THE INTEGRAL CLUSTER CATEGORY

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ABSTRACT. Integral cluster categories of acyclic quivers have recently been used in the representation-theoretic approach to quantum cluster algebras. We show that over a principal ideal domain, such categories behave much better than one would expect: They can be described as orbit categories, their indecomposable rigid objects do not depend on the ground ring and the mutation operation is transitive.

1. INTRODUCTION

Cluster categories were introduced in [3] for acyclic quivers and independently in [5] for Dynkin quivers of type A. They have played an important rôle in the study of Fomin-Zelevinsky's cluster algebras [8], cf. the surveys [2] [12] [15] [19] [20].

Integral cluster categories appear naturally in the study of quantum cluster algebras as defined and studied in [4] and [7]. Indeed, one would like to interpret the quantum parameter q as the cardinality of a finite field [21] and in order to study the cluster categories over all prime fields simultaneously [18], one considers the cluster category over the ring of integers, cf. the appendix to [18]. In this paper, we continue the study begun there: For an acyclic quiver Q and a principal ideal domain R, we construct the cluster category $\mathcal{C}_{RQ}$ using Amiot's method [1] as a triangle quotient of the perfect derived category of the Ginzburg dg algebra [9] associated with the path algebra RQ. On the other hand, we define the category $\mathcal{C}_{RQ}^{orb}$ as the category of orbits of the bounded derived category of RQ under the action of the autoequivalence $\Sigma^{-2}S$, where S is the Serre functor and Σ the suspension functor. Our first main result is the existence of a natural equivalence

$$\mathcal{C}_{RQ}^{orb} \xrightarrow{\sim} \mathcal{C}_{RQ}.$$

This shows in particular that the orbit category is triangulated. For the case where R is a field, this has been known since [14]; in the general case, it is quite surprising since the algebra RQ is of global dimension ≤ 2 and the proof in [14] strongly used the fact that for a field $\mathbb{F}$, the path algebra $\mathbb{F}Q$ is of global dimension ≤ 1.

Our second main result states that all indecomposable rigid objects of $\mathcal{C}_{RQ}$ are either images of rigid indecomposable RQ-modules or direct factors of the image of ΣRQ. If we combine this with Crawley-Boevey's classification

[6] of rigid indecomposable RQ-modules, we obtain that the classification of the rigid indecomposable objects of $\mathcal{C}_{RQ}$ is independent of the principal ideal domain R and that iterated mutation starting from RQ reaches all indecomposable rigid objects. Again, this is well-known in the field case, cf. [3] [10] [11], but quite surprising in the general case.

The paper is structured as follows: In the first section, we recall general adjointness relations between the derived tensor and the derived Hom-functor for dg algebras over any commutative ring R. We define the (relative) Serre functor.

In the second section, we consider the derived category of an R-algebra A (finitely generated projective over R) and an endofunctor F of the derived category of A which is given by the derived tensor product with an A-bimodule complex Θ. The tensor dg algebra associated with Θ is a differential graded algebra Γ. We give sufficient conditions for the orbit category $\mathcal{C}^{orb}$ of $\mathrm{per}(A)$ by F to embed canonically into the triangulated quotient category $\mathrm{per}(\Gamma)/D_{perR}(\Gamma)$. The methods used in this section generalize the approach used in [13] to algebras over arbitrary commutative rings.

In the third section, we consider the orbit category of the perfect derived category $\mathrm{per}(RQ)$ under the action of the auto-equivalence given by $\Sigma^{-2}S$. This functor is given by the total derived functor associated with the RQ-bimodule complex $\Theta = \Sigma^{-2}\mathrm{Hom}_R(RQ, R)$. By a result of [13], the differential graded tensor algebra of Θ is isomorphic to the Ginzburg algebra Γ associated to the quiver Q with the zero potential. So using the results of the second section, we can embed the orbit category into the integral cluster category $\mathrm{per}(\Gamma)/D_{perR}(\Gamma)$. Furthermore, we show in this section that a relative 3-Calabi-Yau property holds for $D(\Gamma)$.

In the fourth section, under the assumption that R is a principal ideal domain, we show that all rigid indecomposable objects in the integral cluster category come from modules and their suspensions and that the embedding given in section 2 is an equivalence of categories. Hence the orbit category $\mathcal{C}^{orb}$ is triangulated and the integral cluster category is relative 2-Calabi-Yau. Using a result by Crawley-Boevey [6] we establish a bijection between the rigid indecomposable objects in the cluster category over a field $\mathbb{F}$ and those over a ring induced by the triangle functor $? \otimes_R^L \mathbb{F}$. It is then an easy corollary that the rigid indecomposable objects are all related by mutations.

2. Derived categories over commutative rings

Let R be a commutative ring. For an associative differential graded R-algebra A which is cofibrant as an R-module (cf. section 2.12 of [17]), we denote by $D(A)$ the *derived category* of dg A-modules, by $\mathrm{per}(A)$ the *perfect derived category*, i.e. the thick subcategory of $D(A)$ generated by A, and by $D_{\mathrm{per}\,R}(A)$ the full subcategory of $D(A)$ whose objects are the dg A-modules whose underlying complex of R-modules is perfect. Throughout this article, we denote by Σ the shift functor in the derived category. If the underlying R-module of A is finitely generated projective over R, we denote by S_R the (relative) Serre functor of $D(A)$ given by the total derived functor of tensoring with the A-bimodule $\mathrm{Hom}_R^{\cdot}(A, R)$. Here, for two dg A-modules L and M, we denote by $\mathrm{Hom}_A^{\cdot}(L, M)$ the dg R-module whose nth component is the R-module of morphisms of graded A-modules $f : L \to M$ homogeneous of degree n and whose differential sends such an f to $d_M \circ f - (-1)^n f \circ d_L$. We denote by RHom the total derived functor of $\mathrm{Hom}^{\cdot}$. We define A^e to be the dg algebra $A \otimes_R A^{op}$. The following well-known isomorphisms will often be used in the rest of the article.

Lemma 2.1. *Let A and B be two dg R-algebras which are cofibrant over R.*

(1) Let $M \in D(B \otimes A^{op})$, $L \in D(A)$ and $N \in D(B)$. There is a bifunctorial isomorphism

$$\mathrm{RHom}_B(L \otimes_A^L M, N) \xrightarrow{\sim} \mathrm{RHom}_A(L, \mathrm{RHom}_B(M, N))$$

in $D(R)$.

(2) For $P \in \mathrm{per}\,B$ and $M \in D(B)$, there is a bifunctorial isomorphism

$$M \otimes_B^L \mathrm{RHom}_B(P, B) \xrightarrow{\sim} \mathrm{RHom}_B(P, M)$$

in $D(B^{op})$.

(3) For all L and M in $D(A)$, there is a bifunctorial isomorphism

$$\mathrm{RHom}_{A^e}(A, \mathrm{RHom}_R(L, M)) \xrightarrow{\sim} \mathrm{RHom}_A(L, M)$$

in $D(A^{op})$.

Proof. We denote by pM a cofibrant replacement of M and by iM a fibrant replacement of M. We have $L \otimes_A^L M \cong pL \otimes_A pM$. Therefore, we have

$$\begin{aligned}
\mathrm{RHom}_B(L \otimes_A^L M, N) &= \mathrm{Hom}_B^{\cdot}(pL \otimes_A pM, N) \\
&\cong \mathrm{Hom}_A^{\cdot}(pL, \mathrm{Hom}_B^{\cdot}(pM, N)) \\
&= \mathrm{RHom}_A(L, \mathrm{RHom}_B(M, N)).
\end{aligned}$$

This proves (1).

For part (2), we observe that we have a bifunctorial morphism from $pM \otimes_B \mathrm{Hom}_B^{\cdot}(pP, B)$ to $\mathrm{Hom}_B^{\cdot}(pP, pM)$, which is invertible in $D(R)$ for $P = A$ hence for all objects in $\mathrm{per}\,A$.

For part (3), we have

$$\mathrm{RHom}_{A^e}(A, \mathrm{RHom}_R(L, M)) = \mathrm{Hom}^{\cdot}_{A^e}(pA, \mathrm{Hom}^{\cdot}_R(pL, iM)) \ ,$$

where pA is a cofibrant as a dg A^e-module. Then pA is also cofibrant as a dg A-module and as a dg A^{op}-module. We have $\mathrm{Hom}^{\cdot}_{A^e}(pA, \mathrm{Hom}^{\cdot}_R(pL, iM)) \subset$ $\mathrm{Hom}^{\cdot}_{A^{op}}(pA, \mathrm{Hom}^{\cdot}_R(pL, iM))$ and by (1), there is a bifunctorial isomorphism between $\mathrm{Hom}^{\cdot}_{A^{op}}(pA, \mathrm{Hom}^{\cdot}_R(pL, iM))$ and $\mathrm{Hom}^{\cdot}_R(pA \otimes_{A^{op}} pL, iM)$. This isomorphism induces a bijection between $\mathrm{Hom}^{\cdot}_{A^e}(pA, \mathrm{Hom}^{\cdot}_R(pL, iM))$, which consists of all the elements of $\mathrm{Hom}^{\cdot}_{A^{op}}(pA, \mathrm{Hom}^{\cdot}_R(pL, iM))$ that commute with the right action of A, and $\mathrm{Hom}^{\cdot}_A(pA \otimes_{A^{op}} pL, iM)$. Now $\mathrm{Hom}^{\cdot}_A(pA \otimes_A pL, iM)$ is isomorphic to $\mathrm{RHom}_A(L, M)$, which finishes the proof. $\qquad\square$

Proposition 2.2. *Suppose that the underlying R-module of A is finitely generated projective. For $L \in D(A)$ and $M \in \mathrm{per}\, A$, we have the following canonical bifunctorial isomorphism*

$$\mathrm{RHom}_R(\mathrm{RHom}_A(M, S_R L), R) \xrightarrow{\sim} \mathrm{RHom}_A(L, M).$$

If $\mathrm{Hom}_R(A, R)$ belongs to $\mathrm{per}\, A$, then S_R is an auto-equivalence of $\mathrm{per}\, A$ with inverse $\mathrm{RHom}_A(\mathrm{RHom}(?, R), A)$.

Proof. By applying part (2) of 2.1 twice we obtain

$$\mathrm{RHom}_A(M, L \otimes^L_A \mathrm{RHom}_R(A, R)) \cong L \otimes^L_A \mathrm{RHom}_A(M, \mathrm{RHom}_R(A, R)).$$

By part (1) of 2.1, we obtain

$$\mathrm{RHom}_A(M, \mathrm{RHom}_R(A, R)) \cong \mathrm{RHom}_R(M, R).$$

Therefore, we have

$$\begin{aligned}
\mathrm{RHom}_R(\mathrm{RHom}_A(M, &L \otimes^L_A \mathrm{RHom}_R(A, R))), R) \\
&\cong \mathrm{RHom}_R(L \otimes^L_A \mathrm{RHom}_R(M, R), R) \\
&\cong \mathrm{RHom}_A(L, \mathrm{RHom}_R(\mathrm{RHom}_R(M, R), R)) \\
&\cong \mathrm{RHom}_A(L, M)
\end{aligned}$$

by (1) of 2.1. This proves the first part.

If we choose $M = A$, we get, by the first statement,

$$S_R L \cong \mathrm{RHom}_R(\mathrm{RHom}_A(L, A), R).$$

Now we use the fact that $\mathrm{RHom}_R(?, R)$ and $\mathrm{RHom}_A(?, A)$ are duality functors on $\mathrm{per}\, A$.

$\qquad\square$

3. EMBEDDINGS OF ORBIT CATEGORIES

Let A be an associative R-algebra which is finitely generated projective as an R-module and let Θ be a complex of A-A-bimodules. We suppose that Θ is cofibrant as a dg A-bimodule and that Θ is perfect as a dg R-module.

We denote by $F : D(A) \to D(A)$ the functor $? \otimes_A^L \Theta$. We define the dg algebra $\Gamma = T_A(\Theta)$ to be the tensor algebra over A given by

$$A \oplus \Theta \oplus (\Theta \otimes_A \Theta) \oplus \cdots \oplus (\Theta \otimes_A \cdots \otimes_A \Theta) \oplus \cdots \quad .$$

Then Γ is homologically smooth over R by [13, 3.7]. For $N \geq 0$, we denote by $\Gamma^{>N}$ the ideal $\bigoplus_{n>N} \Theta^{\otimes_A n}$ of Γ and put $\Gamma^{\leq N} = \Gamma/\Gamma^{>N}$. Then $D_{\mathrm{per}\,R}(\Gamma)$ is contained in $\mathrm{per}(\Gamma)$ and $\Gamma^{\leq N}$ lies in $D_{\mathrm{per}\,R}(\Gamma)$ for all $N \in \mathbb{N}$. We consider the category $\mathcal{C}(\Gamma) = \mathrm{per}(\Gamma)/D_{\mathrm{per}\,R}(\Gamma)$ and compute its morphism spaces. We have a functor $? \otimes_A^L \Gamma : \mathrm{per}(A) \to \mathcal{C}(\Gamma)$ and a restriction functor $\mathrm{per}(\Gamma) \to D(A)$ induced by the natural embedding of A into Γ. For any $Y \in \mathrm{per}(\Gamma)$ and any $N \in \mathbb{N}$, let $F^N(Y) = Y \otimes_A \Theta^{\otimes_A N}$ and let $m_N : F^N(Y) \to Y$ be induced by the multiplication.

We assume that for any $X \in D_{\mathrm{per}\,R}(A)$ there is an $n \in \mathbb{N}$ such that $\mathrm{Hom}_{D(A)}(F^n(A), X)$ vanishes.

Lemma 3.1. *For Y in $\mathrm{per}(\Gamma)$, we have the following isomorphisms*

$$\mathrm{colim}_N \mathrm{Hom}_{D(\Gamma)}(\Gamma^{>N}, Y) \cong \mathrm{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, Y)$$
$$\cong \mathrm{colim}_N \mathrm{Hom}_{D(A)}(F^{N+1}(A), Y|_A).$$

Proof. By definition, the space $\mathrm{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, Y)$ is given by

$$\mathrm{colim}_{M_\Gamma} \mathrm{Hom}_{D(\Gamma)}(\Gamma', Y) \quad,$$

where M_Γ denotes the category of all morphisms $s : \Gamma' \to \Gamma$ in $D(\Gamma)$ such that $\mathrm{cone}(s)$ lies in $D_{\mathrm{per}\,R}(\Gamma)$. We consider the exact sequence

$$0 \longrightarrow \Gamma^{>N} \xrightarrow{e_N} \Gamma \longrightarrow \Gamma^{\leq N} \longrightarrow 0.$$

As $\Gamma^{\leq N}$ vanishes in $\mathcal{C}(\Gamma)$, the embedding e_N is an isomorphism for any N in $\mathbb{N}$. The transition maps in the direct system are induced by the embedding of $\Gamma^{>N+1}$ into $\Gamma^{>N}$ and the maps from $\mathrm{Hom}_{D(\Gamma)}(\Gamma^{>N}, Y)$ to $\mathrm{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, Y)$ by composing with the inverse of e_N.

By a classical result of Verdier, it is sufficient to show that for every morphism s in M_Γ, there is an $N \in \mathbb{N}$ such that e_N factors through s. It is therefore sufficient to show that for every $Y \in D_{\mathrm{per}\,R}(\Gamma)$ there is an $N \in \mathbb{N}$ such that the space $\mathrm{Hom}_{D(\Gamma)}(\Gamma^{>N}, Y)$ vanishes. We have $\Gamma^{>N} = \Theta^{\otimes_A(N+1)} \otimes_A \Gamma$ and by adjunction

$$\mathrm{Hom}_{D(\Gamma)}(\Theta^{\otimes_A(N+1)} \otimes_A \Gamma, Y) \cong \mathrm{Hom}_{D(A)}(\Theta^{\otimes_A(N+1)}, \mathrm{RHom}_\Gamma(\Gamma, Y))$$
$$\cong \mathrm{Hom}_{D(A)}(F^{N+1}(A), Y|_A).$$

The transition maps of the direct system $\mathrm{colim}_N \mathrm{Hom}_{D(A)}(F^{N+1}(A), Y|_A)$ are given by applying F and composing with the multiplication map $m_1 : Y \otimes_A^L \Theta \to Y$. If $Y \in D_{\mathrm{per}\,R}(\Gamma)$, then $Y|_A \in D_{\mathrm{per}\,R}(A)$ and by the assumption there is an $N \in \mathbb{N}$ such that $\mathrm{Hom}_{D(A)}(F^N(A), Y|_A)$ vanishes. By the above isomorphism, any map from $\Gamma^{>N}$ to Y vanishes. Therefore, the colimit

$$\mathrm{colim}_N \mathrm{Hom}_{D(\Gamma)}(\Gamma^{>N}, Y) \cong \mathrm{colim}_N \mathrm{Hom}_{D(A)}(F^{N+1}(A), Y|_A)$$

computes $\operatorname{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, Y)$.   □

Definition 3.2. *Let* $F : \mathcal{C} \to \mathcal{C}$ *be an endofunctor of an additive category* $\mathcal{C}$. *The* orbit category $\mathcal{C}/\langle F \rangle$ *of* $\mathcal{C}$ *by* F *is the category with the same objects as* $\mathcal{C}$ *and the spaces of morphisms*

$$\operatorname{Hom}_{\mathcal{C}/\langle F \rangle}(M, N) = \operatorname{colim}_{l \in \mathbb{N}} \bigoplus_{i \in \mathbb{N}} \operatorname{Hom}_{\mathcal{C}}(F^i(M), F^l(N)).$$

Theorem 3.3. *Let* $Y = Y_0 \otimes_A^L \Gamma$ *for some object* Y_0 *of* $\operatorname{per}(A)$ *and suppose that* Θ *belongs to* $\operatorname{per}(A)$. *Then we have*

$$\operatorname{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, Y) \cong \operatorname{colim}_N \bigoplus_{l \in \mathbb{N}} \operatorname{Hom}_{D^b(A)}(F^N(A), F^l(Y_0))$$

and $? \otimes_A^L \Gamma$ *induces a fully faithful embedding of the orbit category* $\mathcal{C}^{orb}$ *of* $\operatorname{per} A$ *by* F *into* $\mathcal{C}(\Gamma)$. *Furthermore* $\mathcal{C}(\Gamma)$ *equals its smallest thick subcategory containing the orbit category.*

Proof. By the previous lemma, we have

$$\operatorname{Hom}_{\mathcal{C}(\Gamma)}(\Gamma, Y) \cong \operatorname{colim}_N \operatorname{Hom}_{D(A)}(F^{N+1}(A), Y|_A).$$

But

$$Y|_A = (Y_0 \otimes_A^L \Gamma)|_A \cong \bigoplus_{l \in \mathbb{N}} Y_0 \otimes_A^L \Theta^{\otimes_A l} \cong \bigoplus_{l \in \mathbb{N}} F^l(Y_0).$$

This proves the first statement because $F^{N+1}(A)$ is perfect in $D(A)$. Using the fact that $\operatorname{Hom}_{\mathcal{C}(\Gamma)}(? \otimes_A^L \Gamma, Y)$ and

$$\operatorname{colim}_N \bigoplus_{l \in \mathbb{N}} \operatorname{Hom}_{D(A)}(F^N(?), F^l(Y_0))$$

are homological functors on $\operatorname{per}(A)$, we obtain that

$$\operatorname{Hom}_{\mathcal{C}(\Gamma)}(L \otimes_A^L \Gamma, Y) \cong \operatorname{colim}_N \bigoplus_{l \in \mathbb{N}} \operatorname{Hom}_{D(A)}(F^N(L), F^l(Y_0))$$

for all $L \in \operatorname{per} A$. Since we have $\operatorname{colim}_N \bigoplus_{l \in \mathbb{N}} \operatorname{Hom}_{D(A)}(F^N(L), F^l(Y_0)) = \operatorname{Hom}_{\mathcal{C}^{orb}}(L, Y_0)$, the functor $? \otimes_A^L \Gamma$ induces a fully faithful embedding of the orbit category into $\mathcal{C}(\Gamma)$. The functor $? \otimes_A^L \Gamma$ induces a triangle functor from $\operatorname{per}(A)$ to $\operatorname{per}(\Gamma)$ such that A maps to Γ. The triangle closure of the image of $\operatorname{per} A$ is therefore $\operatorname{per} \Gamma$. The last statement now follows from the commutativity of the following diagram

$$
\begin{array}{ccc}
\operatorname{per} A & \xrightarrow{? \otimes_A^L \Gamma} & \operatorname{per} \Gamma \\
\downarrow & & \downarrow \\
\mathcal{C}^{orb} & \xrightarrow[? \otimes_A^L \Gamma]{} & \mathcal{C}(\Gamma).
\end{array}
$$

□

Remark 3.4. *Note that if F is an equivalence, then the colimit in 3.3 is given by*

$$\bigoplus_{l\in\mathbb{Z}} \operatorname{Hom}_{D(A)}(A, F^l(Y_0)).$$

Suppose that $F = ? \otimes_A^L \Theta$ sends per A to itself. Let $\mathbb{F}$ be a field and $\pi : R \to \mathbb{F}$ a ring homomorphism. We denote by $\mathbb{F}A$ the scalar extension $A \otimes_R \mathbb{F}$, by $\mathcal{C}(\mathbb{F}\Gamma)$ the category per $\mathbb{F}\Gamma / D^b(\mathbb{F}\Gamma)$, by $\mathbb{F}F$ the functor $? \otimes_{\mathbb{F}A}^L (\mathbb{F} \otimes^L \Theta)$ on per $\mathbb{F}A$ and by $\mathcal{C}_{\mathbb{F}}^{orb}$ the orbit category of per $\mathbb{F}A$ by $\mathbb{F}F$.

Corollary 3.5. *The following diagram commutes:*

$$
\begin{array}{ccccccc}
\operatorname{per}(A) & \longrightarrow & \mathcal{C}^{orb} & \xrightarrow{\ ?\otimes_A^L\Gamma\ } & \mathcal{C}(\Gamma) & \longleftarrow & \operatorname{per}(\Gamma) \\
{\scriptstyle ?\otimes_R^L\mathbb{F}}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle ?\otimes_R^L\mathbb{F}} \\
\operatorname{per}(\mathbb{F}A) & \longrightarrow & \mathcal{C}_{\mathbb{F}}^{orb} & \xrightarrow[?\otimes_{\mathbb{F}A}^L\mathbb{F}\Gamma]{} & \mathcal{C}(\mathbb{F}\Gamma) & \longleftarrow & \operatorname{per}(\mathbb{F}\Gamma)
\end{array}
$$

Proof. We get the functor from $\mathcal{C}(\Gamma)$ to $\mathcal{C}(\mathbb{F}\Gamma)$ from the fact that $? \otimes_R^L \mathbb{F}$ maps $D_{\operatorname{per} R}(\Gamma)$ into $D^b(\mathbb{F}\Gamma)$. Since A is cofibrant over R, every complex that is cofibrant over A or A^e is also cofibrant over R. Therefore $\mathbb{F} \otimes_R \Theta$ is cofibrant as a dg $\mathbb{F}A^e$-module and the following diagram commutes

$$
\begin{array}{ccc}
\operatorname{per} A & \xrightarrow{\ F\ } & \operatorname{per} A \\
{\scriptstyle ?\otimes_R^L\mathbb{F}}\downarrow & & \downarrow{\scriptstyle ?\otimes_R^L\mathbb{F}} \\
\operatorname{per}\mathbb{F}A & \xrightarrow[\mathbb{F}F]{} & \operatorname{per}\mathbb{F}A.
\end{array}
$$

This proves the existence of a natural functor from $\mathcal{C}^{orb}$ to $\mathcal{C}_{\mathbb{F}}^{orb}$. The commutativity of the middle square follows from the diagram in the proof of 3.3.

$\square$

4. The integral cluster category

Let Q be a finite quiver without oriented cycles. An *RQ-lattice* is a finitely generated RQ-module which is free over R. We denote by $D(RQ)$ the derived category of RQ-modules. We denote by Γ the Ginzburg dg algebra [9] associated to $(Q, 0)$ and by $D(\Gamma)$ the derived category of differential graded Γ-modules. We refer to [17, 2.12] for an introduction to the Ginzburg algebra and its derived category. The *integral cluster category* is defined as the triangle quotient

$$\mathcal{C}_{RQ} = \operatorname{per}\Gamma / D_{\operatorname{per} R}(\Gamma).$$

In analogy with [15], we define $\mathcal{C}^{orb}$ to be the orbit category of $\operatorname{per}(RQ)$ by the auto-equivalence $S_R\Sigma^{-2}$.

Theorem 4.1. *The functor* $? \otimes_{RQ}^L \Gamma$ *induces a fully faithful embedding of* $\mathcal{C}^{orb}$ *into* $\mathcal{C}_{RQ}$. *The category* $\mathcal{C}_{RQ}$ *is the triangulated hull of* $\mathcal{C}^{orb}$.

Proof. Let $\Theta = \Sigma^{-2} \operatorname{RHom}_{RQ^e}(RQ, RQ^e) \cong \Sigma^{-2} \operatorname{Hom}_R(RQ, R)$. The tensor functor $? \otimes_A^L \Theta$ induces the functor $S_R \Sigma^{-2}$ and restricts to an equivalence of per RQ by 2.2 . By [13, 6.3], the tensor algebra $T_A(\Theta)$ is quasi-isomorphic to the Ginzburg algebra Γ. Now Theorem 3.3 yields the statement. $\qquad \square$

Proposition 4.2. *The category* $D(\Gamma)$ *satisfies the relative 3-Calabi-Yau property, i.e. for* $Y \in D(\Gamma)$ *and* $X \in D_{\operatorname{per} R}(\Gamma)$, *there is a bifunctorial isomorphism*

$$\operatorname{RHom}_R(\operatorname{RHom}_{D(\Gamma)}(X, Y), R) \cong \operatorname{RHom}_{D(\Gamma)}(Y, \Sigma^3 X)$$

for any $Y \in D(\Gamma)$ *and* $X \in D_{\operatorname{per} R}(\Gamma)$.

Proof. By [13, 4.8], the dg module $\Omega = \operatorname{RHom}_{\Gamma^e}(\Gamma, \Gamma^e)$ is isomorphic to $\Sigma^{-3}\Gamma$ in $D(\Gamma^e)$. Now using Lemma 2.1 we obtain the required isomorphism by the same proof as in [13, 4.8]. $\qquad \square$

Corollary 4.3. *Suppose that the ring R is hereditary. Let L be an object in $D(\Gamma)$ and let Z be an object in the subcategory $D_{\operatorname{per} R}(\Gamma)$. Then we have*

$$\operatorname{Hom}_R(\operatorname{Hom}_{D(\Gamma)}(Z, L), R) \oplus \operatorname{Ext}^1(\operatorname{Hom}_{D(\Gamma)}(Z, \Sigma^{-1}L), R) \cong \operatorname{Hom}_{D(\Gamma)}(L, \Sigma^3 Z).$$

Proof. As R is hereditary, every object X in $D(R)$ is isomorphic to the sum of its shifted homologies $\bigoplus_{n \in \mathbb{Z}} \Sigma^{-n} H^n(X)$. As $H^n \operatorname{RHom}_{D(\Gamma)}(Z, L) = \operatorname{Hom}_{D(\Gamma)}(Z, \Sigma^n L)$, we have $\operatorname{RHom}_{D(\Gamma)}(Z, L) \cong \bigoplus_{n \in \mathbb{Z}} \Sigma^{-n} \operatorname{Hom}_{D(\Gamma)}(Z, \Sigma^n L)$. Therefore $\operatorname{RHom}_R(\operatorname{RHom}_{D(\Gamma)}(Z, L), R)$ is isomorphic to

$$\prod_{n \in \mathbb{N}} \Sigma^n \operatorname{RHom}_R(\operatorname{Hom}_{D(\Gamma)}(Z, \Sigma^n L), R).$$

Furthermore, we have $\operatorname{RHom}_R(M, R) \cong \operatorname{Hom}_R(M, R) \oplus \Sigma^{-1} \operatorname{Ext}_R^1(M, R)$ for any R-module M. Therefore, the homology of $\operatorname{RHom}_R(\operatorname{RHom}_{D(\Gamma)}(Z, L), R)$ in degree zero is given by

$$\operatorname{Hom}_R(\operatorname{Hom}_{D(\Gamma)}(Z, L), R) \oplus \operatorname{Ext}^1(\operatorname{Hom}_{D(\Gamma)}(Z, \Sigma^{-1}L), R).$$

We obtain the statement by comparing the homology in degree zero in 4.2 and using the fact that the homology of $\operatorname{RHom}_{D(\Gamma)}(L, \Sigma^3 Z)$ in degree zero is given by $\operatorname{Hom}_{D(\Gamma)}(L, \Sigma^3 Z)$. $\qquad \square$

5. RIGID OBJECTS

We assume from now on that R is a hereditary ring. Our goal in this section is to show that, if R is a principal ideal domain, each rigid object of the integral cluster category is either the image of a rigid indecomposable RQ-module or the suspension of the image of an indecomposable projective RQ-module. We also show that the orbit category $\mathcal{C}^{orb}$ and the integral cluster category are equivalent, so that the orbit category is triangulated.

By [6, Theorem 1] all rigid indecomposable RQ-modules are lattices and there is a bijection between the rigid indecomposable lattices and the real Schur roots of the quiver Q given by the rank vector. Following [1], we define the *fundamental domain* to be the R-linear subcategory

$$\mathcal{F} = D_{\leq 0} \cap {}^{\perp}D_{\leq -2} \cap \operatorname{per} \Gamma$$

of $\operatorname{per} \Gamma$, where ${}^{\perp}D_{\leq n}$ denotes the full subcategory whose objects are the $X \in D(\Gamma)$ such that $\operatorname{Hom}_{D(\Gamma)}(X, Y)$ vanishes for all $Y \in D_{\leq n}$. Let $\pi : \operatorname{per} \Gamma \to \mathcal{C}_{RQ}$ be the canonical triangle functor.

Theorem 5.1. *For every object Z of $\mathcal{C}_{RQ}$, there is an $N \in \mathbb{Z}$ and an object $Y \in \mathcal{F}[N]$ such that $\pi(Y)$ is isomorphic to Z.*

Proof. For every object $X \in \operatorname{per}(\Gamma)$, there is an $N \in \mathbb{Z}$ and an $M \in \mathbb{Z}$ such that $X \in D_{\leq N}$ and $X \in {}^{\perp}D_{\leq M}$. This follows from the facts that $\Gamma \in D_{\leq 0}(\Gamma) \cap {}^{\perp}D_{\leq -1}(\Gamma)$ and that the property is stable under taking shifts, extensions and direct factors. So let $X \in {}^{\perp}D_{\leq N-2}$ for some $N \in \mathbb{Z}$. Consider the canonical triangle

$$\tau_{\leq N}(X) \to X \to \tau_{>N}(X) \to \Sigma\tau_{\leq N}(X).$$

As $\tau_{>N}(X) \in D_{\operatorname{per} R}(\Gamma)$ and π is a triangle functor, the objects $\pi(\tau_{\leq N}(X))$ and $\pi(X)$ are isomorphic. It remains to show that $\tau_{\leq N}(X) \in {}^{\perp}D_{\leq N-2}$, which is equivalent to the fact that $\Sigma^{-1}\tau_{>N}(X)$ lies in ${}^{\perp}D_{\leq N-2}$. By 4.3, for each object L of $D(\Gamma)$, the sum

$$\operatorname{Hom}_R(\operatorname{Hom}_{D(\Gamma)}(\Sigma^{-1}\tau_{>N}(X), L), R) \oplus \operatorname{Ext}^1_R(\operatorname{Hom}_{D(\Gamma)}(\Sigma^{-1}\tau_{>N}(X), \Sigma^{-1}L), R)$$

is isomorphic to $\operatorname{Hom}_{D(\Gamma)}(L, \Sigma^2\tau_{>N}(X))$. This group vanishes for all $L \in D_{\leq N-2}$. We fix an object L in $D_{\leq N-2}$. The R-module

$$H = \operatorname{Hom}_{D(\Gamma)}(\Sigma^{-1}\tau_{>N}(X), L)$$

is left orthogonal to R and so has to be a torsion module. Since ΣL also lies in $D_{\leq N-2}$, the group $\operatorname{Ext}^1_R(H, R)$ vanishes and so we have $H = 0$. Therefore, the object $\tau_{\leq N}(X)$ lies in $\mathcal{F}[N]$ and its image is isomorphic to the image of X in the integral cluster category. $\square$

Following [1], we define an $\operatorname{add}(\Gamma)$-*resolution* of an object $M \in \operatorname{per} \Gamma$ to be a triangle

$$P_0 \to P_1 \to M \to \Sigma P_0$$

with P_0, P_1 in $\operatorname{add} \Gamma$.

Lemma 5.2. *An object $X \in \operatorname{per} \Gamma$ has an $\operatorname{add}(\Gamma)$-resolution if and only if X lies in $\mathcal{F}$.*

Proof. If X lies in $\mathcal{F}$, then X has an $\operatorname{add}(\Gamma)$-resolution by the proof of [1, 2.10]. Now let

$$P_1 \to P_0 \to X \to \Sigma P_0$$

be an add(Γ)-resolution. Applying the homology functor to this triangle, we get a long exact sequence. Using the fact that Γ has non-zero homology only in non-positive degree, we see that X lies in $D_{\leq 0}$. Next we apply the functor $\mathrm{Hom}_{D(\Gamma)}(?, Y)$ to the add(Γ)-resolution for any object $Y \in D_{\leq -2}$. Then we obtain a long exact sequence

$$\cdots \to \mathrm{Hom}(\Sigma P_0, Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(P_0, Y) \to \mathrm{Hom}(P_1, Y) \to \cdots \quad .$$

All terms in this sequence have to vanish and therefore X belongs to $^{\perp}D_{\leq -2}$.
$\square$

We have

$$\mathrm{Hom}_{\mathcal{C}_{RQ}}(\Gamma, \Gamma) \cong \mathrm{Hom}_{\mathcal{C}^{orb}}(RQ, RQ) \cong \mathrm{Hom}_{\mathcal{C}^{orb}}(RQ, RQ) \cong RQ$$

by 4.1. Therefore we have a functor $G = \mathrm{Hom}_{\mathcal{C}_{RQ}}(\Gamma, ?)$ from $\mathcal{C}_{RQ}$ to the category of RQ-modules. Note that G vanishes on $\Sigma\Gamma$, hence G factors through the quotient category $\mathcal{C}_{RQ}/\mathrm{add}\,\Sigma\Gamma$.

Example 5.3. *We consider the quiver $Q : 1 \xrightarrow{\alpha} 2$ and $R = \mathbb{Z}$. Let M be the module given by the quiver representation $0 \leftarrow \mathbb{Z}/2\,\mathbb{Z}$. Then M has the projective resolution*

$$0 \longrightarrow P_1 \xrightarrow{[2\ \alpha]^t} P_1 \oplus P_2 \xrightarrow{[\alpha\ -2]} P_2 \longrightarrow M \longrightarrow 0$$

By applying the functor $\mathrm{Hom}_{\mathbb{Z}Q}(?, M)$ to the resolution we see that the group of selfextensions of M is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$ so that M is not rigid. Let M' be the image of M in $\mathrm{per}(\Gamma)$. We have $G(M') \cong M$ but M' does not lie in the fundamental domain as $\mathrm{Hom}_{D(RQ)}(M, \Sigma^2 P_1)$ is isomorphic to $\mathbb{Z}/2\,\mathbb{Z}$. But clearly M' belongs to $^{\perp}D_{\leq -3}(\Gamma)$, hence by the proof of 5.1, we have that $\tau_{\leq -1}(M') \cong M'$ in $\mathcal{C}_{\mathbb{Z}Q}$ and $\Sigma^{-1}\tau_{\leq -1}(M') \in \mathcal{F}$.

Lemma 5.4. *Let M be an object in $\mathcal{C}_{RQ}$. Then $G(GM \otimes^L_{RQ} \Gamma)$ and GM are isomorphic in $\mathcal{C}_{RQ}$. If GM is a lattice, then $G(GM \otimes^L_{RQ} \Gamma)$ viewed as an element of $\mathrm{per}\,\Gamma$ lies in the fundamental domain.*

Proof. We have

$$\begin{aligned}
G(GM \otimes^L_{RQ} \Gamma) &= \mathrm{Hom}_{\mathcal{C}_{RQ}}(\Gamma, GM \otimes^L_{RQ} \Gamma) \\
&\cong \mathrm{Hom}_{\mathcal{C}^{orb}}(RQ, GM) \\
&\cong \mathrm{Hom}_{RQ}(RQ, GM) \cong GM,
\end{aligned}$$

as the embedding of $\mathcal{C}^{orb}$ into $\mathcal{C}_{RQ}$ is fully faithful by 4.1. Let now GM be a lattice. Then $GM \otimes^L_{RQ} \Gamma$ is in $D_{\leq 0}$. We have $\mathrm{Hom}_\Gamma(GM \otimes^L_{RQ} \Gamma, Y) \cong \mathrm{Hom}_{\mathrm{per}\,RQ}(GM, R\,\mathrm{Hom}_\Gamma(\Gamma, Y))$. If $Y \in D_{\leq -2}$, then $R\,\mathrm{Hom}_\Gamma(\Gamma, Y)$ also lies in $D_{\leq -2}$. As GM is a lattice, we have that GM lies in $^{\perp}D_{\leq -2}$ and hence $\mathrm{Hom}_{\mathrm{per}\,RQ}(GM, R\,\mathrm{Hom}_\Gamma(\Gamma, Y))$ vanishes. This finishes the proof. $\square$

Definition 5.5. *We call an indecomposable object X in $\mathcal{C}_{RQ}$ lattice-like, if there is a lattice L such that X is isomorphic to $\Gamma \otimes_{RQ}^L L$ or X is isomorphic to $\Sigma\Gamma \otimes_{RQ}^L P$ for a projective indecomposable RQ-module P.*

All lattice-like objects are images of objects in the orbit category $\mathcal{C}^{orb}$.

Theorem 5.6. *Let M be an indecomposable rigid object of $\mathcal{C}_{RQ}$. There is an $N \in \mathbb{Z}$ such that M is isomorphic to $G(\Sigma^N M) \otimes_{RQ}^L \Sigma^{-N}\Gamma$.*

Proof. By 5.1, there is an $N \in \mathbb{Z}$ and an object $M' \in \mathcal{F}[N]$ such that $\pi(M') = M$. We assume without loss of generality that $N = 0$. By Lemma 5.2, each object of $\mathcal{F}$ admits an add(Γ)-resolution. Therefore, for N' in $\mathcal{F}$, we have, as in Proposition 2.1 c) of [16], the isomorphism

$$\mathrm{Hom}_{\mathcal{C}_{RQ}/\operatorname{add}\Sigma\Gamma}(\pi M', \pi N') \cong \mathrm{Hom}_{RQ}(GM, G\pi N').$$

Note also that $G(GM \otimes_{RQ}^L \Gamma)$ is isomorphic to GM by Lemma 5.4. Let

$$\Sigma^{-1}M \to P_1 \xrightarrow{h} P_0 \to M$$

be an add(Γ)-resolution in the integral cluster category. Then all morphisms from P_1 to M factor through h. Applying G to the triangle gives the start of a projective resolution

$$GP_1 \to GP_0 \to GM \to 0.$$

As P_1, P_0 and M are all images of objects in $\mathcal{F}$, we have that every morphism from GP_1 to GM factors through Gh. Therefore GM is rigid as an RQ-module and hence is a lattice. If GM vanishes, then M lies in add($\Sigma\Gamma$). Since we have equivalences

$$\operatorname{add}(RQ) \xrightarrow{\sim} \operatorname{add}(\Gamma) \xrightarrow{\sim} \operatorname{add}(\pi(\Gamma)) \xrightarrow{\sim} \operatorname{add}(RQ)\ ,$$

we obtain $M \cong G(\Sigma^{-1}M) \otimes_{RQ}^L \Sigma\Gamma$. So let us suppose that GM does not vanish. As GM is a lattice, the object $GM \otimes_{RQ}^L \Gamma$ lies in $\mathcal{F}$. It follows that there are isomorphisms

$$f \in \mathrm{Hom}_{\mathcal{C}_{RQ}/\operatorname{add}\Sigma\Gamma}(M, GM \otimes_{RQ}^L \Gamma) \text{ and } g \in \mathrm{Hom}_{\mathcal{C}_{RQ}/\operatorname{add}\Sigma\Gamma}(GM \otimes_{RQ}^L \Gamma, M).$$

We lift f and g to morphisms $\tilde{f}$ and $\tilde{g}$ in the integral cluster category. Then $\tilde{f}\tilde{g}$ lies in

$$\mathrm{Hom}_{\mathcal{C}_{RQ}}(GM \otimes_{RQ}^L \Gamma, GM \otimes_{RQ}^L \Gamma) \cong \mathrm{Hom}_{\mathcal{C}^{orb}}(GM, GM).$$

Now the functor $\mathrm{Hom}_{\mathcal{C}^{orb}}(RQ, ?)$ induces a surjective ring homomorphism

$$\mathrm{Hom}_{\mathcal{C}^{orb}}(GM, GM) \to \mathrm{Hom}_{RQ}(GM, GM)$$

whose kernel is a radical ideal. Since fg is an isomorphism of RQ-modules, $\tilde{f}\tilde{g}$ is an isomorphism in the integral cluster category. But M is indecomposable and $\tilde{f}\tilde{g}$ factors through M, hence the objects M and $GM \otimes_{RQ}^L \Gamma$ are isomorphic. $\square$

Note that the proof of the previous theorem also holds if $G(\Sigma^N M)$ is a non vanishing lattice. Therefore we have

Lemma 5.7. *Let M be an indecomposable object in $\mathcal{C}_{RQ}$ such that there is a $Z \in \mathcal{F}$ with $\pi(Z)$ is isomorphic to M. If GM is a non vanishing lattice, then M is isomorphic to $GM \otimes_{RQ}^L \Gamma$.*

The next result is well-known for derived categories of hereditary algebras over fields.

Lemma 5.8. *Let R be a principal ideal domain. The Serre functor S_R of $\mathrm{per}(RQ)$ maps shifts of rigid lattices to shifts of rigid lattices.*

Proof. The Serre functor is given by the left derived functor of tensoring with the bimodule $\Theta = \mathrm{Hom}_R(RQ, R)$. As $\mathbb{F}Q$ is hereditary, the Serre functor $S_\mathbb{F}$ maps a non-projective module L to $\Sigma\tau(L)$, where τ denotes the Auslander-Reiten translation of the category of $\mathbb{F}Q$-modules. Hence the Serre functor applied to non projective indecomposable $\mathbb{F}Q$-modules has non-vanishing homology only in degree minus one. Moreover, the functor $S_\mathbb{F}$ maps projective modules to injective modules.

The statement is clear for projective lattices of RQ. Let M be a non-projective indecomposable rigid lattice over RQ and

$$0 \to P_1 \xrightarrow{f} P_0 \to M \to 0$$

a projective resolution of M. Then P_0 and P_1 are lattices and f splits as a map of R-modules. The object $S_R(M)$ is isomorphic to the complex

$$\cdots \to 0 \to P_1 \otimes_{RQ} \Theta \xrightarrow{f \otimes \Theta} P_0 \otimes_{RQ} \Theta \to 0 \to \cdots$$

As Θ is a lattice, so are $P_0 \otimes_{RQ} \Theta$ and $P_1 \otimes_{RQ} \Theta$. The cokernel of $f \otimes \Theta$ is given by $M \otimes_{RQ} \Theta$. We show next that $f \otimes \Theta$ is surjective by proving that $M \otimes_{RQ} \Theta$ vanishes. By the proof of 3.5, S_R commutes with the functor $- \otimes_R^L \mathbb{F}$. Therefore $(\mathbb{F} \otimes_R \Theta) \otimes_{\mathbb{F}Q} (M \otimes_R \mathbb{F})$ and $(\Theta \otimes_{RQ} M) \otimes_R \mathbb{F}$ are isomorphic. By [6, Theorem 2] the module $M \otimes_R \mathbb{F}$ is a rigid indecomposable non-vanishing lattice which is non-projective. Suppose that $M \otimes_{RQ} \Theta$ does not vanish. Then there is a field $\mathbb{F}$ such that $(\Theta \otimes_{RQ} M) \otimes_R \mathbb{F}$ does not vanish. Hence $S_\mathbb{F}(\mathbb{F} \otimes_R M)$ has non-vanishing cohomology in degree zero, which is a contradiction, as $\mathbb{F} \otimes_R M$ is a non-projective $\mathbb{F}Q$-module. Therefore $f \otimes \Theta$ is surjective. Its kernel has to be a lattice as it is a submodule of a lattice. The object $S_R(M)$ is isomorphic to the one-shift of this lattice. $\square$

Next we analyze the relationship between the integral cluster category and the cluster category over the field $\mathbb{F}$. We can strengthen [15, A.8].

Theorem 5.9. *Let R be a principal ideal domain. (1) The rigid indecomposable objects of $\mathcal{C}_{RQ}$ are lattice-like.*

(2) The reduction functor

$$? \otimes_R \mathbb{F} : \mathcal{C}_{RQ} \to \mathcal{C}_{\mathbb{F}Q}$$

induces a bijection from the set of isomorphism classes of rigid indecomposable objects in $\mathcal{C}_{RQ}$ to the set of isomorphism classes of rigid indecomposable objects of $\mathcal{C}_{\mathbb{F}Q}$.

Proof. We denote by F_R the functor $S_R\Sigma^{-2}$ in per RQ and by F the functor $S\Sigma^{-2}$ in per $\mathbb{F}Q$. Let $M \in \mathcal{C}_{RQ}$ be a rigid indecomposable object. By 5.6 and 4.1, we can view M as an object of $\mathcal{C}^{orb}$ and M is isomorphic to the N-shift of the image of a rigid RQ-lattice M'. Then $M \otimes_R^L \mathbb{F}$ is isomorphic to the N-shift of the rigid module $M' \otimes_R \mathbb{F}$ seen as an object in $\mathcal{C}_{\mathbb{F}Q}$. If we view $\Sigma^N M' \otimes_R^L \mathbb{F}$ as an object in per $\mathbb{F}Q$, we see that there is a rigid indecomposable module L in mod $\mathbb{F}Q$ or an indecomposable direct summand P of $\mathbb{F}Q$ such that $\Sigma^N M' \otimes_R \mathbb{F}$ and L lie in the same F-orbit or $\Sigma^N M' \otimes_R^L \mathbb{F}$ and ΣP lie in the same F-orbit. Let $n \in \mathbb{Z}$ be such that $L \cong F^n \Sigma^{-N} M' \otimes_R \mathbb{F}$ or $\Sigma P \cong F^n \Sigma^{-N} M' \otimes_R \mathbb{F}$.

As S_R maps the shift of a rigid lattice to the shift of a rigid lattice by 5.8, we have that $F_R^n \Sigma^N M'$ is also the k-shift of a rigid RQ-lattice, say L' in per RQ for some $k \in \mathbb{Z}$. By 3.5, we have that $\Sigma^k L' \otimes \mathbb{F}$ and L are isomorphic or $\Sigma^k L' \otimes_R F \cong \Sigma P$ in per $\mathbb{F}Q$, hence k vanishes in the first case and k equals one in the second case. Furthermore, in the second case L' is isomorphic to a projective RQ-module. We obtain therefore that $\Sigma^N M'$ is in the F_R-orbit of L' in the first case and is in the F_R-orbit of $\Sigma L'$ in the second case. Hence in the orbit category, we have that M is isomorphic to a lattice or to the one-shift of a projective lattice. This finishes the proof of the first statement. Using Theorem 1 of [6] we then immediately obtain the second statement. $\square$

Note also that all rigid objects satisfy the unique decomposition property, as they are lattice-like and the statement holds by [6, Theorem 2] for rigid lattices in the category of RQ-modules.

Corollary 5.10. *Let R be a principal ideal domain. The cluster tilting objects in $\mathcal{C}_{RQ}$ are all connected via cluster-tilting mutation in the sense of [12] and can therefore be obtained by iterated mutation from the initial object Γ.*

We can also show that the orbit category $\mathcal{C}^{orb}$ and the integral cluster category coincide and hence the orbit category is triangulated.

Theorem 5.11. *The embedding of the orbit category $\mathcal{C}^{orb}$ into $\mathcal{C}_{RQ}$ is an equivalence. Therefore the orbit category $\mathcal{C}^{orb}$ is canonically triangulated.*

Proof. We consider the commutative diagram of functors

$$
\begin{array}{ccc}
\operatorname{per} RQ & \xrightarrow{?\otimes^L_{RQ}\Gamma} & \operatorname{per}\Gamma \\
\downarrow & & \pi\downarrow \\
\mathcal{C}^{orb} & \xrightarrow[?\otimes^L_{RQ}\Gamma]{} & \mathcal{C}_{RQ}.
\end{array}
$$

By 3.3, the bottom functor is fully faithful. Let us show that it is essentially surjective. Let $M \in \mathcal{C}_{RQ}$. By 5.1, there is an $n \in \mathbb{Z}$ and an $M' \in \mathcal{F}$ such that $\Sigma^n \pi M' \cong M$. We assume without loss of generality that $n = 0$ and chose an $\operatorname{add}(\Gamma)$-resolution $P_1 \xrightarrow{h} P_0 \to M' \to \Sigma P_1$. By remark 3.4, the restriction of π to $\operatorname{add}(\Gamma)$ is fully faithful and so is the restriction of $\operatorname{per}(RQ) \to \operatorname{per}(\Gamma)$ to $\operatorname{add}(RQ)$. Thus, the morphism $h : P_1 \to P_0$ is the image of a morphism in $\operatorname{per} RQ$. Since $- \otimes^L_{RQ} \Gamma : \operatorname{per} RQ \to \operatorname{per}\Gamma$ is a triangle functor, M' is also isomorphic to an image of an object in $\operatorname{per} RQ$. By the commutativity of the diagram, we deduce that M, as an object in $\mathcal{C}_{RQ}$, is isomorphic to the image of an object in $\mathcal{C}^{orb}$. Now the objects in $\mathcal{C}_{RQ}$ are identical with the objects in $\operatorname{per}\Gamma$, hence $? \otimes^L_{RQ} \Gamma : \mathcal{C}^{orb} \to \mathcal{C}_{RQ}$ is essentially surjective and hence an equivalence. $\square$

Corollary 5.12. *The integral cluster category satisfies the relative 2-Calabi-Yau property, i.e. X and $Y \in \mathcal{C}_{RQ}$, there is a bifunctorial isomorphism*

$$
\operatorname{RHom}_R(\operatorname{RHom}_{\mathcal{C}_{RQ}}(X, Y), R) \cong \operatorname{RHom}_{\mathcal{C}_{RQ}}(Y, \Sigma^2 X)
$$

in $D(R)$.

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