the map ηL is injective since L is stable. Since the canonical morphism $K_L(L) \to K_R(L)$ equals $(\eta L)(\varepsilon L)$, it follows that its image is contained in that of ηL and we obtain a canonical injection $K_{LR}(M) \to L$. The quotient $L/K_{LR}(M)$ is isomorphic to the image $\overline{L}$ of L in $CK(M) = K_R(M)/K_{LR}(M)$, which is a submodule of dimension $v - v_0$. Conversely, if $U \subset CK(M)$ is a submodule of dimension $v - v_0$, its inverse image $L \subset K_R(M)$ is a stable $\mathcal{R}_C$-module of dimension vector (v, w) such that $\mathrm{res}(L) = M$. $\square$

Lemma 4.21. *Let v be a dimension vector of the mesh category $k(\mathbb{Z}Q)$. The fibre of $\pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w)$ over M is homeomorphic, in the complex-analytic topology, to the quiver Grassmannian* $\mathrm{Gr}_{v-v_0}(CK(M))$ *of* $CK(M) = D\mathcal{D}_Q(\Phi(M), ?)$.

Proof. By Lemma 4.20, we can assume that the fiber is non-empty. Hence $w\sigma - C_q v_0$ has non-negative components: these components indicate the multiplicities of the indecomposable factors of $\Phi(M)$ by Lemma 4.14. Consider the following dimension vector of $\mathcal{S}_C$:

$$w_0 = w - (C_q v_0)\sigma^{-1}.$$

Let S be the semi-simple $\mathcal{S}_C$-module of dimension vector w_0. By Nakajima's slice theorem (cf. [28, Theorem 2.4.9] based on [36, Theorem 3.14] based on [35, Section 3.3]), the fibre of $\pi : \mathcal{M}(v, w) \to \mathcal{M}_0(w)$ over M is homeomorphic, in the complex-analytic topology, to the fibre of

$$\pi : \mathcal{M}(v - v_0, w_0) \to \mathcal{M}_0(w_0)$$

over S. Moreover, it follows from Lemma 4.14 that $CK(M)$ is isomorphic to $CK(S)$. So it remains to prove the assertion for $M = S$. In this case, it was shown by Savage–Tingley in [43, Theorem 5.4], who used input from Shipman's [44] to improve on a bijection constructed in the non-graded case by Lusztig in [32, Theorem 2.26]. □

4.22. Exactness of Φ in the non-Dynkin case. Let Q be a connected non-Dynkin quiver. Our goal in this subsection is to show that the stratifying functor Φ constructed in Section 4.1 is exact in a suitable sense. Let $H : k(\mathbb{Z}Q) \to \mathcal{D}_Q$ be Happel's embedding (cf. Theorem 2.2). Its image consists of the τ-orbits in $\mathcal{D}_Q$ of the indecomposable projective kQ-modules. Recall that $\mathcal{V}$ denotes the category of all finite direct sums of objects in the image. The category $\mathcal{V}$ is the category of 'vector bundles' on the 'non-commutative curve' whose category of coherent sheaves is the heart of the t-structure on $\mathcal{D}_Q$ whose left aisle consists of the objects X such that $H^1(X)$ is a preinjective kQ-module and $H^p(X)$ vanishes for all $p \geq 2$. In particular, $\mathcal{V}$ is an exact category, whose conflations are the sequences which give rise to triangles in $\mathcal{D}_Q$.

Theorem 4.23. *The functor* $\Phi : \mathrm{mod}(\mathcal{S}_C) \to \mathcal{V}$ *is exact.*

Lemma 4.24. *The functors K_L and K_R* $: \mathrm{Mod}(\mathcal{S}_C) \to \mathrm{Mod}(\mathcal{R}_C)$ *are exact.*

Proof of Lemma 4.24. By applying the restriction functor to the sequences of part (b) of Theorem 3.9, we see that $\mathrm{res}(x^\wedge) = \mathrm{res}(P_C(x))$ is projective and $\mathrm{res}(x^\vee) = \mathrm{res}(I_C(x))$ is injective for each non-frozen vertex x. Moreover, the restriction of $\sigma(x)^\wedge$ is clearly projective and that of $\sigma(x)^\vee$ injective. It follows that the restriction functor preserves projectivity and thus its right adjoint K_R is exact. Moreover, we see that res takes finitely cogenerated injective modules to injective modules. Now in order to check whether a sequence is exact, it suffices to check whether its image under $\mathrm{Hom}(?, I)$ is exact for each finitely cogenerated injective module I. Thus, the left adjoint K_L of res is also exact. □

Proof of Theorem 4.23. Let

(4.24.1) $$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of $\operatorname{mod}(\mathcal{S}_C)$. We know (for example from the appendix of [24]) that in order to show that the sequence

$$0 \to \Phi(M') \to \Phi(M) \to \Phi(M'') \to 0$$

is a conflation of the exact category $\mathcal{V}$, it suffices to show that the sequence

$$0 \to \operatorname{Hom}(\Phi(M''), ?) \to \operatorname{Hom}(\Phi(M), ?) \to \operatorname{Hom}(\Phi(M'), ?) \to 0$$

is exact in the abelian category of left exact functors $\operatorname{Lex}(\mathcal{V}^{\mathrm{op}}) \subset \operatorname{Mod}(\mathcal{V}^{\mathrm{op}})$. By Lemma 4.24, the images of the sequence (4.24.1) under K_L and K_R are exact. By the Snake Lemma, we thus have an exact sequence

$$0 \to KK(M') \to KK(M) \to KK(M'') \to CK(M') \to CK(M) \to CK(M'') \to 0.$$

Since we have

$$CK(M)(x) = D\operatorname{Hom}(\Phi(M), H(x)),$$

we deduce that the sequence

$$0 \to \operatorname{Hom}(\Phi(M''), ?) \to \operatorname{Hom}(\Phi(M), ?) \to \operatorname{Hom}(\Phi(M'), ?)$$

is exact in the category $\operatorname{Mod}(\mathcal{V}^{\mathrm{op}})$ and thus in $\operatorname{Lex}(\mathcal{V}^{\mathrm{op}})$. In order to show that the morphism

$$\operatorname{Hom}(\Phi(M), ?) \to \operatorname{Hom}(\Phi(M'), ?)$$

is epimorphic in $\operatorname{Lex}(\mathcal{V}^{\mathrm{op}})$ it suffices to show that its cokernel U in $\operatorname{Mod}(\mathcal{V}^{\mathrm{op}})$ is effaceable, i.e. that for each element f of $U(V)$, $V \in \mathcal{V}$, there is an inflation $V \to V'$ such that the map $U(V) \to U(V')$ takes f to zero. Now the cokernel U is a subfunctor of the k-dual of $KK(M'')$, and $KK(M'')$ is right bounded. Now all the Auslander–Reiten sequences

$$0 \to H(x) \to \bigoplus_{x \to y} H(y) \to H(\tau^{-1}(x)) \to 0,$$

where x is a vertex of $\mathbb{Z}Q_0$ and the sum ranges over all arrows of $\mathbb{Z}Q$ with source x, are conflations of $\mathcal{V}$. In particular, the maps

$$0 \to H(x) \to \bigoplus_{x \to y} H(y)$$

are inflations. This shows that each right bounded left $\mathcal{V}$-module is effaceable. Thus, the k-dual of $KK(M'')$ and its submodule U are effaceable, as claimed. □

Corollary 4.25. *Let U be a finite-dimensional semi-simple $\mathcal{S}_C$-module and let M be a finite-dimensional $\mathcal{S}_C$-module. Then Φ induces a surjection*

$$\operatorname{Ext}^1_{\mathcal{S}_C}(U, M) \to \operatorname{Ext}^1_{\mathcal{D}_Q}(\Phi(U), \Phi(M)).$$

Proof. We may assume that U is a simple $\mathcal{S}_C$-module $S_{\sigma^{-1}(x)}$. If M is also a simple $\mathcal{S}_C$-module $S_{\sigma^{-1}(y)}$, the claim is easy to check using part (b) of Theorem 3.9. For the general case, we use induction on the length of M and the fact that both functors $\operatorname{Ext}^1_{\mathcal{S}_C}(U, ?)$ and $\operatorname{Ext}^1_{\mathcal{D}_Q}(\Phi(U), \Phi(?))$ are right exact. For $\operatorname{Ext}^1_{\mathcal{S}_C}(U, ?)$, this is again a consequence of part (b) of Theorem 3.9, which yields a resolution of length 1. □

5. The Dynkin case

5.1. The singularity category of $\mathcal{S}_C$. Suppose that Q is a connected acyclic quiver. By part (b) of Theorem 3.9, if Q is not a Dynkin quiver, the category $\mathcal{S}_C$ is the path category of an (infinite) quiver and thus is of global dimension one. On the other hand, if Q is a Dynkin quiver, by part (a) of Theorem 3.9, the category $\mathcal{S}_C$ is of infinite global dimension. We will show that its singularity category (defined via Gorenstein projective/injective modules) is equivalent to the derived category of Q. We will then use this equivalence to construct the stratifying functor $\Phi : \operatorname{mod} \mathcal{S}_C \to \mathcal{D}_Q$ as outlined in Section 2.14.

5.2. Construction of resolutions. From now on, we suppose that Q is a Dynkin quiver and C is a configuration in $\mathbb{Z}Q$ satisfying Assumption 3.4. We have the restriction functor and its right and left adjoints, which we denote by the same symbols as in the case where $C = \mathbb{Z}Q_0$ considered in Section 2.10:

$$\begin{array}{c} \operatorname{Mod}(\mathcal{R}_C) \\ K_L \uparrow \quad \downarrow \operatorname{res} \quad \uparrow K_R \\ \operatorname{Mod}(\mathcal{S}_C). \end{array}$$

The Kan extensions K_L and K_R are fully faithful so that res is a localization of abelian categories in the sense of [10]. Recall from Lemma 4.5 that an object M belongs to the image of K_R if and only if, for each $\mathcal{R}_C$-module N with $\operatorname{res}(N) = 0$, we have $\operatorname{Hom}(N, M) = 0$ and $\operatorname{Ext}^1(N, M) = 0$.

Lemma 5.3. *The following hold.*

(a) *Each $\mathcal{R}_C$-module N with* $\operatorname{res}(N) = 0$ *is the union of its submodules of finite length.*

(b) *An $\mathcal{R}_C$-module M belongs to the image of K_R if and only if we have*

$$\operatorname{Hom}(S_x, M) = 0 \quad \textit{and} \quad \operatorname{Ext}^1(S_x, M) = 0$$

for each non-frozen vertex x.

Proof. (a) Let N be an $\mathcal{R}_C$-module such that $\operatorname{res}(N) = 0$. Then N is a module over the quotient of $\mathcal{R}_C$ by the ideal generated by the identities of the objects $\sigma(x)$, $x \in \mathbb{Z}Q_0$. Now this quotient is equivalent to the category of indecomposable objects of the derived category $\mathcal{D}_Q$. Since Q is a Dynkin quiver, the projective $\mathcal{D}_Q$-modules $\mathcal{D}_Q(?, u)$, $u \in \mathcal{D}_Q$, are of finite length. Thus, each $\mathcal{D}_Q$-module is the union of its submodules of finite length and the same holds for the $\mathcal{R}_C$-modules whose restriction to $\mathcal{S}_C$ vanishes. Thus, the claim holds for N.

(b) Of course, the condition is necessary. Suppose conversely that it holds for some $\mathcal{R}_C$-module M. By part (a), it follows that we have $\operatorname{Hom}(N, M) = 0$ for each $\mathcal{R}_C$-module N with $\operatorname{res}(N) = 0$. Thus, the adjunction morphism $M \to K_R \operatorname{res}(M)$ is injective and we have an exact sequence

$$0 \to M \to K_R \operatorname{res}(M) \to M' \to 0,$$

where $\operatorname{res}(M') = 0$. We have the exact sequence

$$\operatorname{Hom}(S_x, K_R \operatorname{res}(M)) \to \operatorname{Hom}(S_x, M') \to \operatorname{Ext}^1(S_x, M),$$

where the first and the last term vanish. Thus, the module M' has no submodules of finite length. By part (a), we must have $M' = 0$. □

Lemma 5.4. *The following hold.*

(a) *For each finitely generated projective $\mathcal{S}_C$-module P, the canonical morphism*

$$K_L P \to K_R P$$

is invertible.

(b) *For each finitely generated projective $\mathcal{R}_C$-module P, the canonical morphism*

$$P \to K_R(\operatorname{res} P)$$

is invertible.

(c) *For each finitely generated projective $\mathcal{R}_C$-module P, the module $K_{LR}(\operatorname{res}(P))$ is isomorphic to the submodule of P generated by the images of all morphisms $\sigma(x)^\wedge \to P$, $x \in C$.*

Proof. (a) It suffices to show that $K_L P$ belongs to the image of K_R. We check the condition of part (b) of Lemma 5.3. Let z be a non-frozen vertex. By part (e) of Corollary 3.6, we have

$$D\,\mathrm{RHom}(S_z, K_L P) = \mathrm{RHom}(K_L P, \Sigma^2 S_{\tau^{-1}(z)}).$$

This last object vanishes since $K_L P$ is a direct sum of projectives $\sigma(y)^\wedge$, $y \in \mathbb{Z}Q_0$, and $\tau^{-1}(z)$ is a non-frozen vertex.

(b) By part (a), it suffices to prove the assertion for $P = \tau(x)^\wedge$ for any vertex x of $\mathbb{Z}Q$. If we apply $K_R \circ \operatorname{res}$ to the exact sequence

$$0 \to \tau(x)^\wedge \to \sigma(x)^\wedge \to S_{\sigma(x)} \to 0,$$

we obtain the exact sequence

$$0 \to K_R(\operatorname{res}(\tau(x)^\wedge)) \to K_R(\sigma(x)^\wedge) \to K_R(S_{\sigma(x)}).$$

By part (a), we have an isomorphism $\sigma(x)^\wedge \xrightarrow{\sim} K_R(\sigma(x)^\wedge)$ and we have a monomorphism $K_R(S_{\sigma(x)}) \to \sigma(x)^\vee$. Thus, we have an exact sequence

$$0 \to K_R(\operatorname{res}(\tau(x)^\wedge)) \to \sigma(x)^\wedge \to \sigma(x)^\vee$$

and one checks that the morphism $\sigma(x)^\wedge \to \sigma(x)^\vee$ is non-zero. Thus, its image is $S_{\sigma(x)}$ and we find that $K_R(\operatorname{res}(\tau(x)^\wedge))$ is the kernel of $\sigma(x)^\wedge \to S_{\sigma(x)}$. But this is $\tau(x)^\wedge$.

(c) By definition, $K_{LR}(\operatorname{res}(P))$ is the sum of the images in $K_R(\operatorname{res}(P))$ of all the morphisms

$$K_L(\operatorname{res}(\sigma(x)^\wedge)) \to K_R(\operatorname{res}(P))$$

induced by morphisms $\sigma(x)^\wedge \to P$. Now trivially, we have $K_L(\operatorname{res}(\sigma(x)^\wedge)) \xrightarrow{\sim} \sigma(x)^\wedge$ and by part a), we have $P \xrightarrow{\sim} K_R(\operatorname{res}(P))$. This implies the claim. □

The following lemma will be of great use.

Lemma 5.5. *Let x be a vertex of $\mathbb{Z}Q$. Let P be a finitely generated projective $\mathcal{S}_C$-module and let I be a finitely cogenerated injective $\mathcal{S}_C$-module.*

(a) *The image under* $\mathrm{Hom}(\mathrm{res}(?), P)$ *of the resolution*

$$0 \to (\Sigma^{-1}x)^\wedge \to P_C(x) \to x^\wedge \to 0 \tag{5.5.1}$$

of $x^\wedge_{\mathcal{D}}$ constructed in Theorem 3.7 *is acyclic.*

(b) *The image under* $\mathrm{Hom}(I, \mathrm{res}(?))$ *of the coresolution*

$$0 \to x^\vee \to I_C(x) \to (\Sigma x)^\vee \to 0 \tag{5.5.2}$$

of $x^\vee_{\mathcal{D}}$ constructed in Theorem 3.7 *is acyclic.*

Proof. (a) We have $\mathrm{Hom}(\mathrm{res}(?), P) = \mathrm{Hom}(?, K_R(P))$ and by Lemma 5.4, we know that $K_R(P)$ is isomorphic to $K_L(P)$, which is a finite direct sum of projective $\mathcal{R}_C$-modules $\sigma(y)^\wedge$ associated with the vertices y of $\mathbb{Z}Q$. So the claim is that $\mathrm{RHom}(x^\wedge_{\mathcal{D}}, \sigma(y)^\wedge)$ vanishes. Since $x^\wedge_{\mathcal{D}}$ is a finite-dimensional module concentrated on non-frozen vertices, it suffices to show that $\mathrm{RHom}(S_z, \sigma(y)^\wedge)$ vanishes for each non-frozen vertex z. Now by part (e) of Corollary 3.6, we have

$$D\,\mathrm{RHom}(S_z, \sigma(y)^\wedge) = \mathrm{RHom}(\sigma(y)^\wedge, \Sigma^2 S_{\tau(z)})$$

and the last object is isomorphic to a shift of $DS_{\tau(z)}(\sigma(y)) = 0$. The proof of (b) is dual. □

5.6. The weak Gorenstein property. The next lemma implies that the category $\mathcal{S}_C$ is *weakly Gorenstein of dimension* 1 in the sense that we have

$$\mathrm{Ext}^p_{\mathcal{S}_C}(M, P) = 0 = \mathrm{Ext}^p_{\mathcal{S}_C}(I, M)$$

for all $p \geq 2$ and each finite-dimensional module M, each finitely generated projective module P and each finitely cogenerated injective module I.

Lemma 5.7. *The following hold.*

(a) *We have* $\mathrm{Ext}^p_{\mathcal{S}_C}(I, M) = 0$ *for all $p \geq 2$, for each finitely cogenerated injective $\mathcal{S}_C$-module I and each pointwise finite-dimensional right bounded $\mathcal{S}_C$-module M.*

(b) *We have* $\mathrm{Ext}^p_{\mathcal{S}_C}(M, P) = 0$ *for all $p \geq 2$, for each finitely generated projective $\mathcal{S}_C$-module P and each pointwise finite-dimensional left bounded $\mathcal{S}_C$-module M.*

Proof. (a) We may and will assume that $I = \sigma(x)^\vee$ for some vertex x in C.

First step: If M is finite-dimensional, then $\mathrm{Ext}^p_{\mathcal{S}_C}(\sigma(x)^\vee, M)$ *is finite-dimensional for all integers p and vanishes for all $p \geq 2$.* It suffices to prove the statement for a simple module $M = S_{\sigma(y)}$. Now the injective resolution of $S_{\sigma(y)}$ in part (a) of Theorem 3.9 is the complex of $\mathcal{S}_C$-modules

$$0 \to \sigma(y)^\vee \to I_C(y) \to I_C(\Sigma y) \to I_C(\Sigma^2 y) \to \cdots,$$

which is spliced together from $\sigma(y)^\vee \to y^\vee$ and the sequences

$$(\Sigma^{p-1}y)^\vee \to I_C(\Sigma^{p-1}y) \to (\Sigma^p y)^\vee, \quad p \geq 1,$$

which are extracted from the co-resolutions

$$0 \to (\Sigma^{p-1} y)^{\vee}_{\mathcal{D}} \to (\Sigma^{p-1} y)^{\vee} \to I_C(\Sigma^{p-1} y) \to (\Sigma^p y)^{\vee} \to 0$$

constructed in part (a) of Theorem 3.7. Now the fact that $\mathrm{Ext}^p_{\mathcal{S}_C}(\sigma(x)^{\vee}, S_{\sigma}(y))$ vanishes for $p \geq 2$ follows from Lemma 5.5.

Second step: The claim. Since M is right-bounded and pointwise finite-dimensional, it is the inverse limit of a countable system

$$\cdots \to M_i \to M_{i-1} \to \cdots \to M_1 \to M_0$$

of finite-dimensional modules. We have

$$\mathrm{RHom}(\sigma(x)^{\vee}, M) = \mathrm{Rlim}\, \mathrm{RHom}(\sigma(x)^{\vee}, M_i).$$

Since the homology of each complex $\mathrm{RHom}(\sigma(x)^{\vee}, M_i)$ is finite-dimensional (by the first step), we obtain that

$$\mathrm{Ext}^p(\sigma(x)^{\vee}, M) = \lim \mathrm{Ext}^p(\sigma(x)^{\vee}, M_i)$$

for each integer p. By the first step, this implies the claim.

The proof of (b) is dual. □

Question 5.8. Are the injective $\mathcal{S}_C$-modules of projective dimension at most 1 and the projective $\mathcal{S}_C$-modules of injective dimension at most 1?

We do not know the answer if C is the set of all vertices of $\mathbb{Z}Q$. On the other hand, in certain cases, the classes of projective and injective $\mathcal{S}_C$-modules coincide, for example when C is chosen so that $\mathcal{S}_C$ is the category of projective modules over the repetitive algebra of an algebra B derived equivalent to the Dynkin quiver Q (in particular if $B = kQ$ as in Leclerc–Plamondon's [31]), cf. also Section 5.24.

5.9. Coherence. We consider the category $\mathcal{S}_C = \mathcal{S}$ associated with the set C of all vertices of $\mathbb{Z}Q$. Let $\mathcal{T}$ be the full subcategory of $\mathcal{R}$ whose objects are all the vertices of $\mathbb{Z}Q$. The following proposition implies in particular part (a) of Proposition 2.15.

Proposition 5.10. *The following hold.*

(a) *The category $\mathcal{T}$ is hereditary and thus coherent.*

(b) *The category $\mathcal{R}$ is coherent.*

(c) *The category $\mathcal{S}$ is coherent.*

Remark 5.11. We do not know under what conditions on C the category $\mathcal{S}_C$ is coherent. Clearly, this holds if $\mathcal{S}_C$ happens to be locally bounded, i.e. if, for each object x of $\mathcal{S}_C$, there are at most finitely many objects y such that $\mathcal{S}_C(x, y) \neq 0$ or $\mathcal{S}_C(y, x) \neq 0$. By the proposition, it also holds for $C = \mathbb{Z}Q_0$.

Proof of Proposition 5.10. (a) Let $\widetilde{\mathcal{R}}$ be the path category of $\mathbb{Z}\widetilde{Q}$ and let $\widetilde{\mathcal{T}}$ be the path category of $\mathbb{Z}Q$. The projection $\widetilde{\mathcal{R}} \to \mathcal{R}$ induces a functor $P : \widetilde{\mathcal{T}} \to \mathcal{T}$. It is not hard to see that there is a well-defined inverse functor $S : \mathcal{T} \to \widetilde{\mathcal{T}}$ such that

$$S(\alpha_x \beta_x) = -\sum_{i=1}^{s} \alpha_i \sigma(\alpha_i)$$

whenever we have a mesh of $\mathbb{Z}\widetilde{Q}$ with arrows

$$\tau(x) \xrightarrow{\beta_x} \sigma(x) \xrightarrow{\alpha_x} x$$

and arrows

$$\tau(x) \xrightarrow{\sigma(\alpha_i)} y_i \xrightarrow{\alpha_i} x, \quad 1 \le i \le s.$$

Thus, the category $\mathcal{T}$ is isomorphic to $\widetilde{\mathcal{T}}$, which is hereditary since it is the path category of a quiver.

(b) Let $f : P_1 \to P_0$ be a morphism in $\operatorname{proj}(\mathcal{R})$. We need to show that its kernel is finitely generated. Since it is a submodule of P_1, it is pointwise finite-dimensional and right bounded. Thus, it has a projective cover and it suffices to show that $\operatorname{Hom}(\ker(f), S_u)$ vanishes for all but finitely many vertices u of $\mathbb{Z}\widetilde{Q}$. We first consider vertices u of the form $\sigma(x)$ for some vertex x of $\mathbb{Z}Q$. We have the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow P_1 \xrightarrow{f} \operatorname{im}(f) \longrightarrow 0$$

and deduce the exactness of the sequence

$$\operatorname{Hom}(P_1, S_u) \to \operatorname{Hom}(\ker(f), S_u) \to \operatorname{Ext}^1(\operatorname{im}(f), S_u) \to 0.$$

Thus, it suffices to show that $\operatorname{Ext}^1(\operatorname{im}(f), S_u)$ vanishes for all but finitely many u. We have

$$0 \to \operatorname{im}(f) \to P_0 \to \operatorname{cok}(f) \to 0$$

and so we have

$$\operatorname{Ext}^1(\operatorname{im}(f), S_u) \xrightarrow{\sim} \operatorname{Ext}^2(\operatorname{cok}(f), S_u).$$

Now for $u = \sigma(x)$, the module S_u is of injective dimension at most 1 and so both terms vanish. We deduce that $\operatorname{Hom}(P_1, S_u) \to \operatorname{Hom}(\ker(f), S_u)$ is surjective. Thus, there are at most finitely many vertices $u = \sigma(x)$ such that $\operatorname{Hom}(\ker(f), S_u)$ is non-zero. It remains to study the case where $u = x$ for some vertex x of $\mathbb{Z}Q$. Now since S_x is a $\mathcal{T}$-module, we have an injection

$$\operatorname{Hom}_{\mathcal{R}}(\ker(f), S_x) \subset \operatorname{Hom}_{\mathcal{T}}(\operatorname{res}_{\mathcal{T}}(\ker(f)), S_x).$$

So it suffices to show that the right hand term vanishes for almost all vertices x of $\mathbb{Z}Q$. Now $\operatorname{res}_{\mathcal{T}}(\ker(f))$ identifies with the kernel of the restriction $\operatorname{res}_{\mathcal{T}}(f) : \operatorname{res}_{\mathcal{T}}(P_1) \to \operatorname{res}_{\mathcal{T}}(P_0)$. The restriction of a module $\sigma(x)^\wedge$ to $\mathcal{T}$ is isomorphic to $\tau(x)^\wedge$ and the restriction of a module $x^\wedge_{\mathcal{R}}$ to $x^\wedge_{\mathcal{T}}$. Thus the restrictions of P_0 and P_1 to $\mathcal{T}$ are finitely generated projective and by part (a), the kernel $\ker(\operatorname{res}_{\mathcal{T}}(f))$ is finitely generated. This shows that $\operatorname{Hom}(\operatorname{res}_{\mathcal{T}}(\ker(f)), S_x)$ vanishes for almost all vertices x of $\mathbb{Z}Q$.

(c) Let $f : P_0 \to P_1$ be a morphism of $\operatorname{proj}(\mathcal{S})$. Then

$$f \otimes_{\mathcal{S}} \mathcal{R} : P_1 \otimes_{\mathcal{S}} \mathcal{R} \to P_0 \otimes_{\mathcal{S}} \mathcal{R}$$

is a morphism of $\mathrm{proj}(\mathcal{R})$ and its restriction to $\mathcal{S}$ identifies with f. We have

$$\ker(f) \xrightarrow{\sim} \mathrm{res}_{\mathcal{S}}(\ker(f \otimes_{\mathcal{S}} \mathcal{R})).$$

By part (b), the module $\ker(f \otimes_{\mathcal{S}} \mathcal{R})$ is finitely generated. Now the claim follows because for each vertex u of $\mathbb{Z}\widetilde{Q}$, the module $\mathrm{res}_{\mathcal{S}}(u^{\wedge}_{\mathcal{R}})$ is finitely generated: This is clear for the vertices $u = \sigma(x)$, $x \in \mathbb{Z}Q_0$; for the vertices $u = x$, $x \in \mathbb{Z}Q$, it follows from part (a) of Theorem 3.7. □

5.12. Two Frobenius categories. Recall that, for a k-category $\mathcal{C}$, a $\mathcal{C}$-module M is *Gorenstein projective* [7] if there is an acyclic complex

$$P : \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots$$

of finitely generated projective modules such that M is isomorphic to the cokernel of $P_1 \to P_0$ and that the complex $\mathrm{Hom}(P, P')$ is still acyclic for each finitely generated projective $\mathcal{C}$-module P'. Dually, a $\mathcal{C}$-module M is *Gorenstein injective* if there is an ayclic complex of finitely cogenerated injective $\mathcal{C}$-modules

$$I : \cdots \to I^{-1} \to I^0 \to I^1 \to \cdots$$

such that M is isomorphic to the kernel of $I^0 \to I^1$ and the complex $\mathrm{Hom}(I', I)$ is still acyclic for each finitely cogenerated injective $\mathcal{S}_C$-module I'. By [2, Proposition 5.1], the full subcategories $\mathrm{gpr}(\mathcal{C})$ and $\mathrm{gin}(\mathcal{C})$ formed by the Gorenstein projective, respectively injective, modules are closed under extensions in $\mathrm{Mod}(\mathcal{C})$. It then follows easily that they are Frobenius exact categories and that their subcategories of projective-injective objects are the subcategory of finitely generated projective $\mathcal{C}$-modules, respectively finitely cogenerated injective $\mathcal{C}$-modules.

For each $\mathcal{S}_C$-module M, choose exact sequences

$$0 \to \Omega M \to P_M \to M \to 0 \quad \text{and} \quad 0 \to M \to I^M \to \Sigma M \to 0,$$

where P_M is projective and I^M injective. For example, if x is a vertex of $\mathbb{Z}Q$, we can use the restrictions to $\mathcal{S}_C$ of the sequences of $\mathcal{R}_C$-modules

$$0 \to x^{\wedge} \to \sigma^{-1}(x)^{\wedge} \to S_{\sigma^{-1}(x)} \to 0 \quad \text{and} \quad 0 \to S_{\sigma(x)} \to \sigma(x)^{\vee} \to x^{\vee} \to 0$$

so that $\Omega S_{\sigma^{-1}(x)} = \mathrm{res}(x^{\wedge})$ and $\Sigma S_{\sigma(x)} = \mathrm{res}(x^{\vee})$.

Lemma 5.13. *If M is a finite-dimensional $\mathcal{S}_C$-module, then the module ΩM is Gorenstein projective and the module ΣM is Gorenstein injective.*

Proof. Since the category $\mathrm{gpr}(\mathcal{S}_C)$ is closed under extensions in $\mathrm{Mod}(\mathcal{S}_C)$, it suffices to prove the claim when M is a simple module associated with a vertex in C. Let P be the complex obtained by splicing together the restrictions to $\mathcal{S}_C$ of the sequences

$$0 \to (\Sigma^{p-1}x)^{\wedge} \to P(\Sigma^p x) \to (\Sigma^p x)^{\wedge} \to 0$$

extracted from the resolution of $(\Sigma^p x)^{\wedge}_{\mathcal{D}}$ from Theorem 3.7, where $p \in \mathbb{Z}$. By Lemma 5.5, the complex $\mathrm{Hom}(P, P')$ is still acyclic for each finitely generated $\mathcal{S}_C$-module P'. Hence $\Omega S_{\sigma^{-1}(x)} = \mathrm{res}(x^{\wedge})$ is Gorenstein projective. The proof for $\Sigma S_{\sigma(x)}$ is dual. □

For future reference, we record the following easy consequences of the definition of Gorenstein modules.

Lemma 5.14. *Let x be a non-frozen vertex. Let L be a Gorenstein projective module and let M be a Gorenstein injective module. We have isomorphisms*

$$\operatorname{Ext}^1(L, \Omega S_{\sigma^{-1}(x)}) = \operatorname{Hom}(L, S_{\sigma^{-1}(x)}) / \operatorname{Hom}(L, \sigma^{-1}(x)^\wedge),$$
$$\operatorname{Ext}^1(\Sigma S_{\sigma(x)}, M) = \operatorname{Hom}(S_{\sigma(x)}, M) / \operatorname{Hom}(\sigma(x)^\vee, M).$$

Proof. This follows at once by applying the functor $\operatorname{Hom}(L, ?)$ to the sequence

$$0 \to \Omega S_{\sigma^{-1}(x)} \to \sigma^{-1}(x)^\wedge \to S_{\sigma^{-1}(x)} \to 0$$

and using the fact that $\operatorname{Ext}^1(L, \sigma^{-1}(x)^\wedge)$ vanishes because L is Gorenstein projective. Similarly for the second isomorphism. □

Lemma 5.15. *Let L and M be finite-dimensional $\mathcal{S}_C$-modules. Let $L \to I$ be an injection into an injective module and let $P \to M$ be a surjection from a projective module. We have canonical isomorphisms*

$$\operatorname{Ext}^1(\Sigma L, \Sigma M) \xrightarrow{\sim} \operatorname{Ext}^1(L, M) / \operatorname{Ext}^1(I, M),$$
$$\operatorname{Ext}^1(\Omega L, \Omega M) \xrightarrow{\sim} \operatorname{Ext}^1(L, M) / \operatorname{Ext}^1(L, P).$$

Proof. We have $\operatorname{Ext}^1(\Sigma L, \Sigma M) \xrightarrow{\sim} \operatorname{Ext}^2(\Sigma L, M)$. If we apply $\operatorname{Ext}^1(?, M)$ to the sequence

$$0 \to L \to I \to \Sigma L \to 0,$$

we get the first isomorphism. The proof of the second one is dual. □

5.16. Link to the derived category. The stable categories $\underline{\operatorname{gpr}}(\mathcal{S}_C)$ and $\underline{\operatorname{gin}}(\mathcal{S}_C)$ of the Frobenius categories $\operatorname{gpr}(\mathcal{S}_C)$ respectively $\operatorname{gin}(\mathcal{S}_C)$ are canonically triangulated. In accordance with our overall convention, we write Σ for their suspension functors. We will show that these categories are triangle equivalent to $\mathcal{D}_Q$.

Lemma 5.17. *For all vertices x and y of $\mathbb{Z}Q$ and all integers p, we have isomorphisms*

$$\begin{aligned}\underline{\operatorname{gin}}(\mathcal{S}_C)(\Sigma S_{\sigma(x)}, \Sigma^p \Sigma S_{\sigma(y)}) &= \mathcal{D}_Q(H(x), \Sigma^p H(y)) \\ &= \underline{\operatorname{gpr}}(\mathcal{S}_C)(\Omega S_{\sigma^{-1}(x)}, \Sigma^p \Omega S_{\sigma^{-1}(y)}).\end{aligned}$$

Proof. First step: The claim for $p \geq 2$. Let us abbreviate $\Sigma S_{\sigma(x)}$ by Fx. We have

$$\underline{\operatorname{gin}}(\mathcal{S}_C)(Fx, \Sigma^p Fy) = \operatorname{Ext}^p_{\mathcal{S}_C}(Fx, \Sigma S_{\sigma(y)}) = \operatorname{Ext}^{p+1}_{\mathcal{S}_C}(\Sigma S_{\sigma(x)}, S_{\sigma(y)}).$$

Now by the sequence

$$0 \to M \to I^M \to \Sigma M \to 0,$$

where $M = S_{\sigma(x)}$, and part (a) of Lemma 5.7, this last space is isomorphic to

$$\operatorname{Ext}^p_{\mathcal{S}_C}(S_{\sigma(x)}, S_{\sigma(y)}) = \mathcal{D}_Q(H(x), \Sigma^p H(y)),$$

where we have used Corollary 3.10 for the last isomorphism.

Second step: The claim for arbitrary p. Let us first note that by the sequences

$$0 \to (\Sigma^{p-1} y)^{\vee} \to I_C(\Sigma^p y) \to (\Sigma^p y)^{\vee} \to 0$$

of Theorem 3.7, which become exact when restricted to $\mathcal{S}_C$, we have isomorphisms in $\underline{\mathrm{gin}}(\mathcal{S}_C)$:

$$\Sigma^m F y = \Sigma^m \Sigma S_{\sigma(y)} = \Sigma^m \operatorname{res}(y^{\vee}) = \operatorname{res}((\Sigma^m y)^{\vee}) = \Sigma S_{\sigma(\Sigma^m y)} = F \Sigma^m y$$

for all $m \in \mathbb{Z}$. We deduce that, for a given $p \in \mathbb{Z}$ and any $q \geq 2 - p$, we have

$$\underline{\mathrm{gin}}(\mathcal{S}_C)(Fx, \Sigma^p F y) = \underline{\mathrm{gin}}(\mathcal{S}_C)(Fx, \Sigma^{p+q} F(\Sigma^{-q} y)).$$

By the first step, this last space is isomorphic to

$$\mathcal{D}_Q(H(x), \Sigma^{p+q}(H(\Sigma^{-q} y))) = \mathcal{D}_Q(H(x), \Sigma^p H(y)).$$

The proof of the second isomorphism is analogous. □

Theorem 5.18. *There are triangle equivalences*

$$F : \mathcal{D}_Q \xrightarrow{\sim} \underline{\mathrm{gin}}(\mathcal{S}_C) \quad \textit{respectively} \quad F' : \mathcal{D}_Q \xrightarrow{\sim} \underline{\mathrm{gpr}}(\mathcal{S}_C)$$

taking $H(x)$ to $\Sigma S_{\sigma(x)}$ respectively $\Omega S_{\sigma(x)}$ (sic!) for each vertex x of $\mathbb{Z}Q$.

Remark 5.19. Let kQ denote the path category of Q considered as a full subcategory of $\mathcal{R}_C$ via the embedding $i \mapsto (i, 0)$. We have a functor $kQ \to \mathrm{gpr}(\mathcal{S}_C)$ taking x to $\operatorname{res}(x^{\wedge})$. It gives rise to a kQ-$\mathcal{S}_C$-bimodule X given by

$$X(u, x) = \operatorname{Hom}(u^{\wedge}, \operatorname{res}(x^{\wedge})), \quad x \in Q_0,\ u \in \sigma(C).$$

Since $\mathrm{gpr}(\mathcal{S}_C)$ is a Frobenius category, we have a canonical triangle functor

$$\mathrm{can} : \mathcal{D}^b(\mathrm{gpr}(\mathcal{S}_C)) \to \underline{\mathrm{gpr}}(\mathcal{S}_C)$$

cf. for example [27, 39]. Now we can describe the equivalence $F' : \mathcal{D}_Q \xrightarrow{\sim} \underline{\mathrm{gpr}}(\mathcal{S}_C)$ as the composition

$$\mathcal{D}_Q \xrightarrow{?\overset{\mathbf{L}}{\otimes}_{kQ} X} \mathcal{D}^b(\mathrm{gpr}(\mathcal{S}_C)) \xrightarrow{\mathrm{can}} \underline{\mathrm{gpr}}(\mathcal{S}_C).$$

Of course, there is an analogous description for F.

Proof. By Lemma 5.17, when x and y are vertices of Q and p is a non-zero integer, we have

$$\underline{\mathrm{gin}}(\mathcal{S}_C)(\Sigma S_{\sigma(x)}, \Sigma^p S_{\sigma(y)}) = \mathcal{D}_Q(H(x), \Sigma^p H(y)) = 0.$$

Moreover, the endomorphism algebra of the sum of the $\Sigma S_{\sigma(x)}$, $x \in Q_0$, is isomorphic to the path algebra kQ. Since $\underline{\mathrm{gin}}(\mathcal{S}_C)$ is an algebraic triangulated category, it follows that we have a fully faithful triangle functor $F : \mathcal{D}_Q \to \underline{\mathrm{gin}}(\mathcal{S}_C)$ taking $H(x)$ to $\Sigma S_{\sigma(x)}$ for each $x \in Q_0$. Now recall that for an arbitrary vertex x of $\mathbb{Z}Q$, the module $\Sigma S_{\sigma(x)}$ is isomorphic to $\operatorname{res}(x^{\vee})$. By restricting the co-resolution

$$0 \to S_x \to x^{\vee} \to \bigoplus_{x \to y} y^{\vee} \to \tau^{-1}(x)^{\vee} \to 0$$

of part (d) of Lemma 3.5 to $\mathcal{S}_C$ we obtain an exact sequence

$$0 \to \operatorname{res}(x^\vee) \to \bigoplus_{x \to y} \operatorname{res}(y^\vee) \to \operatorname{res}(\tau^{-1}(x)^\vee) \to 0.$$

Starting from the vertices of the slice $Q \subset \mathbb{Z}Q$ and 'knitting' to the left and to the right, we use the triangles associated with these sequences to check that F takes each vertex x of $\mathbb{Z}Q$ to $\operatorname{res}(x^\vee) = \Sigma S_{\sigma(x)}$.

Now fix an object M in $\underline{\operatorname{gin}}(\mathcal{S}_C)$. Clearly the functor

$$\operatorname{Hom}(F(?), M) : \mathcal{D}_Q^{\mathrm{op}} \to \operatorname{Mod} k$$

is cohomological. Moreover, the isomorphism

$$\operatorname{Hom}(FH(x), M) = \operatorname{Hom}(S_{\sigma(x)}, M)/\operatorname{Hom}(\sigma(x)^\vee, M),$$

where morphisms on the left are taken in $\underline{\operatorname{gin}}(\mathcal{S}_C)$ and those on the right in $\operatorname{Mod}(\mathcal{S}_C)$, shows that $\operatorname{Hom}(FH(x), M)$ is only non-zero if $S_{\sigma(x)}$ occurs in the socle of M and that its dimension is bounded by the dimension of the socle of M. Since M is a submodule of a finite sum of modules $\sigma(y)^\vee$, $y \in \mathbb{Z}Q_0$, it follows that $\operatorname{Hom}(F(?), M)$ takes values in the finite-dimensional vector spaces and vanishes on all but finitely many indecomposable objects of $\mathcal{D}_Q$. In particular, $\operatorname{Hom}(F(?), M)$ is a finitely generated cohomological functor on $\mathcal{D}_Q$. This implies that it is representable. Thus, the functor F admits a right adjoint F_ρ and for each object M of $\underline{\operatorname{gin}}(\mathcal{S}_C)$, we have a canonical triangle

$$FF_\rho M \to M \to GM \to \Sigma FF_\rho M,$$

where GM is right orthogonal to the image of $\mathcal{D}_Q$ under F. We will show that this right orthogonal subcategory vanishes. Indeed, suppose that M is an object in the right orthogonal. Since M is a submodule of a finite direct sum of modules $\sigma(y)^\vee$, $y \in \mathbb{Z}Q_0$, it has a finite-dimensional socle. We proceed by induction on its dimension. If it is zero, then M has to be zero. So suppose that $S_{\sigma(x)}$ is a simple submodule in the socle of M. Since M is right orthogonal to the image of F, we have

$$0 = \operatorname{Hom}(FH(x), M) = \operatorname{Hom}(S_{\sigma(x)}, M)/\operatorname{Hom}(\sigma(x)^\vee, M).$$

Thus the inclusion $S_{\sigma(x)} \to M$ extends to a map $\sigma(x)^\vee \to M$. This map is injective since it induces an injection in the socles. Since $\sigma(x)^\vee$ is an injective module, it is actually a direct summand and M is the direct sum of $\sigma(x)^\vee$ and a submodule M', whose socle is of strictly smaller dimension than that of M and which still belongs to the right orthogonal of the image of F. By the induction hypothesis, M' must be injective and so M is injective. □

5.20. Description of Φ via Kan extensions. Let

$$\Phi : \operatorname{mod}(\mathcal{S}_C) \to \mathcal{D}_Q$$

be the composition of $\Omega : \operatorname{mod}(\mathcal{S}_C) \to \underline{\operatorname{gpr}}(\mathcal{S}_C)$ with the quasi-inverse of the equivalence $F' : \mathcal{D}_Q \to \underline{\operatorname{gpr}}(\mathcal{S}_C)$ of Theorem 5.18. Notice that Φ is a δ-functor as the composition of the δ-functor Ω with a triangle equivalence. Equivalently, we could define Φ as the composition of $\Sigma : \operatorname{mod}(\mathcal{S}_C) \to \underline{\operatorname{gin}}(\mathcal{S}_C)$ with the quasi-inverse of the equivalence $F : \mathcal{D}_Q \to \underline{\operatorname{gin}}(\mathcal{S}_C)$.

Let us now prove Proposition 2.13, which claims that for $M \in \mathrm{mod}(\mathcal{S})$, we have functorial isomorphisms of $k(\mathbb{Z}Q)$-modules

$$KK(M) = \mathrm{Hom}_{\mathcal{D}_Q}(H(?), \tau\Phi(M)) \text{ and } CK(M) = D\,\mathrm{Hom}_{\mathcal{D}_Q}(\Phi(M), H(?)),$$

where H is Happel's embedding (Theorem 2.2).

Proof of Proposition 2.13. We only prove the second isomorphism, the proof of the first one being similar. Let $P \to M$ be a surjection with finitely generated projective P.

First step: We have a canonical isomorphism

$$\mathrm{cok}(K_R(P) \to K_R(M)) \xrightarrow{\sim} CK(M).$$

Indeed, by definition, $CK(M)$ is the cokernel of the canonical morphism $K_L(M) \to K_R(M)$. Now we have a commutative diagram

$$\begin{array}{ccc} K_L(P) & \longrightarrow & K_R(P) \\ \downarrow & & \downarrow \\ K_L(M) & \longrightarrow & K_R(M). \end{array}$$

Here the top horizontal arrow is an isomorphism by Lemma 5.4 and the left vertical arrow is surjective since K_L is right exact. The claim follows.

Second step: For each vertex x of $\mathbb{Z}Q$, we have a canonical isomorphism

$$(K_R(M)/K_R(P))(x) \xrightarrow{\sim} \mathrm{Ext}^1(S_{\sigma^{-1}(x)}, M)/\,\mathrm{Ext}^1(S_{\sigma^{-1}(x)}, P).$$

Recall the sequence

$$0 \to \mathrm{res}(x^\wedge) \to \sigma^{-1}(x)^\wedge \to S_{\sigma^{-1}(x)} \to 0.$$

It shows that $\Omega(S_{\sigma^{-1}(x)})$ is isomorphic to $\mathrm{res}(x^\wedge)$. Now we have isomorphisms

$$(K_R(M))(x) = \mathrm{Hom}(x^\wedge, K_R(M)) = \mathrm{Hom}(\mathrm{res}(x^\wedge), M) = \mathrm{Hom}(\Omega S_{\sigma^{-1}(x)}, M)$$

and similarly for P instead of M. Now by definition, we have

$$\mathrm{Ext}^1(S_{\sigma^{-1}(x)}, M) = \mathrm{Hom}(\Omega S_{\sigma^{-1}(x)}, M)/\,\mathrm{Hom}(\sigma^{-1}(x)^\wedge, M)$$

and similarly for P instead of M. The claim follows because each morphism $\sigma^{-1}(x)^\wedge \to M$ factors through $P \to M$.

Third step: For each vertex x of $\mathbb{Z}Q$, we have canonical isomorphisms

$$\mathrm{Ext}^1(S_{\sigma^{-1}(x)}, M)/\,\mathrm{Ext}^1(S_{\sigma^{-1}(x)}, P) \xrightarrow{\sim} \underline{\mathrm{gpr}}(\mathcal{S}_C)(\Omega S_{\sigma^{-1}(x)}, \Sigma\Omega M) = D\mathcal{D}_Q(\Phi M, H(x)).$$

Indeed, since $\mathrm{Ext}^2(S_{\sigma^{-1}(x)}, P)$ vanishes (Lemma 5.7), we have an isomorphism

$$\mathrm{Ext}^1(S_{\sigma^{-1}(x)}, M)/\,\mathrm{Ext}^1(S_{\sigma^{-1}(x)}, P) \xrightarrow{\sim} \mathrm{Ext}^2(S_{\sigma^{-1}(x)}, \Omega M).$$

Now we clearly have an isomorphism

$$\mathrm{Ext}^2(S_{\sigma^{-1}(x)}, \Omega M) = \mathrm{Ext}^1(\Omega S_{\sigma^{-1}(x)}, \Omega M)$$

and the last space identifies with $\underline{\mathrm{gpr}}(\mathcal{S}_C)(\Omega S_{\sigma^{-1}(x)}, \Sigma\Omega M)$. Now we have $F'\Phi M = \Omega M$ and

$$F'(\tau^{-1}H(x)) = F'(H(\tau^{-1}(x))) = \Omega S_{\sigma(\tau^{-1}(x))} = \Omega S_{\sigma^{-1}(x)},$$

whence the isomorphism

$$\underline{\mathrm{gpr}}(\mathcal{S}_C)(\Omega S_{\sigma^{-1}(x)}, \Sigma\Omega M) = \mathcal{D}_Q(\tau^{-1}H(x), \Sigma\Phi M).$$

Finally, we get

$$\begin{aligned}\mathcal{D}_Q(\tau^{-1}H(x), \Sigma\Phi M) &= \mathcal{D}_Q(H(x), \tau\Sigma\Phi M)\\ &= \mathcal{D}_Q(H(x), \nu\Phi M)\\ &= D\mathcal{D}_Q(\Phi M, H(x)).\end{aligned}$$

□

5.21. The regular category as a Gorenstein–Auslander category. Our goal in this subsection is to prove Theorem 2.17. We need the following lemma.

Lemma 5.22. *Let x be a vertex of $\mathbb{Z}Q$. The adjunction morphisms fit into exact sequences*

$$0 \to x^\vee_{\mathcal{D}} \to x^\vee \to K_R\,\mathrm{res}(x^\vee) \to (\Sigma x)^\vee_{\mathcal{D}} \to 0$$

and

$$0 \to (\Sigma^{-1}x)^\wedge_{\mathcal{D}} \to K_L\,\mathrm{res}(x^\wedge) \to x^\wedge \to x^\wedge_{\mathcal{D}} \to 0.$$

Proof. To compute $K_R\,\mathrm{res}(x^\vee)$, we use the injective coresolution

$$0 \to \mathrm{res}(x^\vee) \to I_C(x) \to I_C(\Sigma x)$$

obtained by splicing the exact sequences

$$\begin{gathered}0 \to \mathrm{res}(x^\vee) \to I_C(x) \to \mathrm{res}((\Sigma x)^\vee) \to 0,\\ 0 \to \mathrm{res}((\Sigma x)^\vee) \to I_C(\Sigma x) \to \mathrm{res}((\Sigma^2 x)^\vee) \to 0\end{gathered}$$

from part (a) of Theorem 3.7. We find that $K_R\,\mathrm{res}(x^\vee)$ is the kernel of the composition

$$I_C(x) \twoheadrightarrow (\Sigma x)^\vee \twoheadrightarrow (\Sigma x)^\vee/(\Sigma x)^\vee_{\mathcal{D}} \rightarrowtail I_C(\Sigma x).$$

Thus, $K_R\,\mathrm{res}(x^\vee)$ is also the kernel of the composition f of the first two morphisms in this sequence. Now we have the diagram with exact rows and columns

$$\begin{array}{ccccc} x^\vee/x^\vee_{\mathcal{D}} & \rightarrowtail & \ker(f) & \twoheadrightarrow & (\Sigma x)^\vee_{\mathcal{D}} \\ \| & & \downarrow & & \downarrow \\ x^\vee/x^\vee_{\mathcal{D}} & \rightarrowtail & I_C(x) & \twoheadrightarrow & (\Sigma x)^\vee \\ & & {\scriptstyle f}\downarrow & & \downarrow \\ & & (\Sigma x)^\vee/(\Sigma x)^\vee_{\mathcal{D}} & = & (\Sigma x)^\vee/(\Sigma x)^\vee_{\mathcal{D}}. \end{array}$$

This shows the claim. The proof of the second assertion is dual. □

The following theorem implies Theorem 2.17 when we take C to be the set of all vertices of $\mathbb{Z}Q$.

Theorem 5.23. *The restriction functor induces equivalences*

$$\mathrm{proj}(\mathcal{R}_C) \to \mathrm{gpr}(\mathcal{S}_C) \quad \textit{and} \quad \mathrm{inj}(\mathcal{R}_C) \to \mathrm{gin}(\mathcal{S}_C).$$

It yields isomorphisms from the quiver $\mathbb{Z}\widetilde{Q}_C$ *onto the Auslander–Reiten quivers of* $\mathrm{gpr}(\mathcal{S}_C)$ *and* $\mathrm{gin}(\mathcal{S}_C)$ *so that the vertices of* C *correspond to the projective-injective vertices.*

Proof. By Theorem 5.18, each non-injective indecomposable object of $\mathrm{gin}(\mathcal{S}_C)$ is of the form $\mathrm{res}(x^\vee)$ for some non-frozen vertex x of $\mathbb{Z}\widetilde{Q}$ and of course the indecomposable injective object $\sigma(x)^\vee_{\mathcal{S}_C}$ is the restriction of $\sigma(x)^\vee_{\mathcal{R}_C}$. Thus, the restriction functor is essentially surjective. Let us show that it is fully faithful. Let u and v be any vertices of $\mathbb{Z}\widetilde{Q}$. We need to show that the adjunction morphism

$$v^\vee \to K_R(\mathrm{res}\, v^\vee)$$

induces a bijection

$$\mathrm{Hom}(u^\vee, v^\vee) \to \mathrm{Hom}(u^\vee, K_R(\mathrm{res}\, v^\vee)).$$

If $v = \sigma(y)$ for some non-frozen vertex y, then the adjunction morphism $\sigma(v)^\vee \to K_R(\mathrm{res}\, v^\vee)$ is itself invertible. So let us assume that v is a non-frozen vertex y. By Lemma 5.22, the adjunction morphism $y^\vee \to K_R(\mathrm{res}\, y^\vee)$ is the composition of the epimorphism p in the sequence

$$0 \to y^\vee_{\mathcal{D}} \to y^\vee \xrightarrow{p} y^\vee / y^\vee_{\mathcal{D}} \to 0 \tag{5.23.1}$$

with the monomorphism i in the sequence

$$0 \to y^\vee / y^\vee_{\mathcal{D}} \xrightarrow{i} K_R(\mathrm{res}\, y^\vee) \to (\Sigma y)^\vee_{\mathcal{D}} \to 0. \tag{5.23.2}$$

The sequence (5.23.1) yields the exact sequence

$$\mathrm{Hom}(u^\vee, y^\vee_{\mathcal{D}}) \to \mathrm{Hom}(u^\vee, y^\vee) \to \mathrm{Hom}(u^\vee, y^\vee / y^\vee_{\mathcal{D}}) \to \mathrm{Ext}^1(u^\vee, y^\vee_{\mathcal{D}}).$$

Now $y^\vee_{\mathcal{D}}$ is an iterated extension of objects S_z, $z \in \mathbb{Z}Q_0$. We have

$$\mathrm{RHom}(u^\vee, S_z) = D\,\mathrm{RHom}(\Sigma^{-2} S_{\tau^{-1}(z)}, u^\vee)$$

by Corollary 3.6 and so

$$\mathrm{Hom}(u^\vee, S_z) = D\,\mathrm{Ext}^2(S_z, u^\vee) = 0 \quad \text{and} \quad \mathrm{Ext}^1(u^\vee, S_z) = D\,\mathrm{Ext}^1(S_z, u^\vee) = 0.$$

Thus, the map $\mathrm{Hom}(u^\vee, p)$ is bijective. The sequence (5.23.2) yields the exact sequence

$$0 \to \mathrm{Hom}(u^\vee, y^\vee / y^\vee_{\mathcal{D}}) \to \mathrm{Hom}(u^\vee, K_R(\mathrm{res}(y^\vee)) \to \mathrm{Hom}(u^\vee, (\Sigma y)^\vee_{\mathcal{D}}).$$

We have $\mathrm{Hom}(u^\vee, (\Sigma y)^\vee_{\mathcal{D}}) = 0$ because $(\Sigma y)^\vee_{\mathcal{D}}$ is also an extension of simples S_z, $z \in \mathbb{Z}Q_0$. Thus, the map $\mathrm{Hom}(u^\vee, i)$ is also bijective and the functor res is indeed fully faithful on the subcategory $\mathrm{inj}(\mathcal{R}_C)$. The proof for $\mathrm{proj}(\mathcal{R}_C)$ is dual. The last assertion is clear. □

5.24. Frobenius models for derived categories of Dynkin quivers. Let k be an algebraically closed field. In this subsection, by a *Frobenius category*, we mean a k-linear, Hom-finite Krull–Schmidt category $\mathcal{E}$ endowed with the structure of an exact category for which it is Frobenius.

Let Q be a Dynkin quiver. A *Frobenius model for* $\mathcal{D}_Q$ is a Frobenius category $\mathcal{E}$ together with a triangle equivalence $F : \mathcal{D}_Q \xrightarrow{\sim} \underline{\mathcal{E}}$. For example, if $C \subset \mathbb{Z}Q_0$ is a set of vertices satisfying condition (R) of Section 3.3, then the category $\mathcal{E}_C = \mathrm{gpr}(\mathcal{S}_C)$ becomes a Frobenius model of $\mathcal{D}_Q$: It is a Frobenius category by Section 5.12 and its stable category is equivalent to $\mathcal{D}_Q$ by Theorem 5.18. Now for an arbitrary Frobenius category $\mathcal{E}$, consider the following properties:

(P1) For each indecomposable non-projective object X of $\mathcal{E}$, there is an almost split sequence starting and an almost split sequence ending at X.

(P2) For each indecomposable projective object P of $\mathcal{E}$, the $\mathcal{E}$-module $\mathrm{rad}_{\mathcal{E}}(?, P)$ and the $\mathcal{E}^{\mathrm{op}}$-module $\mathrm{rad}_{\mathcal{E}}(P, ?)$ are finitely generated with simple tops.

(P3) $\mathcal{E}$ is standard, i.e. its category of indecomposables is equivalent to the mesh category of its Auslander–Reiten quiver (cf. [42, Section 2.3, p. 63]).

The existence of almost split triangles in the stable category $\underline{\mathcal{E}}$ implies condition (P1) so that this condition holds in particular in all Frobenius models of $\mathcal{D}_Q$. For $\mathcal{E} = \mathcal{E}_C = \mathrm{gpr}(\mathcal{S}_C)$ as above, the category of indecomposables of $\mathcal{E}$ is equivalent to the mesh category $\mathcal{R}_C$, by Theorem 5.23. We deduce that such categories also satisfy (P2) and (P3). We do not know Frobenius models of $\mathcal{D}_Q$ which do not satisfy (P2). On the other hand, in many cases, condition (P3) does not hold. For example, let us assume that Q is the quiver $1 \to 2 \to 3$ and A the algebra given by the quiver

$$1 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3$$

with the relation $\alpha\beta = 0$. Then the category $\mathcal{C}^b(\mathrm{proj}\, A)$ of bounded complexes of finitely generated projective A-modules becomes a Frobenius model for $\mathcal{D}_Q$ because A is derived equivalent to the path algebra kQ. It is not hard to compute the Auslander–Reiten quiver of the category $\mathcal{E} = \mathcal{C}^b(\mathrm{proj}\, A)$ and to check that it satisfies (P2). However, it is not standard because the simple $\mathcal{E}$-module S_P associated with the complex

$$P = (\cdots \to 0 \to P_3 = P_3 \to 0 \to \cdots)$$

is of projective dimension 2 whereas in a standard Frobenius category satisfying (P2), the simple module associated with a projective object is always of projective dimension ≤ 1.

The Frobenius models of $\mathcal{D}_Q$ naturally form a 2-category: If $(\mathcal{E}, F)$ and $(\mathcal{E}', F')$ are two Frobenius models of $\mathcal{D}_Q$, a 1-*morphism* $(\mathcal{E}, F) \to (\mathcal{E}', F')$ is an exact functor $G : \mathcal{E} \to \mathcal{E}'$ together with an isomorphism $\alpha : \underline{G} \circ F \xrightarrow{\sim} F'$. We leave it to the reader to define the 2-morphisms and to show that a 1-morphism is an equivalence in this 2-category iff its underlying exact functor is an equivalence. For example, an inclusion $C \supset C'$ of sets of vertices satisfying (R) yields a 1-morphism $G : \mathcal{E}_C \to \mathcal{E}_{C'}$ which annihilates all indecomposable projectives associated with the vertices in C but not C'. The following corollary results from Section 5.12, Theorem 5.18, Theorem 5.23 and the above discussion.

Corollary 5.25. *The map taking C to $(\mathcal{E}_C, F_C)$ induces a bijection from the set of subsets $C \subset \mathbb{Z}Q_0$ satisfying condition* (R) *of Section* 3.3 *onto the set of equivalence classes of Frobenius models $(\mathcal{E}, F)$ of $\mathcal{D}_Q$ satisfying* (P2)–(P3). *The inverse bijection sends a Frobenius model $(\mathcal{E}, F)$ to the set $C \subset \mathbb{Z}Q_0$ such that the indecomposable projectives of $\mathcal{E}$ correspond to the vertices $\sigma^{-1}(c)$, $c \in C$, of the Auslander–Reiten quiver of $\mathcal{E}$.*

Proof. The only thing left to check is that if $\mathcal{E}$ is a Frobenius model of $\mathcal{D}_Q$ satisfying conditions (P2) and (P3), then the corresponding set C satisfies condition (R). Indeed, let x be a vertex of $\mathbb{Z}Q$. Let X be the corresponding indecomposable object of $\mathcal{E}$. Since $\mathcal{E}$ is Frobenius, we can find an inflation $X \to I$, where I is injective. In particular, there is a non-zero morphism from X to an indecomposable injective object. Thus, there is a path p from x to $\sigma^{-1}(c)$ for some c in C such that the class of p in $\mathcal{R}_C$ is non-zero. Let us assume, as we may, that this path is of minimal length. It is the composition of the canonical arrow $c \to \sigma^{-1}(c)$ with a path p' from x to c. Suppose that the class of p' in $k(\mathbb{Z}Q)$ vanishes. Then the morphism corresponding to p' vanishes in the stable category $\underline{\mathcal{E}}$. This implies that the class of p' in $\mathcal{R}_C$ is a linear combination of paths factoring through vertices $\sigma^{-1}(c')$ which lie between x and c for the ordering given by the existence of a path. But then we obtain a path with non-zero class in $\mathcal{R}_C$ from x to some $\sigma^{-1}(c')$, which contradicts the minimality of the length of p. □

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