Aisles in derived categories

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Bull. Soc. Math. Belg. **40** (1988), 239-253.

Summary

The aim of the present paper is to demonstrate the usefulness of aisles for studying the tilting theory of $\mathcal{D}^b(\operatorname{mod} A)$, where A is a finite-dimensional algebra. In section 1, we establish the equivalence of "aisles" with "t-structures" in the sense of [3] and give a characterization of aisles in molecular categories. Section 2 contains an application to the generalized tilting theory of hereditary algebras. Using aisles, we then give a geometrical proof of the theorem of Happel [7] which states that a finite-dimensional algebra which shares its derived category with a Dynkin-algebra A can be transformed into A by a finite number of reflections. The techniques developed so far naturally lead to the classification of the tilting sets in $\mathcal{D}^b(\operatorname{mod} k\vec{A}_n)$ presented in section 5. Finally, we consider the classification problem for aisles in $\mathcal{D}^b(\operatorname{mod} A)$, where A is a Dynkin-algebra. We reduce it to the classification of the silting sets in $\mathcal{D}^b(\operatorname{mod} A)$, which we carry out for $\Delta = \vec{A}_n$.

We thank P. Gabriel for lectures on these topics and for his helpful criticisms during the preparation of the manuscript.

Notations

Let A be a finite-dimensional algebra over a field k. We denote by

- $\operatorname{mod} A$ the category of finitely generated right A-modules,
- $\operatorname{proj} A$ the full subcategory of $\operatorname{mod} A$ consisting of the projective A-modules,

- $\mathcal{C}^-(\mathrm{proj}\, A)$ the category of differential complexes over $\mathrm{proj}\, A$ which are bounded above,
- $\mathcal{C}^b(\mathrm{proj}\, A)$ the full subcategory of $\mathcal{C}^-(\mathrm{proj}\, A)$ consisting of the complexes which are bounded above and below,
- $\mathcal{C}^-_b(\mathrm{proj}\, A)$ the full subcategory of $\mathcal{C}^-(\mathrm{proj}\, A)$ consisting of the complexes X such that $H^iX = 0$ for almost all $i \in \mathbf{Z}$,
- $\mathcal{H}^b(\mathrm{proj}\, A)$ the homotopy category obtained from $\mathcal{C}^b(\mathrm{proj}\, A)$ by factoring out the ideal of the morphisms homotopic to zero [12],
- $\mathcal{D}^b(A) := \mathcal{D}^b(\mathrm{mod}\, A)$ the bounded derived category [12] of the abelian category $\mathrm{mod}\, A$.

1 Aisles

1.1 Let $\mathcal{T}$ be a triangulated category with suspension functor S. A full additive subcategory $\mathcal{U}$ of $\mathcal{T}$ is called an *aisle* in $\mathcal{T}$ if

a) $S\mathcal{U} \subset \mathcal{U}$,

b) $\mathcal{U}$ is stable under extensions, i.e. for each triangle $X \to Y \to Z \to SX$ of $\mathcal{T}$ we have $Y \in \mathcal{U}$ whenever $X, Z \in \mathcal{U}$,

c) the inclusion $\mathcal{U} \to \mathcal{T}$ admits a right adjoint $\mathcal{T} \to \mathcal{U}$, $X \mapsto X_{\mathcal{U}}$.

For each full subcategory $\mathcal{V}$ of $\mathcal{T}$ we denote by $\mathcal{V}^{\perp}$ (resp. ${}^{\perp}\mathcal{V}$) the full additive subcategory consisting of the objects $Y \in \mathcal{T}$ satisfying $\mathrm{Hom}\,(X, Y) = 0$ (resp. $\mathrm{Hom}\,(Y, X) = 0$) for all $X \in \mathcal{V}$.

The following proposition shows that the assignment $\mathcal{U} \mapsto (\mathcal{U}, S\mathcal{U}^{\perp})$ is a bijection between the aisles $\mathcal{U}$ in $\mathcal{T}$ and the t-structures on $\mathcal{T}$ (in the sense of [3]).

Proposition. *A strictly (=closed under isomorphisms) full subcategory $\mathcal{U}$ of $\mathcal{T}$ is an aisle iff it satisfies a) and c')*

c') for each object X of $\mathcal{T}$ there is a triangle $X_{\mathcal{U}} \to X \to X^{\mathcal{U}^{\perp}} \to SX_{\mathcal{U}}$ with $X_{\mathcal{U}} \in \mathcal{U}$ and $X^{\mathcal{U}^{\perp}} \in \mathcal{U}^{\perp}$.

Proof. Suppose $\mathcal{U}$ satisfies a) and c'). The long exact sequence arising from the triangle in c') shows that $\mathrm{Hom}\,(U, X_{\mathcal{U}}) \xrightarrow{\sim} \mathrm{Hom}\,(U, X)$ for each $U \in \mathcal{U}$. If $U \to X \to V \to SU$ is a triangle and $U, V \in \mathcal{U}$ then $\mathrm{Hom}\,(X, ?)$ vanishes on $\mathcal{U}^{\perp}$ and $X^{\mathcal{U}^{\perp}}$. In particular, the morphism $X^{\mathcal{U}^{\perp}} \to SX_{\mathcal{U}}$ of c') has a retraction, hence $X^{\mathcal{U}^{\perp}} = 0$ and $X \cong X_{\mathcal{U}}$ lies in $\mathcal{U}$. Conversely, let $\mathcal{U}$ satisfy a),b),c). According to b), $\mathcal{U}$ is strictly full. In order to prove c), we form a triangle $X_{\mathcal{U}} \xrightarrow{\varphi} X \xrightarrow{\psi} Y \xrightarrow{\varepsilon} SX_{\mathcal{U}}$ over the adjunction morphism φ. Let $V \in \mathcal{U}$ and $f \in \mathrm{Hom}\,(V, Y)$. We insert f into a morphism of triangles

$$\begin{array}{ccccccc} X_{\mathcal{U}} & \xrightarrow{h} & W & \to & V & \xrightarrow{\varepsilon f} & SX_{\mathcal{U}} \\ \| & & \downarrow g & & \downarrow f & & \| \\ X_{\mathcal{U}} & \xrightarrow{\varphi} & X & \xrightarrow{\psi} & Y & \xrightarrow{\varepsilon} & SX_{\mathcal{U}} \end{array}$$

According to b), W lies in $\mathcal{U}$. By assumption, g factors uniquely through φ. Therefore, h has a retraction and $\varepsilon f = 0$. So f factors through ψ and even through $\psi\varphi = 0$ since $V \in \mathcal{U}$.

1.2 For certain triangulated categories, condition 1.1c) can still be weakened. We call an additive category $\mathcal{T}$ a *molecular category* if each object of $\mathcal{T}$ is a finite direct sum of objects with local endomorphism rings. In particular, if A is a finite-dimensional algebra over a field k, the category $\mathcal{D}^b(A)$ is a molecular category: For all $X, Y \in \mathcal{D}^b(A)$, we have $\dim_k \mathrm{Hom}\,(X, Y) < \infty$ and for each indecomposable U of $\mathcal{D}^b(A)$, $\mathrm{End}\,(U)$ is local since $\mathrm{End}_{\mathcal{C}^-(\mathrm{proj}\,A)}(V)$ is local for each indecomposable V of $\mathcal{C}^-(\mathrm{proj}\,A)$.

If $\mathcal{T}$ is a molecular category, we denote by $\mathrm{ind}\,\mathcal{T}$ a full subcategory of $\mathcal{T}$ whose objects form a system of representatives for the isomorphism classes of indecomposables of $\mathcal{T}$.

1.3 Proposition. *Let $\mathcal{T}$ be a triangulated molecular category and $\mathcal{U}$ a full additive subcategory which is closed under taking direct summands. The subcategory $\mathcal{U}$ is an aisle in $\mathcal{T}$ iff it satisfies a), b) and c").*

- *c") For each object X of $\mathcal{T}$ the functor* $\mathrm{Hom}\,(?, X)|\mathcal{U}$ *is finitely generated, i.e. there is $U \in \mathcal{U}$ and an epimorphism* $\mathrm{Hom}_{\mathcal{U}}(?, U) \to \mathrm{Hom}_{\mathcal{T}}(?, X)|\mathcal{U}$.

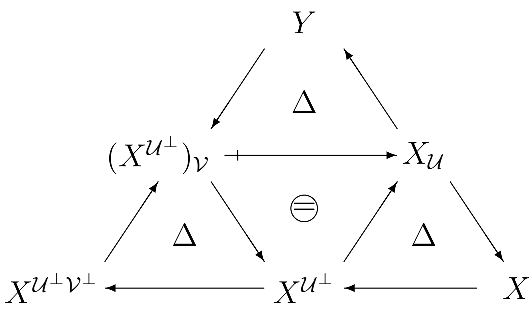
Proof. The claim follows from the

Lemma. *Let $\mathcal{S}$ be a suspended* [9] *molecular category and $F : \mathcal{S}^{\text{op}} \to Ab$ a cohomological functor, i.e. for each triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} SX$ of $\mathcal{S}$ the sequence $FX \xleftarrow{Fu} FY \xleftarrow{Fv} FZ$ is exact. The functor F is representable iff it is finitely generated.*

Proof. Let F be finitely generated. Because $\mathcal{S}$ is a molecular category F has a projective cover $\text{Hom}_{\mathcal{S}}(?, X) \xrightarrow{\varphi} F$ in the abelian category of the additive functors $\mathcal{S}^{\text{op}} \to Ab$. We shall show that φ is a monomorphism. Let $Y \in \mathcal{S}$ and $f \in \text{Hom}\,(Y, X)$ such that $\varphi \circ \text{Hom}\,(?, f) = 0$. We form a triangle $Y \xrightarrow{f} X \xrightarrow{g} Z \xrightarrow{h} SY$ in $\mathcal{S}$. By the "Yoneda Lemma" we conclude from the exactness of $FY \xleftarrow{Ff} FX \xleftarrow{Fg} FZ$ that φ factors through $\text{Hom}\,(?, g)$. By construction of φ, $\text{Hom}\,(?, g)$ admits a retraction. Hence g admits a retraction and $f = 0$.

1.4 Proposition. *Let $\mathcal{U}, \mathcal{V}$ be aisles in a triangulated category $\mathcal{T}$ such that $\mathcal{V} \subset \mathcal{U}^{\perp}$. Then $\mathcal{W} := \mathcal{U} * \mathcal{V} = \{X \in \mathcal{T}$: There is a triangle $U \to X \to V \to SU$ with $U \in \mathcal{U}, V \in \mathcal{V}\}$ is also an aisle in $\mathcal{T}$ (cf.* [3, 1.4]*)*

Proof. We shall verify the conditions of Proposition 1.1. Only c') is not immediate from the definitions. Let $X \in \mathcal{T}$. We form a diagram



where triangles are marked by Δ and tailed arrows denote morphisms of degree 1. By definition, $Y \in \mathcal{W}$, $X^{\mathcal{U}^\perp\mathcal{V}^\perp} \in \mathcal{V}^\perp$. Since $X^{\mathcal{U}^\perp\mathcal{V}^\perp}$ is an extension of $S((X^{\mathcal{U}^\perp})_{\mathcal{V}})$ by $X^{\mathcal{U}^\perp}$ it also lies in $\mathcal{U}^\perp$, hence in $\mathcal{W}^\perp = \mathcal{U}^\perp \cap \mathcal{V}^\perp$. We obtain the required triangle $Y \to X \to X^{\mathcal{U}^\perp \mathcal{V}^\perp} \to SY$ by forming an octahedron with base $X \to X^{\mathcal{U}^\perp} \to X^{\mathcal{U}^\perp \mathcal{V}^\perp}$.

2 Tilting sets

2.1 Let A be a finite-dimensional algebra over a field k. We consider the problem of determining all finite-dimensional k-algebras B such that $\mathcal{D}^b(B)$ is S-equivalent [9] to $\mathcal{D}^b(A)$.

A *tilting set* (cf. [7]) in $\mathcal{D}^b(A)$ is a finite subset $\{T_1, \ldots, T_s\} \subset \operatorname{ind} \mathcal{D}^b(A)$ such that $\operatorname{Hom}(S^l T_i, T_j) = 0$ for all i, j and all integers $l \neq 0$.

Each fully faithful S-functor $F : \mathcal{D}^b(B) \to \mathcal{D}^b(A)$ gives rise to the tilting set $\{T \in \operatorname{ind} \mathcal{D}^b(A) : T \cong FP$ for some indecomposable projective B-module $P\}$. Conversely, let $\{T_1, \ldots, T_s\}$ be a tilting set and $B := \operatorname{End}(\bigoplus_{i=1}^s T_i)$. By [9, 3.2], the obvious functor from $\operatorname{proj} B$ to $\mathcal{D}^b(A)$ extends to a fully faithful S-functor $E : \mathcal{H}^b(\operatorname{proj} B) \to \mathcal{D}^b(A)$ (put $\mathcal{E} := \mathcal{C}_b^-(\operatorname{proj} A)$ in [9, 3.2]). We make the additional assumptions

a) $\operatorname{Hom}(T_i, T_j) = 0 \ \forall\, i > j$ and $\operatorname{Hom}(T_i, T_i)$ is a skew field $\forall i$.

b) $\operatorname{gldim} A < \infty$.

Assumption a) implies $\operatorname{gldim} B < \infty$. By composing E with a quasi-inverse of the equivalence $\mathcal{H}^b(\operatorname{proj} B) \to \mathcal{D}^b(B)$ we obtain a fully faithful S-functor $F : \mathcal{D}^b(B) \to \mathcal{D}^b(A)$.

Proposition. *The essential image of F is an aisle in $\mathcal{D}^b(A)$. In particular, F has a right adjoint.*

Proof. Let $\mathcal{U}_i$ be the strictly full triangulated subcategory of $\mathcal{D}^b(A)$ generated by T_i. Since $\mathcal{D}^b(\operatorname{End}(T_i)) \xrightarrow{\sim} \mathcal{U}_i$, assumption b) and proposition 1.3 imply that $\mathcal{U}_i$ is an aisle in $\mathcal{D}^b(A)$. The essential image of F equals $\mathcal{U}_s * \mathcal{U}_{s-1} * \cdots * \mathcal{U}_2 * \mathcal{U}_1$ (by [3, 1.3.10], $*$ is associative). The claim now follows from proposition 1.4.

2.2 Theorem. *Let A be a hereditary finite-dimensional k-algebra and $\{T_1, \ldots, T_s\}$ a tilting set in $\mathcal{D}^b(A)$. The functor F is an S-equivalence iff s equals the number of isomorphism classes of simple A-modules.*

Proof. By [7, 7.3] assumption a) is satisfied for an appropriate numbering of the T_i and, since A is hereditary, so is assumption b). An S-equivalence $\mathcal{D}^b(B) \xrightarrow{\sim} \mathcal{D}^b(A)$ induces an isomorphism of the Grothendieck-groups $K_0(B) \xrightarrow{\sim} K_0(A)$. Therefore, since s equals the number of isomorphism classes of simple B-modules, the condition is necessary. For the converse, it is enough to show that $\{FX : X \in \mathcal{D}^b(B)\}^\perp = 0$, by proposition 2.1. Let $Y \in \mathcal{D}^b(A)$ be indecomposable and such that $\mathrm{Hom}\,(FX, Y) = 0$, $\forall X \in \mathcal{D}^b(B)$. This implies $< [FX], [Y] >= 0$, where $[.]$ denotes the canonical map $\mathcal{D}^b(A) \to K_0(A)$ and $< .\,, . >$ denotes the canonical bilinear form on $K_0(A)$ [7]. F induces a section $K_0(B) \to K_0(A)$ (the right adjoint of F yields a retraction). Since $\mathrm{rank} K_0(B) = s = \mathrm{rank}\, K_0(A)$, we have $K_0(A) = \{[FX] : X \in \mathcal{D}^b(B)\}$, hence $[Y] = 0$ and, since A is hereditary and Y is indecomposable, $Y = 0$ [7].

3 Dynkin-algebras

3.1 Let B be a basic, connected, finite-dimensional algebra over an algebraically closed field k. Assume that there exists a simple, projective, non-injective right B-module P. Define Q by $B \cong Q \oplus P$. Then $T = \tau^- P \oplus Q$ is a tilting module in $\mathrm{mod}\, B$ (cf. [5], [8]), where $\tau^- P$ denotes a preimage of P under the Auslander-Reiten-translation. The derived functors $F = \underline{\mathrm{R}}\mathrm{Hom}_B(T, ?) : \mathcal{D}^b(B) \to \mathcal{D}^b(B_P)$ and $G = \underline{\mathrm{L}}(? \otimes_{B_P} T)$ are quasi-inverse S-equivalences, where $B_P = \mathrm{End}\,(T_B)$ is obtained from B by *reflection in* P [4]. Thus, the derived category $\mathcal{D}^b(B)$ is "invariant under reflections". Conversely, we have the

Theorem. (Happel) *Let $\mathcal{D}^b(B)$ be S-equivalent to $\mathcal{D}^b(A)$ where A is a Dynkin-algebra (=path algebra* [6, 4.1] *of a Dynkin quiver). Then A is obtained from B by a finite number of reflections.*

Proof. Let $\mathcal{U} = \{X \in \mathcal{D}^b(B) : H^i X = 0\, \forall i > 0\}$ be the *natural aisle* in $\mathcal{D}^b(B)$ [3, 1.3.1], $\mathcal{V}$ the natural aisle in $\mathcal{D}^b(A)$ and $E : \mathcal{D}^b(B) \to \mathcal{D}^b(A)$ an S-equivalence with $[\mathcal{V}] \subset [E\mathcal{U}]$, where $[?]$ denotes the set of isomorphism classes of indecomposables in ?. If $[\mathcal{V}] = [E\mathcal{U}]$, E induces an equivalence of the hearts (=hearts of the corresponding t-structures) of $\mathcal{U}$ and $\mathcal{V}$, i.e. of mod B and mod A. In general, $[\mathcal{V}]$ is obtained from $[E\mathcal{U}]$ by the omission of finitely many isomorphism classes, as it is apparent from Happel's description of ind $\mathcal{D}^b(A)$ [7]. By induction, we are reduced to the case $[\mathcal{V}] = [E\mathcal{U}]\backslash\{EM\}$, where M is a source of ind$\mathcal{U}$, i.e. Hom $(V, M) = 0\,\forall V \in$ ind$\mathcal{U}, V \neq M$ and $S^-M \notin \mathcal{U}$. The sources of ind$\mathcal{U}$ are isomorphic to the simple projectives of mod B. It is therefore enough to prove the

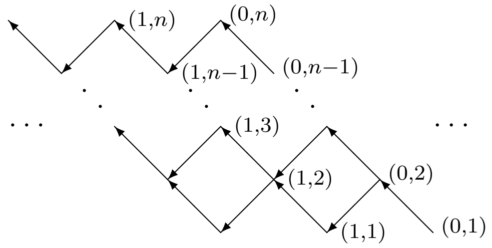
3.2 Lemma. *In the setting of 3.1 let* $\mathcal{U}$ *be the natural aisle in* $\mathcal{D}^b(B)$, $\mathcal{W}$ *the natural aisle in* $\mathcal{D}^b(B_P)$ *and* $\mathcal{W}'$ *its essential image under* G. *Then* $\mathcal{W}' = \{X \in \mathcal{U} : P$ *is not a direct summand of* $X\}$.

Proof. Let $\mathcal{U}_P = \{X \in \mathcal{U} :\ P$ is not a direct summand of $X\}$. It follows from $\mathrm{pdim}_{B_P}T \leq 1$ that $G\mathcal{W}^\perp \subset S\mathcal{U}^\perp$ hence $S\mathcal{U} \subset \mathcal{W}'$. We also have $S\mathcal{U} \subset \mathcal{U}_P$ and $\mathcal{U}_P = (S\mathcal{U}) * (\mathcal{U}_P \cap \mathrm{mod}\, B)$, $\mathcal{W}' = (S\mathcal{U}) * (\mathcal{W}' \cap \mathrm{mod}\, B)$. By [2], $\mathcal{U}_P \cap \mathrm{mod}\, B$ is just the torsion theory generated by T, which by [5] coincides with the essential image of mod B_P under $H^0G \cong ? \otimes_{B_P} T$.

4 Complete tilting sets in $\mathcal{D}^b(k\vec{A}_n)$

Let A be a finite-dimensional algebra over a field k. We define the *spectrum of a tilting set* M in $\mathcal{D}^b(A)$ as the full subcategory of $\mathcal{D}^b(A)$ whose objects are the elements of M. We call a tilting set in $\mathcal{D}^b(A)$ *complete* if its cardinality equals the number of isomorphism classes of simple A-modules.

Let A be the path algebra of the quiver $\vec{A}_n : 1 \to 2 \to \ldots \to n$ [6]. We identify [7] the category ind $\mathcal{D}^b(A)$ with the mesh-category $k(\mathbf{Z}\vec{A}_n)$ [11, 2.1] of the translation quiver $\mathbf{Z}\vec{A}_n$:



By an $\vec{A}_n$-quiver we mean an oriented tree K having n vertices and whose set of arrows is decomposed into a class of α-arrows and a class of β-arrows such that in each vertex of K there terminate at most one α-arrow and one β-arrow and there originate at most one α-arrow and one β-arrow. Now let K be an $\vec{A}_n$-quiver. For each vertex x of K let x^α (resp. x_α) be the number of vertices y of K such that the shortest walk from y to x ends (resp. begins) with an α-arrow terminating (resp. originating) in x. Analogously, we define x^β and x_β. Then there is exactly one map of the underlying sets of vertices $K_0 \to (\mathbf{Z}\vec{A}_n)_0$, $x \mapsto (gx, hx)$ such that

a) $\min_{x\in K} gx = 0$,

b) $(gy, hy) = (gx, hx + x^\beta + y_\beta + 1)$ for each α-arrow $x \xrightarrow{\alpha} y$ and

c) $(gy, hy) = (gx + x^\alpha + y_\alpha + 1, hx - x^\alpha - y_\alpha - 1)$ for each β-arrow $x \xrightarrow{\beta} y$.

Because $hx = 1 + x^\alpha + x_\beta$, this map is indeed well defined. Let M_K denote its image.

Theorem. *The assignment $K \mapsto M_K$ induces a bijection from the isomorphism classes of $\vec{A}_n$-quivers to the complete tilting sets M in* $\operatorname{ind}\mathcal{D}^b(k\vec{A}_n) = k(\mathbf{Z}\vec{A}_n)$ *with* $\min_{(g,h)\in M} g = 0$. *Moreover, the spectrum of M_K is described by the quiver K bound by all possible relations $\alpha\beta = 0$ and $\beta\alpha = 0$ (cf.* [1]*).*

Proof. Let K be an $\vec{A}_n$-tree. We call $x \in K_0$ a knot of K if x is contained in a full subtree of one of the forms

$$\bullet \xrightarrow{\alpha} x \xrightarrow{\alpha} \bullet,\quad \bullet \xrightarrow{\alpha} x \xleftarrow{\beta} \bullet,\quad \bullet \xleftarrow{\beta} x \xrightarrow{\alpha} \bullet,\quad \bullet \xleftarrow{\beta} x \xleftarrow{\beta} \bullet.$$

The other vertices of K are called peaks. We term $(g, h) \in (\mathbf{Z}\vec{A}_n)_0$ a marginal vertex if $h = 1$ or $h = n$ and we call the other vertices of $\mathbf{Z}\vec{A}_n$ inner vertices. We use induction on the number m of knots of K.

If $m = 0$, K has one of the forms

$$K_\alpha = \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\alpha} \dots$$

or

$$K_\beta = \bullet \xrightarrow{\beta} \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \dots$$

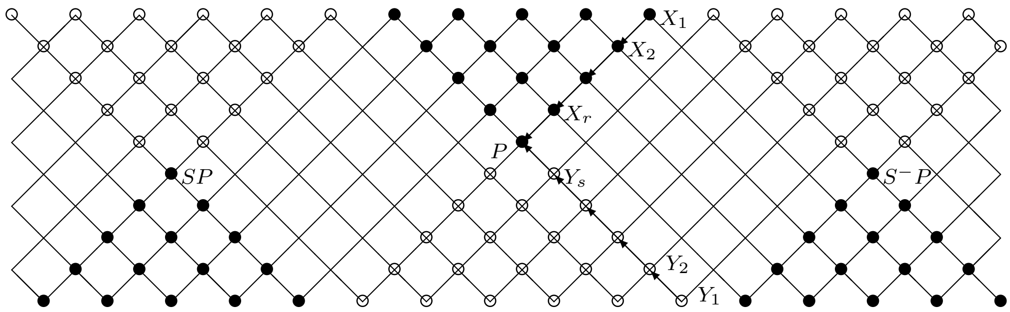


Figure 1: $\mathbf{Z}\vec{A}_n$ with given vertex P

It is easy to see that the corresponding sets M_{K_α}, M_{K_β} are exactly the complete tilting sets M of $\operatorname{ind}\mathcal{D}^b(k\vec{A}_n)$ which only consist of marginal vertices and satisfy $\min_{(g,h)\in M} g = 0$.

Now let $m > 0$. We first describe the complete tilting sets in $k(\mathbf{Z}\vec{A}_n)$ which contain a given inner vertex P. The tilting sets $\{X_1, \dots, X_r, P\}$ and $\{Y_1, \dots, Y_s, P\}$ (cf. Fig. 1) give rise to fully faithful embeddings $j_X : \mathcal{D}^b(k\vec{A}_{r+1}) \to \mathcal{D}^b(k\vec{A}_n)$ and $j_Y : \mathcal{D}^b(k\vec{A}_{s+1}) \to \mathcal{D}^b(k\vec{A}_n)$. We may assume that $j_X(\mathbf{Z}\vec{A}_{r+1})_0 \subset (\mathbf{Z}\vec{A}_n)_0$ and $j_Y(\mathbf{Z}\vec{A}_{s+1})_0 \subset (\mathbf{Z}\vec{A}_n)_0$, in particular $j_X P_X = P$ and $j_Y P_Y = P$ where $P_X = (0, r+1) \in (\mathbf{Z}\vec{A}_{r+1})_0$ and $P_Y = (0, s+1) \in (\mathbf{Z}\vec{A}_{s+1})_0$. With these notations, the complete tilting sets M in $k(\mathbf{Z}\vec{A}_n)$ containing P are exactly the sets $j_X(L) \cup j_Y(N)$ where $L \subset (\mathbf{Z}\vec{A}_{r+1})_0$ and $N \subset (\mathbf{Z}\vec{A}_{s+1})_0$ are complete tilting sets containing P_X and P_Y, respectively. Here, the set of marginal points of M equals $\{j_X(R) :\ R \text{ is a marginal point of } L \setminus P_X\ \} \cup \{j_Y(R) :\ R \text{ is a marginal point of } N \setminus P_Y\ \}$. For the corresponding spectra we have the pushout diagram

$$\begin{array}{ccc} k & \xrightarrow{i_X} & L \\ i_Y \downarrow & & \downarrow j_X \\ N & \xrightarrow{j_Y} & M \end{array}$$

where k is considered as a category with one object and i_X and i_Y are fully faithful with $i_X(k) = P_X$ and $i_Y(k) = P_Y$. Combined with the induction hypothesis, this description shows that the spectra of the complete tilting sets in $k(\mathbf{Z}\vec{A}_n)$ are described by the $\vec{A}_n$-quivers with all relations $\alpha\beta = 0 = \beta\alpha$ and that peaks are mapped to marginal points by the corresponding isomorphisms of categories. So let M be a complete tilting set in $k(\mathbf{Z}\vec{A}_n)$ whose spectrum is described by K. We claim that the corresponding bijection $e : K_0 \to M \subset (\mathbf{Z}\vec{A}_n)_0$ satisfies conditions b) and c). This is obvious if x, y are peaks of K since then ex, ey are marginal points. If for example x is a knot we apply the above construction with $P = ex$ and the claim follows from the induction hypothesis.

5 Aisles in $\mathcal{D}^b(k\Delta)$

5.1 Let $\mathcal{U}$ be an aisle in a triangulated category $\mathcal{T}$. The *heart of* $\mathcal{U}$ is the full subcategory $\mathcal{U}^0 = \mathcal{U} \cap S\mathcal{U}^\perp$ of $\mathcal{T}$; the associated cohomology functor $H^0_\mathcal{U} : \mathcal{T} \to \mathcal{U}^0$ is given by $X \mapsto (X_\mathcal{U})^{S\mathcal{U}^\perp}$ [3, 1.3.1-6]. The aisle $\mathcal{U}$ is *faithful* if the inclusion $\mathcal{U}^0 \to \mathcal{T}$ extends to an S-equivalence $\mathcal{D}^b(\mathcal{U}^0) \to \bigcup_{n\in\mathrm{N}} S^{-n}\mathcal{U}$; it is *separated* if $\bigcap_{n\in\mathrm{N}} S^n\mathcal{U} = 0$.

Let $k\Delta$ be the path algebra [6] of a Dynkin-quiver Δ. For each subset $M \subset \operatorname{ind}\mathcal{D}^b(k\Delta)$ let $\mathcal{F}(M)$ be the smallest strictly full subcategory of $\mathcal{D}^b(k\Delta)$ which contains M and is stable under S and closed under extensions and direct summands. By proposition 1.3, $\mathcal{F}(M)$ is an aisle. The assignment $\{T_1, \ldots, T_s\} \mapsto \mathcal{F}(T_1, \ldots, T_s)$ is a bijection between the tilting sets in $\mathcal{D}^b(k\Delta)$ and the faithful aisles. We shall generalize the concept of a tilting set in order to obtain an analogous description of all separated aisles in $\mathcal{D}^b(k\Delta)$.

A *silting set* in $\mathcal{D}^b(k\Delta)$ is a finite subset $\{R_1, \ldots, R_s\} \subset \operatorname{ind}\mathcal{D}^b(k\Delta)$ such that $\operatorname{Hom}(R_i, S^l R_j) = 0 \; \forall i, j$ and $\forall l > 0$.

Theorem.

a) The assignment $\{R_1, \ldots, R_s\} \mapsto \mathcal{F}(R_1, \ldots, R_s)$ *is a bijection between the silting sets in* $\mathcal{D}^b(k\Delta)$ *and the separated aisles in* $\mathcal{D}^b(k\Delta)$.

If $\{R_1, \ldots, R_s\}$ *is a silting set and* $\mathcal{W} = \mathcal{F}(R_1, \ldots, R_s)$ *we have*

b) $\mathcal{W} \cap {}^\perp(S\mathcal{W}) \cap \operatorname{ind}\mathcal{D}^b(k\Delta) = \{R_1, \ldots, R_s\}$

c) $s \le |\Delta_0|$, *and* $s = |\Delta_0| \Leftrightarrow \bigcup_{n\in\mathrm{N}} S^{-n}\mathcal{W} = \mathcal{D}^b(k\Delta)$

d) $\mathcal{W} = \{X \in \bigcup_{n\in\mathrm{N}} S^{-n}\mathcal{W} : \operatorname{Hom}(R_i, S^l X) = 0 \; \forall i, \; \forall l > 0\}$

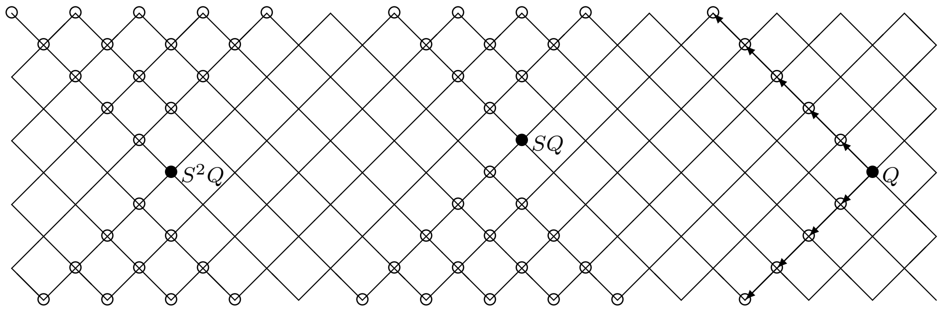


Figure 2: Example of $\mathcal{V}$ ($\bullet$) and $\mathcal{U}$ ($\circ$) in $\mathbf{Z}\vec{A}_n$

e) $H^0_{\mathcal{W}}R_1, \ldots, H^0_{\mathcal{W}}R_s$ *form a system of representatives of the isomorphism classes of indecomposable projectives of* $\mathcal{W}^0$*, and* $H^0_{\mathcal{W}}$ *induces an equivalence between the full subcategories* $\{R_1, \ldots, R_s\}$ *and* $\{H^0_{\mathcal{W}}R_1, \ldots, H^0_{\mathcal{W}}R_s\}$ *of* $\mathcal{D}^b(k\Delta)$.

Proof. 1st step: The following variant of a construction by Parthasarathy [10] allows us to use induction on $|\Delta_0|$.

Let $\mathcal{W}$ be a separated aisle in $\mathcal{D}^b(k\Delta)$ and Q a source (3.1) of $\operatorname{ind}\mathcal{W}$. We may assume that Δ admits a unique source q and that $Q = (0, q)$. Let $\mathcal{Q}$ be the full subcategory of $\mathcal{D}^b(k\Delta)$ whose objects are the direct sums of objects S^nQ, $n \in \mathbf{Z}$. The tilting set $\{(0, r) : r \in \Delta_0, r \neq q\} \subset k(\mathbf{Z}\Delta)$ yields a fully faithful S-functor $\mathcal{D}^b(k\Delta') \to \mathcal{D}^b(k\Delta)$ (2.1), where Δ' is obtained from Δ by omitting the vertex q and all arrows originating in q. The essential image of this S-functor equals ${}^{\perp}\mathcal{Q}$. We claim that $\mathcal{W} = \mathcal{U} * \mathcal{V}$, where $\mathcal{U} = \mathcal{W} \cap^{\perp} \mathcal{Q}$ and $\mathcal{V} = \mathcal{W} \cap \mathcal{Q}$. Obviously, we have $\mathcal{W} \supset \mathcal{U} * \mathcal{V}$. Conversely, let X be an indecomposable in $\mathcal{W}$ which is not isomorphic to Q. We have the triangle $X_{\perp\mathcal{Q}} \to X \to X^{\mathcal{Q}} \to SX_{\perp\mathcal{Q}}$ (1.3). The assumptions on X imply that $X^{\mathcal{Q}}$ lies in $S\mathcal{V}$. Thus, as an extension of X by $S^{-}X^{\mathcal{Q}} \in \mathcal{V} \subset \mathcal{W}$, $X_{\perp\mathcal{Q}}$ lies in $\mathcal{W} \cap^{\perp} \mathcal{Q} = \mathcal{U}$.

2nd step: b) Let the numbering be chosen in such a way that $\operatorname{Hom}(R_j, R_i) = 0 \, \forall i < j$. We apply the construction of the first step to the source $Q = R_1$ of $\operatorname{ind}\mathcal{W}$. Let $X \in \mathcal{W} \cap^{\perp}(S\mathcal{W})$ be indecomposable. We have the triangle $X_{\perp\mathcal{Q}} \to X \to X^{\mathcal{Q}} \to SX_{\perp\mathcal{Q}}$. As in the first step, either $X \cong R_1$ or $X^{\mathcal{Q}} \in S\mathcal{V}$ and in this case we infer $X^{\mathcal{Q}} = 0$ and $X \in \mathcal{U} \cap^{\perp}(S\mathcal{U})$. The claim now follows from the induction hypothesis.

a) Because of b) we only have to show surjectivity. In the setting of the first step let $R_1 = Q$. We complete R_1 to a system of representatives $\{R_1, \ldots, R_s\}$ of the indecomposables of $\mathcal{W} \cap^{\perp}(S\mathcal{W})$. Then $\{R_2, \ldots, R_s\}$ is a system of representatives of the indecomposables of $\mathcal{U} \cap^{\perp}(S\mathcal{U})$. According to the induction hypothesis, we have $\mathcal{U} = \mathcal{F}(R_2, \ldots, R_s)$, and therefore $\mathcal{W} = \mathcal{U} * \mathcal{V} = \mathcal{F}(R_1, \ldots, R_s)$.

The proof of c) is left to the reader. d) Obviously, $\mathcal{W}$ is contained in the aisle given in the assertion. Conversely, let $X = S^{-n}Y$ ($Y \in \mathcal{W}, n \in \mathbf{N}$) and $\mathrm{Hom}\,(R_i, S^l X) = 0\, \forall i, \forall l > 0$. By induction we conclude $S^{-n}Y_{\mathcal{U}} \in \mathcal{U}$. Using $\mathrm{Hom}\,(R_1, S\mathcal{U}) = 0$ and the triangle $Y_{\mathcal{U}} \cong Y_{\perp\mathcal{Q}} \to Y \to Y^{\mathcal{Q}} \to SY_{\perp\mathcal{Q}}$ we infer $\mathrm{Hom}\,(S^{-l}R_1, S^{-n}Y^{\mathcal{Q}}) = 0$ for all $l > 0$ and $S^{-n}Y^{\mathcal{Q}} \in \mathcal{V}$.

e) From $\mathrm{Hom}\,(R_1, S\mathcal{W}) = 0$ it follows that $R_1 \cong H^0_{\mathcal{W}}R_1$ is projective in $\mathcal{W}^0$. The rest of the assertion follows from the

Lemma. *Let $\mathcal{U}, \mathcal{V}$ be aisles in a triangulated category $\mathcal{T}$ such that $\mathcal{U} \subset^{\perp}\mathcal{V}$ and let $\mathcal{W} = \mathcal{U} * \mathcal{V}$.* (cf. proposition 1.4)

a) $\mathcal{V}^0 \subset \mathcal{W}^0$ and $H^0_{\mathcal{V}}|\,\mathcal{W}^0$ is right adjoint to this inclusion.

b) $H^0_{\mathcal{U}}|\,\mathcal{W}^0$ is exact and $H^0_{\mathcal{W}}|\,\mathcal{U}^0$ is left adjoint to $H^0_{\mathcal{U}}|\,\mathcal{W}^0$ and fully faithful.

c) For each $A \in \mathcal{W}^0$ we have an exact sequence

$$H^0_{\mathcal{W}}H^0_{\mathcal{U}}A \to A \to H^0_{\mathcal{W}}A^{\mathcal{U}^{\perp}} \to 0$$

d) $H^0_{\mathcal{W}}|\,\mathcal{W} \cap^{\perp}(S\mathcal{W})$ is fully faithful and for $X \in \mathcal{U}$ we have $H^0_{\mathcal{W}}X \cong H^0_{\mathcal{W}}H^0_{\mathcal{U}}X$.

We leave the proof of the lemma to the reader (compare with [3]).

5.2 In the setting of 5.1 let $\{T_1, \ldots, T_s\}$ be a tilting set in $\mathcal{D}^b(k\Delta)$. Suppose that the numbering has been chosen in such a way that $\mathrm{Hom}\,(T_j, T_i) = 0 \ \forall j > i$. Let $p : \{1, \ldots, s\} \to \mathbf{N}$ be a non-decreasing function with $p(1) = 0$.

Proposition.

a) $R_1 := S^{p(1)}T_1, \ldots, R_s := S^{p(s)}T_s$ *form a silting set in* $\mathcal{D}^b(k\Delta)$.

b) With $\mathcal{U} = \mathcal{F}(T_1, \ldots, T_s)$, $\mathcal{W} = \mathcal{F}(R_1, \ldots, R_s)$ *we have for each* $i \in \mathbf{Z}$

$$\mathcal{U}^0 \cap S^{-i}\mathcal{W} = \{X \in \mathcal{U}^0 : \mathrm{Hom}\,(T_j, X) = 0 \textit{ for each } j \textit{ with } i < p(j)\}$$

$$\mathcal{W} = \{X \in \mathcal{U} : H^i_{\mathcal{U}}X \in \mathcal{U}^0 \cap S^i\mathcal{W},\ \forall i \in \mathbf{Z}\}$$

c) The full subcategory $\{R_1, \ldots, R_s\}$ *of* $\mathcal{D}^b(k\Delta)$ *is isomorphic to the disjoint sum of the full subcategories* $C_j = \{T_i : p(i) = j\}$, $j \in \mathbf{N}$, *of* $\{T_1, \ldots, T_s\}$.

We leave the proof to the reader.

5.3 Theorem. *Each silting set in* $\mathcal{D}^b(k\vec{A}_n)$ (cf. section 4) *is of the form given in 5.2*

Proof. Let $\{R_1, \ldots, R_s\}$ be a silting set in $\mathcal{D}^b(k\vec{A}_n)$.

1st step: $\{R_1, \ldots, R_s\}$ is contained in a complete silting set (= silting set of maximal cardinality).

If $s < n$ we have $\mathcal{T}^\perp \neq 0$ (5.1 c), where $\mathcal{T} = \bigcup_{m \in \mathrm{N}} S^{-m}\mathcal{F}(R_1, \ldots, R_s)$. Because gldim$k\Delta < \infty$ we can find an indecomposable $R_0 \in \mathcal{T}^\perp$ such that $\mathrm{Hom}\,(R_0, S^l R_i) = 0$ for all $l > 0$, $i = 1, \ldots, s$. Then $\{R_0, \ldots R_s\}$ is a silting set. The claim now follows by induction on $n - s$.

2nd step: By the first step we may assume $n = s$. Let the numbering of the R_i be non-decreasing with respect to the order on ind$\,\mathcal{D}^b(k\vec{A}_n)$ generated by the arrows of $\mathbf{Z}\vec{A}_n$. With the notations of the first step of the proof of theorem 5.1 we set $Q = R_1$. The induction hypothesis applied to $\{R_2, \ldots, R_n\} \subset {}^\perp\mathcal{Q}$ yields a tilting set $\{T_2, \ldots, T_n\}$ which corresponds to a complete tilting set in $\mathcal{D}^b(k\Delta')$. Therefore the connected components of the spectrum of $\{T_2, \ldots, T_n\}$ are precisely the intersections of $\{T_2, \ldots, T_n\}$ with the connected components of ind${}^\perp\mathcal{Q}$.

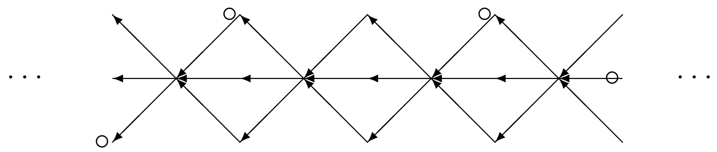


Figure 3: A silting set in $\mathbf{Z}\vec{D}_4$

Let $\mathcal{C}$ be a connected component of $\mathrm{ind}\,{}^{\perp}\mathcal{Q}$. Because $\mathrm{Hom}\,(Q,?)|\,\mathcal{C} \neq 0$ and $\{T_2,\ldots,T_n\} \cap \mathcal{C}$ is a complete tilting set in $\mathcal{C}$, $\mathrm{Hom}\,(Q,?)$ does not vanish on $S^l\{T_2,\ldots,T_n\} \cap \mathcal{C}$ for some $l \in \mathbf{Z}$. Hence we may assume that $\{Q,T_2,\ldots,T_n\}$ is connected. Because $\{X \in \mathcal{C} : \mathrm{Hom}\,(Q,X) \neq 0\}$ is linearly ordered by $X \leq X' :\Leftrightarrow \mathrm{Hom}\,(X,X') \neq 0$ (cf. figure 2), $\{Q,T_2,\ldots,T_n\}$ must be a tilting set in $\mathcal{D}^b(k\vec{A}_n)$. Setting $T_1 = Q$ we have $R_i = S^{p\,(i)}T_i$ for some function $p : \{1,\ldots,n\} \to \mathbf{Z}$ with $p(1) = 0$. It is easy to see that $\mathrm{Hom}\,(T_i,T_j) \neq 0$ implies $p(i) \leq p(j)$.

5.4 Remarks: a) Silting sets can always be completed (cf. the first step of the above proof) but in general, tilting sets cannot : $\{(0,1),(3,3)\} \subset \mathbf{Z}\vec{A}_3$.

b) The silting set of figure 3 is not of the form given in 5.2 since 5.2 c) cannot be satisfied.

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