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Deformed Calabi–Yau completions

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Abstract. We define and investigate deformed n-Calabi–Yau completions of homologically smooth differential graded (= dg) categories. Important examples are: deformed preprojective algebras of connected non-Dynkin quivers, Ginzburg dg algebras associated to quivers with potentials and dg categories associated to the category of coherent sheaves on the canonical bundle of a smooth variety. We show that deformed Calabi–Yau completions do have the Calabi–Yau property and that their construction is compatible with derived equivalences and with localizations. In particular, Ginzburg dg algebras have the Calabi–Yau property. We show that deformed 3-Calabi–Yau completions of algebras of global dimension at most 2 are quasi-isomorphic to Ginzburg dg algebras and apply this to the study of cluster-tilted algebras and to the construction of derived equivalences associated to mutations of quivers with potentials. In the appendix, Michel Van den Bergh uses non-commutative differential geometry to give an alternative proof of the fact that Ginzburg dg algebras have the Calabi–Yau property.

Contents

1. Introduction

1.1. Context and main results. This article is motivated by the theory which links cluster algebras [17] to representations of quivers and finite-dimensional algebras, cf. [23]

for a survey. In this theory, Calabi–Yau algebras and categories play an important rôle. For example, Geiss–Leclerc–Schröer use the 2-Calabi–Yau property of the category of modules over a preprojective algebra (cf. [18]), Iyama–Reiten [21] study mutations using tilting modules over 2- and 3-Calabi–Yau algebras related to singularities [45] and Amiot's construction [1] of generalized cluster categories relies on dg algebras which are 3-Calabi–Yau as bimodules. The Calabi–Yau property is also important in Kontsevich–Soibelman's recent interpretation of cluster transformations in their study of Donaldson–Thomas invariants and stability structures [33].

Let us recall the definition of the Calabi–Yau property for algebras and for triangulated categories: Let A be an (associative, unital) algebra over a field k. We identify A-bimodules with (right) modules over the enveloping algebra $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$. Let n be an integer. Recall that the algebra A is *homologically smooth* if, as a bimodule, it admits a finite resolution by finitely generated projective bimodules. Following Ginzburg and Kontsevich ([19], Definition 3.2.3), it is *n-Calabi–Yau as a bimodule* if it is homologically smooth and, in the derived category of A-bimodules, we have an isomorphism

$$f : A^{\vee} \xrightarrow{\sim} A \quad \text{such that } f^{\vee} = f,$$

where, for a bimodule complex M, we denote by $M^{\vee}$ the derived bimodule dual shifted by n degrees

$$M^{\vee} = \Sigma^n \operatorname{RHom}_{A^{\mathrm{e}}}(M, A^{\mathrm{e}}).$$

The bimodule complex $\operatorname{RHom}_{A^{\mathrm{e}}}(M, A^{\mathrm{e}})$ is the *inverse dualizing complex* of [44]. If A is n-Calabi–Yau as a bimodule, the subcategory $\mathscr{D}_{fd}(A)$ of the derived category $\mathscr{D}(A)$ formed by the modules whose homology is of finite total dimension is *n-Calabi–Yau as a triangulated category*, i.e. we have non-degenerate bifunctorial pairings

$$\langle\, , \rangle : \operatorname{Hom}(M, \Sigma^n L) \times \operatorname{Hom}(L, M) \to k$$

such that, for $p + q = n$, we have

$$\langle \Sigma^p f, g \rangle = (-1)^{pq} \langle \Sigma^q g, f \rangle$$

for all $f : M \to \Sigma^q L$ and $g : L \to \Sigma^p M$, cf. [30].

Let A be any homologically smooth algebra (or more generally: dg category), and let n be an integer. One of the main objects of study of this paper is a canonical dg algebra $\Pi_n(A)$ which we call the *n-Calabi–Yau completion* or the *derived n-preprojective algebra*. If θ denotes a projective resolution of the shifted bimodule dual $A^{\vee}$, we simply put $\Pi_n(A)$ equal to the tensor dg algebra

$$\Pi_n(A) = T_A(\theta) = A \oplus \theta \oplus (\theta \otimes_A \theta) \oplus \cdots.$$

Under Koszul duality, this construction corresponds to Ed Segal's cyclic completion [38]. If A is the path algebra of a connected non-Dynkin quiver and $n = 2$, one can show that $\Pi_n(A)$ is quasi-isomorphic to the preprojective algebra of A, cf. [30], Section 4.2. If A is the endomorphism algebra of a tilting object in the derived category of quasi-coherent sheaves

on a smooth algebraic variety X of dimension $n-1$ (or more generally, the derived endomorphism algebra of any compact generator [6]), then the derived category of $\Pi_n(A)$ is triangle equivalent to the derived category of quasi-coherent sheaves on the total space of the canonical bundle of X, cf. [40]. We will show that $\Pi_n(A)$ is always n-Calabi–Yau as a bimodule and that the construction $A \mapsto \Pi_n(A)$ is equivariant under derived Morita equivalences and compatible with (dg) localizations.

Let c be a Hochschild cycle of degree $n-2$ of A. It yields a canonically defined morphism $\delta : \theta \to A$ of degree 1. We define the *deformed n-Calabi–Yau completion* or *deformed derived n-preprojective algebra* $\Pi_n(A,c)$ to be obtained from $\Pi_n(A)$ by deforming the differential of the tensor algebra using δ. More intrinsically, the dg algebra $\Pi_n(A,c)$ can be constructed as a homotopy pushout from the Calabi–Yau completion $\Pi_{n-1}(A)$ as suggested in [14]. One can show that deformed preprojective algebras [13] of connected non-Dynkin quivers are obtained in this way for $n=2$. For $n=3$, the (non-complete) Ginzburg dg algebra (cf. [19], Section 4.2) associated with a quiver Q and a potential W becomes an example. Indeed, it is quasi-isomorphic to $\Pi_3(kQ,c)$, where c is the image of W, considered as an element of the zeroth cyclic homology of A, under Connes' map B. We refer to [19] for a wealth of examples related to the Ginzburg dg algebra. Our main results state that $\Pi_n(A,c)$ is n-Calabi–Yau as a bimodule and that the construction taking (A,c) to $\Pi_n(A,c)$ is equivariant under derived Morita equivalences and compatible with localizations. In particular, we obtain that the Ginzburg dg algebra is always 3-Calabi–Yau. When informed of this fact, Michel Van den Bergh provided an alternative proof [43], based on non-commutative geometry. He has kindly made his proof available in the appendix to this paper. The Calabi–Yau property of the Ginzburg dg algebra is an important ingredient of Amiot's construction [1] of the generalized cluster category associated to an algebra of global dimension $\leqq 2$ or a Jacobi-finite quiver with potential. This construction in turn is an important ingredient in the proof of the periodicity conjecture sketched in [23], Section 8.

We compute deformed Calabi–Yau completions of most 'homotopically finitely presented dg categories' (cf. Section 6.5 for the definition) and use this to show that deformed 3-Calabi–Yau completions of algebras of global dimension at most 2 are quasi-isomorphic to Ginzburg dg algebras. A related statement was proved independently by Ginzburg in [20]. As a corollary, we obtain that cluster-tilted algebras [10] are Jacobian algebras of quivers with potentials, a result that was proved independently by Buan–Iyama–Reiten–Smith [8] using completely different methods.

As an application of the derived Morita equivariance of the construction of deformed Calabi–Yau completions, we obtain a new construction of the derived equivalence associated [32] to the mutation of a quiver with potential [15]. Our approach also allows to generalize the mutation operation: For a given quiver Q, each tilting module over the path algebra kQ yields a 'generalized mutation' of any quiver with potential of the form (Q,W).

As an example of the localization theorem, we show that deleting a vertex in a quiver with potential translates into a localization of the associated Ginzburg algebra. In the case where the associated Jacobian algebras are finite-dimensional, this localization then yields a Calabi–Yau reduction [22] of the associated generalized cluster categories introduced by Amiot [1]. A related result was recently obtained by Amiot–Oppermann [2].

1.2. Contents. Each anti-involution $\tau : B \xrightarrow{\sim} B^{\mathrm{op}}$ of an algebra B allows one to define a preduality functor $M \mapsto \mathrm{Hom}_B(M, B)$ from the category of right A-modules *to itself* by letting B act on the target via τ. The most important example for us is the case where $B = A \otimes A^{\mathrm{op}}$ and $\tau(a \otimes a') = a' \otimes a$. Bimodule duality is confusing and the general context of an algebra with involution brings some clarification. We develop the necessary material in the setting of dg categories in Section 2.

We then introduce and study the inverse dualizing complex of a homologically smooth dg category in Section 3. We compute it for (most) homotopically finitely presented dg categories (Section 3.6) and show that it behaves well under derived Morita equivalences and localizations (Proposition 3.10). In particular, homological smoothness and the Calabi–Yau property are preserved under localizations.

We define n-Calabi–Yau completions in Section 4 and show that their construction is compatible with derived Morita equivalences and localizations (Proposition 4.2 and Theorem 4.6). We show that Calabi–Yau completions do have the Calabi–Yau property in Theorem 4.8. In Section 5, we construct deformed Calabi–Yau completions, prove that they have the Calabi–Yau property (Theorem 5.2), identify them with homotopy pushouts (Proposition 5.5) and show that their construction is compatible with derived Morita equivalences and localizations (Theorem 5.8).

After a reminder on Hochschild and cyclic homology of tensor categories (Section 6.1), we recall the definition of Ginzburg dg algebras in Section 6.2. We interpret them as deformed Calabi–Yau completions in Theorem 6.3. In Section 6.5, we observe that deformed Calabi–Yau completions of homotopically finitely presented dg categories are closely related to Ginzburg dg algebras. We use this in Theorem 6.10 to show that any deformed 3-Calabi–Yau completion of an algebra of global dimension $\leqq 2$ is a Ginzburg dg algebra. We apply this in Section 6.11 to show that all cluster-tilted algebras are Jacobian algebras.

In the final Section 7, we give two more applications of our general results to the study of mutations and of generalized cluster categories. In Corollary 7.3, we show that deleting a vertex in a quiver Q translates into a localization of the Ginzburg algebra associated with any quiver with potential of the form (Q, W). In Theorem 7.4 we prove that in the associated generalized cluster categories, the localization yields a Calabi–Yau reduction. We establish the link to Amiot–Oppermann's result in Section 7.5. Finally, in Section 7.6, we show that if (Q, W) is a quiver with potential and T any tilting module for the path algebra kQ, there is an associated 'generalized pre-mutation' for (Q, W). In particular, from the classical APR-tilts [4], one obtains the pre-mutation as defined in [15] and the associated derived equivalence of [32].

In the appendix, Michel Van den Bergh uses non-commutative differential geometry to give an alternative proof of the fact that Ginzburg dg algebras have the Calabi–Yau property.

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proof of the Calabi–Yau property for dg Ginzburg algebras as an appendix to this article. I am indebted to Claire Amiot and Steffen Oppermann for letting me know about their recent work and for pointing out a gap in a previous version of the proof of Theorem 7.4. I thank both of them as well as Ben Davison, Osamu Iyama, David Ploog, Raphaël Rouquier, Michel Van den Bergh and Dong Yang for stimulating questions and conversations.

2. Preduality functors

2.1. From involutions to preduality functors. Let k be a commutative ring and A an (associative, unital) k-algebra. Let τ be an *involution on* A, i.e. an isomorphism from A to the opposite algebra A^{op} whose square is the identity. Let $\operatorname{Mod} A$ denote the category of right A-modules. If M is a right A-module, the dual

$$A^* = \operatorname{Hom}_A(M, A)$$

becomes a left A-module via the left action of A on itself, that is to say, for an element $a \in A$ and an A-linear map f from M to A, we define af by

$$(af)(m) = af(m),$$

where m runs through the elements of M. Now for any left A-module N, we define the *conjugate right A-module* $\overline{N}$ to be the abelian group N endowed with the right action by A defined by

$$na = \tau(a)n,$$

where n is an element of N and a an element of A. In particular, if M is a right A-module, we obtain the *dual right A-module*

$$M^\vee = \overline{M^*}.$$

The functor

$$V : \operatorname{Mod} A \to (\operatorname{Mod} A)^{\mathrm{op}}$$

taking M to $VM = M^\vee$ together with the natural transformation

$$\varphi : M \to VVM$$

given by evaluation defines a *preduality functor* on the category $\operatorname{Mod} A$, i.e. the composition

$$V \xrightarrow{\varphi V} VVV \xrightarrow{V\varphi} V$$

is the identity. Equivalently, the map $f \mapsto \varphi \circ f^\vee$ is a bijection

$$\operatorname{Hom}_A(L, M^\vee) \xrightarrow{\sim} \operatorname{Hom}_A(M, L^\vee)$$

bifunctorial in the A-modules L and M. Notice that the left-hand side is in canonical bijection with the set of *sesquilinear forms on* $L \times M$, i.e. maps

$$s : L \times M \to A$$

such that $s(la, m) = \tau(a)s(l, m)$ and $s(l, ma) = s(l, m)a$ for all $l \in L$, $m \in M$ and $a \in A$. Similarly, the right-hand side is in bijection with the set of sesquilinear forms on $M \times L$. The bijection then corresponds to mapping a sesquilinear form s to the form $\tau \circ s \circ \sigma$, where σ exchanges the two factors of the product.

To say that (V, φ) is preduality is also equivalent to saying that the pair

$$\begin{array}{c} \operatorname{Mod} A \\ {\scriptstyle V^{\mathrm{op}}} \uparrow \;\big\downarrow {\scriptstyle V} \\ (\operatorname{Mod} A)^{\mathrm{op}} \end{array}$$

together with the morphisms

$$\varphi : VV^{\mathrm{op}} \to \mathrm{id} \quad \text{in } \operatorname{Mod} A \quad \text{and} \quad \varphi : \mathrm{id} \to V^{\mathrm{op}}V \quad \text{in } (\operatorname{Mod} A)^{\mathrm{op}}$$

is a pair of adjoint functors. So a preduality functor could also be called a self-coadjoint functor.

If (V, φ) is a preduality functor, then so is $(V, -\varphi)$. An A-module M is *reflexive for* V is φM is an isomorphism. For example, all finitely generated projective A-modules are reflexive. A duality functor is a preduality functor (V, φ) with invertible φ. The restriction of a preduality functor to the subcategory of reflexive objects is a duality functor.

2.2. Extension of preduality functors to module categories. Now let $\mathscr{A}$ be a k-category. By definition, the category $\operatorname{Mod}\mathscr{A}$ of (right) $\mathscr{A}$-modules is the category of k-linear functors

$$M : \mathscr{A}^{\mathrm{op}} \to \operatorname{Mod} k.$$

Suppose that V is a preduality functor on $\mathscr{A}$ and $\varphi : \mathrm{id} \to VV$ the corresponding adjunction morphism. A *left* $\mathscr{A}$*-module* is a k-linear functor $N : \mathscr{A} \to \operatorname{Mod} k$. Its *conjugate right module* is the composition $\overline{N} = N \circ V$. The *dual left module* M^* of a right $\mathscr{A}$-module M is the module given by

$$X \mapsto \operatorname{Hom}_{\mathscr{A}}\big(M, \mathscr{A}(?, X)\big),$$

where X runs through the objects of $\mathscr{A}$. The *dual* (or, more precisely, V*-dual*) of a right $\mathscr{A}$-module M is

$$M^\vee = \overline{M^*}.$$

It is given by

$$X \mapsto \mathrm{Hom}_{\mathscr{A}}\big(M, \mathscr{A}(?, VX)\big),$$

where X runs through the objects of $\mathscr{A}$. Let L and M be right modules. Then the set

$$\mathrm{Hom}(L, M^{\vee})$$

is in bijection with the set of *sesquilinear forms on* $L \times M$, i.e. the families of maps

$$s_{X,Y} : LY \times MX \to \mathscr{A}(X, VY)$$

bifunctorial in the objects X and Y of $\mathscr{A}$. By assumption on V, we have a canonical bifunctorial bijection

$$\theta : \mathscr{A}(X, VY) \to \mathscr{A}(Y, VX).$$

By taking $s_{X,Y}$ to $\theta \circ s_{X,Y} \circ \sigma$, where σ exchanges the two factors, we obtain a bifunctorial bijection

$$\mathrm{Hom}(L, M^{\vee}) \to \mathrm{Hom}(M, L^{\vee}).$$

It corresponds uniquely to a natural transformation

$$\tilde{\varphi} : M \to \tilde{V}\tilde{V}M,$$

where $\tilde{V}M = M^{\vee}$. We conclude that $(\tilde{V}, \tilde{\varphi})$ is a preduality functor on $\mathrm{Mod}\,\mathscr{A}$. Notice that for a representable module $\mathscr{A}(?, X)$, we have a canonical isomorphism

$$\tilde{V}\big(\mathscr{A}(?, X)\big) \xrightarrow{\sim} \mathscr{A}(X, V?) \xrightarrow{\sim} \mathscr{A}(?, VX)$$

and $\tilde{\varphi}$ is induced by φ for such modules. Thus the pair $(\tilde{V}, \tilde{\varphi})$ is a preduality functor which canonically extends (V, φ) from the subcategory of representable modules to all of $\mathrm{Mod}\,A$. By abuse of notation, we will often write (V, φ) instead of $(\tilde{V}, \tilde{\varphi})$.

2.3. Dg categories. Concerning dg categories, we follow the terminology and notations of [28]. Let us recall the most important points: We fix a commutative ground ring k. Let $\mathscr{A}$ be small dg k-category, i.e. a small category enriched over the tensor category $\mathscr{C}(k)$ of complexes over k. A dg $\mathscr{A}$*-module* is a dg functor

$$M : \mathscr{A}^{\mathrm{op}} \to \mathscr{C}_{\mathrm{dg}}(k)$$

with values in the dg category of complexes over k. In particular, each object X of $\mathscr{A}$ gives rise to the *free module* (= representable module) $X^{\wedge} = \mathscr{A}(?, X)$. The category of dg modules $\mathscr{C}(\mathscr{A})$ has as morphisms the morphisms of graded $\mathscr{A}$-modules, homogeneous of degree 0 which commute with the differential. It is endowed with a structure of Frobenius category whose conflations are the short exact sequences of dg modules which split as sequences of graded modules. The projective-injectives are the contractible dg modules. The associated stable category is the homotopy category $\mathscr{H}(\mathscr{A})$. It is triangulated and its suspension func-

tor takes a dg module M to $\Sigma M = M[1]$ whose underlying graded module has components $\big(M[1](X)\big)^p = M(X)^{p+1}$ and whose differential is $d_{M[1]} = -d_M$. The category of *strictly perfect* dg *modules* is the smallest subcategory of the Frobenius category $\mathscr{C}(\mathscr{A})$ which contains the free dg modules and is stable under shifts, extensions and passage to direct summands. The *derived category* $\mathscr{D}(\mathscr{A})$ is the localization of the category $\mathscr{H}(\mathscr{A})$ with respect to the class of quasi-isomorphisms. It is a triangulated category with suspension functor Σ. For each dg module M and each free module $X^\wedge$, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(X^\wedge, \Sigma^n M) = H^n\big(M(X)\big).$$

The derived category is compactly generated, in the sense of [35], by the free modules $X^\wedge$, $X \in \mathscr{A}$. An object of $\mathscr{D}(\mathscr{A})$ is defined to be *perfect* if it is a compact object. The *perfect derived category* $\mathsf{per}(\mathscr{A})$ is the full subcategory of perfect objects of $\mathscr{D}(\mathscr{A})$. A dg functor $F : \mathscr{A} \to \mathscr{B}$ is a *Morita functor* if restriction along F is an equivalence from $\mathscr{D}\mathscr{B}$ to $\mathscr{D}\mathscr{A}$. Equivalently, the total left derived functor of the induction along F is an equivalence. Still equivalently, the morphisms

$$\mathscr{A}(X, Y) \to \mathscr{B}(FX, FY)$$

are quasi-ismorphisms for all X, Y in $\mathscr{A}$ and the objects $F_*\mathscr{A}(?, X) = \mathscr{B}(?, FX)$ generate the perfect derived category $\mathsf{per}(\mathscr{B})$ as an idempotent complete triangulated category. In the localization of the category of dg categories with respect to the class of Morita functors, the set of morphisms from a dg category $\mathscr{A}$ to a dg category $\mathscr{B}$ is in canonical bijection with the set of isomorphism classes in $\mathscr{D}(\mathscr{A}^{\mathrm{op}} \otimes \mathscr{B})$ of dg $\mathscr{A}$-$\mathscr{B}$-bimodules X such that $X(?, A)$ is perfect as a dg $\mathscr{B}$-module for each object A of $\mathscr{A}$, cf. [41]. Two dg categories are *derived Morita equivalent* if they become isomorphic in this localization. Equivalently, they are linked by a chain of Morita functors.

2.4. Preduality functors on dg categories. Let $\mathscr{A}$ be a small dg category and (V, φ) a preduality dg functor on $\mathscr{A}$. Thus, V is a dg functor $\mathscr{A} \to \mathscr{A}^{\mathrm{op}}$ and $\varphi : \mathrm{id} \to VV$ a natural transformation such that the map $f \mapsto V(f) \circ \varphi$ is a bijection

$$\mathscr{A}(X, VY) \to \mathscr{A}(Y, VX)$$

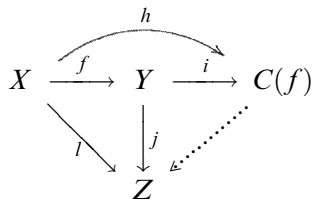
for all objects X and Y of $\mathscr{A}$. As in the case of the module category over a k-linear category treated in Section 2.2, we have a natural extension of (V, φ) to the category $\mathscr{C}_{\mathrm{dg}}(\mathscr{A})$ of (right) dg $\mathscr{A}$-modules.

Suppose from now on that $\mathscr{A}$ is an exact dg category. Recall that this means that the dg Yoneda functor

$$\mathscr{A} \to \mathscr{C}_{\mathrm{dg}}(\mathscr{A}), \quad X \mapsto X^\wedge$$

induces an equivalence onto a full subcategory which is stable under shifts and under graded split extensions. In particular, the category $\mathscr{A}$ then has a canonical shift functor Σ and each morphism f of $Z^0\mathscr{A}$ has a cone $C(f)$ whose image under the Yoneda functor is the cone on $f^\wedge$. In the underlying graded category $\mathscr{A}_{\mathrm{gr}}$, the cone on a morphism f from X to Y splits as $C(f) = Y \oplus \Sigma X$. Let $i : Y \to C(f)$ be the inclusion and $h : X \to C(f)$ the inclusion considered as a morphism of degree -1. Then the pair (i, h) is universal among

the pairs consisting of a closed morphism $j : Y \to Z$ and a morphism $l : X \to Z$ of degree -1 such that $j \circ f = d(l)$.



Since $\mathscr{A}$ is exact, the opposite dg category $\mathscr{A}^{\mathrm{op}}$ is also exact. If $f : X \to Y$ is a closed morphism in $\mathscr{A}$, we can form its cone $C'(f)$ in $\mathscr{A}^{\mathrm{op}}$. In $\mathscr{A}$, it is endowed with morphisms $i' : C(f) \to Y$ and $h' : C'(f) \to Y$ such that $f \circ i' = d(h')$ and which are universal with this property. It follows that $C'(f)$ splits as $\Sigma^{-1} Y \oplus X$ and that its differential is given by the matrix

$$\begin{bmatrix} -d_Y & f \\ 0 & d_X \end{bmatrix}.$$

Thus, the shift $\Sigma C'(f)$ endowed with the canonical morphisms $Y \to \Sigma C'(f)$ and $X \to \Sigma C'(f)$ is uniquely isomorphic to the cone $C(-f)$ on the *opposite of* f.

Since V is a dg functor, it preserves cones. So if $f : X \to Y$ is a closed morphism, we obtain a canonical isomorphism

$$\Sigma V C_{\mathscr{A}}(f) \xrightarrow{\sim} C(-Vf)$$

compatible with the (closed) inclusion i of VX and the inclusion h (homogeneous of degree -1) of VY.

Let n be an integer. Since V is a dg functor from $\mathscr{A}$ to $\mathscr{A}^{\mathrm{op}}$, we have a canonical isomorphism

$$V\Sigma^n \xrightarrow{\sim} \Sigma^{-n} V.$$

From φ, we get a canonical isomorphism

$$\psi : \mathrm{id} \to (\Sigma^n V)(\Sigma^n V)$$

and it is not hard to check that $(\Sigma^n V, \psi)$ is still a preduality dg functor.

Let X be an object of $\mathscr{A}$ and $f : X \to VX$ a closed morphism. The morphism f is (V, φ)-*symmetric* (respectively *antisymmetric*) if

$$f = V(f) \circ \varphi \quad (\text{respectively } f = -V(f) \circ \varphi).$$

The object X is *reflexive* (respectively *homotopy reflexive*) if $\varphi : X \to VVX$ is an isomorphism (respectively if $H^0(\varphi)$ is an isomorphism). The analogue of the following proposition in a triangulated setting is due to Balmer [5], Theorem 1.6:

Proposition 2.5. *The cone on a V-antisymmetric closed morphism carries a canonical ΣV-symmetric form. More precisely, let $f : X \to X^\vee$ be a closed and (V, φ)-antisymmetric morphism. Let*

$$g : C(f) \to \Sigma V\big(C(f)\big)$$

be given by the matrix

$$\begin{bmatrix} 0 & \mathrm{id} \\ \Sigma\varphi & 0 \end{bmatrix} : VX \oplus \Sigma X \to \Sigma VVX \oplus \Sigma V\Sigma X.$$

Then g is a closed $(\Sigma V, \psi)$-symmetric morphism. If X is (homotopy) reflexive, then g is invertible (up to homotopy).

Proof. By the above discussion and the assumption that $f = -V(f) \circ \varphi$, the morphism g is indeed well-defined and closed.

$$\begin{array}{ccccc} & & \overset{h}{\frown} & & \\ X & \xrightarrow{f} & VX & \xrightarrow{i} & C(f) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle g} \\ VVX & \xrightarrow{-Vf} & VX & \xrightarrow{i} & \Sigma VC(f) \\ & & \underset{h}{\smile} & & \end{array}$$

Clearly it is symmetric. We have a morphism of graded split exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & VX & \longrightarrow & C(f) & \longrightarrow & \Sigma X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle\Sigma\varphi} & & \\ 0 & \longrightarrow & VX & \longrightarrow & \Sigma VC(f) & \longrightarrow & \Sigma VVX & \longrightarrow & 0. \end{array}$$

This implies that $C(f)$ is reflexive if X is. By considering the corresponding triangles in $H^0(\mathscr{A})$, we obtain that $H^0(g)$ is an isomorphism if $H^0(\varphi X)$ is an isomorphism. □

Now let $g : Y \to VY$ be a closed symmetric morphism and suppose that $f : X \to Y$ is a closed morphism such that

$$(Vf) \circ g \circ f = 0.$$

We then have a complex of closed morphisms

$$X \xrightarrow{f} Y \xrightarrow{(Vf)\circ g} VX,$$

and we can form its totalization, i.e. the object Z such that for U in $\mathscr{A}$, the complex $\mathscr{A}(U, Z)$ is functorially isomorphic to the total complex of

$$\mathscr{A}(U, X) \xrightarrow{f_*} \mathscr{A}(U, Y) \xrightarrow{(Vf)_*\circ g_*} \mathscr{A}(U, VX),$$

where we think of $\mathscr{A}(U, Y)$ as the zeroth column of the double complex. The underlying graded object of Z is isomorphic to $\Sigma^{-1} VX \oplus Y \oplus \Sigma X$.

Proposition 2.6. *The graded morphism $h : Z \to VZ$ given by $\mathrm{id}_{VX} \oplus g \oplus \varphi_X$ is closed and V-symmetric. It is invertible* (*respectively invertible up to homotopy*) *if g is.*

Proof. We have a commutative diagram of complexes

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{(VF)\circ g} & VX \\ {\scriptstyle \varphi_X}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_X} \\ VVX & \xrightarrow[(VG)(VVf)]{} & VY & \xrightarrow[VF]{} & VX. \end{array}$$

Therefore the morphism h is closed. It is symmetric because g and $\mathrm{id}_X \oplus \varphi$ are symmetric. □

2.7. Induction and preduality. Let $\mathscr{A}$ and $\mathscr{B}$ be two dg categories each endowed with a dg preduality functor denoted by (V, φ). Let $F : \mathscr{A} \to \mathscr{B}$ be a dg functor. For a dg $\mathscr{A}$-module M, we denote by

$$F_* M \quad \text{or} \quad M \otimes_{\mathscr{A}} \mathscr{B},$$

its induction along F. We assume that we are given a morphism of dg functors

$$FV \to VF.$$

We wish to extend it to a compatibility morphism between induction along F and preduality with respect to V.

For each object X of $\mathscr{A}$, we have the representable left $\mathscr{A}$-module $\mathscr{A}(X, ?)$. Its image under induction along F is $\mathscr{B}(FX, ?)$ and the predual of the image is

$$\mathscr{B}(FX, V?) \xrightarrow{\sim} \mathscr{B}(?, VFX).$$

On the other hand, the predual of $\mathscr{A}(X, ?)$ is $\mathscr{A}(?, VX)$ and its image under induction is $\mathscr{B}(?, FVX)$. Thus, the given morphism $FV \to VF$ yields a natural transformation

$$F_*(M^\vee) \to (F_* M)^\vee$$

defined at first for representable and then for arbitrary dg $\mathscr{A}$-modules M.

If M is a right dg $\mathscr{A}$-module, then its dual

$$M^* : X \mapsto \mathrm{Hom}_{\mathscr{A}}\big(M, \mathscr{A}(?, X)\big)$$

is a left dg $\mathscr{A}$-module and we have a natural transformation

$$F_*(M^*) \to (F_* M)^*.$$

By composing the natural transformations constructed so far, we obtain, for each dg right $\mathscr{A}$-module M, a natural transformation

$$F_*(M^\vee) \to (F_* M^\vee)$$

or, in the other notation,

$$M^\vee \otimes_{\mathscr{A}} \mathscr{B} \to (M \otimes_{\mathscr{A}} \mathscr{B})^\vee. \tag{2.7.1}$$

Lemma 2.8. (a) *Under the natural transformation* (2.7.1), *an element* $f \otimes b$ *is sent to the map*

$$m \otimes x \mapsto (-1)^{|f||b|} V(b) f(m) x. \tag{2.8.1}$$

(b) *If the underlying graded* $\mathscr{A}$*-module of* M *is finitely generated projective, the transformation* (2.7.1) *is invertible and its inverse sends an element* g *to*

$$\sum m_i^* \otimes V\big(g(m_i \otimes \mathrm{id})\big),$$

where $\sum m_i \otimes m_i^*$ *is the Casimir element for* M, *i.e. the pre-image of the identity under the canonical isomorphism*

$$M \otimes_{\mathscr{A}} \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_{\mathscr{A}}(M, M).$$

Proof. These are straightforward verifications. □

Let $\mathscr{D}\mathscr{A}$ denote the derived category of $\mathscr{A}$. We still denote by $M \mapsto M^\vee$ the total derived functor of the duality functor and by $? \otimes_{\mathscr{A}} \mathscr{B}$ the total derived functor $\mathscr{D}\mathscr{A} \to \mathscr{D}\mathscr{B}$ of the induction functor.

Lemma 2.9. *Suppose that* $FV \to VF$ *is a pointwise homotopy equivalence. Then the morphism*

$$M^\vee \otimes_A \mathscr{B} \to (M \otimes_{\mathscr{A}} \mathscr{B})^\vee$$

is a quasi-isomorphism for all perfect M. *It is a quasi-isomorphism for all* M *if* $\mathscr{B}(F?, X)$ *is perfect over* $\mathscr{A}$ *for all* X *in* $\mathscr{B}$, *for example if* F *is a Morita functor.*

Proof. The canonical morphism

$$\varphi M : M^\vee \otimes_A \mathscr{B} \to (M \otimes_{\mathscr{A}} \mathscr{B})^\vee$$

is a quasi-isomorphism for each representable dg module $M = \mathscr{A}(?, X)$, by the assumption on F and V. Since φ is a morphism between triangle functors, it is still a quasi-isomorphism for each perfect dg module M. Finally, if $\mathscr{B}(F?, X)$ is perfect over $\mathscr{A}$ for all X in $\mathscr{B}$, then the derived tensor product $? \otimes_{\mathscr{A}} \mathscr{B}$ preserves arbitrary products. Then φ is a morphism between triangle functors taking arbitrary sums to products and hence is a quasi-isomorphism for each object M of $\mathscr{D}\mathscr{A}$. □

Now for a given right dg $\mathscr{A}$-module M, we wish to study the dg k-module

$$\mathrm{Hom}_{\mathscr{B}}\big(M \otimes_{\mathscr{A}} \mathscr{B}, (M \otimes_{\mathscr{A}} \mathscr{B})^{\vee}\big)$$

(whose n-th component is formed by the maps of graded $\mathscr{B}$-modules which are homogeneous of degree n). We can think of its elements as sesquilinear forms on $M \otimes_{\mathscr{A}} \mathscr{B}$. We have an isomorphism

$$\mathrm{Hom}_{\mathscr{B}}\big(M \otimes_{\mathscr{A}} \mathscr{B}, (M \otimes_{\mathscr{A}} \mathscr{B})^{\vee}\big) = \mathrm{Hom}_{\mathscr{A}}\big(M, (M \otimes_{\mathscr{A}} \mathscr{B})^{\vee}\big)$$

and the right-hand side is the target of a natural transformation with source

$$(M \otimes_{\mathscr{A}} \mathscr{B})^{\vee} \otimes_{\mathscr{A}} M^{*}.$$

Thus we obtain a natural transformation

$$M^{\vee} \otimes_{\mathscr{A}} \mathscr{B} \otimes_{\mathscr{A}} M^{*} \to \mathrm{Hom}_{\mathscr{B}}\big(M \otimes_{\mathscr{A}} \mathscr{B}, (M \otimes_{\mathscr{A}} \mathscr{B})^{\vee}\big). \tag{2.9.1}$$

Notice that the right-hand side carries a natural involution, namely the map taking f to $f^{\vee} \circ \varphi$. The left-hand side also carries a natural involution, namely the one which on tensors of homogeneous elements is given by

$$m_1 \otimes b \otimes m_2 \mapsto (-1)^{pq+pr+qr} m_2 \otimes Vb \otimes m_1,$$

where p, q, r are the degrees of m_1, f and m_2, respectively.

Lemma 2.10. *The map* (2.9.1) *is strictly compatible with these involutions.*

Proof. This is a straightforward verification. □

3. The inverse dualizing complex

3.1. Duality for bimodules. Let k be a commutative ring and $\mathscr{A}$ a dg k-category. We may and will assume that $\mathscr{A}$ is cofibrant over k, i.e. each morphism complex $\mathscr{A}(X, Y)$ is cofibrant in the category of dg k-modules. This always holds if k is a field. Let $\mathscr{A}^{\mathrm{e}}$ be the dg category $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$. We endow it with the involution V taking a pair of objects (X, Y) to (Y, X) and given on morphisms by

$$f \otimes g \mapsto (-1)^{pq} g \otimes f,$$

where f is of degree p and g of degree q. Note that (V, φ), where the morphism φ is the identity, is a preduality on $\mathscr{A}^{\mathrm{e}}$ in the sense of Section 2.4.

By a *bimodule* we mean a right dg module M over $\mathscr{A}^{\mathrm{e}}$. Via the morphism

$$M \otimes \mathscr{A}^{\mathrm{e}} = M \otimes (\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}) \xrightarrow{\sim} \mathscr{A}^{\mathrm{op}} \otimes M \otimes \mathscr{A}$$

taking $m \otimes (a \otimes b)$ to $(-1)^{|b|(|m|+|a|)} b \otimes m \otimes a$, the right $\mathscr{A}^{\mathrm{e}}$-module structure yields left and right $\mathscr{A}$-module structures on M. The right module structure on $\mathscr{A}^{\mathrm{e}}$ itself is given by the multiplication of $\mathscr{A}^{\mathrm{e}}$:

$$(f \otimes g)(f' \otimes g') = ff' \otimes g'g.$$

So right multiplication yields the 'inner' bimodule structure on $\mathscr{A}^{\mathrm{e}}$, whereas the left $\mathscr{A}^{\mathrm{e}}$-module structure on $\mathscr{A}^{\mathrm{e}}$ yields the 'outer' bimodule structure.

As we have seen in Section 2.4, from (V, φ), we obtain a natural preduality on the exact dg category of dg $\mathscr{A}^{\mathrm{e}}$-modules which takes a dg module M to the conjugate $M^{\vee}$ of the dual M^* defined by

$$M^* : (X, Y) \mapsto \mathrm{Hom}_{\mathscr{A}^{\mathrm{e}}}\big(M, \mathscr{A}^{\mathrm{e}}\big(?, (X, Y)\big)\big).$$

Lemma 3.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a* dg *functor and P an $\mathscr{A}$-bimodule. We identify $F_*P = P \otimes_{\mathscr{A}^{\mathrm{e}}} \mathscr{B}^{\mathrm{e}}$ with $\mathscr{B} \otimes_{\mathscr{A}} P \otimes_{\mathscr{A}} \mathscr{B}$ via the map $p \otimes (x \otimes y) \mapsto (-1)^{|y|\,|p \otimes x|} y \otimes p \otimes x$.*

(a) *The canonical morphism constructed in Section* 2.4

$$\mathscr{B} \otimes_{\mathscr{A}} P^{\vee} \otimes_{\mathscr{A}} \mathscr{B} \to (\mathscr{B} \otimes_{\mathscr{A}} P \otimes_{\mathscr{A}} \mathscr{B})^{\vee}$$

takes $b_1 \otimes f \otimes b_2$ to the map

$$x_1 \otimes p \otimes x_2 \mapsto \sum \pm b_1 f(p)_1 x_2 \otimes x_1 f(p)_2 b_2,$$

where the sign is given by the Koszul sign rule and $f(p) = \sum f(p)_1 \otimes f(p)_2$.

(b) *If the underlying graded module of P is finitely generated projective, the inverse*

$$(\mathscr{B} \otimes_{\mathscr{A}} P \otimes_{\mathscr{A}} \mathscr{B})^{\vee} \to \mathscr{B} \otimes_{\mathscr{A}} P^{\vee} \otimes_{\mathscr{A}} \mathscr{B}$$

of the morphism in (a) *takes a map g to*

$$\sum \pm g(p_i)_1 \otimes p_i^* \otimes g(p_i)_2,$$

where the sign is given by the Koszul sign rule, we have $g(p_i) = \sum g(p_i)_1 \otimes g(p_i)_2$ and $\sum p_i \otimes p_i^*$ is the Casimir element for P.

Proof. This is a special case of Lemma 2.8. □

3.3. Definition of the inverse dualizing complex. As in Section 3.1, we let k be a commutative ring and $\mathscr{A}$ a dg k-category which is cofibrant over k. We endow $\mathscr{A}^{\mathrm{e}} = \mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$ with the preduality (V, φ) of Section 3.1. By $\mathscr{A}$, we also denote the bimodule

$$(X, Y) \mapsto \mathscr{A}(X, Y).$$

We define the *inverse dualizing complex* $\Theta_{\mathscr{A}}$ to be any cofibrant replacement of the image of the bimodule $\mathscr{A}$ under the total derived functor of the preduality functor $M \mapsto M^{\vee}$ de-

fined in Section 3.1. Thus, if $\mathscr{A}$ is given by a dg algebra A, then $\Theta_{\mathscr{A}}$ is a cofibrant replacement of

$$\mathrm{RHom}_{A^e}(A, A^e)$$

considered as an object of $\mathscr{D}(A^e)$, i.e. a right dg A^e-module, via the canonical involution on A^e. Thus, the morphism set is computed using the 'inner' bimodule structure of A^e and the right A^e-action on Θ_A comes from the twisted right multiplication

$$(a \otimes b).(x \otimes y) = V(x \otimes y)(a \otimes b) = (y \otimes x)(a \otimes b) = ya \otimes bx,$$

which corresponds to the 'outer' bimodule structure. In this case, the homology $H^1\Theta_{\mathscr{A}}$ is the space of outer double derivations of A, i.e. the quotient of the space of derivations of A with values in A^e by the subspace of inner derivations. The inverse dualizing complex owes its name to the following lemma. Let $\mathscr{D}_{fd}(\mathscr{A})$ denote the full subcategory of $\mathscr{D}(\mathscr{A})$ formed by the dg modules M such that each dg k-module $M(X)$, $X \in \mathscr{A}$, is perfect. If k is a field and $\mathscr{A}$ is given by a dg algebra, this means that the sum $\sum_p \dim H^p(M)$ is finite.

Lemma 3.4. *Suppose that k is a field and $\mathscr{A}$ is homologically smooth. For any* dg *module L and any* dg *module M in $\mathscr{D}_{fd}(\mathscr{A})$, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathscr{D}\mathscr{A}}(L \otimes_{\mathscr{A}} \Theta_{\mathscr{A}}, M) \xrightarrow{\sim} D\,\mathrm{Hom}_{\mathscr{D}\mathscr{A}}(M, L),$$

where $D = \mathrm{Hom}_k(?, k)$. In particular, if $\Theta_{\mathscr{A}}$ is isomorphic to $\Sigma^{-n}\mathscr{A}$ in $\mathscr{D}(\mathscr{A}^e)$, then $\mathscr{D}_{fd}(\mathscr{A})$ is n-Calabi–Yau as a triangulated category.

Proof. This is a small variation of [30], Lemma 4.1. □

3.5. Quivers, tensor categories, cyclic derivatives. In this section, we collect preliminary material for the computation in Section 3.6. Let Q be a *graded k-quiver*, i.e. Q consists of a set of objects Q_0 and, for all objects x and y, a $\mathbb{Z}$-graded k-module $Q(x, y)$. Let $\mathscr{R}$ be the discrete k-category on Q_0: It has the set of objects Q_0, each endomorphism algebra is isomorphic to k and all morphisms between different objects vanish. By abuse of notation, we also denote by Q the $\mathscr{R}$-bimodule $(x, y) \mapsto Q(x, y)$. Recall that the tensor product $L \otimes_{\mathscr{R}} M$ of a right by a left $\mathscr{R}$-module is given by

$$(L \otimes_{\mathscr{R}} M)(x, y) = \coprod_z L(z, y) \otimes M(x, z),$$

where z ranges over the objects of $\mathscr{R}$. The *path category* of Q is the *tensor category* $T_{\mathscr{R}}(Q)$: It has the set of objects Q_0 and the bimodule of morphisms

$$\mathscr{R} \oplus Q \oplus (Q \otimes_{\mathscr{R}} Q) \oplus \cdots$$

with the natural composition. We put $\mathscr{A} = T_{\mathscr{R}}(Q)$.

Now assume that Q is finitely generated and free as an $\mathscr{R}^e$-module. Fix a basis α_i, $1 \leqq i \leqq n$, of Q and let $\sum \alpha_i \otimes \alpha_i^*$ be the *Casimir element* of the $\mathscr{R}^e$-bimodule Q, i.e. the preimage of the identity under the canonical isomorphism

$$Q \otimes_{\mathscr{R}^e} \mathrm{Hom}_{\mathscr{R}^e}(Q, \mathscr{R}^e) \to \mathrm{Hom}_{\mathscr{R}^e}(Q, Q).$$

The *cyclic derivative with respect to* α_i [37] is the unique map

$$\partial_{\alpha_i} : T_{\mathscr{R}}(Q) \to T_{\mathscr{R}}(Q),$$

taking a composition $\beta_1 \dots \beta_s$ of elements of Q to the sum

$$\sum_j \alpha_i^*(\beta_j)\beta_{j+1} \dots \beta_s\beta_1 \dots \beta_{j-1}.$$

3.6. Computation for a homotopically finitely presented dg category. Let k be a commutative ring and Q a graded k-quiver whose set of objects is finite and whose bimodule of morphisms is finitely generated and projective over k. Let $\mathscr{R}$ be the k-category with the same objects as Q and whose only non-zero morphisms are the scalar multiples of the identities. Let $\mathscr{A}$ be a dg category of the form $\big(T_{\mathscr{R}}(Q), d\big)$, where $T_{\mathscr{R}}(Q)$ is the tensor dg category (cf. Section 3.5)

$$\mathscr{R} \oplus Q \oplus (Q \otimes_{\mathscr{R}} Q) \oplus \dots \oplus (Q \otimes_{\mathscr{R}} \dots \otimes_{\mathscr{R}} Q) \oplus \dots$$

and the differential d is such that Q admits a finite filtration

$$F_0 \subset F_1 \subset F_2 \subset \dots \subset F_N = Q, \tag{3.6.1}$$

such that all F_p have the same objects as Q, the bimodule of arrows of F_0 vanishes and $d(F_p)$ is contained in $T_{\mathscr{R}}(F_{p-1})$ for all $p \geqq 1$. As shown in [42], cf. also [28], in the Morita homotopy category of dg categories, the dg category $\big(T_{\mathscr{R}}(Q), d\big)$ is homotopically finitely presented and every homotopically finitely presented dg category is a retract of such a dg category. Our aim in this section is to compute the inverse dualizing complex $\Theta_{\mathscr{A}}$ for $\mathscr{A} = \big(T_{\mathscr{R}}(Q), d\big)$. For this, we first need to construct a cofibrant resolution of $\mathscr{A}$ over $\mathscr{A}^{\mathrm{e}}$. Let $\tilde{\beta}$ be the unique bimodule derivation

$$\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A},$$

which takes an element $v : x \to y$ of Q to $\mathrm{id}_y \otimes v \otimes \mathrm{id}_x$. Notice that $\tilde{\beta}$ vanishes on $\mathscr{R} \subset \mathscr{A}$. If we have $n \geqq 1$ and $a = v_1 \dots v_n$ for elements $v_i : x_i \to x_{i-1}$ of Q, we have

$$\tilde{\beta}(a) = 1_{x_0} \otimes v_1 \otimes v_2 \dots v_n + \sum_{i=2}^{n-1} v_1 \dots v_{i-1} \otimes v_i \otimes v_{i+1} \dots v_n + v_1 \dots v_{n-1} \otimes v_n \otimes 1_{x_n}.$$

Let us denote by

$$\rho : \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \to \mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A}$$

the $\mathscr{A}$-bilinear extension of $\tilde{\beta}$. Notice that ρ is a retraction of the inclusion of $\mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A}$ into $\mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A}$. Let δ be the composition

$$\mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A} \xrightarrow{d} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \xrightarrow{\rho} \mathscr{A} \otimes_{\mathscr{R}} \otimes\, Q \otimes_{\mathscr{R}} \mathscr{A}.$$

Proposition 3.7. (a) *We have* $\delta^2 = 0$ *and* $\mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A}$ *endowed with* δ *is a cofibrant dg bimodule.*

(b) *The diagram*

$$0 \to \mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A} \xrightarrow{\tilde{\alpha}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \to \mathscr{A} \to 0,$$

where $\mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A}$ *is endowed with* δ *and*

$$\tilde{\alpha}(u \otimes v \otimes w) = uv \otimes w - u \otimes vw,$$

is a complex of dg *modules. The cone* $\mathbf{p}\mathscr{A}$ *over the morphism*

$$\mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A} \xrightarrow{\tilde{\alpha}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \tag{3.7.1}$$

is a cofibrant resolution of $\mathscr{A}$ *and is strictly perfect* (*cf. Section* 2.3). *In particular, the* dg *category* $\mathscr{A}$ *is homologically smooth.*

Remark 3.8. If instead of the finite filtration (3.6.1), we have a countable exhaustive filtration $F_0 \subset F_1 \subset \cdots \subset Q$ satisfying the same conditions, then the cone $\mathbf{p}\mathscr{A}$ of part (b) is still a cofibrant resolution of $\mathscr{A}$ (but $\mathscr{A}$ is no longer homologically smooth in general).

Proof. (a) Let us consider the commutator $d \circ \rho - \rho \circ d$ as a graded map from $\mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A}$ to itself. Its restriction to

$$\mathscr{A} \xrightarrow{\sim} \mathscr{R} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{R}$$

is a bimodule derivation. Since ρ is bilinear, the composition $\rho(d \circ \rho - \rho \circ d)$ still restricts to a bimodule derivation on $\mathscr{A}$. For $v \in Q$, we have

$$\rho(d \circ \rho - \rho \circ d)(v) = \rho d(v) - \rho^2 d(v) = 0.$$

Thus, the composition $\rho(d \circ \rho - \rho \circ d)$ vanishes on Q, thus on $\mathscr{A}$ and thus on $\mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A}$. It follows that we have

$$\delta^2 = \rho d \rho d = \rho^2 d^2 = 0.$$

To check that $(\mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A}, \delta)$ is cofibrant it suffices to observe that δ takes $\mathscr{A} \otimes_{\mathscr{R}} F_p \otimes_{\mathscr{R}} \mathscr{A}$ to $\mathscr{A} \otimes_{\mathscr{R}} F_{p-1} \otimes_{\mathscr{R}} \mathscr{A}$ for each $p \geqq 1$ and that the subquotient is a finitely generated free dg bimodule. Since the filtration by the F_p is finite, it also follows that $(\mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A}, \delta)$ is perfect. Since $\mathscr{R}$ is perfect over $\mathscr{R}^{\mathrm{e}}$ and

$$\mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} = \mathscr{R} \otimes_{\mathscr{R}^{\mathrm{e}}} \mathscr{A}^{\mathrm{e}},$$

it follows that the cone over

$$0 \to \mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A} \xrightarrow{\tilde{\alpha}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \to 0$$

is indeed cofibrant and perfect in $\mathscr{D}(\mathscr{A}^{\mathrm{e}})$. □

Let $\Theta_{\mathscr{A}} = (\mathbf{p}\mathscr{A})^\vee$ be the image under the preduality functor $M \mapsto M^\vee$ defined in Section 3.1 of the cofibrant resolution $\mathbf{p}\mathscr{A}$ given by the cone over the morphism

$$\mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A} \xrightarrow{\tilde{\alpha}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A}$$

of (3.7.1). Since the cone is stricly perfect, so is $\Theta_{\mathscr{A}}$. In particular, it is cofibrant and is therefore (homotopy equivalent to) the inverse dualizing complex. Let us make $\Theta_{\mathscr{A}}$ more explicit. By definition, $\Sigma\Theta_{\mathscr{A}}$ is isomorphic to the cone of the induced morphism

$$\mathrm{Hom}_{\mathscr{A}^{\mathrm{e}}}(\mathscr{A} \otimes_{\mathscr{R}} \mathscr{A}, \mathscr{A}^{\mathrm{e}}) \to \mathrm{Hom}_{\mathscr{A}^{\mathrm{e}}}(\mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A}, \mathscr{A}^{\mathrm{e}})$$

endowed with the bimodule structure coming from the 'outer' structure on $\mathscr{A}^{\mathrm{e}}$. Using Lemma 3.2, we obtain that $\Sigma\Theta$ is isomorphic to the cone over the morphism of dg modules

$$\mathscr{A} \otimes_{\mathscr{R}} \mathscr{R}^\vee \otimes_{\mathscr{R}} \mathscr{A} \to \mathscr{A} \otimes_{\mathscr{R}} Q^\vee \otimes_{\mathscr{R}} \mathscr{A}.$$

which takes an element $\mathrm{id}_x \otimes \mathrm{id}_x^* \otimes \mathrm{id}_x$ of $\mathscr{A} \otimes_{\mathscr{R}} \mathscr{R}^\vee \otimes_{\mathscr{R}} \mathscr{A}$ to

$$\mathrm{id}_x \big(\textstyle\sum (-1)^{|\alpha_i^*|} \alpha_i^{\otimes} \alpha_i \otimes \mathrm{id}_{x_i} - \mathrm{id}_{x_i} \otimes \alpha_i^* \otimes \alpha_i \big) \, \mathrm{id}_x,$$

where $\sum \mathrm{id}_x \otimes \mathrm{id}_x^*$ is the Casimir element of the $\mathscr{R}^{\mathrm{e}}$-module $\mathscr{R}$ and $\sum \alpha_i \otimes \alpha_i^*$ is the Casimir element of the $\mathscr{R}^{\mathrm{e}}$-module Q and $\alpha_i : x_i \to y_i$. The differential of $\mathscr{A} \otimes_{\mathscr{R}} \mathscr{R}^\vee \otimes_{\mathscr{R}}$ is that of the tensor product (where $\mathscr{R}^\vee$ carries the zero differential). To describe the differential of $\mathscr{A} \otimes_{\mathscr{R}} Q^\vee \otimes_{\mathscr{R}} \mathscr{A}$, we consider $\mathscr{A} \otimes_{\mathscr{R}} Q^\vee \otimes_{\mathscr{R}} \mathscr{A}$ as a dg submodule of the tensor algebra over $\mathscr{R}$ of $Q \oplus Q^\vee$. Then the differential of an element $\mathrm{id}_{x_i} \otimes \alpha_i^* \otimes \mathrm{id}_{y_i}$ equals the cyclic derivative (cf. Section 3.5) with respect to α_i of

$$W = \sum_j (-1)^{|\alpha_j|} \alpha_j^* d(\alpha_j).$$

This determines the differential because $\mathscr{A} \otimes_{\mathscr{R}} Q^\vee \otimes_{\mathscr{R}} \mathscr{A}$ is a dg $\mathscr{A}$-bimodule whose underlying graded module is generated by the elements $\mathrm{id}_{x_i} \otimes \alpha_i^* \otimes \mathrm{id}_{y_i}$.

3.9. Compatibility with Morita functors and localizations. Keep the hypotheses of Section 3.3. Let $\mathscr{B}$ be another dg category and $F : \mathscr{A} \to \mathscr{B}$ a dg functor. The dg functor F is a *localization functor* if the (total left derived functor of) induction along F induces an equivalence

$$(\mathscr{D}\mathscr{A})/\mathscr{N} \xrightarrow{\sim} \mathscr{D}\mathscr{B}$$

for some localizing subcategory $\mathscr{N}$ of $\mathscr{D}\mathscr{A}$ (namely the kernel of the induction functor). Equivalently, restriction along F is an equivalence from $\mathscr{D}\mathscr{B}$ onto a full subcategory of $\mathscr{D}\mathscr{A}$ (whose inclusion admits a left adjoint given by the induction functor). The localizations $F : \mathscr{A} \to \mathscr{B}$ such that the kernel $\mathscr{N}$ of the induced functor $F_* : \mathscr{D}(\mathscr{A}) \to \mathscr{D}(\mathscr{B})$ is compactly generated are precisely the dg quotients in the sense of Drinfeld [16], [26].

Proposition 3.10. *Assume that $F : \mathscr{A} \to \mathscr{B}$ is a localization functor.*

(a) *The functor $F^{\mathrm{e}} : \mathscr{A}^{\mathrm{e}} \to \mathscr{B}^{\mathrm{e}}$ induced by F is still a localization functor. It sends the bimodule $\mathscr{A}$ to the bimodule $\mathscr{B}$.*

(b) *The restriction* $(F^{\mathrm{e}})^*$ *along* F^{e} *is monoidal for the derived functors of the tensor products* $\otimes_{\mathscr{A}}$ *and* $\otimes_{\mathscr{B}}$ (*but does not preserve the unit in general*).

(c) *If* $\mathscr{A}$ *is homologically smooth, then so is* $\mathscr{B}$ *and the left derived functor of induction along* $F^{\mathrm{e}} : \mathscr{A}^{\mathrm{e}} \to \mathscr{B}^{\mathrm{e}}$ *sends* $\Theta_{\mathscr{A}}$ *to* $\Theta_{\mathscr{B}}$. *In particular, for each* dg $\mathscr{B}$*-module* L, *we have the projection formula*

$$F_*\big((F^*L) \overset{L}{\otimes}_{\mathscr{A}} \Theta_{\mathscr{A}}\big) \xrightarrow{\sim} L \overset{L}{\otimes}_{\mathscr{B}} \Theta_{\mathscr{B}}. \tag{3.10.1}$$

(d) *If the* dg *category* $\mathscr{A}$ *is homologically smooth and* n*-Calabi–Yau as a bimodule for some integer* n (*cf. Section* 4.7), *then* $\mathscr{B}$ *has the same properties.*

(e) *If* F *is even a Morita functor, so is* $F^{\mathrm{e}} : \mathscr{A}^{\mathrm{e}} \to \mathscr{B}^{\mathrm{e}}$ *and the induced equivalence* $\mathscr{D}(\mathscr{A}^{\mathrm{e}}) \to \mathscr{D}(\mathscr{B}^{\mathrm{e}})$ *is naturally a monoidal functor for the derived functors of the tensor products* $\otimes_{\mathscr{A}}$ *and* $\otimes_{\mathscr{B}}$. *It commutes with the total derived functors of the preduality functors and sends* $\Theta_{\mathscr{A}}$ *to* $\Theta_{\mathscr{B}}$.

Remark 3.11. If A is an (ordinary) algebra and $A \to B$ a localization of A in the sense that the induced functor

$$\mathsf{proj}(A) \to \mathsf{proj}(B)$$

between the categories of finitely generated projective modules is a localization of categories, it may well happen that A is homologically smooth but B is not. For example, if A is the path algebra of the quiver

$$1 \underset{x_2}{\overset{\varepsilon_2}{\rightrightarrows}} 2 \underset{x_1}{\overset{\varepsilon_1}{\rightrightarrows}} 3$$

over a field k, then A is finite-dimensional and of global dimension 2 but its localization B obtained by inverting x_1 and x_2 is the 2×2-matrix algebra over the algebra $k[\varepsilon]/(\varepsilon^2)$ of dual numbers. More generally, as shown in [36], every finitely presented k-algebra can be obtained in a similar way from a finite-dimensional algebra of global dimension at most 2. This is not in contradiction with part (c) of the proposition, because there, we consider *derived* localizations. In fact, in our example, the algebra B is the zeroth homology of the dg quotient $\tilde{\boldsymbol{B}}$ obtained from A by inverting x_1 and x_2 and this generalizes to the setup of [36].

Proof. Let us first describe the induction functor $\mathscr{D}(\mathscr{A}^{\mathrm{e}}) \to \mathscr{D}(\mathscr{B}^{\mathrm{e}})$ induced by F. For this, let us denote by X the $\mathscr{A}$-$\mathscr{B}$-bimodule $(A, B) \mapsto \mathscr{B}(B, FA)$ and by X' the $\mathscr{B}$-dual bimodule $(B, A) \mapsto \mathscr{B}(FA, B)$, which is isomorphic to $\mathrm{RHom}_{\mathscr{B}}(X, \mathscr{B})$. Then the induction along F is isomorphic to the derived tensor product with X and the restriction along F is isomorphic to the derived tensor product with X'. From the fact that F is a localization functor, it follows that the canonical morphism

$$X' \overset{L}{\otimes}_{\mathscr{A}} X \to \mathscr{B}$$

is an isomorphism in $\mathscr{D}(\mathscr{B}^{\mathrm{e}})$. Moreover, since X is perfect over $\mathscr{B}$, the canonical morphism

$$X \overset{L}{\otimes}_{\mathscr{B}} X' \to \mathrm{RHom}_{\mathscr{B}}(X, X)$$

is an isomorphism in $\mathscr{D}(\mathscr{A}^{\mathrm{e}})$. The action of $\mathscr{A}$ on X yields a bimodule morphism $\mathscr{A} \to \mathrm{RHom}_{\mathscr{B}}(X, X)$ and thus a morphism

$$\mathscr{A} \to X \overset{L}{\otimes}_{\mathscr{B}} X'$$

in $\mathscr{D}(\mathscr{A}^{\mathrm{e}})$. Now we can describe the induction functor $\mathscr{D}(\mathscr{A}^{\mathrm{e}}) \to \mathscr{D}(\mathscr{B}^{\mathrm{e}})$: It is isomorphic to

$$M \mapsto X' \overset{L}{\otimes}_{\mathscr{A}} M \overset{L}{\otimes}_{\mathscr{A}} X.$$

In particular, the bimodule $M = \mathscr{A}$ is sent to $X' \overset{L}{\otimes}_{\mathscr{A}} X \xrightarrow{\sim} \mathscr{B}$. The restriction functor $\mathscr{D}(\mathscr{B}^{\mathrm{e}}) \to \mathscr{D}(\mathscr{A}^{\mathrm{e}})$ is isomorphic to

$$N \mapsto X \overset{L}{\otimes}_{\mathscr{B}} N \overset{L}{\otimes}_{\mathscr{B}} X'.$$

Since $X' \overset{L}{\otimes}_{\mathscr{A}} X$ is isomorphic to $\mathscr{A}$, this shows part (b): the restriction functor is monoidal. If we compose it with the induction functor, we find the identity functor because $X' \overset{L}{\otimes}_{\mathscr{A}} X \xrightarrow{\sim} \mathscr{B}$. It follows that the induction functor $\mathscr{D}(\mathscr{A}^{\mathrm{e}}) \to \mathscr{D}(\mathscr{B}^{\mathrm{e}})$ is a localization functor and sends $\mathscr{A}$ to $\mathscr{B}$, which is part (a). If $\mathscr{A}$ is homologically smooth, then $\mathscr{A}$ is perfect in $\mathscr{D}(\mathscr{A}^{\mathrm{e}})$ and so its dual $\Theta_{\mathscr{A}}$ is sent to the dual $\Theta_{\mathscr{B}}$ of its image $\mathscr{B}$, by Lemma 2.9. Thus, we have

$$X^* \otimes_{\mathscr{A}} \Theta_{\mathscr{A}} \otimes_{\mathscr{A}} X \xrightarrow{\sim} \Theta_{\mathscr{B}}.$$

By applying $L \overset{L}{\otimes}_{\mathscr{A}} ?$ to this isomorphism, we get the projection formula (3.10.1). This ends the proof of (c). Part (d) is immediate from (c) and (a).

Let us prove (e): If F is a Morita functor, the canonical morphism $\mathscr{A} \to X \overset{L}{\otimes}_{\mathscr{B}} X'$ is also invertible and then the description of the induction functor via X and X' shows that it is monoidal. The commutation of the induction functor with the preduality functor follows from Lemma 2.9. Now the last assertion follows from (a). □

4. Calabi–Yau completions

4.1. Definition and Morita equivariance. Let k be a commutative ring and $\mathscr{A}$ a dg k-category whose morphism complexes are cofibrant over k. Let n be an integer and $\Theta = \Theta_{\mathscr{A}}$ the inverse dualizing complex of Section 3.3. Put $\theta = \Sigma^{n-1}\Theta_{\mathscr{A}}$. The *n-Calabi–Yau completion of* $\mathscr{A}$ is the tensor dg category

$$\Pi_n(\mathscr{A}) = T_{\mathscr{A}}(\theta) = \mathscr{A} \oplus \theta \oplus (\theta \otimes_{\mathscr{A}} \theta) \oplus \cdots.$$

We also call it the *derived n-preprojective* dg *category of* $\mathscr{A}$ (whence the notation Π_n). Notice that we have canonical inclusion and projection functors

$$\mathscr{A} \to \Pi_n(\mathscr{A}) \to \mathscr{A}.$$

Up to a quasi-isomorphism (canonical up to homotopy), it is independent of the choice of cofibrant replacement made in the definition of $\Theta_{\mathscr{A}}$.

Proposition 4.2. *Let $F : \mathscr{A} \to \mathscr{B}$ be a Morita functor. Then F yields a canonical Morita functor $\Pi_n(F) : \Pi_n(\mathscr{A}) \to \Pi_n(\mathscr{B})$ such that we have a commutative diagram*

$$\begin{array}{ccccc} \mathscr{A} & \longrightarrow & \Pi_n(\mathscr{A}) & \longrightarrow & \mathscr{A} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Pi_n(F)} & & \downarrow{\scriptstyle F} \\ \mathscr{B} & \longrightarrow & \Pi_n(\mathscr{B}) & \longrightarrow & \mathscr{B}. \end{array}$$

Proof. Let F^{e} be the induced functor from $\mathscr{A}^{\mathrm{e}}$ to $\mathscr{B}^{\mathrm{e}}$ and denote by $F^{\mathrm{e}*}$ the restriction along F^{e}. By part (e) of Proposition 3.10, we can find a quasi-isomorphism $\varphi : \Theta_{\mathscr{A}} \to F^{\mathrm{e}*}\Theta_{\mathscr{B}}$ and by part (a), it induces quasi-isomorphisms between the (derived) tensor powers

$$\theta_{\mathscr{A}}^{\otimes_{\mathscr{A}} n} \to F^*(\theta_{\mathscr{B}}^{\otimes_{\mathscr{B}} n})$$

for all $n \geqq 1$. Thus, the pair (F, φ) yields a dg functor

$$\Pi_n(F) : T_{\mathscr{A}}(\theta_{\mathscr{A}}) \to T_{\mathscr{B}}(\theta_{\mathscr{B}}),$$

which is quasi-fully faithful. It remains to be shown that the image generates the derived category of $\Pi_n(\mathscr{B})$. Now clearly the image contains all representable functors $\Pi_n(\mathscr{B})(?, FX)$ associated with objects FX in the image of F. But for an arbitrary object M of the derived category of $\Pi_n(\mathscr{B})$, we have

$$\mathrm{Hom}\big(\Pi_n(\mathscr{B})(?, FX), M\big) = \mathrm{Hom}_{\mathscr{B}}\big(\mathscr{B}(?, FX), M \,|\, \mathscr{B}\big) = M(FX).$$

Now since F is a Morita functor, the object M vanishes iff $M(FX)$ is acyclic for all X in $\mathscr{A}$. Thus, the right orthogonal of the image of $\Pi_n(F)$ vanishes and so the image is all of the derived category. □

4.3. Morphisms between restrictions. We keep the notations and assumptions of Section 4.1. Let $i : \mathscr{D}(\mathscr{A}) \to \mathscr{D}\big(\Pi_n(\mathscr{A})\big)$ be the restriction along the projection onto the first component $\Pi_n(\mathscr{A}) \to \mathscr{A}$.

Lemma 4.4. *Let L and M be in $\mathscr{D}\mathscr{A}$.*

(a) *We have a canonical isomorphism*

$$\mathrm{RHom}_{\Pi_n(\mathscr{A})}(iL, iM) = \mathrm{RHom}_{\mathscr{A}}(L, M) \oplus \Sigma^{-n}\,\mathrm{RHom}_{\mathscr{A}}(L \otimes_{\mathscr{A}} \Theta_{\mathscr{A}}, M),$$

where $\Theta_{\mathscr{A}}$ is the inverse dualizing complex (*Section* 3.3).

(b) *If k is a field, $\mathscr{A}$ is homologically smooth and M belongs to $\mathscr{D}_{fd}(\mathscr{A})$* (*cf. Section* 3.3), *we have a canonical isomorphism*

$$\mathrm{RHom}_{\Pi_n(\mathscr{A})}(iL, iM) = \mathrm{RHom}_{\mathscr{A}}(L, M) \oplus \Sigma^{-n} D\,\mathrm{RHom}_{\mathscr{A}}(M, L),$$

where D is the duality functor $\mathrm{Hom}_k(?, k)$.

Proof. We may and will assume that L is cofibrant over $\mathscr{A}$. Then we have an exact sequence of dg modules over $\Pi_n(\mathscr{A}) = T_{\mathscr{A}}(\theta)$:

$$0 \to (iL) \otimes_{\mathscr{A}} \theta \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) \xrightarrow{\alpha} (iL) \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) \xrightarrow{\beta} iL \to 0, \tag{4.4.1}$$

where α takes $l \otimes x \otimes u$ to $lx \otimes u - l \otimes xu$ and β is the multiplication of iL. Clearly the cone over α is a cobrant resolution $\mathbf{p}(iL)$ of iL over $T_{\mathscr{A}}(\theta)$. Since θ acts by zero in iL and iM, the morphism α induces zero in $\mathrm{Hom}_{T_{\mathscr{A}}(\theta)}(?, iM)$. So we find a canonical isomorphism in the derived category of k-modules

$$\mathrm{Hom}_{T_{\mathscr{A}}(\theta)}\big(\mathbf{p}(iL), iM\big) = \mathrm{Hom}_{\mathscr{A}}(L, M) \oplus \Sigma^{-1} \mathrm{Hom}_{\mathscr{A}}(L \otimes_{\mathscr{A}} \theta, M).$$

This implies part (a). Part (b) follows from part (a) and Lemma 3.4. □

4.5. Compatibility with localizations. We keep the notations and assumptions of Section 4.1. We say that a sequence of dg categories

$$0 \to \mathscr{N} \xrightarrow{G} \mathscr{A} \xrightarrow{F} \mathscr{B} \to 0$$

is *exact* if the induced sequence

$$0 \to \mathscr{D}(\mathscr{N}) \xrightarrow{G_*} \mathscr{D}(\mathscr{A}) \xrightarrow{F_*} \mathscr{D}(\mathscr{B}) \to 0$$

is exact, i.e. the composition vanishes, $\mathscr{D}(\mathscr{N})$ identifies with a full triangulated subcategory of $\mathscr{D}(\mathscr{A})$ and the triangle quotient of $\mathscr{D}(\mathscr{A})$ by $\mathscr{D}(\mathscr{N})$ identifies via F_* with $\mathscr{D}(\mathscr{B})$. In this case, the dg functor $F : \mathscr{A} \to \mathscr{B}$ is a localization in the sense of Section 3.9 (but not each localization is obtained in this way as shown in [24]).

Theorem 4.6. *Assume that $\mathscr{A}$ is homologically smooth.*

(a) *Let $F : \mathscr{A} \to \mathscr{B}$ be a localization functor. Then F yields a canonical localization functor $\Pi_n(F) : \Pi_n(\mathscr{A}) \to \Pi_n(\mathscr{B})$ such that we have a commutative diagram*

$$\begin{array}{ccccc} \mathscr{A} & \longrightarrow & \Pi_n(\mathscr{A}) & \longrightarrow & \mathscr{A} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \Pi_n(F)} & & \downarrow{\scriptstyle F} \\ \mathscr{B} & \longrightarrow & \Pi_n(\mathscr{B}) & \longrightarrow & \mathscr{B}. \end{array}$$

(b) *If we have an exact sequence of* dg *categories*

$$0 \to \mathscr{N} \xrightarrow{G} \mathscr{A} \xrightarrow{F} \mathscr{B} \to 0,$$

then the kernel of the functor $\Pi_n(F)_* : \mathscr{D}\big(\Pi_n(\mathscr{A})\big) \to \mathscr{D}\big(\Pi_n(\mathscr{B})\big)$ is the localizing subcategory generated by the objects $\Pi_n(\mathscr{A})(?, N)$, $N \in \mathscr{N}$.

Proof. We may and will assume that $F : \mathscr{A} \to \mathscr{B}$ is the identity on the set of objects. Let $(F^{\mathrm{e}})^* : \mathscr{C}(\mathscr{B}^{\mathrm{e}}) \to \mathscr{C}(\mathscr{A}^{\mathrm{e}})$ be the restriction functor. Let us put $\Theta'_{\mathscr{B}} = (F^{\mathrm{e}})^*(\Theta_{\mathscr{B}})$. Notice that for any objects A, A' of $\mathscr{A}$ (equivalently: $\mathscr{B}$), we have $\Theta'_{\mathscr{B}}(A, A') = \Theta_{\mathscr{B}}(A, A')$ and that in $\Theta'_{\mathscr{B}}$, the morphisms of $\mathscr{A}$ act via $F : \mathscr{A} \to \mathscr{B}$. According to part (c) of Proposition 3.10,

we have a canonical morphism of dg modules $\varphi : \theta_{\mathscr{A}} \to \theta'_{\mathscr{B}}$ whose image under the induction along F^{e} is invertible in $\mathscr{D}(\mathscr{B}^{\mathrm{e}})$. The morphism φ yields morphisms of dg modules between the tensor powers

$$\theta_{\mathscr{A}} \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \theta_{\mathscr{A}} \to \theta'_{\mathscr{B}} \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \theta'_{\mathscr{B}} \to \theta_{\mathscr{B}} \otimes_{\mathscr{B}} \cdots \otimes_{\mathscr{B}} \theta_{\mathscr{B}}.$$

Thus, the pair (F, φ) yields a dg functor $G : \Pi_n(\mathscr{A}) \to \Pi_n(\mathscr{B})$. Clearly, G is compatible with the canonical inclusion and projection functors. It remains to be shown that the restriction along G is a fully faithful functor

$$\mathscr{D}\big(\Pi_n(\mathscr{B})\big) \to \mathscr{D}\big(\Pi_n(\mathscr{A})\big).$$

Let L be a dg $\Pi_n(\mathscr{B})$-module. It is given by its underlying dg $\mathscr{B}$-module and a morphism of dg $\mathscr{B}$-modules

$$\lambda : L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \to L.$$

The dg module G^*L is given by the restriction of L to $\mathscr{A}$ and the morphism of dg $\mathscr{A}$-modules deduced from λ and φ

$$L \otimes_{\mathscr{A}} \theta_{\mathscr{A}} \xrightarrow{\mathrm{id}\otimes\varphi} L \otimes_{\mathscr{A}} \theta_{\mathscr{B}} \xrightarrow{\mathrm{can}} L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \xrightarrow{\lambda} L.$$

Let us use this description of G^* to show that it is fully faithful. Let L be a dg $\Pi_n(\mathscr{B})$-module. We may and will assume that L is cofibrant. Since $\Pi_n(\mathscr{B})$ is cofibrant as a right dg $\mathscr{B}$-module, the restriction of L to $\mathscr{B}$ is then cofibrant. We have an exact sequence of cofibrant dg $\Pi_n(\mathscr{B})$-modules

$$0 \to L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}}) \xrightarrow{\alpha} L \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}}) \to L \to 0,$$

where $\alpha(l \otimes x \otimes u) = lx \otimes u - l \otimes xu$. This makes it clear that the cone over the morphism

$$L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}}) \to L \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}})$$

is homotopy equivalent to L. Let M be another dg $\Pi_n(\mathscr{B})$-module. By applying $\mathrm{Hom}_{\mathscr{B}}(?, M)$ to the above morphism, we obtain a morphism of dg k-modules

$$\mathrm{Hom}_{\mathscr{B}}(L, M) \to \mathrm{Hom}_{\mathscr{B}}(L \otimes_{\mathscr{B}} \theta_{\mathscr{B}}, M)$$

whose cone (shifted by one degree to the right) computes morphisms from L to M in the derived category of $\Pi_n(\mathscr{B})$. An analogous reasoning yields the morphisms between G^*L and G^*M in the derived category of $\Pi_n(\mathscr{A})$. Thus, to conclude that G^* is fully faithful, it suffices to check that for all M, F^* induces bijections

$$\mathrm{Hom}_{\mathscr{D}(\mathscr{B})}(L, M) \to \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(F^*L, F^*M)$$

and

$$\mathrm{Hom}_{\mathscr{D}(\mathscr{B})}(L \overset{L}{\otimes}_{\mathscr{B}} \theta_{\mathscr{B}}, M) \to \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}\big(F^*(L) \overset{L}{\otimes}_{\mathscr{A}} \theta_{\mathscr{A}}, F^*M\big).$$

The first bijection follows from the full faithfulness of F^*. The second one is a consequence of the full faithfulness of F^* and of the projection formula (3.10.1). This ends the proof of (a). To prove (b), it suffices to show that the image of $\Pi_n(F)^*$ is exactly the full subcategory of the dg modules over $\Pi_n(\mathscr{A})$ which are right orthogonal to all the representable dg modules $\Pi_n(\mathscr{A})(?, N)$ for N in $\mathscr{N}$. We have

$$\mathrm{RHom}_{\Pi_n(\mathscr{A})}\big(\Pi_n(\mathscr{A})(?, N), M\big) = \mathrm{RHom}_{\mathscr{A}}\big(\mathscr{A}(?, N), M\big),$$

which shows that if M is in the image of $\Pi_n(F)^*$, it is right orthogonal to the $\Pi_n(\mathscr{A})(?, N)$. Conversely, if M satisfies this condition, then the underlying $\mathscr{A}$-module of M is quasi-isomorphic to F^*L for some dg $\mathscr{B}$-module L. The structural morphism

$$M \otimes_{\mathscr{A}} \theta_{\mathscr{A}} \to M,$$

then yields a morphism $F^*L \otimes_{\mathscr{A}} \theta_{\mathscr{A}} \to F^*L$ hence a morphism

$$F_*(F^*L \otimes_{\mathscr{A}} \theta_{\mathscr{A}}) \to L$$

and thus by the projection formula (3.10.1), a morphism

$$L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \to L.$$

Thus, L carries a canonical structure of dg module over $\Pi_n(\mathscr{B})$ and it is clear that M is isomorphic to the image under $\Pi_n(F)^*$ of L endowed with this structure. □

4.7. The Calabi–Yau property. We keep the notations and assumptions of Section 4.1. In particular, the symbol n denotes a fixed integer. On the category of dg $\mathscr{A}^{\mathrm{e}}$-modules, we consider the composition V_n of the preduality functor V with the shift Σ^n. It is part of a canonical preduality functor (V_n, φ_n) (by Section 2.4). We also use the notation V_n for the derived functor of V_n. Slightly modifying the terminology of Ginzburg and Kontsevich (cf. [19], Definition 3.2.3), we say that the dg category $\mathscr{A}$ is *n-Calabi–Yau as a bimodule* if, in $\mathscr{D}(\mathscr{A}^{\mathrm{e}})$, there is an isomorphism

$$f : \mathscr{A} \to V_n\mathscr{A},$$

which is (V_n, φ_n)-symmetric, i.e. such that $V_n(f)\varphi_n = f$.

Theorem 4.8. *If $\mathscr{A}$ is homologically smooth, its n-Calabi–Yau completion $\Pi_n(\mathscr{A})$ is homologically smooth and n-Calabi–Yau as a bimodule.*

Proof. Let $\mathscr{B}$ be the n-Calabi–Yau completion. We have a short exact sequence of $\mathscr{B}^{\mathrm{e}}$-modules

$$0 \to T_{\mathscr{A}}(\theta) \otimes_{\mathscr{A}} \theta \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) \xrightarrow{\alpha} T_{\mathscr{A}}(\theta) \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) \to T_{\mathscr{A}}(\theta) \to 0,$$

where the morphism α takes an element f of $\theta(X, Y)$ to $1_Y \otimes f - f \otimes 1_X$ and the second map is composition. Thus, in the derived category of $\mathscr{B}^{\mathrm{e}}$, the bimodule $T_{\mathscr{A}}(\theta)$ is isomorphic to the cone on the morphism α. We deduce first that $T_{\mathscr{A}}(\theta)$ is perfect as a bimodule: Indeed,

the objects

$$T_{\mathscr{A}}(\theta) \otimes_{\mathscr{A}} \theta \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) = \theta \otimes_{\mathscr{A}^e} \mathscr{B}^e \quad \text{and} \quad T_{\mathscr{A}}(\theta) \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) = \mathscr{A} \otimes_{\mathscr{A}^e} \mathscr{B}^e$$

are perfect since they are induced from perfect $\mathscr{A}^e$-modules (all tensor products are also derived tensor products since $\mathscr{B}^e$ is cofibrant over $\mathscr{A}^e$).

To prove the second part of the assertion, we first notice that θ is the V_{n-1}-dual of $\mathscr{A}$. Since the bimodule $\mathscr{A}$ is perfect, it is homotopically V_{n-1}-reflexive and so, up to homotopy, $\mathscr{A}$ is also the V_{n-1}-dual of θ. By Lemma 2.9, for perfect modules, the induction functor $? \otimes_{\mathscr{A}^e} \mathscr{B}^e$ commutes with the preduality V_n up to isomorphism in the derived category. Thus, in $\mathscr{D}(\mathscr{B}^e)$, the objects

$$\theta \otimes_{\mathscr{A}^e} \mathscr{B} \quad \text{and} \quad \mathscr{A} \otimes_{\mathscr{A}^e} \mathscr{B}$$

are still V_{n-1}-dual to each other. So by Proposition 2.5, in order to show that $\mathscr{B}$ is n-Calabi–Yau as a bimodule, it suffices to show that α is V_{n-1}-antisymmetric. Now as seen in Section 2.7, we have a natural homotopy equivalence

$$V_{n-1}(\theta) \otimes_{\mathscr{A}^e} \mathscr{B}^e \otimes_{\mathscr{A}^e} \theta^* \to \mathrm{Hom}_{\mathscr{B}^e}\big(\theta \otimes_{\mathscr{A}^e} \mathscr{B}^e, V_{n-1}(\theta \otimes_{\mathscr{A}^e} \mathscr{B}^e)\big).$$

The right-hand side is quasi-isomorphic to the following dg k-modules:

$$\mathrm{Hom}_{\mathscr{A}^e}\big(\theta, V_{n-1}(\theta) \otimes_{\mathscr{A}^e} \mathscr{B}^e\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{A}^e}(\theta, \mathscr{A} \otimes_{\mathscr{A}^e} \mathscr{B}^e) = \mathrm{Hom}_{\mathscr{A}^e}(\theta, \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B}),$$

where we use the fact that θ is cofibrant. So we get a natural quasi-isomorphism

$$V_{n-1}(\theta) \otimes_{\mathscr{A}^e} \mathscr{B}^e \otimes_{\mathscr{A}^e} \theta^* \to \mathrm{Hom}_{\mathscr{A}^e}(\theta, \mathscr{B} \otimes_{\mathscr{A}} \mathscr{B}).$$

Let us lift the morphism $\lambda : x \mapsto 1 \otimes x$ along this quasi-isomorphism: Let c be the Casimir element in $\theta \otimes_{\mathscr{A}^e} \theta^*$, i.e. the image of $1 \in k$ under the morphism

$$k \to \mathrm{Hom}_{\mathscr{A}^e}(\theta, \theta) \xrightarrow{\sim} \theta \otimes_{\mathscr{A}^e} \theta^*.$$

We let $\tilde{\lambda}$ be the image of $\mathrm{id} \otimes c$ under the composition

$$(V_{n-1}\theta) \otimes_{\mathscr{A}^e} (\mathscr{A}^{\mathrm{op}} \otimes_k \theta) \otimes_{\mathscr{A}^e} \theta^* \to (V_{n-1}\theta) \otimes_{\mathscr{A}^e} (\mathscr{B}^{\mathrm{op}} \otimes_k \mathscr{B}) \otimes_{\mathscr{A}^e} \theta^*.$$

Then clearly $\tilde{\lambda}$ maps to λ and the transpose conjugate of $\tilde{\lambda}$ maps to $\rho : x \mapsto x \otimes 1$. Since α equals $\rho - \lambda$, it follows that α is indeed V_{n-1}-antisymmetric. □

5. Deformed Calabi–Yau completions

5.1. Construction and Calabi–Yau property. Let k be a commutative ring and $\mathscr{A}$ a dg k-category such that $\mathscr{A}(X, Y)$ is cofibrant as a dg k-module for all objects X and Y of $\mathscr{A}$. We assume that $\mathscr{A}$ is homologically smooth. Let Θ be the inverse dualizing complex (cf. Section 3.3), n an integer, $\theta = \Sigma^{n-1}\Theta$ and $\Pi_n(\mathscr{A}) = T_{\mathscr{A}}(\theta)$ the n-Calabi–Yau completion. It is natural to deform $\Pi_n(\mathscr{A})$ by adding an $\mathscr{A}$-bilinear (super-)derivation D of degree 1 to

its differential. Such a derivation is determined by its restriction to the generating bimodule θ. It has to satisfy

$$0 = (d + D)^2 = d(D) + D^2.$$

Since the right-hand side is a degree 2 derivation, it suffices to check this identity on the generating bimodule θ. Assume that D takes θ to $\mathscr{A} \subset T_{\mathscr{A}}(\theta)$. Then D^2 vanishes and the condition reduces to $d(D) = 0$. Thus, we see that each closed bimodule morphism c of degree 1 from θ to $\mathscr{A}$ gives rise to a 'deformation'

$$\Pi_n(\mathscr{A}, c)$$

of $\Pi_n(\mathscr{A})$, obtained by adding c to the differential of $\Pi_n(\mathscr{A})$. A standard argument shows that two homotopic morphisms c and c' yield quasi-isomorphic dg categories $\Pi_n(\mathscr{A}, c)$ and $\Pi_n(\mathscr{A}, c')$. Thus, up to quasi-ismorphism, the deformation $\Pi_n(\mathscr{A}, c)$ only depends on the image of c in the derived category of bimodules (recall that θ is cofibrant). Now notice that since the bimodule $\mathscr{A}$ is perfect, we have the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}(\mathscr{A}^{\mathrm{e}})}(\Sigma^{n-1}\Theta, \Sigma\mathscr{A}) &= \mathrm{Hom}_{\mathscr{D}(\mathscr{A}^{\mathrm{e}})}(\mathscr{A}^{\vee}, \Sigma^{2-n}\mathscr{A}) = H^{2-n}(\mathscr{A} \overset{L}{\otimes}_{\mathscr{A}^{\mathrm{e}}} \mathscr{A}^{\vee\vee}) \\ &= H^{2-n}(\mathscr{A} \overset{L}{\otimes}_{\mathscr{A}^{\mathrm{e}}} \mathscr{A}) = \mathrm{Tor}^{\mathscr{A}^{\mathrm{e}}}_{n-2}(\mathscr{A}, \mathscr{A}) = HH_{n-2}(\mathscr{A}),\end{aligned}$$

where HH denotes Hochschild homology.

Theorem 5.2. *The deformed n-Calabi–Yau completion $\Pi_n(\mathscr{A}, c)$ associated with an element c of $HH_{n-2}(\mathscr{A})$ is homologically smooth and n-Calabi–Yau.*

Proof. This is a variation on the proof of Theorem 4.8 where we have to take into account the new component of the differential of $T_{\mathscr{A}}(\theta)$. Let $\mathscr{B}$ be the deformed n-Calabi–Yau completion. We still have a short exact sequence of $\mathscr{B}^{\mathrm{e}}$-modules

$$0 \to T_{\mathscr{A}}(\theta) \otimes_{\mathscr{A}} \theta \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) \xrightarrow{\alpha} T_{\mathscr{A}}(\theta) \otimes_{\mathscr{A}} T_{\mathscr{A}}(\theta) \to T_{\mathscr{A}}(\theta) \to 0,$$

where the morphism α takes an element f of $\theta(X, Y)$ to $1_Y \otimes f - f \otimes 1_X$ and the second map is composition. Notice that here the differentials of the tensor algebras $T_{\mathscr{A}}(\theta)$ are deformed but that the one of the middle factor θ on the left is not! The map α is indeed compatible with the differential: For an element x of θ, we have

$$d\big(\alpha(x)\big) = d(1 \otimes x - x \otimes 1) = 1 \otimes (dx + cx) - (dx + cx) \otimes 1 = 1 \otimes dx - dx \otimes 1,$$

where the last equality holds because cx belongs to $\mathscr{A}$ and the tensor product is over $\mathscr{A}$. Now we can proceed as in the proof of Theorem 4.8. We obtain that for arbitrary c, the dg category $\mathscr{B}$ is smooth and n-Calabi–Yau. □

Remark 5.3. The formulas in Lemma 4.4 remain true when we replace the Calabi–Yau completion $\Pi_n(\mathscr{A})$ with the deformed Calabi–Yau completion $\Pi_n(\mathscr{A}, c)$. Indeed, the sequence (4.4.1) in the proof of the lemma remains well-defined and exact when we replace $T_{\mathscr{A}}(\theta)$ with $\Pi_n(\mathscr{A}, c)$.

5.4. Deformed Calabi–Yau completions as homotopy pushouts. The slightly ad hoc construction of the deformed Calabi–Yau completion given in Section 5.1 can be viewed more intrinsically as a homotopy pushout. Let us explain this in more detail. Let k, $\mathscr{A}$ and Θ be as in Section 5.1 and let c be an element of $HH_{n-2}(\mathscr{A})$. We may lift c to a morphism of dg bimodules

$$\tilde{c} : \Theta[n-2] \to \mathscr{A}.$$

This morphism extends uniquely to a morphism of dg categories

$$[\mathrm{id}, \tilde{c}] : \Pi_{n-1}(\mathscr{A}) \to \mathscr{A}$$

which is the identity on $\mathscr{A}$ and given by $\tilde{c}$ on $\Theta[n-2]$. We also have the projection

$$[\mathrm{id}, 0] : \Pi_{n-1}(\mathscr{A}) \to \mathscr{A}.$$

Now let $i : \mathscr{A} \to \Pi_n(\mathscr{A}, c)$ be the canonical inclusion.

Proposition 5.5. *The square*

$$\begin{array}{ccc} \Pi_{n-1}(\mathscr{A}) & \xrightarrow{[\mathrm{id},0]} & \mathscr{A} \\ {\scriptstyle [\mathrm{id},\tilde{c}]}\downarrow & & \downarrow{\scriptstyle i} \\ \mathscr{A} & \xrightarrow[i]{} & \Pi_n(\mathscr{A}, c) \end{array}$$

is a homotopy pushout square for the model category structure on the category of dg *categories introduced in* [39].

Notice that the square is not commutative in the category of dg categories. The proof will show in particular that it becomes commutative in the homotopy category.

The proposition is a special case of the following general fact: Let $\mathscr{A}$ be any (small) dg category and X a cofibrant $\mathscr{A}$-bimodule. Let $f : X \to \mathscr{A}$ be a bimodule morphism. We also view f as a morphism of degree 1 from $X[1]$ to $\mathscr{A}$. Let $T_A(X[1])$ denote the tensor category $T_{\mathscr{A}}(X[1])$ whose differential has been deformed using $f : X[1] \to \mathscr{A}$ as an additional component. Let the morphisms $[\mathrm{id}, f]$, $[\mathrm{id}, 0]$ from $T_{\mathscr{A}}(X)$ to $\mathscr{A}$ and $i : \mathscr{A} \to T_{\mathscr{A}}(X[1], f)$ be defined analogously to the above morphisms. Proposition 5.5 is now clearly a special case of the following:

Proposition 5.6. *The square*

$$\begin{array}{ccc} T_{\mathscr{A}}(X) & \xrightarrow{[\mathrm{id},0]} & \mathscr{A} \\ {\scriptstyle [\mathrm{id},f]}\downarrow & & \downarrow{\scriptstyle i} \\ \mathscr{A} & \xrightarrow[i]{} & T_{\mathscr{A}}(X[1], f) \end{array}$$

is a homotopy pushout square for the model category structure on the category of dg *categories introduced in* [39].

Proof. We may and will assume that $\mathscr{A}$ is cofibrant and that X is cofibrant as a bimodule. To compute the homotopy pushout of the angle

$$\begin{array}{ccc} T_{\mathscr{A}}(X) & \xrightarrow{[\mathrm{id},0]} & \mathscr{A} \\ {\scriptstyle [\mathrm{id},f]}\downarrow & & \\ \mathscr{A}, & & \end{array}$$

it is then enough to replace the morphism $[\mathrm{id}, 0]$ by a cofibration and to compute the pushout in the category of dg categories. To replace $[\mathrm{id}, 0]$ by a homotopy pushout, we consider the natural inclusion

$$j : X \to IX$$

of X into the cone IX over the identity of X. Clearly, the morphism $[\mathrm{id}, 0]$ factors as the cofibration $T_{\mathscr{A}}(X) \to T_{\mathscr{A}}(IX)$ followed by the trivial fibration $T_{\mathscr{A}}(IX) \to \mathscr{A}$. So to compute the homotopy pushout, it is enough to compute the homotopy pushout of the angle

$$\begin{array}{ccc} T_{\mathscr{A}}(X) & \longrightarrow & T_{\mathscr{A}}(IX) \\ {\scriptstyle [\mathrm{id},f]}\downarrow & & \\ \mathscr{A}. & & \end{array}$$

We claim that this is given by the commutative square

$$\begin{array}{ccc} T_{\mathscr{A}}(X) & \longrightarrow & T_{\mathscr{A}}(IX) \\ {\scriptstyle [\mathrm{id},f]}\downarrow & & \downarrow \\ \mathscr{A} & \longrightarrow & T_{\mathscr{A}}(X[1], f). \end{array}$$

Indeed, we have a pushout diagram of dg bimodules

$$\begin{array}{ccc} X & \xrightarrow{j} & IX \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathscr{A} & \longrightarrow & \mathscr{A} \oplus X[1], \end{array}$$

where $\mathscr{A} \oplus X[1]$ is endowed with the differential of the mapping cone over f. Using this one easily checks that $T_{\mathscr{A}}(X[1], f)$ has the correct universal property. □

5.7. Compatibility with Morita functors and localizations. As in Section 5.1, let n be an integer, k a commutative ring and $\mathscr{A}$ a homologically smooth dg k-category such that $\mathscr{A}(X, Y)$ is cofibrant as a dg k-module for all objects X and Y of $\mathscr{A}$. Consider the deformed n-Calabi–Yau completion $\mathscr{B} = \Pi_n(\mathscr{A}, c)$ associated with an element c of $HH_{n-2}(\mathscr{A})$.

Now let $\mathscr{B}$ be another dg k-category satisfying the same hypotheses as $\mathscr{A}$. Assume that we have a localization functor $F : \mathscr{A} \to \mathscr{B}$ and let c' be the element of $HH_{n-2}(\mathscr{B})$ obtained as the image of c under the map induced by F, cf. [25].

Theorem 5.8. (a) *Under the above hypotheses, there is a canonical localization functor* $G : \Pi_n(\mathscr{A}, c) \to \Pi_n(\mathscr{B}, c')$ *such that we have a commutative diagram*

$$\begin{array}{ccc} \mathscr{A} & \longrightarrow & \Pi_n(\mathscr{A}, c) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathscr{B} & \longrightarrow & \Pi_n(\mathscr{B}, c'). \end{array}$$

The functor G *is a Morita functor if* F *is.*

(b) *If we have an exact sequence of* dg *categories* (*cf. Section* 4.5)

$$0 \to \mathscr{N} \xrightarrow{G} \mathscr{A} \xrightarrow{F} \mathscr{B} \to 0,$$

then the kernel of the induced functor

$$G_* : \mathscr{D}\big(\Pi_n(\mathscr{A}, c)\big) \to \mathscr{D}\big(\Pi_n(\mathscr{B}, c')\big)$$

is the localizing subcategory generated by the dg *modules* $\Pi_n(\mathscr{A}, c)(?, N)$, *where* N *belongs to* $\mathscr{N}$.

Proof. We have a commutative square of isomorphisms

$$\begin{array}{ccc} H_{n-2}(\mathscr{A} \overset{L}{\otimes}_{\mathscr{A}^{\mathrm{e}}} \mathscr{A}) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}^{\mathrm{e}}}(\theta_{\mathscr{A}}, \mathscr{A}) \\ \downarrow & & \downarrow \\ H_{n-2}(\mathscr{B} \overset{L}{\otimes}_{\mathscr{B}^{\mathrm{e}}} \mathscr{B}) & \longrightarrow & \mathrm{Hom}_{\mathscr{B}^{\mathrm{e}}}(\theta_{\mathscr{B}}, \mathscr{B}), \end{array}$$

where the vertical arrows are induced by F. This yields a commutative square in $\mathscr{D}(\mathscr{A}^{\mathrm{e}})$, where we also write F^* for $(F^{\mathrm{e}})^*$,

$$\begin{array}{ccc} \theta_{\mathscr{A}} & \xrightarrow{c} & \mathscr{A} \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle F} \\ F^*\theta_{\mathscr{B}} & \xrightarrow{F^*c'} & F^*\mathscr{B}. \end{array}$$

We would like to lift it to a strictly commutative square of dg modules. We choose an arbitrary lift $\tilde{c}$ of c. After replacing $\theta_{\mathscr{B}}$ by a homotopy equivalent cofibrant dg module, we may choose a dg module morphism $\tilde{c}' : \theta_{\mathscr{B}} \to \mathscr{B}$ lifting c' such that $\tilde{c}'$ induces a split surjection of graded $\mathscr{B}^{\mathrm{e}}$-modules. The same then holds for the morphism $F^*\tilde{c}'$ of dg $\mathscr{A}^{\mathrm{e}}$-modules. Therefore, we can choose a lift $\tilde{\varphi}$ of φ such that the square of dg modules

$$\begin{array}{ccc} \theta_{\mathscr{A}} & \xrightarrow{\tilde{c}} & \mathscr{A} \\ {\scriptstyle \tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle F} \\ F^*\theta_{\mathscr{B}} & \xrightarrow{F^*\tilde{c}'} & F^*\mathscr{B} \end{array}$$

commutes strictly. As in the proof of Theorem 4.6, the morphisms F and $\tilde{\varphi}$ then induce a dg functor

$$G : \Pi_n(\mathscr{A}, c) \to \Pi_n(\mathscr{B}, c').$$

It remains to be checked that the restriction G^* is a fully faithful functor from $\mathscr{D}\big(\Pi_n(\mathscr{B}, c')\big)$ to $\mathscr{D}\big(\Pi_n(\mathscr{A}, c)\big)$. Let L be a dg $\Pi_n(\mathscr{B}, c')$-module. It is given by its underlying dg $\mathscr{B}$-module and a morphism of *graded modules* homogeneous of degree 0

$$\lambda : L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \to L$$

such that

$$(d\lambda)(l \otimes x) = lc'(x)$$

for all l in L and x in $\theta_{\mathscr{B}}$. Suppose that L is cofibrant as a $\Pi_n(\mathscr{B}, c')$-module. Since the underlying $\mathscr{B}$-module of $\Pi_n(\mathscr{B}, c')$ is cofibrant (even with the deformed differential), the underlying $\mathscr{B}$-module of L is cofibrant. We have an exact sequence of cofibrant dg $\Pi_n(\mathscr{B})$-modules

$$0 \to L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}}) \xrightarrow{\alpha} L \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}}) \to L \to 0,$$

where $\alpha(l \otimes x \otimes u) = lx \otimes u - l \otimes xu$. Notice that the map α is a morphism of dg modules despite the deformation of the differential on $T_{\mathscr{B}}(\theta_{\mathscr{B}})$, analogously to what we have seen in the proof of Theorem 5.2. The sequence shows that the cone over the morphism

$$L \otimes_{\mathscr{B}} \theta_{\mathscr{B}} \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}}) \to L \otimes_{\mathscr{B}} T_{\mathscr{B}}(\theta_{\mathscr{B}})$$

is homotopy equivalent to L. Let M be another dg $\Pi_n(\mathscr{B}, c')$-module. By applying $\mathrm{Hom}_{\mathscr{B}}(?, M)$ to the above morphism, we obtain a morphism of dg k-modules

$$\mathrm{Hom}_{\mathscr{B}}(L, M) \to \mathrm{Hom}_{\mathscr{B}}(L \otimes_{\mathscr{B}} \theta_{\mathscr{B}}, M),$$

whose cone (shifted by one degree to the right) computes morphisms from L to M in the derived category of $\Pi_n(\mathscr{B}, c')$. An analogous reasoning yields the morphisms between G^*L and G^*M in the derived category of $\Pi_n(\mathscr{A}, c)$. Thus, to conclude that G^* is fully faithful, it suffices to check that for all M, the dg functor F^* induces bijections

$$\mathrm{Hom}_{\mathscr{D}(\mathscr{B})}(L, M) \to \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(F^*L, F^*M)$$

and

$$\mathrm{Hom}_{\mathscr{D}(\mathscr{B})}(L \overset{L}{\otimes}_{\mathscr{B}} \theta_{\mathscr{B}}, M) \to \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}\big(F^*(L) \overset{L}{\otimes}_{\mathscr{A}} \theta_{\mathscr{A}}, F^*M\big).$$

As in the proof of Theorem 4.6, the first bijection follows from the full faithfulness of F^* and the second one is a consequence of the full faithfulness of F^* and of the projection formula (3.10.1). This ends the proof of (a). The proof of (b) is entirely analogous to that of part (b) of Theorem 4.6 and left to the reader. □

6. Ginzburg dg categories

6.1. Reminder on Hochschild and cyclic homology. Let k be a commutative ring and Q a graded k-quiver, cf. Section 3.5. We put $\mathscr{A} = T_{\mathscr{R}}(Q)$. The bimodule $\mathscr{A}$ has the *small resolution*

$$0 \to \mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A} \xrightarrow{\tilde{\alpha}} \mathscr{A} \otimes_{\mathscr{R}} \mathscr{A} \to \mathscr{A} \to 0, \tag{6.1.1}$$

where the map $\tilde{\alpha}$ takes a tensor $u \otimes v \otimes w$ to $uv \otimes w - u \otimes vw$ and the right-hand map is composition. By tensoring this resolution with $\mathscr{A}$ over $\mathscr{A}^{\mathrm{e}}$ we obtain the following complex which computes Hochschild homology:

$$0 \to (Q \otimes_{\mathscr{R}} \mathscr{A}) \otimes_{\mathscr{R}^{\mathrm{e}}} \mathscr{R} \xrightarrow{\alpha} \mathscr{A} \otimes_{\mathscr{R}^{\mathrm{e}}} \mathscr{R} \to 0,$$

where α takes a tensor $v \otimes u$ with factors of degree p and q to $vu - (-1)^{pq}uv$.

Let $\tilde{\beta}$ be the unique bimodule derivation

$$\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A},$$

which takes an element $v : x \to y$ of Q to $\mathrm{id}_y \otimes v \otimes \mathrm{id}_x$. If we have $n \geqq 1$ and $a = v_1 \dots v_n$ for elements $v_i : x_i \to x_{i-1}$ of Q, we have

$$\tilde{\beta}(a) = 1_{x_0} \otimes v_1 \otimes v_2 \dots v_n + \sum_{i=2}^{n-1} v_1 \dots v_{i-1} \otimes v_i \otimes v_{i+1} \dots v_n + v_1 \dots v_{n-1} \otimes v_n \otimes 1_{x_n}$$

and

$$\tilde{\alpha}\tilde{\beta}(a) = -1_{x_0} \otimes a + a \otimes 1_{x_n}.$$

The map $\tilde{\beta}$ induces a (unique) map β making the following square commutative:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\tilde{\beta}} & \mathscr{A} \otimes_{\mathscr{R}} Q \otimes_{\mathscr{R}} \mathscr{A} \\ \downarrow & & \downarrow \\ \mathscr{A} \otimes_{\mathscr{R}^{\mathrm{e}}} \mathscr{R} & \xrightarrow{\beta} & (Q \otimes_{\mathscr{R}} \mathscr{A}) \otimes_{\mathscr{R}^{\mathrm{e}}} \mathscr{R}, \end{array}$$

where the left-vertical map takes a path a from x to y to $a \otimes 1_x 1_y$ and the right-vertical map takes $a \otimes v \otimes b$ to $(-1)^{pq}(v \otimes ba) \otimes 1_x$, where a is of degree p and vb is of degree q. Note that the tensor product $M \otimes_{\mathscr{R}^{\mathrm{e}}} \mathscr{R}$ of an $\mathscr{R}$-bimodule M with $\mathscr{R}$ over $\mathscr{R}^{\mathrm{e}}$ identifies with the quotient of M by the dg submodule generated by all differences $m1_x - 1_x m$ for $m \in M$ and x an object of $\mathscr{R}$. If we make this identification, the map β takes a path $v_1 \dots v_n$ of Q to the sum

$$\sum_i \pm v_i \otimes v_{i+1} v_{i+2} \dots v_n v_1 \dots v_{i-1},$$

where the sign is computed by the Koszul sign rule from the degrees of the v_j. We clearly have $\alpha \circ \beta = 0$. The following complex is to be continued in a 2-periodic fashion to the

left:

$$\cdots \xrightarrow{\alpha} \mathscr{A} \otimes_{\mathscr{R}^e} \mathscr{R} \xrightarrow{\beta} (Q \otimes_{\mathscr{R}} \mathscr{A}) \otimes_{\mathscr{R}^e} \mathscr{R} \xrightarrow{\alpha} \mathscr{A} \otimes_{\mathscr{R}^e} \mathscr{R} \to 0. \tag{6.1.2}$$

It is the *small cyclic complex* $C_{\mathrm{sm}}(\mathscr{A})$ and computes cyclic homology (cf. [34], Chapter 3). We sometimes consider its components as columns. If $\mathscr{A} = \mathscr{R}$, cyclic homology is two-periodic, the module $HC_1(\mathscr{R})$ vanishes and $HC_0(\mathscr{R})$ is a sum of copies of k indexed by Q_0. If k contains $\mathbb{Q}$, and $\mathscr{A}$ is arbitrary, then the *reduced small cyclic complex* $C_{\mathrm{sm}}(\mathscr{A})/C_{\mathrm{sm}}(\mathscr{R})$ is quasi-isomorphic to the quotient of its rightmost column by the image of α, i.e. to the cokernel of the map

$$(Q \otimes_{\mathscr{R}} \mathscr{A}) \otimes_{\mathscr{R}^e} \mathscr{R} \xrightarrow{\alpha} \mathscr{A} \otimes_{\mathscr{R}^e} \mathscr{R}.$$

The inclusion of the subcomplex of the two rightmost terms induces the canonical morphism from Hochschild to cyclic homology. The corresponding quotient complex is isomorphic to the original complex shifted by two degrees to the left. The short exact sequence thus obtained induces the long exact sequence (known as the SBI-sequence)

$$HH_n(\mathscr{A}) \xrightarrow{I} HC_n(\mathscr{A}) \xrightarrow{S} HC_{n-2}(\mathscr{A}) \xrightarrow{B} HH_{n-1}(\mathscr{A}).$$

In particular, the rightmost arrow β of the small cyclic complex induces Connes' connecting map

$$B : HC_n(\mathscr{A}) \to HH_{n+1}(\mathscr{A}).$$

If the ring k contains $\mathbb{Q}$ and the quiver Q is concentrated in degree 0, then in the exact sequence

$$HH_2(\mathscr{A}) \to HC_2(\mathscr{A}) \to HC_0(\mathscr{A}) \to HH_1(\mathscr{A}) \to HC_1(\mathscr{A}),$$

the terms $HH_2(\mathscr{A})$ and $HC_1(\mathscr{A})$ vanish (as we see by considering the small cyclic complex), the map S induces an isomorphism $HC_2(\mathscr{A}) \xrightarrow{\sim} HC_0(\mathscr{R})$, and the map B induces an isomorphism from the reduced zeroth cyclic homology of $\mathscr{A}$ to its first Hochschild homology.

6.2. Ginzburg dg categories. Let Q be a graded k-quiver such that the set of objects Q_0 is finite and $Q(x, y)$ is a finitely generated graded projective k-module for all objects x and y. We fix an integer n and a *superpotential of degree* $n-3$, i.e. an element W in $(\mathscr{A} \otimes_{\mathscr{R}^e} \mathscr{R})/\operatorname{im}\alpha$ of degree $n-3$. So W is a linear combination of cycles considered up to cyclic permutation 'with signs'. Notice that W need not be homogeneous with respect to the grading by path length. We can view W as an element in $HC_{n-3}(\mathscr{A})$ and if the ring k contains $\mathbb{Q}$, every element of $HC_{n-3}(\mathscr{A})$ has such a representative. Let $\mathscr{R}$ be the discrete category on Q_0 and $Q^\vee$ the dual of the $\mathscr{R}$-bimodule Q over $\mathscr{R}^e$ (endowed with the canonical involution). Let $\sum v_i \otimes v_i^*$ be the Casimir element of $Q \otimes_{\mathscr{R}^e} Q^\vee$, i.e. the element which, under the canonical isomorphism

$$Q \otimes_{\mathscr{R}^e} Q^\vee \to \operatorname{Hom}_{\mathscr{R}^e}(Q, Q),$$

corresponds to the identity of Q.

The *Ginzburg* dg *category* $\Gamma_n(Q,W)$, due to V. Ginzburg ([19], Section 4.2) for a quiver Q concentrated in degree 0 and $n=3$, is defined as the tensor category over $\mathscr{R}$ of the bimodule

$$\tilde{Q} = Q \oplus Q^\vee[n-2] \oplus \mathscr{R}[n-1]$$

endowed with the unique differential which

(a) vanishes on Q,

(b) takes the element v_i^* of $Q^\vee[n-2]$ to the cyclic derivative $\partial_{v_i} W$ (cf. Section 3.5),

(c) takes the element id_x of $\mathscr{R}[n-1]$ to $(-1)^n \mathrm{id}_x(\sum[v_i, v_i^*])\,\mathrm{id}_x$, where $[\,,\,]$ denotes the supercommutator.

Let $\mathscr{A}$ be the path category of Q and $c=\beta(W)$ the image of W in

$$HH_{n-2}(\mathscr{A}) = \mathrm{Tor}_{n-2}^{\mathscr{A}^e}(\mathscr{A},\mathscr{A}).$$

Thanks to the small resolution (6.1.1), the path category $\mathscr{A}$ is homologically smooth. By Theorem 5.2, the associated deformed n-Calabi–Yau completion $\Pi_n(\mathscr{A},c)$ is homologically smooth and n-Calabi–Yau.

Theorem 6.3. *The deformed n-Calabi–Yau completion* $\Pi_n(\mathscr{A},c)$ *is quasi-isomorphic to the Ginzburg* dg *category* $\Gamma_n(\mathscr{A},W)$. *In particular, the Ginzburg* dg *category is homologically smooth and n-Calabi–Yau.*

Remark 6.4. If we use the theorem and Proposition 5.5, we obtain that the Ginzburg dg category is given, up to isomorphism in the homotopy category of dg categories in the sense of [39], by the homotopy pushout square

$$\begin{array}{ccc} \Pi_{n-1}(\mathscr{A}) & \xrightarrow{[\mathrm{id},0]} & \mathscr{A} \\ {\scriptstyle [\mathrm{id},\bar{c}]}\downarrow & & \downarrow{\scriptstyle i} \\ \mathscr{A} & \xrightarrow[i]{} & \Gamma_n(\mathscr{A},W). \end{array}$$

I thank Ben Davison [14] for suggesting this statement.

Proof. We first apply the computation of the inverse dualizing complex of Section 3.6 to the special case where $\mathscr{A}=T_{\mathscr{R}}(Q)$ with $d=0$. We obtain that the non-deformed CY-completion is quasi-isomorphic to the tensor category over $\mathscr{R}$ of the bimodule $Q \oplus Q^\vee[n-2] \oplus \mathscr{R}[n-1]$ endowed with the unique differential which vanishes on Q and $Q^\vee$ and takes the element id_x of $\mathscr{R}[n-1]$ to $(-1)^{n-2}\mathrm{id}_x(\sum[v_i,v_i^*])\,\mathrm{id}_x$. The deforming component of the differential of $\Pi_n(\mathscr{A},c)$ is the map $\theta \to \mathscr{A}$ given by the contraction with $c=\beta(W)$ in

$$\Sigma^{n-1}\,\mathrm{RHom}_{\mathscr{A}^e}(\mathscr{A},\mathscr{A}^e) \otimes (\mathscr{A} \overset{L}{\otimes}_{\mathscr{A}^e} \mathscr{A}) \to \Sigma\mathscr{A}.$$

This last map identifies with

$$\Sigma^{n-1}\operatorname{Hom}_{\mathscr{A}^e}(P,\mathscr{A}^e)\otimes(P\otimes_{\mathscr{A}^e}\mathscr{A})\to\Sigma\mathscr{A},$$

where P is the cofibrant resolution of $\mathscr{A}$ constructed in Proposition 3.7. The complex $P\otimes_{\mathscr{A}^e}\mathscr{A}$ is isomorphic to

$$0\to(Q\otimes_{\mathscr{R}}\mathscr{A})\otimes_{\mathscr{R}^e}\mathscr{R}\xrightarrow{\alpha}\mathscr{A}\otimes_{\mathscr{R}^e}\mathscr{R}\to 0,$$

and c lies in the subcomplex $(Q\otimes_{\mathscr{R}}\mathscr{A})\otimes_{\mathscr{R}^e}\mathscr{R}$. The complex $\Sigma^{n-1}\operatorname{Hom}_{\mathscr{A}^e}(P,\mathscr{A}^e)$ is isomorphic to

$$0\to\mathscr{A}\otimes_{\mathscr{R}}\mathscr{A}\to\mathscr{A}\otimes_{\mathscr{R}}Q^\vee\otimes_{\mathscr{R}}\mathscr{A}\to 0.$$

Therefore, the deforming component of the differential vanishes on the left-hand component $\mathscr{A}\otimes_{\mathscr{R}}\mathscr{A}$. Now it is clear that the deforming component of the differential vanishes on $\mathscr{R}[n-1]$ and takes an element v^* of $Q^\vee[n-2]$ to $(v^*\otimes\mathrm{id})\circ\beta(W)$. For $v=v_i^*$, clearly this equals the cyclic derivative $\partial_{v_i}W$. □

6.5. Deformed Calabi–Yau completions of homotopically finitely presented dg categories. Let k be a commutative ring and Q a graded k-quiver whose set of objects is finite and whose bimodule of morphisms is finitely generated and projective over k. Let $\mathscr{A}$ be a dg category of the form $\big(T_{\mathscr{R}}(Q),d\big)$, where the differential d satisfies the condition of Section 3.6. Let n be an integer, $Q^\vee=\operatorname{Hom}_{\mathscr{R}^e}(Q,\mathscr{R}^e)$ and

$$\tilde{Q}=Q\oplus Q^\vee[n-2]\oplus\mathscr{R}[n-1].$$

Let $\sum\alpha_j\otimes\alpha_j^*$ be the Casimir element of Q and let W be the element

$$W=\sum(-1)^{|\alpha_j|}\alpha_j^*d(\alpha_j)$$

of $T_{\mathscr{R}}(\tilde{Q})$. Let W' be an element of $HC_{n-3}(\mathscr{A})$ and $c\in HH_{n-2}(\mathscr{A})$ its image under Connes' map B.

Proposition 6.6. *The deformed n-Calabi–Yau completion $\Pi_n(\mathscr{A},c)$ is isomorphic to the tensor category $T_{\mathscr{R}}(\tilde{Q})$, endowed with the unique differential d such that for each i, we have*

$$d(\alpha_i)=\partial_{\alpha_i^*}(W+W')\quad\text{and}\quad d(\alpha_i^*)=\partial_{\alpha_i}(W+W'),$$

and for an object x of Q, the element id_x of $\Sigma^{n-1}\mathscr{R}$ is taken to

$$d(\mathrm{id}_x)=(-1)^n\,\mathrm{id}_x\big(\textstyle\sum[\alpha_i,\alpha_i^*]\big)\,\mathrm{id}_x$$

where $[\,,\,]$ is the supercommutator.

Proof. This follows from the description of the inverse dualizing complex of $\mathscr{A}$ in Section 3.6. The details of the computation are similar to those in the proof of Theorem 6.3 and left to the reader. □

6.7. 3-Calabi–Yau completions of 2-dimensional dg categories. Let k be a commutative ring and $\mathscr{A}$ a dg category Morita equivalent to $\big(T_{\mathscr{R}}(V), d\big)$ for a graded k-quiver V whose set of objects is finite and whose bimodule of arrows is finitely generated free over k and concentrated in degrees -1 and 0 (the differential d is arbitrary). The following proposition shows in particular that $\Pi_3(\mathscr{A})$ is Morita-equivalent to a Ginzburg dg category.

Let $\mathscr{B}$ be the path category $\mathscr{B} = T_{\mathscr{R}}\big(V^0 \oplus (V^{-1})^\vee\big)$ of the sum of the 0th component of V with the $\vee$-dual of V^{-1} placed in degree 0. Let W be the class in $HC_0(\mathscr{B})$ of the element

$$\sum_j v_j^* d(v_j),$$

where $\sum v_j \otimes v_j^*$ is a Casimir element for V^{-1}. Let $W' \in HC_0(\mathscr{A})$ and $c' \in HH_1(\mathscr{A})$ its image under Connes' map B. For example we can have $W' = 0$ and $c = 0$.

Proposition 6.8. *The deformed* 3*-Calabi–Yau completion* $\Pi_3(\mathscr{A}, c)$ *is derived Morita-equivalent to the deformed* 3*-Calabi–Yau completion* $\Pi_3(\mathscr{B}, W + W')$ *and thus to the Ginzburg algebra* $\Gamma_3\big(V^0 \oplus (V^{-1})^\vee, W + W'\big)$.

Proof. This is a special case of 6.6. □

6.9. 3-CY completions of algebras of global dimension 2. Let k be a field and A an algebra given as the quotient kQ'/I of the path algebra of a finite quiver Q' by an ideal I contained in the square of the ideal J generated by the arrows of Q'. Assume that A is of global dimension $\leqq 2$ (but not necessarily of finite dimension over k). We construct a quiver Q and a superpotential W as follows: Let R be the union over all pairs of vertices (i, j) of a set of representatives of the vectors belonging to a basis of

$$\mathrm{Tor}_2^A(S_j, DS_i) = e_j\big(I/(IJ + JI)\big)e_i,$$

where $D = \mathrm{Hom}_k(?, k)$ and S_i is the simple right module associated with the vertex i. We think of these representatives as 'minimal relations' from i to j, cf. [10]. For each such representative r, let ρ_r be a new arrow from j to i. We define Q to be obtained from Q' by adding all the arrows ρ_r. We define a potential by

$$W = \sum_{r \in R} r\rho_r.$$

Now let $W' \in HC_0(A)$ and $c \in HH_1(A, A)$ its image under Connes' map B. Let $\tilde{W}'$ be an element of $HC_0(kQ)$ which lifts W' along the canonical surjection $kQ \to kQ' \to A$ taking all arrows ρ_r to 0. For example, we can have $W' = 0$ and $\tilde{W}' = 0$.

Theorem 6.10. *The deformed* 3*-Calabi–Yau completion* $\Pi_3(A, c)$ *is quasi-isomorphic to the Ginzburg* dg *algebra* $\Gamma_3(Q, W + \tilde{W}')$.

A very similar result was independently obtained by Ginzburg [20] in a slightly different setting.

Proof. For each vertex i of A let P_i be the indecomposable projective e_iA. Let $\mathscr{A}$ be the full subcategory of the module category formed by the P_i. By induction, one constructs a graded $\mathscr{R}$-bimodule V and a differential d on $T_{\mathscr{R}}(V)$ such that

(1) V^n vanishes in degrees $n \geqq 1$, V^0 is free with basis Q' and V^{-1} is free with basis R;

(2) the differential d sends the basis element $r \in R$ of V^{-1} to the element r of $T_{\mathscr{R}}(V^0)$;

(3) for all $n \geqq 1$, the differential d maps V^{-n-1} to T_n and induces an isomorphism from V^{-n-1} onto $H^{-n}(T_n)$, where T_n denotes the dg category $T_{\mathscr{R}}(V^0 \oplus \cdots \oplus V^{-n})$.

Notice that (a) the image $d(V)$ lies in the *square* of the ideal generated by V in $T_{\mathscr{R}}(V)$ and that (b) we have a canonical quasi-isomorphism between $\mathscr{F} = \big(T_{\mathscr{R}}(V), d\big)$ and $\mathscr{A}$. The point (a) implies that we have isomorphisms

$$V^{-n}(i, j) \cong \mathrm{Tor}^{\mathscr{F}}_{1+n}(S_i, DS_j)$$

for all i, j and n (thanks to Remark 3.8, we can use the bimodule resolution of part (b) of Proposition 3.7). The point (b) implies that we have isomorphisms

$$\mathrm{Tor}^{\mathscr{F}}_{1+n}(S_i, DS_j) \cong \mathrm{Tor}^{\mathscr{A}}_{1+n}(S_i, DS_j).$$

Thus, we have $V^n = 0$ for all n different from 0 and -1. Now we can apply Proposition 6.8 to conclude. □

6.11. Application to cluster-tilted algebras. Let k be an algebraically closed field. If A is a finite-dimensional k-algebra of finite global dimension, its *generalized cluster-category* $\mathscr{C}_A$ is defined as the full triangulated subcategory of the triangle quotient

$$\mathscr{D}^b\big(A \oplus (DA)[-3]\big)/\mathsf{per}\big(A \oplus (DA)[-3]\big)$$

generated by the image of the free module A, cf. [27] and [1]. Here, the dg algebra $A \oplus (DA)[-3]$ is the trivial extension of A by the dg bimodule $(DA)[-3]$, where $D = \mathrm{Hom}_k(?, k)$. In general, the category $\mathscr{C}_A$ has infinite-dimensional morphism spaces. As shown in [27], if A is the path algebra of a quiver Q without oriented cycles, then $\mathscr{C}_A$ is triangle equivalent to the cluster category $\mathscr{C}_Q$ as defined in [9], cf. also [11] for the case where Q is Dynkin of type A.

The *generalized cluster category* $\mathscr{C}_{(Q,W)}$ of a finite quiver Q with potential W is defined as the triangle quotient

$$\mathsf{per}\big(\Gamma_3(Q, W)\big)/\mathscr{D}^b\big(\Gamma_3(Q, W)\big),$$

cf. [1]. In general, it has infinite-dimensional morphism spaces. If Q does not have oriented cycles (and so $W = 0$), then $\mathscr{C}_{(Q,0)}$ is equivalent to the cluster category $\mathscr{C}_Q$, cf. [1]. For arbitrary (Q, W), the endomorphism algebra of the image of the free module $\Gamma_3(Q, W)$ in $\mathscr{C}_{(Q,W)}$ is isomorphic to the *Jacobian algebra* $H^0\big(\Gamma_3(Q, W)\big)$.

Recall [29] that a *tilting module* over an algebra B is a B-module T such that the total derived functor of the tensor product by T over the endomorphism algebra $\mathrm{End}_B(T)$ is an equivalence

$$\mathscr{D}\big(\mathrm{End}_B(T)\big) \xrightarrow{\sim} \mathscr{D}(B).$$

The endomorphism algebra A of a tilting module T over a hereditary algebra B is of global dimension at most 2. A module M is *basic* if each indecomposable module occurs with multiplicity at most 1 as a direct factor of M. If T is a basic tilting module over the path algebra $B = kQ''$ of a finite quiver without oriented cycles, the endomorphism algebra $\tilde{A}$ of the image of T in $\mathscr{C}_{Q''}$ is called the *cluster-tilted algebra* associated with T, cf. [10].

Theorem 6.12. *Let $A = kQ'/I$ be a k-algebra of global dimension at most 2 as in Section 6.9 and define (Q, W) as there. Let $\Gamma = \Gamma_3(Q, W)$.*

(a) *The category $\mathscr{C}_{(Q,W)}$ is canonically triangle equivalent to the cluster category $\mathscr{C}_A$. The equivalence takes Γ to the image $\pi(A)$ of A in $\mathscr{C}_A$ and thus induces an isomorphism from the Jacobian algebra $\mathscr{P}(Q, W)$ onto the endomorphism algebra $\tilde{A}$ of the image of A in $\mathscr{C}_A$.*

(b) *If T is a basic tilting module over kQ'' for a quiver without oriented cycles Q'' and A is the endomorphism algebra of T, then $\mathscr{C}_{(Q,W)}$ is triangle equivalent to $\mathscr{C}_{Q''}$ by an equivalence which takes Γ to the image of T in $\mathscr{C}_{Q''}$. Thus, the endomorphism algebra $\tilde{A}$ of T in $\mathscr{C}_{Q''}$ is isomorphic to the Jacobian algebra $H^0(\Gamma)$.*

The quiver of $\tilde{A}$ in part (b) was first described by Assem–Brüstle–Schiffler [3]. The fact that cluster-tilted algebras are Jacobian algebras was independently proved by Buan–Iyama–Reiten–Smith [8] using an entirely different method.

Proof. (a) By Theorem 6.10, the 3-Calabi–Yau completion $\Pi = \Pi_3(A)$ is quasi-isomorphic to $\Gamma = \Gamma_3(Q, W)$. Thus we have an equivalence of triangulated categories

$$\mathscr{C}_{(Q,W)} \xrightarrow{\sim} \mathsf{per}(\Pi)/\mathscr{D}_{fd}(\Pi),$$

taking the free module Γ to Π. Moreover, we have an equivalence of triangulated categories

$$\mathsf{per}(\Pi)/\mathscr{D}_{fd}(\Pi) \xrightarrow{\sim} \mathscr{C}_A,$$

taking the free module Π to the image $\pi(A)$ of the free module A, cf. [27], the proof of Theorem 7.1, or [1], Lemmas 4.13 to 4.15. The claim follows because $H^0(\Gamma)$ is isomorphic to the endomorphism algebra of Γ in $\mathscr{C}_{(Q,W)}$ by [1], Theorem 3.6.

(b) If A is the endomorphism algebra of T, then A is derived equivalent to the path algebra kQ'' and therefore $\mathscr{C}_A$ is equivalent to $\mathscr{C}_{Q''}$. The claim now follows from part (a). □

7. Particular cases of localization and Morita equivalence

7.1. Deleting a vertex is localization. Let k be a field and Q a finite quiver (possibly with oriented cycles). Let A be the path algebra kQ. Notice that A may be of infinite dimension. Let i be a vertex of Q and e_i the associated idempotent. Let $P_i = e_iA$ be the associated projective indecomposable. Let $\mathscr{N} \subset \mathscr{D}(A)$ be the localizing subcategory generated by P_i. Let $B = A/Ae_iA$. Notice that B is the path algebra of the quiver Q' obtained from Q by deleting the vertex i and all arrows starting or ending at this vertex.

Lemma 7.2. *The functor*

$$? \overset{L}{\otimes}_A B : \mathscr{D}(A) \to \mathscr{D}(B)$$

induces an equivalence from $\mathscr{D}(A)/\mathscr{N}$ *onto* $\mathscr{D}(B)$. *Thus, the morphism* $A \to B$ *is a localization of* dg *categories* (*cf. Section* 3.9).

Proof. Since $\mathscr{N}$ is generated by a compact object, we know (see for example [35]) that for each object X of $\mathscr{D}(A)$, there is a triangle, unique up to unique isomorphism,

$$X_{\mathscr{N}} \to X \to X^{\mathscr{N}^\perp} \to \Sigma X_{\mathscr{N}} \tag{7.2.1}$$

with $X_{\mathscr{N}}$ in $\mathscr{N}$ and $X^{\mathscr{N}^\perp}$ in the right orthogonal subcategory $\mathscr{N}^\perp$. Moreover, the projection functor $\mathscr{D}(A) \to \mathscr{D}(A)/\mathscr{N}$ induces an equivalence from $\mathscr{N}^\perp$ onto $\mathscr{D}(A)/\mathscr{N}$. Let us compute the triangle (7.2.1) for $X = P_j$, where $P_j = e_jA$ is the projective associated with a vertex j of Q. If we have $j = i$, the morphism $X_{\mathscr{N}} \to X$ is the identity of P_i. If we have $j \neq i$, let $\mathscr{M}_j$ be the set of minimal elements of the set of paths p from i to j, where we have $p \leqq p'$ if $p' = pu$ for a path u from i to i. Then each morphism $P_i \to P_j$ uniquely factors through the morphism

$$\bigoplus_{\mathscr{M}_j} P_i \to P_j,$$

whose component associated with $p \in \mathscr{M}_j$ is the left multiplication by p. Moreover, this morphism is injective. It follows easily that it induces a bijection

$$\mathrm{Hom}_{\mathscr{D}(A)}\left(\Sigma^m P_i, \bigoplus_{\mathscr{M}_j} P_i\right) \to \mathrm{Hom}_{\mathscr{D}(A)}(\Sigma^m P_i, P_j)$$

for each $m \in \mathbb{Z}$ and this implies that it induces a bijection

$$\mathrm{Hom}_{\mathscr{D}(A)}\left(N, \bigoplus_{\mathscr{M}_j} P_i\right) \to \mathrm{Hom}_{\mathscr{D}(A)}(N, P_j)$$

for each $N \in \mathscr{N}$. It follows that the morphism

$$\bigoplus_{\mathscr{M}_j} P_i \to P_j$$

is the universal morphism $X_{\mathscr{N}} \to X$ for $X = P_j$. Therefore, the object $P_j^{\mathscr{N}^\perp}$ is the cokernel of

$$\bigoplus_{\mathscr{M}_j} P_i \to P_j.$$

Now it is easy to check that for all vertices j and l, the morphism space

$$\mathrm{Hom}_{\mathscr{D}(A)}(P_j^{\mathscr{N}^\perp}, \Sigma^m P_l^{\mathscr{N}^\perp})$$

vanishes for $m \neq 0$ and is canonically isomorphic to $e_l(A/Ae_iA)e_j$ for $m = 0$. This shows that the functor $? \overset{L}{\otimes}_A (A/Ae_iA) : \mathscr{D}(A) \to \mathscr{D}(A/Ae_iA)$ induces an equivalence from the subcategory of compact objects of $\mathscr{D}(A)/\mathscr{N}$ onto the perfect derived category of $\mathscr{D}(B) = \mathscr{D}(A/Ae_iA)$. Since this functor commutes with arbitrary coproducts, it does indeed induce an equivalence from $\mathscr{D}(A)/\mathscr{N}$ onto $\mathscr{D}(B)$. □

Recall that Q is a finite quiver, possibly with oriented cycles, k is a field and A is the path algebra kQ. The quiver Q' is obtained from Q by deleting the vertex i and all arrows starting or ending at i and $B = A/Ae_iA$. Now let W be a potential on Q, i.e. an element of $HC_0(A)$ and let W' be the image of W in $HC_0(B)$.

Corollary 7.3. *The canonical functor*

$$\Gamma_3(Q, W) \to \Gamma_3(Q', W')$$

is a localization.

Proof. By the functoriality of Connes' map B, the class $c' = B(W')$ is the image of $c = B(W)$ under the map $HH_1(A, A) \to HH_1(B, B)$ induced by $A \to B$. By the localization Theorem 5.8 and the above Lemma 7.2, we have an induced localization functor

$$\Pi_3(A, c) \to \Pi_3(B, c')$$

and by Theorem 6.3, this yields a localization functor between the Ginzburg dg algebras. □

Let us put $\Gamma = \Gamma_3(Q, W)$ and $\Gamma' = \Gamma_3(Q', W')$. Notice that in zeroth homology, the induced morphism between the Jacobian algebras is the natural quotient map

$$\mathscr{P}(Q, W) \to \mathscr{P}(Q', W').$$

Let us compare the generalized cluster categories

$$\mathscr{C}_{(Q,W)} = \mathsf{per}(\Gamma)/\mathscr{D}_{fd}(\Gamma),$$

and $\mathscr{C}_{(Q',W')}$ under the assumption that these categories have finite-dimensional morphism spaces. We refer to [1] for a thorough analysis of this situation. Let $\tilde{P}_i = e_i\Gamma$ and let $\bar{P}_i$ be the image of $\tilde{P}_i$ under the projection functor $\pi : \mathsf{per}(\Gamma) \to \mathscr{C}$.

Theorem 7.4. *The triangulated category $\mathscr{C}_{(Q',W')}$ is triangle equivalent to the Calabi–Yau reduction in the sense of Iyama–Yoshino* ([22], *Section* 4) *of $\mathscr{C}_{(Q,W)}$ at $\bar{P}_i$.*

Proof. Let us put $\mathscr{C} = \mathscr{C}_{(Q,W)}$ and $\mathscr{C}' = \mathscr{C}_{(Q',W')}$. Let $\mathscr{Z}$ be the full subcategory of $\mathscr{C}$ formed by the objects M such that $\mathrm{Ext}^1(\bar{P}_i, M)$ vanishes. By definition, the Calabi–Yau reduction at $\bar{P}_i$ is the quotient $\mathscr{Z}/(\bar{P}_i)$ of $\mathscr{Z}$ by the ideal of morphisms factoring through a finite direct sum of copies of $\bar{P}_i$. To construct a functor from $\mathscr{Z}$ to $\mathscr{C}'$, we consider the fundamental domain $\mathscr{F} \subset \mathsf{per}(\Gamma)$ as defined in [1], Section 2.2. Thus, the subcategory $\mathscr{F}$ can be described as the full subcategory

$$\mathsf{per}(\Gamma) \cap \mathscr{D}_{\leqq 0} \cap {}^{\perp}(\mathscr{D}_{\leqq -2}),$$

where $\mathscr{D}_{\leqq 0}$ is the left aisle of the canonical t-structure on $\mathscr{D}(\Gamma)$. Alternatively, the subcategory $\mathscr{F}$ can be described as the full subcategory whose objects are the cones on morphisms between objects of the closure $\mathrm{add}(\Gamma)$ of the free module Γ under finite direct sums and direct factors. We know from [loc. cit.] that the projection induces a k-linear equivalence $\mathscr{F} \xrightarrow{\sim} \mathscr{C}$. Now we consider the composition

$$\mathscr{Z} \subset \mathscr{C} \xrightarrow{\sim} \mathscr{F} \to \mathscr{F}' \xrightarrow{\sim} \mathscr{C}',$$

where $\mathscr{F}'$ is the fundamental domain for $\mathscr{C}'$. Let us denote this functor by F. Its restriction to the full subcategory $\mathscr{T}$ whose objects are the $\bar{P}_j$ associated with all vertices j identifies with the canonical projection functor

$$\mathscr{P}(Q, W) \to \mathscr{P}(Q', W').$$

In particular, since $\mathscr{P}(Q, W)$ is isomorphic to $\mathscr{T}$ by [1], Theorem 2.1, the restriction induces an equivalence

$$\mathscr{T}/(\bar{P}_i) \to \mathscr{T}',$$

where $\mathscr{T}' \subset \mathscr{C}'$ is the full subcategory of the $\bar{P}_j$, $j \neq i$. We will show below that the functor $\mathscr{Z}/(\bar{P}_i) \to \mathscr{C}'$ induced by F is naturally a triangle functor. Since this triangle functor induces an equivalence between the cluster-tilting subcategories

$$\mathscr{T}/(\bar{P}_i) \to \mathscr{T}',$$

it is itself an equivalence by [31], Lemma 4.5.

It remains to be shown that the functor $\bar{F} : \mathscr{Z}/(\bar{P}_i) \to \mathscr{C}'$ induced by F is naturally a triangle functor. Let $q : \mathscr{C} \to \mathscr{F}$ be a k-linear quasi-inverse of the projection $\mathscr{F} \to \mathscr{C}$. Let

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

be a triangle of $\mathscr{C}$ such that X, Y and Z lie in $\mathscr{Z}$. Notice that v induces a surjection

$$\mathscr{C}(\bar{P}_i, Y) \to \mathscr{C}(\bar{P}_i, Z).$$

Form a triangle in $\mathsf{per}(\Gamma)$

$$X' \longrightarrow q(Y) \xrightarrow{q(v)} q(Z) \longrightarrow \Sigma X'.$$

Claim. *The object $\tau_{\leqq 0}X'$ lies in $\mathscr{F}$. The object $q(X)$ is isomorphic to $\tau_{\leqq 0}X'$ by an isomorphism canonical up to a morphism factoring through $q(Y)$. Moreover, the image of the morphism $\tau_{\leqq 0}X' \to X'$ under the composed functor $\mathsf{per}(\Gamma) \to \mathsf{per}(\Gamma') \to \mathscr{C}'$ is invertible.*

Indeed, from the triangle

$$\Sigma^{-1}q(Z) \longrightarrow X' \longrightarrow q(Y) \xrightarrow{q(v)} q(Z),$$

we see that X' is left orthogonal to $\mathscr{D}_{\leqq -2}$. If M belongs to $\mathscr{D}_{\leqq 0}$, we have, using the Calabi–Yau property and the fact that $\tau_{>0}X'$ belongs to $\mathscr{D}_{fd}(\Gamma)$, the isomorphisms

$$\operatorname{Hom}(\Sigma^{-1}\tau_{>0}X', \Sigma^2 M) = D\operatorname{Hom}(\Sigma^{-1}M, \Sigma^{-1}\tau_{>0}X') = 0.$$

Now from the triangle

$$\Sigma^{-1}\tau_{>0}X' \to \tau_{\leqq 0}X' \to X' \to \tau_{>0}X',$$

we see that $\tau_{\leqq 0}X'$ belongs to ${}^{\perp}\mathscr{D}_{\leqq -2}$ and of course, it belongs to $\mathscr{D}_{\leqq 0}$. Thus, it belongs to $\mathscr{F}$. By our assumption, the object $\tau_{>0}X'$ has finite-dimensional homology. Thus, the image of $\tau_{\leqq 0}X'$ in $\mathscr{C}$ is isomorphic to $\pi(X')$. By the uniqueness of the triangle on the morphism $v : Y \to Z$, we obtain that X is isomorphic to $\pi(\tau_{\leqq 0}X')$ by a morphism canonical up to a morphism factoring through Y. Thus, since $\tau_{\leqq 0}X'$ belongs to $\mathscr{F}$, the object $q(X)$ is isomorphic to $\tau_{\leqq 0}X'$ by an isomorphism canonical up to a morphism factoring through $q(Y)$. Finally, the homology of $\tau_{>0}X'$ is concentrated in degree 1, and we have an exact sequence

$$H^0\big(q(Y)\big) \to H^0\big(q(X)\big) \to H^1(\tau_{>0}X') \to 0.$$

In particular, we have an exact sequence

$$\operatorname{Hom}\big(\tilde{P}_i, q(Y)\big) \to \operatorname{Hom}\big(\tilde{P}_i, q(Z)\big) \to \operatorname{Hom}(\tilde{P}_i, \tau_{>0}X') \to 0.$$

Since $\operatorname{Hom}\big(\tilde{P}_i, q(U)\big)$ is isomorphic to $\operatorname{Hom}_{\mathscr{C}}(\overline{P}_i, U)$ for each U in $\mathscr{C}$, it follows that $\tau_{>0}X'$ is right orthogonal to $\Sigma^m \tilde{P}_i$ for all $m \in \mathbb{Z}$. Thus it is right orthogonal to the kernel of the localization functor $L : \mathscr{D}\Gamma \to \mathscr{D}\Gamma'$. Therefore, for each object M of $\mathscr{D}\Gamma$, the localization functor induces a bijection

$$\operatorname{Hom}(M, \tau_{>0}X') \to \operatorname{Hom}(LM, L\tau_{>0}X').$$

If, for M, we take the objects $\Sigma^m \tilde{P}_j$ associated with the vertices of Q, we obtain that $L\tau_{>0}X'$ has its homology of finite total dimension. This implies the last part of the claim.

Now let us show that the functor $\overline{F} : \mathscr{Z}/(\overline{P}_i) \to \mathscr{C}'$ induced by F is naturally a triangle functor. In any triangulated category, by default, we denote the suspension functor

by Σ and a quasi-inverse of Σ by Ω. However, we denote the desuspension functor of the 'reduced' category $\overline{\mathscr{X}} = \mathscr{X}/(\overline{P}_i)$ by Ω_r. We will construct a natural isomorphism $\varphi : \Omega\overline{F} \xrightarrow{\sim} \overline{F}\Omega_r$ and show that the pair $(\overline{F}, \varphi)$ transforms triangles into triangles. Let Z be an object of $\mathscr{X}$ and $P \to Z$ a right approximation of Z by $\mathrm{add}(\overline{P}_i)$. Form the triangle

$$\Omega_r Z \to P \to Z \to \Sigma\Omega_r Z$$

of $\mathscr{C}$. The object $\Omega_r Z$ still belongs to $\mathscr{X}$ and its image in $\overline{\mathscr{X}}$ is the desuspension of the image of Z. Now form a triangle of $\mathsf{per}(\Gamma)$:

$$O \to q(P) \to q(Z) \to \Sigma O.$$

Let us denote the composition of the localization functor $L : \mathsf{per}(\Gamma) \to \mathsf{per}(\Gamma')$ with the projection $\mathsf{per}(\Gamma') \to \mathscr{C}'$ by $L' : \mathsf{per}(\Gamma) \to \mathscr{C}'$. By the claim, we have an isomorphism

$$q(\Omega_r Z) \xrightarrow{\sim} \tau_{\leqq 0} O$$

canonical up to a morphism factoring through $q(P)$ and the morphism $L'\tau_{\leqq 0} O \to L'O$ is invertible. The triangle

$$\Omega q(Z) \to O \to q(P) \to q(Z),$$

and the triangle structure on L' yield an isomorphism $\Omega L' q(Z) \to L'\Omega q(Z) \to L'O$. Thus, we obtain a canonical composed isomorphism

$$\Omega\overline{F}Z = \Omega L' q(Z) \xrightarrow{\sim} L'\Omega q(Z) \xrightarrow{\sim} L'O \xleftarrow{\sim} L'(\tau_{\leqq 0} O) \xleftarrow{\sim} L' q\Omega_r(Z) = \overline{F}\Omega_r(Z),$$

and we define $\varphi(Z)$ to be this isomorphism. One checks that $\varphi(Z)$ is natural in the object Z of $\overline{\mathscr{X}}$. Now let a standard triangle of $\overline{\mathscr{X}}$ be given. Then in $\mathscr{C}$, with $P \to Z$ as above, we have a morphism of triangles, where the first and fourth vertical morphisms are identities:

$$\begin{array}{ccccccc} \Omega Z & \longrightarrow & \Omega_r Z & \longrightarrow & P & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega Z & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z. \end{array}$$

Notice that the second morphism is not canonical; in fact, any morphism making the first square commutative lifts the given morphism in $\overline{\mathscr{X}}$. We will show that $(\overline{F}, \varphi)$ takes the triangle $\Omega_r Z \to X \to Y \to Z$ of $\overline{\mathscr{X}}$ to a triangle of $\mathscr{C}'$. For this, we form a morphism of triangles in $\mathsf{per}(\Gamma)$:

$$\begin{array}{ccccccc} \Omega q(Z) & \longrightarrow & O & \longrightarrow & qP & \longrightarrow & qZ \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega q(Z) & \longrightarrow & X' & \longrightarrow & qY & \longrightarrow & qZ. \end{array}$$

Its image under $\pi : \mathsf{per}(\Gamma) \to \mathscr{C}$ becomes isomorphic to the given morphism after possibly adding a morphism factoring through $\Omega Z \to P$ to the given morphism $\Omega Z \to X$. Thus, we may assume that the image under π is isomorphic to the given morphism. By the claim, the image of this morphism under $L'q$ is then isomorphic to

$$\begin{array}{ccccccc}
\Omega L'q(Z) & \longrightarrow & L'\tau_{\leqq 0}O & \longrightarrow & L'qP & \longrightarrow & L'qZ \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega L'q(Z) & \longrightarrow & L'\tau_{\leqq 0}X' & \longrightarrow & L'qY & \longrightarrow & L'qZ.
\end{array}$$

We deduce that $(\bar{F}, \varphi)$ takes the triangle $\Omega Z \to X \to Y \to Z$ to the triangle

$$\Omega L'q(Z) \to L'\tau_{\leqq 0}X' \to L'qY \to L'qZ$$

of $\mathscr{C}'$. □

7.5. Deleting a sink in global dimension 2. As a second example of localization, let us consider a finite-dimensional basic algebra A over an algebraically closed field k. Assume that P_i is the indecomposable projective module corresponding to a sink i of the quiver of A. Let e_i be the corresponding idempotent of A. Let $B = A/Ae_iA$. Then it is easy to check that the projection map

$$A \to B$$

is a localization of dg categories. Indeed, the localizing subcategory $\mathscr{N}$ of $\mathscr{D}(A)$ generated by P_i consists of all coproducts of shifted copies of P_i and its right orthogonal subcategory $\mathscr{N}^{\perp}$ is the localizing subcategory generated by the P_j, $j \neq i$. Clearly, this subcategory is equivalent to $\mathscr{D}(B)$ by the functor $? \overset{L}{\otimes}_A B$.

From now on, let us assume that A (and thus B) are of global dimension at most 2. Then A is in particular homologically smooth and by Theorem 4.6, we obtain a localization of the corresponding 3-Calabi–Yau completions

$$\Pi_3(A) \to \Pi_3(B).$$

Using Theorem 6.10, we can identify these dg algebras with Ginzburg algebras $\Gamma_3(Q, W)$ and $\Gamma_3(Q', W')$. It is not hard to check that Q' is obtained from Q by omitting the vertex corresponding to i and all arrows starting or ending at it and that W' is obtained from W by deleting all cycles passing through this vertex. Thus, the results of Section 7.1 apply and we obtain that if $\mathscr{C}(Q', W')$ is Hom-finite, then it is the Calabi–Yau reduction [22] of $\mathscr{C}(Q, W)$ at the image of $e_i\Gamma_3(Q, W)$. This example was treated previously by Amiot–Oppermann [2] using different methods.

7.6. Generalized mutations. Let k be an algebraically closed field and Q a finite quiver (possibly with oriented cycles). Let W be a potential on Q. Let T be a tilting module over kQ, i.e. a module such that if B is the endomorphism algebra of T, the derived functor

$$? \overset{L}{\otimes}_B T : \mathscr{D}(B) \to \mathscr{D}(kQ)$$

is an equivalence, cf. [29]. If X is a projective resolution of T as a B-kQ-bimodule, then $? \otimes_B X$ is a Morita functor from the dg category of bounded complexes of finitely generated projective B-modules to the corresponding category of kQ-modules. This functor yields an isomorphism

$$HC_0(B) \xrightarrow{\sim} HC_0(kQ).$$

We let $W_B \in HC_0(B)$ be the element corresponding to $W \in HC_0(kQ)$. Let c_B and c be the images in Hochschild homology of W_B and W under Connes' map B. Then by Theorem 5.8, we have an induced Morita functor

$$\Pi_3(B, c_B) \to \Pi_3(kQ, c),$$

and by Theorem 6.3 and Theorem 6.10, we obtain an induced Morita functor between Ginzburg algebras

$$\Gamma_3(Q', W' + W'') \to \Gamma_3(Q, W),$$

where the quiver Q' is obtained from the quiver of B by adding a new arrow $\rho_r : j \to i$ for each minimal relation $r : i \to j$, the potential W' is

$$W' = \sum \rho_r r,$$

and the potential W'' lifts W_B along the surjection $kQ' \to B$ taking all arrows ρ_r to zero. This construction is linked to mutation of quivers with potentials in the sense of [15] as follows: Let i be a vertex of Q which is the source of at least one arrow and let T be the direct sum of the projectives P_j, $j \neq i$, and of T_i defined by the exact sequence

$$0 \to P_i \to \bigoplus_{\alpha : i \to j} P_j \to T_i \to 0,$$

where the sum is taken over all arrows α with source i and the corresponding component of the map from P_i to the sum is the left multiplication by α. Then the passage from $B = \mathrm{End}(T)$ to kQ is given by an APR-tilt [4]. In this case, one can check that (Q', W') is the 'pre-mutation' of Q at i in the sense of [15], i.e. Q' is obtained from Q by

(1) adding an arrow $[\alpha\beta] : j \to l$ for each subquiver

$$l \xrightarrow{\beta} i \xrightarrow{\alpha} l$$

of Q and

(2) replacing each arrow $\beta : l \to i$ by an arrow $\beta^* : i \to l$ and each arrow $\alpha : i \to j$ by an arrow $\alpha^* : j \to i$;

and the potential W' is equal to $[W] + \sum[\alpha\beta]\beta^*\alpha^*$ where $[W]$ is obtained from W by replacing each occurrence of a composition $\alpha\beta$ in a cycle passing through i by $[\alpha\beta]$.

Appendix A. Ginzburg's algebra is Calabi–Yau of dimension three

A.1. Introduction. To a quasi-free algebra A and an element $z \in A$ (a "super potential") Ginzburg associates in [19] a certain DG-algebra $\mathfrak{D}(A, z)$. He proves that if $\mathfrak{D}(A, z)$ has no negative cohomology then it is 3-Calabi–Yau (see [19], Remark 5.3.2, but beware that Ginzburg uses homological grading). It was recently observed by Keller that $\mathfrak{D}(A, z)$ is *always* 3-Calabi–Yau. Below we give a proof of this fact using the formalism of non-commutative differential geometry.

A.2. Notations and conventions. Throughout we work over the semi-simple base ring $l = ke_1 + \cdots + ke_d$ where $e_i^2 = e_i$ and k is a field. In other words all our rings R are implicitly equipped with a ring homomorphism $l \to R$. Unadorned tensor products are *over* k.

A.3. Pairings of bimodules. Duality for bimodules is confusing so here we write out our conventions. This is a copy of [47], §3.1. Let B be an arbitrary graded k-algebra. We equip $B \otimes B$ with the outer B-bimodule structure. If Q is a graded B-bimodule then Q^* is by definition $\mathrm{Hom}_{B^e}(Q, B \otimes B)$. This is still a B-bimodule through the surviving inner bimodule structure on $B \otimes B$.

A pairing (or bilinear map) between graded B-bimodules P, Q is a homogeneous map of degree n,

$$\langle -, - \rangle : P \times Q \to B \otimes B \tag{A.3.1}$$

such that $\langle p, - \rangle$ is linear for the outer bimodule structure on $B \otimes B$ and $\langle -, q \rangle$ is linear for the inner bimodule structure on $B \otimes B$. The obvious example is of course when P is the bimodule dual Q^* of Q and $\langle -, - \rangle$ is the evaluation pairing. We say that the pairing is non-degenerate if P, Q are finitely generated graded projective bimodules and the pairing induces an isomorphism $P \cong \Sigma^n(Q^*)$.

Example A.4. Let $P = \Sigma^n(B \otimes_l B)$, $Q = B \otimes_l B$. It is easy to see that the pairing

$$\langle a \otimes b, c \otimes d \rangle = (-1)^{|a|\,|b|+|a|\,|c|+|b|\,|c|-n|c|} \sum_i ce_ib \otimes ae_id$$

for $a, b, c, d \in B$ is well-defined and non-degenerate of degree n.

The opposite pairing of $\langle -, - \rangle$ is defined by

$$\langle -, - \rangle^{\mathrm{opp}} : Q \times P \to B \otimes B : (q, p) \mapsto (-1)^{(n+|p|)(n+|q|)} \sigma \langle p, q \rangle,$$

where "σ" denotes the interchange operator: $\sigma(a \otimes b) = (-1)^{|a|\,|b|}(b \otimes a)$. So although the definition of a pairing of bimodules is asymmetric it is not important which bimodule appears on the left or right.

If $P = Q$ then we say that a pairing $\langle -, - \rangle$ is (anti-)symmetric if

$$\langle p, p' \rangle = (-) \langle p, p' \rangle^{\mathrm{opp}}.$$

If B is a DG-algebra and P, Q are DG-bimodules then we say that (A.3.1) is a DG-pairing if it is compatible with the differential, i.e. if

$$d\langle p, q\rangle = \langle dp, q\rangle + (-1)^{|p|+n}\langle p, dq\rangle.$$

If a DG-pairing is non-degenerate then obviously it induces an isomorphism of DG-*modules* $P \cong \Sigma^n(Q^*)$.

A.5. Differentials and double derivations. If B is a graded algebra then we denote by $\Omega_{B/l}$ the bimodule of relative differentials for B/l. $\Omega_{B/l}$ fits in an exact sequence

$$0 \to \Omega_{B/l} \xrightarrow{\varphi} B \otimes_l B \to B \to 0. \tag{A.5.1}$$

We denote the generators of $\Omega_{B/l}$ by Db, $b \in B$ where $\varphi(Db) = b \otimes 1 - 1 \otimes b$.

With respect to signs *we assume that D has homological degree zero*. If B is equipped with a differential d then we extend it to $\Omega_{B/l}$ by putting $d(Db) = D(db)$.

Assume that B is equipped with a graded double Poisson bracket of degree n (see [46], §2.1). Then there is a well-defined anti-symmetric pairing on $\Omega_{B/l}$ of degree n which is determined by

$$\langle D\eta, D\xi\rangle = \{\{\eta, \xi\}\}.$$

We define $\mathbb{T}_{B/l} = \Omega^*_{B/l}$. We may identify $\mathbb{T}_{B/l}$ with the bimodule of *double derivations*

$$\mathbb{T}_{B/l} = \mathrm{Der}_{B/l}(B, B \otimes B).$$

If $b \in B$ and $\delta \in \mathbb{T}_{B/l}$ then we write $\delta(b) = \delta(b)' \otimes \delta(b)''$. $\mathbb{T}_{B/l}$ contains a canonical element E given by

$$E(a) = \sum_i ae_i \otimes e_i - e_i \otimes e_i a.$$

Remark A.6. We may write $E(a) = [a, \xi]$ where $\xi = \sum_i e_i \otimes e_i \in l \otimes l$. If, as in [12], one works over a more general separable k-algebra then one must replace ξ by the separability idempotent in l^e.

A.7. The graded cotangent bundle. Now let A be a quasi-free finitely generated k-algebra and put $\mathbb{T}A = T_A(\Sigma\mathbb{T}_{A/l})$. According to [46], §3.2, $\mathbb{T}A$ carries a canonical graded double Poisson bracket of degree 1: the so-called double Schouten–Nijenhuis bracket.[1)] Thus according to §A.5 we get an induced anti-symmetric pairing on $\Omega_{\mathbb{T}A/l}$ of degree 1.

Lemma A.8. *This pairing is non-degenerate.*

[1)] In [46] this bracket had degree -1 since we used the opposite grading.

Proof. This can be deduced from the fact that the double Schouten–Nijenhuis bracket is actually induced from a bisymplectic form [12], [46]. To help the reader let us give a proof here. We have a standard exact sequence

$$0 \to \mathbb{T}A \otimes_A \Omega_{A/l} \otimes_A \mathbb{T}A \xrightarrow{\alpha} \Omega_{\mathbb{T}A/l} \xrightarrow{\beta} \Sigma(\mathbb{T}A \otimes_A \mathbb{T}_{A/l} \otimes_A \mathbb{T}A) \to 0$$

with for $\omega \in \Omega_{A/l}$, $\delta \in \mathbb{T}_{A/l}$

$$\alpha(1 \otimes \omega \otimes 1) = \omega,$$
$$\beta(D\delta) = 1 \otimes \delta \otimes 1.$$

Hence we have for $a \in A$

$$\langle 1 \otimes D\delta \otimes 1, \alpha(Da)\rangle = \langle D\delta, Da\rangle = \{\{\delta, a\}\} = \delta(a) = \langle \beta(D\delta), Da\rangle,$$

where on the right we have the standard (non-degenerate) pairing between $\mathbb{T}_{A/l}$ and $\Omega_{A/l}$, extended to a (still non-degenerate) pairing between $\mathbb{T}A \otimes_A \mathbb{T}_{A/l} \otimes_A \mathbb{T}A$ and $\mathbb{T}A \otimes_A \Omega_{A/l} \otimes_A \mathbb{T}A$. It follows that α and β are adjoint.

Thus one gets a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{T}A \otimes_A \Omega_{A/l} \otimes_A \mathbb{T}A & \xrightarrow{\alpha} & \Omega_{\mathbb{T}A/l} & \xrightarrow{\beta} & \Sigma(\mathbb{T}A \otimes_A \mathbb{T}_{A/l} \otimes_A \mathbb{T}A) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{T}A \otimes_A \mathbb{T}^*_{A/l} \otimes_A \mathbb{T}A & \xrightarrow[\beta^*]{} & \Sigma(\Omega^*_{\mathbb{T}A/l}) & \xrightarrow[\alpha^*]{} & \Sigma(\mathbb{T}A \otimes_A \Omega^*_{A/l} \otimes_A \mathbb{T}A) & \longrightarrow & 0.
\end{array}$$

Hence the middle arrow is an isomorphism. □

Now fix a "super potential" $z \in \sum e_i A e_i$. Contraction with Dz defines a differential d on $\mathbb{T}A$ [19] (see also [47], §3.1). On generators we have

$$da = 0 \quad \text{for } a \in A,$$
$$d\delta = \delta(z)''\delta(z)' \quad \text{for } \delta \in \mathbb{T}_{A/l}.$$

We will denote resulting DG-algebra by $\mathbb{T}(A, z)$.

In the commutative case it is well known that contraction with a 1-form is a derivation for the Gerstenhaber structure on the graded cotangent bundle and hence in particular it is compatible with the Schouten bracket. A similar result is true in the non-commutative case.

Lemma A.9. *$\mathbb{T}(A, z)$ is a* DG-*Gerstenhaber algebra with product of degree zero and double bracket of degree one.*

Proof. We only need to check compatibility of the differential with the double bracket. This can be done on generators. The only non-trivial verification is

$$d\{\{\delta,\Delta\}\} = \{\{d\delta,\Delta\}\} + \{\{\delta,d\Delta\}\} \tag{A.9.1}$$

for $\delta,\Delta \in \mathbb{T}_{A/l}$.

Following the notations of [46], §3.2, we have

$$\{\{\delta,\Delta\}\} = \{\{\delta,\Delta\}\}_l + \{\{\delta,\Delta\}\}_r$$

with

$$\{\{\delta,\Delta\}\}_l = \{\{\delta,\Delta\}\}_l' \otimes \{\{\delta,\Delta\}\}_l'' \in \mathbb{T}_{A/l} \otimes A,$$
$$\{\{\delta,\Delta\}\}_r = \{\{\delta,\Delta\}\}_r' \otimes \{\{\delta,\Delta\}\}_r'' \in A \otimes \mathbb{T}_{A/l},$$

so that we have

$$d\{\{\delta,\Delta\}\} = d\{\{\delta,\Delta\}\}_l + d\{\{\delta,\Delta\}\}_r$$

with

$$d\{\{\delta,\Delta\}\}_l = \{\{\delta,\Delta\}\}_l'(z)''\{\{\delta,\Delta\}\}_l'(z)' \otimes \{\{\delta,\Delta\}\}_l'',$$
$$d\{\{\delta,\Delta\}\}_r = \{\{\delta,\Delta\}\}_r' \otimes \{\{\delta,\Delta\}\}_r''(z)''\{\{\delta,\Delta\}\}_r''(z)'.$$

By definition we have

$$\{\{\delta,\Delta\}\}_l = \sigma_{23} \circ \big((\delta \otimes 1)\Delta - (1 \otimes \Delta)\delta\big),$$
$$\{\{\delta,\Delta\}\}_r = \sigma_{12} \circ \big((1 \otimes \delta)\Delta - (\Delta \otimes 1)\delta\big),$$

which after inspection becomes

$$d\{\{\delta,\Delta\}\}_l = \Delta(z)''\delta\big(\Delta(z)'\big)' \otimes \delta\big(\Delta(z)'\big)'' - \Delta\big(\delta(z)''\big)''\delta(z)' \otimes \Delta\big(\delta(z)''\big)',$$
$$d\{\{\delta,\Delta\}\}_r = \delta\big(\Delta(z)''\big)' \otimes \delta\big(\Delta(z)''\big)''\Delta(z)' - \Delta\big(\delta(z)'\big)'' \otimes \delta(z)''\Delta\big(\delta(z)'\big)'.$$

On the other hand, we have

$$\begin{aligned}\{\{d\delta,\Delta\}\} &= -\sigma\Delta\big(\delta(z)''\delta(z)'\big)\\ &= -\Delta\big(\delta(z)''\big)''\delta(z)' \otimes \Delta\big(\delta(z)''\big)' - \Delta\big(\delta(z)'\big)'' \otimes \delta(z)''\Delta\big(\delta(z)'\big)'\end{aligned}$$

and

$$\begin{aligned}\{\{\delta,d\Delta\}\} &= \delta\big(\Delta(z)''\Delta(z)'\big)\\ &= \delta\big(\Delta(z)''\big)' \otimes \delta\big(\Delta(z)''\big)''\Delta(z)' + \Delta(z)''\delta\big(\Delta(z)'\big)' \otimes \delta\big(\Delta(z)'\big)''\end{aligned}$$

so that (A.9.1) indeed holds. □

We immediately deduce

Lemma A.10. *The pairing on $\Omega_{\mathbb{T}A/l}$ is compatible with d.*

Proof. We have to prove for $\omega, \omega' \in \Omega_{\mathbb{T}A/l}$

$$d\langle \omega, \omega' \rangle = \langle d\omega, \omega' \rangle + (-1)^{|\omega'|+1} \langle \omega, d\omega' \rangle.$$

One verifies that it is sufficient to check this on $\mathbb{T}A$-bimodule generators of $\Omega_{\mathbb{T}A/l}$. The only interesting case to consider is $\omega = D\delta$, $\omega' = D\Delta$ and $\delta, \Delta \in \mathbb{T}_{A/l}$. In that case the result is a direct concequence of Lemma A.9 and in particular (A.9.1). □

A.11. Ginzburg's algebra. Let A, z, $\mathbb{T}A$ be as in the previous section. We have $E \in \mathbb{T}_{A/l} \subset \mathbb{T}A$. We immediately check that $dE = 0$. So E defines a (presumably always non-trivial) cohomology class in $\mathbb{T}A$. Ginzburg's idea is to kill this class through adjunction of an extra variable c of degree -2 commuting with l. So Ginzburg's algebra is

$$\mathfrak{D}(A, z) = \mathbb{T}(A, z) *_l l[c],$$

where $|c| = -2$ and $dc = E$. To simplify the notations we will write $\mathbb{T} = \mathbb{T}(A, z)$ and $\mathfrak{D} = \mathfrak{D}(A, z)$ in this section.

We have a presentation

$$0 \to \Omega_{\mathfrak{D}/l} \xrightarrow{\varphi} \mathfrak{D} \otimes_l \mathfrak{D} \to \mathfrak{D} \to 0,$$

where φ is as in (A.5.1). It is easy to see that as graded $\mathfrak{D}$-bimodule we have

$$\Omega_{\mathfrak{D}/l} = (\mathfrak{D} \otimes_{\mathbb{T}} \Omega_{\mathbb{T}/l} \otimes_{\mathbb{T}} \mathfrak{D}) \oplus (\mathfrak{D} \otimes_l lDc \otimes_l \mathfrak{D}).$$

Put $\mathbb{I} = \sum_i e_i \otimes e_i$. Then $\mathfrak{D}$ is quasi-isomorphic to $\operatorname{cone} \varphi$ and $\operatorname{cone} \varphi$ is given by

$$P = (\mathfrak{D} \otimes_l l\mathbb{I} \otimes_l \mathfrak{D}) \oplus \Sigma(\mathfrak{D} \otimes_{\mathbb{T}} \Omega_{\mathbb{T}/l} \otimes_{\mathbb{T}} \mathfrak{D}) \oplus \Sigma(\mathfrak{D} \otimes_l lDc \otimes_l \mathfrak{D})$$

with total differential

$$\begin{aligned} d_P \mathbb{I} &= 0, \\ d_P \omega &= \varphi_{\mathbb{T}}(\omega) - d_{\mathbb{T}} \omega \quad \text{for } \omega \in \Omega_{\mathbb{T}}, \\ d_P(Dc) &= [c, \mathbb{I}] - D(E). \end{aligned}$$

We define a *symmetric* pairing of degree 3 on P by putting

$$\langle Dc, \mathbb{I} \rangle_P = \sum_i e_i \otimes e_i,$$

$$\langle \omega, \omega' \rangle_P = (-1)^{|\omega|_{\mathbb{T}}-1} \langle \omega, \omega' \rangle_{\mathbb{T}},$$

and assigning the value zero on other combinations of generators of P taken from $\mathbb{I}$, $\Omega_{\mathbb{T}/l}$, Dc. Note that in P we have $|\mathbb{I}| = 0$, $|Dc| = -3$ and $|\omega|_P = |\omega|_{\mathbb{T}} - 1$ for $\omega \in \Omega_{\mathbb{T}/l}$. The requirement of symmetry yields

$$\begin{aligned}\langle \mathbb{I}, Dc\rangle_P &= (-1)^{(|\mathbb{I}|+3)(|Dc|+3)}\sigma\langle Dc, \mathbb{I}\rangle_P \\ &= \sum_i e_i \otimes e_i.\end{aligned}$$

By combining Example A.4 with Lemma A.8 we see that $\langle -, -\rangle_P$ is non-degenerate.

We claim that $\langle -, -\rangle_P$ is compatible with the differential. By symmetry this amounts to six verifications which we now carry out.

Case 1. One has

$$d_{\mathfrak{D}}\langle Dc, Dc\rangle_P = 0,$$

and

$$\begin{aligned}\langle d_P Dc, Dc\rangle_P &= \langle [c, \mathbb{I}] - D(E), Dc\rangle_P \\ &= \sum_i (e_i \otimes ce_i - e_i c \otimes e_i) \\ &= \sum_i (e_i \otimes e_i c - ce_i \otimes e_i),\end{aligned}$$

and

$$\begin{aligned}\langle Dc, d_P Dc\rangle_P &= \langle Dc, [c, \mathbb{I}] - D(E)\rangle_P \\ &= \sum_i (ce_i \otimes e_i - e_i \otimes e_i c)\end{aligned}$$

so that

$$d_{\mathfrak{D}}\langle Dc, Dc\rangle_P = \langle d_P Dc, Dc\rangle_P + (-1)^{|Dc|+3}\langle Dc, d_P Dc\rangle_P.$$

Case 2. One has for $u \in \mathbb{T}$

$$d_{\mathfrak{D}}\langle Dc, Du\rangle_P = 0,$$

and

$$\begin{aligned}\langle d_P Dc, Du\rangle_P &= \langle [c, \mathbb{I}] - D(E), Du\rangle_P \\ &= -(-1)^{|E|_{\mathbb{T}}-1}\langle D(E), Du\rangle_{\mathbb{T}} \\ &= -\{\{E, u\}\} \\ &= -\sum_i (ue_i \otimes e_i - e_i \otimes e_i u),\end{aligned}$$

and

$$\begin{aligned}\langle Dc, d_P Du\rangle_P &= \langle Dc, [u, \mathbb{l}] - Dd_{\mathbb{T}}u\rangle \\ &= \sum_i ue_i \otimes e_i - e_i \otimes e_i u\end{aligned}$$

so that

$$d_{\mathfrak{D}}\langle Dc, Du\rangle_P = \langle d_P Dc, Du\rangle_P + (-1)^{|Dc|+3}\langle Dc, d_P Du\rangle_P.$$

Case 3. One has

$$d_{\mathfrak{D}}\langle Dc, \mathbb{l}\rangle_P = 0,$$

and

$$\begin{aligned}\langle d_P Dc, \mathbb{l}\rangle_P &= \langle [c, \mathbb{l}] - D(E), \mathbb{l}\rangle_P \\ &= 0,\end{aligned}$$

and

$$\langle Dc, d_P \mathbb{l}\rangle_P = 0.$$

Hence this case is trivial.

Case 4. One has for $\omega, \omega' \in \Omega_{\mathbb{T}/l}$

$$\begin{aligned}d_{\mathfrak{D}}\langle \omega, \omega'\rangle_P &= (-1)^{|\omega|_{\mathbb{T}}-1} d_{\mathbb{T}}\langle \omega, \omega'\rangle_{\mathbb{T}} \\ &= (-1)^{|\omega|_{\mathbb{T}}-1}\langle d_{\mathbb{T}}\omega, \omega'\rangle_{\mathbb{T}} + (-1)^{|\omega|_{\mathbb{T}}-1}(-1)^{|\omega|_{\mathbb{T}}+1}\langle \omega, d_{\mathbb{T}}\omega'\rangle_{\mathbb{T}} \\ &= (-1)^{|\omega|_{\mathbb{T}}-1}\langle d_{\mathbb{T}}\omega, \omega'\rangle_{\mathbb{T}} + \langle \omega, d_{\mathbb{T}}\omega'\rangle_{\mathbb{T}},\end{aligned}$$

and

$$\begin{aligned}\langle d_P\omega, \omega'\rangle_P &= \langle \varphi_{\mathbb{T}}(\omega) - d_{\mathbb{T}}\omega, \omega'\rangle_P \\ &= -(-1)^{|\omega_{\mathbb{T}}|+1-1}\langle d_{\mathbb{T}}\omega, \omega'\rangle_{\mathbb{T}} \\ &= (-1)^{|\omega_{\mathbb{T}}|-1}\langle d_{\mathbb{T}}\omega, \omega'\rangle_{\mathbb{T}},\end{aligned}$$

and

$$\begin{aligned}\langle \omega, d_P\omega'\rangle_P &= \langle \omega, \varphi_{\mathbb{T}}(\omega') - d_{\mathbb{T}}(\omega')\rangle_P \\ &= -(-1)^{|\omega|_{\mathbb{T}}-1}\langle \omega, d_{\mathbb{T}}(\omega')\rangle_{\mathbb{T}} \\ &= (-1)^{|\omega|_{\mathbb{T}}}\langle \omega, d_{\mathbb{T}}(\omega)\rangle_{\mathbb{T}}\end{aligned}$$

so that we get

$$d_{\mathfrak{D}}\langle\omega,\omega'\rangle_P = \langle d_P\omega,\omega'\rangle_P + (-1)^{|\omega|_{\mathbb{T}}}\langle\omega,d_P\omega'\rangle_P,$$

which is correct since $|\omega|_{\mathbb{T}} = |\omega|_P + 3(\mathrm{mod}\,2)$.

Case 5. One has for $\omega \in \Omega_{\mathbb{T}/l}$

$$d_{\mathfrak{D}}\langle\omega,\mathbb{I}\rangle_P = 0,$$

and

$$\begin{aligned}\langle d_P\omega,\mathbb{I}\rangle_P &= \langle\varphi_{\mathbb{T}}(\omega) - d_c(\omega),\mathbb{I}\rangle\\ &= 0,\end{aligned}$$

and

$$\langle\omega,d_P\mathbb{I}\rangle_P = 0.$$

So nothing to prove here!

Case 6. The last case is about $\langle\mathbb{I},\mathbb{I}\rangle_P$ but this is trivial.

We can now conclude

Theorem A.12. *The Ginzburg algebra* $\mathfrak{D}$ *is* 3*-Calabi–Yau.*

Proof. We need to prove

$$\mathrm{RHom}_{\mathfrak{D}^e}(\mathfrak{D},\mathfrak{D}\otimes\mathfrak{D}) \cong \Sigma^{-3}\mathfrak{D} \tag{A.12.1}$$

in $D(\mathfrak{D}^e)$ and moreover this isomorphism must be self dual. We have

$$\begin{aligned}\mathrm{RHom}_{\mathfrak{D}^e}(\mathfrak{D},\mathfrak{D}\otimes\mathfrak{D}) &\cong \mathrm{Hom}_{\mathfrak{D}^e}(P,\mathfrak{D}\otimes\mathfrak{D})\\ &\cong \Sigma^{-3}P\\ &\cong \Sigma^{-3}\mathfrak{D},\end{aligned}$$

where the second isomorphism is obtained from the pairing $\langle-,-\rangle_P$. Self duality follows from the fact that $\langle-,-\rangle_P$ is symmetric. □

A.13. A word on quivers. Assume now that V is a finitely generated l-bimodule and put $A = T_l V$. Thus A is the path algebra of a quiver. We remind the reader on the concrete interpretation of $\mathfrak{D}(A,z)$ in this case. This is taken from [19]. Let $(t_i)_i$ be a k-basis of V where for each i we have $t(i)$, $h(i)$ such that $t_i \in e_{t(i)}Ve_{h(i)}$.

Then we may define operations

$$\left(\frac{\partial}{\partial t^i}\right)^+ : A/[A,A] \to A,$$

$$\frac{\partial}{\partial t^i} : A \to A \otimes A,$$

where the second one is the element of $\mathbb{T}_{A/l}$ with the property

$$\frac{\partial t^j}{\partial t^i} = \delta^{ij}(e_{t(i)} \otimes e_{h(i)}),$$

and the first one is obtained from the first by the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & A/[A,A] \\ {\scriptstyle \frac{\partial}{\partial t^i}}\downarrow & & \downarrow{\scriptstyle \left(\frac{\partial}{\partial t^i}\right)^+} \\ A \otimes A & \xrightarrow{a\otimes b \mapsto ba} & A. \end{array}$$

By [46], Proposition 6.2.2(2), we have

$$E = \sum_i \left[\frac{\partial}{\partial t^i}, t^i\right] \tag{A.13.1}$$

as elements of $\mathbb{T}_{A/l}$.

Pick $z \in \bigoplus_i e_i A e_i$.

Lemma A.14 ([19]). *As graded algebras there is a canonical isomorphism*

$$\mathfrak{D}(A,z) = T_l(V \oplus \Sigma V^* \oplus kc).$$

Furthermore, if t^i is the dual basis to t_i, then the differential on $\mathfrak{D}(A,z)$ is given by

$$\begin{aligned} dt^i &= 0, \\ dt_i &= \left(\frac{\partial z}{\partial x^i}\right)^+, \\ dc &= \sum_i [t_i, t^i]. \end{aligned} \tag{A.14.1}$$

Proof. Put $t_i = \partial/\partial t^i$. We get $\mathbb{T}(A,z) = T_l(V \oplus V^*)$ where (t^i) is the basis for V^*, dual to $(t_i)_i$.

The differential d on $\mathbb{T}(A,z)$ has the property

$$dt^i = 0,$$
$$dt_i = \left(\frac{\partial z}{\partial x^i}\right)^+.$$

Finally, the algebra $\mathfrak{D}(A,z)$ is obtained by adjoining c such that

$$dc = E = \sum_i [t_i, t^i],$$

where we have used (A.13.1). □

A.15. A word on Ext-algebras. The advantage of the presentation (A.14.1) is that we can immediately read off the A_∞-structure on the Ext-algebra of $\mathfrak{D}(A,z)$. This works more generally as follows. Assume that W is a finite dimensional l-bimodule and we have a DG-algebra structure on $B = T_l W$ compatible with the canonical augmentation $B \to l$. Then for $w \in W$ we may write

$$dw = \sum_{n=1}^{\infty} b_n^*(w),$$

where the b_n^* are maps

$$b_n^* : W \to W^{\otimes n}$$

of degree 1. Dualizing we get maps of degree 1

$$b_n : (W^*)^{\otimes n} \to W^*,$$

which define an A_∞ structure on $\Sigma^{-1}(W^*)$ (without unit). It follows from tbe bar-cobar machinery that the A_∞-algebra $l \oplus \Sigma^{-1}(W^*)$ corresponds to $\mathrm{RHom}_B(l,l)$.

Now let V, A, z, $\mathfrak{D}(A,z)$ be as before and assume that z contains no linear terms. We put $W = V \oplus \Sigma V^* \oplus kc$. Thus $\mathfrak{D}(A,z) = T_l W$ and the Ext-algebra of $\mathfrak{D}(A,z)$ as a graded vector space[2)] is $l \oplus \Sigma^{-1} W^* = l \oplus \Sigma^{-1} V^* \oplus \Sigma^{-2} V \oplus k\Sigma^{-1}(c^*)$.

One checks that the A_∞-operations are the pairings $V^* \otimes V \to l$ and $V \otimes V^* \to l$ as well n-ary operations $(V^*)^{\otimes n} \to \Sigma^{-1} V$ which are obtained from the degree $n+1$-part $z_{n+1} \in V^{\otimes(n+1)}$ of the superpotential z.

[2)] If z contains quadratic terms then this algebra has a non-trivial differential so it is not strictly speaking the Ext-algebra.

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