

Cluster algebras and derived categories

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Abstract. This is an introductory survey on cluster algebras and their (additive) categorification using derived categories of Ginzburg algebras. After a gentle introduction to cluster combinatorics, we review important examples of coordinate rings admitting a cluster algebra structure. We then present the general definition of a cluster algebra and describe the interplay between cluster variables, coefficients, c -vectors and g -vectors. We show how c -vectors appear in the study of quantum cluster algebras and their links to the quantum dilogarithm. We then present the framework of additive categorification of cluster algebras based on the notion of quiver with potential and on the derived category of the associated Ginzburg algebra. We show how the combinatorics introduced previously lift to the categorical level and how this leads to proofs, for cluster algebras associated with quivers, of some of Fomin–Zelevinsky’s fundamental conjectures.

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1. Introduction

Cluster algebras, invented [44] by Sergey Fomin and Andrei Zelevinsky around the year 2000, are commutative algebras whose generators and relations are constructed in a recursive manner. Among these algebras, there are the algebras of homogeneous coordinates on the Grassmannians, on the flag varieties and on many other varieties which play an important role in geometry and representation theory. Fomin and Zelevinsky's main aim was to set up a combinatorial framework for the study of the so-called canonical bases which these algebras possess [78] [97] and which are closely related to the notion of total positivity [98] [41] in the associated varieties. It has rapidly turned out that the combinatorics of cluster algebras also appear in many other subjects, for example in

- Poisson geometry [60] [61] [62] [63] [10];
- discrete dynamical systems [27] [47] [74] [79] [84] [81] [91];

- higher Teichmüller spaces [35] [36] [37] [38] [39] ...;
- combinatorics and in particular the study of polyhedra like the Stasheff associahedra [21] [22] [42] [43] [71] [90] [102] [104] [103] [130] ...;
- commutative and non commutative algebraic geometry and in particular the study of stability conditions in the sense of Bridgeland [11], Calabi-Yau algebras [64] [75], Donaldson-Thomas invariants in geometry [76] [89] [88] [124] [137] ... and in string theory [1] [2] [14] [15] [16] [49] [50] [51] ...;
- in the representation theory of quivers and finite-dimensional algebras, cf. for example the survey articles [3] [5] [57] [84] [95] [126] [125] [127] ...

as well as in mirror symmetry [67], KP solitons [87], hyperbolic 3-manifolds [109], We refer to the introductory articles [41] [46] [144] [145] [146] [147] and to the cluster algebras portal [40] for more information on cluster algebras and their links with other subjects in mathematics (and physics).

In these notes, we give a concise introduction to cluster algebras and survey their (additive) categorification via derived categories of Ginzburg dg (=differential graded) algebras. We prepare the ground for the formal definition of cluster algebras by giving an approximate description and the first examples in section 2. In section 3, we introduce the central construction of quiver mutation and define the cluster algebra associated with a quiver and, more generally, with a valued quiver (section 3.3). We extend the definition to that of cluster algebras of geometric type and present several examples in section 4. Here we also review results on ring-theoretic properties of cluster algebras (finite generation and factoriality). In section 5, we give the general definition of cluster algebras with coefficients in an arbitrary semifield. In this general framework, the symmetry between cluster variables and coefficients becomes apparent, for example in the separation formulas in Theorem 5.7 but also, at the ‘tropical level’, in the duality Theorem 5.4. In section 6, we present the construction of quantum cluster algebras and its link with the quantum dilogarithm function. We show how cluster algebras allow one to construct identities between products of quantum dilogarithm series. This establishes the link to Donaldson–Thomas theory, as we will see for example in section 7.11.

In section 7, we turn to the (additive) categorification of cluster algebras. In section 5 of [84], the reader will find a gentle introduction to this subject along the lines of the historical development. We will not repeat this here but restrict ourselves to a description of the most recent framework, which applies to arbitrary symmetric cluster algebras (of geometric type). The basic idea is to lift the cluster variables in the cluster algebra associated with a quiver Q to suitable representations of Q . These representations have to satisfy certain relations, which are encoded in a potential on the quiver. We review quivers with potentials and their mutations following Derksen–Weyman–Zelevinsky [25] in section 7.1. A conceptual framework for the study of the representations of a quiver with potential is provided by the derived category of the associated Ginzburg dg algebra (section 7.4). Here mutations of quivers with potential yield equivalences between derived categories of Ginzburg algebras (section 7.5). In fact, each mutation canonically lifts to *two*

equivalences. Thus, in trying to compose the categorical lifts of N mutations, we are forced to choose between 2^N possibilities. The canonical choice was discovered by Nagao [106] and is presented in section 7.7. The framework thus created allows for the categorification of all the data associated with a commutative cluster algebra (Theorem 7.9). A recent extension to quantum cluster algebras (under suitable technical assumptions) is due to Efimov [30]. Surprisingly, the combinatorial data determine the categorical data to a very large extent (sections 7.8 and 7.9). We end by linking our formulation of the ‘decategorification Theorem’ 7.9 to the statements available in the literature (sections 7.10) and by proving Theorem 6.5 on quantum dilogarithm identities (section 7.11).

This introductory survey leaves out a number of important recent developments, notably monoidal categorification, as developed by Hernandez-Leclerc [70] [95] and Nakajima [111], the theory of cluster algebras associated with marked surfaces [43] [104] [103] [20] ... and recent progress on the links between (quantum) cluster algebras and Lie theory [59] [58] [53]

Acknowledgment

This survey expands on a talk given by the author at the GCOE Conference ‘Derived Categories 2011 Tokyo’, where Theorem 6.5 was presented. He deeply thanks the organizers for their invitation and their kind hospitality. He is very grateful to B. Leclerc for correcting an error in a previous version of example 4.3 and to H. Nakajima for a helpful conversation on the results of [30].

2. Description and first examples of cluster algebras

2.1. Description. A *cluster algebra* is a commutative \mathbb{Q} -algebra endowed with a set of distinguished generators (the *cluster variables*) grouped into overlapping subsets (the *clusters*) of constant cardinality (the *rank*) which are constructed recursively via *mutation* from an *initial cluster*. The set of cluster variables can be finite or infinite.

Theorem 2.1 ([45]). *The cluster algebras having only a finite number of cluster variables are parametrized by the finite root systems.*

Thus, the classification is analogous to the one of semi-simple complex Lie algebras. We will make the theorem more precise in section 3 below (for simply laced root systems).

2.2. First example. In order to illustrate the description and the theorem, we present [147] the cluster algebra \mathcal{A}_{A_2} associated with the root system A_2 . By definition, it is generated as a \mathbb{Q} -algebra by the cluster variables x_m , $m \in \mathbb{Z}$, submitted to the *exchange relations*

$$x_{m-1}x_{m+1} = 1 + x_m, \quad m \in \mathbb{Z}.$$

Its clusters are by definition the pairs of consecutive cluster variables $\{x_m, x_{m+1}\}$, $m \in \mathbb{Z}$. The initial cluster is $\{x_1, x_2\}$ and two clusters are linked by a mutation if and only if they share exactly one variable.

The exchange relations allow one to write each cluster variable as a rational expression in the initial variables x_1, x_2 and thus to identify the algebra \mathcal{A}_{A_2} with a subalgebra of the field $\mathbb{Q}(x_1, x_2)$. In order to make this subalgebra explicit, let us compute the cluster variables x_m for $m \geq 3$. We have:

$$x_3 = \frac{1 + x_2}{x_1} \quad (1)$$

$$x_4 = \frac{1 + x_3}{x_2} = \frac{x_1 + 1 + x_2}{x_1 x_2} \quad (2)$$

$$x_5 = \frac{1 + x_4}{x_3} = \frac{x_1 x_2 + x_1 + 1 + x_2}{x_1 x_2} \div \frac{1 + x_2}{x_1} = \frac{1 + x_1}{x_2}. \quad (3)$$

Notice that, contrary to what one might expect, the denominator in (3) remains a monomial! In fact, each cluster variable in an arbitrary cluster algebra is a Laurent polynomial, cf. Theorem 3.1 below. Let us continue the computation:

$$x_6 = \frac{1 + x_5}{x_4} = \frac{x_2 + 1 + x_1}{x_2} \div \frac{x_1 + 1 + x_2}{x_1 x_2} = x_1 \quad (4)$$

$$x_7 = (1 + x_1) \div \frac{1 + x_1}{x_2} = x_2. \quad (5)$$

It is now clear that the sequence of the x_m , $m \in \mathbb{Z}$, is 5-periodic and that the number of cluster variables is indeed finite and equal to 5. In addition to the two initial variables x_1 and x_2 , we have three non initial variables x_3 , x_4 and x_5 . By examining their denominators we see that they are in natural bijection with the positive roots $\alpha_1, \alpha_1 + \alpha_2, \alpha_2$ of the root system A_2 . This generalizes to an arbitrary Dynkin diagram, cf. Theorem 3.1.

2.3. Cluster algebras of rank 2. To each pair of positive integers (b, c) , there is associated a cluster algebra $\mathcal{A}_{(b,c)}$. It is defined in analogy with \mathcal{A}_{A_2} by replacing the exchange relations with

$$x_{m-1}x_{m+1} = \begin{cases} x_m^b + 1 & \text{if } m \text{ is odd,} \\ x_m^c + 1 & \text{if } m \text{ is even.} \end{cases}$$

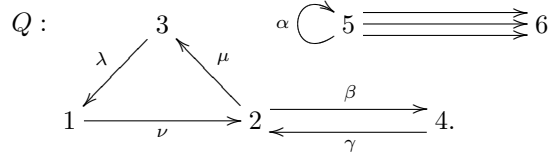
The algebra $\mathcal{A}_{(b,c)}$ has only a finite number of cluster variables if and only if we have $bc \leq 3$. In other words, if and only if the matrix

$$\begin{bmatrix} 2 & -b \\ -c & 2 \end{bmatrix}$$

is the Cartan matrix of a finite root system Φ of rank 2. The reader is invited to check that in this case, the non initial cluster variables are still in natural bijection with the positive roots of Φ .

3. Cluster algebras associated with quivers

3.1. Quiver mutation. A *quiver* is an oriented graph, i.e. a quadruple $Q = (Q_0, Q_1, s, t)$ formed by a set of vertices Q_0 , a set of arrows Q_1 and two maps s and t from Q_1 to Q_0 which send an arrow α respectively to its source $s(\alpha)$ and its target $t(\alpha)$. In practice, a quiver is given by a picture as in the following example



An arrow α whose source and target coincide is a *loop*; a *2-cycle* is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$. Similarly, one defines *n-cycles* for any positive integer n . A vertex i of a quiver is a *source* (respectively a *sink*) if there is no arrow with target i (respectively with source i).

By convention, in the sequel, by a quiver we always mean a finite quiver without loops nor 2-cycles whose set of vertices is the set of integers from 1 to n for some $n \geq 1$. Up to an isomorphism fixing the vertices such a quiver Q is given by the *skew-symmetric matrix* $B = B_Q$ whose coefficient b_{ij} is the difference between the number of arrows from i to j and the number of arrows from j to i for all $1 \leq i, j \leq n$. Conversely, each skew-symmetric matrix B with integer coefficients comes from a quiver.

Let Q be a quiver and k a vertex of Q . The *mutation* $\mu_k(Q)$ is the quiver obtained from Q as follows:

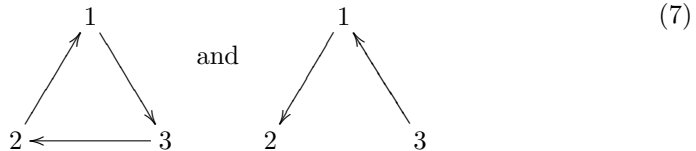
- 1) for each subquiver $i \xrightarrow{\beta} k \xrightarrow{\alpha} j$, we add a new arrow $[\alpha\beta] : i \rightarrow j$;
- 2) we reverse all arrows with source or target k ;
- 3) we remove the arrows in a maximal set of pairwise disjoint 2-cycles.

For example, if k is a source or a sink of Q , then the mutation at k simply reverses all the arrows incident with k . In general, if B is the skew-symmetric matrix associated with Q and B' the one associated with $\mu_k(Q)$, we have

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k ; \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(0, b_{ik}b_{kj}) & \text{else.} \end{cases} \quad (6)$$

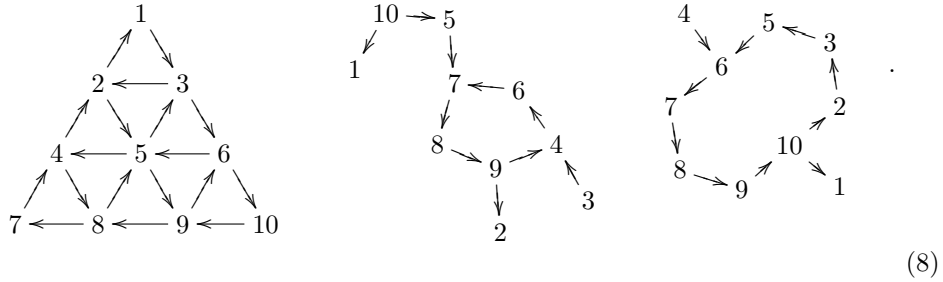
This is the *matrix mutation rule* for skew-symmetric (more generally: skew-symmetrizable) matrices introduced by Fomin-Zelevinsky in [44], cf. also [48].

One checks easily that μ_k is an involution. For example, the quivers



are linked by a mutation at the vertex 1. Notice that these quivers are drastically different: The first one is a cycle, the second one the Hasse diagram of a linearly ordered set.

Two quivers are *mutation equivalent* if they are linked by a finite sequence of mutations. For example, it is an easy exercise to check that any two orientations of a tree are mutation equivalent. Using the quiver mutation applet [82] or the package [105] one can check that the following three quivers are mutation equivalent



The common *mutation class* of these quivers contains 5739 quivers (up to isomorphism). The mutation class of ‘most’ quivers is infinite. The classification of the quivers having a finite mutation class was achieved by by Felikson-Shapiro-Tumarkin [34] [33]: in addition to the quivers associated with triangulations of surfaces (with boundary and marked points, cf. [43]) the list contains 11 exceptional quivers, the largest of which is in the mutation class of the quivers (8).

3.2. Seed mutation, cluster algebras. Let $n \geq 1$ be an integer and \mathcal{F} the field $\mathbb{Q}(x_1, \dots, x_n)$ generated by n indeterminates x_1, \dots, x_n . A *seed* (more precisely: *X-seed*) is a pair (R, u) , where R is a quiver and u a sequence u_1, \dots, u_n of elements of \mathcal{F} which freely generate the field \mathcal{F} . If (R, u) is a seed and k a vertex of R , the *mutation* $\mu_k(R, u)$ is the seed (R', u') , where $R' = \mu_k(R)$ and u' is obtained from u by replacing the element u_k by the element u'_k defined by the *exchange relation*

$$u'_k u_k = \prod_{s(\alpha)=k} u_{t(\alpha)} + \prod_{t(\alpha)=k} u_{s(\alpha)}, \tag{9}$$

where the sums range over all *arrows* of R with source k respectively target k . Notice that, if B is the skew-symmetric matrix associated with R , we can rewrite the exchange relation as

$$u'_k u_k = \prod_i u_i^{[b_{ik}]_+} + \prod_i u_i^{[-b_{ik}]_+}, \tag{10}$$

where, for a real number x , we write $[x]_+$ for $\max(x, 0)$. One checks that $\mu_k^2(R, u) = (R, u)$. For example, the mutations of the seed

$$(1 \longrightarrow 2 \longrightarrow 3, \{x_1, x_2, x_3\})$$

with respect to the vertices 1 and 2 are the seeds

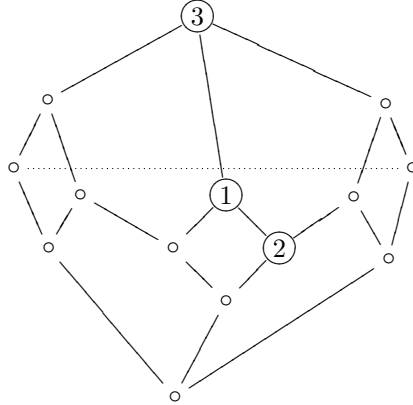
$$(1 \longleftarrow 2 \longrightarrow 3, \left\{ \frac{1+x_2}{x_1}, x_2, x_3 \right\}) \quad \text{and} \quad (11)$$

$$(1 \xleftarrow{\quad} 2 \xleftarrow{\quad} 3, \left\{ x_1, \frac{x_1+x_3}{x_2}, x_3 \right\}). \quad (12)$$

Let us fix a quiver Q . The *initial seed* of Q is $(Q, \{x_1, \dots, x_n\})$. A *cluster* associated with Q is a sequence u which appears in a seed (R, u) obtained from the initial seed by iterated mutation. The *cluster variables* are the elements of the clusters. The *cluster algebra* \mathcal{A}_Q is the \mathbb{Q} -subalgebra of \mathcal{F} generated by the cluster variables. Clearly, if (R, u) is a seed associated with Q , the natural isomorphism

$$\mathbb{Q}(u_1, \dots, u_n) \xrightarrow{\sim} \mathbb{Q}(x_1, \dots, x_n)$$

induces an isomorphism of \mathcal{A}_R onto \mathcal{A}_Q which preserves the cluster variables and the clusters. Thus, the cluster algebra \mathcal{A}_Q is an invariant of the mutation class of Q . It is useful to introduce a combinatorial object which encodes the recursive construction of the seeds: the *exchange graph*. By definition, its vertices are the isomorphism classes of seeds (isomorphisms of seeds renumber the vertices and the variables simultaneously) and its edges correspond to mutations. For example, the exchange graph obtained from the quiver $Q : 1 \longrightarrow 2 \longrightarrow 3$ is the 1-skeleton of the Stasheff associahedron [136]:



Here the vertex 1 corresponds to the initial seed and the vertices 2 and 3 to the seeds (11) and (12). For analogous polytopes associated with the other Dynkin diagrams, we refer to [22].

Let Q be a connected quiver. If its underlying graph is a simply laced Dynkin diagram Δ , we say that Q is a *Dynkin quiver of type Δ* .

Theorem 3.1 ([45]). a) *Each cluster variable of \mathcal{A}_Q is a Laurent polynomial with integer coefficients [44].*

b) *The cluster algebra \mathcal{A}_Q has only a finite number of cluster variables if and only if Q is mutation equivalent to a Dynkin quiver R . In this case, the*

underlying graph Δ of R is unique up to isomorphism and is called the cluster type of Q .

- c) If Q is a Dynkin quiver of type Δ , then the non initial cluster variables of \mathcal{A}_Q are in bijection with the positive roots of the root system Φ of Δ ; more precisely, if $\alpha_1, \dots, \alpha_n$ are the simple roots, then for each positive root $\alpha = d_1\alpha_1 + \dots + d_n\alpha_n$, there is a unique non initial cluster variable X_α whose denominator is $x_1^{d_1} \dots x_n^{d_n}$.

Statement a) is usually referred to as the *Laurent phenomenon*. A *cluster monomial* is a product of non negative powers of cluster variables belonging to the same cluster. The construction of a ‘canonical basis’ of the cluster algebra \mathcal{A}_Q is an important and largely open problem, cf. for example [44] [134] [29] [18] [59] [103] [92] [93] [69]. It is expected that such a basis should contain all cluster monomials. Whence the following conjecture.

Conjecture 3.2 ([44]). *The cluster monomials are linearly independent over the field \mathbb{Q} .*

The conjecture was recently proved in [19] using the additive categorification of [117] and techniques from [17] [20]. It is expected to hold more generally for cluster algebras associated with valued quivers, cf. section 3.3 below. It is shown for a certain class of valued quivers by L. Demonet [23] [24]. For special classes of quivers, a basis containing the cluster monomials is known: If Q is a Dynkin quiver, one knows [13] that the cluster monomials form a basis of \mathcal{A}_Q . If Q is *acyclic*, i.e. does not have any oriented cycles, then Geiss-Leclerc-Schröer [52] show the existence of a ‘generic basis’ containing the cluster monomials.

Conjecture 3.3 ([45]). *The cluster variables are Laurent polynomials with non negative integer coefficients in the variables of each cluster.*

For quivers with two vertices, an explicit and manifestly positive formula for the cluster variables is given in [96]. The technique of monoidal categorification developed by Leclerc [94] and Hernandez-Leclerc [70] has recently allowed to prove the conjecture first for the quivers of type A_n and D_4 , cf. [70], and then for each bipartite quiver [111], i.e. a quiver where each vertex is a source or a sink. The positivity of all cluster variables with respect to the initial seed of an acyclic quiver is shown by Fan Qin [119] and by Nakajima [111, Appendix]. This is also proved by Efimov [30], who moreover shows the positivity of all cluster variables belonging to an acyclic seed with respect to the initial variables of an arbitrary quiver. Efimov combines the techniques of [88] with those of [106]. A proof of the full conjecture for acyclic quivers using Nakajima quiver varieties is announced by Kimura–Qin [86]. The conjecture has been shown in a combinatorial way by Musiker-Schiffler-Williams [104] for all the quivers associated with triangulations of surfaces (with boundary and marked points) and by Di Francesco-Kedem [27] for the quivers and the cluster variables associated with the T -system of type A , with respect to the initial cluster.

We refer to [46] and [48] for numerous other conjectures on cluster algebras and to [26], cf. also [106] and [118] [117], for the solution of a large number of them using additive categorification.

3.3. Cluster algebras associated with valued quivers. A *valued quiver* is a quiver Q endowed with a function $v : Q_1 \rightarrow \mathbb{N}^2$ such that

- a) there are no loops in Q ,
- b) there is at most one arrow between any two vertices of Q and
- c) there is a function $d : Q_0 \rightarrow \mathbb{N}$ such that $d(i)$ is strictly positive for all vertices i and, for each arrow $\alpha : i \rightarrow j$, we have

$$d(i)v(\alpha)_1 = v(\alpha)_2 d(j),$$

where $v(\alpha) = (v(\alpha)_1, v(\alpha)_2)$.

For example, we have the valued quivers (we omit the labels $(1, 1)$ from our pictures)

$$\vec{B}_3 : 1 \longrightarrow 2 \xrightarrow{(1,2)} 3 \quad \text{and} \quad \vec{C}_3 : 1 \longrightarrow 2 \xrightarrow{(2,1)} 3,$$

where possible functions d are given by $d(1) = d(2) = 2, d(3) = 1$ respectively $d(1) = d(2) = 1, d(3) = 2$. A valued quiver (Q, v) is *equally valued* if we have $v(\alpha)_1 = v(\alpha)_2$ for each arrow α . If Q is an ordinary quiver without loops nor 2-cycles, the *associated valued quiver* is the equally valued quiver which has an arrow $\alpha : i \rightarrow j$ if there is at least one arrow $i \rightarrow j$ in Q and where $v(\alpha) = (m, m)$, where m is the number of arrows from i to j in Q . For example, the equally valued quiver

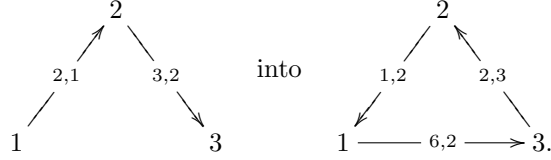
$$1 \xrightarrow{(2,2)} 2 \quad \text{corresponds to the Kronecker quiver} \quad 1 \rightrightarrows 2.$$

In this way, the ordinary quivers without loops nor 2-cycles correspond bijectively to the equally valued quivers (up to isomorphism fixing the vertices). Let Q be a valued quiver with vertex set I . We associate an integer matrix $B = (b_{ij})_{i,j \in I}$ with it as follows

$$b_{ij} = \begin{cases} 0 & \text{if there is no arrow between } i \text{ and } j; \\ v(\alpha)_1 & \text{if there is an arrow } \alpha : i \rightarrow j; \\ -v(\alpha)_2 & \text{if there is an arrow } \alpha : j \rightarrow i. \end{cases}$$

If D is the diagonal $I \times I$ -matrix with diagonal entries $d_{ii} = d(i)$, $i \in I$, then the matrix DB is skew-symmetric. The existence of such a matrix D means that the matrix B is *skew-symmetrizable*. It is easy to check that in this way, we obtain a bijection between the skew-symmetrizable $I \times I$ -matrices B and the valued quivers with vertex set I (up to isomorphism fixing the vertices). Using this bijection, we

define the *mutation of valued quivers* using Fomin-Zelevinsky's matrix mutation rule (6). For example, the mutation at 2 transforms the valued quiver



We extend the notion of an (X) -seed (R, u) by now allowing the first component R to be any valued quiver and we extend the construction of seed mutation by using the rule (10), where B is the skew-symmetrizable matrix associated with R . For example the mutations of the seed

$$(1 \longrightarrow 2 \xrightarrow{(1,2)} 3 , \{x_1, x_2, x_3\})$$

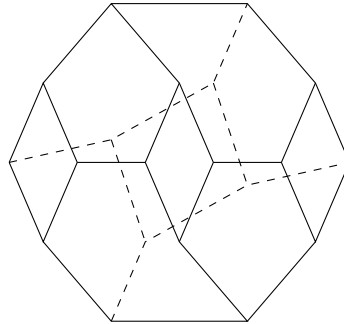
at the vertices 1 and 2 are the seeds

$$(1 \longleftarrow 2 \xrightarrow{(1,2)} 3 , \{ \frac{1+x_2}{x_1}, x_2, x_3 \})$$

and

$$(1 \xleftarrow{(1,2)} 2 \xleftarrow{(2,1)} 3 , \{ x_1, \frac{x_1+x_3^2}{x_2}, x_3 \}).$$

Given a valued quiver Q , we define its associated *clusters*, *cluster variables*, *cluster monomials*, the *cluster algebra* \mathcal{A}_Q and the *exchange graph* in complete analogy with the constructions in section 3.2. For example, the exchange graph of the above quivers \vec{B}_3 and \vec{C}_3 is the 3rd cyclohedron [22], with 4 quadrilateral, 4 pentagonal and 4 hexagonal faces:



Let (Q, v) be a valued quiver with vertex set $I = Q_0$. Its *associated Cartan matrix* is the Cartan companion [45] of the skew-symmetrizable matrix B associated with Q . Explicitly, it is the $I \times I$ -matrix C whose coefficient c_{ij} vanishes if there are no arrows between i and j , equals 2 if $i = j$, equals $-v(\alpha)_1$ if there is

an arrow $\alpha : i \rightarrow j$ and equals $-v(\alpha)_2$ if there is an arrow $\alpha : j \rightarrow i$. Thus, the Cartan matrix associated with the above valued quiver \bar{B}_2 equals

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

Fomin-Zelevinsky have shown in [45] that the the analogue of Theorem 3.1 holds for valued quivers. In particular, the Laurent phenomenon holds and the cluster algebra associated with a valued quiver Q has only finitely many cluster variables iff Q is mutation-equivalent to a valued quiver whose associated Cartan matrix corresponds to a finite root system.

For valued quivers, the independence conjecture 3.2 is open except for the valued quivers treated by Demonet [23] [24]. The positivity conjecture 3.3 is open except in rank two, where it was shown by Dupont in [28].

4. Cluster algebras of geometric type

We will slightly generalize the definition of section 3 in order to obtain the class of ‘skew-symmetrizable cluster algebras of geometric type’. This class contains many algebras of geometric origin which are equipped with ‘dual semi-canonical bases’ [99]. The construction of a large part of such a basis in [55] is one of the most remarkable applications of cluster algebras so far.

We refer to section 5.7 for the definition of the ‘skew-symmetrizable cluster algebras with coefficients in a semi-field’, which constitute so far the most general class considered.

4.1. Definition. Let $1 \leq n \leq m$ be integers. Let \tilde{Q} be an *ice quiver of type* (n, m) , i.e. a quiver with m vertices and which does not have any arrows between vertices i, j which are both strictly greater than n . The *principal part* of \tilde{Q} is the full subquiver Q whose vertices are $1, \dots, n$ (a subquiver is *full* if, with any two vertices, it contains all the arrows linking them). The vertices $n + 1, \dots, m$ are called the *frozen vertices*. The *cluster algebra associated with the ice quiver* \tilde{Q}

$$\mathcal{A}_{\tilde{Q}} \subset \mathbb{Q}(x_1, \dots, x_m)$$

is defined in the same manner as the cluster algebra associated with a quiver (section 3) but

- only mutations with respect to non frozen vertices are allowed and no arrows between frozen vertices are added in the mutations;
- the variables x_{n+1}, \dots, x_m , which belong to all clusters, are called *coefficients* rather than cluster variables;
- the *cluster type* of the ice quiver is that of its principal part (if it is defined).

Notice that the datum of \tilde{Q} is equivalent to that of the integer $m \times n$ -matrix \tilde{B} whose coefficient b_{ij} is the difference of the number of arrows from i to j minus the number of arrows from j to i for all $1 \leq i \leq m$ and all $1 \leq j \leq n$. The top $n \times n$ part B of \tilde{B} is called its *principal part*. In complete analogy, one defines the cluster algebra associated with a *valued ice quiver* respectively with an integer $m \times n$ -matrix whose principal part is skew-symmetrizable.

We have the following sharpening of the Laurent phenomenon proved in Proposition 11.2 of [45].

Theorem 4.1 ([45]). *Each cluster variable in $\mathcal{A}_{\tilde{Q}}$ is a Laurent polynomial in the initial variables x_1, \dots, x_n with coefficients in $\mathbb{Z}[x_{n+1}, \dots, x_m]$.*

Often one considers localizations of $\mathcal{A}_{\tilde{Q}}$ obtained by inverting some or all of the coefficients. If K is an extension field of \mathbb{Q} and A a commutative K -algebra without zero divisors, a *cluster structure of type \tilde{Q} on A* is given by an isomorphism φ from $\mathcal{A}_{\tilde{Q}} \otimes_{\mathbb{Q}} K$ onto A . Such an isomorphism is determined by the images of the coefficients and of the initial cluster variables $\varphi(x_i)$, $1 \leq i \leq m$. We call the datum of \tilde{Q} and of the $\varphi(x_i)$ an *initial seed for A* . The following proposition is a reformulation of Proposition 11.1 of [45], cf. also Proposition 1 of [132]:

Proposition 4.2. *Let X be a rational quasi-affine irreducible algebraic variety over \mathbb{C} . Let \tilde{Q} be an ice quiver of type (m, n) . Assume that we are given a regular function φ_c on X for each coefficient $c = x_i$, $n < i \leq m$, and a regular function φ_x on X for each cluster variable x of $\mathcal{A}_{\tilde{Q}}$ such that*

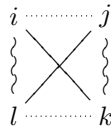
- a) *the dimension of X equals m ;*
- b) *the functions φ_x and φ_c generate the coordinate algebra $\mathbb{C}[X]$;*
- c) *the correspondence $x \mapsto \varphi_x$, $c \mapsto \varphi_c$ takes each exchange relation of $\mathcal{A}_{\tilde{Q}}$ to an equality in $\mathbb{C}[X]$.*

Then the correspondence $x \mapsto \varphi_x$, $c \mapsto \varphi_c$ extends to an algebra isomorphism $\varphi : \mathcal{A}_{\tilde{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}[X]$ so that $\mathbb{C}[X]$ carries a cluster algebra structure of type \tilde{Q} with initial seed φ_{x_i} , $1 \leq i \leq m$.

4.2. Example: Planes in a vector space. Let $n \geq 1$ be an integer. Let A be the algebra of polynomial functions on the cone over the Grassmannian of planes in \mathbb{C}^{n+3} . This algebra is generated by the Plücker coordinates x_{ij} , $1 \leq i < j \leq n+3$, subject to the Plücker relations: for each quadruple of integers $i < j < k < l$ between 1 and $n+3$, we have

$$x_{ik}x_{jl} = x_{ij}x_{kl} + x_{jk}x_{il}. \tag{13}$$

Notice that the monomials in this relation are naturally associated with the diagonals and the sides of the square



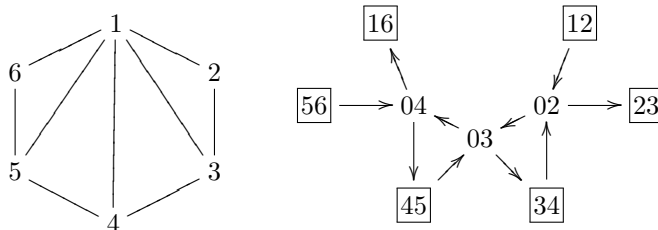
The idea is to interpret this relation as an exchange relation in a cluster algebra with coefficients. In order to describe this algebra, let us consider, in the euclidean plane, a regular polygon P whose vertices are numbered from 1 to $n + 3$. Consider the variable x_{ij} as associated with the segment $[ij]$ which links the vertices i and j .

Proposition 4.3 ([45, Example 12.6]). *The algebra A has a cluster algebra structure such that*

- the coefficients are the variables x_{ij} associated with the sides of P ;
- the cluster variables are the variables x_{ij} associated with the diagonals of P ;
- the clusters are the n -tuples of cluster variables corresponding to diagonals which form a triangulation of P .

Moreover, the exchange relations are exactly the Plücker relations and the cluster type is A_n .

A triangulation of P determines an initial seed for the cluster algebra and the exchange relations satisfied by the initial cluster variables determine the ice quiver \tilde{Q} . For example, one can check that in the following picture, the triangulation and the ice quiver (whose frozen vertices are in boxes) correspond to each other.



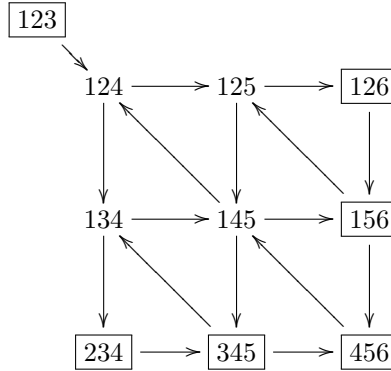
The hypotheses of proposition 4.2 are straightforward to check in this example. Many other (homogeneous) coordinate algebras of classical algebraic varieties admit cluster algebra structures (or ‘upper cluster algebra structures’) and in particular the Grassmannians [132], cf. section 4.3 below, and the double Bruhat cells [9]. Some of these algebras have only finitely many cluster variables and thus a well-defined cluster type. Here is a list of some examples of varieties and their cluster type extracted from [46], where N denotes a maximal unipotent subgroup of the corresponding reductive algebraic group:

$Gr_{2,n+3}$	$Gr_{3,6}$	$Gr_{3,7}$	$Gr_{3,8}$
A_n	D_4	E_6	E_8

SL_3/N	SL_4/N	SL_5/N	Sp_4/N	SL_2	SL_3
A_1	A_3	D_6	B_2	A_1	D_4

A theorem analogous to proposition 4.3 for ‘reduced double Bruhat cells’ is due to Yang and Zelevinsky [140]. They thus obtain a cluster algebra (with principal coefficients) with an explicit description of the cluster variables for each Dynkin diagram.

4.3. Example: The Grassmannian $Gr(3, 6)$. Let us consider the cone X over the Plücker embedding of the variety $Gr(3, 6)$ of 3-dimensional subspaces in 6-dimensional complex space \mathbb{C}^6 , considered as a space of rows. The Plücker coordinates of the subspace generated by the rows of a complex 3×6 -matrix are the 3×3 -minors of the matrix, *i.e.* the determinants $D(\mathbf{j})$ of the 3×3 -submatrices formed by the columns with indices in a 3-element subset \mathbf{j} of $\{1, \dots, 6\}$. It is a particular case of Scott's theorem [132], cf. also Example 10.3 of [56], that the algebra $\mathbb{C}[X]$ admits a cluster algebra structure of the type



whose initial seed is given by the minors $D(\mathbf{j})$ associated with the vertices \mathbf{j} of this quiver (frozen vertices appear in boxes). If we mutate the principal part of this quiver at the vertex 124, we obtain a Dynkin quiver of type D_4 , which is thus the cluster type of this cluster algebra. It admits $4 + 12 = 16$ cluster variables. As shown in [132], fourteen among these are minors and the remaining two are

$$X_1 = |P_1 \wedge Q_1, P_2 \wedge Q_2, P_3 \wedge Q_3| \quad \text{and} \quad X_2 = |Q_1 \wedge P_2, Q_2 \wedge P_3, Q_3 \wedge P_1|,$$

where we denote the columns of our matrix by $P_1, Q_1, P_2, Q_2, P_3, Q_3$ (in this order) and write $||$ for the determinant. In this cluster algebra, we have the remarkable identity [141]

$$|P_1 P_2 Q_2| |P_2 P_3 Q_3| |P_3 P_1 Q_1| - |P_1 P_2 Q_1| |P_2 P_3 Q_2| |P_3 P_1 Q_3| = |P_1 P_2 P_3| X_1, \quad (14)$$

which we can rewrite as

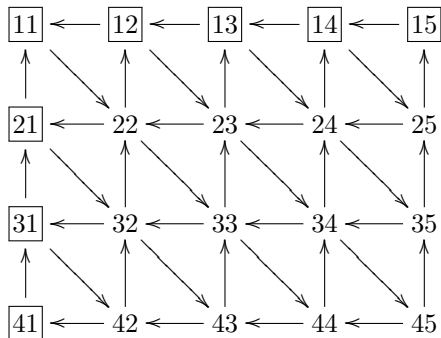
$$D(134)D(356)D(125) + D(123)D(345)D(156) = D(135)X_1.$$

This is in fact an exchange relation in our cluster algebra (many thanks to B. Leclerc for pointing this out): Indeed, if we successively mutate the initial seed at the vertices 124 and 145, we obtain the cluster

$$D(135), D(125), D(356), D(134)$$

(exercise: compute the corresponding quiver!) and if we now mutate at the variable $D(135)$, we obtain X_1 and the exchange relation (14). This relation appears implicitly in [65] and finding a suitable generalization to higher dimensions would be of interest in view of Zagier's conjecture [142].

4.4. Example: Rectangular matrices. Polynomial algebras admit many interesting cluster algebra structures. As a representative example, let us consider such a structure on the algebra A of polynomial functions on the space of complex 4×5 -matrices. For $1 \leq i \leq 4$ and $1 \leq j \leq 5$, let $D(ij)$ be the determinant of the largest square submatrix of a 4×5 -matrix whose upper left corner is the (i, j) -coefficient. Then the algebra A admits a cluster structure of type \tilde{Q}



whose initial seed is formed by the functions $D(ij)$ associated with the vertices of the quiver \tilde{Q} . This is a particular case of a theorem of Geiss–Leclerc–Schröer [58]. Perhaps the most remarkable fact is that iterated mutations of the initial seed still produce polynomials in the matrix coefficients (and not fractions). Geiss–Leclerc–Schröer’s proof of this fact in [58] is ultimately based on Lusztig’s results [99]. They sketch a more elementary approach in section 7.3 of [54], cf. section 4.6 below. It is not obvious either that the cluster variables generate the polynomial ring. To prove it, we first notice that the variables $x_{25}, x_{35}, x_{42}, x_{43}, x_{44}, x_{45}$ already belong to the initial seed. Now, following [58], we consider the sequence of mutations at the vertices

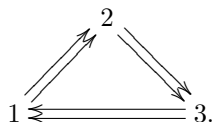
$$45, 44, 43, 42, 35, 25; 34, 33, 32, 24; 23, 22; 45, 44, 43, 35; 34, 33; 45, 44.$$

The sequence naturally splits into ‘hooks’, which we have separated by semicolons. The cluster variables which appear successively under this sequence of mutations are

$$x_{34}, x_{33}, x_{32}, \dots, x_{24}, \dots, x_{23}, x_{22}, \dots,$$

where we have only indicated those variables associated with mutations at the vertices of the lower right rim: 25, 35, 42, 43, 44, 45. So we see that in fact all the functions x_{ij} are cluster variables.

4.5. Finite generation. In general, cluster algebras are not finitely generated as algebras. For example, consider the cluster algebra \mathcal{A}_Q associated with the quiver



Let us show, following [101], that \mathcal{A}_Q is not even Noetherian. Indeed, up to isomorphism, the quiver Q is invariant under mutations. Hence all exchange relations are of the form

$$u_k u'_k = u_i^2 + u_j^2$$

for three pairwise distinct indices i, j and k . It follows that \mathcal{A}_Q admits a grading such that all cluster variables have degree 1. Since Q is not mutation-equivalent to a Dynkin quiver, by Theorem 3.1, there are infinitely many cluster variables and by Conjecture 3.2, proved in [19], they are linearly independent over the field \mathbb{Q} , which is the degree 0 part of \mathcal{A}_Q . But a positively graded commutative algebra whose degree 1 part is not a finitely generated module over its degree 0 part cannot be Noetherian. Many more examples are provided by the following theorem

Theorem 4.4 (Th. 1.24 of [9]). *If Q is a valued quiver with three vertices, the cluster algebra \mathcal{A}_Q is finitely generated over the rationals if and only if Q is mutation-equivalent to an acyclic valued quiver.*

For an acyclic valued quiver with n vertices, the cluster algebra \mathcal{A}_Q admits a set of $2n$ generators. More precisely, we have the following theorem.

Theorem 4.5 (Cor. 1.21 of [9]). *If Q is acyclic with n vertices, the cluster algebra \mathcal{A}_Q is generated over the rationals by the initial variables x_1, \dots, x_n and the cluster variables x'_j , $1 \leq j \leq n$, obtained by mutating the initial seed at each vertex j .*

Moreover, by Cor. 1.21 of [9], if Q is acyclic, the generators $x_1, \dots, x_n, x'_1, \dots, x'_n$ together with the exchange relations between x_j and x'_j , $1 \leq j \leq n$, form a presentation of \mathcal{A}_Q and the monomials in the generators not containing any product $x_j x'_j$ form a \mathbb{Q} -basis.

The class of ‘locally acyclic’ cluster algebras is introduced in [101]. It contains all acyclic cluster algebras. As shown in [101], each locally acyclic cluster algebra is finitely generated, integrally closed and locally a complete intersection.

4.6. Factoriality. In general, cluster algebras need not be factorial, even when the exchange matrix is of full rank. The following example, based on an idea of P. Lampe, is given in [54]. Let Q be the generalized Kronecker quiver

$$1 \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} 2$$

and x'_1 the cluster variable obtained by mutating the initial seed at the vertex 1. Then we have

$$x_1 x'_1 = 1 + x_2^3 = (1 + x_2)(1 - x_2 + x_2^2)$$

and one can show that these are essentially different factorizations of the product $x_1 x'_1$ in \mathcal{A}_Q , cf. Prop. 6.3 of [54].

Now let \tilde{Q} be a valued ice quiver of type (n, m) and let $n \leq p \leq m$ be an integer. Let \mathcal{P} be the polynomial ring $\mathbb{Z}[x_{n+1}, \dots, x_m]$ and \mathcal{L} its localization at x_{n+1}, \dots, x_p . Let

$$\mathcal{A} = \mathcal{A}_{\tilde{Q}} \otimes_{\mathcal{P}} \mathcal{L}$$

be the localization of the cluster algebra $\mathcal{A}_{\tilde{Q}}$ at x_{n+1}, \dots, x_p . Notice that the invertible elements of \mathcal{L} are the Laurent monomials in x_{n+1}, \dots, x_p multiplied by ± 1 .

Theorem 4.6 ([54]). a) *The invertible elements of \mathcal{A} are those of \mathcal{L} .*

b) *Each cluster variable of \mathcal{A} is irreducible and two cluster variables are associate iff they are equal.*

As an application, let us show that the cluster algebra associated with a Dynkin quiver of type A_3 is not factorial. Indeed, consider the cluster algebra \mathcal{A} associated with the quiver

$$Q : 1 \longrightarrow 2 \longrightarrow 3.$$

Let x'_1 and x'_3 be the cluster variables obtained from the initial seed by mutating respectively at the vertices 1 and 3. We have

$$x'_1 = \frac{1+x_2}{x_1} \quad \text{and} \quad x'_3 = \frac{1+x_2}{x_3}$$

and therefore

$$x'_1 x_1 = x'_3 x_3.$$

Since x_1, x'_1, x_3, x'_3 are pairwise distinct cluster variables, it follows from the theorem that these are essentially distinct factorizations.

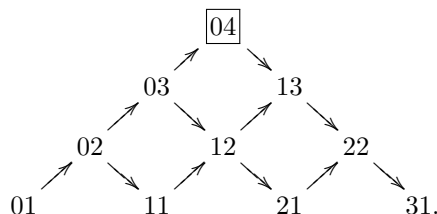
Despite these examples, many cluster algebras appearing ‘in nature’ are in fact factorial. The following theorem often allows to check this.

Theorem 4.7 ([54]). *As above, let \mathcal{A} be the cluster algebra associated with a valued ice quiver of type (n, m) localized at a subset x_{n+1}, \dots, x_p of the set of coefficients. Let y and z be disjoint clusters and $U \subset \mathcal{A}$ a subalgebra which is factorial and contains y, z and the localized coefficient algebra \mathcal{L} . Then \mathcal{A} equals U and an element x of the ambient field $\mathbb{Q}(x_1, \dots, x_m)$ belongs to \mathcal{A} iff it is a Laurent polynomial with coefficients in \mathcal{L} both in y and in z .*

As a prototypical example, consider the ice quiver

$$\tilde{Q} : 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \boxed{4}.$$

We will parametrise its coefficient and its cluster variables by the vertices of the following quiver



Namely, to a vertex ij , we associate a cluster variable $x_{i,j}$ in such a way that $x_{0,j}$ equals x_j , $1 \leq j \leq 4$, and each ‘mesh’ gives rise to an exchange relation: We have

$$x_{i,1}x_{i+1,1} = x_{i,2} + 1 \quad \text{for } 0 \leq i \leq 2$$

and

$$x_{i,j}x_{i+1,j} = x_{i,j+1}x_{i+1,j-1} + 1$$

for all vertices ij among 02, 03, 12. Then the set of cluster variables is the set of the $x_{i,j}$, where ij runs through the vertices other than 04. The variables at the bottom are

$$x_{0,1} = x_1, \quad x_{1,1} = \frac{1+x_2}{x_1}, \quad x_{2,1} = \frac{x_1+x_3}{x_2}, \quad x_{3,1} = \frac{x_2+x_4}{x_3}$$

They are algebraically independent and the polynomial ring

$$U = \mathbb{Z}[x_{0,1}, x_{1,1}, x_{2,1}, x_{3,1}]$$

contains the disjoint clusters $y = \{x_1, x_2, x_3\}$ and $z = \{x_{1,3}, x_{2,2}, x_{3,1}\}$ appearing on the left and the right rim. We see from the theorem that U equals the cluster algebra and that an element of the ambient field belongs to the cluster algebra iff it is a Laurent polynomial with coefficients in $\mathbb{Z}[x_4]$ both in y and in z . We refer to section 7.3 of [54] for more elaborate examples arising as coordinate algebras of unipotent cells in Kac-Moody groups.

5. General cluster algebras

5.1. Parametrization of seeds by the n -regular tree. Let us introduce a convenient parametrization of the seeds in the mutation class of a given initial seed. Let $1 \leq n \leq m$ be integers and \tilde{Q} a valued ice quiver of type (n, m) . Let $X = \{x_1, \dots, x_m\}$ be the initial cluster and (\tilde{Q}, X) the initial seed. Let \mathbb{T}_n be the n -regular tree: Its edges are labeled by the integers $1, \dots, n$ such that the n edges emanating from each vertex carry different labels, cf. figure 1. Let t_0 be a vertex of \mathbb{T}_n . To each vertex t of \mathbb{T}_n we associate a seed $(\tilde{Q}(t), X(t))$ such that at $t = t_0$, we have the initial seed and whenever t is linked to t' by an edge labeled k , the seeds associated with t and t' are related by the mutation at k . We write $x_i(t)$, $1 \leq i \leq n$, for the cluster variables in the seed $X(t)$. If \tilde{B} is the $m \times n$ -matrix associated with \tilde{Q} , we write $\tilde{B}(t)$ for the matrix associated with $\tilde{Q}(t)$.

5.2. Principal coefficients. Let $n \geq 1$ be an integer and Q a valued quiver with n vertices. Let B be the associated skew-symmetrizable integer $n \times n$ -matrix. In the next subsections, following [48], we will define data associated with Q which are relevant for all cluster algebras with coefficients associated with valued ice quivers \tilde{Q} whose principal part is Q . This will become apparent from a general formula expressing the cluster variables in terms of these data, cf. section 5.7.

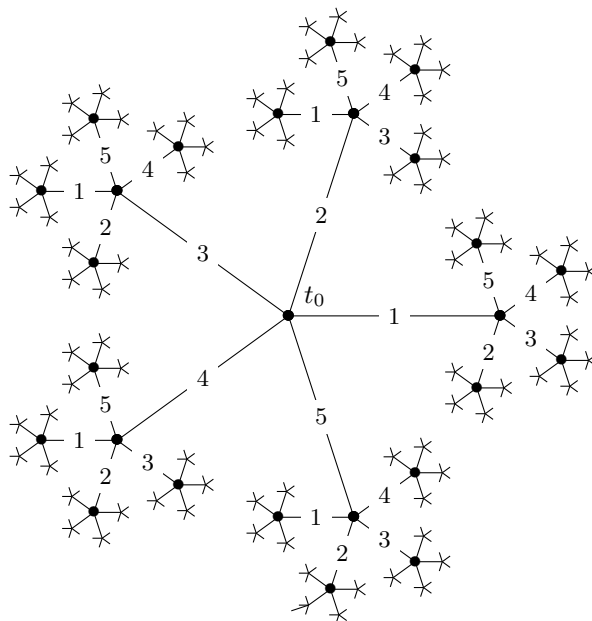


Figure 1. A picture, up to depth 4, of the 5-regular tree

5.3. Principal coefficients: c -vectors. Let Q_{pr} be the *principal extension* of Q , i.e. the valued quiver obtained from Q by adding new vertices $n + 1, \dots, 2n$ and new arrows $i + n \rightarrow i$, $1 \leq i \leq n$, for each vertex i of Q . For example, if we have

$$Q : 1 \longrightarrow 2, \quad \text{then} \quad Q_{pr} : \begin{array}{ccc} & 1' & 2' \\ & \downarrow & \downarrow \\ & 1 & \longrightarrow 2, \end{array}$$

where we write i' for $i + n$. The *cluster algebra with principal coefficients associated with Q* is the cluster algebra associated with Q_{pr} . We write B_{pr} for the corresponding integer $2n \times n$ -matrix. It is obtained from B by appending an $n \times n$ identity matrix at the bottom:

$$B_{pr} = \begin{bmatrix} B \\ I_n \end{bmatrix}.$$

For a vertex t of the n -regular tree, the *matrix of c -vectors* $C(t)$ is by definition the $n \times n$ -matrix appearing in the bottom part of $B_{pr}(t)$, so that we have

$$B_{pr}(t) = \begin{bmatrix} B(t) \\ C(t) \end{bmatrix}.$$

Its columns are the c -vectors at t . When necessary, we will denote the matrix $C(t)$ by $C(B, t_0, t)$ to clarify its dependence on B and the sequence of mutations linking

The F -polynomial

$$F_j(t) \in \mathbb{Z}[x_{n+1}, \dots, x_{2n}]$$

is by definition the specialization of $x_j(t)$ at $x_1 = 1, x_2 = 1, \dots, x_n = 1$.

To define the g -vectors, let us endow the ring

$$\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}, \dots, x_{2n}]$$

with the \mathbb{Z}^n -grading such that

$$\deg(x_j) = e_j \text{ and } \deg(x_{n+j}) = -Be_j \text{ for } 1 \leq j \leq n.$$

For each vertex t of the n -regular tree and each $1 \leq j \leq n$, the cluster variable $x_j(t)$ of $\mathcal{A}(Q_{pr})$ is in fact homogeneous for this grading (Prop. 6.1 of [48]). Its degree is by definition the g -vector $g_j(t)$. The *matrix of g -vectors* $G(t)$ has as its columns the vectors $g_j(t)$. When necessary, we will denote this matrix by $G(B, t_0, t)$ to clarify its dependence on B and the sequence of mutations linking t_0 to t .

For example, if B is associated with $Q : 1 \rightarrow 2$ and we mutate along the path

$$t_0 \xrightarrow{1} t_1 \xrightarrow{2} t_2 \xrightarrow{1} t_3 \xrightarrow{2} t_4 \xrightarrow{1} t_5$$

in the 2-regular tree, then, in addition to the g -vectors $g_1(t_0) = e_1$ and $g_2(t_0) = e_2$ and the F -polynomials $F_1(t_0) = F_2(t_0) = 1$ associated with the initial variables, we successively find the following cluster variables in $\mathcal{A}(B_{pr})$ and the corresponding F -polynomials and g -vectors:

$$\begin{aligned} x_1(t_1) &= \frac{x_2 + x_3}{x_1}, & F_1(t_1) &= 1 + x_3, & g_1(t_1) &= e_2 - e_1, \\ x_2(t_2) &= \frac{x_2 + x_3 + x_1 x_3 x_4}{x_1 x_2}, & F_2(t_2) &= 1 + x_3 + x_3 x_4, & g_2(t_2) &= -e_1, \\ x_1(t_3) &= \frac{1 + x_1 x_4}{x_2}, & F_1(t_3) &= 1 + x_4, & g_1(t_3) &= -e_2 \\ x_2(t_4) &= x_1, & F_2(t_4) &= 1, & g_2(t_4) &= e_1 \\ x_1(t_5) &= x_2, & F_1(t_5) &= 1, & g_1(t_5) &= e_2. \end{aligned}$$

The associated G -matrices are

$$\begin{aligned} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \\ & \tag{17} \end{aligned}$$

If we let α_1 and α_2 be the simple roots of the root system A_2 , then clearly the linear map which takes e_1 to α_1 and e_2 to $\alpha_1 + \alpha_2$ yields a bijection from the set of the g -vectors to the *set of almost positive roots*, i.e. the union of the set of positive roots with the set of opposites of the simple roots, cf. Figure 2.

This statement generalizes to an acyclic (equally valued) quiver Q as follows: For two vertices i, j of Q , let p_{ij} be the number of paths from i to j (i.e. formal compositions of ≥ 0 arrows). Let $\alpha_1, \dots, \alpha_n$ be the simple roots of the root system corresponding to the underlying graph of Q . The following theorem is a consequence of the results of [12].

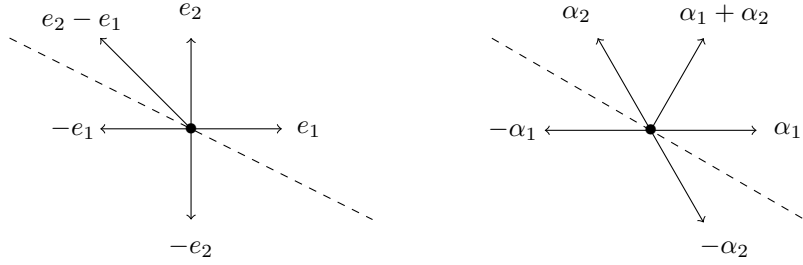


Figure 2. g -vectors and almost positive roots for A_2

Theorem 5.2. *The linear map taking e_j to $\sum_{i=1}^n p_{ij}\alpha_i$, $1 \leq j \leq n$, is a bijection from the set of g -vectors of Q other than $-e_i$, $1 \leq i \leq n$, to the set of real (positive) Schur roots.*

5.5. Tropical duality. Let Q be a valued quiver, B the associated skew-symmetrizable $n \times n$ -matrix and D a diagonal integer $n \times n$ -matrix with strictly positive diagonal entries such that the transpose $(DB)^T$ of DB equals $-DB$. The *opposite valued quiver* Q^{op} corresponds to the matrix $-B$. For example, the opposite valued quiver of

$$\vec{B}_3 : 1 \longrightarrow 2 \xrightarrow{(1,2)} 3 \quad \text{is} \quad \vec{B}_3^{op} : 1 \longleftarrow 2 \xleftarrow{(2,1)} 3 ,$$

which is in fact mutation equivalent to \vec{B}_3 (we mutate at 1 and 3).

Theorem 5.3 ([115]). *Suppose that the main conjecture 5.1 holds for Q . Then for each vertex t of the n -regular tree, we have*

$$G(t)^T D C(t) = D , \tag{18}$$

$$C(t)^{-1} = C(Q(t)^{op}, t, t_0) \quad \text{and} \quad G(t)^{-1} = G(Q(t)^{op}, t, t_0). \tag{19}$$

To check the equality (18) in the example of the quiver $1 \rightarrow 2$, the reader may inspect the C - and G -matrices given in (16) and (17). The equalities (19) are given in Theorem 1.2 of [115]. The equality (18) is equation (3.11) from [115], cf. also Prop. 3.2 of [113]. For skew-symmetric matrices B , it was first proved using Plamondon’s results [117] in Prop. 4.1 of [112] by T. Nakanishi, who had discovered the statement by combining in Cor. 6.10 and 6.11 of [81].

Let $v : Q_1 \rightarrow \mathbb{N}^2$ denote the valuation of the valued quiver Q , cf. section 3.3. Following [37], we define the *Langlands dual* Q^\vee as the valued quiver whose underlying oriented graph equals that of Q and whose valuation v^\vee is defined by reversing the valuation of Q : For each arrow α , we put

$$v^\vee(\alpha) = (v(\alpha)_2, v(\alpha)_1). \tag{20}$$

The corresponding skew-symmetrizable matrix B^\vee equals $-B^T$. For example, if Q is the valued quiver

$$\vec{B}_3 : 1 \longrightarrow 2 \xrightarrow{(1,2)} 3 ,$$

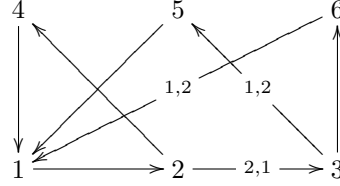
then Q^\vee is

$$\vec{C}_3 : 1 \longrightarrow 2 \xrightarrow{(2,1)} 3 .$$

Theorem 5.4 (Th. 1.2 of [115]). *Suppose that the main conjecture 5.1 holds for Q . Then for each vertex t of the n -regular tree, we have*

$$G(Q, t_0, t)^T = C(Q^\vee, t_0, t)^{-1}.$$

For example, if we successively mutate the principal extension of the above valued quiver \vec{C}_3 at the vertices 1, 2, 3, 1, 2, 3, we find the valued quiver



and hence the C -matrix

$$C(\vec{C}_3, t_0, t) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}.$$

On the other hand, if successively mutate the initial seed of the principal extension of \vec{B}_3 at 1, 2, 3, 1, 2, 3, we find the cluster

$$\begin{aligned} x_1(t) &= \frac{1}{x_2}(x_3^2 + x_1x^5), \\ x_2(t) &= \frac{1}{x_1x_2^2x_3^2}(x_1^2x_2^2x_4x_5^2 + 2x_1^2x_2x_4x_5^2x_6 + \cdots + x_3^4x_4), \\ x_3(t) &= \frac{1}{x_2x_3}(x_3^2 + x_1x_5 + x_1x_2x_5x_6) \end{aligned}$$

and thus the G -matrix

$$G(\vec{B}_3, t_0, t) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix}.$$

This is indeed the inverse transpose of $C(\vec{B}_3, t_0, t)$. This was to be expected by theorem 5.4 since the main conjecture holds for \vec{C}_3 by Demonet's work [24] [23].

5.6. Product formulas for c -matrices and g -matrices. We will give a key ingredient for the proof of theorem 5.4 which is also useful in the investigation of quantum cluster algebras (section 6.2). Let Q be a valued quiver, B the associated skew-symmetrizable $n \times n$ -matrix and D a diagonal integer $n \times n$ -matrix with strictly positive diagonal entries such that the transpose $(DB)^T$ of DB equals $-DB$. $1 \leq k \leq n$ be an integer. Choose a sign ε equal to 1 or -1 . Let $F_\varepsilon = F_{k,\varepsilon}(Q)$ be the $n \times n$ -matrix which differs from the identity matrix only in its k th row, whose coefficients are given by

$$(F_\varepsilon)_{kj} = \begin{cases} -1 & \text{if } j = k; \\ [\varepsilon b_{kj}]_+ & \text{if } j \neq k. \end{cases}$$

Let $E_\varepsilon = E_{k,\varepsilon}(Q)$ be the $n \times n$ -matrix which differs from the identity matrix only in its k th column, whose coefficients are given by

$$(E_\varepsilon)_{ik} = \begin{cases} -1 & \text{if } i = k; \\ [-\varepsilon b_{ik}]_+ & \text{if } i \neq k. \end{cases}$$

Notice that both E_ε and F_ε square to the identity matrix. Parts a) to d) of the following lemma become natural in the categorical picture to be developed in section 7, cf. Corollary 7.5. Part e) seems harder to interpret.

Lemma 5.5. a) We have $E_\varepsilon \mu_k(B) = BF_\varepsilon$ and $E_\varepsilon^T DF_\varepsilon = D$.

b) We have $E_{k,-\varepsilon}(\mu_k(Q)) = E_{k,\varepsilon}(Q)^{-1}$ and $F_{k,-\varepsilon}(\mu_k(Q)) = F_{k,\varepsilon}(Q)^{-1}$.

c) For $1 \leq k \leq n$, let $T_k = E_{k,\varepsilon}(\mu_k(Q))E_{k,\varepsilon}(Q)$. Then for two vertices i, j , the matrices T_i and T_j satisfy the braid relation associated with the full valued subquiver whose vertices are i and j , i.e. we have

$$\underbrace{T_i T_j T_i \dots}_{m \text{ factors}} = \underbrace{T_j T_i T_j \dots}_{m \text{ factors}}, \quad (21)$$

where the number of factors m equals 2, 3, 4 or 6 depending on whether $|b_{ij}b_{ji}|$ equals 0, 1, 2 or 3.

d) We have $E_{k,\varepsilon}(Q^{op}) = E_{k,-\varepsilon}(Q)$, $F_{k,\varepsilon}(Q^{op}) = F_{k,-\varepsilon}(Q)$.

e) We have $E_{k,\varepsilon}(Q^\vee)^T = F_{k,\varepsilon}(Q)$.

Now let

$$t_0 \xrightarrow{i_1} t_1 \xrightarrow{i_2} t_2 \xrightarrow{i_3} \dots \xrightarrow{i_N} t_N .$$

be a path in the n -regular tree, let ε_s be the sign of the c -vector $C(t_{s-1})e_s$ and let $E_{i_s,\varepsilon_s}(t_s)$ resp. $F_{i_s,\varepsilon_s}(t_s)$ be the matrix E_{ε_s} resp. F_{ε_s} associated with the quiver $Q(t_{s-1})$ and the vertex i_s , $1 \leq s \leq N$.

Theorem 5.6 ([115]). *If the main conjecture 5.1 holds for Q , we have*

$$G(t_N) = E_{i_1,\varepsilon_1}(t_1) \dots E_{i_N,\varepsilon_N}(t_N) \text{ and } C(t_N) = F_{i_1,\varepsilon_1}(t_1) \dots F_{i_N,\varepsilon_N}(t_N).$$

5.7. Cluster algebras with coefficients in a semifield. A *semifield* is an abelian group \mathbb{P} endowed with an additional binary operation $\oplus : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{P}$ which is commutative, associative and distributive with respect to the group law of \mathbb{P} . For example, the *tropical semifield* $\text{Trop}(u_1, \dots, u_n)$ is the free (multiplicative) abelian group generated by the indeterminates u_i endowed with the operation \oplus defined by

$$\left(\prod u_i^{l_i}\right) \oplus \left(\prod u_i^{m_i}\right) = \prod u_i^{\min(l_i, m_i)}.$$

Clearly, it is isomorphic to $\mathbb{Z}_{\text{trop}}^n$, where \mathbb{Z}_{trop} is the abelian group \mathbb{Z} endowed with the operation \oplus defined by $x \oplus y = \min(x, y)$. It is shown in Lemma 2.1.6 of [8] that the *universal semifield* $\mathbb{Q}_{sf}(x_1, \dots, x_n)$ on given indeterminates x_1, \dots, x_n is the closure, in $\mathbb{Q}(x_1, \dots, x_n)$, of the set $\{x_1, \dots, x_n\}$ under multiplication, division and addition. Notice that this closure contains polynomials whose coefficients are not all positive; for example, the polynomial

$$x^2 - x + 1 = \frac{x^3 + 1}{x + 1}$$

belongs to $\mathbb{Q}_{sf}(x)$. The abelian group underlying a semifield \mathbb{P} is torsion-free. Indeed, if an element x satisfies $x^m = 1$, then

$$x = \frac{x^m \oplus x^{m-1} \oplus \dots \oplus x}{x^{m-1} \oplus x^{m-2} \oplus \dots \oplus 1} = \frac{1 \oplus x^{m-1} \oplus \dots \oplus x}{x^{m-1} \oplus x^{m-2} \oplus \dots \oplus 1} = 1.$$

Thus, the group ring $\mathbb{Z}\mathbb{P}$ is integral.

Let us fix a semifield \mathbb{P} and an integer $n \geq 1$. A *Y-seed* of rank n with values in \mathbb{P} is a pair (Q, Y) formed by a valued quiver Q with n vertices and by a sequence $Y = (y_1, \dots, y_n)$ of elements of \mathbb{P} . Let B be the skew-symmetrizable matrix corresponding to Q . If (Q, Y) is a *Y-seed* and k a vertex of Q , the *mutated Y-seed* $\mu_k(Q, Y)$ is the *Y-seed* (Q', Y') where $Q' = \mu_k(Q)$ and, for $1 \leq j \leq n$, we have

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k; \\ y_j y_k^{[b_{kj}]_+} (1 \oplus y_k)^{-b_{kj}} & \text{if } j \neq k. \end{cases} \quad (22)$$

One checks that $\mu_k^2(Q, Y) = (Q, Y)$. For example, the following *Y-seeds* are related by a mutation at the vertex 1

$$\begin{array}{ccc} y_1 & \longrightarrow & y_2 \\ \uparrow & \searrow & \uparrow \\ y_3 & \longrightarrow & y_4 \end{array} \quad , \quad \begin{array}{ccc} 1/y_1 & \longleftarrow & y_2/(1 \oplus y_1^{-1}) \\ \downarrow & & \uparrow \\ y_3(1 \oplus y_1) & \longrightarrow & y_4 \end{array}$$

where we write the element y_i in place of the vertex i .

Let $\mathbb{Q}\mathbb{P}$ be the fraction field of the group ring $\mathbb{Z}\mathbb{P}$ and \mathcal{F} any field obtained from $\mathbb{Q}\mathbb{P}$ by adjoining n indeterminates. A *seed* with coefficients in \mathbb{P} is a triple (Q, Y, X) , where (Q, Y) is a *Y-seed* of rank n with values in \mathbb{P} and X is a sequence (x_1, \dots, x_n) of elements of \mathcal{F} which freely generate the field \mathcal{F} . If (Q, Y, X) is

a seed and k a vertex of Q , the mutation $\mu_k(Q, Y, X)$ is the seed formed by the mutation $\mu_k(Q, Y)$ and the sequence X' with $x'_j = x_j$ for $j \neq k$ and x'_k defined by the exchange relation

$$x'_k x_k (1 \oplus y_k) = y_k \prod_i x_i^{[b_{ik}]_+} + \prod_i x_i^{[-b_{ik}]_+}. \quad (23)$$

A *seed pattern* is the datum, for each vertex t of the n -regular tree, of a seed $(Q(t), Y(t), X(t))$ such that if t and t' are linked by an edge labeled k , then the seeds corresponding to t and t' are linked by the mutation at k . The *cluster algebra* is the $\mathbb{Z}\mathbb{P}$ -subalgebra of the field \mathcal{F} generated by the cluster variables.

We recover the cluster algebra of geometric type associated with an $m \times n$ -matrix \tilde{B} as follows: We let B be the principal part of \tilde{B} ; we define the semifield \mathbb{P} to be the tropical semifield $\text{Trop}(x_{n+1}, \dots, x_m)$ and the initial Y -variables to be

$$y_j = \prod_{i=n+1}^m x_i^{b_{ij}}, \quad 1 \leq j \leq n.$$

As a simple example of a cluster algebra of ‘non geometric’ type, consider the case where $n = 1$, $\mathbb{P} = \mathbb{Q}_{sf}(y_1, y_2)$ and $Q : 1 \rightarrow 2$. Then the sequence of mutations

$$t_0 \xrightarrow{1} t_1 \xrightarrow{2} t_2 \xrightarrow{1} t_3 \xrightarrow{2} t_4 \xrightarrow{1} t_5$$

starting from the initial seed $(1 \rightarrow 2, \{x_1, x_2\}, \{y_1, y_2\})$ yields

$$\begin{aligned} y_1(t_1) &= \frac{1}{y_1}, & y_2(t_1) &= \frac{y_1 y_2}{1 + y_1}, & x_1(t_1) &= \frac{y_1 + x_2}{x_1(1 + y_1)} \\ y_1(t_2) &= \frac{y_2}{1 + y_1 + y_1 y_2}, & y_2(t_2) &= \frac{1 + y_1}{y_1 y_2}, & x_2(t_2) &= \frac{x_1 y_1 y_2 + x_2 + y_1}{x_1 x_2 (1 + y_1 + y_2 y_1)} \\ y_1(t_3) &= \frac{1 + y_1 + y_1 y_2}{y_2}, & y_2(t_3) &= \frac{1}{y_1(1 + y_2)}, & x_1(t_3) &= \frac{x_1 y_2 + 1}{x_2(1 + y_2)} \\ y_1(t_4) &= \frac{1}{y_2}, & y_2(t_4) &= y_1(1 + y_2), & x_2(t_4) &= x_1 \\ y_1(t_5) &= y_2, & y_2(t_5) &= y_1, & x_1(t_5) &= x_2. \end{aligned}$$

5.8. The separation formulas. Let a seed pattern be given and let us write (Q, Y, X) for the initial seed $(Q(t_0), Y(t_0), X(t_0))$ associated with the chosen root t_0 of the n -regular tree. Let us write $c_{ij}(t)$ for the coefficients of the c -matrix $C(t)$ and $g_{ij}(t)$ for those of the g -matrix $G(t)$ associated with a vertex t of the n -regular tree. Recall that

$$F_j(t) \in \mathbb{Z}[x_{n+1}, \dots, x_{2n}], \quad 1 \leq j \leq n,$$

are the F -polynomials at the vertex t . By construction, they belong to the universal semifield $\mathbb{Q}_{sf}(x_{n+1}, \dots, x_{2n})$ and thus it makes sense to consider their evaluations

$$F_j(t)(y_1, \dots, y_n)$$

at the elements y_1, \dots, y_n of \mathbb{P} and more generally, at an n -tuple of elements of any semifield.

Theorem 5.7 (Prop. 3.13 and Cor. 6.3 of [48]). *For each vertex t of the n -regular tree and each $1 \leq j \leq n$, we have*

$$y_j(t) = y_1^{c_{1j}(t)} \dots y_n^{c_{nj}(t)} \prod_i F_i(t)(y_1, \dots, y_n)^{b_{ij}(t)} \quad (24)$$

$$x_j(t) = x_1^{g_{1j}(t)} \dots x_n^{g_{nj}(t)} \frac{F_j(t)(\widehat{y}_1, \dots, \widehat{y}_n)}{F_j(t)(y_1, \dots, y_n)}, \quad (25)$$

where $\widehat{y}_l = y_j \prod_i x_i^{b_{il}}$, $1 \leq l \leq n$.

6. Quantum cluster algebras and quantum dilogarithms

6.1. The quantum dilogarithm. Let $q^{1/2}$ be an indeterminate. We will denote its square by q . The (exponential of) the *quantum dilogarithm series* is

$$\mathbb{E}(y) = \mathbb{E}_q(y) = 1 + \frac{q^{1/2}}{q-1}y + \dots + \frac{q^{n^2/2}}{(q^n-1)(q^{n-1}-1)\dots(q-1)} + \dots$$

It is a series in the indeterminate y with coefficients in the field $\mathbb{Q}(q^{1/2})$. It is related to the classical dilogarithm

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = - \int_0^x \frac{\log(1-y)}{y} dy, \quad |x| < 1,$$

by the asymptotic expansion

$$\mathbb{E}_q(y) \sim \exp\left(-\frac{\mathrm{Li}_2(-y)}{\log(q)}\right)$$

when q goes to 1^- . An easy computation shows that we have the functional equation

$$(1 + q^{1/2}y) \mathbb{E}(y) = \mathbb{E}(qy). \quad (26)$$

The quantum dilogarithm is related to the classical q -exponential function by the substitution $y \mapsto \frac{q^{1/2}}{q-1}y$. Therefore, as discovered by Schützenberger [131], if y_1 and y_2 are two indeterminates which q -commute, i.e. $y_1y_2 = qy_2y_1$, then we have

$$\mathbb{E}(y_1 + y_2) = \mathbb{E}(y_2)\mathbb{E}(y_1). \quad (27)$$

In 1993, Faddeev, Kashaev and Volkov [32] [31] discovered that (26) and (27) together imply the *pentagon identity*

$$y_1y_2 = qy_2y_1 \implies \mathbb{E}(y_1)\mathbb{E}(y_2) = \mathbb{E}(y_2)\mathbb{E}(q^{-1/2}y_1y_2)\mathbb{E}(y_1), \quad (28)$$

cf. [139] for a recent account. Their main result states that this identity implies the classical five-term identity

$$L(x) + L(y) - L(xy) = L\left(\frac{x-xy}{1-xy}\right) + L\left(\frac{y-xy}{1-xy}\right)$$

for the Rogers dilogarithm

$$L(x) = \operatorname{Li}_2(x) + \log(1-x)\log(x)/2.$$

We refer to [112] [77] for more information on the many recent developments around this subject and to [143] for more information on the dilogarithm function.

6.2. Quantum mutations and quantum cluster algebras. We will construct quantum cluster algebras following Berenstein–Zelevinsky [10]. Quantum cluster algebras are certain non commutative deformations of cluster algebras of geometric type. Let $1 \leq n \leq m$ be integers, \tilde{B} an integer $m \times n$ -matrix with skew-symmetrizable principal part B and Λ a skew-symmetric integer $m \times m$ -matrix. Let \tilde{Q} and Q be the associated valued ice quivers. Recall from section 4 that the datum of \tilde{B} gives rise to a cluster algebra of geometric type. Let us assume that (Λ, \tilde{B}) is a *compatible pair*, i.e. we have

$$\tilde{B}^T \Lambda = [D \ 0], \quad (29)$$

where D is a diagonal $n \times n$ -matrix whose diagonal coefficients are strictly positive integers. This will ensure that Λ gives rise to a (non commutative) deformation of the cluster algebra associated with \tilde{B} . We first need to define the mutation of compatible pairs: Let $1 \leq k \leq n$ be an integer and choose a sign ε equal to 1 or -1 . In the notations of section 5.6, let F_ε be the $n \times n$ -matrix $F_{k,\varepsilon}(Q)$ and E_ε the $m \times m$ -matrix $E_{k,\varepsilon}(\tilde{Q})$. The *mutation* $\mu_k(\tilde{B}, \Lambda)$ is defined to be the compatible pair (\tilde{B}', Λ') with

$$\tilde{B}' = E_\varepsilon \tilde{B} F_\varepsilon \quad \text{and} \quad \Lambda' = E_\varepsilon^T \Lambda E_\varepsilon.$$

One checks that \tilde{B}' equals $\mu_k(\tilde{B})$ and that (Λ', \tilde{B}') does not depend on the choice of ε and is again a compatible pair (with the same matrix D). One checks that mutation of compatible pairs is an involution. Thus, given a compatible pair (\tilde{B}, Λ) , we can assign a compatible pair $(\tilde{B}(t), \Lambda(t))$ to each vertex t of the n -regular tree such that the given pair is assigned to t_0 and, whenever t and t' are linked by an edge labeled k , the corresponding pairs are related by the mutation at k .

The *quantum affine space* \mathbb{A}_Λ associated with Λ is by definition the $\mathbb{Z}[q^{\pm 1/2}]$ -algebra generated by all symbols x^α , $\alpha \in \mathbb{N}^m$, subject to the relations

$$x^\alpha x^\beta = q^{\frac{1}{2} \alpha^T \Lambda \beta} x^{\alpha+\beta}.$$

The *quantum torus* \mathbb{T}_Λ is defined similarly on generators x^α , $\alpha \in \mathbb{Z}^m$. One checks that the underlying $\mathbb{Z}[q^{\pm 1/2}]$ -module of \mathbb{A}_Λ resp. \mathbb{T}_Λ is free on the basis formed by the x^α , $\alpha \in \mathbb{N}^m$ resp. $\alpha \in \mathbb{Z}^m$. The *completed quantum affine space* $\hat{\mathbb{A}}_\Lambda$ is the completion of \mathbb{A}_Λ with respect to the kernel of the projection $\mathbb{A}_\Lambda \rightarrow \mathbb{Z}[q^{\pm 1/2}]$. The algebras \mathbb{A}_Λ and \mathbb{T}_Λ are Ore domains (cf. the Appendix to [10]) and so have a field of fractions \mathbb{F}_Λ whose elements are given by right fractions (or left fractions).

The *initial quantum seed* is (\tilde{B}, Λ, X) , where X is the sequence of the $x_i = x^{e_i}$. Its *mutation at k* , where $1 \leq k \leq n$, is $(\tilde{B}', \Lambda', X')$, where the sequence X' is

formed by the x_i , $i \neq k$, and by the element x'_k defined by the *quantum exchange relation*

$$x'_k = x^{E_+ e_k} + x^{E_- e_k}. \quad (30)$$

By part (3) of Prop. 4.7 of [10], there is a unique morphism of $\mathbb{Z}[q^{\pm 1/2}]$ -algebras

$$\mu_k^\# : \mathbb{A}_{\Lambda'} \rightarrow \mathbb{T}_\Lambda$$

taking x_i to x'_i , $1 \leq i \leq m$; moreover it is injective and induces an isomorphism

$$\mu_k^\# : \mathbb{F}_{\Lambda'} \xrightarrow{\sim} \mathbb{F}_\Lambda.$$

One checks that mutation of quantum seeds is an involution. Thus, with each vertex t of the n -regular tree, one can associate a quantum seed $(\tilde{B}(t), \Lambda(t), X(t))$ such that the initial quantum seed is associated with t_0 and seeds with vertices t and t' linked by an edge labeled k are related by a quantum mutation. The *quantum cluster variables* are the $x_j(t)$, $1 \leq j \leq n$, associated with the vertices t of the n -regular tree. The *quantum cluster algebra* is the $\mathbb{Z}[q^{\pm 1/2}]$ -subalgebra of \mathbb{F}_Λ generated by the quantum cluster variables. We have the quantum Laurent phenomenon:

Theorem 6.1 (Cor. 5.2 of [10]). *The quantum cluster variables are contained in the quantum torus \mathbb{T}_Λ .*

We refer to [10] [66] [53] for examples of quantum cluster algebras. The *exchange graph* of quantum seeds associated with (\tilde{B}, Λ) is defined in analogy with the exchange graph of (classical) seeds associated with \tilde{B} , cf. section 3.2. The specialization map

$$\mathbb{Z}[q^{\pm 1/2}] \rightarrow \mathbb{Z}$$

taking $q^{1/2}$ to 1 yields a morphism of $\mathbb{Z}[q^{\pm 1/2}]$ -modules

$$\mathbb{T}_\Lambda \rightarrow \mathbb{Z}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$$

which takes quantum cluster variables to classical ones and induces a map from the quantum exchange graph to the classical exchange graph.

Theorem 6.2 (Th. 6.1 of [10]). *The specialization at $q^{1/2} = 1$ yields an isomorphism from the quantum to the classical exchange graph.*

6.3. Fock-Goncharov's separation formula. Recall that the numbers d_i are the coefficients of the diagonal matrix D appearing in the compatibility condition (29). We consider the mutation at k of a given initial quantum seed (\tilde{B}, Λ, X) .

Lemma 6.3 ([37]). *We have the separation formulas*

$$\mu_k^\# = \text{Ad}'(\mathbb{E}_{q^{d_k}}(y_k)) \circ \varphi_{k,+} = \text{Ad}'(\mathbb{E}_{q^{d_k}}(y_k^{-1}))^{-1} \circ \varphi_{k,-}, \quad (31)$$

where the right adjoint action $\text{Ad}'(u)$ takes an element v to $u^{-1}vu$, we put

$$y_k = x^{\tilde{B}e_k}$$

and $\varphi_{k,\varepsilon} : \mathbb{T}_{\Lambda'} \rightarrow \mathbb{T}_\Lambda$ is the unique morphism of $\mathbb{Z}[q^{\pm 1/2}]$ -algebras taking x^α to $x^{E_\varepsilon \alpha}$.

Thus, we have separated the mutation isomorphism into a ‘tropical’ part and a ‘transcendental’ part. Notice that in order to give meaning to the formulas (31), we need to embed the quantum tori into suitable localizations of completions of quantum affine space. Using formula (26) one then checks the claim. Of course, one would like to iterate this formula. The iteration should be meaningful in (at least) two ways:

- (1) the product of the appearing power series should have a meaning, i.e. all the series should live in a common completion of quantum affine space;
- (2) the composition of the ‘tropical parts’ should have a meaning from the point of view of ‘tropical’ cluster theory, as we have seen it in sections 5.3 and 5.4.

In order to obtain both, it is essential to choose the sign \pm in each factor carefully. This can be achieved using the main conjecture 5.1.

6.4. The quantum separation formula. To simplify the notations, let us assume from now on that (\tilde{B}, Λ) is *unitally compatible*, i.e. equation (29) holds with D the $n \times n$ -identity matrix. Let $\mathbf{i} = (i_1, \dots, i_N)$ be a sequence of vertices in $\{1, \dots, n\}$. Consider the corresponding path in the n -regular tree

$$t_0 \xrightarrow{i_1} t_1 \xrightarrow{i_2} t_2 \xrightarrow{i_3} \dots \xrightarrow{i_N} t_N .$$

It yields a chain of mutation isomorphisms between the associated quantum tori:

$$\mathbb{T}_\Lambda \xleftarrow{\mu_{i_1}^\#} \mathbb{T}_{\Lambda(t_1)} \xleftarrow{\mu_{i_2}^\#} \mathbb{T}_{\Lambda(t_2)} \xleftarrow{\mu_{i_3}^\#} \dots \xleftarrow{\mu_{i_N}^\#} \mathbb{T}_{\Lambda(t_N)} .$$

Let us write $\Phi(\mathbf{i})$ for the composition of these isomorphisms. We would like to write down a separation formula for $\Phi(\mathbf{i})$ which generalizes (31). We need some more notation: For $1 \leq s \leq N$, let β_s be the c -vector $C(t_{s-1})e_{i_s}$ and let ε_s be the common sign of the components of β_s (cf. section 5.4). For a vector α in \mathbb{Z}^n , let us write $\mathbb{E}(\alpha)$ for $\mathbb{E}(y^\alpha)$, where $y^\alpha = x^{\tilde{B}\alpha}$.

Theorem 6.4 ([106]). *Put*

$$\mathbb{E}(\mathbf{i}) = \mathbb{E}(\varepsilon_N \beta_N)^{\varepsilon_N} \dots \mathbb{E}(\varepsilon_1 \beta_1)^{\varepsilon_1} \quad (32)$$

$$\varphi(\mathbf{i}) = \varphi_{i_1, \varepsilon_1} \circ \dots \circ \varphi_{i_N, \varepsilon_N} . \quad (33)$$

Then we have

$$\Phi(\mathbf{i}) = \text{Ad}'(\mathbb{E}(\mathbf{i})) \circ \varphi , \quad (34)$$

the isomorphism φ sends x^α to $x^{G(t_N)\alpha}$, where $G(t_N)$ is the g -matrix at t_N (section 5.4), and $\text{Ad}'(\mathbb{E}(\mathbf{i}))$ acts on x^{e_i} by multiplication with the quantum F -polynomial of [138].

Notice that by construction all the vectors $\varepsilon_s \beta_s$ have non negative components so that all the series $\mathbb{E}(\varepsilon_s \beta_s)^{\varepsilon_s}$ belong to the same completion of quantum affine

space. If we replace the right adjoint action of $\mathbb{E}(\mathbf{i})$ by the multiplication with the quantum F -polynomials, we obtain Tran's formula (Theorem 6.1 of [138]), which is the quantum analogue of Fomin-Zelevinsky's [48] separation formula (25). The theorem is due, in a different language, to Nagao [106] (cf. also Theorem 5.1 in [30]). Alternatively, using Theorem 5.6 and Tran's formula, it is not hard to prove the analogous theorem for arbitrary valued quivers Q for which the main conjecture 5.1 holds.

Let us keep the notations from Theorem 6.4. It is not hard to check that there is a unique $\mathbb{Q}(q^{1/2})$ -algebra embedding

$$\mathbb{A}_B \rightarrow \mathbb{T}_\Lambda$$

taking an element x^α to $x^{\tilde{B}\alpha}$ (if D is not the identity matrix, it is an embedding $\mathbb{A}_{DB} \rightarrow \mathbb{T}_\Lambda$). Thus, by construction, the product $\mathbb{E}(\mathbf{i})$ lies in a completed quantum affine subspace isomorphic to $\widehat{\mathbb{A}}_B$ and independent of the choice of the non principal part in \tilde{B} . For example, we can always choose $\tilde{B} = B_{pr}$, cf. section 5.3 and

$$\Lambda = \begin{bmatrix} 0 & -I \\ I & B^T \end{bmatrix}.$$

Theorem 6.5 ([80] [108]). a) *If $C(t_N)$ is a permutation matrix, then $\mathbb{E}(\mathbf{i}) = 1$.*

b) *If the opposite matrix $-C(t_N)$ is a permutation matrix, then $\mathbb{E}(\mathbf{i}) \in \widehat{\mathbb{A}}_B$ is Kontsevich-Soibelman's non commutative Donaldson-Thomas invariant [89] associated with the quiver corresponding to B (when this invariant is defined, cf. section 7.11).*

Remark 6.6. One can sharpen part a) as follows: Let \mathbf{i} and \mathbf{i}' be two sequences of vertices in $\{1, \dots, n\}$ and let t and t' be the end points of the corresponding paths in the n -regular tree. Suppose that we have $PC(t) = C(t')$ for a permutation matrix P . We will show in section 7.11 that we then have $\mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}')$. Thus, if Q admits some sequence \mathbf{i} such that $-C(t)$ is a permutation matrix, then the series $\mathbb{E}(\mathbf{i}) \in \widehat{\mathbb{A}}_B$ is independent of the choice of the sequence \mathbf{i} with this property. We then call this series the *combinatorial DT-invariant* associated with Q .

We will give a proof of the theorem and the remark in section 7.11, cf. also Theorem 3.5 in [77]. Let us illustrate the theorem on the example of the mutation sequence $\mathbf{i} = (1, 2, 1, 2, 1)$ of the quiver $\vec{A}_2 : 1 \rightarrow 2$. We have computed the sequence of c -matrices $C(t_s)$, $1 \leq s \leq 5$, in equation (16). We obtain

$$\beta_1 = e_1, \beta_2 = e_1 + e_2, \beta_3 = e_2, \beta_4 = -e_1, \beta_5 = -e_2.$$

Since $C(t_5)$ is the matrix of the transposition $e_1 \leftrightarrow e_2$, part a) of the theorem yields the identity

$$\mathbb{E}(e_2)^{-1} \mathbb{E}(e_1)^{-1} \mathbb{E}(e_2) \mathbb{E}(e_1 + e_2) \mathbb{E}(e_1) = 1,$$

which is of course equivalent to the pentagon identity (28). Since $C(t_3)$ is the opposite of the transposition matrix, we find that Kontsevich-Soibelman's DT invariant equals

$$\mathbb{E}(e_2)\mathbb{E}(e_1 + e_2)\mathbb{E}(e_1)$$

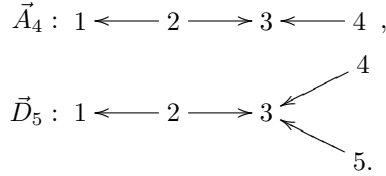
for the quiver \vec{A}_2 , as is well-known, cf. Example 2) in section 6.4 of [89]. This example can be generalized to any Dynkin quiver, which yields a family of quantum dilogarithm identities due to Reineke [123], cf. also Cor. 1.7 in [80] and [120] [121]. Namely, let Δ be a simply laced Dynkin diagram and let Q be an alternating quiver (i.e. each vertex is a source or a sink) whose underlying graph is Δ . Let i_+ be the sequence of sources of Q and i_- its sequence of sinks (in any order). Let

$$\mathbf{i} = \underbrace{i_+ i_- i_+ \dots}_{h \text{ factors}},$$

where h is the Coxeter number of Δ and let $\mathbf{i}' = i_- i_+$. Let $\mu_{\mathbf{i}}(t_0)$ be the final vertex in the path in the regular tree which starts at t_0 and runs through the sequence of mutations \mathbf{i} starting at the leftmost vertex in the sequence. Then one can show that both $-C(\mu_{\mathbf{i}}(t_0))$ and $-C(\mu_{\mathbf{i}'}(t_0))$ are permutation matrices and so the Kontsevich-Soibelman invariant associated with Q equals

$$\mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}'),$$

which is Reineke's identity associated with Q . One can further generalize this class as follows: Let Δ and Δ' be two simply laced Dynkin diagrams and $\vec{\Delta}$ and $\vec{\Delta}'$ alternating quivers with underlying graphs Δ and Δ' . Let Q be the square product $\vec{\Delta} \square \vec{\Delta}'$ as defined in section 3.3 of [81]. For example, the square product of the quivers



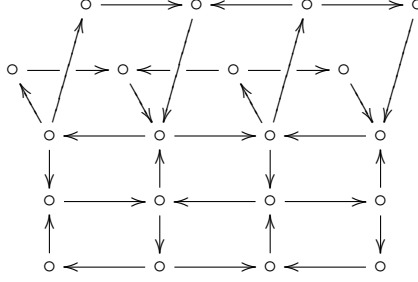
is depicted in Figure 3. Let i_+ be the sequence of all source-sinks of $\vec{\Delta} \square \vec{\Delta}'$ (i.e. vertices (u, v) such that u is a source in the full subquiver $p_2^{-1}(v)$ and v a sink the full subquiver $p_1^{-1}(u)$, where the p_i are the projections) and let i_- be the sequence of all sink-sources. Let

$$\mathbf{i} = \underbrace{i_+ i_- i_+ \dots}_{h \text{ factors}} \quad \text{and} \quad \mathbf{i}' = \underbrace{i_- i_+ i_- \dots}_{h' \text{ factors}},$$

where h is the Coxeter number of Δ and h' that of Δ' . Again one can show that both $-C(\mu_{\mathbf{i}}(t_0))$ and $-C(\mu_{\mathbf{i}'}(t_0))$ are permutation matrices and so the Kontsevich-Soibelman invariant associated with $\vec{\Delta} \square \vec{\Delta}'$ equals

$$\mathbb{E}(\mathbf{i}) = \mathbb{E}(\mathbf{i}').$$

In physics, a related method for computing this invariant is the *mutation method* developed and applied in [2].

Figure 3. The quiver $\vec{A}_4 \square \vec{D}_5$

7. Categorification

The setup we will describe uses triangulated 3-Calabi-Yau categories (derived categories of Ginzburg dg algebras). It is due to Kontsevich-Soibelman [88] and Nagao [106]. It is closely related to that of Plamondon [118], who uses triangulated 2-Calabi-Yau categories (cluster categories). Both build on work by Derksen-Weyman-Zelevinsky on quivers with potentials [25], who first proved a statement equivalent to the main theorem 7.9 using decorated representations of quivers with potentials [26].

7.1. Mutation of quivers with potential. We follow Derksen-Weyman-Zelevinsky's fundamental article [25]. Let Q be a finite quiver. Let $\widehat{\mathbb{C}Q}$ be the *completed path algebra*, i.e. the completion of the path algebra at the ideal generated by the arrows of Q . Thus, $\widehat{\mathbb{C}Q}$ is a topological algebra and the paths of Q form a topological basis so that the underlying vector space of $\widehat{\mathbb{C}Q}$ is

$$\prod_{p \text{ path}} kp$$

and the multiplication is induced from the composition of paths (we compose paths in the same way as we compose morphisms). The *continuous zeroth Hochschild homology* of $\widehat{\mathbb{C}Q}$ is the vector space $HH_0(\widehat{\mathbb{C}Q})$ obtained as the quotient of $\widehat{\mathbb{C}Q}$ by the closure of the subspace generated by all commutators. It admits a topological basis formed by the *cycles* of Q , i.e. the orbits of paths $p = (i|\alpha_m| \dots |\alpha_1|i)$ of any length $m \geq 0$ with identical source and target under the action of the cyclic group of order m . In particular, the space $HH_0(\widehat{\mathbb{C}Q})$ is a product of copies of \mathbb{C} indexed by the vertices if Q does not have oriented cycles. For each arrow a of Q , the *cyclic derivative with respect to a* is the unique continuous \mathbb{C} -linear map

$$\partial_a : HH_0(\widehat{\mathbb{C}Q}) \rightarrow \widehat{\mathbb{C}Q}$$

which takes the class of a path p to the sum

$$\sum_{p=uaav} vu$$

taken over all decompositions of p as a concatenation of paths u, a, v , where u and v are of length ≥ 0 . A *potential* on Q is an element W of $HH_0(\widehat{\mathbb{C}Q})$ whose expansion in the basis of cycles does not involve cycles of length ≤ 1 . A potential is *reduced* if it does not involve cycles of length ≤ 2 . The *Jacobian algebra* $J(Q, W)$ associated to a quiver Q with potential W is the quotient of the completed path algebra by the closure of the 2-sided ideal generated by the cyclic derivatives of the elements of W . If the potential W is reduced and the Jacobian algebra $J(Q, W)$ is finite-dimensional, its quiver is isomorphic to Q .

As typical examples, we may consider the quiver Q

$$\begin{array}{ccc}
 & 2 & \\
 b \nearrow & & \searrow a \\
 1 & \xleftarrow{c} & 3
 \end{array} \tag{35}$$

with the potential $W = abc$ or with the potential $W = (abc)^2$.

In order to define the mutation of a quiver with potential (Q, W) at a vertex k , we need to recall the construction of a reduced quiver with potential from an arbitrary quiver with potential.

Two quivers with potential (Q, W) and (Q', W') are *right equivalent* if $Q_0 = Q'_0$ and there exists a \mathbb{C} -algebra isomorphism $\varphi : \widehat{kQ} \rightarrow \widehat{kQ'}$ such that φ induces the identity on the subalgebra $\prod_{Q_0} \mathbb{C}$ and the induced map in topological Hochschild homology takes W to W' . A quiver with potential (Q, W) is *trivial* if W is a (possibly infinite) linear combination of 2-cycles and $J(Q, W)$ is isomorphic to $\prod_{Q_0} \mathbb{C}$. If (Q, W) and (Q', W') are two quivers with potential such that the sets of vertices of Q and Q' coincide, their *direct sum* $(Q, W) \oplus (Q', W')$ is defined as the pair consisting of the quiver with the same vertex set, with set of arrows the disjoint union of those of Q and Q' , and with the potential equal to the sum $W \oplus W'$.

Theorem 7.1 ([25], Theorem 4.6 and Proposition 4.5). *Any quiver with potential (Q, W) is right equivalent to the direct sum of a reduced one (Q_{red}, W_{red}) and a trivial one (Q_{triv}, W_{triv}) , both unique up to right equivalence. Moreover, the inclusion induces an isomorphism from $J(Q_{red}, W_{red})$ onto $J(Q, W)$.*

The quiver with potential (Q_{red}, W_{red}) is the *reduced part* of (Q, W) .

We can now define the mutation of a quiver with potential. Let (Q, W) be a quiver with potential such that Q does not have loops. Let k be a vertex of Q not lying on a 2-cycle. The *mutation* $\mu_k(Q, W)$ is defined as the reduced part of the quiver with potential $\tilde{\mu}_k(Q, W) = (Q', W')$, which is defined as follows:

- a) (i) To obtain Q' from Q , add a new arrow $[\alpha\beta]$ for each pair of arrows $\alpha : k \rightarrow j$ and $\beta : i \rightarrow k$ of Q and
 - (ii) replace each arrow γ with source or target k by a new arrow γ^* with $s(\gamma^*) = t(\gamma)$ and $t(\gamma^*) = s(\gamma)$.
- b) Put $W' = [W] + \Delta$, where

- (i) $[W]$ is obtained from W by replacing, in a representative of W without cycles passing through k , each occurrence of $\alpha\beta$ by $[\alpha\beta]$, for each pair of arrows $\alpha : i \rightarrow k$ and $\beta : k \rightarrow j$ of Q ;
- (ii) Δ is the sum of the cycles $[\alpha\beta]\beta^*\alpha^*$ taken over all pairs of arrows $\alpha : k \rightarrow j$ and $\beta : i \rightarrow k$ of Q .

Then k is not contained in a 2-cycle of $\mu_k(Q, W)$ and $\mu_k(\mu_k(Q, W))$ is right equivalent to (Q, W) (Theorem 5.7 of [25]). As examples, consider the mutation at 2 of the cyclic quiver (35) endowed with the potential $W = abc$ and with $W' = (abc)^2$. For $W = abc$, the mutated quiver with potential is the acyclic quiver

$$\begin{array}{ccc}
 & 2 & \\
 b^* \swarrow & & \nwarrow a^* \\
 1 & & 3
 \end{array} \tag{36}$$

with the zero potential. But for $W = (abc)^2$, the mutated quiver with potential is

$$\begin{array}{ccc}
 & 2 & \\
 b^* \swarrow & & \nwarrow a^* \\
 1 & \xleftrightarrow{e} & 3 \\
 & \xleftarrow{c} &
 \end{array} \tag{37}$$

with the potential $ec ec + eb^*a^*$.

The general construction implies that if neither Q nor the quiver Q' in $(Q', W') = \mu_k(Q, W)$ have loops or 2-cycles, then Q and Q' are linked by the quiver mutation rule (cf. Prop. 7.1 of [25]). Thus, if we want to ‘extend’ this rule to quivers with potentials, it is important to ensure that no 2-cycles appear during the mutation process.

Let Q be a finite quiver. A *continuous quotient* of $HH_0(\widehat{\mathbb{C}Q})$ is linear surjection $q : HH_0(\widehat{\mathbb{C}Q}) \rightarrow V$ such that for some $N \gg 0$, all potentials involving only cycles of length $> N$ lie in the kernel of q . A *polynomial function* $HH_0(\widehat{\mathbb{C}Q}) \rightarrow \mathbb{C}$ is the composition of a continuous quotient $HH_0(\widehat{\mathbb{C}Q}) \rightarrow V$ with a polynomial map $V \rightarrow \mathbb{C}$. A *hypersurface* in $HH_0(\widehat{\mathbb{C}Q})$ is the set of zeroes of a non zero polynomial function.

Theorem 7.2 ([25], Cor. 7.4). *Let Q be a finite quiver without loops nor 2-cycles. There is a countable union of hypersurfaces $C \subset HH_0(\widehat{\mathbb{C}Q})$ such that for each W not belonging to C , no 2-cycles appear in any iterated mutation of (Q, W) .*

A potential W not belonging to C is called *generic*. So if Q is a quiver without loops nor 2-cycles and W a generic potential, we can indefinitely mutate the quiver with potential (Q, W) and the mutation of the underlying quivers is given by the quiver mutation rule. Notice that the potential $W = (abc)^2$ on the quiver (35) is not generic, which is compatible with the appearance of a 2-cycle in (37).

7.2. Ginzburg algebras. Let Q be a finite quiver and W a potential on Q (cf. section 7.1). Let Γ be the *Ginzburg [64] dg algebra* of (Q, W) . It is constructed as follows: Let \overline{Q} be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (they all have degree 0),
- an arrow $a^* : j \rightarrow i$ of degree -1 for each arrow $a : i \rightarrow j$ of Q ,
- a loop $t_i : i \rightarrow i$ of degree -2 for each vertex i of Q .

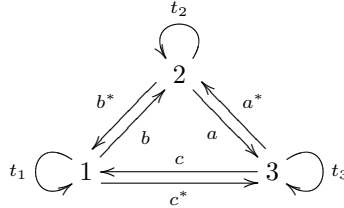
The underlying graded algebra of $\Gamma(Q, W)$ is the completion of the graded path algebra $\mathbb{C}\overline{Q}$ in the category of graded vector spaces with respect to the ideal generated by the arrows of \overline{Q} . Thus, the n -th component of $\Gamma(Q, W)$ consists of elements of the form $\sum_p \lambda_p p$, where p runs over all paths of degree n . The differential of $\Gamma(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule

$$d(uv) = (du)v + (-1)^p u dv,$$

for all homogeneous u of degree p and all v , and takes the following values on the arrows of \overline{Q} :

- $da = 0$ for each arrow a of Q ,
- $d(a^*) = \partial_a W$ for each arrow a of Q ,
- $d(t_i) = e_i(\sum_a [a, a^*])e_i$ for each vertex i of Q , where e_i is the lazy path at i and the sum runs over the set of arrows of Q .

One checks that $d^2 = 0$. For example, for the cyclic quiver 35 with the potential $W = abc$, the graded quiver \overline{Q} is



and the differential is given by

$$d(a^*) = bc, \quad d(b^*) = ca, \quad d(c^*) = ab, \quad d(t_1) = cc^* - b^*b, \dots$$

The Ginzburg algebra should be viewed as a refined version of the Jacobian algebra $J(Q, W)$. It is concentrated in (cohomological) degrees ≤ 0 and $H^0(\Gamma)$ is isomorphic to $J(Q, W)$. Two right equivalent quivers with potential have isomorphic Ginzburg algebras (Lemma 2.9 of [85]). If $(Q, W) = (Q_{triv}, W_{triv}) \oplus (Q_{red}, W_{red})$ is the direct sum of a trivial and a reduced quiver with potential, then the projection from Q onto Q_{red} induces a quasi-isomorphism $\Gamma(Q, W) \rightarrow \Gamma(Q_{red}, W_{red})$ (Lemma 2.10 of [85]).

7.3. Derived categories of dg algebras. Let us recall the construction of the *derived category* $\mathcal{D}(A)$ of a dg (=differential graded) algebra A : A (*right*) *dg module* M over A is a graded A -module equipped with a differential d such that

$$d(ma) = d(m)a + (-1)^{|m|}md(a)$$

where m in M is homogeneous of degree $|m|$, and $a \in A$.

Given two dg A -modules M and N , we define the *morphism complex* to be the graded \mathbb{C} -vector space $\mathcal{H}om_A(M, N)$ whose i -th component $\mathcal{H}om_A^i(M, N)$ is the subspace of the product $\prod_{j \in \mathbb{Z}} \text{Hom}_{\mathbb{C}}(M^j, N^{j+i})$ consisting of the morphisms f such that

$$f(ma) = f(m)a,$$

for all m in M and all a in A , together with the differential d given by

$$d(f) = f \circ d_M - (-1)^{|f|}d_N \circ f$$

for a homogeneous morphism f of degree $|f|$.

The *category* $\mathcal{C}(A)$ of dg A -modules is the category whose objects are the dg A -modules, and whose morphisms are the 0-cycles of the morphism complexes. This is an abelian category and a Frobenius category for the conflations which are split exact as sequences of graded A -modules. Its stable category $\mathcal{H}(A)$ is called the *homotopy category* of dg A -modules, which is equivalently defined as the category whose objects are the dg A -modules and whose morphism spaces are the 0-th homology groups of the morphism complexes. The homotopy category $\mathcal{H}(A)$ is a triangulated category whose suspension functor Σ is the shift of dg modules $M \mapsto \Sigma M = M[1]$. The *derived category* $\mathcal{D}(A)$ of dg A -modules is the localization of $\mathcal{H}(A)$ at the full subcategory of acyclic dg A -modules. A short exact sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

in $\mathcal{C}(A)$ yields a triangle

$$M \longrightarrow N \longrightarrow L \longrightarrow \Sigma M$$

in $\mathcal{D}(A)$. A dg A -module P is *cofibrant* if

$$\text{Hom}_{\mathcal{C}(A)}(P, L) \xrightarrow{s_*} \text{Hom}_{\mathcal{C}(A)}(P, M)$$

is surjective for each quasi-isomorphism $s : L \rightarrow M$ which is surjective in each component. We use the term “cofibrant” since these are actually the objects which are cofibrant for a certain structure of Quillen model category on the category $\mathcal{C}(A)$, cf. [83, Theorem 3.2]. For an explicit description of the cofibrant dg A -modules, cf. Prop. 2.13 of [85].

The derived category $\mathcal{D}(A)$ admits arbitrary (set-indexed) coproducts. An object P of $\mathcal{D}(A)$ is *compact* or if the functor $\text{Hom}_{\mathcal{D}(A)}(P, ?) : \mathcal{D}(A) \rightarrow \mathcal{D}(\mathbb{C})$ commutes with arbitrary coproducts. For example, the functor

$$\text{Hom}_{\mathcal{D}(A)}(A, ?) = H^0(?)$$

commutes with arbitrary sums and so the free A -module of rank 1 is compact. An arbitrary object of $\mathcal{D}(A)$ is compact iff it is *perfect*, i.e if it belongs to the closure of A under left and right shifts, extensions and passage to direct factors. The *perfect derived category* $\text{per}(A) \subset \mathcal{D}(A)$ is the full subcategory on the perfect objects.

7.4. The derived category of the Ginzburg algebra. As in section 7.2, let Q be a finite quiver. Assume that the vertex set of Q is $\{1, \dots, n\}$. Let W be a reduced potential on Q . Let Γ be the Ginzburg dg algebra of the *opposite* quiver with potential (Q^{op}, W^{op}) . Let $\mathcal{D}(\Gamma)$ be the derived category of Γ and $\text{per}(\Gamma)$ the perfect derived category. By Lemma 2.17 of [85], the category $\text{per}(\Gamma)$ is a Krull-Schmidt category, i.e. each object decomposes into a finite direct sum of indecomposable objects and each indecomposable object has a local endomorphism algebra. In particular, the free module of rank one $\Gamma \in \text{per}(\Gamma)$ decomposes into the indecomposable summands $P_i = e_i \Gamma$ associated with the vertices i of Q . The Grothendieck group $K_0(\text{per}(\Gamma))$ is free on the basis formed by the classes $[P_i]$, $1 \leq i \leq n$.

Now let $\mathcal{D}_{fd}(\Gamma)$ the *finite-dimensional derived category* of Γ , i.e. the full subcategory of $\mathcal{D}(\Gamma)$ formed by the dg modules whose homology is of finite total dimension. An object M belongs to $\mathcal{D}_{fd}(\Gamma)$ if and only if for each object P of $\text{per}(\Gamma)$, the space $\text{Hom}_{\mathcal{D}(\Gamma)}(P, \Sigma^i M)$ vanishes for almost all $i \in \mathbb{Z}$ and is finite-dimensional for all $i \in \mathbb{Z}$. The category $\mathcal{D}_{fd}(\Gamma)$ is in fact contained in $\text{per}(\Gamma)$. It is triangulated and has finite-dimensional morphism spaces. More precisely, for L and M in $\mathcal{D}_{fd}(\Gamma)$, the spaces $\text{Hom}_{\mathcal{D}(\Gamma)}(L, \Sigma^i M)$ are finite-dimensional for all $i \in \mathbb{Z}$ and vanish for all but finitely many $i \in \mathbb{Z}$. Thus, the Grothendieck group $K_0(\mathcal{D}_{fd}(\Gamma))$ carries a well-defined Euler form:

$$\langle L, M \rangle = \sum_{p \in \mathbb{Z}} (-1)^p \dim \text{Hom}_{\mathcal{D}(\Gamma)}(L, \Sigma^p M).$$

For a vertex i of Q , the simple \overline{Q}^{op} -representation S_i concentrated at the vertex i yields a simple dg Γ -module still denoted by S_i . The S_i generate the triangulated category $\mathcal{D}_{fd}(\Gamma)$. The Grothendieck group $K_0(\mathcal{D}_{fd}(\Gamma))$ is free on the basis given by the classes $[S_i]$, $1 \leq i \leq n$. We have a well-defined non degenerate pairing

$$\langle \cdot, \cdot \rangle : K_0(\text{per}(\Gamma)) \times K_0(\mathcal{D}_{fd}(\Gamma)) \rightarrow \mathbb{Z}$$

given again as a Euler form

$$\langle P, M \rangle = \sum_{p \in \mathbb{Z}} (-1)^p \dim \text{Hom}_{\mathcal{D}(\Gamma)}(P, \Sigma^p M).$$

We have

$$\langle P_i, S_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n,$$

so that the basis of the $[S_j]$, $1 \leq j \leq n$, is dual to that of the $[P_i]$, $1 \leq i \leq n$.

Let j be a vertex of Q . It follows from the cofibrant resolution of S_j given at the beginning of the proof of Lemma 3.12 in [85] that the image of the class of S_j

in $K_0(\text{per}(\Gamma))$ equals

$$[P_j] - \sum_{\alpha:t(\alpha)=j} [P_{s(\alpha)}] + \sum_{\beta:s(\beta)=j} [P_{t(\alpha)}] - [P_j] = \sum_i b_{ij} [P_i],$$

where the source and target maps refer to Q^{op} and $B = (b_{ij})$ is the antisymmetric matrix associated with the quiver Q . Thus, the matrix of the linear map

$$K_0(\mathcal{D}_{fd}(\Gamma)) \rightarrow K_0(\text{per}(\Gamma))$$

in the bases of the $[S_j]$ and the $[P_i]$ is B . It follows that we have

$$\langle S_i, S_j \rangle = -b_{ij}$$

so that $-B$ is the matrix of the Euler form $\langle \cdot, \cdot \rangle$ on $K_0(\mathcal{D}_{fd}(\Gamma))$ in the basis of the $[S_i]$, $1 \leq i \leq n$.

The category $\mathcal{D}_{fd}(\Gamma)$ is 3-Calabi-Yau, by which we mean that we have bifunctorial isomorphisms

$$D\text{Hom}(X, Y) \xrightarrow{\sim} \text{Hom}(Y, \Sigma^3 X),$$

where D is the duality functor $\text{Hom}_{\mathbb{C}}(\cdot, \mathbb{C})$ and Σ the shift functor. The simple modules S_i are 3-spherical objects in $\mathcal{D}_{fd}(\Gamma)$, i.e. we have an isomorphism

$$\text{Ext}_{\Gamma}^*(S_i, S_i) \cong H^*(S^3, \mathbb{C})$$

where the left hand side denotes the graded vector space whose p th component is $\text{Hom}_{\mathcal{D}(\Gamma)}(S_i, \Sigma^p S_i)$ and the right hand side is the (singular) cohomology of the 3-sphere with complex coefficients. The spherical objects S_i yield the Seidel-Thomas [133] twist functors tw_{S_i} . These are autoequivalences of $\mathcal{D}(\Gamma)$ such that each object X fits into a triangle

$$\text{RHom}(S_i, X) \otimes_k S_i \rightarrow X \rightarrow \text{tw}_{S_i}(X) \rightarrow \Sigma \text{RHom}(S_i, X) \otimes_k S_i.$$

By [133], the twist functors give rise to a (weak) action on $\mathcal{D}(\Gamma)$ of the braid group associated with Q , i.e. the group with generators σ_i , $i \in Q_0$, and relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ if i and j are not linked by an arrow and

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$$

if there is exactly one arrow between i and j (no relation if there are two or more arrows).

The category $\mathcal{D}(\Gamma)$ admits a natural t -structure whose truncation functors are those of the natural t -structure on the category of complexes of vector spaces (because Γ is concentrated in degrees ≤ 0). Thus, we have an induced natural t -structure on $\mathcal{D}_{fd}(\Gamma)$. Its heart \mathcal{A} is canonically equivalent to the category $\text{nil}(J(Q, W))$ of finite-dimensional right modules over $J(Q, W)$ where all sufficiently long paths act by 0. In particular, the inclusion of \mathcal{A} into $\mathcal{D}_{fd}(\Gamma)$ induces an isomorphism in the Grothendieck groups

$$K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathcal{D}_{fd}(\Gamma)).$$

The *cluster category* is the triangle quotient

$$\mathcal{C}_\Gamma = \text{per}(\Gamma)/\mathcal{D}_{fd}(\Gamma) \quad (38)$$

For acyclic quivers Q , Amiot [4] has shown that it is equivalent to the cluster category $\mathcal{C}_{Q^{op}}$ (we pass to the opposite because Γ is associated with Q^{op}) in the sense of [6]. For arbitrary quivers, there is also a close link between $\text{per}(\Gamma)$ and \mathcal{C}_Γ : For a triangulated category \mathcal{T} and an object X of \mathcal{T} , let us denote by $\text{pr}_{\mathcal{T}}(X)$ the subcategory of *X-presentable objects of \mathcal{T}* , i.e. the objects Y which occur in a triangle

$$X'' \rightarrow X' \rightarrow Y \rightarrow \Sigma X'' , \quad (39)$$

where X'' and X' belong to the closure $\text{add}(X)$ of X under taking (finite) direct sums and direct summands.

Proposition 7.3 (Prop. 2.10 of [118]). *The projection $\text{per}(\Gamma) \rightarrow \mathcal{C}_\Gamma$ induces a \mathbb{C} -linear equivalence*

$$\text{pr}_{\text{per}(\Gamma)}(\Gamma) \xrightarrow{\sim} \text{pr}_{\mathcal{C}_\Gamma}(\Gamma) ,$$

Plamondon also relates the extension groups computed in the two categories (Prop. 2.19 of [118]).

7.5. Derived equivalences from mutations. As in the preceding section, let Q be a finite quiver without loops nor 2-cycles with vertex set $\{1, \dots, n\}$ and let W be a generic potential on Q . Let Γ denote the Ginzburg algebra associated with the *opposite* quiver with potential (Q^{op}, W^{op}) . Let k be a vertex of Q and Γ' the Ginzburg algebra associated with the opposite of the mutated quiver with potential $\mu_k(Q, W)$. Put $P_i = e_i \Gamma$ and $P'_i = e_i \Gamma'$, $1 \leq i \leq n$.

Theorem 7.4 ([85]). *There are two canonical equivalences*

$$\Phi_{\pm} : \mathcal{D}(\Gamma') \rightarrow \mathcal{D}(\Gamma)$$

related by an isomorphism

$$\text{tw}_{S_k} \circ \Phi_- \xrightarrow{\sim} \Phi_+ .$$

Both Φ_+ and Φ_- send P'_i to P_i for $i \neq k$ and the images of P'_k under the two functors fit into triangles

$$P_k \longrightarrow \bigoplus_{k \rightarrow i} P_i \longrightarrow \Phi_-(P'_k) \longrightarrow \Sigma P_k \quad (40)$$

and

$$\Sigma^{-1} P_k \longrightarrow \Phi_+(P'_k) \longrightarrow \bigoplus_{j \rightarrow k} P_j \longrightarrow P_k , \quad (41)$$

where the sums are taken over the arrows in Q^{op} .

The intrinsic characterizations of the subcategories $\text{per}(\Gamma)$ and $\mathcal{D}_{fd}(\Gamma)$ show that the equivalences Φ_{\pm} induce equivalences

$$\text{per}(\Gamma') \rightarrow \text{per}(\Gamma) \quad \text{and} \quad \mathcal{D}_{fd}(\Gamma') \rightarrow \mathcal{D}_{fd}(\Gamma) .$$

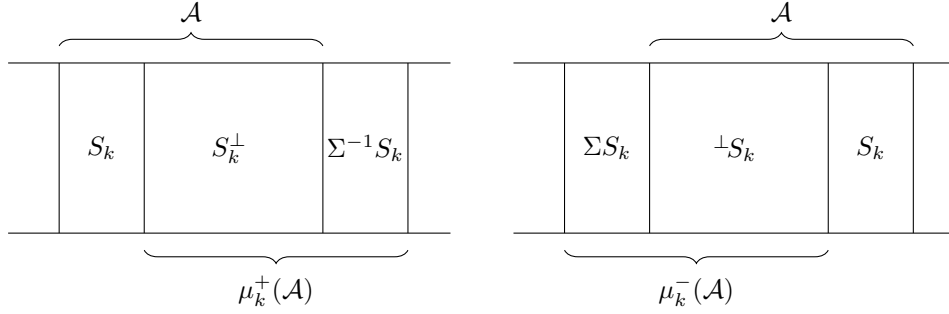


Figure 4. Right and left mutation of a heart

and thus isomorphisms in the associated Grothendieck groups. By the triangles (40) and (41), we get the first statement of the following corollary; the second one follows by passage to the duals. We use the matrices E_ε and F_ε associated with Q in section 6.2 (remember however that Γ is the Ginzburg algebra of the opposite of (Q, W)).

Corollary 7.5. *Let ε be equal to 1 or -1 . Under the assumptions of the theorem, the matrix of the induced isomorphism*

$$K_0(\Phi_\varepsilon) : K_0(\text{per}(\Gamma')) \rightarrow K_0(\text{per}(\Gamma))$$

in the bases $[P'_j]$ and $[P_i]$ is E_ε and the matrix of the induced isomorphism

$$K_0(\Phi_\varepsilon) : K_0(\mathcal{D}_{fd}(\Gamma')) \rightarrow K_0(\mathcal{D}_{fd}(\Gamma))$$

in the bases $[S'_j]$ and $[S_i]$ is F_ε .

Let \mathcal{A}' be the heart of the canonical t -structure on $\mathcal{D}_{fd}(\Gamma')$. The equivalences Φ_\pm send \mathcal{A}' onto the hearts $\mu_k^\pm(\mathcal{A})$ of two new t -structures. These can be described in terms of \mathcal{A} and the subcategory $\text{add}S_k$ as follows (cf. figure 4): Let S_k^\perp be the right orthogonal subcategory of S_k in \mathcal{A} , whose objects are the M with $\text{Hom}(S_k, M) = 0$. Then $\mu_k^+(\mathcal{A})$ is formed by the objects X of $\mathcal{D}_{fd}(\Gamma)$ such that the object $H^0(X)$ belongs to S_k^\perp , the object $H^1(X)$ belongs to $\text{add}S_k$ and $H^p(X)$ vanishes for all $p \neq 0, 1$. Similarly, the subcategory $\mu_k^-(\mathcal{A})$ is formed by the objects X such that the object $H^0(X)$ belongs to the left orthogonal subcategory ${}^\perp S_k$, the object $H^{-1}(X)$ belongs to $\text{add}(S_k)$ and $H^p(X)$ vanishes for all $p \neq -1, 0$. The subcategory $\mu_k^+(\mathcal{A})$ is the *right mutation of \mathcal{A}* and $\mu_k^-(\mathcal{A})$ is its *left mutation*.

By construction, we have

$$\text{tw}_{S_k}(\mu_k^-(\mathcal{A})) = \mu_k^+(\mathcal{A}).$$

Since the categories \mathcal{A} and $\mu^\pm(\mathcal{A})$ are hearts of bounded, non degenerate t -structures on $\mathcal{D}_{fd}(\Gamma)$, their Grothendieck groups identify canonically with that of $\mathcal{D}_{fd}(\Gamma)$.

They are endowed with canonical bases given by the simples. Those of \mathcal{A} identify with the simples S_i , $i \in Q_0$, of $\text{nil}(J(Q, W))$. The simples of $\mu_k^+(\mathcal{A})$ are $\Sigma^{-1}S_k$, the simples S_i of \mathcal{A} such that $\text{Ext}^1(S_k, S_i)$ vanishes and the objects $\text{tw}_{S_k}(S_i)$ where $\text{Ext}^1(S_k, S_i)$ is of dimension ≥ 1 . By applying $\text{tw}_{S_k}^{-1}$ to these objects we obtain the simples of $\mu_k^-(\mathcal{A})$.

7.6. Torsion subcategories and intermediate t -structures. In order to investigate the effect on hearts of suitable compositions of the equivalences Φ_{\pm} of Theorem 7.4, let us recall the construction of ‘tilted hearts’, a variation on a construction of [68]. Let \mathcal{D} be a triangulated category (for example the category $\mathcal{D}_{fd}(\Gamma)$). Let $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ be a bounded non degenerate t -structure on \mathcal{D} and \mathcal{A} its heart. A *torsion pair* on \mathcal{A} is a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that

- a) we have $\text{Hom}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$ and
- b) for each object M of \mathcal{A} , there is a short exact sequence

$$0 \longrightarrow M_{\mathcal{T}} \longrightarrow M \longrightarrow M^{\mathcal{F}} \longrightarrow 0$$

with $M_{\mathcal{T}}$ in \mathcal{T} and $M^{\mathcal{F}}$ in \mathcal{F} .

In this case, the subcategories \mathcal{T} and \mathcal{F} determine each other: we have $\mathcal{F} = \mathcal{T}^{\perp}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$, where the orthogonal subcategories are taken in \mathcal{A} .

For two full subcategories \mathcal{U} and \mathcal{V} of \mathcal{D} , let us write $\mathcal{U} \star \mathcal{V}$ for the full subcategory whose objects X occur in triangles

$$U \rightarrow X \rightarrow V \rightarrow \Sigma X$$

with U in \mathcal{U} and V in \mathcal{V} . Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathcal{A} . Then the subcategory $\mathcal{D}_{\leq 0} \star (\Sigma^{-1}\mathcal{F})$ is the left aisle of a new t -structure, whose heart $\mathcal{A}(\mathcal{F}, \Sigma^{-1}\mathcal{T})$ equals $\mathcal{F} \star \Sigma^{-1}\mathcal{T}$. It is called the *right tilt* of \mathcal{A} at $(\mathcal{T}, \mathcal{F})$. Dually, the subcategory $(\Sigma\mathcal{F}) \star \mathcal{D}_{\geq 0}$ is the right aisle of a new t -structure on \mathcal{D} , whose heart $\mathcal{A}(\Sigma\mathcal{F}, \mathcal{T})$ equals $(\Sigma\mathcal{F}) \star \mathcal{T}$. It is called the *left tilt* of \mathcal{A} at $(\mathcal{T}, \mathcal{F})$. The right tilt $\mathcal{A}(\mathcal{F}, \Sigma^{-1}\mathcal{T})$ admits the torsion pair $(\mathcal{F}, \Sigma^{-1}\mathcal{T})$ and its left tilt with respect to this pair equals the original category $\mathcal{A} = \mathcal{T} \star \mathcal{F}$. Similarly, the left tilt $\mathcal{A}(\Sigma\mathcal{F}, \mathcal{T})$ admits the torsion pair $(\Sigma\mathcal{F}, \mathcal{T})$ and we recover \mathcal{A} as its right tilt with respect to this pair.

Clearly, the left aisle $\mathcal{D}_{\leq 0} \star (\Sigma^{-1}\mathcal{F})$ is an *intermediate left aisle*, i.e. we have

$$\mathcal{D}_{\leq 0} \subset \mathcal{D}_{\leq 0} \star (\Sigma^{-1}\mathcal{F}) \subset \mathcal{D}_{\leq 1}.$$

It is not hard to check that each intermediate left aisle is of this form. Dually, each right aisle between $\mathcal{D}_{\geq -1}$ and $\mathcal{D}_{\geq 0}$ is of the form $(\Sigma\mathcal{F}) \star \mathcal{D}_{\geq 0}$.

Of course, in the situation of section 7.5, the heart $\mu_k^+(\mathcal{A})$ is the right tilt of \mathcal{A} with respect to $(\text{add}S_k, S_k^{\perp})$ and $\mu_k^-(\mathcal{A})$ is its left tilt. The following lemma will allow us to iterate such mutations. The point is that given a suitable torsion pair in a right tilt \mathcal{A}' of \mathcal{A} , either the left tilt or the right tilt of \mathcal{A}' with respect to this pair is again a right tilt of \mathcal{A} , cf. figure 5.

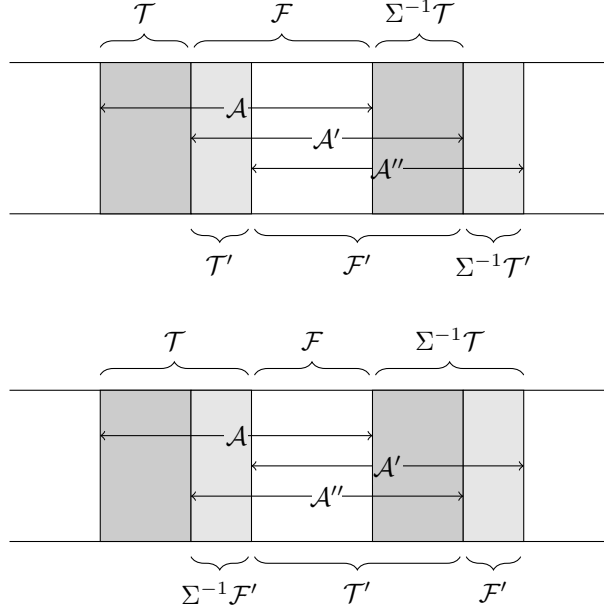


Figure 5. Composition of a right tilt with a right tilt (top) resp. a left tilt (bottom)

Lemma 7.6 (Bridgeland [106]). *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in \mathcal{A} and $(\mathcal{T}', \mathcal{F}')$ a torsion pair in $\mathcal{A}' = \mathcal{A}(\mathcal{F}, \Sigma^{-1}\mathcal{T})$.*

a) *If $\mathcal{T}' \subset \mathcal{F}$, then the right tilt $\mathcal{A}'' = \mathcal{A}'(\mathcal{F}', \Sigma^{-1}\mathcal{T}')$ equals the right tilt*

$$\mathcal{A}(\mathcal{T} \star \mathcal{T}', \mathcal{F} \cap \mathcal{F}').$$

b) *If $\mathcal{F}' \subset \Sigma^{-1}\mathcal{T}$, then the left tilt $\mathcal{A}'' = \mathcal{A}'(\Sigma\mathcal{T}', \mathcal{F}')$ equals the right tilt*

$$\mathcal{A}(\Sigma\mathcal{F}' \star \mathcal{T}, \mathcal{T}' \cap \Sigma^{-1}\mathcal{T}).$$

The lemma is not hard to check. The following easy lemma is a key point for the main conjecture 5.1:

Lemma 7.7. *Let $\mathcal{A}' = \mathcal{A}(\mathcal{F}, \Sigma^{-1}\mathcal{T})$ be the right tilt of \mathcal{A} with respect to a torsion pair $(\mathcal{T}, \mathcal{F})$. Then each simple object of \mathcal{A}' either lies in \mathcal{A} or in $\Sigma^{-1}\mathcal{A}$.*

Indeed, let S be the given simple object. Since $(\mathcal{F}, \Sigma^{-1}\mathcal{T})$ forms a torsion pair in \mathcal{A}' , we have the exact sequence

$$0 \longrightarrow S_{\mathcal{F}} \longrightarrow S \longrightarrow S^{\Sigma^{-1}\mathcal{T}} \longrightarrow 0$$

where $S_{\mathcal{F}}$ belongs to $\mathcal{F} \subset \mathcal{A}$ and $S^{\Sigma^{-1}\mathcal{T}}$ to $\Sigma^{-1}\mathcal{F} \subset \Sigma^{-1}\mathcal{A}$. Since S is simple, we either have $S_{\mathcal{F}} \xrightarrow{\sim} S$ or $S \xrightarrow{\sim} S^{\Sigma^{-1}\mathcal{T}}$.

Now let (Q, W) and (Q', W') be two quivers with reduced potentials and let Γ and Γ' be the associated Ginzburg dg algebras. Suppose that

$$\Phi : \mathcal{D}(\Gamma') \rightarrow \mathcal{D}(\Gamma)$$

is a triangle equivalence. Let $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$ be the natural t -structure on $\mathcal{D}_{fd}(\Gamma)$ and let \mathcal{A} be its heart. Similarly, let $(\mathcal{D}'_{\leq 0}, \mathcal{D}'_{\geq 0})$ be the natural t -structure on $\mathcal{D}_{fd}(\Gamma')$ and let \mathcal{A}' be its heart. Let us denote by $H_{\mathcal{A}}^p$, $p \in \mathbb{Z}$, the homology functors with respect to the natural t -structure on $\mathcal{D}_{fd}(\Gamma)$.

Proposition 7.8. *The following are equivalent:*

- (i) *the subcategory $\Phi(\mathcal{A}') \subset \mathcal{D}_{fd}(\Gamma)$ is the right tilt of \mathcal{A} with respect to a torsion pair $(\mathcal{T}, \mathcal{F})$;*
- (ii) *the object $\Phi(\Gamma')$ is $(\Sigma^{-1}\Gamma)$ -presentable (cf. the end of section 7.4);*
- (iii) *we have $\mathcal{D}_{\leq 0} \subset \Phi(\mathcal{D}'_{\leq 0}) \subset \mathcal{D}_{\leq 1}$.*

If these conditions hold, then the torsion subcategory \mathcal{T} of (i) is formed by the finite-dimensional quotients of the object $H_{\mathcal{A}}^1(\Phi(\Gamma'))$.

7.7. Patterns of tilts and decategorification. As in section 7.5, let Q be a finite quiver without loops nor 2-cycles with vertex set $\{1, \dots, n\}$ and let W be a generic potential on Q . Let Γ denote the Ginzburg algebra associated with the *opposite* quiver with potential (Q^{op}, W^{op}) .

As we have seen in section 7.1, we can indefinitely mutate (Q, W) . Thus, with each vertex t of the n -regular tree, we can associate a quiver with potential $(Q(t), W(t))$ such that (Q, W) is associated with t_0 and, whenever t and t' are linked by an edge labeled k , the corresponding quivers with potential are linked by a mutation. We write $\Gamma(t)$ for the Ginzburg dg algebra associated with the *opposite* of $(Q(t), W(t))$.

Now we will use induction on the distance of a vertex t of n -regular tree from the root t_0 to define a triangle equivalence

$$\Phi(t) : \mathcal{D}(\Gamma(t)) \longrightarrow \mathcal{D}(\Gamma)$$

such that $\Phi(t)$ satisfies the equivalent conditions of Proposition 7.8. The construction follows an idea of Bridgeland [106]. By definition, $\Phi(t_0)$ is the identity. Now suppose that $\Phi(t)$ has been defined and that t' is linked to t by an edge labeled k and is at greater distance from t_0 than t . Let $\mathcal{A}(t) \subset \mathcal{D}_{fd}(\Gamma)$ be the image under $\Phi(t)$ of the heart of the natural t -structure of $\mathcal{D}_{fd}(\Gamma(t))$ and let $S_i(t)$, $1 \leq i \leq n$, be the simple objects of $\mathcal{A}(t)$. By assumption, the subcategory $\mathcal{A}(t)$ is the right tilt of $\mathcal{A} = \mathcal{A}(t_0)$ with respect to some torsion theory $(\mathcal{T}(t), \mathcal{F}(t))$. Thus, by Lemma 7.7, the simple object $S_k(t)$ either lies in $\mathcal{F}(t) \subset \mathcal{A}$ or in $\Sigma^{-1}\mathcal{T}(t) \subset \Sigma^{-1}\mathcal{A}$. In the first case, we put

$$\Phi(t') = \Phi(t) \circ \Phi_{k,+} \text{ and in the second case } \Phi(t') = \Phi(t) \circ \Phi_{k,-}.$$

Then in the first case, as we have seen in section 7.5, $\mathcal{A}(t')$ is the right tilt of $\mathcal{A}(t)$ with respect the torsion pair

$$(\text{add}(S_k(t)), S_k(t)^\perp)$$

and in the second case, it is the left tilt with respect to

$$({}^\perp S_k(t), \text{add}(S_k(t))).$$

In both cases, Lemma 7.6 shows that $\mathcal{A}(t')$ is again a right tilt of \mathcal{A} and so $\Phi(t')$ again satisfies the conditions of Proposition 7.8.

Notice that at the same time, this construction produces a sign $\varepsilon(e)$ for each edge $e : t \rightarrow t'$ of the n -regular tree. For each vertex t of \mathbb{T}_n , and for $1 \leq i \leq n$, let $T_i(t)$ be the image of $e_i \Gamma(t)$ under $\Phi(t)$.

Theorem 7.9. *Let t be a vertex of the n -regular tree and let $1 \leq j \leq n$.*

- a) *The j th column of the c -matrix $C(t)$ contains the coordinates of $[S_j(t)]$ in the basis $[S_1], \dots, [S_n]$ of $K_0(\mathcal{D}_{fd}(\Gamma))$.*
- b) *The object $S_j(t)$ lies in \mathcal{A} or $\Sigma^{-1}\mathcal{A}$. Therefore, each c -vector is non zero and has either all its components non negative or all its components non positive (i.e. the main conjecture 5.1 holds for Q).*
- c) *The j th column of the g -matrix $G(t)$ contains the coordinates of $[T_j(t)]$ in the basis $[P_1], \dots, [P_n]$ of $K_0(\text{per}(\Gamma))$.*
- d) *The (left) $J(Q, W)$ -module $H_{\mathcal{A}}^1(T_j(t))$ is finite-dimensional and the F -polynomial $F_j(t)$ equals*

$$\sum_e \chi(\text{Gr}_e(H_{\mathcal{A}}^1(T_j(t)))) y^e,$$

where e runs through \mathbb{N}^n , Gr_e denotes the variety of submodules whose quotient has dimension vector e , χ is the Euler characteristic (with respect to singular cohomology with rational coefficients) and

$$y^e = \prod_{i=1}^n x_{n+i}^{e_i}.$$

To make sure that our conventions are coherent, let us consider the example of the quiver $Q : 1 \rightarrow 2$ and the vertex t linked to t_0 by the mutation at 1. We have to consider the Ginzburg algebra Γ associated with $Q^{op} : 2 \rightarrow 1$ and perform a right mutation at the vertex 1. We get the exchange triangle

$$\Sigma^{-1}P_1 \rightarrow T_1(t) \rightarrow P_2 \rightarrow P_1. \quad (42)$$

Thus, the class of $T_1(t)$ in $K_0(\text{per}(\Gamma))$ equals $-[P_1] + [P_2]$, which does correspond to the g -vector

$$g_1(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The new simple modules are $S_1(t) = \Sigma^{-1}S_1$ and $S_2(t)$ given by the universal extension

$$S_2 \rightarrow S_2(t) \rightarrow S_1 \rightarrow \Sigma S_2.$$

So in $K_0(\mathcal{D}_{fd}(\Gamma))$, we have $[S_1(t)] = -[S_1]$ and $[S_2(t)] = [S_1] + [S_2]$, which does correspond to the c -matrix

$$C(t) = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Using the exchange triangle (42), we easily check that $\text{Hom}(\Sigma^{-1}P_1, T_1(t)) = \mathbb{C}$ and $\text{Hom}(\Sigma^{-1}P_2, T_1(t)) = 0$ so that the module $H_{\mathcal{A}}^1(T_1(t))$ is the simple at the vertex 1. The associated generating series of Euler characteristics is indeed equal to

$$F_1(t) = 1 + y^{e_1}.$$

Notice that since each $S_j(t)$ belongs to \mathcal{A} or $\Sigma^{-1}\mathcal{A}$ (Lemma 7.7), part a) implies part b). Thanks to parts a) and c), the duality between the bases formed by the $[T_i(t)]$ and the $[S_j(t)]$ corresponds to the first part of the tropical duality theorem 5.5.

Parts a), b) and c) are proved in Nagao's [106], and part d) is proved there under an additional technical assumption. Parts a), b), c) and d) follow from the results of Plamondon [117], cf. section 7.10 (and when $H^0(\Gamma)$ is finite-dimensional from [116]). Using his dictionary between objects of the cluster category and decorated representations, the theorem also can also be deduced from the results of [26].

For acyclic quivers Q , part d) was extended to the quantum case by Qin [119] and (for prime powers q) by Rupel [129] [128], who also obtained an analogous result for acyclic valued quivers. Under certain technical assumptions, Efimov [30] has recently been able to extend part d) to the quantum case for arbitrary quivers (without loops nor 2-cycles). He mainly builds on the work of Kontsevich-Soibelman [89] [88] and Nagao [106].

7.8. Reign of the tropics. The following theorem and its corollary are the basis of the 'tropicalization method' which is used in applications of cluster algebras to discrete dynamical systems and to dilogarithm identities, cf. [114] [72] [73] [112] [113].

Theorem 7.10 ([118]). *Let Γ' and Γ'' be two Ginzburg dg algebras and let $\Phi' : \mathcal{D}(\Gamma') \rightarrow \mathcal{D}(\Gamma)$ and $\Phi'' : \mathcal{D}(\Gamma'') \rightarrow \mathcal{D}(\Gamma)$ be triangle equivalences satisfying the conditions of Proposition 7.8. For $1 \leq j \leq n$, let us write S'_j for the image of the j th simple module under Φ' and P'_j for the image of the module $e_j\Gamma'$. Similarly, we define S''_j and P''_j . Suppose that for each $1 \leq j \leq n$, we have $[S'_j] = [S''_j]$ in $K_0(\mathcal{D}_{fd}(\Gamma))$. Then for each $1 \leq j \leq n$, we have $P'_j \cong P''_j$ and $S'_j \cong S''_j$.*

We first notice that for each $1 \leq j \leq n$, we have the equality $[P'_j] = [P''_j]$ in $K_0(\text{per}(\Gamma))$. Indeed, this follows from the duality between the bases $[P'_i]$ and $[S'_j]$ in $K_0(\text{per}(\Gamma))$ and $K_0(\mathcal{D}_{fd}(\Gamma))$. Now the first isomorphism follows from the fact, proved in section 3.1 of [117], that an object X of $\text{pr}(\Gamma)$ which is rigid, i.e.

$\text{Hom}(X, \Sigma X) = 0$, is determined by its class in $K_0(\mathbf{per}(\Gamma))$. The objects S'_i are the simples of the abelian subcategory of $\mathcal{D}(\Gamma)$ formed by the objects U such that $\text{Hom}(P'_j, \Sigma^p U)$ vanishes for all $p \neq 0$ and all $1 \leq j \leq n$. Among these simples, S'_i is the only one which receives a non zero morphism from P'_i . Now it is clear that the isomorphisms for the P'_j imply those for the S'_j .

Corollary 7.11. *Let t and t' be vertices of the n -regular tree such that there is a permutation π of $\{1, \dots, n\}$ with $C(t') = P_\pi C(t)$, where P_π is the permutation matrix associated with π . Then we have $G(t) = P_\pi G(t')$, the permutation π yields an isomorphism $Q(t) \xrightarrow{\sim} Q(t')$ and for each $1 \leq j \leq n$, we have*

- a) $T_j(t') = T_{\pi(j)}(t)$ and $S_j(t') = S_{\pi(j)}(t)$;
- b) $F_j(t') = F_{\pi(j)}(t)$;
- c) $x_j(t') = x_{\pi(j)}(t)$ and $y_j(t') = y_{\pi(j)}(t)$.

In particular, the seeds associated with t and t' are isomorphic via π .

To prove the corollary, we apply the theorem to the equivalences $\Phi(t)$ and $\Phi(t')$. We immediately obtain part a). This implies the statement on the g -matrices and the quivers. By Theorem 7.9, it also implies parts b) and c).

A proof of the corollary based on the study of stability conditions on $\mathcal{D}_{fd}(\Gamma)$ is given in section 4.2 of [106]. It can also be deduced from the results of [26].

7.9. Rigid objects and cluster monomials. Let \tilde{Q} be an ice quiver (equally valued). Let Γ be the Ginzburg dg algebra associated with the opposite of (\tilde{Q}, W) , where W is a generic potential.

For each object L of $\text{pr}(\Sigma^{-1}\Gamma)$ such that $H^1(L)$ is finite-dimensional, we define a Laurent polynomial

$$X(L) = \sum_e \chi(\text{Gr}_e(H^1_{\mathcal{A}}(L))) \hat{y}^e,$$

where $\hat{y}_l = \prod_{i=1}^m x_i^{\tilde{b}_{li}}$, $1 \leq l \leq n$, and $\hat{y}^e = \prod_l \hat{y}_l^{e_l}$. By part c) of Theorem 7.9 and by the separation formula of Theorem 5.7, when $L = T_i(t)$ for some $1 \leq i \leq n$ and some vertex t of the n -regular tree, then $X(L)$ equals the cluster variable $x_i(t)$. It is not hard to check that for two objects L and L' of $\text{pr}(\Gamma)$, we have

$$X(L \oplus L') = X(L)X(L').$$

So if we apply the map $L \mapsto X(L)$ to direct sums of objects $T_i(t)$, $1 \leq i \leq n$, for a fixed vertex t , we recover the cluster monomials associated with t .

Let us call a rigid object L of $\text{pr}(\Sigma^{-1}\Gamma)$ *reachable* if there is a vertex t of the n -regular tree such that L is a direct sum of copies of the objects $T_i(t)$, $1 \leq i \leq n$.

Theorem 7.12 ([117] [19]). a) *If L_1, \dots, L_N are pairwise non-isomorphic reachable rigid objects, then the Laurent polynomials $X(L_1), \dots, X(L_N)$ are linearly independent.*

- b) *The map $L \mapsto X(L)$ induces a bijection from the set of isomorphism classes of reachable rigid objects onto the set of cluster monomials.*

The surjectivity in b) is proved by Plamondon [117]. The linear independence in a), and hence the injectivity in b), is proved in [19].

7.10. Proof of decategorification. We will sketch a proof of Theorem 7.9. We prove a) and b) simultaneously by induction on the distance of t from t_0 . For $t = t_0$, there is nothing to prove. Now assume we have proved the claim for t and that t' is at greater distance from t_0 and linked to t by an edge labeled k . Then the coefficients of the c -matrix at t' can be computed as

$$c_{ij}(t') = \begin{cases} -c_{ij}(t) & \text{if } j = k; \\ c_{ij}(t) + c_{ik}(t)[\varepsilon b_{kj}(t)]_+ + [-\varepsilon c_{ik}(t)]_+ b_{kj}(t) & \text{else,} \end{cases}$$

where $1 \leq i, j \leq n$ and ε is any sign, cf. Prop. 5.8 of [44] and formula (3.3) in [113]. We know that $b_{kj}(t)$ equals the number of arrows from k to j in $Q(t)$ minus the number of arrows from j to k in $Q(t)$. Thus, we have

$$b_{kj}(t) = \dim \operatorname{Ext}_{\Gamma}^1(S_k(t), S_j(t)) - \dim \operatorname{Ext}_{\Gamma}^1(S_j(t), S_k(t)).$$

By the induction hypothesis, the coordinates of $[S_k(t)]$ in the basis of the $[S_i]$ are the components $c_{ik}(t)$, $1 \leq i \leq n$, of the c -vector $C(t)e_k$. By Lemma 7.7, they are all of the same sign. Let us choose ε equal to this sign. Then the formula for the $c_{ij}(t')$ simplifies as follows:

$$c_{ij}(t') = \begin{cases} -c_{ij}(t) & \text{if } j = k; \\ c_{ij}(t) + c_{ik}(t)[\varepsilon b_{kj}(t)]_+ & \text{else.} \end{cases}$$

Now assume that $\varepsilon = 1$. This means that $S_k(t)$ lies in \mathcal{A} and that $S_k(t')$ is $\Sigma^{-1}S_k(t)$. Let us put $m = b_{kj}(t)$. If we have $m \leq 0$, then the space $\operatorname{Ext}_{\Gamma}^1(S_k(t), S_j(t))$ vanishes and we have $S_j(t') = S_j(t)$. If we have $m \geq 0$, then we get $m = \dim \operatorname{Ext}_{\Gamma}^1(S_k(t), S_j(t))$ and the object $S_j(t')$ is constructed as a universal extension:

$$(\Sigma^{-1}S_k(t))^m \rightarrow S_j(t) \rightarrow S_j(t') \rightarrow S_k(t)^m.$$

In both cases, the formula for $c_{ij}(t')$ gives the correct multiplicity of $[S_i]$ in $[S_j(t')]$. Now suppose that $\varepsilon = -1$. Then $S_k(t)$ belongs to $\Sigma^{-1}\mathcal{A}(t)$ and $S_k(t')$ is $\Sigma S_k(t)$. Let us put $m = -b_{kj}(t) = b_{jk}(t)$. If we have $m \leq 0$, then the space $\operatorname{Ext}_{\Gamma}^1(S_j(t), S_k(t))$ vanishes and $S_j(t') = S_j(t)$. If we have $m \geq 0$, then we get $m = \dim \operatorname{Ext}_{\Gamma}^1(S_j(t), S_k(t))$ and $S_j(t')$ is constructed as a universal extension

$$S_k(t)^m \rightarrow S_j(t') \rightarrow S_j(t) \rightarrow \Sigma S_k(t)^m.$$

Again, in both cases, the formula for $c_{ij}(t')$ gives the correct multiplicity of $[S_i]$ in $[S_j(t')]$.

We get part c) as a consequence: Indeed, by part b) the main conjecture 5.1 holds for Q and so we have $G(t)^T C(t) = I$ for all vertices t of the n -regular tree,

by the tropical duality theorem 5.3. On the other hand, the basis of the $[P_i(t)]$ is dual to that of the $[S_j(t)]$. Clearly, this implies c).

We deduce d) from Plamondon's results [118]. Indeed, both $T_j(t)$ and $\Sigma^{-1}\Gamma$ belong to $\text{pr}(\Sigma^{-1}\Gamma)$. Thus, by Proposition 7.3, we have

$$H_{\mathcal{A}}^1(T_j(t)) = \text{Hom}_{\text{per}(\Gamma)}(\Sigma^{-1}\Gamma, T_j(t)) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}_{\Gamma}}(\pi(\Sigma^{-1}\Gamma), \pi(T_j(t))),$$

where $\mathcal{C}_{\Gamma} = \text{per}(\Gamma)/\mathcal{D}_{fd}(\Gamma)$ is the cluster category and π the projection functor. Let us omit this functor from the notations. Since $T_j(t) \in \mathcal{C}_{\Gamma}$ is obtained by iterated mutation from Γ , it belongs to Plamondon's category $\mathcal{D} \subset \mathcal{C}_{\Gamma}$ and so the space

$$\text{Hom}_{\mathcal{C}_{\Gamma}}(T_j(t), \Sigma\Gamma)$$

is finite-dimensional. By Prop. 2.16 of [118], this space is in duality with

$$\text{Hom}_{\mathcal{C}_{\Gamma}}(\Sigma^{-1}\Gamma, T_j(t))$$

which therefore also finite-dimensional. So we find that $H_{\mathcal{A}}^1(T_j(t))$ is finite-dimensional and in duality with $\text{Hom}_{\mathcal{C}_{\Gamma}}(T_j(t), \Sigma\Gamma)$. Now let $P \mapsto P^{\vee}$ denote the canonical equivalence

$$\text{per}(\Gamma)^{op} \xrightarrow{\sim} \text{per}(\Gamma^{op}), \quad P \mapsto P^{\vee} = \text{RHom}_{\Gamma}(P, \Gamma).$$

It induces an equivalence $\mathcal{C}_{\Gamma}^{op} \xrightarrow{\sim} \mathcal{C}_{\Gamma^{op}}$ still denoted by the same symbol. We have

$$\text{Hom}_{\mathcal{C}_{\Gamma}}(T_j(t), \Sigma\Gamma) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}_{\Gamma^{op}}}(\Sigma\Gamma)^{\vee}, T_j(t)^{\vee}) = \text{Hom}_{\mathcal{C}_{\Gamma^{op}}}(\Sigma^{-1}\Gamma, T_j(t)^{\vee}).$$

Notice that $\Gamma^{op} = \Gamma(Q^{op}, W^{op})^{op} = \Gamma(Q, W)$. So we get that the left $J(Q, W)$ -module $H_{\mathcal{A}}^1(T_j(t))$ is in duality with the right $J(Q, W)$ -module

$$\text{Hom}_{\mathcal{C}_{\Gamma^{op}}}((\Sigma\Gamma)^{\vee}, T_j(t)^{\vee}) = \text{Hom}_{\mathcal{C}_{\Gamma^{op}}}(\Sigma^{-1}\Gamma, T'_j(t)),$$

where $T'_j(t)$ denotes the object obtained from Γ in $\mathcal{C}_{\Gamma^{op}}$ by the sequence of mutations linking t_0 to t . Thus, the Grassmannian of e -dimensional quotients of $H_{\mathcal{A}}^1(T_j(t))$ identifies with the Grassmannian of e -dimensional submodules of the above $J(Q, W)$ -module. The generating series of their Euler characteristics is the F -polynomial associated with $T'_j(t)$ in Def. 3.14 of [117] and it equals the F -polynomial $F_j(t)$, as shown in section 4.2 of [117].

7.11. Proof of the quantum dilogarithm identities. We will sketch a proof of Theorem 6.5. We start with part a). We prove the stronger statement given in Remark 6.6. So suppose that, in the notations of the remark, we have $PC(t) = C(t')$ for the permutation matrix $P = P_{\pi}$ associated with a permutation π of $\{1, \dots, n\}$. By Corollary 7.11, we find that the seeds associated with t and t' are isomorphic via π in any cluster algebra associated with a matrix \tilde{B} whose principal part B corresponds to Q . Now by Theorem 6.2, we find that the quantum seeds associated with t and t' are isomorphic via π in any quantum cluster algebra associated with a compatible pair (\tilde{B}, Λ) , where the principal part of \tilde{B} is the given matrix B . Thus, in the notations of Theorem 6.4, we have $\Phi(\mathbf{i}) = \Phi(\mathbf{i}')$. Now by

the duality theorem 5.3, we also have $P_\pi G(t) = G(t')$ for the same permutation π . By the equality (34) in Theorem 6.4, we obtain

$$\text{Ad}'(\mathbb{E}(\mathbf{i})) = \text{Ad}'(\mathbb{E}(\mathbf{i}')).$$

Now let us choose $\tilde{B} = B_{pr}$. We find that the power series $\mathbb{E}(\mathbf{i})\mathbb{E}(\mathbf{i}')^{-1}$ in the variables x_1, \dots, x_n commutes with the variables x_{n+i} , $1 \leq i \leq n$. Now by our choice of $\tilde{B} = B_{pr}$, we have

$$x_{n+i}x_j = q^{\delta_{ij}}x_jx_{n+i}$$

for all $1 \leq i, j \leq n$. This implies that for any power series $f(x_1, \dots, x_n)$, we have

$$x_{n+i}f(x_1, \dots, x_n)x_{n+i}^{-1} = f(x_1, \dots, qx_i, \dots, x_n).$$

So a power series in x_1, \dots, x_n which commutes with all the x_{n+i} , $1 \leq i \leq n$, must be constant. Since the constant term of $\mathbb{E}(\mathbf{i})\mathbb{E}(\mathbf{i}')^{-1}$ is 1, we find $\mathbb{E}(\mathbf{i})\mathbb{E}(\mathbf{i}')^{-1} = 1$ as claimed.

For part b), we have to invoke Donaldson-Thomas theory in its form pioneered by Kontsevich-Soibelman [89] [88]. This theory is not yet completely developed for formal potentials and therefore, for the moment, does not apply to arbitrary quivers (cf. [7] [100] [107] for recent progress on special classes). However, in its final form, the theory should yield the following: Let $\hat{\mathcal{A}}_Q$ denote the completed quantum affine space associated with Q . Let \mathcal{A} be the category of finite-dimensional (hence nilpotent) right modules over the completed Jacobian algebra $J(Q, W)$ of the quiver Q endowed with a generic potential. Let \mathcal{T}_1 and \mathcal{T}_2 be torsion subcategories of \mathcal{A} . Following the explanation after Remark 21 on page 90 of [89] we define \mathcal{T}_1 to be *constructibly less than or equal to* \mathcal{T}_2 if \mathcal{T}_1 is contained in \mathcal{T}_2 and for each dimension vector d , the subset of the variety of (contravariant) representations of $J(Q, W)$ with dimension vector d formed by the points corresponding to modules in $\mathcal{T}_1^\perp \cap \mathcal{T}_2$ is constructible. In this case, following [89] we write

$$\mathcal{T}_1 \leq_{\text{constr}} \mathcal{T}_2. \quad (43)$$

What the fully fledged version of Kontsevich-Soibelman's theory should yield is a *DT-character on \mathcal{A}* , i.e. the datum of an element $A_{\mathcal{T}_1, \mathcal{T}_2}$ of the group $\hat{\mathcal{A}}_Q^\times$ for each pair of torsion theories $\mathcal{T}_1, \mathcal{T}_2$ satisfying (43) such that the following hold

- a) whenever we have three torsion theories $\mathcal{T}_1, \mathcal{T}_2$ and \mathcal{T}_3 such that

$$\mathcal{T}_1 \leq_{\text{constr}} \mathcal{T}_2, \mathcal{T}_2 \leq_{\text{constr}} \mathcal{T}_3 \text{ and } \mathcal{T}_1 \leq_{\text{constr}} \mathcal{T}_3,$$

we have

$$A_{\mathcal{T}_1, \mathcal{T}_2} A_{\mathcal{T}_2, \mathcal{T}_3} = A_{\mathcal{T}_1, \mathcal{T}_3}; \quad (44)$$

- b) if we have $\mathcal{T}_2 = \mathcal{T}_1 \star \text{add}(L)$, where L is a module in \mathcal{T}_1^\perp satisfying $\text{Hom}(L, L) = \mathbb{C}$ and $\text{Ext}^1(L, L) = 0$, we have

$$A_{\mathcal{T}_1, \mathcal{T}_2} = \mathbb{E}(x^\alpha), \quad (45)$$

where α is the dimension vector of L .

The *non commutative DT invariant* associated with Q and the given DT-character is then the power series

$$DT_Q = A_{0,\mathcal{A}} \in \widehat{\mathbb{A}}_Q. \quad (46)$$

Via the duality functor $\mathrm{Hom}_{\mathbb{C}}(\cdot, \mathbb{C})$ and the canonical isomorphism $\widehat{\mathbb{A}}_Q^{op} \xrightarrow{\sim} \widehat{\mathbb{A}}_{Q^{op}}$ taking x^α to x^α , a DT-character for Q yields one for Q^{op} and $DT_Q \in \widehat{\mathbb{A}}_Q$ is mapped to $DT_{Q^{op}} \in \widehat{\mathbb{A}}_{Q^{op}}$.

Now assume that we have a quiver Q whose *non commutative DT-invariant is defined*, i.e. it admits a DT-character. Then this also holds for Q^{op} . Suppose that we are in the situation of part b) of Theorem 6.5 so that $-C(t_N)$ is a permutation matrix. Then by Theorem 7.10, the simples of $\mathcal{A}(t_N)$ lie in $\Sigma^{-1}\mathcal{A}$ and so we must have $\mathcal{A}(t_N) = \Sigma^{-1}\mathcal{A}$ and $\mathcal{T}(t_N) = \mathcal{A}$. Now the torsion subcategories

$$\{0\} = \mathcal{T}(t_0), \mathcal{T}(t_1), \dots, \mathcal{T}(t_N) = \mathcal{A}$$

form a sequence such that for each $1 \leq s \leq N$, we either have

- (1) $S_{i_s}(t_{s-1}) \in \mathcal{A}$ and then

$$\mathcal{T}(t_{s-1}) \leq_{\text{constr}} \mathcal{T}(t_s) \text{ and } \mathcal{T}(t_s) = \mathcal{T}(t_{s-1}) \star \mathrm{add}(S_{i_s}(t_{s-1}))$$

or

- (2) $S_{i_s}(t_{s-1}) \in \Sigma^{-1}\mathcal{A}$ and then

$$\mathcal{T}(t_s) \leq_{\text{constr}} \mathcal{T}(t_{s-1}) \text{ and } \mathcal{T}(t_{s-1}) = \mathcal{T}(t_s) \star \mathrm{add}(\Sigma S_{i_s}(t_{s-1})).$$

depending on the sign of the c -vector $\beta_s = C(t_{s-1})e_{i_s}$, which is just the (signed) dimension vector of $S_{i_s}(t_{s-1})$, by Theorem 7.9. By induction on s , one now proves that

$$\mathbb{E}(\varepsilon_1 \beta_1)^{\varepsilon_1} \dots \mathbb{E}(\varepsilon_s \beta_s)^{\varepsilon_s} = A_{0,\mathcal{T}(t_s)}.$$

For $s = N$, we obtain the equality

$$\mathbb{E}(\varepsilon_1 \beta_1)^{\varepsilon_1} \dots \mathbb{E}(\varepsilon_s \beta_s)^{\varepsilon_s} = A_{0,\mathcal{A}} = DT_{Q^{op}}$$

in $\widehat{\mathbb{A}}_{Q^{op}}$. Its image under the canonical isomorphism $\widehat{\mathbb{A}}_Q^{op} \xrightarrow{\sim} \widehat{\mathbb{A}}_{Q^{op}}$ is the claimed equality in $\widehat{\mathbb{A}}_Q$.

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