Acyclic Calabi-Yau categories are cluster categories

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(joint work with Idun Reiten)

Let k be a field and Q a finite quiver without oriented cycles. Let kQ be the path algebra of Q and $\operatorname{mod} kQ$ the category of k-finite-dimensional right kQ-modules. The cluster category $\mathcal{C}_Q$ was introduced in [1] (for general Q) and, independently, in [4] (for Q of type A_n). It is defined as the orbit category of the bounded derived category $\mathcal{D}^b(\operatorname{mod} kQ)$ under the action of the automorphism $\Sigma^{-1} \circ S^2$, where S is the suspension (=shift) functor of the derived category and Σ its Serre functor, characterized by the Serre duality formula

$$D \operatorname{Hom}(X, Y) = \operatorname{Hom}(Y, \Sigma X) \ ,$$

where D is the duality functor $\operatorname{Hom}_k(?, k)$. The motivation behind this definition was to find a 'categorification' of the cluster algebras introduced by Fomin-Zelevinsky in [6]. This program has been quite successful, *cf. e.g.* [3] [5] and the references given there. The cluster category has the following properties (explained below in more detail):

a) $\mathcal{C}_Q$ is a triangulated category. In fact, it is even an algebraic triangulated category, *i.e.* there is a triangle equivalence between $\mathcal{C}_Q$ and the stable category $\underline{\mathcal{E}}$ of a Frobenius category $\mathcal{E}$.
b) $\mathcal{C}_Q$ *is* Hom-*finite* (*i.e.* all its morphism spaces are finite-dimensional) and *Calabi-Yau of CY-dimension* 2. By this, one means that it admits a Serre functor Σ (which is induced by that of the derived category) and that there is an isomorphism of triangle functors between Σ and S^2. Note that this last property holds almost by definition of $\mathcal{C}_Q$.
c) If T_Q denotes the image of the free module kQ under the projection from the derived category to the cluster category, then T_Q is a *cluster-tilting object* in $\mathcal{C}_Q$, *i.e.* we have
  c1) $\operatorname{Hom}(T_Q, ST_Q) = 0$ and
  c2) for each object X, if we have $\operatorname{Hom}(T_Q, SX) = 0$, then X vanishes.
d) The endomorphism algebra of T_Q is isomorphic to kQ. In particular, its ordinary quiver does not admit oriented cycles.

These properties were proved in [1] except for a), which was proved in [9]. We say that a k-linear category is a 2-*Calabi-Yau category* if it satisfies a) and b). Our main result is that properties a) to d) characterize the cluster category if k is algebraically closed:

Theorem. *Suppose that k is algebraically closed. If $\mathcal{C}$ is an algebraic 2-Calabi-Yau category and admits a cluster-tilting object T such that the ordinary quiver Q of the endomorphism algebra of T does not contain oriented cycles, then there is a triangle equivalence from $\mathcal{C}_Q$ to $\mathcal{C}$ which takes the object T_Q to T.*

The theorem allows one to show that cluster categories, whose definition may seem artifical at first glance, do occur in nature: Let k be an algebraically closed field of characteristic 0, S the completed power series algebra $k[[X,Y,Z]]$ and G the cyclic group of order three acting linearly on S such that a generator of G multiplies the three variables by the same primitive third root of unity. It is not hard to show that the fixed point algebra $R = S^G$ is a Gorenstein complete local normal domain that has an isolated singularity. We consider the Frobenius category $\mathcal{E} = CM(R)$ of its maximal Cohen-Macaulay modules. By a theorem of Auslander's, the stable category $\mathcal{C} = \underline{\mathcal{E}}$ is 2-Calabi-Yau. Work of Iyama [8] shows that $T = S$ considered as an R-module is a cluster-tilting object in $\mathcal{C}$. Its ring of R-linear endomorphisms is isomorphic to the skew group algebra $S * G$ and its endomorphism ring in $\underline{\mathcal{E}}$ is the path algebra of the generalized Kronecker quiver Q with three arrows. Thus the hypotheses of the theorem are satisfied and we obtain a triangle equivalence between $\mathcal{C}_Q$ and $\underline{CM}(R)$. In particular, this allows us to compute the Auslander-Reiten quiver of $\underline{CM}(R)$. It also shows that Yoshino's classification of the rigid Cohen-Macaulay modules [12] in $\underline{CM}(R)$ is equivalent to the classification of the cluster-tilting objects in the cluster category of the generalized Kronecker quiver with three arrows and thus [5] to that of the cluster variables in the corresponding cluster algebra [11].

Assume that k is algebraically closed. Let $\mathcal{C}$ be a 2-Calabi-Yau category admitting a cluster-tilting object T. One can show that the number of pairwise indecomposable non isomorphic direct factors of T does not depend on the choice of T, *cf.* [10]. We call this number the *rank* of $\mathcal{C}$. We say that $\mathcal{C}$ is *acyclic* if it admits a cluster tilting object the quiver of whose endomorphism algebra does not have oriented cycles (or equivalently, if it is triangle equivalent to a cluster category).

Conjecture. *If $\mathcal{C}$ is of rank at most three and the quivers of the endomorphism algebras of its tilting objects admit neither loops nor 2-cycles, then it is acyclic.*

It was shown in [7] that if $\mathcal{C}$ is the stable module category $\underline{\mathrm{mod}}\,\Lambda(\Delta)$ of a preprojective algebra associated with a simply laced Dynkin diagram Δ, then the quivers of the endomorphism algebras of its cluster-tilting objects admit neither loops nor 2-cycles. It follows that this property also holds if $\mathcal{C}$ is constructed as a 'CY-subquotient' of $\underline{\mathrm{mod}}\,\Lambda(\Delta)$ (for the 'CY-subquotient' construction, *cf.* section 2 of [2] and section 5.4 of [5]). Thus the conjecture implies that any CY-subquotient of rank ≤ 3 of $\underline{\mathrm{mod}}\,\Lambda(\Delta)$ is acyclic. This holds indeed in all the examples we have checked.

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