Calabi-Yau triangulated categories

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Abstract. We review the definition of a Calabi-Yau triangulated category and survey examples coming from the representation theory of quivers and finite-dimensional algebras. Our main motivation comes from the links between quiver representations and Fomin-Zelevinsky's cluster algebras.

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1. Introduction

These notes reflect the contents of three lectures given at the workshop preceding the XII International Conference on Representations of Algebras (ICRA XII) held in August 2007 at Toruń.

The notion of Calabi-Yau triangulated category was introduced by Kontsevich in the late nineties [41]. It appears in

- mathematical physics, notably string theory and conformal field theory,
- algebraic geometry, notably mirror symmetry,
- integrable systems,
- $\vdots$
- representation theory of quivers and finite-dimensional algebras.

In representation theory, triangulated Calabi-Yau categories have become popular thanks to their application in the categorification of Fomin-Zelevinsky's cluster algebras, *cf.* the surveys [2] [32] [49] [55].

In this brief account, we review basic notions on triangulated categories, discuss the Calabi-Yau property and, most importantly, describe two classes of examples: Calabi-Yau categories arising as orbit categories and Calabi-Yau categories arising as (subcategories of) derived categories.

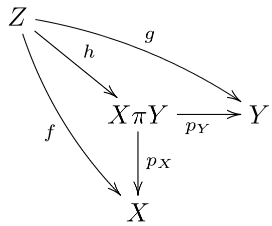
2. Triangulated categories, Serre functors

2.1. *k*-categories. Let k be a commutative ring, for example the ring of integers. A *k-category* is a category $\mathcal{C}$ where each set of morphisms $\mathcal{C}(X, Y)$ is endowed with a structure of k-module in such a way that the composition maps

$$\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

are bilinear for all objects X, Y and Z of $\mathcal{C}$. For example, if R is a k-algebra (associative, with 1), then we have a k-category $\mathcal{C}$ with one object whose endomorphism algebra is R. Clearly, up to isomorphism, all k-categories with one object arise in this way. A general k-category should simply be thought of as a 'ring with several objects' [48].

Let $\mathcal{C}$ be a k-category. For two objects X and Y of $\mathcal{C}$, a *product of* X *by* Y is an object $X\pi Y$ endowed with morphisms $p_X : X\pi Y \to X$ and $p_Y : X\pi Y \to Y$ such that for each pair of morphisms (f,g) from an object Z to X respectively Y, there is a unique morphism h from Z to $X\pi Y$ such that $p_X \circ h = f$ and $p_Y \circ h = g$.



A product of two objects may or may not exist but if it exists, it is unique up to a unique isomorphism. It is best to formulate such universal properties using the concept of a representable functor: A functor

$$F : \mathcal{C}^{op} \to \mathsf{Mod}\, k$$

from the opposite category of $\mathcal{C}$ to the category of k-modules is *representable* if there is an object U of $\mathcal{C}$ and an isomorphism of functors

$$\mathcal{C}(?,U) \xrightarrow{\sim} F.$$

For example, if two objects X and Y of $\mathcal{C}$ admit a product, then the map

$$\mathcal{C}(Z, X\pi Y) \to \mathcal{C}(Z,X) \times \mathcal{C}(Z,Y)\,,\ h \mapsto (h \circ p_X, h \circ p_Y)$$

is bijective for each object Z and its existence means that the product $X\pi Y$ represents the product functor

$$\mathcal{C}(?,X) \times \mathcal{C}(?,Y) : \mathcal{C}^{op} \to \mathsf{Mod}\, k.$$

We will often use the formalism of representable functors to transfer notions and constructions from the category of k-modules to an arbitrary k-category. For instance, we define an object N of $\mathcal{C}$ to be a *zero object* if $\mathcal{C}(?,N)$ is the zero functor. A functor

$$G : \mathcal{C} \to \mathsf{Mod}\, k$$

is *corepresentable* if there is an object U of $\mathcal{C}$ and an isomorphism of functors

$$\mathcal{C}(U,?) \xrightarrow{\sim} G.$$

For instance, the *coproduct* $X \sqcup Y$ defined to be a corepresentative of the product (!) functor

$$\mathcal{C}(X,?) \times \mathcal{C}(Y,?).$$

A *k-linear category* is a k-category $\mathcal{C}$ such that $\mathcal{C}$ has a zero object and any two objects of $\mathcal{C}$ have a product. For example, if R is a k-algebra, the category of free (right) R-modules is k-linear. So are the categories of respectively, all R-modules, all projective R-modules, all flat R-modules ....

It is a useful exercise to show that if $\mathcal{C}$ is a k-linear category, then the canonical morphism from the coproduct to the product

$$X \sqcup Y \to X \pi Y$$

is an isomorphism for any objects X and Y. One therefore writes $X \oplus Y$ for both. It is also instructive to show that the structure of abelian group on $\mathcal{C}(X,Y)$ is fully determined by the underlying category of $\mathcal{C}$.

2.2. Triangulated categories. As before, let k be a commutative ring. A *triangulated k-category* is a k-linear category $\mathcal{T}$ endowed with

a) an autoequivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ called the *suspension functor* (or shift, or translation functor);

b) a class of sequences

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

called *triangles* which is stable under *isomorphism* in the sense of the commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle a} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma a} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

whose vertical arrows are isomorphisms.

These data have to satisfy the following axioms:

T0 For each object X of $\mathcal{T}$, the sequence

$$0 \longrightarrow X \xrightarrow{\mathbf{1}_X} X \longrightarrow \Sigma 0$$

is a triangle;

T1 for each morphism $u : X \to Y$ of $\mathcal{T}$, there is a triangle

$$X \xrightarrow{u} Y \longrightarrow Z \longrightarrow \Sigma X \ ;$$

T2 a sequence of three morphisms (u, v, w) is a triangle if and only if the sequence $(v, w, -\Sigma(u))$ is a triangle;

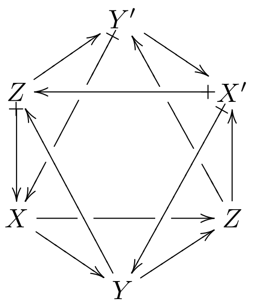
T3 if (u, v, w) and (u', v', w') are triangles and a, b morphisms such that $bu = u'a$, then there is a morphism c which makes the following diagram commutative

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Sigma a} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & \Sigma X'; \end{array}$$

T4 for all composable morphisms

$$X \xrightarrow{a} Y \xrightarrow{b} Z \ ,$$

there is an octahedron



where an arrow $U \,+\!\!\longrightarrow V$ denotes a morphism $U \to \Sigma V$, the cyclically oriented triangles are triangles of $\mathcal{T}$, the triangles with poset orientation are commutative, and so are the two squares containing the center.

A whole little theory can be deduced from these axioms, *cf.* [59] [60]. In developping this theory, one may assume that Σ is not only an autoequivalence but in fact an automorphism, *cf.* [40]. In the rest of this paragraph, we make this assumption. The most important consequence of the axioms (which follows from T0-T3 alone) is that for each triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \ ,$$

the long induced sequences

$$\cdots \xrightarrow{\mathcal{T}(U,\Sigma^{-1}w)} \mathcal{T}(U, X) \xrightarrow{\mathcal{T}(U,u)} \mathcal{T}(U, Y) \xrightarrow{\mathcal{T}(U,v)} \mathcal{T}(U, Z) \xrightarrow{\mathcal{T}(U,w)} \cdots$$

and

$$\cdots \longleftarrow \mathcal{T}(X,U) \longleftarrow \mathcal{T}(Y,U) \longleftarrow \mathcal{T}(Z,U) \longleftarrow \cdots$$

are exact. It is also important to notice that if (u,v,w) is a triangle, then $(u,v,-w)$ is *not* a triangle, in general. Finally, note that by applying T2 three times, we find that if (u,v,w) is a triangle, then so is $(-\Sigma u, -\Sigma v, -\Sigma w)$. This last sequence is clearly isomorphic to $(\Sigma u, \Sigma v, -\Sigma w)$, which is therefore a triangle. We will need this observation below.

2.3. Stable categories. Suppose that A is a finite-dimensional algebra over a field k and that A is selfinjective, *i.e.* injective as a right module over itself. For example, this happens if A is the group algebra of a finite group. Let $\mathsf{mod}\, A$ denote the category of finite-dimensional right modules over A. For two A-modules L and M, let $\mathcal{P}(L,M)$ be the space of morphisms from L to M which factor through a projective A-module. Then the *stable category* $\underline{\mathsf{mod}}A$, whose objects are the same as those of $\mathsf{mod}\, A$ and whose morphisms are given by the quotient spaces

$$\underline{\mathsf{Hom}}(L,M) = \mathsf{Hom}(L,M)/\mathcal{P}(L,M)$$

carries a canonical structure of triangulated category. Its suspension functor is obtained (on objects) by choosing, for each finite-dimensional A-module L, a short exact sequence

$$0 \longrightarrow L \longrightarrow IM \longrightarrow \Sigma M \longrightarrow 0\,.$$

Its triangles are defined to be the sequences isomorphic to standard triangles, *i.e.* images in $\underline{\mathsf{mod}}A$ of sequences (a,b,e) obtained from short exact sequences of modules by fitting them into diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{a} & M & \xrightarrow{b} & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathbf{1}_L} & & \downarrow & & \downarrow{\scriptstyle e} & & \\ 0 & \longrightarrow & L & \longrightarrow & IL & \longrightarrow & \Sigma L & \longrightarrow & 0. \end{array}$$

This construction generalizes from categories of finite-dimensional modules over selfinjective algebras to arbitrary Frobenius categories, *cf.* [24] [25] [40]. Here, an important example is the following: Let $\mathcal{A}$ be an additive category and $\mathcal{E}$ the category of complexes of objects of $\mathcal{A}$. Then $\mathcal{E}$ is an additive category. We endow it with the class of all componentwise split short exact sequences. Then $\mathcal{E}$ becomes an exact category in the sense of Quillen, it has enough projectives and an object is projective iff it is injective iff it admits a contracting homotopy. Thus $\mathcal{E}$ is a Frobenius category. The associated stable category is the *homotopy category* $\mathcal{HA}$. It is triangulated and its suspension functor is (up to isomorphism) the functor taking a complex X to the complex $\Sigma X = X[1]$, where $X[1]^p = X^{p+1}$, $p \in \mathbb{Z}$, and $d_{X[1]} = -d_X$.

2.4. Derived categories. Let k be a commutative ring and A a k-algebra (associative, with 1). Let $\mathsf{Mod}\, A$ denote the category of all right A-modules. The *derived category* $\mathcal{D}(A)$ of the abelian category $\mathsf{Mod}\, A$ has as its objects all complexes

$$\cdots \longrightarrow M^p \xrightarrow{d} M^{p+1} \longrightarrow \cdots$$

of A-modules; its morphisms are obtained from the morphisms of complexes by formally inverting all quasi-isomorphisms. It takes some work to deduce the fundamental properties of the derived category from this quick definition, *cf.* [60] [35]. In particular, one shows that the k-linear structure of the category of complexes is inherited by the derived category. Thus, the derived category has direct sums and they are given by direct sums of complexes. Even better, the derived category is triangulated: Its suspension functor takes a complex M to the complex ΣM with components $(\Sigma M)^p = M^{p+1}$ and with differential $-d_M$. Its triangles are those sequences isomorphic to standard triangles and the standard triangles

$$L \longrightarrow M \longrightarrow N \longrightarrow \Sigma L$$

are canonically associated with short exact sequences of complexes (L, M, N). The triangles of the derived category are the 'mothers' of all the long exact sequences appearing in homological algebra.

The canonical functor $\mathcal{C}A \to \mathcal{D}A$ factors canonically through a triangle functor $\mathcal{H}A \to \mathcal{D}A$.

If k is a field, we denote by $\mathcal{D}^b(A)$ the full triangulated subcategory of $\mathcal{D}(A)$ whose objects are the complexes whose homology modules are finite-dimensional over k and vanish for all but finitely many indices.

2.5. Triangle functors. We will denote the suspension functors of all triangulated categories by Σ. Let $\mathcal{S}$ and $\mathcal{T}$ be triangulated k-categories. A *triangle functor* from $\mathcal{S}$ to $\mathcal{T}$ is a pair (F, ϕ), where $F : \mathcal{S} \to \mathcal{T}$ is a k-linear functor and

$$\phi : F\Sigma \to \Sigma F$$

an isomorphism of functors such that for each triangle (u, v, w) of $\mathcal{S}$, the sequence

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{(\phi X)(Fw)} \Sigma FX$$

is a triangle of $\mathcal{T}$. The pair $(\mathbf{1}_{\mathcal{T}}, \mathbf{1}_{\Sigma})$ is the *identity triangle functor.* If (F, ϕ) and (G, ψ) are two triangle functors, their *composition* $(FG, (\phi G)(F\psi))$ is a triangle functor. A triangle functor (F, ϕ) is *strict* if ϕ is the identity.

An important example is the following: We have seen that if (u, v, w) is a triangle of , then $(\Sigma u, \Sigma v, -\Sigma w)$ is always a triangle. This means that the pair $(\Sigma, -\mathbf{1}_{\Sigma^2})$ formed by the suspension functor and the *opposite* of the identity of its square is a triangle functor. Notice that the pair $(\Sigma, \mathbf{1}_{\Sigma^2})$ is *not* a triangle functor in general. Often, one simply writes Σ for the triangle functor $(\Sigma, -\mathbf{1}_{\Sigma^2})$. This sometimes leads to confusion because of the implicit sign.

Suppose that (F, ϕ) and (G, γ) are triangle functors from $\mathcal{S}$ to $\mathcal{T}$. A *morphism of triangle functors* is a morphism of functors $\alpha : F \to G$ such that the square

$$\begin{array}{ccc} F\Sigma & \xrightarrow{\phi} & \Sigma F \\ {\scriptstyle \alpha\Sigma}\downarrow & & \downarrow{\scriptstyle \Sigma\alpha} \\ G\Sigma & \xrightarrow[\psi]{} & \Sigma G \end{array}$$

commutes. Clearly, the identity morphism is a morphism of triangle functors and so is the composition of two morphisms of triangle functors. Thus, the triangle functors from $\mathcal{S}$ to $\mathcal{T}$ form a category.

In fact, as one easily checks, triangulated categories, triangle functors and their morphisms together form a 2-category (*i.e.* a category enriched in categories), and more precisely a sub-2-category of the 2-category of categories. Now in any 2-category, one has natural notions of adjoint and equivalence. For the 2-category of triangulated categories, these give rise to the notion of *triangle adjoint* and *triangle equivalence*. Fortunately, to check whether a triangle functor has a triangle adjoint (respectively is a triangle equivalence) it suffices to check the corresponding property for the underlying k-linear functor, *cf.* [40] [60]. It is not hard to show that for each triangulated category $\mathcal{T}$, there is a natural triangle equivalence $\mathcal{T} \to \mathcal{T}'$, where the suspension functor of $\mathcal{T}'$ is an automorphism (and not just an autoequivalence), *cf.* [40].

2.6. Serre functors and the Calabi-Yau condition. Suppose that k is a field and that $\mathcal{T}$ is a triangulated k-category which is Hom-*finite*, *i.e.* for any two objects X and Y of $\mathcal{T}$, the morphism space $\mathcal{T}(X, Y)$ is finite-dimensional. Let D denote the duality functor $\mathsf{Hom}_k(?, k)$. A *right Serre functor for* $\mathcal{T}$ is given by a triangle functor $(S, \sigma) : \mathcal{T} \to \mathcal{T}$ together with a family of isomorphisms (called *trace maps*)

$$t_X : \mathcal{T}(?, SX) \to D\mathcal{T}(X, ?)$$

functorial in $X \in \mathcal{T}$ and such that[1] for all $X \in \mathcal{T}$, we have

$$t_X \circ \Sigma^{-1} \circ (\sigma X)_* = -(D\Sigma) \circ (t_{\Sigma X}\Sigma).$$

Notice the minus sign. It is needed in the proof of the following proposition.

Proposition 2.1 (Bondal-Kapranov [9], [58]). a) *$\mathcal{T}$ admits a right Serre functor iff, for each object X of $\mathcal{T}$, the functor*

$$D\mathcal{T}(X, ?) : \mathcal{T}^{op} \to \mathsf{Mod}\, k$$

is representable.

[1]The author thanks Guodong Zhou, Bill Crawley-Boevey and Andrew Hubery for pointing out that a compatibility condition is needed here if 'weakly Calabi-Yau' is to be different from 'Calabi-Yau'. The condition was unfortunately omitted in the published version.

(ii) *If $\mathcal{T}$ admits a right Serre functor, it is unique up to canonical isomorphism of triangle functors.*

A *Serre functor for $\mathcal{T}$* is a right Serre functor which moreover is an equivalence. We then say that $\mathcal{T}$ *has Serre duality.* For example, if X is a smooth projective variety of dimension d over a field, then the bounded derived category of coherent sheaves on X is Hom-finite and admits a Serre functor given by $\mathcal{F} \mapsto \mathcal{F} \otimes \omega[d]$, where ω is the canonical bundle.

Let d be an integer. The triangulated category $\mathcal{T}$ is *weakly d-Calabi-Yau* if it admits a Serre functor S and there is an isomorphism of k-linear functors

$$\Sigma^d \xrightarrow{\sim} S.$$

It is *d-Calabi-Yau* if it admits a Serre functor and there is an isomorphism of triangle functors

$$(S, \sigma) \xrightarrow{\sim} (\Sigma, -\mathbf{1}_{\Sigma^2})^d \ ,$$

where $(\Sigma, -\mathbf{1}_{\Sigma^2})$ is the suspension triangle functor defined in section 2.5.

It is helpful to translate these conditions in terms of trace forms. Without restriction of generality, let us suppose that the suspension functor of $\mathcal{T}$ is an automorphism (and not just an autoequivalence).

Proposition 2.2. *Suppose that $\mathcal{T}$ admits a Serre functor.*

a) *$\mathcal{T}$ is weakly d-Calabi-Yau iff there is a family of linear forms*

$$t_X : \mathcal{T}(X, \Sigma^d X) \to k \ , \ X \in \mathcal{T} \ ,$$

such that for all objects X and Y, the induced pairing $(f, g) \mapsto t_X(f \circ g)$ between $\mathcal{T}(X, Y)$ and $\mathcal{T}(Y, \Sigma^d X)$ is non degenerate and, for all morphisms $g : X \to Y$ and $f : Y \to \Sigma^d X$ of $\mathcal{T}$, we have

$$t_X(f \circ g) = t_Y((\Sigma^d g) \circ f).$$

b) *$\mathcal{T}$ is d-Calabi-Yau iff there is a family of linear forms $t_X : \mathcal{T}(X, \Sigma^d X) \to k$ satisfying the conditions of a) and such that moreover, for all morphisms $g : X \to \Sigma^p Y$ and $f : Y \to \Sigma^q X$ of $\mathcal{T}$ with $p + q = d$, we have*

$$t_X((\Sigma^p f) \circ g) = (-1)^{pq}\, t_Y((\Sigma^q g) \circ f).$$

For each object X of $\mathcal{T}$, we have the graded algebra

$$A = \mathsf{Ext}^*(X, X) = \oplus_{p \in \mathbb{Z}} \mathcal{T}(X, \Sigma^p X)$$

whose multiplication is given by $f \cdot g = (\Sigma^p f) \circ g$ where g is supposed homogeneous of degree p. Suppose that $\mathcal{T}$ is d-Calabi-Yau. Then we have the linear form $t : A \to k$ whose restriction to the component A^d is

$$t_X : \mathsf{Ext}^d(X, X) \to k$$

and which vanishes on all other components. Then the pairing

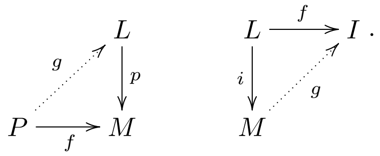
$$\langle a, b\rangle = t(ab)$$

is non degenerate on A and supersymmetric in the sense that, for a of degree p and b of degree q, we have

$$\langle a, b\rangle = (-1)^{pq}\langle b, a\rangle.$$

For example, if $d = 2$, then the finite-dimensional vector space $\mathsf{Ext}^1(X, X)$ carries a non degenerate antisymmetric form and thus has to be even dimensional. Notice that in order to deduce this, we need $\mathcal{T}$ to be 2-Calabi-Yau and not just weakly 2-Calabi-Yau.

2.7. Derived functors. In practice, Serre functors (and other triangle functors) are often given by total derived functors. Let k be a commutative ring and A a k-algebra. A complex of right A-modules P is *cofibrant* if for each quasi-isomorphism $p : L \to M$ with surjective components and each morphism $f : P \to M$, there is a lifting $g : P \to L$ such that $pg = f$.



Dually, a complex of right A-modules I is *fibrant* if for each quasi-isomorphism $i : L \to M$ with injective components and each morphism $f : L \to I$, there is an extension $g : M \to I$ such that $gi = f$.

One can show (*cf. e.g.* [34]) that for each complex L, there are quasi-isomorphisms

$$\mathbf{p}L \to L \text{ and } L \to \mathbf{i}L$$

where $\mathbf{p}L$ is cofibrant and $\mathbf{i}L$ is fibrant. For example, if L is an A-module (considered as a complex concentrated in degree 0) and we have a projective resolution

$$\ldots \to P_1 \to P_0 \to M \to 0\,,$$

then $\mathbf{p}M$ is homotopy equivalent to the complex

$$\ldots \to P_1 \to P_0 \to 0 \to 0 \to \ldots.$$

One can show that the assignments $\mathbf{p}$ and $\mathbf{i}$ give rise to functors from the derived category $\mathcal{D}A$ to the homotopy category $\mathcal{H}A$ which are fully faithful and left (respectively right) adjoint to the quotient functor $\mathcal{H}A \to \mathcal{D}A$.

Now suppose that k is a field and A and B are k-algebras. Let X be a complex of A-B-bimodules, *i.e.* an object of the derived category $\mathcal{D}(A^{op} \otimes B)$ (the symbol

$\otimes$ stands for the tensor product over the ground field). For a complex L of right A-modules, we write $L\otimes_A X$ for the complex of right B-modules whose nth component is

$$(L \otimes_A X)^n = \bigoplus_{p+q=n} L^p \otimes_A X^q$$

and whose differential is defined by

$$d(m \otimes x) = (dm) \otimes x + (-1)^p m \otimes (dx) \, ,$$

where $m \in L^p$, $x \in X^q$. Clearly, the complex $L \otimes_A X$ is functorial in L. The functor $? \otimes_A X$ admits a right adjoint: For a complex M of right B-modules, we write $\mathsf{Hom}_B(X, M)$ for the complex of right A-modules whose nth component is formed by the morphisms $f : X \to M$ of graded B-modules homogeneous of degree n (and which are not required to commute with the differential) and whose differential is defined by

$$d(f) = d_M \circ f - (-1)^n f \circ d_X.$$

It is not hard to check that the functors $? \otimes_A X$ and $\mathsf{Hom}_B(X, ?)$ induce a pair of adjoint functors between the homotopy categories of A- and B-modules. The *left derived functor*

$$? \overset{L}{\otimes}_A X : \mathcal{D}(A) \to \mathcal{D}(B)$$

takes a complex L to $(\mathbf{p}L) \otimes_A X$ and the *right derived functor*

$$\mathsf{RHom}_B(X, ?) : \mathcal{D}(B) \to \mathcal{D}(A)$$

takes a complex M to $\mathsf{Hom}_B(X, \mathbf{i}M)$. These are triangle functors (since they are compositions of triangle functors) and it is not hard to show that they are adjoints: We have a canonical isomorphism

$$\mathsf{Hom}_{\mathcal{D}B}(L \overset{L}{\otimes}_A X, M) = \mathsf{Hom}_{\mathcal{D}A}(L, \mathsf{RHom}_B(X, M)).$$

3. Examples: Orbit categories

3.1. Serre functors for finite-dimensional algebras. Let k be a field and A a finite-dimensional k-algebra (associative, with 1). Then the bounded derived category $\mathcal{D}^b(A)$ is known to be Hom-finite and the *decomposition theorem holds in* $\mathcal{D}^b(A)$: indecomposable objects have local endomorphism rings and each object is a finite direct sum of indecomposables [25]. We refer to [loc. cit.] and [50] for the notion of almost split triangle.

Theorem 3.1. *The following are equivalent*

(i) $\mathcal{D}^b A$ *has a Serre functor* S.

(ii) $\mathcal{D}^b A$ *has almost split triangles.*

(iii) *A is of finite global dimension.*

The equivalence between (i) and (ii) is proved in [50] and the equivalence between (ii) and (iii) in [26]. If the conditions of the theorem hold, then the Serre functor of $\mathcal{D}^b(A)$ is given by the left derived functor

$$S = ? \overset{L}{\otimes}_A DA \,,$$

where DA denotes the A-bimodule $\mathsf{Hom}_k(A, k)$ and the Auslander-Reiten translation is given by

$$\tau = \Sigma^{-1} \circ S.$$

3.2. Cluster categories. Now assume that the algebra A considered in the preceding paragraph is the path algebra kQ of a finite quiver Q without oriented cycles. Then A is finite-dimensional and of global dimension 1. Let d be an integer. Suppose that $d \geq 2$ or that $d = 1$ and Q is a Dynkin quiver (*i.e.* its underlying graph is a disjoint union of Dynkin diagrams of type A, D or E). It is natural to try and 'force' the triangulated category $\mathcal{D}^b(A)$ to become a Calabi-Yau category by 'quotienting' $\mathcal{D}^b(A)$ by the action of the autoequivalence $\Sigma^d S^{-1}$. Surprisingly, this actually works: The *d-cluster category of* Q is the k-linear category

$$\mathcal{C}_Q^{(d)} = \mathcal{D}^b(kQ)/(\Sigma^d S^{-1})^{\mathbb{Z}} \,,$$

obtained as the orbit category of the bounded derived category under the action of the automorphism group generated by $\Sigma^d S^{-1}$. By definition, this means that its objects are the same as those of $\mathcal{D}^b(kQ)$ and its morphisms are given by

$$\mathsf{Hom}_{\mathcal{C}_Q^{(d)}}(L, M) = \bigoplus_{p \in \mathbb{Z}} \mathsf{Hom}_{\mathcal{D}(kQ)}(L, (\Sigma^d S^{-1})^p M)$$

with the natural composition. Let us write $\pi : \mathcal{D}^b(kQ) \to \mathcal{C}_Q^{(d)}$ for the projection functor. Clearly we have an isomorphism of k-linear functors

$$\pi \Sigma^d S^{-1} \xrightarrow{\sim} \pi$$

and π is universal among the k-linear functors defined on $\mathcal{D}^b(kQ)$ and endowed with such an isomorphism. It is not hard to check that the d-cluster category is Hom-finite.

The *cluster category* $\mathcal{C}_Q$ is defined as the d-cluster category with $d = 2$. In the case where the underlying graph of Q is a Dynkin diagram of type A_n, the cluster category was introduced by Caldero-Chapoton-Schiffler [15] with a very different, more geometric description. In the general case, it was introduced independently by Buan-Marsh-Reineke-Reiten-Todorov [3]. The d-cluster category was introduced in [37] and first analyzed in [57].

Theorem 3.2 ([37]). *Suppose that $d \geq 2$ or $d \geq 1$ and Q is a Dynkin quiver. Then the d-cluster category has a natural structure of triangulated category such that the projection functor π becomes a strict triangle functor. Moreover, the d-cluster category is d-Calabi-Yau.*

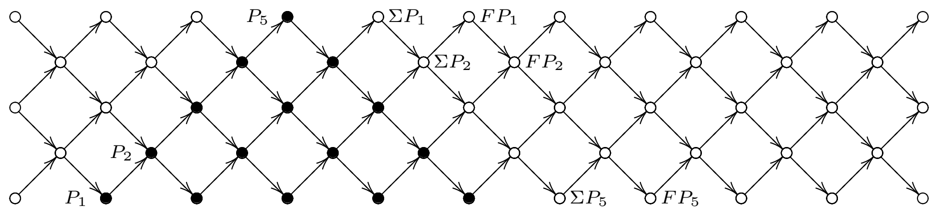


Figure 1. The Auslander-Reiten quiver of the derived category of A_5

In general, the orbit category of a triangulated category under an autoequivalence no longer admits a structure of triangulated category. The proof of the theorem heavily relies on the fact that kQ is of global dimension 1. The construction of the d-cluster category (and the theorem) generalize to bounded derived categories of hereditary abelian categories (satisfying suitable finiteness conditions). We refer to [63] for the case where $d = 2$.

Theorem 3.3 ([3]). *The decomposition theorem holds in the d-cluster category and its Auslander-Reiten quiver identifies with the quotient of that of the derived category $\mathcal{D}^b(kQ)$ under the action of the automorphism induced by $\Sigma^d S^{-1} = \Sigma^{d-1}\tau^{-1}$.*

We recall from [24] [25] that if Q is a Dynkin quiver, the Auslander-Reiten quiver of $\mathcal{D}^b(kQ)$ is the repetition $\mathbb{Z}Q$ and that $\mathcal{D}^b(kQ)$ is standard, *i.e.* its category of indecomposables admits a presentation by the quiver $\mathbb{Z}Q$ together with the mesh relations. For the quiver Q obtained by endowing the Dynkin diagram A_5 with the linear orientation, the quiver $\mathbb{Z}Q$ is recalled in Figure 1. Vertices corresponding to modules (identified with complexes concentrated in degree 0) are marked by $\bullet$. We have denoted the ith indecomposable projective module by P_i and the functor $\Sigma^2 S^{-1} = \tau^{-1}\Sigma$ by F. We obtain a 'fundamental domain' for the action of F by taking the full subquiver whose vertices lie between the sclices formed by the P_i and the FP_i. According to the theorem, we obtain the Auslander-Reiten quiver of the cluster category $\mathcal{C}_{A_5}$ by identifying the vertices P_i and FP_i in this subquiver. Thus, we obtain a Moebius strip.

3.3. Cluster categories and cluster algebras. The motivation for introducing the cluster category in [3] was to explain the similarities, discovered in [46], between the combinatorics of Fomin-Zelevinsky's cluster algebras and those of tilting theory over hereditary algebras. The following theorem shows that indeed, the link is very close. Let us assume that the ground field k is algebraically closed. As above, we denote by Q a finite quiver without oriented cycles and by $\mathcal{C}_Q$ its cluster category. We suppose that the set of vertices of Q is $\{1, \ldots, n\}$. An object M of $\mathcal{C}_Q$ is *rigid* if $\mathsf{Ext}^1_{\mathcal{C}_Q}(M, M) = 0$. We refer to [21] [22] for the cluster algebra associated with an antisymmetric matrix B. By definition, the cluster algebra $\mathcal{A}_Q$ associated with a quiver Q with vertex set $\{1, \ldots, n\}$ is the one associated with the matrix B whose (i, j)-coefficient equals the number of arrows from i to j minus the number of arrows from j to i. In [32], the reader can find a translation of Fomin-Zelevinsky's construction of $\mathcal{A}_Q$ into the quiver language.

Theorem 3.4. *There is a canonical bijection $M \mapsto X_M$ between the isomorphism classes of indecomposable rigid objects of the cluster category and the cluster variables of the cluster algebra associated with Q. Moreover, under this bijection, the clusters of $\mathcal{A}_Q$ correspond exactly to the n-tuples of indecomposable rigid objects whose direct sum is rigid.*

The theorem is proved in [16] on the basis of the previous results obtained by many authors notably Buan-Marsh-Reiten-Todorov [13], Buan-Marsh-Reiten [4], Buan-Marsh-Reineke-Reiten-Todorov [3], Caldero-Chapoton [14], Marsh-Reineke-Zelevinsky [46], ... The two main ingredients of the proof are the Calabi-Yau property of the cluster category and an explicit formula for X_M proved by Caldero-Chapoton in [14]. An alternative proof was given by A. Hubery [29] for quivers whose underlying graph is an extended simply laced Dynkin diagram.

The combinatorics of d-cluster categories with finitely many indecomposables are closely related to those of the generalized Coxeter complexes introduced in [20]. This was shown in [57], *cf.* also [39] [62] [64] [61].

3.4. A characterization of cluster categories. As above, let us assume that k is an algebraically closed field. An object T of a triangulated 2-Calabi-Yau category $\mathcal{C}$ is *cluster-tilting* [39] (or *maximal* 1*-orthogonal* in the terminology of [30]) if it is rigid and each object X satisfying $\mathsf{Ext}^1(T, X) = 0$ is a direct factor of a finite direct sum of copies of T. Now let Q be a finite quiver without oriented cycles. Then the cluster category $\mathcal{C}_Q$ has the following properties:

a) it is a triangulated weakly 2-Calabi-Yau category,

b) it contains a cluster-tilting object T whose endomorphism algebra has its quiver without oriented cycles (one can take $T = \pi(kQ)$ and then has $\mathsf{End}_{\mathcal{C}_Q}(T) = kQ$),

c) it is algebraic, *i.e.* triangle equivalent to the stable category of some Frobenius category (this is shown in [37]).

Theorem 3.5 ([38]). *If $\mathcal{C}$ is a triangulated category with the properties a), b) and c) and Q the quiver of a cluster-tilting object as in b), then $\mathcal{C}$ is triangle equivalent to the cluster category $\mathcal{C}_Q$.*

3.5. Additively finite Calabi-Yau triangulated categories. Let us now assume that Q is a Dynkin quiver and k is algebraically closed. Let $d \geq 1$ be an integer. Then the d-cluster category $\mathcal{C}_Q^{(d)}$ has the following properties:

a) it is a weakly d-Calabi-Yau triangulated category,

b) it has only finitely many isomorphism classes of indecomposables,

c) it is algebraic and standard.

Theorem 3.6 ([1]). *If $\mathcal{C}$ is a triangulated weakly 2-Calabi-Yau category with these properties, then $\mathcal{C}$ is an orbit category $\mathcal{C}_Q^{(d)}/G$ for some cyclic group of automorphisms G of $\mathcal{C}_Q^{(d)}$.*

In fact, Amiot [1] gives the classification of all the algebraic standard triangulated categories with finitely many indecomposables. She also shows that if an algebraic triangulated category has 'enough' indecomposables, it is automatically standard. However, if there are 'too few' indecomposables, it may be non standard. Namely, as shown in [1], the k-linear categories underlying the 1-Calabi-Yau triangulated categories with finitely many isomorphism classes of indecomposables are precisely the categories of finite-dimensional projective modules over deformed preprojective algebras of generalized Dynkin type introduced by Białkowsky-Erdmann-Skowroński [5]. Using this one deduces that there are non standard 1-Calabi-Yau triangulated categories in characteristic 2 (by [5]) and also in characteristic 3 (by [6]).

It is instructive to review Riedtmann's classification of representation-finite selfinjective algebras [51] [52] [53] [54] from the point of view of Calabi-Yau triangulated categories: In [27] and [28], Holm and Jørgensen determine which stable module categories are actually d-cluster categories. In [7], Białkowski and Skowroński extract the Calabi-Yau categories from Riedtmann's lists.

4. Examples: Derived categories

4.1. Serre functors: A key lemma. Let k be a field and A a k-algebra (associative, with 1). We do not assume that A is of finite dimension over k. Recall that $\mathcal{D}A$ denotes the (unbounded) derived category of the category of right A-modules and $\mathcal{D}^b(A)$ its full subcategory formed by the complexes whose homology is of finite total dimension. We write $\mathsf{per}(A)$ for the full triangulated subcategory of $\mathcal{D}A$ formed by the *perfect complexes*, *i.e.* those quasi-isomorphic to a bounded complex of finitely generated projective modules. An object P of $\mathcal{D}A$ lies in $\mathsf{per}(A)$ iff the functor $\mathsf{Hom}_{\mathcal{D}A}(P, ?)$ commutes with infinite direct sums. The algebra A is *homologically smooth* if A, considered as a bimodule over itself, belongs to $\mathsf{per}(A^{op} \otimes A)$. In other words, A is homologically smooth iff the bimodule A admits a finite resolution by finitely generated projective bimodules.

If M is a right module over an algebra B, then $\mathsf{Hom}_B(M, B)$ is a left B-module, *i.e.* a B^{op}-module. If we are given a morphism $\tau : B \to B^{op}$, we can convert $\mathsf{Hom}_B(M, B)$ again into a right B-module using the restriction along τ. This applies in particular to the algebra $B = A^{op} \otimes A$, which we endow with the morphism

$$\tau : A^{op} \otimes A \to (A^{op} \otimes A)^{op} \ , \ x \otimes y \mapsto y \otimes x.$$

This amounts to viewing $\mathsf{Hom}_{A^{op} \otimes A}(M, A^{op} \otimes A)$ as a bimodule using the 'inner' bimodule structure on $A^{op} \otimes A$. We write D for the k-dual $\mathsf{Hom}_k(?, k)$.

Lemma 4.1. *Suppose that A is homologically smooth. Define*

$$\Omega = \mathsf{RHom}_{A^{op}\otimes A}(A, A^{op}\otimes A)$$

and view it as an object of $\mathcal{D}(A^{op}\otimes A)$. Then for all objects L of $\mathcal{D}A$ and M of $\mathcal{D}^b A$, we have a canonical isomorphism

$$D\,\mathsf{Hom}_{\mathcal{D}A}(M, L) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{D}A}(L \overset{L}{\otimes}_A \Omega, M).$$

If we have an isomorphism $\Omega \xrightarrow{\sim} \Sigma^{-d} A$ in $\mathcal{D}(A^{op}\otimes A)$, then $\mathcal{D}^b A$ is d-Calabi-Yau.

Proof. Let us write A^e for $A^{op}\otimes A$, DM for $\mathsf{Hom}_k(M,k)$, Hom for RHom and $\otimes$ for $\overset{L}{\otimes}$. Since M is perfect in $\mathcal{D}(k)$ and A is perfect in $\mathcal{D}(A^e)$, the following canonical morphisms are invertible in $\mathcal{D}(k)$

$$\begin{aligned} L\otimes_A \Omega \otimes_A DM &= (L\otimes_k DM)\otimes_{A^e}\Omega \\ &\to \mathsf{Hom}_k(M,L)\otimes_{A^e}\Omega \\ &\to \mathsf{Hom}_{A^e}(A, \mathsf{Hom}_k(M,L)) = \mathsf{Hom}_A(M,L). \end{aligned}$$

If we use again that M is perfect in $\mathcal{D}(k)$, we obtain the isomorphisms

$$\begin{aligned} \mathsf{Hom}_k(\mathsf{Hom}_A(M,L),k) &\xrightarrow{\sim} \mathsf{Hom}_k(L\otimes_A\Omega\otimes_A DM, k) \\ &= \mathsf{Hom}_A(L\otimes_A\Omega, \mathsf{Hom}_k(DM,k)) \\ &\xrightarrow{\sim} \mathsf{Hom}_A(L\otimes\Omega, M). \end{aligned}$$

We obtain the first claim by taking zeroth homology. For the second claim, we first have to check that $\mathcal{D}^b A$ is Hom-finite. For this, we notice first that $\mathcal{D}^b A$ is contained in $\mathsf{per}(A)$. Indeed, A is contained in the thick subcategory of $\mathcal{D}(A^e)$ generated by A^e. This implies that $M = M\overset{L}{\otimes}_A A$ is contained in the thick triangulated subcategory generated by

$$M\overset{L}{\otimes}_A (A^{op}\otimes A) = M\overset{L}{\otimes}_k A$$

for each M in $\mathcal{D}(A)$. Now if M is perfect in $\mathcal{D}(k)$, then $M\otimes_k A$ is perfect in $\mathcal{D}(A)$ and so M is perfect in $\mathcal{D}(A)$. Clearly, if $M\in\mathcal{D}(A)$ is perfect in $\mathcal{D}(k)$, then $M = \mathsf{Hom}_A(A,M)$ is perfect in $\mathcal{D}(k)$ and so $\mathsf{Hom}_A(P,M)$ is perfect in $\mathcal{D}(k)$ for each P in $\mathsf{per}(A)$. In particular, $\mathsf{Hom}_A(L,M)$ is perfect in $\mathcal{D}(k)$ for all L, M in $\mathcal{D}^b(A)$. According to the first statement, the category $\mathcal{D}^b(A)$ admits a left Serre functor (in the sense of [50]) given by $?\otimes_A\Omega$. If Ω is isomorphic to $A[-d]$ in $\mathcal{D}(A^e)$, then $?\otimes_A\Omega$ is isomorphic to $?\otimes_A A[-d] = \Sigma^{-d}$ when restricted to $\mathcal{D}^b(A)$ and this is what we had to show. □

4.2. Two examples. Let V be an n-dimensional vector space and A the symmetric algebra on V. We compute the complex Ω using the Koszul bimodule resolution $\mathbf{p}A$, which is of the form

$$0\to A\otimes\Lambda^n\otimes A\to\ldots\to A\otimes\Lambda^0\otimes A\to A\to 0$$

where we put $\Lambda^i = \Lambda^i V$. We find that $\mathsf{Hom}_{A^e}(\mathbf{p}A, A^e)$ is isomorphic, as a complex of bimodules, to $(\mathbf{p}A)[-n]$ and hence quasi-isomorphic to $A[-n]$. Thus the category $\mathcal{D}^b(A)$ is n-Calabi-Yau (recall that the objects of this category are the complexes of A-modules whose homology is of finite total dimension).

Now let Q be a finite connected non Dynkin quiver. The *double quiver* $\overline{Q}$ is obtained by adjoining an arrow $a^* : j \to i$ for each arrow $a : i \to j$ of Q. The *preprojective algebra* $A = \Pi(Q)$ is defined to be the quotient of the path algebra of $\overline{Q}$ by the ideal generated by the sum of the commutators

$$\sum [a, a^*] \, ,$$

where a runs through the arrows of Q. Since Q is not Dynkin, the preprojective algebra is a Koszul algebra (this was shown for quivers Q with bipartite orientation in [47], for quivers without oriented cycles in [10, Cor. 4.3] following notes by B. Crawley-Boevey and in [45] for general quivers). Moreover, it is not hard to show that if $A^!$ denotes the Koszul dual algebra of A, then there is an isomorphism of graded $A^!$-modules

$$DA^! \xrightarrow{\sim} A^! \langle -2 \rangle \, ,$$

where D denotes the graded dual and $\langle\rangle$ the degree shift. Now using again the Koszul bimodule resolution to compute Ω, one obtains that Ω is quasi-isomorphic to $A[-2]$. Thus $\mathcal{D}^b(\Lambda(Q))$ is 2-Calabi-Yau.

Recall that according to Crawley-Boevey's description of the preprojective algebra [18], we have

$$\Pi(Q) = T_B(\mathsf{Ext}^1_{B^e}(B, B^e) \, ,$$

where $B = kQ$ and T_B denotes the tensor algebra in the category of B-B-bimodules. If Q is not a Dynkin quiver, then Ω has its homology concentrated in degree 1, and if we use a cofibrant resolution of Ω to compute the tensor algebra, we find a quasi-isomorphism of differential graded algebras

$$\Pi(Q) = T_B(\Omega[1]) \, ,$$

where T_B now denotes the tensor algebra in the category of complexes of B-B-bimodules. This construction generalizes: If B is any homologically smooth dg algebra and n any integer, we can form the 'derived preprojective algebra'

$$\mathbf{\Pi}_n(B) = T_B(\Omega[n-1])$$

and show that $\mathcal{D}^b(\mathbf{\Pi}_n(B))$ is n-Calabi-Yau, *cf.* [33].

4.3. Calabi-Yau quotients of path algebras. Let k be a field of characteristic 0. Let Q be a finite connected quiver and I an ideal of the path algebra kQ which is homogeneous (for path length) and generated in degrees ≥ 2. Let $A = kQ/I$ be the quotient of the path algebra by the ideal I.

Theorem 4.2 ([8]). *If $\mathcal{D}^b(A)$ is weakly 2-Calabi-Yau, then Q is isomorphic to the double quiver of some non Dynkin quiver R and kQ/I is isomorphic to the preprojective algebra of R.*

As we have seen in the preceding paragraph, the converse also holds. Let $[kQ, kQ]$ be the k-linear subspace of kQ generated by all commutators $uv - vu$, where u, v belong to kQ. A *potential on* Q is an element $W \in kQ/[kQ, kQ]$. Equivalently, a potential is a linear combination of cycles (=cyclic equivalence classes of cyclic paths). For an arrow a of Q, the *cyclic derivative with respect to* a is the unique linear map

$$\frac{\partial}{\partial a} : kQ/[kQ, kQ] \to kQ$$

which takes the class of a path p to the sum

$$\sum_{p=uav} vu$$

taken over all decompositions of the path p (where u and v are paths of length ≥ 0).

Theorem 4.3 ([8]). *If $\mathcal{D}^b(A)$ is weakly 3-Calabi-Yau, there is a homogeneous potential W in $kQ/[kQ, kQ]$ such that I is generated by the cyclic derivatives*

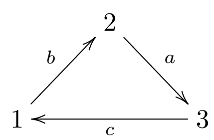
$$\frac{\partial}{\partial a} W \ , \ a \text{ an arrow of } Q.$$

Let us consider the following basic example: We have seen that the polynomial algebra $A = k[x, y, z]$ yields a 3-Calabi-Yau category $\mathcal{D}^b(A)$. Now we can write A as the quotient of the path algebra of a quiver Q with 3 loops x, y, z by a homogeneous ideal I generated in degree 2. According to the theorem, there must be a potential whose cyclic derivatives generate the ideal I. Indeed, if we take

$$W = xyz - xzy$$

the three cyclic derivatives yield three commutators which generate I.

The converse of the theorem is not true: Consider the cyclic quiver Q



with the potential $W = abc$. The quotient of the path algebra by the cyclic derivatives bc, ca, ab is a 6-dimensional self-injective algebra whose bounded derived category does not have a Serre functor (by Theorem 3.1) and a fortiori is not 3-Calabi-Yau. Nevertheless, there is a canonical 3-Calabi-Yau category associated with this quiver potential, as we will see in the next section.

5. 3-Calabi-Yau categories from potentials, after Kontsevich-Soibelman

5.1. A_∞-categories. We refer to [36] for an introduction to A_∞-structures and to [44] [56] for more detailed studies. An A_∞-*category* $\mathcal{A}$ is given by

- a set of objects $\mathsf{obj}(\mathcal{A})$,
- for all objects X and Y a $\mathbb{Z}$-graded vector space

$$\mathcal{A}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathcal{A}(X,Y)^p ,$$

- for all sequences $X_0, \dots, X_n$ of objects a linear map

$$m_n : \mathcal{A}(X_{n-1}, X_n) \otimes \mathcal{A}(X_{n-2}, X_{n-1}) \otimes \dots \otimes \mathcal{A}(X_0, X_1) \to \mathcal{A}(X_0, X_n)$$

  homogeneous of degree $2-n$

such that the following hold

- m_1 is a differential on $\mathcal{A}(X,Y)$, for all objects X, Y,
- m_2 is a derivation for m_1, *i.e.* we have

$$m_2 \circ (m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1) = 0$$

  on $\mathcal{A}(X_1, X_2) \otimes \mathcal{A}(X_0, X_1)$ for all triples of objects X_0, X_1, X_2,
- more generally, for each $n \geq 0$ and all $(n+1)$-tuples of objects $X_0, \dots, X_n$, we have the identity

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t} \circ (\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$

  on $\mathcal{A}(X_{n-1}, X_n) \otimes \mathcal{A}(X_{n-2}, X_{n-1}) \otimes \dots \otimes \mathcal{A}(X_0, X_1)$,
- each graded space $H^*(\mathcal{A}(U,U))$, $U \in \mathsf{obj}(\mathcal{A})$, contains a twosided unit for the composition maps on the $H^*(\mathcal{A}(X,Y))$, $X, Y \in \mathsf{obj}(\mathcal{A})$ induced by m_2.

If $\mathcal{A}$ is an A_∞-algebra, it has a well-defined *homology category* $H^*\mathcal{A}$, whose morphism spaces are the $H^*(\mathcal{A}(X,Y))$, $X, Y \in \mathsf{obj}(\mathcal{A})$ and whose composition is induced by m_2. If $\mathcal{A}$ is *minimal* (*i.e.* $m_1 = 0$), then $\mathcal{A}$ can be viewed as a deformation of its homology category.

For example, if B is a finite-dimensional algebra and $S_1, \dots, S_n$ are the simple B-modules (up to isomorphism), then there is a minimal A_∞-category $\mathcal{S}$, canonical up to *A_∞-isomorphism*, whose objects are the S_i, whose morphism spaces are the $\mathsf{Ext}^*_B(S_i, S_j)$, whose m_2 is the Yoneda composition and whose higher m_i encode more subtle information. One can define the *derived category* $\mathcal{DA}$ of an A_∞-category $\mathcal{A}$ and show that in this example, there is a triangle equivalence $\mathcal{DS} \to \mathcal{D}B$.

5.2. Cyclic structures and potentials. We follow section 10 of [42]. Let $\mathcal{A}$ be a minimal A_∞-category whose morphism spaces are of finite total dimension and let d be an integer. A *cyclic structure* on $\mathcal{A}$ is the datum of bilinear forms

$$\langle , \rangle : \mathcal{A}(X,Y) \times \mathcal{A}(Y,X) \to k$$

homogeneous of degree $-d$ and such that

a) the form $\langle , \rangle$ is non degenerate for all X, Y and

b) for each $n \geq 0$ and all $X_0, \ldots, X_n$, the map

$$w_{n+1} : \mathcal{A}(X_{n-1}, X_n) \otimes \mathcal{A}(X_{n-1}, X_{n-2}) \otimes \ldots \otimes \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_n, X_0) \to k$$

taking $(a_1, \ldots, a_{n+1})$ to $\langle (m_n(a_1, \ldots, a_n), a_{n+1}) \rangle$ is *cyclically invariant*, *i.e.* we have

$$w_{n+1}(a_1, \ldots, a_{n+1}) = \pm w_{n+1}(a_2, \ldots, a_{n+1}, a_1)$$

where the sign depends on n and the parities of the homogeneous elements a_i in the natural way.

Notice that if we fix the bilinear form $\langle , \rangle$, then the datum of the compositions m_i, $i \geq 2$, is equivalent to that of the linear forms w_i, $i \geq 3$.

One can define the perfect derived category $\mathsf{per}(\mathcal{A})$ as the thick triangulated subcategory of $\mathcal{D}\mathcal{A}$ generated by the representable A_∞-modules $\mathcal{A}(?, X)$, $X \in \mathcal{A}$, and show that $\mathsf{per}(\mathcal{A})$ is Hom-finite.

Proposition 5.1. *If $\mathcal{A}$ has a cyclic structure of degree d, then $\mathsf{per}(\mathcal{A})$ is d-Calabi-Yau.*

If Q is a quiver without loops or 2-cycles and W a potential on Q, the idea is now to construct an A_∞-category $\mathcal{A}(Q, W)$ with a cyclic structure of degree 3 (whose objects are simply the vertices of Q). By the proposition, the perfect derived category $\mathsf{per}(\mathcal{A}(Q, W))$ is then a 3-Calabi-Yau category associated with (Q, W). We will sketch the construction of $\mathcal{A} = \mathcal{A}(Q, W)$ below. One can show that $\mathsf{per}(\mathcal{A})$ carries a canonical t-structure whose heart has as its simples the representable modules associated with the vertices of Q. The heart is in fact equivalent to the category of finite-dimensional modules over the Jacobi algebra associated with (Q, W). The mutations of the quiver potential (Q, W) (in the sense of [19]) can be interpreted as tiltings of the t-structure on the 3-Calabi-Yau category $\mathsf{per}(\mathcal{A})$ similar to those used by Bridgeland [11] [12], *cf.* also Iyama-Reiten's study [31] of mutation versus tilting in the 3-dimensional case and forthcoming work by Chuang-Rouquier [17] and Kontsevich-Soibelman [43].

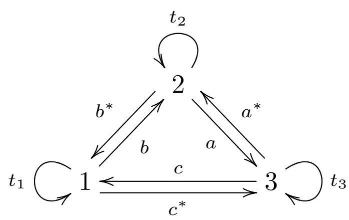
5.3. Construction of $\mathcal{A}(Q, W)$. Let Q be a finite quiver and W a potential on Q. We would like to construct an A_∞-category $\mathcal{A}(Q, W)$ endowed with a cyclic structure of degree 3 associated with (Q, W). This will be done directly in [43]. Here, we present an alternative approach via the A_∞-Koszul dual: The graded morphism spaces of $\mathcal{A} = \mathcal{A}(Q, W)$ will be finite-dimensional so that the datum of $\mathcal{A}$ will be equivalent to that of the A_∞-*cocategory* with morphism spaces $D\mathcal{A}(X,Y)$, $X,Y \in \mathsf{obj}(\mathcal{A})$. Now the datum of this A_∞-cocategory is equivalent to the datum of its *completed cobar category*, whose objects are those of $\mathcal{A}$, whose morphisms are obtained by forming the completed path category over the k-quiver with morphism spaces $\Sigma D\mathcal{A}(X,Y)$ (where Σ is the shift of grading) and whose differential has components given on the generators by the cocompositions Dm_n. We will describe this completed cobar category with its differential. It turns out to be isomorphic to the (completed) differential graded category $\mathcal{G} = \widehat{\mathfrak{D}}_{\bullet}(kQ, W)$ introduced by Ginzburg in [23]: Its objects are the vertices of Q. Its morphism spaces are those of the completed graded path category which is generated by

- the arrows of Q (they all have degree 0),
- an arrow $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q,
- loops $t_i : i \to i$ of degree -2 associated with each vertex i of Q.

The differential of $\mathcal{G}$ is defined on the generators as follows:

- $da = 0$ for each arrow a of Q,
- $d(a^*) = \frac{\partial}{\partial a} W$ for each arrow a of Q,
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex i of Q, where e_i is the idempotent associated with i and the sum runs over the set of arrows of Q.

One checks that $d^2 = 0$, which is equivalent to the A_∞-conditions. Here is the quiver of the Ginzburg dg category associated with the cyclic quiver at the end of section 4 with the potential $W = abc$:



The differential is given by

$$d(a^*) = bc\ ,\ d(b^*) = ca\ ,\ d(c^*) = ab\ ,\ d(t_1) = cc^* - b^*b\ ,\ \ldots$$

We obtain a triangulated 3-Calabi-Yau category equivalent to $\mathsf{per}(\mathcal{A}(Q,W))$ by taking the full subcategory $\mathcal{D}^b_{nil}(\mathcal{G})$ formed by the dg modules whose homologies are of finite total dimension and are nilpotent as modules over the Jacobi algebra.

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