LINEAR RECURRENCE RELATIONS FOR CLUSTER VARIABLES OF AFFINE QUIVERS

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ABSTRACT. We prove that the frieze sequences of cluster variables associated with the vertices of an affine quiver satisfy linear recurrence relations. In particular, we obtain a proof of a recent conjecture by Assem-Reutenauer-Smith.

1. INTRODUCTION

Caldero and Chapoton noted in [6] that one obtains natural generalizations of Coxeter-Conway's frieze patterns [11] [9] [10] when one constructs the bipartite belt of the Fomin-Zelevinsky cluster algebra [17] associated with a (connected) acyclic quiver Q. Such a generalized frieze pattern consists of a family of sequences of cluster variables, one sequence for each vertex of the quiver. For simplicity, we call these sequences the *frieze sequences* associated with the vertices of Q. Recently, they have been studied by Assem-Reutenauer-Smith [2] and by Assem-Dupont [1] for affine quivers Q. They also appear implicitly in the work of Di Francesco and Kedem, cf. for example [14] [13].

Our main motivation in this paper comes from a conjecture formulated by Assem, Reutenauer and Smith [2]: They proved that if the frieze sequences associated with a (valued) quiver Q satisfy linear recurrence relations, then Q is necessarily affine or Dynkin. They conjectured that conversely, the frieze sequences associated with a quiver of Dynkin or affine type always satisfy linear recurrence relations. For Dynkin quivers, this is immediate from Fomin-Zelevinsky's classification theorem for the finite-type cluster algebras [18]. In [2], Assem-Reutenauer-Smith gave an ingenious proof for the affine types $\tilde{A}$ and $\tilde{D}$ as well as for the non simply laced types obtained from these by folding. For the exceptional affine types, the conjecture remained open.

In this paper, we prove Assem-Reutenauer-Smith's conjecture in full generality using the representation-theoretic approach to cluster algebras pioneered in [26]. More precisely, our main tool is the categorification of acyclic cluster algebras via cluster categories (cf. e.g. [24]) and especially the cluster multiplication formula of [7]. Our method also yields a new proof for

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$\tilde{A}$ and $\tilde{D}$. It leads to linear recurrence relations which are explicit for the frieze sequences associated with the extending vertices and which allow us to conjecture explicit minimal linear recurrence relations for all vertices.

Notice that in addition to their intrinsic interest, linear recurrence relations have proved useful in establishing links between cluster algebras and canonical bases [25] and in the study of BPS spectra [8]. Finally, let us point out two links of our results to the theory of discrete integrable systems:

1) Our main theorem can be viewed as generalizing a result from section 9 of [20] on linearisations arising from 'period 1 quivers', cf. section 12 of [20]. Here the linear recurrence relations are obtained using first integrals.

2) As pointed out in [2], the frieze equations (1) of section 2, which define the frieze sequences, play the exact same role as the corresponding T- and Q-systems in [14] [13]. Those systems are integrable and this property was used in [14] [13] to derive linear recurrence relations. Namely, the (constant) coefficients of these relations were constructed as conserved quantities for the system. Similarly, in our context, the crucial element X_δ (defined just before Theorem 5.5) plays the role of a conserved quantity. We thank the referee for raising the interesting question of whether the equations (1) of section 2 may also be interpreted as an integrable system. We hope to come back to this in a future article.

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2. Main result and plan of the paper

Let Q be a finite quiver without oriented cycles. We assume that its vertices are numbered from 0 to n in such a way that each vertex i is a sink in the full subquiver on the vertices 1, ..., i. We introduce a total order on the set $\mathbb{N} \times Q_0$ by requiring that $(j, i) \leq (j', i')$ if we have $j < j'$ or both $j = j'$ and $i \geq i'$ hold.

The *generalized frieze pattern associated with* Q is a family of sequences $(X_j^i)_{j \in \mathbb{N}}$ of elements of the field $\mathbb{Q}(x_0, \ldots, x_n)$, where i runs through the vertices of Q. We recursively define these *frieze sequences* as follows: we set

$X_0^i = x_i$ for all vertices i of the quiver. Once $X_{j'}^{i'}$ has been defined for every pair $(j', i') < (j+1, i)$, we define X_{j+1}^i by the equality

$$X_{j+1}^i X_j^i = 1 + \prod_{s \to i} X_{j+1}^s \prod_{i \to d} X_j^d. \tag{1}$$

Note that the elements of the frieze sequences defined in this way are cluster variables of the cluster algebra $\mathcal{A}_Q$ associated with Q, cf. [6] [24]. The aim of this paper is to show the following result:

Theorem 2.1. *If Q is an affine quiver, then every frieze sequence $(X_j^i)_{j \in \mathbb{N}}$ satisfies a linear recurrence relation.*

This confirms the main conjecture of Assem-Reutenauer-Smith's [2]. They proved it for the case where Q is of type $\tilde{A}$ or $\tilde{D}$ (and for the non simply laced types obtained from these by folding). We will provide a new proof for these types and an extension to the exceptional types. Following [2] we show in section 9, using the folding technique, that the theorem also holds for affine valued quivers.

Our proof is based on the additive categorification of the cluster algebra $\mathcal{A}_Q$ by the cluster category of Q as introduced in [4]. In addition to the cluster category, the main ingredient of our proof is the Caldero-Chapoton map [6], which takes each object of the cluster category to an element of the field $Q(x_0, \ldots, x_n)$. Under this map, the exchange relations used to define the cluster variables are related to certain pairs of triangles in the cluster category, called exchange triangles. We will obtain linear recurrence relations from 'generalized exchange relations' obtained via the Caldero-Chapoton map from 'generalized exchange triangles'.

The main steps of the proof are as follows:

Step 1. We describe the action of the Coxeter transformation on the root system of an affine quiver.

Step 2. We show that the frieze sequence associated with a vertex i of the quiver is the image under the Caldero-Chapoton map of the τ-orbit of the projective indecomposable module associated with the vertex i.

Step 3. We prove the existence of generalized exchange triangles in the cluster category of an affine quiver using Step 1.

Step 4. By Step 2 we can deduce relations between the frieze sequences associated with vertices of the quiver from the generalized exchange triangles constructed in Step 3.

Step 5. The relations between frieze sequences obtained in Step 4 are either linear recurrence relations or they show that a fieze sequence is a product or sum of sequences that satisfy a linear recurrence. Hence all frieze sequences satisfy a linear recurrence relation.

In section 1, we study the action of the Coxeter transformation c of an affine quiver on the roots corresponding to preprojective indecomposables.

We use this result to determine, for every affine quiver, the minimal strictly positive integers b and m such that c satisfies $c^b = \mathrm{id} + m\langle -, \delta\rangle\delta$, where $\langle -, -\rangle$ denotes the Euler form of the quiver. Let us stress that b is *not* the Coxeter number of the associated finite root system.

In section 2, we briefly recall the cluster category of a quiver without oriented cycles. We introduce the Caldero-Chapoton map from the class of objects of the cluster category to $\mathbb{Q}(x_0, \ldots, x_n)$ and define exchange triangles and generalized exchange triangles of the cluster category. We state a result which describes how a pair of exchange triangles determines an equation between the images of the objects appearing in the triangles under the Caldero-Chapoton map. Then we show that the frieze sequence associated with a vertex i is obtained by applying the Caldero-Chapoton map to the τ-orbit of the projective indecomposable module associated with i viewed as an object in the cluster category.

In the third section, we give conditions, in the case of an affine quiver, for the existence of certain generalized exchange triangles. We deduce linear recurrence relations from these generalised exchange triangles, using the results of the previous section.

In the next three sections, we show that the main theorem holds for affine quivers. In doing so we use results of section 1 to show that the conditions of section 3 are satisfied. The exchange triangles yield relations between frieze sequences that prove that the sequences satisfy linear recurrence relations.

In section 6, we prove the main theorem for affine quivers of type $\tilde{D}$ and in section 7 for all exceptional affine quivers. Here the linear recurrence relations are given explicitly for frieze sequences associated with extending vertices. For all other frieze sequences, the existence of a linear recurrence is proven by showing that every sequence associated with a vertex can be written as a product or a linear combination of sequences satisfying a linear recurrence.

In section 8, we prove the main theorem for affine quivers of type $\tilde{A}_{p,q}$. Here the explicit linear recurrence relations are given only if p equals q. Otherwise, the existence of a linear recurrence relation is shown simultaneously for all frieze sequences by considering the sequence of vectors in $\mathbb{Q}(x_0, \ldots, x_n)^{n+1}$ whose ith coordinate is given by the entries of the frieze sequence associated with the vertex i for all vertices i of the quiver.

In section 9, we extend the main theorem to valued quivers using the folding technique and in the final section 10, we conjecture explicit minimal linear recurrence relations.

3. On the Coxeter transformation of an affine quiver

We first fix the notation and recall some basic facts. We refer to [12] and [3] for an introduction to quivers and their representations.

Let Q be an affine quiver, i.e. a quiver whose underlying graph is an extended simply laced Dynkin diagram $\tilde{\Delta}$. The *type of* Q is the diagram $\tilde{\Delta}$ except if we have $\tilde{\Delta} = \tilde{A}_n$, in which case the type of Q is $\tilde{A}_{p,q}$ where, for a chosen cyclic orientation of the underlying graph of Q, the number of positively (respectively negatively) oriented arrows equals p (respectively q). We number the vertices of Q from 0 to n and define the *Euler form* of Q as the bilinear form $\langle , \rangle$ on $\mathbb{Z}^{n+1}$ such that, for a and b in $\mathbb{Z}^{n+1}$, we have

$$\langle a, b\rangle = \sum_{i=0}^{n} a_i b_i - \sum_{i,j=0}^{n} c_{ji} a_i b_j \ ,$$

where c_{ij} is the number of arrows from i to j in Q. The *symmetrized Euler form* $(,)$ is defined by

$$(a, b) = \langle a, b\rangle + \langle b, a\rangle$$

for a and b in $\mathbb{Z}^{n+1}$. A *root* for Q is a non zero vector α in $\mathbb{Z}^{n+1}$ such that $(\alpha, \alpha)/2 \leq 1$; it is *real* if we have $(\alpha, \alpha)/2 = 1$ and *imaginary* if $(\alpha, \alpha) = 0$. It is *positive* if all of its components are positive. The *root system* Φ is the set of all roots. There is a unique root δ with strictly positive coefficients whose integer multiples form the radical of the form $(,)$ (cf. Chapter 4 of [12]). A vertex i of Q is an *extending vertex* if we have $\delta_i = 1$. If α is a real root, the *reflection* at α is the automorphism s_α of $\mathbb{Z}^{n+1}$ defined by

$$s_\alpha(x) = x - (\alpha, x)\alpha.$$

For each vertex i, the *simple root* α_i is the $(i + 1)$th vector of the standard basis of $\mathbb{Z}^{n+1}$. Let us number the vertices in such a way that each vertex i is a sink of the full subquiver of Q on the vertices $0, \ldots, i$. Using this ordering, we define the *Coxeter transformation* of Q to be the composition

$$c = s_{\alpha_0} s_{\alpha_1} \cdots s_{\alpha_n}.$$

We have

$$\langle x, y\rangle = -\langle y, cx\rangle$$

for all x and y in $\mathbb{Z}^{n+1}$.

Let k be an algebraically closed field and kQ the path algebra of Q over k. Let $\operatorname{mod} kQ$ be the category of k-finite-dimensional right kQ-modules. For a vertex i of Q, we denote the simple module supported at i by S_i, its projective cover by P_i and its injective hull by I_i. The map taking a module M to its *dimension vector*

$$\underline{\dim}\, M = (\dim \operatorname{Hom}(P_i, M))_{i=0\ldots n}$$

induces an isomorphism from the Grothendieck group of $\operatorname{mod} kQ$ to $\mathbb{Z}^{n+1}$. By Kac's theorem, the dimension vectors of the indecomposable modules are precisely the positive roots. For two modules L and M, we have

$$\langle \underline{\dim}\, L, \underline{\dim}\, M\rangle = \dim \operatorname{Hom}(L, M) - \dim \operatorname{Ext}^1(L, M).$$

For an indecomposable non injective module M, we have

$$c^{-1}\,\underline{\dim}\,M = \underline{\dim}\,\tau_m^{-1}M \quad ,$$

where τ is the Auslander-Reiten translation of the module category $\operatorname{mod} kQ$.

Theorem 3.1. *There exist a strictly positive integer b and a non zero integer m such that $c^b = \mathrm{id} - m\langle -, \delta\rangle\delta$. The integer b is a multiple of the width of the tubes in the Auslander-Reiten quiver of Q.*

1. *For Q of type $\tilde{E}_t$, the minimal b is given by $b = 6$ for $t = 6$; $b = 12$ for $t = 7$ and $b = 30$ for $t = 8$. In all those cases m is equal to 1.*
2. *For Q of type $\tilde{D}_n$, we have for even n that $b = n - 2$ and $m = 1$; if n is odd, we have $b = 2n - 4$ and $m = 2$.*
3. *For Q of type $\tilde{A}_{p,q}$, the minimal b is the least common multiple of p and q and m is the order of the class of q in the additive group $\mathbb{Z}/(p+q)\,\mathbb{Z}$.*

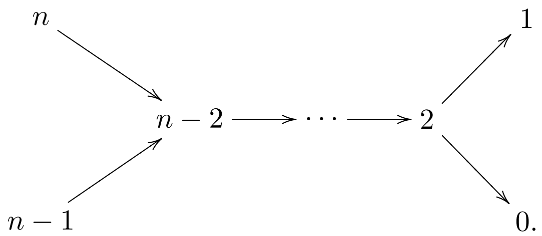
We will give a uniform interpretation of the integer b in Lemma 3.2 below. Let us stress that, contrary to a common misconception, it is *not* the Coxeter number of the corresponding finite root system.

Proof. The automorphism induced by c permutes the elements of the image of $\Phi \cup \{0\}$ in $\mathbb{Z}^{n+1}/\mathbb{Z}\,\delta$. This image is finite (see [12, 7]) and generates $\mathbb{Z}^{n+1}/\mathbb{Z}\,\delta$. Therefore there exists a strictly positive integer b such that c^b induces the identity on $\mathbb{Z}^{n+1}/\mathbb{Z}\,\delta$. It follows that there is a linear form $f : \mathbb{Z}^{n+1} \to \mathbb{Z}$ such that $c^b - \mathrm{id}$ is equal to $\langle f, -\rangle\delta$. In order to show that f is a multiple of $\langle -, \delta\rangle$, as $\langle -, \delta\rangle$ is primitive, it is sufficient to show that f vanishes on the kernel of $\langle -, \delta\rangle$. By [12, 7] the kernel is generated by the dimension vectors of the regular modules. Clearly it is enough to verify that f vanishes on the dimension vectors of the regular simple modules. Let M be such a module. If M lies in a homogenous tube, its dimension vector is δ and $f(\delta)$ vanishes by construction. Let us therefore assume that M is in an exceptional tube of width $s > 1$. The dimension vectors of $\underline{\dim}\,M$, $\underline{\dim}\,\tau M, \ldots, \underline{\dim}\,\tau^{s-1}M$ are non-zero and have sum δ. It follows that they are real roots and two by two distinct. Moreover the difference between two of these vectors is not a non-zero multiple of δ. Therefore their images in $\mathbb{Z}^{n+1}/\mathbb{Z}\,\delta$ are pairwise distinct. We must therefore have $c^b(\underline{\dim}\,M) = \underline{\dim}\,M$ and $f(\underline{\dim}\,M)$ vanishes. This argument also shows that the widths of the tubes divide b.

(1) The values of b and m for the exceptional quivers can be verified by direct computation using for example the cluster mutation applet [22].

For the other cases, we need a more detailed description of the roots and of the Coxeter transformation. Let Q' be the Dynkin quiver obtained from Q by deleting the extending vertex 0 and all arrows adjacent to it. Let $\alpha_1, \ldots, \alpha_n$ be the root basis of Q' consisting of the dimension vectors of the simples and let θ be the highest root of Q'. Via the inclusion of Q' into Q we identify the roots of Q' with their image in $\mathbb{Z}^{n+1}$. Then the dimension vector of the simple at the vertex 0 is $\alpha_0 = \delta - \theta$.

(2) We choose the following labeling and orientation on $\tilde{D}_n$:



Let $e_1, \dots, e_n$ be the vectors in $\mathbb{R}^{n+1}$ defined by

$$\alpha_i = e_i - e_{i+1} \text{ for } 1 \le i \le n-1 \text{ and } \alpha_n = e_n + e_{n-1}.$$

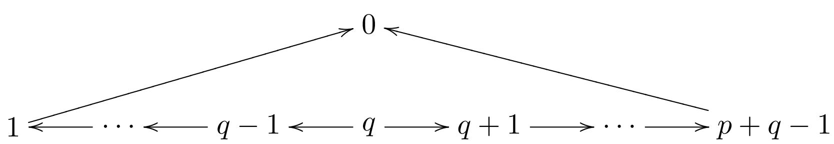
Then if we extend the form $(-,-)$ to $\mathbb{R}^{n+1}$, we have $(e_i, e_j) = \delta_{i,j}$ and $(e_i, \delta) = 0$. Furthermore θ equals $e_1 + e_2$ and α_0 equals $\delta - e_1 - e_2$. The reflections s_{α_i} for $1 \le i \le n-1$ act as the transposition of e_i and e_{i+1}. The reflection at α_n maps e_n to $-e_{n-1}$ and e_{n-1} to $-e_n$. The reflection at α_0 is given by $e_1 \mapsto -e_2 + \delta$ and $e_2 \mapsto -e_1 + \delta$.

We see that $c = s_{\alpha_0} \cdots s_{\alpha_n}$ acts up to multiples of δ as the $(n-2)$-cycle on $e_2, \dots, e_{n-1}$ and inverses the sign of e_1 and e_n. So c maps e_i to e_{i+1} for $2 \le i < n-1$ and e_{n-1} to $e_2 - \delta$ and e_1 to $-e_1 + \delta$ and e_n to $-e_n$. Then c^{n-2} corresponds to the action

$$e_1 \mapsto \begin{cases} e_1 & \text{if } n \text{ is even} \\ -e_1 + \delta & \text{else.} \end{cases}$$

and $e_i \mapsto e_i - \delta$ for $2 \le i \le n-1$ and $e_n \mapsto (-1)^{n-2} e_n$. Therefore we have that b equals $n-2$ if n is even and b equals $2n-4$ if n is odd. We see that c^{n-2} maps α_n to $\alpha_n - \delta$ if n is even and it maps α_n to $\alpha_{n-1} - \delta$ if n is odd. As $\langle \alpha_n, \delta \rangle = 1$, this shows that m is equal to 1 for n even and m is equal to 2 if n is odd.

(3) We consider the case $\tilde{A}_{p,q}$ with $q \le p$. We choose the following orientation and labeling on $\tilde{A}_{p,q}$:



Let E be a real vector space with basis $e_1, \dots, e_{n+1}, d$. We endow E with a symmetric bilinear form $(-,-)$ such that $(e_i, e_j) = \delta_{i,j}$ and $(d, e_i) = (d, d) = 0$. We have an isometric embedding of $\mathbb{Z}^{n+1}$ into E taking

$$\alpha_i \mapsto e_i - e_{i+1} \text{ for } 1 \le i \le n \text{ and } \delta \mapsto d.$$

Then θ is mapped to $e_1 - e_{n+1}$ and α_0 is mapped to $d - e_1 + e_{n+1}$. From now on, we identify $\mathbb{Z}^{n+1}$ with a subset of E using this embedding. The reflection s_{α_i} acts as the transposition of e_i and e_{i+1} for $1 \leq i \leq n$. The reflection at α_0 maps e_1 to $e_{n+1}+\delta$ and e_{n+1} to $e_1-\delta$. Then c is given by the product $s_{\alpha_0} \cdots s_{\alpha_{q-1}} s_{\alpha_{p+q-1}} \cdots s_{\alpha_q}$. The action of c is up to multiples of δ the product of the q-cycle on $e_1, \ldots, e_q$ and the p-cycle on $e_{p+q}, \ldots, e_{q+1}$. More concretely, we have $e_i \mapsto e_{i+1}$ for $1 \leq i \leq q-1$, $e_q \mapsto e_1 - \delta$ and $e_i \mapsto e_{i-1}$ for $q+2 \leq i \leq p+q$ and $e_{q+1} \mapsto e_{p+q} + \delta$. Then $c^{\mathrm{lcm}(p,q)}$ corresponds to the action $e_i \mapsto e_i - (\mathrm{lcm}(p,q)/q)\delta$ for $1 \leq i \leq q$ and $e_i \mapsto e_i + (\mathrm{lcm}(p,q)/p)\delta$ for $q+1 \leq i \leq p+q$. Therefore we have $b = \mathrm{lcm}(p,q)$. We verify that $c^{\mathrm{lcm}(p,q)}$ maps α_q to $\alpha_q - (\mathrm{lcm}(p,q)/p + \mathrm{lcm}(p,q)/q)\delta$. As $\langle \alpha_q, \delta \rangle = 1$, this shows that m is equal to $\mathrm{lcm}(p,q)/p + \mathrm{lcm}(p,q)/q$ which is the order of the class of q in $\mathbb{Z}/(p+q)\mathbb{Z}$. □

We have more information when Q is of type $\tilde{E}_t$ for $t = 6$ and $t = 7$. Then, for each $i \in Q_0$ there are positive integers k_i such that $k_i \delta_i = b$. These k_i satisfy $c^{k_i} \underline{\dim} P_i = \underline{\dim} P_i - \delta$.

Let us give a uniform interpretation of the integer b of the Theorem. We use the notations of the above proof. Let c' denote the Coxeter transformation of the Dynkin quiver Q'. Let $\bar{c}$ be the automorphism on $\mathbb{Z}^{n+1}/\mathbb{Z}\,\delta$ induced by c.

Lemma 3.2. *The automorphism $\bar{c}$ equals $s_\theta c'$. Hence b equals the order of the element $s_\theta c'$ in the Weyl group of Q'.*

Proof. The embedding of the root system of Q' in Q given in the proof of 3.1 yields an embedding of the Weyl group of Q' into the Weyl group of Q such that every reflection at a root of Q' fixes δ. Hence c equals $s_{\delta-\theta} c'$. We can write every element $y \in \mathbb{Z}^{n+1}$ as a linear combination of the roots of Q' and δ. Let $y = j + t\delta$ where j is a linear combination of the roots of Q' and $t \in \mathbb{Z}$. Then

$$s_{\delta-\theta}(j + t\delta) = j - (\theta, j)\theta + (t + (\theta, j))\delta = s_\theta(j) + (t + (\theta, j))\delta.$$

Therefore the action of $s_{\delta-\theta}$ modulo δ equals the action of s_θ. As c' fixes δ, we have $\bar{c} = s_\theta c'$ and b is the order of $s_\theta c'$. □

We denote by σ the automorphism on $\tilde{D}_n$ with $\sigma 1 = 0$, $\sigma 0 = 1$ and $\sigma n = n-1$, $\sigma(n-1) = n$ and σ fixes all other vertices of $\tilde{D}_n$. Recall that the extending vertices of $\tilde{D}_n$ are precisely 0, 1, $n-1$ and n.

Lemma 3.3. *(a) Let Q be of type $\tilde{D}_n$. Suppose that n is odd and i is an extending vertex of Q. Then $c^{n-2}(\underline{\dim} P_i) = \underline{\dim} P_{\sigma i} - \delta$.*

(b) For every vertex i of $\tilde{A}_{p,q}$ we have $c^{l_i}(\underline{\dim} P_{i-q}) = \underline{\dim} P_i + \delta$, where $l_i = i - q$ for $0 \leq i \leq q$ and $l_i = max\{q - i, -q\}$ for $q < i$.

Proof. (a) We have $\langle \underline{\dim} P_i, \delta \rangle = \delta_i$, which equals one as i is extending. By the proof of 3.1, we have $c^{n-2}(\underline{\dim} P_n) = c^{n-2}(\alpha_n) = \alpha_{n-1} - \delta = \underline{\dim} P_{n-1} - \delta$. If we apply c^{n-2} to this equation, we obtain $\underline{\dim} P_n - 2\delta = c^{2n-4}(\underline{\dim} P_n) = c^{n-2}(\underline{\dim} P_{n-1}) - \delta$ and thus $\underline{\dim} P_n - \delta = c^{n-2}(\underline{\dim} P_{n-1})$.

Furthermore $c^{n-2}(\underline{\dim} P_1) = c^{n-2}(\sum_{i=1}^{n} \alpha_i) = c^{n-2}(e_1 + e_{n-1}) = -e_1 + \delta + e_{n-1} - \delta = e_2 + e_{n-1} + \delta - \theta - \delta = \sum_{i=2}^{n} \alpha_i + \alpha_0 = \underline{\dim} P_0 - \delta$. Analogously to the first case, we apply c^{n-2} to the equation and obtain $\underline{\dim} P_1 - \delta$ equals $c^{n-2}(\underline{\dim} P_0)$.

(b) We first assume that i satisfies $0 < i < q$. By the proof of 3.1, we have $c^{q-i}(\underline{\dim} P_i) = c^{q-i}(\sum_{l=i}^{q} \alpha_l) = c^{q-i}(e_i - e_{q+1}) = e_q - e_{p+i+1} - \delta = \underline{\dim} P_{p+i} - \delta$. For $i = q$ we have $\underline{\dim} P_q = e_q - e_{q+1} = \underline{\dim} P_0 - \delta$ and for $i = 0$ we have $c^q(\underline{\dim} P_0) = c^q(e_q - e_{q+1} + \delta) = e_q - \delta - e_{p+1} - \delta + \delta = \underline{\dim} P_p - \delta$.

Let $q+1 \leq i \leq 2q$, then $c^{q-i}(\underline{\dim} P_i) = c^{q-i}(e_q - e_{i+1}) = e_{i-q} - e_{q+1} - \delta = \underline{\dim} P_{i-q} - \delta$.

Let finally $2q \leq i \leq p+q-1$ if $p > q$, then $c^q(\underline{\dim} P_i) = c^q(e_q - e_{i+1}) = e_q - e_{i-q+1} - \delta = \underline{\dim} P_{i-q} - \delta$, which finishes the proof. □

4. Frieze sequences of cluster variables

Let $\mathcal{F}$ denote the field $\mathbb{Q}(x_0, \ldots, x_n)$. A sequence $(a_j)_{j\in\mathbb{N}}$ of elements in $\mathcal{F}$ *satisfies a linear recurrence* if for some integer $s \geq 1$, there exist elements $\alpha_0, \ldots, \alpha_{s-1}$ in $\mathcal{F}$ such that for all $j \in \mathbb{N}$, one has $a_{j+s} = \alpha_0 a_j + \ldots \alpha_{s-1} a_{j+s-1}$. Equivalently, the generating series

$$\sum_{j\in\mathbb{N}} a_j \lambda^j$$

in $\mathcal{F}[[\lambda]]$ is rational and its denominator is a multiple of the polynomial $P(\lambda) = \lambda^s - \alpha_{s-1}\lambda^{s-1} - \ldots - \alpha_0$. We say that the polynomial *annihilates* the sequence.

Lemma 4.1. *(a) Let $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ be two sequences in $\mathcal{F}$ that satisfy a linear recurrence relation. Then the sequences $(a_j + b_j)_{j\in\mathbb{N}}$ and $(a_j b_j)_{j\in\mathbb{N}}$ satisfy a linear recurrence relation.*

(b) Let $m \geq 1$ be an integer and for each $1 \leq i \leq m$, let $(a_j^i)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{F}$. We consider the sequence of vectors $(v_j)_{j\in\mathbb{N}}$ defined by $v_j = (a_j^1, \ldots, a_j^m)^t$ for all $j \in \mathbb{N}$. Suppose there exist $m \times m$ matrices $A_0, \ldots, A_{s-1}$ over $\mathcal{F}$ such that for every $j \in \mathbb{N}$ we have $v_{j+s} = A_0 v_j + \ldots + A_{s-1} v_{j+s-1}$. Then each sequence $(a_j^i)_{j\in\mathbb{N}}$ satisfies a linear recurrence.

Proof. We refer to [5] for complete proofs of these fundamental facts. Let us record however, that if the two series are annihilated by polynomials P and Q, then their sum is annihilated by PQ and their Hadamard product $(a_j b_j)_{j\in\mathbb{N}}$ by the characteristic polynomial of $C_P \otimes_{\mathcal{F}} C_Q$, where C_P is the companion matrix of P. In b), the sequences are annihilated by the determinant of the matrix $\lambda^s - \lambda^{s-1} A_{s-1} - \ldots - \lambda A_1 - A_0$. □

We refer to [24] for an introduction to the links between cluster algebras and quiver representations which we now briefly recall. Let $\mathcal{D}_Q$ denote the bounded derived category of kQ-modules. It is a triangulated category and we denote its suspension functor by $\Sigma : \mathcal{D}_Q \to \mathcal{D}_Q$. As kQ has finite global dimension, Auslander-Reiten triangles exist in $\mathcal{D}_Q$ by [21, 1.4]. We denote the Auslander-Reiten translation of $\mathcal{D}_Q$ by τ. On the non projective modules, it coincides with the Auslander-Reiten translation of $\operatorname{mod} kQ$. The *cluster category* [4]

$$\mathcal{C}_Q = \mathcal{D}_Q/(\tau^{-1}\Sigma)^{\mathbb{Z}}$$

is the orbit category of $\mathcal{D}_Q$ under the action of the cyclic group generated by $\tau^{-1}\Sigma$. One can show [23] that $\mathcal{C}_Q$ admits a canonical structure of triangulated category such that the projection functor $\pi : \mathcal{D}_Q \to \mathcal{C}_Q$ becomes a triangle functor.

From now on, we assume that the field k has characteristic 0. We refer to [7] for the definition of the Caldero-Chapoton [6] map $L \mapsto X_L$ from the set of isomorphism classes of objects L of $\mathcal{C}_Q$ to the field $\mathcal{F}$. We have $X_{\tau P_i} = x_i$ for all vertices i of Q and $X_{M\oplus N} = X_M X_N$ for all objects M and N of $\mathcal{C}_Q$. We call an object M in $\mathcal{C}_Q$ *rigid* if it has no self-extensions, that is if the space $\operatorname{Ext}^1_{\mathcal{C}_Q}(M,M)$ vanishes.

Theorem 4.2 ([7]). a) *The map $L \mapsto X_L$ induces a bijection from the set of isomorphism classes of rigid indecomposables of the cluster category $\mathcal{C}_Q$ onto the set of cluster variables of the cluster algebra $\mathcal{A}_Q$.*

b) *If L and M are indecomposables such that the space* $\operatorname{Ext}^1(L,M)$ *is one-dimensional, then we have the generalized exchange relation*

$$X_L X_M = X_E + X_{E'} \tag{2}$$

where E and E' are the middle terms of 'the' non split triangles

$$L \longrightarrow E \longrightarrow M \longrightarrow \Sigma L \quad \text{and} \quad M \longrightarrow E' \longrightarrow L \longrightarrow \Sigma M\ .$$

Let L and M be two indecomposable objects in the cluster category such that $\operatorname{Ext}^1_{\mathcal{C}_Q}(M,L)$ is one dimensional. If both L and M are rigid, then so are E and E' and the sequence (2) is an exchange relation of the cluster algebra $\mathcal{A}_Q$ associated with Q. Therefore in this case, we call the triangles in (4.2) *exchange triangles.* If L or M is not rigid, we call them *generalized exchange triangles.*

Corollary 4.3. *For each vertex i of Q_0 and each j in $\mathbb{N}$, we have $X^i_j = X_{\tau^{-j+1}P_i}$.*

Proof. By the definition, the initial variables $x_0, \ldots, x_n$ are the images under the Caldero-Chapoton map of $\tau P_0, \ldots, \tau P_n$. The Auslander-Reiten component of $\mathcal{D}_Q$ containing the projective indecomposable modules is isomorphic to $\mathbb{Z}Q$, where the vertex (j,i) of $\mathbb{Z}Q$ corresponds to the isomorphism class of $\tau^{-j+1}P_i$ for all vertices i of Q and $j \in \mathbb{Z}$. To prove the statement, we use induction on the ordered set $\mathbb{N} \times \mathbb{Q}_0$. The claim holds for all vertices of Q and $j = 0$. Now let (j,w) be a vertex of $\mathbb{N} \times \mathbb{Q}_0$ such that $j > 0$. By the induction hypothesis, we have $X_{\tau^{-j+2}P_i} = X^i_{j-1}$ for all vertices i of the quiver and $X_{\tau^{-j+1}P_i} = X^i_j$ for all $i > w$. We consider the Auslander-Reiten triangle ending in $\tau^{-j+1}P_w$

$$\tau^{-j+2}P_w \to (\bigoplus_{s\to w} \tau^{-j+2}P_s) \oplus (\bigoplus_{w \to d} \tau^{-j+1}P_d) \to \tau^{-j+1}P_w \to \Sigma\tau^{-j+2}P_w.$$

The three terms of this triangle are rigid and the space of extensions of $\tau^{-1}P_w$ by P_w is one-dimensional. By 4.2 part b), this yields the exchange relation

$$X_{\tau^{-j+2}P_w} X_{\tau^{-j+1}P_w} = 1 + \prod_{w\to s} X_{\tau^{-j+2}P_s} \prod_{d \to w} X_{\tau^{-j+1}P_d}.$$

By the induction hypothesis, this translates into the relation $X^w_{j-1}X_{\tau^{-j+1}P_w} = 1+\prod_{w\to s} X^s_{j-1}\prod_{d\to w} X^d_j$. Therefore $X_{\tau^{-j+1}P_w}$ equals X^w_j, which proves the statement. □

5. Generalized exchange triangles in the cluster category

Let Q be an affine quiver. In this section, we construct some generalized exchange triangles in the cluster category $\mathcal{C}_Q$.

Lemma 5.1. *Let L and N be two indecomposable preprojective kQ-modules of defect minus one satisfying the equation $\underline{\dim}\, L = \underline{\dim}\, N + \delta$. Then, for every regular simple kQ-module M of dimension vector δ, there exists an exact sequence*

$$0 \to N \to L \to M \to 0$$

and $\dim_k \operatorname{Ext}^1_{kQ}(M,N) = 1$.

Proof. As N has defect minus one, we have

$$\begin{aligned} -1 = \langle \delta, \underline{\dim}\, N\rangle &= \dim \operatorname{Hom}(M,N) - \dim \operatorname{Ext}^1_{kQ}(M,N) \\ &= -\dim \operatorname{Ext}^1_{kQ}(M,N). \end{aligned}$$

By the assumption, we have $\underline{\dim}\, L = \underline{\dim}\, N+\delta$ and therefore $1 = \langle \underline{\dim}\, L, \delta\rangle = \dim \operatorname{Hom}(L,M)$ as $\operatorname{Ext}^1_{kQ}(L,M)$ vanishes. Since M is regular simple, every submodule of M that is not equal to M is preprojective. Every submodule of L is preprojective hence of defect at most -1. Thus, every quotient of L is of defect ≥ 0. Since the proper submodules of M are preprojective, every non zero map from L to M is surjective. The kernel of such a map has defect -1 and is preprojective. Therefore the kernel is indecomposable and its dimension vector equals $\underline{\dim}\, N$. Any preprojective indecomposable module is determined by its dimension vector. Thus, the kernel of every non zero map is isomorphic to N. This proves the existence of the exact sequence. □

Lemma 5.2. *Let N and M be two kQ-modules. Then we have a canonical isomorphism $\operatorname{Ext}^1_{\mathcal{C}_Q}(M,N) \cong \operatorname{Ext}^1_{kQ}(M,N) \oplus D\operatorname{Ext}^1_{kQ}(N,M)$.*

Proof. This is Proposition 1.7 c) of [4]. □

Theorem 5.3. *Let $i \in Q_0$ be an extending vertex and suppose there is a positive integer b such that P_i satisfies the equation $\underline{\dim}\,\tau^{-b}P_i = \underline{\dim}\,P_i + \delta$. Then, for every regular simple kQ-module M of dimension vector δ, there exist generalized exchange triangles in $\mathcal{C}_Q$*

$$P_i \to \tau^{-b}P_i \to M \to \Sigma P_i \quad \text{and} \quad M \to \tau^b P_i \to P_i \to \Sigma M.$$

Proof. The defect of P_i is $\langle \delta, \underline{\dim}\,P_i \rangle = -\delta_i$, which equals -1 since i is an extending vertex. Therefore, the defect of $\tau^{-b}P_i$ also equals -1 and the existence of the first triangle follows from 5.1. If we rotate the first triangle, we obtain a triangle $\Sigma^{-1}M \to P_i \to \tau^{-b}P_i \to M$. If we apply τ^b to it and use the fact that $\Sigma^{-1}M \cong \tau^{-1}M \cong M$ in $\mathcal{C}_Q$, we get the second triangle. By 5.1 and 5.2 the vector space $\operatorname{Ext}^1_{\mathcal{C}_Q}(M, P_i)$ is one-dimensional. □

Note that no indecomposable module with dimension vector δ is rigid.

Lemma 5.4. [16, 3.14] *Let N and M be two regular simple kQ-modules whose dimension vectors equal δ. Then X_M equals X_N.*

We set $X_\delta = X_M$ for any regular simple module M with dimension vector δ. By the previous Lemma, X_δ does not depend on the choice of M.

Theorem 5.5. *Let $i \in Q_0$ be an extending vertex and suppose that there is a positive integer b such that P_i satisfies the equation $\underline{\dim}\,\tau^{-b}P_i = \underline{\dim}\,P_i + \delta$. Then the frieze sequence $(X^i_j)_{j\in\mathbb{Z}}$ satisfies the linear recurrence relation $X_\delta X^i_j = X^i_{j-b} + X^i_{j+b}$ for all $j \in \mathbb{Z}$.*

Proof. Applying τ^{-j} to the generalized exchange triangles of 5.3 gives new generalized exchange triangles of the form

$$\tau^{-j}P_i \to \tau^{-j-b}P_i \to M \to \tau^{-j}\Sigma P_i \quad \text{and} \quad M \to \tau^{b-j}P_i \to \tau^{-j}P_i \to \Sigma M$$

since τM is isomorphic to M. These generalized exchange triangles yield the linear recurrence relation $X_\delta X^i_j = X^i_{j-b} + X^i_{j+b}$ for all $j \geq b$ by 4.2 b). □

6. Type $\tilde{D}$

Let Q be of type $\tilde{D}_n$. We use the same orientation and labeling of $\tilde{D}_n$ as in the proof of 3.1.

Theorem 6.1. *Let n be even and let i be an extending vertex of Q. Then the frieze sequence $(X^i_j)_{j\in\mathbb{Z}}$ satisfies the linear recurrence relation $X_\delta X^i_j = X^i_{j-n+2} + X^i_{j+n-2}$ for all $j \geq n-2$.*

Proof. This result follows immediately from 3.1 and 5.5. □

Theorem 6.2. *Suppose that n is odd and i is an extending vertex of Q.*

a) For every regular simple kQ-module M with dimension vector δ, there exist generalized exchange triangles

$$P_i \to \tau^{2-n} P_{\sigma i} \to M \to \Sigma P_i \quad \text{and} \quad M \to \tau^{n-2} P_{\sigma i} \to P_i \to \Sigma M.$$

b) The frieze sequence $(X^i_j)_{j\in\mathbb{Z}}$ satisfies the linear recurrence relation

$$X_\delta^2 X_j^i = 2X_j^i + X^i_{j-4+2n} + X^i_{j-2n+4}$$

for all $j \geq 2n-4$.

Proof. a) Using 5.1 and 3.3 there exist triangles

$$P_i \to \tau^{2-n} P_{\sigma i} \to M \to \Sigma P_i$$

and

$$P_{\sigma i} \to \tau^{2-n} P_i \to M \to \Sigma P_{\sigma i}.$$

Rotating the second triangle, we get a triangle

$$\Sigma^{-1} M \to P_{\sigma i} \to \tau^{2-n} P_i \to M.$$

If we apply τ^{n-2} to it and use the fact that $M \cong \Sigma^{-1}M$ in $\mathcal{C}_Q$ and M is τ-periodic of period one, we get a triangle in $\mathcal{C}_Q$ of the form

$$M \to \tau^{n-2} P_{\sigma i} \to P_i \to M.$$

By 5.2 and 5.1, these are generalized exchange triangles.

b) As in the proof of 5.5 we can apply powers of τ to the triangles of a) and we get the triangles

$$\tau^{-j} P_i \to \tau^{2-n-j} P_{\sigma i} \to M \to \tau^{-j} \Sigma P_i$$

and

$$M \to \tau^{n-2-j} P_{\sigma i} \to \tau^{-j} P_i \to M$$

for all $j \in \mathbb{Z}$. By 5.5 these triangles are generalized exchange triangles and we obtain the relations $X_\delta X^i_j = X^{\sigma i}_{n-2+j} + X^{\sigma i}_{2-n+j}$ and $X_\delta X^{\sigma i}_j = X^i_{n-2+j} + X^i_{2-n+j}$. Multiplying the first equation with X_δ and substituting using the second equation gives the stated recurrence relation. □

Thus we have obtained linear recurrence relations for the frieze sequences associated with all extending vertices of the quiver Q of type $\tilde{D}_n$. Using Auslander-Reiten triangles we will now deduce the existence of linear recurrence relations for the frieze sequences associated with neighbours of extending vertices. There is an Auslander-Reiten triangle

$$P_n \to P_{n-2} \to \tau^{-1} P_n \to \Sigma P_n.$$

This gives the recurrence relation for the vertex $n-2$. Similarly, using the Auslander-Reiten triangle

$$\tau^{-1}P_1 \to P_2 \to P_1 \to \Sigma\tau^{-1}P_1$$

we obtain the recurrence relation for the vertex 2. For the vertex $n-3$ we will use the following exchange triangles

$$P_{n-2} \to P_{n-3} \to S_{n-3} \to \Sigma P_{n-2}$$

$$S_{n-3} \to P_n \oplus P_{n-1} \to P_{n-2} \to \Sigma S_{n-3}$$

and for $2 < i < n-3$ we will use the exchange triangles

$$P_{i+1} \to P_i \to S_i \to \Sigma P_{i+1}$$

$$S_i \to P_{i+2} \to P_{i+1} \to \Sigma S_i.$$

These are indeed exchange triangles since we have

$$-1 = \langle \alpha_i, \alpha_{i+1} \rangle = \langle \alpha_i, \sum_{t=i+1}^{n} \alpha_t \rangle = \langle \underline{\dim}\, S_i, \underline{\dim}\, P_{i+1} \rangle$$

$$= \dim \operatorname{Hom}(S_i, P_{i+1}) - \dim \operatorname{Ext}^1_{kQ}(S_i, P_{i+1}) = -\dim_k \operatorname{Ext}^1_{kQ}(S_i, P_{i+1})$$

for all $2 < i < n-1$. We therefore obtain the relations

$$X_j^{n-3} = X_j^{n-2} X_{\tau^j S_{n-3}} - X_j^n X_j^{n-1}$$

for all $j \in \mathbb{Z}$ and

$$X_j^i = X_j^{i+1} X_{\tau^j S_i} - X_j^{i+2}$$

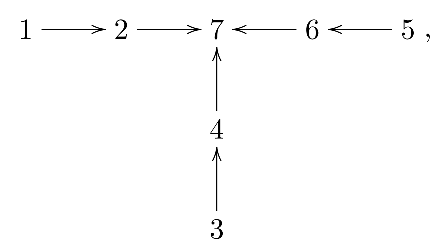
for all $j \in \mathbb{Z}$ and $2 < i < n-2$.

Note that the S_i are regular modules for $2 < i < n-3$ lying in the exceptional tube of length $n-2$ (cf. the tables at the end of [15]). Therefore the corresponding frieze sequence $X_{\tau^j S_i}$ is periodic. We now use descending induction on the vertices: we can recover linear recursion formulas for the frieze sequence associated to a vertex i with $3 < i < n-1$ from the linear recursion formulas of sequences associated to vertices $i' > i$ and the periodic sequences $X_{\tau^j S_i}$.

7. The exceptional types

Let Q be of type $\tilde{E}_t$ for $t \in \{6, 7, 8\}$. We will use the following labeling and orientation:

for $\tilde{E}_6$:



the vector δ is given by

$$\begin{array}{ccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longleftarrow & 2 & \longleftarrow & 1\ ; \\ & & & & \uparrow & & & & \\ & & & & 2 & & & & \\ & & & & \uparrow & & & & \\ & & & & 1 & & & & \end{array}$$

$$\text{for } \tilde{E}_7: \quad \begin{array}{ccccccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 8 & \longleftarrow & 6 & \longleftarrow & 5 & \longleftarrow & 4\ , \\ & & & & & & \uparrow & & & & & & \\ & & & & & & 7 & & & & & & \end{array}$$

the vector δ is given by

$$\begin{array}{ccccccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longleftarrow & 3 & \longleftarrow & 2 & \longleftarrow & 1\ ; \\ & & & & & & \uparrow & & & & & & \\ & & & & & & 2 & & & & & & \end{array}$$

$$\text{for } \tilde{E}_8: \quad \begin{array}{ccccccccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 9 & \longleftarrow & 8 & \longleftarrow & 7\ , \\ & & & & & & & & & & \uparrow & & & & \\ & & & & & & & & & & 6 & & & & \end{array}$$

the vector δ is given by

$$\begin{array}{ccccccccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 6 & \longleftarrow & 4 & \longleftarrow & 2\ . \\ & & & & & & & & & & \uparrow & & & & \\ & & & & & & & & & & 3 & & & & \end{array}$$

Theorem 7.1. *Let i be an extending vertex of Q. Then the frieze sequence $(X_j^i)_{j\in\mathbb{Z}}$ satisfies the linear recurrence relation*

$$X_\delta X_j^i = X_{j-b}^i + X_{j+b}^i$$

for all $j \in \mathbb{Z}$ where b is as in 3.1.

Proof. This result follows immediately by 3.1 and 5.5. □

Let $l \in Q_0$ be a vertex attached to an extending vertex i. Then the projective indecomposable module associated with l appears in an Auslander-Reiten triangle

$$P_i \to P_l \to \tau^{-1} P_i \to \Sigma P_i.$$

This gives us the following relation between the frieze sequence associated with the vertex l and the sequence associated with the extending vertex

$$X_j^l = X_j^i X_{j+1}^i - 1$$

for all $j \in \mathbb{Z}$. By 4.1, the sequence X_j^l satisfies a linear recursion relation.

Let now $l \in Q_0$ be a vertex such that there is an oriented path $i_0 = s, \ldots, i_t = l$ from an extending vertex $s \in Q_0$ to l in Q of length at least two. Then there are exchange triangles of the form

$$P_s \to P_l \to \tau^{-1} P_{i_{t-1}} \to \Sigma P_s$$

and

$$\tau^{-1} P_{i_{t-1}} \to \tau^{-2} P_{i_{t-2}} \to P_s \to \Sigma \tau^{-1} P_{i_{t-1}}.$$

This gives the following relation between the sequences associated with the vertices appearing in the oriented path

$$X_j^s X_{j-1}^{i_{t-1}} = X_j^l + X_{j-2}^{i_{t-2}}$$

for all $j \in \mathbb{Z}$. As all vertices connected to an extending vertex satisfy a linear recurrence relation by the previous case, we can assume that the sequences $X_j^{i_{t-1}}$ and $X_j^{i_{t-2}}$ satisfy a linear recursion using induction on the path length. Then the sequence $(X_j^l)_{j \in \mathbb{Z}}$ also satisfies a linear recursion relation. In the case $\tilde{E}_6$, for every non-extending vertex l of Q, there is an extending vertex and an oriented path from the extending vertex to l. Therefore, we obtain linear recurrence relations for all vertices of the quiver Q of type $\tilde{E}_6$.

In the case $\tilde{E}_7$, only the vertex labeled 7 can not be reached by an oriented path starting in an extending vertex. In this case, we consider the exchange triangles

$$P_1 \to \tau^{-1} P_7 \to \tau^{-4} P_4 \to \tau P_1$$

and

$$\tau^{-4} P_4 \to N \to P_1 \to \tau^{-3} P_4 \ ,$$

where N is the cokernel of any non-zero morphism $\tau^{-1} P_1 \to \tau^{-4} P_4$. Then τN is the cokernel of the map $P_1 \to \tau^{-3} P_4$. It is the indecomposable regular simple module of dimension vector 001100011 which belongs to the mouth of the tube of width 4 (cf. the tables at the end of [15]).

For $\tilde{E}_8$ we use an analogous method. The vertices 6, 7 and 8 can not be reached by an oriented path starting in an extending vertex. Therefore we consider the following exchange triangles

$$P_1 \to \tau^{-2} P_7 \to \tau^{-7} P_1 \to \tau P_1$$

and

$$\tau^{-7} P_1 \to N \to P_1 \to \tau^{-6} P_1 \ ,$$

where N is the cokernel of any non-zero morphism $\tau^{-1} P_1 \to \tau^{-7} P_1$. It is the regular simple module of dimension vector 001111001 which belongs to the mouth of the tube of width 5 by [15, page 49]. Hence the sequence $X_{\tau^i N}$ is periodic. These triangles give the relation $X_j^7 = X_{j+2}^1 X_{j-5}^1 - X_{\tau^{j-2} N}$, which proves that the sequence at the vertex 7 satisfies a linear recurrence relation. The next exchange triangles are given by

$$P_1 \to \tau^{-1}P_6 \to \tau^{-3}P_7 \to \tau P_1$$

and

$$\tau^{-3}P_7 \to \tau^{-8}P_1 \to P_1 \to \tau^{-2}P_1.$$

Here the relation is $X_j^6 = X_{j+1}^1 X_{j-2}^7 - X_{j-7}^1$. Finally, the last exchange triangles can be taken as

$$P_1 \to \tau^{-1}P_8 \to \tau^{-2}P_6 \to \tau P_1$$

and

$$\tau^{-2}P_6 \to \tau^{-4}P_7 \to P_1 \to \tau^{-1}P_6.$$

This gives the exchange relation $X_j^8 = X_{j+1}^1 X_{j-1}^6 - X_{j-3}^7$. This proves that all frieze sequences associated with vertices of the exceptional quivers satisfy linear recurrence relations.

8. Type $\tilde{A}_{p,q}$

We choose $p, q \in \mathbb{N}$ such that $q \leq p$ and use the same labeling and orientation for Q as in the proof of 3.1. We view the labels of vertices modulo $p+q$. Note that all vertices of Q are extending vertices.

Theorem 8.1. *(a) For every vertex $i \in Q_0$ and every regular simple module M with dimension vector δ, there are generalized exchange triangles*

$$P_i \to \tau^{l_i}P_{i-q} \to M \to \Sigma P_i$$

and

$$M \to \tau^{r_i}P_{i+q} \to P_i \to \Sigma M \quad ,$$

where $l_i = i - q$ for $0 \leq i \leq q$ and $l_i = max\{q-i, -q\}$ for $q < i$ and $r_i = -l_{i+q}$.

(b) We obtain relations between the frieze sequences associated to the vertices i, $i+q$ and $i-q$ of the form

$$X_\delta X_j^i = X_{j+l_i}^{i+q} + X_{j+r_i}^{i-q}$$

for all $i \in Q_0$ and $j \geq n$.

Proof. Using 5.1 and 3.3 we obtain the existence of the first triangle. If we replace i by $i+q$ in the first triangle and perform a rotation, we obtain the triangle

$$M \to P_{i+q} \to \tau^{l_{i+q}}P_i \to \Sigma M.$$

Applying $\tau^{-l_{i+q}}$ to this triangle gives the second triangle. Exactly as in the proof of 6.2 we can apply powers of τ to the generalized exchange triangles and we obtain the recurrence relations stated. □

If p equals q the relation from the previous Theorem yields

$$X_\delta X_j^i = X_{j+l_i}^{i+q} + X_{j+r_i}^{i+q}$$

for all $i \in Q_0$ and $j \geq n$ as $i + q = i - q$ seen modulo $2q$. If we iterate once, we obtain

$$(X_\delta^2 - 2)X_j^i = X_{j-q}^i + X_{j+q}^i,$$

using the fact that $r_i - l_i = q$ for all $i \in Q_0$ and $j \geq q$. Hence we can see immediately that all frieze sequences associated to vertices of the quiver Q of type $\tilde{A}_{q,q}$ satisfy a linear recurrence relation. In the case $p \neq q$ we need a different argument. We consider the sequence of vectors $(v(j))_{j\in\mathbb{N}}$ given by $v(j) = (X_j^0, \ldots, X_j^n)$. Then by 8.1, there are matrices $A_0, \ldots, A_n$ such that $v(j+n+1) = \sum_{t=0}^n A_t v(j+t)$ for all $j \in \mathbb{N}$. Using 4.1 b), it is clear that the frieze sequence associated with any vertex i satisfies a linear recurrence relation.

9. Non simply laced types

If Q is a finite quiver without oriented cycles which is endowed with a valuation (cf. [15]), one can define frieze sequences in a natural way. We refer to chapter 3, equation (1) of [2] for the exact definition and to the appendix of [2] for the list of affine Dynkin diagrams, which underlie the affined valued quivers. As in section 7.3 of [2], we can obtain the linear recurrence relation for a frieze sequence associated with a vertex of a valued quiver of affine type from the linear recurrence relation of a frieze sequence associated with the vertices of a non valued affine quiver. This can be done using the folding technique. An introduction to the folding technique can for example be found in section 2.4 of [19].

Theorem 9.1. *The frieze sequences associated with vertices of a quiver of the type $\tilde{G}_{21}$, $\tilde{G}_{22}$, $\tilde{F}_{41}$, $\tilde{F}_{42}$, $\tilde{A}_{11}$ or $\tilde{A}_{12}$ satisfy linear recurrence relations.*

We obtain the linear recurrence relation for a frieze sequence associated to a vertex of a quiver of type $\tilde{G}_{22}$ or $\tilde{F}_{42}$ by folding $\tilde{E}_6$ using the obvious action by $\mathbb{Z}/3\mathbb{Z}$ respectively $\mathbb{Z}/2\mathbb{Z}$. The linear recurrence relations in the case $\tilde{F}_{41}$ are obtained by folding $\tilde{E}_7$ using a natural action by $\mathbb{Z}/2\mathbb{Z}$. In the cases $\tilde{G}_{21}$, $\tilde{A}_{11}$ or $\tilde{A}_{12}$, they are obtained by folding $\tilde{D}_4$ using actions of $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ respectively.

10. On the minimal linear recurrence relations

For type $\tilde{D}_4$ (with the vertices numbered as in the proof of Theorem 3.1), one checks that the following are the polynomials of the minimal linear recurrence relations:

vertices 0, 1, 3, 4 : $\lambda^4 - X_\delta\lambda^2 + 1$, vertex 2 : $(\lambda - 1)(\lambda^2 - X_\delta\lambda + 1)$, where

$$X_\delta = \frac{x_0^2x_1^2 + 2x_0^2x_1^2x_2 + x_0^2x_1^2x_2^2 + 4x_0x_1x_2x_3x_4 + 2x_0x_1x_3x_4}{x_0x_1x_2^2x_3x_4} + \frac{x_2^2x_3^2x_4^2 + 2x_2x_3^2x_4^2 + x_3^2x_4^2}{x_0x_1x_2^2x_3x_4}.$$

Most of the recurrence relations one obtains from our proofs are not minimal. However, we conjecture that for type $\tilde{A}$, they are. In the following tables, for each vertex i of a quiver Q of type $\tilde{D}$ or $\tilde{E}$, we exhibit a polynomial which we conjecture to be associated with the minimal linear recurrence relation for the frieze sequence $(X_j^i)_{j\in\mathbb{N}}$. Our conjecture is based on the relations we have found and on numerical evidence obtained using Maple. For an integer d and an element c of the field $\mathcal{F} = \mathbb{Q}(x_0, \ldots, x_n)$, where $n+1$ is the number of vertices of Q, we put

$$P(2d, c) = \lambda^{2d} - c\lambda^d + 1.$$

The element X_δ of the field $\mathcal{F}$ is always defined as after Lemma 5.4. For type $\tilde{D}_n$, we number the vertices as in the proof of Theorem 3.1 and for the exceptional types as in section 7. For type $\tilde{D}_n$, where $n > 4$ is even, we conjecture the following minimal polynomials. Notice that the polynomials for $\tilde{D}_4$ are not obtained by specializing n to 4 in this table.

vertex	degree	polynomial
$0, 1, n-1, n$	$2n-4$	$P(2n-4, X_\delta)$
$2, \ldots, n/2-1$	$2n-4$	$(\lambda^{n-2} - 1)P(n-2, X_\delta)P(n-2, -X_\delta)$
$n/2$	$3n/2-3$	$(\lambda^{n/2-1} - 1)P(n-2, X_\delta)$

For type $\tilde{D}_n$, where $n > 3$ is odd, we conjecture the following minimal polynomials.

vertex	degree	polynomial
$0, 1, n-1, n$	$4n-8$	$P(4n-8, X_\delta)$
$n/2$	$2n-4$	$(\lambda^{n-2} - 1)P(n-2, X_\delta)$

For $\tilde{E}_6$, we conjecture the following minimal polynomials.

vertex	degree	polynomial
$1, 3, 5$	12	$P(12, X_\delta)$
$2, 4, 6$	9	$(\lambda^3 - 1)P(6, X_\delta)$
7	16	$P(4, X_\delta)P(12, X_\delta)$

For $\tilde{E}_7$, we conjecture the following minimal polynomials.

vertex	degree	polynomial
$1, 4$	24	$P(24, X_\delta)$
$2, 5$	36	$(\lambda^{12} - 1)P(12, X_\delta)P(12, -X_\delta)$
$3, 6$	32	$P(24, X_\delta)P(8, X_\delta)$
7	18	$(\lambda^6 - 1)P(12, X_\delta)$
8	24	$(\lambda^6 - 1)P(12, X_\delta)P(6, -X_\delta)$

For $\tilde{E}_8$, we conjecture the following minimal polynomials.

vertex	degree	polynomial
1	60	$P(60, X_\delta)$
2, 7	45	$(\lambda^{15} - 1)P(30, X_\delta)$
3, 6	80	$P(60, X_\delta)P(20, X_\delta)$
4, 8	75	$(\lambda^{15} - 1)P(30, X_\delta)P(30, X_\delta^2 - 2)$
5	132	$P(60, X_\delta)P(12, X_\delta)P(60, X_\delta^3 - 3X_\delta)$
9	85	$(\lambda^{15} - 1)P(30, X_\delta)P(30, X_\delta^2 - 2)P(10, X_\delta)$

Notice that for $\tilde{E}_6$ and $\tilde{E}_8$, the polynomial associated with a vertex i only depends on the coefficient δ_i of the root δ but that the analogous statement for $\tilde{E}_7$ does not hold since the polynomials associated with the vertices 3 and 6 are different.

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