ON TWO EXAMPLES BY IYAMA AND YOSHINO

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ABSTRACT. In a recent paper Iyama and Yoshino consider two interesting examples of isolated singularities over which it is possible to classify the indecomposable maximal Cohen-Macaulay modules in terms of linear algebra data. In this paper we present two new approaches to these examples. In the first approach we give a relation with cluster categories. In the second approach we use Orlov's result on the graded singularity category.

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1. INTRODUCTION

Throughout k is a field. In [17] Iyama and Yoshino consider the following two settings.

Example 1.1. Let $S = k[[x_1, x_2, x_3]]$ and let $C_3 = \langle\sigma\rangle$ be the cyclic group of three elements. Consider the action of C_3 on S via $\sigma x_i = \omega x_i$ where $\omega^3 = 1$, $\omega \neq 1$. Put $R = S^{C_3}$.

Example 1.2. Let $S = k[[x_1, x_2, x_3, x_4]]$ and let $C_2 = \langle\sigma\rangle$ be the cyclic group of two elements. Consider the action of C_2 on S via $\sigma x_i = -x_i$. Put $R = S^{C_2}$.

In both examples Iyama and Yoshino reduce the classification of maximal Cohen-Macaulay modules over R to the representation theory of certain generalized Kronecker quivers. They use this to classify the rigid Cohen-Macaulay modules over R. As predicted by deformation theory, the latter are described by discrete data.

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The explicit description of the stable category of maximal Cohen-Macaulay modules over a commutative Gorenstein ring (also known as the singularity category [10, 9, 29]) is a problem that has received much attention over the years. This appears to be in general a difficult problem and perhaps the best one can hope for is a reduction to linear algebra, or in other words: the representation theory of quivers. This is precisely what Iyama and Yoshino have accomplished.

The proofs of Iyama and Yoshino are based on the machinery of mutation in triangulated categories, a general theory developed by them. In the current paper we present two alternative approaches to the examples. Hopefully the additional insight obtained in this way may be useful elsewhere.

Our first approach applies to Example 1.2 and is inspired by the treatment in [21] of Example 1.1 where the authors used the fact that in this case the stable category $\underline{\mathrm{MCM}}(R)$ of maximal Cohen-Macaulay R-modules is a 2-Calabi-Yau category which has a cluster tilting object whose endomorphism ring is the path algebra kQ_3 of the Kronecker quiver with 3 arrows. Then they invoke their acyclicity result (slightly specialized):

Theorem 1.3. [21, §1, Thm] *Assume that $\mathcal{T}$ is k-linear algebraic Krull-Schmidt 2-Calabi-Yau category with a cluster tilting object T such that $A = \mathrm{End}(T)$ is hereditary. Then there is an exact equivalence betweem $\mathcal{T}$ and the orbit category $D^b(\mathrm{mod}(A))/(\tau[-1])$.*

From this result they obtain immediately that $\underline{\mathrm{MCM}}(R)$ is the orbit category $D^b(\mathrm{mod}(kQ_3))/(\tau[-1])$. This gives a very satisfactory description of $\underline{\mathrm{MCM}}(R)$ and implies in particular the results by Iyama and Yoshino.

In the first part of this paper we show that Example 1.2 is amenable to a similar approach. Iyama and Yoshino prove that $\underline{\mathrm{MCM}}(R)$ is a 3-Calabi-Yau category with a 3-cluster tilting object T such that $\mathrm{End}(T) = k$ [17, Theorem 9.3]. We show that under these circumstances there is an analogue of the acyclicity result of the first author and Reiten.

Theorem 1.4. *(see §4.4) Assume that $\mathcal{T}$ is k-linear algebraic Krull-Schmidt 3-Calabi-Yau category with a 3-cluster tilting object T such that $\mathrm{End}(T) = k$. Then there is an exact equivalence of $\mathcal{T}$ with the orbit category $D^b(\mathrm{mod}(kQ_n))/(\tau^{1/2}[-1])$, $n = \dim \mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)$, where Q_n is the generalized Kronecker quiver with n arrows and $\tau^{1/2}$ is a natural square root of the Auslander-Reiten translate of $D^b(\mathrm{mod}(kQ_n))$, which on the pre-projective/pre-injective component corresponds to "moving one place to the left".*

In the second part of this paper, which is logically independent of the first we give yet another approach to the examples 1.1,1.2 based on the following observation which might have independent interest.

Proposition 1.5. *(see Prop. A.8) Let $A = k + A_1 + A_2 \cdots$ be a finitely generated commutative graded Gorenstein k-algebra with an isolated singularity. Let $\widehat{A}$ be the completion of A at $A_{\geq 1}$. Let $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ be the stable category of graded maximal Cohen-Macaulay A-modules. Then the obvious functor $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to \underline{\mathrm{MCM}}(\widehat{A})$ induces an equivalence*

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1) \cong \underline{\mathrm{MCM}}(\widehat{A}) \tag{1.1}$$

where $M \mapsto M(1)$ is the shift functor for the grading.

In this proposition the quotient $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1)$ has to be understood as the triangulated/Karoubian hull (as explained in [20]) of the naive quotient of $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ by the shift functor $?(1)$. This result is similar in spirit to [3] which treats the finite representation type case. Note however that one of the main results in loc. cit. is that in case of finite representation type case *every* indecomposable maximal Cohen-Macaulay $\hat{A}$-module is gradable. This does not seem to be a formal consequence of Proposition 1.5. It would be interesting to investigate this further.

In §7 we show that at least rigid Cohen-Macaulay modules are always gradable so they are automatically in the image of $\mathrm{MCM}_{\mathrm{gr}}(A)$. We expect this to be well known in some form but we have been unable to locate a reference.

Hence in order to understand $\mathrm{MCM}(\widehat{A})$ it is sufficient to understand $\mathrm{MCM}_{\mathrm{gr}}(A)$. The latter is the graded singularity category [28] of A and by [28, Thm 2.5] it is related to $D^b(\mathrm{coh}(X))$ where $X = \operatorname{Proj} A$.

In Examples 1.1,1.2 R is the completion of a graded ring A which is the Veronese of a polynomial ring. Hence $\operatorname{Proj} A$ is simply a projective space. Using Orlov's results and the existence of exceptional collections on projective space we get very quickly in Example 1.1

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \cong D^b(\mathrm{mod}(kQ_3))$$

and in Example 1.2

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \cong D^b(\mathrm{mod}(kQ_6))$$

(where here and below $\cong$ actually stands for a quasi-equivalence between the underlying DG-categories). Finally it suffices to observe that in Example 1.1 we have $?(-1) = \tau[-1]$ and in Example 1.2 we have $?(-1) = \tau^{1/2}[-1]$ (see §6 below).

Finally we mention the following interesting side result

Proposition 1.6. *Let* (R, m) *be a Gorenstein local "G-ring"* (*for example* R *may be essentially of finite type over a field*) *with an isolated singularity. Then the natural functor*

$$\widehat{R}\otimes_R ? : \underline{\mathrm{MCM}}(R) \to \underline{\mathrm{MCM}}(\widehat{R}) \tag{1.2}$$

is an equivalence up to direct summands. In partular every maximal Cohen-Macaulay module over $\widehat{R}$ *is a direct summand of the completion of a maximal Cohen-Macaulay module over* R.

The original proof (by the first and the third author) of this result was unnecessarily complicated. After the paper was put on the arXiv Daniel Murfet (who has become the second author) informed us about the existence of a much nicer proof in the context of singularity categories (see Proposition A.1). The same argument also applies to Proposition 1.5. So we dropped our original proofs and put the new argument in an appendix to which we refer.

Meanwhile Orlov [30] has proved (independently and using different methods) a very general result which implies in particular Proposition 1.6.

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3. Notations and conventions

We hope most notations are self explanatory but nevertheless we list them here. If R is a ring then $\mathrm{Mod}(R)$ and $\mathrm{mod}(R)$ denote respectively the category of all left R-modules and the full subcategory of finitely generated R-modules. The derived category of all R-modules is denoted by $D(R)$. If R is graded then we use $\mathrm{Gr}(R)$ and $\mathrm{gr}(R)$ for the category of graded left modules and its subcategory of finitely generated modules. The shift functor on $\mathrm{Gr}(R)$ is denoted by $?(1)$. Explicitly $M(1)_i = M_{i+1}$. If we want to refer to right modules then we use R° instead of R. If X is a scheme then $\mathrm{Qch}(X)$ is the category of quasi-coherent $\mathcal{O}_X$-modules. If X is noetherian then $\mathrm{coh}(X)$ is the category of coherent $\mathcal{O}_X$-modules. We are generally very explicit about which categories we use. E.g. we write $D^b(\mathrm{mod}(R))$ rather than something like $D^b_f(R)$. If R is graded and M, N are graded R-modules then $\mathrm{Ext}^i_R(M,N)$ is the ungraded Ext between M and N. If we need Ext in the category of graded R-modules then we write $\mathrm{Ext}^i_{\mathrm{Gr}(R)}(M,N)$.

4. First approach to the second example

4.1. **Some preliminaries on tilting complexes.** Let C, E be rings. We denote the unbounded derived category of right C-modules by $D(C^\circ)$. We let $\mathrm{Eq}(D(C^\circ), D(E^\circ))$ be the set of triangle equivalences of $D(C^\circ) \to D(E^\circ)$ modulo natural isomorphisms. Define $\mathrm{Tilt}(C,E)$ as the set of pairs (ϕ, T) where T is a perfect complex generating $D(E^\circ)$ and ϕ is an isomorphism $C \to \mathrm{RHom}_E(T)$. Associated to $(\phi,T) \in \mathrm{Tilt}(C,E)$ there is a canonical equivalence $\theta : D(C^\circ) \to D(E^\circ)$ such that $\theta(C) = T$. It may be constructed either directly [34] or using DG-algebras [19]. The induced map

$$\mathrm{Tilt}(C,E) \to \mathrm{Eq}(D(C^\circ), D(E^\circ))$$

is obviously injective (as it is canonically split), but not known to be surjective. Below we will informally refer to the elements of $\mathrm{Tilt}(C,E)$ as tilting complexes.

4.2. **A square root of τ for a generalized Kronecker quiver.** Let W be a finite dimensional k-vector space and let C be the path algebra of the quiver[1]

$$\underset{1}{\bullet} \xleftarrow{\quad W \quad} \underset{2}{\bullet} \tag{4.1}$$

Let E be the path algebra of the quiver

$$\underset{1}{\bullet} \xleftarrow{\quad W^* \quad} \underset{2}{\bullet}$$

which we think of as being obtained from (4.1) by "inverting the arrows" and renumbering the vertices $(1,2) \mapsto (2,1)$.

Let P_i,I_i, S_i be respectively the projective, injective and simple right C-module corresponding to vertex i. For E we use P'_i, I'_i, S'_i. Let $r_i : \mathrm{mod}(C^\circ) \to \mathrm{mod}(E^\circ)$ be the reflection functor at vertex i. Recall that if (U,V) is a representation of C then $r_1(U,V)$ is given by (V,U') where $U' = \ker(V \otimes W \to U)$ (taking into account the renumbered vertices).

[1]We use the convention that multiplication in the path algebra is concatenation. So representations correspond to right modules.

The right derived functor Rr_1 of r_1 defines an equivalence $D(C^\circ) \to D(E^\circ)$. It is obtained from the tilting complex $S_2'[-1] \oplus P_1'$ [2]. One has (see [15])

$$Rr_1 \circ Rr_1 = \tau_C \tag{4.2}$$

where τ_C is the Auslander Reiten translate on $D(C^\circ)$. Assume now that W is equipped with an isomorphism $\pi : W \to W^*$. Then π yields an equivalence $D(E^\circ) \cong D(C^\circ)$, which we denote by the same symbol. We use the same convention for the transpose isomorphism $\pi^* : W \to W^*$.

Lemma 4.2.1. *We have $r_1 \circ \pi^{-1} = \pi^* \circ r_1$ as functors $D(C^\circ) \to D(C^\circ)$.*

Proof. Let (U, V) be a representation of C determined by a linear map $c : V \otimes W \to U$ and put $(V, U'') = (r_1 \circ \pi^{-1})(U, V)$. Then one checks that U'' is given by the exact sequence

$$0 \to U'' \to V \otimes W^* \xrightarrow{c\circ(\pi^{-1}\otimes \mathrm{id})} U \to 0$$

where the first non-trivial map induces the action $U'' \otimes W \to V$. Similarly if we put $(V, U') = (\pi^* \circ r_1)(U, V)$ then one gets the same sequence

$$0 \to U' \to V \otimes W^* \xrightarrow{c\circ(\pi^{-1}\otimes \mathrm{id})} U$$

where the first non-trivial map again yields the action $U' \otimes W \to V$. Thus we have $(V, U') = (V, U'')$. □

Below we put $a = \pi \circ Rr_1$.

Lemma 4.2.2. *One has $(\pi^* \circ \pi^{-1}) \circ a^2 = \tau$. In particular $\tau \cong a^2$ if and only if π is self-adjoint or anti self-adjoint.*

Proof. This is a straightforward verification using Lemma 4.2.1 and (4.2). □

For use below we record

$$\begin{aligned} aP_2 &= P_1 \\ aP_1 &= I_2[-1] \\ aI_2 &= I_1 \end{aligned}$$

4.3. **A 3-Calabi-Yau category with a 3-cluster tilting object.** We let the notations be as in the previous section,

Put $\mathcal{H} = D^b(\mathrm{mod}(C^\circ))$, $\mathcal{D} = \mathcal{H}/a[-1]$. As $\mathcal{H}$ is hereditary we have

$$\mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}(\mathcal{H})/a[-1]$$

Inspection reveals that

$$\mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}(\mathcal{H}) \cup \{I_2[-1]\} \tag{4.3}$$

Lemma 4.3.1. *$\mathcal{D}$ is 3-Calabi-Yau if and only if π is self-adjoint or anti self-adjoint.*

Proof. Let S be the Serre-functor for $\mathcal{H}$. Being canonical S commutes with the auto-equivalence $a[-1]$. Hence S induces an autoequivalence on $\mathcal{D}$ which is easily seen to be the Serre functor of $\mathcal{D}$.

In $\mathcal{D}$ we have $S = \tau[1] = (\pi^* \circ \pi^{-1}) \circ a^2[1] = (\pi^* \circ \pi^{-1})[3]$. Thus $\mathcal{D}$ is 3-Calabi-Yau if and only if $\pi^* \circ \pi^{-1}$ is isomorphic to the identity functor. It is easy to see that this is the case if and only is $\pi^* \circ \pi^{-1} = \pm 1$ in $\mathrm{End}_k(W)$. □

Lemma 4.3.2. *The object P_1 in $\mathcal{D}$ satisfies*

$$\mathrm{Ext}^i_{\mathcal{D}}(P_1, P_1) = 0 \text{ for } i = 1, 2, \quad \mathrm{Hom}_{\mathcal{D}}(P_1, P_1) = k \text{ and } \mathrm{Ext}^{-1}_{\mathcal{D}}(P_1, P_1) = W.$$

Proof. For $N \in \mathrm{Ind}(\mathcal{H}) \cup \{I_2[-1]\}$ one computes

$$\mathrm{Hom}_{\mathcal{D}}(P_1, N) = \mathrm{Hom}_{\mathcal{H}}(P_1, N) \tag{4.4}$$

Thus we find

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, P_1[-1]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, a^{-1}P_1)\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, P_2)\\ &= W\end{aligned}$$

$$\mathrm{Hom}_{\mathcal{D}}(P_1, P_1) = k$$

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, P_1[1]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, aP_1)\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_2[-1])\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, P_1[2]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, aP_1[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_2)\\ &= 0 \quad \square\end{aligned}$$

The following lemma is not used explicitly.

Lemma 4.3.3. *The object P_1 in $\mathcal{D}$ has the properties of a 3-cluster tilting object, i.e. if $\mathrm{Ext}^i_{\mathcal{D}}(P_1, N) = 0$ for $i = 1, 2$ then N is a sum of copies of P_1.*

Proof. Assume that $N \in \mathrm{Ind}(\mathcal{H}) \cup \{I_2[-1]\}$ is such that $\mathrm{Hom}_{\mathcal{D}}(P_1, N[1]) = \mathrm{Hom}_{\mathcal{D}}(P_1, N[2]) = 0$. We have to prove $N = P_1$.

We may rewrite

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, N[2]) &= \mathrm{Hom}_{\mathcal{D}}(P_1[-1], N[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(a^{-1}P_1, N[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(P_2, N[1])\end{aligned}$$

Thus we find $\mathrm{Hom}_{\mathcal{D}}(P_1, aN) = \mathrm{Hom}_{\mathcal{D}}(P_2, aN) = 0$. Hence $aN \notin \mathrm{Ind}(\mathcal{H})$. We deduce $N \in \{P_1, I_2[-1]\}$.

But if $N = I_2[-1]$ then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(P_1, N[2]) &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_2[1])\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, aI_2)\\ &= \mathrm{Hom}_{\mathcal{D}}(P_1, I_1)\\ &\neq 0\end{aligned}$$

So we are left with the possibility $N = P_1$ which finishes the proof. $\square$

4.4. **Proof of Theorem 1.4.** Let $\mathcal{T}$ be an algebraic Ext-finite Krull-Schmidt 3-Calabi-Yau category containing a 3-cluster tilting object T such that $\mathrm{End}_{\mathcal{T}}(T) = k$.

Lemma 4.4.1. *Let $N \in \mathcal{T}$. Then there exists a distinguished triangle in $\mathcal{T}$*

$$T^a \to T^b \oplus T[-1]^c \to N[1] \to \tag{4.5}$$

Proof. Let Y be defined (up to isomorphism) by the following distinguished triangle[2]

$$Y \to T^{\mathrm{Ext}^1_{\mathcal{T}}(T,N)} \oplus T[-1]^{\mathrm{Ext}^2_{\mathcal{T}}(T,N)} \to N[1] \to$$

A quick check reveals that $\mathrm{Ext}^1_{\mathcal{T}}(T,Y) = \mathrm{Ext}^2_{\mathcal{T}}(T,Y) = 0$. Hence $Y = T^a$ for some a. □

We need to consider the special case $N = T[1]$. Then the distinguished triangle (4.5) (constructed as in the proof) has the form

$$T^{\mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)} \xrightarrow{\phi} T[-1] \xrightarrow{\alpha} T[2] \xrightarrow{\beta} \tag{4.6}$$

where ϕ is the universal map (this follows from applying $\mathrm{Hom}_{\mathcal{T}}(T,-)$). Since $\mathrm{End}_{\mathcal{C}}(T[2]) = k$ it follows that α,β are determined up to (the same) scalar.

This has a surprizing consequence. Applying $\mathrm{Hom}_{\mathcal{T}}(-,T)$ to the triangle (4.6) we find that $\mathrm{Hom}_{\mathcal{T}}(\beta[-1],T)^{-1}$ defines an isomorphism

$$\pi : \mathrm{Ext}^{-1}_{\mathcal{T}}(T,T) \to \mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)^*$$

Thus $W \overset{\text{def}}{=} \mathrm{Ext}^{-1}_{\mathcal{C}}(T,T)$ comes equipped with an isomorphism $\pi : W \to W^*$ which is canonical up to a scalar. In other words we are in the setting of §4.2 and we now use the notations introduced in sections 4.2 and 4.3.

As a is obtained from the reflection in vertex 1, one verifies (see §4.2) that a is associated to the element of $\mathrm{Tilt}(C,C)$ given by $(\theta, I_2[-1] \oplus P_1)$ where $\theta : C \to \mathrm{End}_C(I_2[-1] \oplus P_1)$ is the composition

$$C = \begin{pmatrix} k & 0 \\ W & k \end{pmatrix} \xrightarrow{\pi} \begin{pmatrix} k & 0 \\ W^* & k \end{pmatrix} = \mathrm{End}_C(I_2[-1] \oplus P_1) \tag{4.7}$$

Since the autoequivalence a is a derived functor that commutes with coproducts it is isomorphic to a derived tensor functor $- \overset{L}{\otimes}_C X$ for some $X \in D(C^e)$, by [18, 6.4]. As a right C-module we have $X \cong I_2[-1] \oplus P_1$.

Now we use the assumption that $\mathcal{H}$ is algebraic and we proceed more or less as in the appendix to [21]. By [18, Thm. 4.3] we may assume that $\mathcal{T}$ is a strict (= closed under isomorphism) triangulated subcategory of a derived category $D(\mathcal{A})$ for some DG-category $\mathcal{A}$. We denote by ${}_C\mathcal{T}$ the full subcategory of $D(C \otimes \mathcal{A})$ whose objects are differential graded $C \otimes \mathcal{A}$-modules which are in $\mathcal{T}$ when considered as $\mathcal{A}$-modules. Clearly ${}_C\mathcal{T}$ is triangulated. By [21, Lemma A.2.1(a)] T may be lifted to an object in ${}_C\mathcal{T}$, which we also denote by T. Put $S = T \oplus T[-1]$.

Lemma 4.4.2. *One has an isomorphism in ${}_C\mathcal{T}$*

$$X \overset{L}{\otimes}_B S \cong S[1]$$

[2]It would be more logical to write e.g. $\mathrm{Ext}^1_{\mathcal{T}}(T,N) \otimes_k T$ for $T^{\mathrm{Ext}^1_{\mathcal{T}}(T,N)}$ but this would take a lot more space.

Proof. As objects in $\mathcal{T}$ we have

$$\begin{aligned} X \overset{L}{\otimes}_C S &= (I_2[-1] \oplus P_1) \overset{L}{\otimes}_C S \\ &= I_2 \overset{L}{\otimes}_C S[-1] \oplus P_1 \overset{L}{\otimes}_C S \end{aligned}$$

Clearly $P_1 \overset{L}{\otimes}_C S \cong T$. To compute $I_2 \overset{L}{\otimes}_C S$ we use the resolution

$$0 \to P_1^{\mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)} \to P_2 \to I_2 \to 0$$

Tensoring with S we get a distinguished triangle

$$T^{\mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)} \to T[-1] \to I_2 \overset{L}{\otimes}_C S \to$$

Comparing with (4.6) we find $I_2 \overset{L}{\otimes}_C S \cong T[2]$. Thus, we have indeed an isomorphism

$$\varphi : X \overset{L}{\otimes}_B S \to S[1]$$

in $\mathcal{T}$.

Now we check that φ is C-equivariant in $\mathcal{T}$. The left C-module structure on $X \overset{L}{\otimes}_B S$ is obtained from the (homotopy) C-action on $I_2[-1] \oplus P_1$ as given in (4.7).

Let μ be an element of $W = \mathrm{Hom}_C(P_1, P_2) = \mathrm{Ext}^{-1}_{\mathcal{T}}(T,T)$. We need to prove that the following diagram is commutative in $\mathcal{T}$.

$$\begin{array}{ccc} I_2[-1] \overset{L}{\otimes}_B S & \xrightarrow{\cong} & T[1] \\ {\scriptstyle \pi(\mu)\overset{L}{\otimes}_B \mathrm{id}_S}\downarrow & & \downarrow{\scriptstyle \mu} \\ P_1 \overset{L}{\otimes}_B S & \xrightarrow[\cong]{} & T \end{array}$$

We write this out in triangles

$$\begin{array}{ccccccc} T^{\mathrm{Ext}^{-1}(T,T)} & \xrightarrow{\phi} & T[-1] & \xrightarrow{\alpha} & T[2] & \xrightarrow{\beta} & \\ {\scriptstyle \pi(\mu)}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mu} & & \\ T & \longrightarrow & 0 & \longrightarrow & T[1] & \xrightarrow[\mathrm{id}]{} & \end{array}$$

Rotating the triangles we need to prove that the following square is commutative

$$\begin{array}{ccc} T[1] & \xrightarrow{\beta[-1]} & T^{\mathrm{Ext}^{-1}(T,T)} \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \pi(\mu)} \\ T & = & T \end{array}$$

This commutivity holds precisely because of the definition of π. So ϕ is indeed C-equivariant.

But according to [21, Lemma A.2.2], any C-equivariant morphism in $\mathcal{T}$ between objects in ${}_C\mathcal{T}$ may be lifted to a morphism in ${}_C\mathcal{T}$. This finishes the proof. □

We now have a functor

$$? \overset{L}{\otimes}_C T : \mathcal{C} \to \mathcal{T}$$

and by Lemma 4.4.2 one finds that $a[-1](?) \overset{L}{\otimes}_C T$ is isomorphic to $? \overset{L}{\otimes}_C T$. By the universal property of orbit categories [20] we obtain a triangulated functor

$$Q : \mathcal{D} \to \mathcal{T}$$

which sends P_1 to T.

Lemma 4.4.3. *Q is an equivalence.*

Proof. We observe that analogues of the distinguished triangles (4.5) exist in $\mathcal{D}$ (with P_1 replacing T). Indeed, let $N \in \operatorname{Ind}(\mathcal{D})$. By (4.3) we have $N \in \operatorname{Ind}(\mathcal{H}) \cup \{I_2[-1]\}$. If $N \in \operatorname{Ind}(\mathcal{H})$ then $N[1] \cong aN$ and the analog of (4.5) is simply the image in $\mathcal{D}$ of the projective resolution of aN in $\mathcal{H}$ (taking into account that $P_2 = a^{-1}P_1 = P_1[-1]$).

If $N = I_2[-1]$ then $N[1] = I_2$ and the analog of (4.5) is the image in $\mathcal{D}$ of the projective resolution of I_2 in $\mathcal{H}$.

To prove that Q is fully faithful we have to prove that Q induces an isomorphism $\operatorname{Hom}_{\mathcal{D}}(M,N) \to \operatorname{Hom}_{\mathcal{T}}(QM,QN)$. Using the analogues of (4.5) we reduce to $M = P_1[i]$. But since $\operatorname{Hom}_{\mathcal{D}}(P_1[i],N) = \operatorname{Hom}_{\mathcal{D}}(P_1[-1],N[-i-1])$ we reduce in fact to $M = P_1[-1]$. It now suffices to apply $\operatorname{Hom}_{\mathcal{D}}(P_1[-1],-)$ to

$$P_1^a \to P_1^b \oplus P_1[-1]^c \to N[1] \to$$

taking into account that $\operatorname{Hom}_{\mathcal{D}}(M,N) \to \operatorname{Hom}_{\mathcal{T}}(QM,QN)$ is an isomorphism for $M = P_1$, $N = P_1$, $P_1[1]$, $P_2[2]$ by Lemma 4.3.2.

As a last step we need to prove that Q is essentially surjective. But this follows from the distinguished triangles (4.5) together with the fact that $QP_1 = T$. □

To finish the proof of Theorem 1.4 we observe that since $\mathcal{T}$ is 3-Calabi-Yau, so is $\mathcal{D}$. Hence by Lemma 4.3.1 π is either self-adjoint or anti self-adjoint. By Lemma 4.2.2 we deduce $a^2 \cong \tau$ and hence we may write $a = \tau^{1/2}$.

Remark 4.4.4. It would be interesting to deduce the fact that π is (anti) self-adjoint directly from the Calabi-Yau property of $\mathcal{T}$, without going through the construction of $\mathcal{D}$ first. This would have made our arguments above more elegant.

Remark 4.4.5. Iyama and Yoshino also consider $2n+1$-Calabi-Yau categories $\mathcal{T}$ equipped with a $2n+1$-cluster tilting object T such that $\operatorname{End}(T) = k$ and $\operatorname{Ext}^{-i}(T,T) = 0$ for $0 < i < n$. They relate such $\mathcal{T}$ to the representation theory of the generalized Kronecker quiver Q_m where $m = \dim \operatorname{Ext}^{-n}(T,T)$.

One may show that our techniques are applicable to this case as well and yield $\mathcal{T} \cong D^b(\operatorname{mod}(kQ_m))/(\tau^{1/2}[-n])$. We thank Osamu Iyama for bringing this point to our attention.

5. The singularity category of graded Gorenstein rings

5.1. **Orlov's results.** Let $A = k + A_1 + A_2 + \cdots$ be a commutative finitely generated graded k-algebra. As in [1] we write $\operatorname{qgr}(A)$ for the quotient of $\operatorname{gr}(A)$ by the Serre subcategory of graded finite length modules. We write $\pi : \operatorname{gr}(A) \to \operatorname{qgr}(A)$ for the quotient functor. If A is generated in degree one and $X = \operatorname{Proj} A$ then by Serre's theorem [37] we have $\operatorname{coh}(X) = \operatorname{qgr}(A)$.

Now assume that A is Gorenstein. Then we have $\operatorname{RHom}_A(k,A) \cong k(a)[-d]$ where d is the Krull dimension of R and $a \in \mathbb{Z}$. The number a is called the Gorenstein parameter of A (see [28, Definition 2.1]).

Example 5.1.1. If A is a polynomial ring in n variables (of degree one) then $d = n$, $a = n$.

For use below we record another incarnation of the Gorenstein parameter. Let A' be the graded k-dual of A. Then

$$R\Gamma_{A_{>0}}(A) \cong A'(a)[-d] \tag{5.1}$$

where $R\Gamma_{A_{>0}}$ denotes cohomology with support in the ideal $A_{>0}$.

The following is a particular case of [28, Thm 2.5].

Theorem 5.1.2. *If $a \geq 0$ then there are fully faithful functors*

$$\Phi_i : \underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to D^b(\mathrm{qgr}(A))$$

such that for $\mathcal{T}_i = \Phi_i \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ there is a semi-orthogonal decomposition

$$D^b(\mathrm{qgr}(A)) = \langle \pi A(-i-a+1), \ldots, \pi A(-i), \mathcal{T}_i \rangle$$

Hence under the hypotheses of the theorem we obtain in particular that

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \cong {}^{\perp}\langle \pi A(-i-a+1), \ldots, \pi A(-i) \rangle \subset D^b(\mathrm{qgr}(A))$$

for arbitrary i.

5.2. **The action of the shift functor on the singularity category.** Unfortunately the functors Φ_i introduced in the previous section are not compatible with $?(1)$. Our aim in this section is to understand how $?(1)$ acts on the image of Φ_i. This requires us to dig deeper into Orlov's construction which has the unusual feature of depending on the category $D^b(\mathrm{gr}_{\geq i} A)$ where $\mathrm{gr}_{\geq i} A$ are the finitely generated graded A-modules with non zero components concentrated in degrees $\geq i$. The quotient functor

$$D^b(\mathrm{gr}_{\geq i} A) \hookrightarrow D^b(\mathrm{gr}\, A) \xrightarrow{\pi} D^b(\mathrm{qgr}\, A)$$

has a right adjoint $R\omega_i A$. Its image is denoted by $\mathcal{D}_i$.

We let P_i be the graded projective A-module of rank one generated in degree i (i.e. $P_i = A(-i)$). Likewise S_i is the simple A-module concentrated in degree i. As in [28] we put $\mathcal{P}_{\geq i} = \langle (P_j)_{j \geq i} \rangle$, $\mathcal{S}_{\geq i} = \langle (S_j)_{j \geq i} \rangle$ and obvious variants with other types of inequality signs. In [28] it is proved that the image $\mathcal{T}_i$ of Φ_i is the left orthogonal to $\mathcal{P}_{\geq i}$ inside $D^b(\mathrm{gr}_{\geq i} A)$. The identification of $\mathcal{T}_i$ with the graded singularity category is through the composition

$$\mathcal{T}_i \cong D^b(\mathrm{gr}_{\geq i} A)/\mathcal{P}_{\geq i} \cong D^b(\mathrm{gr}\, A)/\,\mathrm{perf}(A) \cong \underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \tag{5.2}$$

Assume $a \geq 0$. Then the relation between $\mathcal{T}_i$, $\mathcal{D}_i$ is given by the following semi-orthogonal decompositions

$$D^b(\mathrm{gr}\, A) = \langle \mathcal{S}_{<i}, \overbrace{\mathcal{P}_{\geq i+a}, \underbrace{P_{i+a-1}, \ldots, P_i, \mathcal{T}_i}_{\mathcal{D}_i \cong D^b(\mathrm{qgr}(A))}}^{D^b(\mathrm{gr}_{\geq i} A)} \rangle$$

This is a refinement of Theorem 5.1.2.

The category $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ comes equipped with the shift functor $?(1)$. We denote the induced endofunctor on $\mathcal{T}_i$ by σ_i. We will now compute it.

Lemma 5.2.1. *For $M \in \mathcal{T}_i \subset D^b(\mathrm{qgr}(A))$ we have*

$$\sigma_i M = \mathrm{cone}(\mathrm{RHom}_{\mathrm{qgr}(A)}(\pi A(-i), M) \otimes_k \pi A(-i+1) \to M(1)) \tag{5.3}$$

where the symbol "cone" is to be understood in a functorial sense, for example by computing it on the level of complexes after first replacing M by an injective resolution.

Proof. Let $N \in \mathcal{T}_i \subset D^b(\mathrm{gr}(A))$. To compute $\sigma_i N$ we see by (5.2) that we have to find $\sigma_i N \in \mathcal{T}_i$ such that $\sigma_i N \cong N(1)$ up to projectives. It is clear we should take

$$\begin{aligned}\sigma_i N &= \mathrm{cone}(\mathrm{RHom}_{\mathrm{gr}(A)}(P_{i-1}, N(1)) \otimes_k P_{i-1} \to N(1)) \\ &= \mathrm{cone}(\mathrm{RHom}_{\mathrm{gr}(A)}(P_i, N) \otimes_k P_{i-1} \to N(1))\end{aligned}$$

Now we note that the RHom can be computed in $\mathcal{D}_i \cong D^b(\mathrm{qgr}(A))$. Furthermore since the result lies in $\mathcal{T}_i \subset \mathcal{D}_i$ we can characterize it uniquely by applying π to it. Since π commutes with ?(1) we obtain (5.3) with $M = \pi N$. □

5.3. **The Serre functor for a graded Gorenstein ring.** Let A, a, d be as above but now assume that A has an isolated singularity and let $M, N \in \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$. Then by a variant of [17, Thm 8.3] we have a canonical graded isomorphism

$$\mathrm{Ext}^d_A(\underline{\mathrm{Hom}}_A(M,N), A) \cong \underline{\mathrm{Hom}}_A(N, M[d-1])$$

and furthermore an appropriate version of local duality yields

$$\mathrm{Ext}^d_A(\underline{\mathrm{Hom}}_A(M,N), A) = \underline{\mathrm{Hom}}_A(M,N)^*(a)$$

In other words we find

$$\underline{\mathrm{Hom}}_A(M,N)^* = \underline{\mathrm{Hom}}_A(N, M[d-1](-a))$$

and hence the Serre functor S on $\underline{\mathrm{MCM}}(A)$ is given by $?[d-1](-a)$.

It is customary to write $S = \tau[1]$ so that we have the usual formula

$$\underline{\mathrm{Hom}}_A(M,N)^* = \mathrm{Ext}^1(N, \tau M)$$

In this setting we find

$$\tau = ?[d-2](-a) \tag{5.4}$$

5.4. **The Gorenstein parameter of a Veronese subring.** We remind the reader of the following well-known result.

Proposition 5.4.1. *Let B be a polynomial ring in n variables of degree one. Assume $m \mid n$ and let $B^{(m)}$ be the corresponding Veronese subring of B. I.e. $B^{(m)}_i = B_{mi}$. Then $B^{(m)}$ is Gorenstein with Gorenstein parameter n/m.*

Proof. The Gorenstein property is standard. To compute the Gorenstein invariant we first let A be the "blown up" Veronese. I.e.

$$A_i = \begin{cases} B_i & \text{if } m \mid i \\ 0 & \text{otherwise}\end{cases}$$

Let $a, b = n$ be respectively the Gorenstein parameters of A and B. If M is a B-module write M^+ for $\oplus_i M_{mi}$, considered as graded A-module. We have

$$\begin{aligned} A'(a)[-n] &= R\Gamma_{A_{>0}}(A) \qquad \text{(see (5.1))} \\ &= R\Gamma_{A_{>0}}(B)^+ \\ &= R\Gamma_{B_{>0}}(B)^+ \\ &= (B'(b)[-n])^+ \\ &= A'(b)[-n] \end{aligned}$$

In the 3rd equality we have used that local homology is insensitive to finite extensions. We deduce $a = b = n$. Since $B^{(m)}$ is obtained from A by dividing the grading by m obtain n/m as Gorenstein parameter for $B^{(m)}$. □

Remark 5.4.2. In characteristic zero we could have formulated the result for invariant rings of finite subgroups of $\mathrm{Sl}_n(k)$ (with the same proof). However in finite characteristic Veronese subrings are not always invariant rings (consider the case where the characteristic divides m).

6. The Iyama-Yoshino examples (again)

6.1. **Example 1.1.** Let $B = k[x_1, x_2, x_3]$ and $A = B^{(3)}$. We have $X \overset{\text{def}}{=} \operatorname{Proj} A = \operatorname{Proj} B = \mathbb{P}^2$. By Proposition 5.4.1 A has Gorenstein invariant 1.

Unfortunately we have to deal with the unpleasant notational problem that the shift functors on $\mathrm{coh}(\mathbb{P}^2)$ coming from A and B do not coincide. To be consistent with the sections 5.1,5.2 we will denote them respectively by $?(1)$ and $?\{1\}$. Thus $?(1) = ?\{3\}$. Note that this choice is rather unconventional.

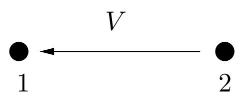
According to Theorem 5.1.2 we have a semi-orthogonal decomposition

$$D^b(\mathrm{coh}(X)) = \langle \mathcal{O}_{\mathbb{P}^2}, \mathcal{T}_0 \rangle$$

From the fact that $D^b(\mathrm{coh}(X))$ has a strong exceptional collection $\mathcal{O}_{\mathbb{P}^2}$, $\mathcal{O}_{\mathbb{P}^2}\{1\}$. $\mathcal{O}_{\mathbb{P}^2}\{2\}$ we deduce that there is a semi-orthogonal decomposition

$$\mathcal{T}_0 = \langle \mathcal{O}_{\mathbb{P}^2}\{1\}, \mathcal{O}_{\mathbb{P}^2}\{2\} \rangle$$

In particular $\mathrm{RHom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}\{1\} \oplus \mathcal{O}_{\mathbb{P}^2}\{2\}, -)$ defines an equivalence between $\mathcal{T}_0$ and the representations of the quiver Q_3



where $V = kx_1 + kx_2 + kx_3$ and where $\mathcal{O}_{\mathbb{P}^2}\{i\}$ corresponds to the vertex labeled by i. By (5.4) the Auslander-Reiten translate on $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ is given by $?[1](-1)$. In other words: the shift functor on $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ is given by $(\tau[-1])^{-1}$. By Proposition A.8 we find (using $R = \widehat{A}$)

$$\underline{\mathrm{MCM}}(R) \cong \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1) \cong D^b(\mathrm{mod}(kQ_3))/(\tau[-1])$$

which is what we wanted to show.

Remark 6.1.1. Note that this in this example we had no need for the somewhat subtle formula (5.3).

6.2. **Example 1.2.** We use similar conventions as in the previous section, Let $B = k[x_1, x_2, x_3, x_4]$ and $A = B^{(2)}$. We have $X = \operatorname{Proj} A \cong \operatorname{Proj} B = \mathbb{P}^3$ and we denote the corresponding shift functors by ?(1), ?{1} so that ?(1) =?{2}. By Proposition 5.4.1 A has Gorenstein invariant 2. By Theorem 5.1.2 we have a semi-orthogonal decomposition

$$D^b(\operatorname{coh}(X)) = \langle \mathcal{O}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}\{2\}, \mathcal{T}_{-1}\rangle$$

Now $D^b(\operatorname{coh}(X))$ has a strong exceptional collection $\mathcal{O}_{\mathbb{P}^3}$, $\mathcal{O}_{\mathbb{P}^3}\{1\}$. $\mathcal{O}_{\mathbb{P}^3}\{2\}$, $\mathcal{O}_{\mathbb{P}^3}\{3\}$. This sequence is geometric [7, Prop. 3.3] and hence by every mutation is strongly exceptional [7, Thm. 2.3]. We get in particular the following strongly exceptional collection

$\mathcal{O}_{\mathbb{P}^3}$, $\mathcal{O}_{\mathbb{P}^3}\{2\}$. $\Omega^*_{\mathbb{P}^3}\{1\}$, $\mathcal{O}_{\mathbb{P}^3}\{3\}$ where $\Omega_{\mathbb{P}^3}$ is defined by the exact sequence

$$0 \to \Omega_{\mathbb{P}^3} \to V \otimes \mathcal{O}_{\mathbb{P}^3}\{-1\} \to \mathcal{O}_{\mathbb{P}^3} \to 0 \tag{6.1}$$

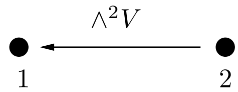
where $V = kx_1 + kx_2 + kx_3 + kx_4$. Thus there is a semi-orthogonal decomposition

$$\mathcal{T}_{-1} = \langle \Omega^*_{\mathbb{P}^3}\{1\}, \mathcal{O}_{\mathbb{P}^3}\{3\}\rangle$$

An easy computation yields

$$\operatorname{RHom}_{\mathbb{P}^3}(\Omega^*_{\mathbb{P}^3}\{1\}, \mathcal{O}_{\mathbb{P}^3}\{3\}) = \wedge^2 V$$

$\operatorname{RHom}_{\mathbb{P}^3}(\Omega^*_{\mathbb{P}^3}\{1\} \oplus \mathcal{O}_{\mathbb{P}^3}\{3\}, -)$ defines an equivalence between $\mathcal{T}_{-1}$ and the representations of the quiver Q_6



Put $W = \wedge^2 V$ and choose an arbitrary trivialization $\wedge^4 V \cong k$. Let $\pi : W \to W^*$ be the resulting (self-adjoint) isomorphism. We are in the setting of §4.2 and hence can define $\tau^{1/2}$ as acting on the derived category of Q_6.

We will now compute $\sigma_{-1}(\Omega^*_{\mathbb{P}^3}\{1\})$, $\sigma_{-1}(\mathcal{O}_{\mathbb{P}^3}\{3\})$. An easy computation yields

$$\operatorname{RHom}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}\{2\}, \Omega^*_{\mathbb{P}^3}\{1\}) = V^*$$
$$\operatorname{RHom}_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}\{2\}, \mathcal{O}_{\mathbb{P}^3}\{3\}) = V$$

Using the formula (5.3) we find

$$\sigma_{-1}(\mathcal{O}_{\mathbb{P}^3}\{3\}) = \operatorname{cone}(V \otimes \mathcal{O}_{\mathbb{P}^3}\{4\} \to \mathcal{O}_{\mathbb{P}^3}\{5\}) = \Omega_{\mathbb{P}^3}\{5\}[1] \tag{6.2}$$

$$\sigma_{-1}(\Omega^*_{\mathbb{P}^3}\{1\}) = \operatorname{cone}(V^* \otimes \mathcal{O}_{\mathbb{P}^3}\{4\} \to \Omega^*_{\mathbb{P}^3}\{3\}) = \mathcal{O}_{\mathbb{P}^3}\{3\}[1] \tag{6.3}$$

where in the second line we have used the dual version of (6.1).

Let P_i be the projective representation of Q_6 generated in vertex i. The endofunctor on $D^b(\operatorname{mod}(kQ_6))$ induced by σ_{-1} will be denoted by the same letter. We will now compute it. From (6.3) we deduce immediately $\sigma_{-1}(P_1) = P_2[1]$. To analyze (6.2) we note that a suitably shifted slice of the Koszul sequence has the form

$$0 \to \wedge^4 V \otimes \Omega^*_{\mathbb{P}^3}\{1\} \to \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^3}\{3\} \to \Omega_{\mathbb{P}^3}\{5\} \to 0$$

Thus $\Omega_{\mathbb{P}^3}\{5\}$ corresponds to the cone of

$$\wedge^4 V \otimes P_1 \to \wedge^2 V \otimes P_2$$

which is easily seen to be equal to $\wedge^4 V \otimes \tau^{-1} P_1$.

If we use our chosen trivialization $\wedge^4 V \cong k$ then we see that at least on objects σ_{-1} coincides with $\tau^{-1/2}[1]$. It is routine to extend this to an isomorphism of functors by starting with a bounded complex of projectives in $\mathrm{mod}(kQ_6)$.

By Proposition A.8 we find (using $R = \widehat{A}$)

$$\underline{\mathrm{MCM}}(R) \cong \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1) \cong D^b(\mathrm{mod}(kQ_6))/(\tau^{1/2}[-1])$$

which is what we wanted to show.

7. A remark on gradability of rigid modules

We keep notations as in the previous section. Since in the Iyama-Yoshino examples $\underline{\mathrm{MCM}}_{\mathrm{gr}}(A)$ is the derived category of a hereditary category the functor

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to \underline{\mathrm{MCM}}_{\mathrm{gr}}(A)/(1)$$

is essentially surjective [20] and hence

$$\underline{\mathrm{MCM}}_{\mathrm{gr}}(A) \to \underline{\mathrm{MCM}}_{\mathrm{gr}}(\widehat{A})$$

is also essentially surjective. In more complicated examples there is no reason however why this should be the case. Nevertheless we have the following result which is probably well-known.

Proposition 7.1. *Assume that k has characteristic zero. Let $A = k + A_1 + A_2 + \cdots$ be a left noetherian graded k-algebra. Put $R = \widehat{A}$. Let $M \in \mathrm{mod}(R)$ be such that $\mathrm{Ext}^1_R(M, M) = 0$. Then M is the completion of a finitely generated graded A-module N.*

In the rest of this section we let the notations and hypotheses be as in the statement of the proposition (in particular k has characteristic zero). We denote the maximal ideal of R by m.

Let E be the Euler derivation on A and R. I.e. on A we have $E(a) = (\deg a)a$ and we extend E to R in the obvious way. If $M \in \mathrm{mod}(R)$ then we will define an Euler connection as a k-linear map $\nabla : M \to M$ such that $\nabla(am) = E(a)m + a\nabla(m)$. If $M = \widehat{N}$ for N a graded A-module then M has an associated Euler connection by extending $\nabla(n) = (\deg n)n$ for n a homogeneous element of N.

Lemma 7.2. *Let M be a finitely generated R module. Then M has an Euler connection if and only if M is the completion of a finitely generated graded A-module.*

Proof. We have already explained the easy direction. Conversely assume that M has an Euler connection. For each n we have that $M/m^n M$ is finite dimensional and hence it decomposes into generalized eigenspaces for ∇.

$$M/m^n M = \prod_{\alpha \in k} (M/m^n M)_\alpha \qquad \text{(finite product)}$$

Considering right exact sequences

$$(m/m^2)^{\otimes n} \otimes M/mM \to M/m^{n+1}M \to M/m^n M \to 0$$

we easily deduce that the multiplicity of a fixed generalized eigenvalue in $M/m^n M$ stabilizes a $n \to \infty$. Thus $M = \prod_{\alpha \in k} M_\alpha$ where M_α is a generalized eigenspace with eigenvalue α. We put $N' = \oplus_\alpha M_\alpha$. Then N' is noetherian since obviously any ascending chain of graded submodules of N' can be transformed into an ascending chain of submodules in M. If particular N' is finitely generated and we have $M = \widehat{N}'$.

Now N' is k-graded and not $\mathbb{Z}$-graded. But we can decompose N' along $\mathbb{Z}$-orbits and then by taking suitable shifts we obtain a $\mathbb{Z}$-graded module with the same completion as N'. □

Proof or Proposition 7.1. Let $\epsilon^2 = 0$ and consider $M[\epsilon]$ where A acts via $a \cdot m = (a + E(a)\epsilon)m$. We have a short exact sequence of A-modules

$$0 \to M\epsilon \to M[\epsilon] \to M \to 0$$

which is split by hypotheses. Denote the splitting by $m + \nabla(m)\epsilon$. For $a \in A$ we have

$$am + \nabla(am)\epsilon = (a + E(a)\epsilon)(m + \nabla(m)\epsilon)$$

and hence

$$\nabla(am) = E(a)m + a\nabla(m)$$

Hence ∇ is an Euler connection and so we may invoke Lemma 7.2 to show that $M = \widehat{N}$. □

Appendix A. Generators of singularity categories

Throughout $(A, \mathfrak{m}, k)$ is a (commutative) local noetherian ring, with maximal ideal $\mathfrak{m}$ and residue field k. The *singularity category* of A is the Verdier quotient

$$D_{\mathrm{Sg}}(A) := D^b(\operatorname{mod} A)/K^b(\operatorname{proj} A)$$

of the bounded derived category of finitely generated A-modules by the full subcategory of perfect complexes. Recall that a functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ is an *equivalence up to direct summands* if F is fully faithful and every object $X \in \mathcal{S}$ is a direct summand of $F(Y)$ for some $Y \in \mathcal{T}$. We say that A is a *G-ring* if the canonical morphism from A to its $\mathfrak{m}$-adic completion $A \longrightarrow \widehat{A}$ is regular [24, §32], and that *A has an isolated singularity* if $A_{\mathfrak{p}}$ is regular for every non-maximal prime ideal $\mathfrak{p}$ of A. Our main result about singularity categories is the following:

Proposition A.1. *Let A be a local noetherian ring with an isolated singularity, which is also a G-ring (e.g. A is essentially of finite type over a field). Then the canonical functor*

$$\gamma := - \otimes_A \widehat{A} : D_{\mathrm{Sg}}(A) \longrightarrow D_{\mathrm{Sg}}(\widehat{A})$$

is an equivalence up to direct summands.

This is a special case of a general result by Orlov [31] (which was obtained indepedently). Our methods are quite different however.

When A is Gorenstein there is an equivalence, due to Buchweitz [9], between $D_{\mathrm{Sg}}(A)$ and the stable category of maximal Cohen-Macaulay A-modules $\underline{\mathrm{MCM}}(A)$, so in this case we obtain Proposition 1.6. We remark that, in general, γ is not an equivalence (see e.g. Example A.5).

Let us outline the proof of the proposition. Recall that a *thick subcategory* of a triangulated category $\mathcal{T}$ is a triangulated subcategory closed under retracts. Given an object C of $\mathcal{T}$, we say that an object X is *finitely built* from C if it belongs to the smallest thick subcategory of $\mathcal{T}$ containing C. If every object of $\mathcal{T}$ has this property, that is, if there are no proper thick subcategories of $\mathcal{T}$ containing C, then C is said to *classically generate* $\mathcal{T}$.

The local ring A and its completion $\widehat{A}$ have the same residue field k, and it is not difficult to see that γ induces an equivalence between the respective subcategories consisting of objects finitely built from k. The subtlety lies in showing that, because A has an isolated singularity, *every* object can be finitely built from k. Our proof of this fact uses homotopy colimits, which presents a technical problem since $D_{\mathrm{Sg}}(A)$ lacks infinite coproducts. One approach is to enlarge the category $D_{\mathrm{Sg}}(A)$ by considering the Verdier quotient

$$D'_{\mathrm{Sg}}(A) := D^b(\operatorname{Mod} A)/K^b(\operatorname{Proj} A)$$

of the bounded derived category of all A-modules by the full subcategory of bounded complexes of projective A-modules. By [30, Proposition 1.13] the canonical functor $D_{\mathrm{Sg}}(A) \longrightarrow D'_{\mathrm{Sg}}(A)$ is fully faithful, and $D'_{\mathrm{Sg}}(A)$ turns out to contain enough coproducts (and thus homotopy colimits) for our purposes. Throughout $D(A)$ denotes the (unbounded) derived category of A-modules.

The next proposition follows immediately from the work of Schoutens [36], but we give a direct proof in the special case of an isolated singularity. The result also follows from the general result by Orlov [31] and Dyckerhoff [12] has given a proof based on the theory of matrix factorizations in the hypersurface case.

Proposition A.2. *A local noetherian ring $(A, \mathfrak{m}, k)$ has an isolated singularity if and only if $D_{\mathrm{Sg}}(A)$ is classically generated by k.*

Proof. We begin with the easy direction. Suppose that $D_{\mathrm{Sg}}(A)$ is classically generated by k, and let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal. The canonical functor $- \otimes_A A_{\mathfrak{p}} : D_{\mathrm{Sg}}(A) \longrightarrow D_{\mathrm{Sg}}(A_{\mathfrak{p}})$ is identically zero, because it sends the generator k to zero. The image of this functor contains the residue field $\kappa(\mathfrak{p}) = A/\mathfrak{p} \otimes_A A_{\mathfrak{p}}$, from which we deduce that $\kappa(\mathfrak{p})$ has finite projective dimension over $A_{\mathfrak{p}}$. Hence $A_{\mathfrak{p}}$ is regular, and we may conclude that A has an isolated singularity.

Now suppose that A has an isolated singularity, and let M in $D^b(\operatorname{mod} A)$ be given. The idea is to write M as a homotopy colimit[3] of a sequence of bounded complexes with finite length cohomology; it follows that M is a direct summand of one of the terms in this sequence, from which we conclude that k classically generates. First, we set up some notation. Given $a \in A$, define complexes

$$\mathrm{K}[a] := A \xrightarrow{a} A, \text{ and } \mathrm{E}[a] := A \xrightarrow{\mathrm{can}} A[a^{-1}],$$

both concentrated in degrees zero and one, and observe that the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \xrightarrow{1} & A & \xrightarrow{1} & \cdots \\ {\scriptstyle a}\downarrow & & {\scriptstyle a^2}\downarrow & & {\scriptstyle a^3}\downarrow & & \\ A & \xrightarrow[a]{} & A & \xrightarrow[a]{} & A & \xrightarrow[a]{} & \cdots \end{array}$$

is a direct system of complexes $\mathrm{K}[a] \longrightarrow \mathrm{K}[a^2] \longrightarrow \mathrm{K}[a^3] \longrightarrow \cdots$ with colimit $\mathrm{E}[a]$. More generally, given a sequence $\mathbf{a} = \{a_1, \ldots, a_d\}$ in A, we define

[3]To be precise, we do not consider homotopy colimits in $D'_{\mathrm{Sg}}(A)$, since coproducts in this category are rather subtle. Instead, we consider the image under the quotient functor $D^b(\operatorname{Mod} A) \longrightarrow D'_{\mathrm{Sg}}(A)$ of homotopy colimits in $D^b(\operatorname{Mod} A)$.

$\mathrm{K}[\mathbf{a}] := \otimes_{j=1}^{d} \mathrm{K}[a_j]$ and $\mathrm{E}[\mathbf{a}] := \otimes_{j=1}^{d} \mathrm{E}[a_j]$. Setting $\mathbf{a}^i = \{a_1^i, \dots, a_d^i\}$, there is a canonical isomorphism $\mathrm{E}[\mathbf{a}] \cong \varinjlim_i \mathrm{K}[\mathbf{a}^i]$ and thus a triangle

$$\bigoplus_{i\geq 1} \mathrm{K}[\mathbf{a}^i] \xrightarrow{1-\mathrm{shift}} \bigoplus_{i\geq 1} \mathrm{K}[\mathbf{a}^i] \longrightarrow \mathrm{E}[\mathbf{a}] \longrightarrow \tag{A.1}$$

in the derived category $D(A)$. This triangle expresses the fact that $\mathrm{E}[\mathbf{a}]$ is the homotopy colimit of the $\mathrm{K}[\mathbf{a}^i]$ in $D(A)$. For background on homotopy colimits, see [6, 27].

Now let $\mathbf{a}$ be a system of parameters for A, and extend the augmentation morphism $\mathrm{E}[\mathbf{a}] \xrightarrow{\varepsilon} A$ to a triangle $\mathrm{E}[\mathbf{a}] \longrightarrow A \longrightarrow \check{\mathrm{C}}[\mathbf{a}] \longrightarrow$, where the complex $\check{\mathrm{C}}[\mathbf{a}] := \Sigma \ker(\varepsilon)$ is given in each degree by $\check{\mathrm{C}}[\mathbf{a}]^t = \oplus_{i_0<\cdots<i_t} A[a_{i_0}^{-1}, \dots, a_{i_t}^{-1}]$. Tensoring with M, we obtain a triangle

$$\mathrm{E}[\mathbf{a}] \otimes_A M \longrightarrow M \longrightarrow \check{\mathrm{C}}[\mathbf{a}] \otimes_A M \longrightarrow \tag{A.2}$$

in $D(A)$. Since A has an isolated singularity, $M[a_{i_0}^{-1} \cdots a_{i_t}^{-1}]$ has finite projective dimension over $A[a_{i_0}^{-1} \cdots a_{i_t}^{-1}]$, and hence also over A, for every sequence of indices $i_0 < \cdots < i_t$ in $\{1, \dots, d\}$. Here we use the fact that $A[a_{i_0}^{-1} \cdots a_{i_t}^{-1}]$ has finite projective dimension as an A-module[4]. We conclude that $\check{\mathrm{C}}[\mathbf{a}] \otimes_A M$ is, up to isomorphism in $D(A)$, a bounded complex of projective A-modules, whence the triangle (A.2) gives rise to an isomorphism $\mathrm{E}[\mathbf{a}] \otimes_A M \cong M$ in $D'_{\mathrm{Sg}}(A)$. Note that the coproduct $\oplus_{i\geq 1} \mathrm{K}[\mathbf{a}^i] \otimes_A M$ is bounded, so tensoring (A.1) with M yields a triangle in $D'_{\mathrm{Sg}}(A)$ of the form

$$\bigoplus_{i\geq 1} \mathrm{K}[\mathbf{a}^i] \otimes_A M \xrightarrow{1-\mathrm{shift}} \bigoplus_{i\geq 1} \mathrm{K}[\mathbf{a}^i] \otimes_A M \longrightarrow M \longrightarrow . \tag{A.3}$$

In what follows, let $\mathrm{Hom}(-,-)$ denote morphism sets in $D'_{\mathrm{Sg}}(A)$. One can check (see Lemma A.4 below) that $\mathrm{Hom}(M,-)$ commutes with coproducts coming from $D^b(\mathrm{Mod}\, A)$ via the quotient functor, so applying $\mathrm{Hom}(M,-)$ to (A.3) and using the argument of [26, Lemma 2.8] we deduce that

$$\mathrm{Hom}(M,M) \cong \varinjlim_i \mathrm{Hom}(M, \mathrm{K}[\mathbf{a}^i] \otimes_A M).$$

In particular, the identity $1_M : M \longrightarrow M$ corresponds to a split monomorphism $M \longrightarrow \mathrm{K}[\mathbf{a}^k] \otimes_A M$ in $D'_{\mathrm{Sg}}(A)$ for some $k \geq 1$. The functor $D_{\mathrm{Sg}}(A) \longrightarrow D'_{\mathrm{Sg}}(A)$ is fully faithful, so M is also a direct summand of $\mathrm{K}[\mathbf{a}^k] \otimes_A M$ in $D_{\mathrm{Sg}}(A)$. The cohomology modules of $\mathrm{K}[\mathbf{a}^k] \otimes_A M$ have finite length ($\mathbf{a}$ is a system of parameters) so this complex is an iterated extension in $D^b(\mathrm{mod}\, A)$ of finite direct sums of copies of k. It is now clear that any thick subcategory of $D_{\mathrm{Sg}}(A)$ containing k must contain M, and since M was arbitrary, this completes the proof. □

Lemma A.3. *A morphism $\varphi : M \longrightarrow C$ in $D(A)$ with $M \in D^b(\mathrm{mod}\, A)$ and $C \in K^b(\mathrm{Proj}\, A)$ factors, in $D(A)$, as $M \longrightarrow Q \longrightarrow C$ for some $Q \in K^b(\mathrm{proj}\, A)$.*

Proof. We may, without loss of generality, assume that M is a bounded above complex of finitely generated projective A-modules, that C is a bounded complex of free A-modules, and that φ is a morphism of complexes. Let $n \in \mathbb{Z}$ be such that $C^i = 0$ for $i < n$. The image of $\varphi^n : M^n \longrightarrow C^n$ is finitely generated, so let Q^n be a finite free submodule of C^n with the property that φ^n factors as

[4] By induction this reduces to the observation that $\mathrm{pd}_A A[a^{-1}] \leq 1$, which holds because $A[a^{-1}] = A[X]/(aX - 1)$.

$M^n \longrightarrow Q^n \longrightarrow C^n$. Similarly, let Q^{n+1} be a finite free submodule of C^{n+1} with the property that $\operatorname{Im}(\varphi^{n+1}) + \partial(Q^n) \subseteq Q^{n+1}$, where ∂ is the differential. Then φ^{n+1} factors as $M^{n+1} \longrightarrow Q^{n+1} \longrightarrow C^{n+1}$ and the differential restricts to a map $\partial|_Q : Q^n \longrightarrow Q^{n+1}$. Proceeding in this way we construct a bounded complex Q of finite free A-modules and a factorization $M \longrightarrow Q \longrightarrow C$, as required. □

Lemma A.4. *Let $\{X_i\}_{i\in I}$ be a family of bounded complexes of A-modules such that there exist $a, b \in \mathbb{Z}$ with $X_i^k = 0$ for all $k \notin [a, b]$ and $i \in I$. Then, given $M \in D^b(\operatorname{mod} A)$, the canonical map*

$$\oplus_i \operatorname{Hom}_{D'_{\mathrm{Sg}}(A)}(M, X_i) \longrightarrow \operatorname{Hom}_{D'_{\mathrm{Sg}}(A)}(M, \oplus_i X_i)$$

is an isomorphism, where $\oplus_i X_i$ denotes the degree-wise coproduct of complexes.

Proof. By a standard argument, it is enough to prove that any morphism $M \longrightarrow \oplus_i X_i$ in $D'_{\mathrm{Sg}}(A)$ factors through a finite subcoproduct. Such a morphism is defined by a roof

$$M \xleftarrow{\ f\ } Y \longrightarrow \oplus_i X_i \tag{A.4}$$

in $D^b(\operatorname{Mod} A)$, where the cone of f is a bounded complex C_f of projective A-modules. Extending f to a triangle $Y \longrightarrow M \longrightarrow C_f \longrightarrow$ in $D^b(\operatorname{Mod} A)$ we deduce from Lemma A.3 that $M \longrightarrow C_f$ factors as $M \longrightarrow Q \longrightarrow C_f$ for some $Q \in K^b(\operatorname{proj} A)$. Let C' denote the cone of $M \longrightarrow Q$. From the octahedral axiom applied to the pair $(M \longrightarrow Q, Q \longrightarrow C_f)$ we obtain a commutative diagram in $D^b(\operatorname{Mod} A)$ of the form

$$\begin{array}{ccc} \Sigma^{-1}C' & & \\ {\scriptstyle h}\downarrow & \searrow & \\ Y & \xrightarrow{\ f\ } & M \end{array}$$

where the cone of h belongs to $K^b(\operatorname{Proj} A)$. The upshot is that the morphism in $D'_{\mathrm{Sg}}(A)$ represented by the roof in (A.4) may also be represented by a roof with $Y \in D^b(\operatorname{mod} A)$ (replace Y with $\Sigma^{-1}C'$). In this case Y is compact in $D^b(\operatorname{Mod} A)$ by [35, Proposition 6.15], so the morphism $Y \longrightarrow \oplus_i X_i$ in the roof factors through a finite subcoproduct, which implies that $M \longrightarrow \oplus_i X_i$ factors through a finite subcoproduct in $D'_{\mathrm{Sg}}(A)$. □

Proof of Proposition A.1. To begin with, let A denote an arbitrary local noetherian ring, and consider the canonical functor

$$\gamma' := - \otimes_A \widehat{A} : D'_{\mathrm{Sg}}(A) \longrightarrow D'_{\mathrm{Sg}}(\widehat{A}).$$

Restriction of scalars defines a functor $(-)_A : D^b(\operatorname{Mod} \widehat{A}) \longrightarrow D^b(\operatorname{Mod} A)$ which sends a bounded complex of projective $\widehat{A}$-modules to a bounded complex of flat A-modules. Since flat A-modules have finite projective dimension by [33, Part II, Corollary 3.2.7], there is an induced functor

$$(-)_A : D'_{\mathrm{Sg}}(\widehat{A}) \longrightarrow D'_{\mathrm{Sg}}(A)$$

right adjoint to γ'. The unit of this adjunction is the canonical morphism

$$1 \longrightarrow (- \otimes_A \widehat{A})_A,$$

which is obviously an isomorphism on k, and thus also an isomorphism on the smallest thick subcategory $\mathcal{S}$ of $D'_{\mathrm{Sg}}(A)$ containing k. By a standard argument of category theory, the restriction of γ' to $\mathcal{S}$ is fully faithful. In particular, γ induces an equivalence of the smallest triangulated subcategory of $D_{\mathrm{Sg}}(A)$ containing k with the smallest triangulated subcategory of $D_{\mathrm{Sg}}(\widehat{A})$ containing k.

Now we assume that A is a G-ring with an isolated singularity. The (only) reason for assuming that A is a G-ring is that this guarantees that the completion $\widehat{A}$ has an isolated singularity [38, Lemma 2.7]. By Proposition A.2 the subcategory $\mathcal{S}$ includes the image of $D_{\mathrm{Sg}}(A)$ under the canonical embedding $D_{\mathrm{Sg}}(A) \longrightarrow D'_{\mathrm{Sg}}(A)$, from which we infer that γ is fully faithful. It follows from a second application of Proposition A.2 that the thick closure of $D_{\mathrm{Sg}}(A)$ in $D_{\mathrm{Sg}}(\widehat{A})$ is all of $D_{\mathrm{Sg}}(\widehat{A})$. Since the thick closure of a triangulated subcategory is just the class of all direct summands of objects in the subcategory [27, Remark 2.1.39], γ is an equivalence up to direct summands. □

It is easy to construct examples where γ is not an equivalence. It suffices to give a Cohen-Macauly module over the completion of a Gorenstein local ring $\hat{A}$ which is not *extended* from A, i.e. which is not of the form $\hat{M}$ for a Cohen-Macaulay A-module.

Example A.5. Let $A = \mathbb{C}[X,Y]_{(X,Y)}/(X^3+X^2-Y^2)$ be the local ring of a node, so the completion of A is isomorphic to the reduced ring $S = \mathbb{C}[[U,V]]/(UV)$. This is a singularity of type (A_1) and by [39, (9.9)] there are up, to isomorphism, exactly three indecomposable maximal Cohen-Macaulay S-modules, which are

$$S,\ \mathfrak{p} = US,\ \text{and}\ \mathfrak{q} = VS.$$

Clearly $S/\mathfrak{p} \cong \mathfrak{q}$, whence $\mathfrak{q} \cong \Sigma\mathfrak{p}$ in $D_{\mathrm{Sg}}(S)$. Since $\mathfrak{p}, \mathfrak{q}$ are minimal prime ideals, $S_\mathfrak{p}$ and $S_\mathfrak{q}$ are fields, and it follows from a result of Levy and Odenthal [23, Theorem 6.2] that a finitely generated S-module M is extended if and only if $\mathrm{rank}_{S_\mathfrak{p}}(M_\mathfrak{p}) = \mathrm{rank}_{S_\mathfrak{q}}(M_\mathfrak{q})$. Hence $\mathfrak{p}$ and $\mathfrak{q}$ are not extended, and thus not in the essential image of γ, but their direct sum $\mathfrak{p} \oplus \mathfrak{q}$ is extended. This corresponds to the fact that the nodal curve is irreducible, while the curve $XY = 0$ has two irreducible components. Another argument that $\mathfrak{p} \oplus \mathfrak{q} \cong \mathfrak{p} \oplus \Sigma\mathfrak{p}$ in $D_{\mathrm{Sg}}(S)$ belongs to the essential image of γ uses K-theory: simply apply [27, Corollary 4.5.12].

Note that $\{U - V\}$ is a system of parameters for S. It follows from the proof of Proposition A.2 that $\mathfrak{p}$ is a direct summand in $D_{\mathrm{Sg}}(S)$ of $\mathrm{K}[(U-V)^n] \otimes \mathfrak{p}$ for some $n \geq 1$. In fact, $\mathrm{K}[(U-V)^n] \otimes \mathfrak{p} = \mathfrak{p} \xrightarrow{U^n} \mathfrak{p}$ is quasi-isomorphic to $\Sigma^{-1}\mathfrak{p}/\mathfrak{p}^{n+1}$, and $\mathfrak{p}$ is a direct summand of $\Sigma^{-1}\mathfrak{p}/\mathfrak{p}^2$ in $D_{\mathrm{Sg}}(S)$. To see this, observe that there is a triangle in the derived category

$$\mathfrak{p} \xrightarrow{U} \mathfrak{p} \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \longrightarrow \Sigma\mathfrak{p},$$

and $U : \mathfrak{p} \longrightarrow \mathfrak{p}$ is zero in $D_{\mathrm{Sg}}(S)$ (as it factors via S) so we may conclude that $\mathfrak{p} \oplus \Sigma\mathfrak{p} \cong \mathfrak{p}/\mathfrak{p}^2$ in $D_{\mathrm{Sg}}(S)$. Since $\mathfrak{p}/\mathfrak{p}^2$ is isomorphic as an S-module to the residue field $\mathbb{C}$, we see for a third time that $\mathfrak{p} \oplus \Sigma\mathfrak{p} \cong \mathbb{C}$ is in the essential image of γ.

Remark A.6. Denoting by A^h the Henselization of A, the ring homomorphisms $A \longrightarrow A^h \longrightarrow \widehat{A}$ give rise to a factorization of γ as the composite

$$D_{\mathrm{Sg}}(A) \xrightarrow{\gamma_1} D_{\mathrm{Sg}}(A^h) \xrightarrow{\gamma_2} D_{\mathrm{Sg}}(\widehat{A}),$$

where $\gamma_1 = - \otimes_A A^h$ and $\gamma_2 = - \otimes_{A^h} \widehat{A}$. In the situation of Proposition A.1, γ_2 is an equivalence: up to a shift, every object of $D_{\mathrm{Sg}}(\widehat{A})$ is a finitely generated module M free on the punctured spectrum, and by Elkik's theorem [13, Théorème 3] such modules can be descended to the Henselization; that is, there exists a finitely generated A^h-module N such that $M \cong \widehat{N}$. In particular, γ is an honest equivalence (not just up to direct summands) when A is Henselian.

Now we give the proof of Proposition 1.5. In [22] Krause produces an embedding $\mu : D_{\mathrm{Sg}}(A) \hookrightarrow K_{\mathrm{ac}}(\mathrm{Inj}\, A)$, where $K_{\mathrm{ac}}(\mathrm{Inj}\, A)$ is the homotopy category of $C_{\mathrm{ac}}(\mathrm{Inj}\, A)$ of acyclic complexes of injective A-modules. This category is compactly generated, and μ induces an equivalence up to direct summands between $D_{\mathrm{Sg}}(A)$ and the full subcategory of compact objects in $K_{\mathrm{ac}}(\mathrm{Inj}\, A)$.

The embedding μ produces a DG-enhancement for $D_{\mathrm{Sg}}(A)$ where for $M, N \in D_{\mathrm{Sg}}(A)$ we put

$$\mathrm{RHom}_{D_{\mathrm{Sg}}(A)}(M, N) = \underline{\mathrm{Hom}}_{C_{\mathrm{ac}}(\mathrm{Inj}\, A)}(\mu(M), \mu(N))$$

If A is a noetherian $\mathbb{Z}$-graded ring (not necessarily commutative) then we may define the graded singularity category $D^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ in the obvious way.

Since $D^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ has an analogous DG-enhancement as $D_{\mathrm{Sg}}(A)$ we may define the orbit category $D^{\mathrm{gr}}_{\mathrm{Sg}}(A)/(1)$ (see [20]). By construction $D^{\mathrm{gr}}_{\mathrm{Sg}}(A)/(1)$ is a triangulated category (with a DG-enhancement) equipped with an exact functor

$$\sigma : D^{\mathrm{gr}}_{\mathrm{Sg}}(A) \to D^{\mathrm{gr}}_{\mathrm{Sg}}(A)/(1)$$

such that $D^{\mathrm{gr}}_{\mathrm{Sg}}(A)/(1)$ is classically generated by its essential image and such that for $M, N \in D^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ we have

$$\mathrm{Hom}_{D^{\mathrm{gr}}_{\mathrm{Sg}}(A)/(1)}(\sigma M, \sigma N) = \bigoplus_i \mathrm{Hom}_{D^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(M, N(i))$$

Forgetting the grading yields an exact functor

$$F : D^{\mathrm{gr}}_{\mathrm{Sg}}(A) \to D_{\mathrm{Sg}}(A)$$

which makes the shift (1) isomorphic to the identity functor. Hence by the universal property of orbit categories F factors canonically through

$$\tilde{F} : D^{\mathrm{gr}}_{\mathrm{Sg}}(A)/(1) \to D_{\mathrm{Sg}}(A)$$

Lemma A.7. *The functor $\tilde{F}$ is fully faithful.*

Proof. We have to prove that for $M, N \in D^{\mathrm{gr}}_{\mathrm{Sg}}(A)$ we have

$$\mathrm{Hom}_{D_{\mathrm{Sg}}(A)}(M, N) = \bigoplus_i \mathrm{Hom}_{D^{\mathrm{gr}}_{\mathrm{Sg}}(A)}(M, N(i))$$

By considering cones over suitable truncated projective resolutions we may assume that M, N are finitely generated A-modules.

We then use the well-known formula

$$\mathrm{Hom}_{D_{\mathrm{Sg}}(A)}(M, N) = \operatorname{inj\,lim}_n \mathrm{Hom}_{D(A)}(\Omega^n M, \Omega^n N)$$

and the corresponding formula in the graded case. This reduces us to proving

$$\mathrm{Hom}_{D(A)}(\Omega^n M, \Omega^n N) = \bigoplus_i \mathrm{Hom}_{D^{\mathrm{gr}}(A)}(\Omega^n M, \Omega^n N(i))$$

This follows easily by replacing M by a projective resolution. $\square$

Proposition A.8. *Let $A = k + A_1 + A_2 \cdots$ be a finitely generated commutative graded k-algebra with the augmentation ideal $m = A_{>0}$ defining an isolated singularity. Then we have equivalences*

$$D^{gr}_{\mathrm{Sg}}(A)/(1) \xrightarrow{\widetilde{F}} D_{\mathrm{Sg}}(A) \xrightarrow{(-)_m} D_{\mathrm{Sg}}(A_m) \xrightarrow{\widehat{A}\otimes_A -} D_{\mathrm{Sg}}(\hat{A})$$

Proof. The third functor is an equivalence because of Proposition A.1. The second functor is an equivalence because of [30]. Finally in Lemma A.7 we have shown that $\tilde{F}$ is fully faithful. So we have to show that it is essentially surjective. This is clear by Proposition A.2 since k lies in the essential image of $\tilde{F}$. □

Again we obtain Proposition 1.5 by invoking Buchweitz's equivalence $D_{\mathrm{Sg}}(A) \cong \underline{\mathrm{MCM}}(A)$.

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