WEIGHT STRUCTURES AND SIMPLE DG MODULES FOR POSITIVE DG ALGEBRAS

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Abstract. Using techniques due to Dwyer–Greenlees–Iyengar we construct weight structures in triangulated categories generated by compact objects. We apply our result to show that, for a dg category whose homology vanishes in negative degrees and is semi-simple in degree 0, each simple module over the homology lifts to a dg module which is unique up to isomorphism in the derived category. This allows us, in certain situations, to deduce the existence of a canonical t-structure on the perfect derived category of a dg algebra. From this, we can obtain a bijection between hearts of t-structures and sets of so-called simple-minded objects for some dg algebras (including Ginzburg algebras associated to quivers with potentials). In three appendices, we elucidate the relation between Milnor colimits and homotopy colimits and clarify the construction of t-structures from sets of compact objects in triangulated categories as well as the construction of a canonical weight structure on the unbonded derived category of a non positive dg category.

Contents

1. Introduction

Finite-dimensional modules over an associative unital algebra may be described as built up from simple modules or as presented by projective modules. The interplay between these two descriptions is at the heart of the interpretation of Koszul duality for dg algebras (and categories) given in [18], cf. also [12] [13]. However, in order to apply this theory a dg algebra A, we need the 'simple dg A-modules' as an additional datum. Clearly, a necessary condition for the existence of such dg modules is that the homology $H^*(A)$ should be equipped with a suitable set of graded simple modules. One may ask whether this condition is also sufficient. Now realizing modules over the homology $H^*(A)$ as homologies of dg modules over A is in general a hard problem, *cf.* for example [5]. In this paper, we treat one class of dg algebras where the problem

Date: October 1, 2011.

1991 *Mathematics Subject Classification.* 18E30.

The second named author has done part of this work while being a postdoctoral Fellow of the Fundación Séneca de la Región de Murcia.

of realizing the simple homology modules has a satisfactory solution. We define this class by merely imposing conditions on the homology $H^*(A)$: It should be concentrated in degrees ≥ 0 and semi-simple in degree 0. Let us point out that if $H^0(A)$ is a field, our result follows from Propositions 3.3 and 3.9 of [13], as kindly explained to us by Srikanth Iyengar [16]. The class we consider contains the Koszul duals of smooth dg algebras B whose homology is concentrated in non positive degrees and finite-dimensional in each degree. Important examples of these are the Ginzburg dg algebras associated to quivers with potential [14] [22]. The proof of our result is based on the construction of canonical weight structures on suitable triangulated categories (section 4) in analogy to results obtained by Pauksztello (Theorem 2.4 of [29]). These weight structures are also useful in a second application, namely the construction of a t-structure on the perfect derived category of a dg algebra A in our class (section 8). This t-structure has as its left aisle the closure under extensions, positive shifts and direct summands of the free module A. Its heart is a length category whose simple objects are the indecomposable factors of A in $\mathsf{per}\, A$. Let us point out that the existence of this t-structure also follows from a recent result by Rickard-Rouquier [30].

As another application, we establish a bijection between families of 'simple-minded objects' (a piece of terminology due to J. Rickard and, independently, to König-Liu [23]) and hearts of t-structures in suitable triangulated categories (section 9). Further applications will be given in the forthcoming paper [20].

In establishing our main theorem, we need foundational results on the precise link between Milnor colimits and homotopy colimits (in the sense of derivators) and on the construction of t-structures from sets of compact objects. We prove these in two appendices. In another appendix, we prove the existence of a canonical weight structure on the (unbounded) derived category of a non positive dg category, in analogy with a result by Bondarko [7, §6].

2. Acknowledgments

The authors thank Chris Brav and David Paukszteilo for stimulating conversations on the material of this paper. They are indebted to Srikanth Iyengar for pointing out reference [13] and explaining how Propositions 3.3 and 3.9 in that paper imply Corollary 5.7 below in the case where H^0A is a field. They are grateful to Dong Yang for his careful reading of the first version of this article.

3. Terminology and notations

In this article, 'graded' will always mean '$\mathbf{Z}$-graded', and 'small' will be frequently used to mean 'set-indexed'. A *length category* is an abelian category where each object has finite length.

We write Σ for the shift functor of any triangulated category. Let

$$L_0 \stackrel{f_0}{\to} L_1 \stackrel{f_1}{\to} L_2 \to \dots$$

be a sequence of morphisms in a triangulated category $\mathcal{D}$. Its *Milnor colimit* [26] is an object, denoted by $\mathsf{Mcolim}_{n\geq 0} L_n$, which fits into the *Milnor triangle*,

$$\coprod_{n\geq 0} L_n \xrightarrow{\mathbf{1}-\sigma} \coprod_{n\geq 0} L_n \longrightarrow \mathsf{Mcolim}_{n\geq 0} L_n \longrightarrow \Sigma \coprod_{n\geq 0} L_n,$$

where σ is the morphism with components

$$L_n \xrightarrow{f_n} L_{n+1} \xrightarrow{\mathrm{can}} \coprod_{n\geq 0} L_n.$$

Thus, the Milnor colimit is determined up to a (non unique) isomorphism. The notion of Milnor colimit has appeared in the literature under the name of *homotopy colimit* (see [6, Definition 2.1], [27, Definition 1.6.4]). However, Milnor colimits are not functorial and, in general, they do not take a sequence of triangles to a triangle of $\mathcal{D}$. Thus, we think it is better to keep this terminology for the notions appearing in the theory of derivators [24, 25, 11]. For a study of the relationship between Milnor colimits and homotopy colimits see Appendix 2.

Let

$$\ldots \to L_2 \stackrel{f_1}{\to} L_1 \stackrel{f_0}{\to} L_0$$

be a sequence of morphism in a triangulated category $\mathcal{D}$. Its *Milnor limit* is an object, denoted by $\mathsf{Mlim}_{n\geq 0} L_n$, which fits into the triangle,

$$\Sigma^{-1} \prod_{n\geq 0} L_n \longrightarrow \mathsf{Mlim}_{n\geq 0} L_n \longrightarrow \prod_{n\geq 0} L_n \stackrel{\mathbf{1}-\sigma}{\longrightarrow} \prod_{n\geq 0} L_n,$$

where σ is the morphism with components

$$L_n \stackrel{f_{n-1}}{\longrightarrow} L_{n-1} \stackrel{\mathrm{can}}{\longrightarrow} \prod_{n\geq 0} L_n$$

for $n \neq 0$, and the zero map in the component 0

$$0 : L_0 \to \prod_{n\geq 0} L_n.$$

As in the case of the Milnor colimit, the Milnor limit is determined up to a (non unique) isomorphism.

If $\mathcal{D}$ is a triangulated category and $\mathcal{S}$ is a set of objects of $\mathcal{D}$, we denote by $\mathsf{thick}_{\mathcal{D}}(\mathcal{S})$ the smallest full subcategory of $\mathcal{D}$ containing $\mathcal{S}$ and closed under extensions, shifts and direct summands.

Let k be a field and A a dg k-algebra. We denote the derived category of A by $\mathcal{D}A$, *cf.* [18]. The *perfect derived category* of A, denoted by $\mathsf{per}\, A$, is $\mathsf{thick}_{\mathcal{D}A}(A)$. The *finite-dimensional derived category* of A, denoted by $\mathcal{D}_{fd}A$, is the full subcategory of $\mathcal{D}A$ formed by those dg modules M whose homology is of finite total dimension:

$$\sum_{p\in\mathbf{Z}} \mathsf{dim}_k\, H^p(M) < \infty.$$

4. Weight structures from compact objects

Let us recall the definition of a weight structure from [7] and [29] (it is called co-t-structure in [29]): A *weight structure* on a triangulated category $\mathcal{T}$ is a pair of full additive subcategories $\mathcal{T}^{>0}$ and $\mathcal{T}^{\leq 0}$ of $\mathcal{T}$ such that

- w0) both $\mathcal{T}^{>0}$ and $\mathcal{T}^{\leq 0}$ are stable under taking direct factors;
- w1) the subcategory $\mathcal{T}^{>0}$ is stable under Σ^{-1} and the subcategory $\mathcal{T}^{\leq 0}$ is stable under Σ;
- w2) we have $\mathcal{T}(X, Y) = 0$ for all X in $\mathcal{T}^{>0}$ and all Y in $\mathcal{T}^{\leq 0}$;
- w3) for each object X of $\mathcal{T}$, there is a *truncation triangle*

$$\sigma_{>0}(X) \to X \to \sigma_{\leq 0}(X) \to \Sigma\sigma_{>0}(X)$$

with $\sigma_{>0}(X)$ in $\mathcal{T}^{>0}$ and $\sigma_{\leq 0}(X)$ in $\mathcal{T}^{\leq 0}$.

Notice that the objects $\sigma_{>0}(X)$ and $\sigma_{\leq 0}(X)$ in the truncation triangle are not functorial in X. The following theorem and its proof are based on Propositions 3.3 and 3.9 of [13]. Compared to the main result of [28], the theorem has stronger hypotheses: assumption c) is not present in [loc. cit.]; but it also has a stronger conclusion: the description of the weight structure in terms of homology.

Theorem 4.1. *Suppose that $\mathcal{T}$ is a triangulated category with small coproducts and that $\mathcal{S} \subset \mathcal{T}$ is a full additive subcategory stable under taking direct summands such that*

- a) $\mathcal{S}$ compactly generates $\mathcal{T}$, i.e. *the functors $\mathcal{T}(S, ?) : \mathcal{T} \to \mathsf{Mod}\,\mathbf{Z}$, $S \in \mathcal{S}$, commute with small coproducts, and if $M \in \mathcal{T}$ satisfies $\mathcal{T}(\Sigma^p S, M) = 0$ for all $p \in \mathbf{Z}$, $S \in \mathcal{S}$, then $M = 0$;*
- b) *we have $\mathcal{T}(L, \Sigma^p M) = 0$ for all L and M in $\mathcal{S}$ and all integers $p < 0$;*
- c) *the category $\mathsf{Mod}\,\mathcal{S}$ of additive functors $\mathcal{S}^{op} \to \mathsf{Mod}\,\mathbf{Z}$ is semi-simple.*

For X in $\mathcal{T}$ and $p \in \mathbf{Z}$, we write $H^p X$ for the object $L \mapsto \mathcal{T}(L, \Sigma^p X)$ of $\mathsf{Mod}\,\mathcal{S}$. Then we have:

1) *There is a unique weight structure* $(\mathcal{T}^{>0}, \mathcal{T}^{\leq 0})$ *on* $\mathcal{T}$ *such that* $\mathcal{T}^{\leq 0}$ *is formed by the objects* X *with* $H^p X = 0$ *for all* $p > 0$ *and* $\mathcal{T}^{>0}$ *is formed by the objects* X *with* $H^p X = 0$ *for all* $p \leq 0$.
2) *For each object* X*, there is a truncation triangle*

$$\sigma_{>0}(X) \to X \to \sigma_{\leq 0}(X) \to \Sigma\sigma_{>0}(X) \tag{4.1}$$

such that the morphism $X \to \sigma_{\leq 0}(X)$ *induces an isomorphism in* H^p *for* $p \leq 0$ *and the morphism* $\sigma_{>0}(X) \to X$ *induces an isomorphism in* H^p *for* $p > 0$.

Proof. Let $\mathsf{Sum}(\mathcal{S})$ be the closure under small coproducts of $\mathcal{S}$ in $\mathcal{T}$.

1st step: The functor $H^0 : \mathsf{Sum}(\mathcal{S}) \to \mathsf{Mod}\,\mathcal{S}$ *is an equivalence.* Indeed, this functor is fully faithful because the objects of $\mathcal{S}$ are compact in $\mathcal{T}$. It is an equivalence because $\mathsf{Mod}\,\mathcal{S}$ is semi-simple and $\mathcal{S}$ stable under direct factors.

2nd step: For each object X *of* $\mathcal{T}$ *and each integer* m*, there is a morphism* $V_m(X) \to X$ *such that* $V_m(X)$ *belongs to* $\Sigma^{-m}\mathsf{Sum}(\mathcal{S})$ *and the induced map*

$$H^m(V_m(X)) \to H^m X$$

is an isomorphism. Indeed, by the first step, the module $H^m X$ is isomorphic to $H^m(V_m(X))$ for some $V_m(X)$ lying in $\Sigma^{-m}\mathsf{Sum}(\mathcal{S})$.

3rd step: For each object X *of* $\mathcal{T}$*, there is a triangle*

$$V(X) \to X \to C(X) \to \Sigma V(X)$$

such that $V(X)$ *is a sum of objects* $\Sigma^p L$*, where* $L \in \mathcal{S}$ *and* $p < 0$*, the map*

$$H^p X \to H^p(C(X))$$

is bijective for all $p \leq 0$*, and the map*

$$H^p(V(X)) \to H^p X$$

is surjective for all $p > 1$. Indeed, we define

$$V(X) = \coprod_{m>0} V_m(X)$$

and $C(X)$ to be the cone over the natural morphism $V(X) \to X$. Then we obtain the claim because the functors H^p commute with coproducts, the $H^p M$ vanishes for all $M \in \mathcal{S}$ and all $p < 0$ and the morphism $V(X) \to X$ induces an isomorphism

$$H^1(V(X)) \to H^1 X.$$

4th step: For each object X *of* $\mathcal{T}$*, there is a triangle*

$$\sigma_{>0}(X) \to X \to \sigma_{\leq 0} X \to \Sigma\sigma_{>0}(X)$$

such that $\sigma_{>0}(X)$ *lies in* $\mathcal{T}^{>0}$ *and* $\sigma_{\leq 0} X$ *lies in* $\mathcal{T}^{\leq 0}$. We iterate the construction of the third step to obtain a direct system

$$X \to C(X) \to C^2(X) \to \cdots \to C^p(X) \to \cdots$$

and define

$$\sigma_{\leq 0}(X) = \mathsf{Mcolim}\, C^p(X).$$

We define $\sigma_{>0}(X)$ by the above triangle. The compactness of the objects of $\mathcal{S}$ in $\mathcal{T}$ implies that each functor H^n, $n \in \mathbf{Z}$, takes Milnor colimits to colimits in $\mathsf{Mod}\,\mathcal{S}$. Let us show that $\sigma_{\leq 0}(X)$ belongs to $\mathcal{T}^{\leq 0}$. Indeed, for $n > 0$, by construction, the morphisms $C^p(X) \to C^{p+1}(X)$ induce the zero map in H^n. Thus, the module

$$H^n(\sigma_{\leq 0}(X)) = H^n(\mathsf{Mcolim}\, C^p(X)) = \mathsf{colim}\, H^n(C^p(X))$$

vanishes for $n > 0$. Let us show that $\sigma_{>0}(X)$ belongs to $\mathcal{T}^{>0}$. Indeed, by induction on p, we see that the object $K_p(X)$ defined by the triangle

$$K_p(X) \to X \to C^p(X) \to \Sigma K_p(X)$$

belongs to $\mathcal{T}^{>0}$. By considering the exact sequence

$$H^{n-1}X \to H^{n-1}(C^p(X)) \to H^n(K_p(X)) \to H^nX \to H^n(C^p(X))$$

we see that for each $n \leq 0$, the morphism

$$H^{n-1}X \to H^{n-1}(C^p(X))$$

is surjective and the morphism

$$H^nX \to H^n(C^p(X))$$

is injective. By passing to the colimit over p, we obtain that for each $n \leq 0$, the morphism

$$H^{n-1}X \to H^{n-1}(\sigma_{\geq 0}(X))$$

is surjective and the morphism

$$H^nX \to H^n(\sigma_{\geq 0}(X))$$

is injective. By the exact sequence

$$H^{n-1}X \to H^{n-1}(\sigma_{\geq 0}(X)) \to H^n(\sigma_{>0}(X)) \to H^nX \to H^n(\sigma_{\geq 0}(X))$$

associated with the truncation triangle, this implies that for each $n \leq 0$, the module $H^n(\sigma_{>0}(X))$ vanishes.

5th step: For each object X of $\mathcal{T}$ and each $n \leq 0$, the map $H^nX \to H^n(\sigma_{\leq 0}(X))$ is an isomorphism and for $n > 0$, the map $H^n(\sigma_{>0}(X)) \to H^nX$ is an isomorphism. Indeed, the first claim follows from the fact that $X \to C^p(X)$ induces an isomorphism in H^n for all $n \leq 0$, which we obtain by induction from the third step. For the second claim, we consider the exact sequence

$$H^{n-1}X \to H^{n-1}(\sigma_{\leq 0}(X)) \to H^n(\sigma_{>0}(X)) \to H^nX \to H^n(\sigma_{\leq 0}(X)).$$

For $n = 1$, the first map is an isomorphism and the last term vanishes; for $n \geq 2$, the second and the last term vanish.

6th step: If X is an object of $\mathcal{T}$ and Y an object of $\mathcal{T}^{\leq 0}$, each morphism $X \to Y$ factors through $X \to \sigma_{\leq 0}(X)$. Indeed, since $V(X)$ is a coproduct of objects $\Sigma^{-m}L$, $m > 0$, $L \in \mathcal{S}$, by the triangle

$$V(X) \to X \to C(X) \to \Sigma V(X)\,,$$

the given morphism factors through $C(X)$. By induction, one constructs a compatible system of factorizations

$$X \longrightarrow C^p(X) \xrightarrow{f_p} Y.$$

This system lifts to a factorization $X \to \mathsf{Mcolim}(C^p(X)) \to Y$, which proves the claim since $\sigma_{\geq 0}(X) = \mathsf{Mcolim}(C^p(X))$.

7th step: For $X \in \mathcal{T}^{>0}$ and $Y \in \mathcal{T}_{\leq 0}$, we have $\mathcal{T}(X, Y) = 0$. Indeed, let $f : X \to Y$ be a morphism. By the 6th step, it factors through $X \to \sigma_{\leq 0}(X)$. We claim that $Z = \sigma_{\leq 0}(X)$ vanishes. Indeed, by the 4th step, we have $H^nZ = 0$ for $n > 0$ and by the 5th step, we have $H^nZ = 0$ for $n \leq 0$ since H^nX vanishes for $n \leq 0$.

8th step: the conclusion. Axioms w0) and w1) are clear, axiom w2) has been shown in the 7th step and axiom w3) in the 4th step. Claim b) has been shown in the 5th step. $\surd$

Although the assignment $X \mapsto \sigma_{\leq 0}X$ in part 2) of Theorem 4.1 is not uniquely defined up to isomorphism and it is not functorial, we have the following useful result:

Lemma 4.2. *In the situation of Theorem 4.1, we have:*

1) $\sigma_{\leq 0}(X \oplus Y) = \sigma_{\leq 0}(X) \oplus \sigma_{\leq 0}(Y)$,
2) $\sigma_{\leq 0}(\Sigma^p X) = \Sigma^p \sigma_{\leq 0}(X)$.

5. Positive dg algebras

Corollary 5.1. *Let k be a commutative associative ring with unit. Let $\mathcal{A}$ be a small dg k-linear category such that:*

a) *$H^p\mathcal{A}$ vanishes for $p < 0$,*
b) *$\mathsf{Mod}\, H^0(\mathcal{A})$ is a semisimple abelian category.*

Then we have:

1) *There exists a weight structure $w = ((\mathcal{DA})^{w>0}, (\mathcal{DA})^{w\leq 0})$ on $\mathcal{DA}$ such that $(\mathcal{DA})^{w>0}$ is formed by those modules X such that $H^pX = 0$ for $p \leq 0$ and $(\mathcal{DA})^{w\leq 0}$ is formed by those modules X such that $H^pX = 0$ for $p > 0$.*
2) *For each module X there exists a truncation triangle*

$$\sigma_{>0}(X) \to X \to \sigma_{\leq 0}(X) \to \Sigma\sigma_{>0}(X)$$

such that the morphism $X \to \sigma_{\leq 0}(X)$ induces an isomorphism in H^p for $p \leq 0$ and the morphism $\sigma_{>0}(X) \to X$ induces an isomorphism in H^p for $p > 0$.

Proof. We apply Theorem 4.1 by taking $\mathcal{T} = \mathcal{DA}$ and $\mathcal{S}$ to be the full subcategory of $\mathcal{DA}$ formed by the direct summands of finite direct sums of modules of the form $A^\wedge = \mathcal{A}(?, A)$ where A is an object of $\mathcal{A}$. Thanks to [18] we know that $\mathcal{D}$ is compactly generated by $\mathcal{S}$ and that condition a) implies $\mathsf{Hom}_{\mathcal{DA}}(L, \Sigma^p M) = 0$ for all L and M in $\mathcal{S}$ and all integers $p < 0$. After restricting scalars along the functor $H^0\mathcal{A} \to \mathcal{S}$ we get an equivalence

$$\mathsf{Mod}\, H^0(\mathcal{A}) \xrightarrow{\sim} \mathsf{Mod}\, \mathcal{S}.$$

Thus, condition b) implies that $\mathsf{Mod}\, \mathcal{S}$ is semisimple. √

Non-example 5.2. If H^0A is not semisimple we do not have a triangle as the one in part 2) of Corollary 5.1. We can take, for example, the algebra of dual numbers $A = k[\varepsilon]$ with $\varepsilon^2 = 0$ over field k and consider the complex M equal to the cone over the map $\varepsilon : A \to A$. Let S be the simple A-module. If there was a triangle

$$\sigma_{\geq 0}(M) \to M \to \sigma_{<0}(M) \to \Sigma\sigma_{\geq 0}(M),$$

the object $\sigma_{\geq 0}(M)$ would have to be isomorphic to S and the object $\sigma_{<0}(M)$ to ΣS (because the homology of M is concentrated in degrees 0 and -1 and isomorphic to S in both degrees). Then the connecting morphism

$$\sigma_{<0}(M) \to \Sigma\sigma_{\geq 0}(M)$$

would be a morphism $\Sigma S \to \Sigma S$ and thus would have to be 0 or an isomorphism. In the first case, we find that M is decomposable, a contradiction, and in the second case, we find that M is a zero object, a contradiction as well.

Notation 5.3. In analogy with the case of t-structures, we say that the weight structure of the Corollary 5.1 is the *canonical weight structure*. If $\mathcal{A}$ is in fact a dg algebra A, we write $S_A = \sigma_{\leq 0}A$.

Lemma 5.4. *Let A be an arbitrary dg algebra. If $M \in \mathcal{D}A$ and P is a direct summand of a small coproduct of copies of A, then the morphism of k-modules induced by H^0*

$$\mathsf{Hom}_{\mathcal{D}A}(P, M) \to \mathsf{Hom}_{H^0A}(H^0P, H^0M)$$

is an isomorphism.

Proof. The full subcategory of $\mathcal{D}A$ formed by the objects P satisfying the assertion contains A and is closed under small coproducts and direct summands. √

Lemma 5.5. *Let A be a dg algebra such that in $\mathsf{Mod}\, H^0(A)$, the module H^0A admits a finite decomposition into indecomposables (*e.g. *H^0A is semisimple). There exists a decomposition into indecomposables $A = \bigoplus_{i=1}^r A_i$ of A in $\mathcal{D}A$ such that $H^0A = \bigoplus_{i=1}^r H^0(A_i)$ is a decomposition into indecomposables of H^0A in $\mathsf{Mod}\, H^0(A)$.*

Proof. A decomposition of H^0A into indecomposables in the category of H^0A-modules gives us a complete family $\{e'_1, \dots, e'_r\}$ of primitive orthogonal idempotents of the ring $\mathsf{End}_{H^0A}(H^0A)$. Now, by using the ring isomorphism

$$H^0 : \mathsf{End}_{\mathcal{D}A}(A) \xrightarrow{\sim} \mathsf{End}_{H^0A}(H^0A)$$

we find a complete family $\{e_1, \dots, e_r\}$ of primitive orthogonal idempotents of the ring $\mathsf{End}_{\mathcal{D}A}(A)$. Since idempotents split in $\mathcal{D}A$, each e_i has an image A_i in $\mathcal{D}A$ and we obtain that $A = \bigoplus_{i=1}^r A_i$ is a decomposition of A into indecomposables in $\mathcal{D}A$. $\surd$

Proposition 5.6. *Let A be a dg algebra with homology concentrated in non negative degrees and such that H^0A is a semi-simple ring.*

1) *Let X be an object of $\mathcal{D}A$ with bounded homology and such that each H^nX , $n \in \mathbf{Z}$, is a finitely generated H^0A-module. If $p \in \mathbf{Z}$ is an integer such that $H^nX = 0$ for $n > p$ and $H^pX \neq 0$, then X belongs to the smallest full subcategory $\mathsf{susp}^{\oplus}(\Sigma^{-p}S_A)$ of $\mathcal{D}A$ containing $\Sigma^{-p}S_A$ and closed under extensions, positive shifts and direct summands.*
2) *Assume that each H^nA, $n \in \mathbf{Z}$, is a finitely generated H^0A-module. Then if $M \in \mathsf{per}\, A$, for any truncation triangle*

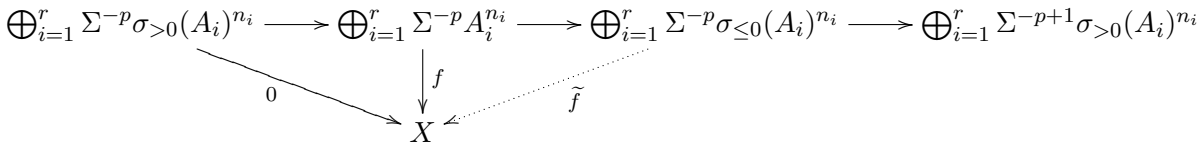
$$\sigma_{>p}(M) \to M \to \sigma_{\leq p}(M) \to \Sigma\sigma_{>p}(M)$$

we have $\sigma_{\leq p}M \in \mathsf{susp}^{\oplus}(\Sigma^{-p}S_A)$.

Proof. 1) We will use induction on the width of the interval delimited by those degrees with non-vanishing homology. By Lemmas 5.4 and 5.5, there are direct summands A_1 , ... , A_r of A in $\mathcal{D}A$, natural numbers n_1 , ... , n_r, and a morphism $f : \bigoplus_{i=1}^r \Sigma^{-p}A_i^{n_i} \to X$ in $\mathcal{D}A$ such that H^pf is an isomorphism in $\mathsf{Mod}\, H^0A$. Consider truncation triangles

$$\sigma_{>0}(A_i) \to A_i \to \sigma_{\leq 0}(A_i) \to \Sigma\sigma_{>0}(A_i),$$

as the ones in part b) of Theorem 4.1. After Lemma 4.2 we know that the objects $\sigma_{\leq 0}A_i$ can be taken to be direct summands of S_A in $\mathcal{D}A$. In particular, the $\Sigma^{-p}A_i$ are objects of $\mathsf{susp}^{\oplus}(S_A)$. Now notice that $X \in (\mathcal{D}A)^{w\leq p}$, and so it is right orthogonal to the objects of the wing $(\mathcal{D}A)^{w>p}$. Hence the morphism f factors through the morphism $\bigoplus_{i=1}^r \Sigma^{-p}A_i^{n_i} \to \bigoplus_{i=1}^r \Sigma^{-p}\sigma_{\leq 0}(A_i)^{n_i}$:



Since $H^p(\widetilde{f})$ is an isomorphism, for the mapping cone X' of $\widetilde{f}$ the width of the interval delimited by those degrees with non-vanishing homology is strictly smaller than that of X, and $H^n(X') = 0$ for $n > p-1$. By induction hypothesis we get $X' \in \mathsf{susp}^{\oplus}(\Sigma^{-p+1}S_A)$, which implies that $X \in \mathsf{susp}(\Sigma^{-p}S_A)$.

2) Since A has homology concentrated in non negative degrees, then $M \in \mathcal{D}^+A$. Therefore, $X = \sigma_{\leq p}M$ has bounded homology. Note that the hypothesis implies that each H^nM , $n \in \mathbf{Z}$, is finitely generated as a module over H^0A. This implies that each H^nX , $n \in \mathbf{Z}$, is finitely generated as a module over H^0A. Now we can use part 1) of the proposition. $\surd$

Corollary 5.7. *Let k be a commutative associative ring with unit. Let A be a dg k-algebra such that:*

a) *H^pA vanishes for $p < 0$,*
b) *$\mathsf{Mod}\, H^0(A)$ is a semisimple abelian category.*

Then for each graded simple module S over the graded ring H^*A, there is a dg A-module $\widetilde{S}$, unique up to isomorphism in the derived category $\mathcal{D}A$, such that the graded H^*A-module $H^*(\widetilde{S})$ is isomorphic to S.

Proof. First step: The graded simple modules over H^*A are precisely the simple modules over H^0A, regarded as graded H^*A-modules (concentrated in degree 0) by restricting scalars along $H^*A \to H^0A$. Clearly, simple H^0A-modules become simple graded H^*A-modules. Conversely, if S is a graded simple H^*A-module, then it has to be concentrated in degree 0. This implies that it is killed by $\bigoplus_{p>0} H^pA$. In other words, it is a (necessarily simple) H^0A-module.

Second step: There exists a decomposition into indecomposables $A = \bigoplus_{i=1}^r A_i$ of A in $\mathcal{D}A$ such that $H^0A = \bigoplus_{i=1}^r H^0(A_i)$ is a decomposition into simples of H^0A in $\mathsf{Mod}\, H^0(A)$. This is Lemma 5.5.

Third step: the graded H^*A-modules $H^*(\sigma_{\le 0}A_i)$, $1 \le i \le r$, are graded simple H^*A-modules, and every graded simple H^*A-module is of this form. Thanks to the first step, it suffices to prove that $H^p(\sigma_{\le 0}A_i) = 0$ for $p \neq 0$, and that with $H^0(\sigma_{\le 0}A_i)$, $1 \le i \le r$, we get all the simple H^0A-modules. This follows from the particular properties of the weight structure we are considering.

Fourth step: if $\widetilde{S} \in \mathcal{D}A$ is a module such that $H^*(\widetilde{S})$ is a graded H^*A-module isomorphic to $H^*(\sigma_{\le 0}A_i)$ for some $1 \le i \le r$, then $\widetilde{S}$ is isomorphic to $\sigma_{\le 0}(A_i)$ in $\mathcal{D}A$. Indeed, the proof of part 1 of Proposition 5.6 can be used to show that the map $\widetilde{f} : \sigma_{\le 0}(A_i) \to \widetilde{S}$ there is an isomorphism. √

Remark 5.8. The result above remains valid for small dg categories $\mathcal{A}$ such that $H^p\mathcal{A} = 0$ for $p < 0$ and $\mathsf{Mod}\, H^0(\mathcal{A})$ is semi-simple and each simple is compact.

6. The Koszul dual

Throughout this section A will be a dg algebra with homology concentrated in non negative degrees and such that H^0A is a semi-simple ring. Recall from Notation 5.3 that $S_A = \sigma_{\le 0}(A)$.

Notation 6.1. We write $B = \mathsf{REnd}(S_A)$. It should be thought thought of as the 'Koszul dual' of A.

Lemma 6.2. *B has homology concentrated in non positive degrees.*

Proof. We have to prove that

$$H^p\, \mathsf{RHom}(S_A, S_A) = \mathsf{Hom}_{\mathcal{D}A}(S_A, \Sigma^p S_A) = 0$$

for $p > 0$. After applying $\mathsf{Hom}_{\mathcal{D}A}(?, \Sigma^p S_A)$ to the triangle

$$\sigma_{>0}(A) \to A \to S_A \to \Sigma\sigma_{>0}(A)$$

we get the exact sequence

$$\mathsf{Hom}(\sigma_{>0}(A), \Sigma^{p-1}S_A) \to \mathsf{Hom}(S_A, \Sigma^p S_A) \to H^p(S_A).$$

Of course, $H^p(S_A) = 0$ for $p > 0$. On the other hand, by definition of weight structure we have

$$\mathsf{Hom}(\sigma_{>0}(A), \Sigma^{p-1}S_A) = 0$$

for $p > 0$. √

Lemma 6.3. 1) *For each $X \in \mathcal{D}A$ we have $X \cong \mathsf{Mlim}_{p\ge 0}\, \sigma_{\le p}X$.*

2) *For every pair of objects X and Y of $\mathcal{D}A$ we have*

$$\mathsf{Hom}(X, Y) = \mathsf{lim}_q\, \mathsf{colim}_p\, \mathsf{Hom}(\sigma_{\le p}X, \sigma_{\le q}Y).$$

Proof. 1) Given $X \in \mathcal{D}A$ we can form triangles

$$\sigma_{>0}(X) \to X \to \sigma_{\le 0}(X) \to \Sigma\sigma_{>0}(X),$$

$$\sigma_{>1}(\sigma_{>0}X) \to \sigma_{>0}X \to \sigma_{\le 1}(\sigma_{>0}X) \to \Sigma\sigma_{>1}(\sigma_{>0}X),$$

$$\dots$$

Thanks to statement (2) of Theorem 4.1, we can take all these triangles so that the maps induce isomorphisms at the level of convenient homologies. Using the octahedron axiom of triangulated categories we prove that in the triangle

$$\sigma_{>1}\sigma_{>0}X \to X \to C \to \Sigma\sigma_{>1}\sigma_{>0}X,$$

over the composition

$$\sigma_{>1}(\sigma_{>0}X) \to \sigma_{>0}(X) \to X,$$

the object C belongs to $(\mathcal{D}A)^{w\leq 1}$. Thus

$$\sigma_{>1}\sigma_{>0}X \to X \to C \to \Sigma\sigma_{>1}\sigma_{>0}X$$

is the truncation triangle corresponding to the weight structure $((\mathcal{D}A)^{w\leq 1}, (\mathcal{D}A)^{w\geq 1})$, and we can write $C = \sigma_{\leq 1}(X)$ and $\sigma_{>1}(\sigma_{>0})X = \sigma_{>1}(X)$. Moreover, we still have an isomorphism

$$H^pX \xrightarrow{\sim} H^p(\sigma_{\leq 1}X)$$

for $p \leq 1$. Indeed, for $p \leq 0$ we have the following diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^pX & \longrightarrow & H^p(\sigma_{\leq 0}X) & \longrightarrow & H^{p+1}(\sigma_{>0}X) \\
 & & \downarrow & & \| & & \downarrow\wr \\
0 & \longrightarrow & H^p(\sigma_{\leq 1}X) & \longrightarrow & H^p(\sigma_{\leq 0}X) & \longrightarrow & H^{p+1}(\sigma_{\leq 1}\sigma_{>0}X),
\end{array}$$

and for $p = 1$ we have the following diagram with exact rows

$$\begin{array}{ccccccccc}
H^1(\sigma_{>1}X) & \longrightarrow & H^1(\sigma_{>0}X) & \longrightarrow & H^1(\sigma_{\leq 1}\sigma_{>0}X) & \longrightarrow & H^2(\sigma_{>1}X) & \longrightarrow & H^2(\sigma_{>0}X) \\
\| & & \downarrow\wr & & \downarrow & & \| & & \downarrow\wr \\
H^1(\sigma_{>1}X) & \longrightarrow & H^1X & \longrightarrow & H^1(\sigma_{\leq 1}X) & \longrightarrow & H^2(\sigma_{>1}X) & \longrightarrow & H^2X,
\end{array}$$

which implies that $H^1(\sigma_{\leq 1}\sigma_{>0}X) \to H^1(\sigma_{\leq 1}X)$ is an isomorphism, and so from the square

$$\begin{array}{ccc}
H^1(\sigma_{>0}X) & \xrightarrow{\sim} & H^1(\sigma_{\leq 1}\sigma_{>0}X) \\
\downarrow\wr & & \downarrow\wr \\
H^1X & \longrightarrow & H^1(\sigma_{\leq 1}X)
\end{array}$$

we deduce that $H^1X \to H^1(\sigma_{\leq 1}X)$ is an isomorphism.

Repeating this construction we get a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \sigma_{\leq 2}X & \longrightarrow & \sigma_{\leq 1}X & \longrightarrow & \sigma_{\leq 0}X \\
 & & \uparrow{\scriptstyle g_2} & & \uparrow{\scriptstyle g_1} & & \uparrow{\scriptstyle g_0} \\
\cdots & \longrightarrow & X & = & X & = & X \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \sigma_{>2}X & \longrightarrow & \sigma_{>1}X & \longrightarrow & \sigma_{>0}X
\end{array}$$

where the morphisms $H^n(g_p) : H^nX \to H^n(\sigma_{\leq p}X)$ are isomorphisms for $n \leq p$. Consider now the induced map

$$X \to \mathsf{Mlim}_{p\geq 0}\, \sigma_{\leq p}X.$$

For each $n \in \mathbf{Z}$ we get a map

$$H^nX \to H^n(\mathsf{Mlim}_{p\geq 0}\, \sigma_{\leq p}X) = \mathsf{lim}_{p\geq 0}\, H^n(\sigma_{\leq p}X)$$

induced by

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^n(\sigma_{\leq 2}X) & \longrightarrow & H^n(\sigma_{\leq 1}X) & \longrightarrow & H^n(\sigma_{\leq 0}X) \\
 & & \uparrow{\scriptstyle H^n g_2} & & \uparrow{\scriptstyle H^n g_1} & & \uparrow{\scriptstyle H^n g_0} \\
\cdots & \longrightarrow & H^n X & = & H^n X & = & H^n X \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & H^n(\sigma_{>2}X) & \longrightarrow & H^n(\sigma_{>1}X) & \longrightarrow & H^n(\sigma_{>0}X)
\end{array}$$

For each $n \in \mathbf{Z}$, almost every map $H^n(g_p)$ is an isomorphism, and so the map $H^n X \to H^n(\mathsf{Mlim}_{p\geq 0}\, \sigma_{\leq p}X)$ is an isomorphism.

2) Given X , $Y \in \mathcal{D}A$, we have $Y = \mathsf{Mlim}_{q\geq 0}\, \sigma_{\leq q}Y$, and so

$$\mathsf{Hom}(X, Y) = \mathsf{Hom}(X, \mathsf{Mlim}_q\, \sigma_{\leq q}Y) = \mathsf{lim}_q\, \mathsf{Hom}(X, \sigma_{\leq q}Y).$$

After applying $\mathsf{Hom}(?, \sigma_{\leq q}Y)$ to the commutative diagram (see the proof of part 1))

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Sigma\sigma_{>2}X & \longrightarrow & \Sigma\sigma_{>1}X & \longrightarrow & \Sigma\sigma_{>0}X \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \sigma_{\leq 2}X & \longrightarrow & \sigma_{\leq 1}X & \longrightarrow & \sigma_{\leq 0}X \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & X & = & X & = & X \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & \sigma_{>2}X & \longrightarrow & \sigma_{>1}X & \longrightarrow & \sigma_{>0}X
\end{array}$$

we get the commutative diagram

$$\begin{array}{ccccccc}
\mathsf{Hom}(\Sigma\sigma_{>0}X, \sigma_{\leq q}Y) & \longrightarrow & \mathsf{Hom}(\Sigma\sigma_{>1}X, \sigma_{\leq q}Y) & \longrightarrow & \mathsf{Hom}(\Sigma\sigma_{>2}X, \sigma_{\leq q}Y) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathsf{Hom}(\sigma_{\leq 0}X, \sigma_{\leq q}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{\leq 1}X, \sigma_{\leq q}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{\leq 2}X, \sigma_{\leq q}Y) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathsf{Hom}(X, \sigma_{\leq q}Y) & = & \mathsf{Hom}(X, \sigma_{\leq q}Y) & = & \mathsf{Hom}(X, \sigma_{\leq q}Y) & = & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathsf{Hom}(\sigma_{>0}X, \sigma_{\leq q}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{>1}X, \sigma_{\leq q}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{>2}X, \sigma_{\leq q}Y) & \longrightarrow & \cdots
\end{array}$$

For $p \gg 0$ we have $\mathsf{Hom}(\sigma_{>p}X, \sigma_{\leq q}Y) = 0 = \mathsf{Hom}(\Sigma\sigma_{>p}X, \sigma_{\leq q}Y)$, and so the map $\mathsf{Hom}(\sigma_{\leq p}X, \sigma_{\leq q}Y) \to \mathsf{Hom}(X, \sigma_{\leq q}Y)$ is an isomorphism. Hence,

$$\mathsf{Hom}(X, Y) = \mathsf{lim}_{q\geq 0}\, \mathsf{Hom}(X, \sigma_{\leq q}Y) = \mathsf{lim}_{q\geq 0}\, \mathsf{colim}_{p\geq 0}\, \mathsf{Hom}(\sigma_{\leq p}X, \sigma_{\leq q}Y).$$

$\surd$

Proposition 6.4. *Assume that each $H^p A$, $p \in \mathbf{Z}$, is a finitely generated $H^0 A$-module. Then the functor*

$$\mathsf{RHom}(?, S_A) : (\mathsf{per}\, A)^{op} \to \mathcal{D}(B^{op}),$$

which has its image in $\mathcal{D}^-(B^{op})$, is fully faithful.

Proof. For the first claim it suffices to notice that

$$\mathsf{Hom}_{\mathcal{D}A}(\Sigma^{-p}X, \sigma_{\leq 0}A) = 0$$

for $X \in \mathsf{per}\, A$ and $p \gg 0$, since every object in $\mathsf{per}\, A$ has left bounded homology.

We prove the second claim in several steps.

First step: The functor $\mathsf{RHom}(?, S_A) : \mathsf{thick}(S_A)^{op} \to \mathcal{D}(B^{op})$ *is fully faithful.* Indeed, we can do finite dévissage using the fact that the map

$$\mathsf{RHom}(?, S_A) : \mathsf{Hom}_{\mathcal{D}A}(S_A, S_A) \to \mathsf{Hom}_{\mathcal{D}(B^{\mathrm{op}})}(B, B)$$

is an isomorphism.

Second step: preservation of truncation of perfect objects. Here we will use both the weight structure on $\mathcal{D}A$ (see Corollary 5.1) and the canonical weight structure on $\mathcal{D}(B^{\mathrm{op}})$ (see Appendix 1). The truncation triangle for A corresponding to the weight structure of Corollary 5.1 is

$$\sigma_{>0}(A) \to A \to S_A \to \Sigma\sigma_{>0}(A).$$

After applying $\mathsf{RHom}(?, S_A)$ and rotating we gt the triangle

$$B \to \mathsf{RHom}(A, S_A) \to \mathsf{RHom}(\sigma_{>0}A, S_A) \to \mathsf{RHom}(S_A, \Sigma S_A),$$

where $B \in \mathcal{D}^-(B^{\mathrm{op}})^{w\geq 0}$ and $\mathsf{RHom}(\sigma_{>0}A, S_A) \in \mathcal{D}^-(B^{\mathrm{op}})^{w<0}$. If X is an arbitrary perfect module, then one can prove that $\mathsf{RHom}(\sigma_{\leq p}X, S_A)$ belongs to $\mathcal{D}^-(B^{\mathrm{op}})^{w\geq -p}$ by using part 2) of Proposition 5.6 together with Remark 10.1, and one can prove that $\mathsf{RHom}(\sigma_{>p}X, S_A) \in \mathcal{D}^-(B^{\mathrm{op}})^{w<p}$ by using the orthogonality property of weight structures.

Third step: the claim. Put $F = \mathsf{RHom}(?, S_A)$. Let X and Y be two objects of $\mathsf{per}\, A$. Thanks to Lemma 6.3, Theorem 10.2, step 2 and step 1 of this proof, and Proposition 5.6, we have the following commutative diagram

$$\begin{array}{ccc}
\mathsf{Hom}(X, Y) & \longrightarrow & \mathsf{Hom}(FY, FX) \\
\| & & \| \\
 & & \mathsf{lim}_q\, \mathsf{colim}_p\, \mathsf{Hom}(\sigma_{\geq -q}FY, \sigma_{\geq -p}FX) \\
 & & \| \\
\mathsf{lim}_q\, \mathsf{colim}_p\, \mathsf{Hom}(\sigma_{\leq p}X, \sigma_{\leq q}Y) & \xrightarrow{\sim} & \mathsf{lim}_q\, \mathsf{colim}_p\, \mathsf{Hom}(F\sigma_{\leq q}Y, F\sigma_{\leq p}X)
\end{array}$$

$\surd$

7. Reminder on t-structures

A *t-structure* [3] on a triangulated category $\mathcal{D}$ is a pair $t = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of strictly full triangulated subcategories of $\mathcal{D}$ such that:

1) $\mathcal{D}^{\leq 0}$ is closed under Σ and $\mathcal{D}^{\geq 0}$ is closed under Σ^{-1},
2) $\mathsf{Hom}_{\mathcal{D}}(M, \Sigma^{-1}N) = 0$ for each $M \in \mathcal{D}^{\leq 0}$ and $N \in \mathcal{D}^{\geq 0}$,
3) for each $M \in \mathcal{D}$ there exists a triangle in $\mathcal{D}$

$$M^{\leq 0} \to M \to M^{\geq 1} \to \Sigma M^{\leq 0},$$

with $M^{\leq 0} \in \mathcal{D}^{\leq 0}$ and $\Sigma(M^{\geq 1}) \in \mathcal{D}^{\geq 0}$.

It is easy to prove that each one of the two subcategories completely determines the other one in the following sense: an object $N \in \mathcal{D}$ belongs to $\mathcal{D}^{\geq 0}$ (resp. $\mathcal{D}^{\leq 0}$) if and only if we have

$$\mathsf{Hom}_{\mathcal{D}}(M, \Sigma^{-1}N) = 0$$

for each $M \in \mathcal{D}^{\leq 0}$ (resp. for each $N \in \mathcal{D}^{\geq 0}$).

It is also easy to prove that the triangle above is unique up to a unique isomorphism extending the identity morphism $\mathbf{1}_M$. Hence, for each $M \in \mathcal{D}$ we can make choices of the objects $M^{\leq 0}$ and $M^{\geq 1}$ so that the map $M \mapsto M^{\leq 0}$ underlies a functor $(?)^{\leq 0} : \mathcal{D} \to \mathcal{D}^{\leq 0}$ right adjoint to the inclusion, and the map $M \mapsto \Sigma((\Sigma^{-1}M)^{\geq 1})$ underlies a functor $(?)^{\geq 0} : \mathcal{D} \to \mathcal{D}^{\geq 0}$ left adjoint to the inclusion.

The *heart* of t is the full subcategory $\mathcal{H}(t)$ of $\mathcal{D}$ formed by those objects which are in $\mathcal{D}^{\leq 0}$ and also in $\mathcal{D}^{\geq 0}$. It is an abelian category, and the functor

$$H^0 : \mathcal{D} \to \mathcal{H}(t)\ ,\ M \mapsto (M^{\leq 0})^{\geq 0},$$

which is said to be the 0*th homology functor of* t, is homological, *i.e.* takes triangles to long exact sequences.

A t-structure $t = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is *non degenerate* if we have

$$\bigcap_{n\in\mathbf{Z}} \Sigma^n \mathcal{D}^{\leq 0} = \{0\} = \bigcap_{n\in\mathbf{Z}} \Sigma^n \mathcal{D}^{\geq 0}.$$

This property implies that an object M of $\mathcal{D}$:

- vanishes if and only if $H^0(\Sigma^n M) = 0$ for each $n \in \mathbf{Z}$,
- belongs to $\mathcal{D}^{\leq 0}$ if and only if $H^0(\Sigma^n M) = 0$ for $n > 0$,
- belongs to $\mathcal{D}^{\geq 0}$ if and only if $H^0(\Sigma^n M) = 0$ for $n < 0$.

The t-structure t is *bounded* if we have:

$$\bigcup_{n\in\mathbf{Z}} \Sigma^n \mathcal{D}^{\leq 0} = \mathcal{D} = \bigcup_{n\in\mathbf{Z}} \Sigma^n \mathcal{D}^{\geq 0}.$$

Note that any bounded t-structure t is non degenerate. Indeed, if t is bounded, any object M is a finite extension of shifts of objects of the form $H^0(\Sigma^n M)\,,\ n \in \mathbf{Z}$. But if $M \in \bigcap_{n\in\mathbf{Z}} \Sigma^n \mathcal{D}^{\leq 0}$ or $M \in \bigcap_{n\in\mathbf{Z}} \Sigma^n \mathcal{D}^{\geq 0}$, then we have $H^0(\Sigma^n M) = 0$ for each $n \in \mathbf{Z}$.

A *left aisle* (resp. *right aisle*) in a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{U}$ containing a zero object 0 of $\mathcal{D}$, closed under Σ (resp. Σ^{-1}), closed under extensions, and such that the inclusion functor $\mathcal{U} \to \mathcal{D}$ admits a right (resp. left) adjoint. We have already mentioned that if $t = (\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$, then $\mathcal{D}^{\leq 0}$ is a left aisle in $\mathcal{D}$ and $\mathcal{D}^{\geq 0}$ is a right aisle in $\mathcal{D}$. Moreover, it is proved in [21, §1] that the map $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \mapsto \mathcal{D}^{\leq 0}$ underlies a bijection between the set of t-structures on $\mathcal{D}$ and the set of left aisles in $\mathcal{D}$, and similarly for right aisles. We will refer to $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$) as *the left (resp. right) aisle of* t.

Example 7.1. It is shown in Appendix 3 that if A is a dg algebra, there exists a t-structure t_A on its unbounded derived category $\mathcal{D}A$ such that $\mathcal{D}^{\geq 0}$ is formed by those modules whose ordinary homology is concentrated in non negative degrees, and $\mathcal{D}^{\leq 0}$ is formed by those modules M which fit into a triangles

$$\coprod_{i\geq 0} L_i \to \coprod_{i\geq 0} L_i \to M \to \Sigma \coprod_{i\geq 0} L_i\,,$$

where L_i is an i-fold extension of small coproducts of non negative shifts of A. Therefore, if A has homology concentrated in non positive degrees, it is not difficult to prove that $\mathcal{D}^{\leq 0}$ is formed by those modules whose ordinary homology is concentrated in non positive degrees. In this case, if we assume moreover, as we may, that the components of A vanish in strictly positive degrees, the functors $(?)^{\leq 0}$ and $(?)^{\geq 0}$ are given by the usual intelligent truncations, and the associated 0th homology functor gives the ordinary homology in degree 0. Therefore, we say that the t-structure t_A is the *canonical* one. It is a non degenerate t-structure, whose heart is equivalent to the category of unital right modules over the ring $H^0(A)$:

$$H^0 : \mathcal{H}(t_A) \xrightarrow{\sim} \mathsf{Mod}\, H^0(A)$$

(see for example [22, Lemma 5.2.b)]).

Assume now that A is a dg algebra over a field k, and let us consider the finite-dimensional derived category $\mathcal{D}_{fd}A$ (see § 3). The canonical t-structure on $\mathcal{D}A$ restricts to a bounded t-structure t_A^{fd} on $\mathcal{D}_{fd}A$, whose heart is equivalent to the category of finite-dimensional unital right modules over the k-algebra $H^0(A)$:

$$H^0 : \mathcal{H}(t_A^{fd}) \xrightarrow{\sim} \mathsf{mod}\, H^0(A).$$

In particular, $\mathcal{H}(t_A^{fd})$ is a length category. If, moreover, $H^0(A)$ is finite-dimensional, then $\mathcal{H}(t_A^{fd})$ has a finite number of isoclasses of simple objects.

8. Application to the construction of t-structures

Theorem 8.1. *Let k be a commutative associative ring with unit, and let A be a dg k-algebra such that:*

a) $H^pA = 0$ *for* $p < 0$,
b) H^0A *is a semi-simple k-algebra, and*
c) *each* H^pA , $p \in \mathbf{Z}$, *is a finitely generated H^0A-module.*

Then the perfect derived category $\mathsf{per}\, A$ *admits a bounded t-structure whose left (resp. right) aisle is the smallest full subcategory containing A and closed under extensions, positive (resp. negative) shifts and direct summands. Its heart is a length category whose simple objects are the indecomposable direct summands of A in* $\mathsf{per}\, A$.

Remark 8.2. Suppose k is a field and H^0A is isomorphic to a product of copies of k. Then the theorem follows from Proposition 3.4 of Rickard-Rouquier's [30] applied to the triangulated category $\mathcal{T} = \mathsf{per}\, A$ and to the set $\mathcal{S}$ formed by a system of representatives of the indecomposable direct factors of A in $\mathsf{per}\, A$.

Proof. Consider the functor $\mathsf{RHom}(?, S_A) : (\mathcal{D}A)^{\mathrm{op}} \to \mathcal{D}(B^{\mathrm{op}})$. Thanks to Proposition 6.4, we know its restriction to $(\mathsf{per}\, A)^{\mathrm{op}}$ is fully faithful. Notice that the obvious morphism of dg algebras $B \to H^0B$ (using the intelligent truncation) and the isomorphism of ordinary algebras $H^0B \to H^0A$ allow us to regard H^0A as a dg B-module. Moreover, we have isomorphisms

$$\mathsf{RHom}(A, S_A) \xrightarrow{\sim} S_A \xleftarrow{\sim} H^0A$$

compatible with the structure of left dg B-modules of $\mathsf{RHom}(A, S_A)$ and H^0A. Thus

$$\mathsf{RHom}(?, S_A) : (\mathsf{per}\, A)^{\mathrm{op}} \xrightarrow{\sim} \mathsf{thick}_{\mathcal{D}(B^{\mathrm{op}})}(H^0A)$$

is an equivalence. The picture of the situation is the following:

$$\begin{array}{ccc} (\mathcal{D}A)^{\mathrm{op}} & \longleftarrow & (\mathsf{per}\, A)^{\mathrm{op}} \\ {\scriptstyle \mathsf{RHom}(?,S_A)}\downarrow & & \downarrow{\scriptstyle \wr} \\ \mathcal{D}(B^{\mathrm{op}}) & \longleftarrow & \mathsf{thick}(H^0A) \end{array}$$

Let us consider a full subcategory $\mathcal{A}$ of the heart $\mathcal{H}$ of the canonical t-structure on $\mathcal{D}(B^{\mathrm{op}})$ formed by those objects with a finite composition series in which the composition factors are direct summands of H^0A. It is not difficult to prove that $\mathsf{thick}(H^0A)$ is precisely the full subcategory $\mathcal{T}$ of $\mathcal{D}(B^{\mathrm{op}})$ formed by those modules M such that $H^pM = 0$ for almos every $p \in \mathbf{Z}$ and $H^pM \in \mathcal{A}$ for each $p \in \mathbf{Z}$. With this description it is easy to check that the canonical t-structure restricts to a t-structure on $\mathcal{T}$ whose heart is $\mathcal{A}$. The simple objects of this heart are given by the simple H^0A-modules, *i.e.* the indecomposable direct summands of H^0A, which corresponds bijectively to the indecomposable direct summands of A. $\surd$

A triangulated category can be recovered from the heart of a bounded t-structure by closing under extensions and shifts. Taking this into account, we have:

Corollary 8.3. *Let A be as in Theorem 8.1. Then* $\mathsf{per}\, A$ *is the smallest full triangulated subcategory of $\mathcal{D}A$ closed under extensions, shifts and containing the indecomposable direct summands of A .*

Remark 8.4. Notice that the simple objects of the heart are also in bijection with the simple modules over $H^0(A)$.

Corollary 8.5. *Let A be an algebra as in Theorem 8.1. If we assume moreover that A is formal, then* $\mathsf{per}(H^*A)$ *admits a canonical t-structure whose left (resp. right) aisle is the smallest full subcategory containing H^*A and closed under extensions, positive (resp. negative) shifts and direct summands. Its heart is a length category whose simples are the indecomposable direct summands of H^*A in* $\mathsf{per}(H^*A)$.

Remark 8.6. Theorem 8.1 should be compared with a result by O. Schnürer [32] which states the existence of a canonical t-structure on the perfect derived category of a dg algebra B positively graded, with B^0 semi-simple and whose differential vanishes on B^0. The main motivation for Schnürer's theorem was to prove that certain categories of sheaves, endowed with a perverse t-structure, are t-equivalent to the perfect derived category of a certain dg algebra B endowed with a canonical t-structure (see [31]). In practice, B is the homology algebra H^*A of a formal dg algebra A satisfying conditions of Theorem 8.1, and so the existence of a canonical t-structure on $\mathsf{per}\, B$ follows from Theorem 8.1 and Corollary 8.5.

Example 8.7. Let A be a dg algebra over a field k such that in each degree its homology is of finite dimension and vanishes for large degrees. Let S_1 , ... , S_r, be a family of perfect A-modules such that:

a) $\mathsf{Hom}_{\mathcal{D}A}(S_i, S_j) = \begin{cases} 0 & \text{if } i \neq j, \\ k \cdot \mathbf{1}_{S_i} & \text{if } i = j. \end{cases}$

b) $\mathsf{Hom}_{\mathcal{D}A}(S_i, \Sigma^p S_j) = 0$ for each $p < 0$.

Then the derived endomorphism dg algebra $B = \mathsf{REnd}_A(\bigoplus_{i=1}^r S_i)$ satisfies the conditions of Theorem 8.1. Indeed, the homology groups of B vanish in degrees < 0 by condition b) and they are finite-dimensional and vanish in degrees $\gg 0$ because the S_i are perfect.

Non-example 8.8. Here we show that condition b) of our theorem is not redundant. Indeed, let A be a finite-dimensional algebra of infinite global dimension over a field k. We will show that $\mathsf{per}\, A$ does not admit a canonical t-structure. Indeed, assume $\mathsf{per}\, A$ admits a t-structure t such that $\mathsf{per}(A)^{t\leq 0}$ is the smallest full subcategory of $\mathsf{per}\, A$ containing A and closed under extensions, shifts and direct summands. Then, by dévissage, we deduce that $\mathsf{per}(A)^{t\geq 0}$ is the full subcategory of $\mathsf{per}\, A$ formed by those objects with ordinary homology concentrated in non negative degrees. On the other hand, it is clear that the objects of $\mathsf{per}(A)^{t\leq 0}$ have ordinary homology concentrated in non positive degrees. Thus, if P belongs to $\mathsf{per}\, A$, then in the triangle

$$P^{t\leq 0} \to P \to P^{t\geq 1} \to \Sigma(P^{t\leq 0}) \,,$$

the object $P^{t\leq 0}$ only has homology in non positive degrees and the object $P^{t\geq 1}$ only has homology in strictly positive degrees. Therefore, this is the triangle for the natural t-structure and so the truncation functors of the given t-structure t on $\mathsf{per}\, A$ coincide with those of the natural t-structure. It follows that $\mathsf{per}\, A$ is stable under the natural truncation functor $P \mapsto \tau_{\geq 0} P$. This is a contradiction since we may take $P = (P_1 \to P_0)$ to be the beginning of a projective resolution of an A-module of infinite projective dimension. Thus, $\mathsf{per}\, A$ does not admit a canonical t-structure.

9. Application to hearts and simple-minded objects

Let k be an algebraically closed field, and let A be a dg k-algebra such that:

1) in each degree its homology is of finite dimension,
2) its homology vanishes for large degrees,
3) A is *homologically smooth*, *i.e.* A is a compact object of the unbounded derived category of dg A-A-bimodules.

Remark 9.1. Note that these conditions are invariant under derived Morita equivalence. The reader can find the proof of the invariance of condition 3) in [33, Lemma 2.6].

Example 9.2. Let A be an ordinary finite-dimensional algebra over a perfect field k. Then A is homologically smooth if and only if it has finite global dimension. That the finiteness of the global dimension is necessary already appeared in Cartan-Eilenberg's book [9, Proposition IX.7.6]. That it is a sufficient condition can be proved by using, for example, the ideas of the proof of [15, Lemma 1.5].

Example 9.3. We can also take A to be the non complete Ginzburg dg algebra associated to a Jacobi-finite quiver with potential [14] [22]. The fact that in this case A satisfies condition 3) has been proved in [17]. That condition 1) also holds has been proved in [2].

Following Rickard (unpublished) and Koenig-Liu [23], we define a *family of simple-minded objects* to be a finite family S_1 , ... , S_r of objects of $\mathcal{D}_{fd}A$ such that:

a) $\mathsf{Hom}_{\mathcal{D}A}(S_i, S_j) = \begin{cases} 0 & \text{if } i \neq j, \\ k \cdot \mathbf{1}_{S_i} & \text{if } i = j. \end{cases}$

b) $\mathsf{Hom}_{\mathcal{D}A}(S_i, \Sigma^t S_j) = 0$ for each $t < 0$.

c) $\mathcal{D}_{fd}A$ is the smallest full triangulated subcategory of $\mathcal{D}A$ containing the objects S_1, ... , S_r.

Example 9.4. Let t be a bounded t-structure on $\mathcal{D}_{fd}A$ whose heart $\mathcal{H}(t)$ is a length category with a finite number of isoclasses of simple objects. Then we can take S_1 , ... , S_r to be a family of representatives of those isoclasses.

Two families S_1 , ... , S_r and S'_1 , ... , $S'_{r'}$ of simple-minded objects of $\mathcal{D}_{fd}A$ are *equivalent* if they have the same closure under extensions.

Corollary 9.5. *Taking representatives of the isoclasses of the simple objects of the heart yields a bijection between:*

1) *Bounded t-structrures on $\mathcal{D}_{fd}A$ whose heart is a length category with a finite number of isoclasses of simple objects.*
2) *Equivalence classes of families of simple-minded objects of $\mathcal{D}_{fd}(A)$.*

Proof. First step: from t-structures to simple-minded objects. We have already observed in Example 9.4 that, from such a t-structure on $\mathcal{D}_{fd}A$, one gets a family of simple-minded objects of $\mathcal{D}_{fd}A$ by considering the simples of the corresponding heart.

Second step: from simple-minded objects to t-structures. Conversely, let S_1 , ... , S_r be a family of simple-minded objects of $\mathcal{D}_{fd}A$. Put $S = \bigoplus_{i=1}^{r} S_i$ and $B = \mathsf{REnd}_A(S)$. The adjoint pair

$$\begin{array}{c} \mathcal{D}A \\ {\scriptstyle ?\otimes^{\mathbf{L}}_B S}\uparrow \;\downarrow {\scriptstyle \mathsf{RHom}_A(S,?)} \\ \mathcal{D}B \end{array}$$

induces mutually quasi-inverse triangle functors

$$\begin{array}{c} \mathcal{D}_{fd}A \\ {\scriptstyle ?\otimes^{\mathbf{L}}_B S}\uparrow \;\downarrow {\scriptstyle \mathsf{RHom}_A(S,?)} \\ \mathsf{per}\, B. \end{array}$$

Under these equivalences, the objects S_i correspond to the indecomposable direct summands of B in $\mathsf{per}\, B$. As noticed in Example 8.7, B satisfies the hypothesis of Theorem 8.1. Therefore, there exists a bounded t-structure on $\mathsf{per}\, B$ whose heart is a length category such that the indecomposable direct summands of B in $\mathsf{per}\, B$ are the representatives of the isoclasses of the simple objects. This t-structure is mapped by $? \otimes^{\mathbf{L}}_B S$ to a bounded t-structure on $\mathcal{D}_{fd}A$ whose heart is a length category such that the simple-minded objects we started with are the representatives of the isoclasses of the simple objects.

Third step: the bijection. By using that a bounded t-structure is completely determined by its heart (see for example [8, Lemma 2.3]) it is easy to check that steps 1 and 2 define a bijection. √

Corollary 9.6. *S_1 , ... , S_r and S'_1 , ... , $S'_{r'}$ are two equivalent families of simple-minded objects of $\mathcal{D}_{fd}A$ if and only if $r = r'$ and, up to reordering, $S_i \cong S'_i$.*

Proof. After Corollary 9.5, two equivalent families of simple-minded objects are families of representatives of the isoclasses of the simple modules of the same length category. √

10. Appendix 1: A weight structure for negative dg algebras

Let B be a dg algebra with homology concentrated in non positive degrees. Consider the following full subcategories of $\mathcal{D}B$:

- $\mathcal{D}^{w\leq 0}$, formed by those modules with homology concentrated in non positive degrees,
- $\mathcal{D}^{w\geq 0}$, formed by those modules X satisfying $\mathsf{Hom}(X,Y)=0$ for each $Y\in\mathcal{D}^{w<0}=\Sigma\mathcal{D}^{w\leq 0}$.

Remark 10.1. Note that $\Sigma^p B\in\mathcal{D}^{w\geq 0}$ for each $p\leq 0$.

The following result is an unbounded analogue of a result by Bondarko, cf. §6 of [7].

Theorem 10.2. 1) *The pair $(\mathcal{D}^{w\leq 0},\mathcal{D}^{w\geq 0})$ is a weight structure on $\mathcal{D}B$.*

2) *$\mathcal{D}^{w\leq 0}$ is the smallest full subcategory $\mathsf{Susp}(B)$ of $\mathcal{D}B$ containing B and closed under positive shifts, extensions and arbitrary coproducts.*

3) *For any object X of $\mathcal{D}B$ we have $X\cong\mathsf{Mcolim}_{p\geq 0}\,\sigma_{\geq -p}X$.*

4) *For any pair X and Y of objects of $\mathcal{D}B$ we have*

$$\mathsf{Hom}(X,Y)=\mathsf{lim}_{q\geq 0}\,\mathsf{colim}_{p\geq 0}\,\mathsf{Hom}(\sigma_{\geq -q}X,\sigma_{\geq -p}Y).$$

Proof. 2) It is clear that $B\in\mathcal{D}^{w\leq 0}$, and that $\mathcal{D}^{w\leq 0}$ is closed under extensions, positive shifts and arbitrary coproducts. Therefore $\mathsf{Susp}(B)$ is contained in $\mathcal{D}^{w\leq 0}$. Now, for an object M of $\mathcal{D}^{w\leq 0}$ we can form a sequence of triangles

$$B_0\stackrel{u}{\to}M\stackrel{v}{\to}M_0\to\Sigma B_0,$$

$$B_1\to M_0\to M_1\to\Sigma B_1,$$

$$\cdots$$

by taking $B_p=\coprod_{q\geq p}\coprod_{\mathsf{Hom}(\Sigma^q B,M_p)}\Sigma^q B$ and defining $B_p\to M_p$ as the obvious map. This yields a diagram

$$\begin{array}{ccccccc}
M_0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots \\
{\scriptstyle v}\uparrow & & \uparrow & & \uparrow & & \\
M & \xrightarrow{\mathbf{1}} & M & \xrightarrow{\mathbf{1}} & M & \longrightarrow & \cdots \\
{\scriptstyle u}\uparrow & & \uparrow & & \uparrow & & \\
L_0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & \cdots
\end{array}$$

where each L_p is a p-fold extension of coproducts of non negative shifts of B. Thanks to Verdier's 3×3 lemma (see [3, Proposition 1.1.11]) we know there exists a triangle

$$L\to M\to\mathsf{Mcolim}\,M_p\to\Sigma L,$$

where L fits in a triangle of the form

$$\coprod_{p\geq 0}L_p\to L\to\coprod_{p\geq 0}\Sigma L_p\to\Sigma\coprod_{p\geq 0}L_p.$$

Thus, it is clear that $L\in\mathsf{Susp}(B)$. On the other hand, for each $n\geq 0$ we have

$$\mathsf{Hom}(\Sigma^n B,\mathsf{Mcolim}\,M_p)=\mathsf{colim}\,\mathsf{Hom}(\Sigma^n B,\mathsf{Mcolim}\,M_p)=0$$

because the morphisms

$$\mathsf{Hom}(\Sigma^n B,M_p)\to\mathsf{Hom}(\Sigma^n B,M_{p+1})$$

vanish. Thus $\mathsf{Mcolim}\,M_p$ has homology concentrated in degrees ≥ 1. But, in fact, for each $n\geq 1$ we have an exact sequence

$$H^n M\to H^n(\mathsf{Mcolim}\,M)\to H^{n+1}L,$$

where $H^n M=0$ by hypothesis and $H^{n+1}L=0$ because B has homology concentrated in non positive degrees. This proves that $\mathsf{Mcolim}\,M_p=0$, and so $M\cong L\in\mathsf{Susp}(B)$.

1) It is clear that $\mathcal{D}^{w\leq 0}$ and $\mathcal{D}^{w\geq 0}$ are closed under finite coproducts and direct summands. It is also clear that $\mathcal{D}^{w\leq 0}$ is closed under positive shifts and $\mathcal{D}^{w\geq 0}$ is closed under negative shifts. The ortohogonality axiom hols by definition of $\mathcal{D}^{w\geq 0}$. It remains to prove the existence of a truncation triangle. Let M be an object of $\mathcal{D}B$. Thanks to [18, §3.1] we can assume that M has a filtration

$$0 = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n \cdots \subset M$$

in the category $\mathcal{C}B$ of dg B-modules such that

F1) $M = \mathsf{colim}_{n\geq 0} M_n$,
F2) each $M_{n-1} \to M_n$ in an inflation in $\mathcal{C}B$, *i.e.* it is a degreewise split-injection,
F3) M_n/M_{n-1} is a small coproduct of (positive or negative) shifts of B.

Using the fact that B has homology concentrated in non positive degrees, we can form a commutative square

$$\begin{array}{ccc} L'_1 & \longrightarrow & \Sigma L_0 \\ \downarrow & & \downarrow \\ M_1/M_0 & \longrightarrow & \Sigma M_0 \end{array}$$

where the vertical morphisms are degree-wise split injections and L'_1 (resp. L_0) is the direct summand of M_1/M_0 (resp. M_0) formed by the non positive shifts of B. Taking the co-cone L_1 of $L'_1 \to \Sigma L_0$ we get a morphism of degree-wise split short exact sequences of dg B-modules

$$\begin{array}{ccccc} L_0 & \longrightarrow & L_1 & \longrightarrow & L'_1 \\ \downarrow & & \downarrow & & \downarrow \\ M_0 & \longrightarrow & M_1 & \longrightarrow & M_1/M_0, \end{array}$$

where the vertical arrows are degree-wise split injections. We write $L'_1 = L_1/L_0$. In this way, we can form morphisms of degree-wise split short exact sequences of dg B-modules

$$\begin{array}{ccccc} L_{n-1} & \longrightarrow & L_n & \longrightarrow & L_n/L_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ M_{n-1} & \longrightarrow & M_n & \longrightarrow & M_n/M_{n-1} \end{array}$$

for each $n \geq 0$, where the vertical arrows are degree-wise split injections and L_n/L_{n-1} is the direct summand of M_n/M_{n-1} formed by the non positive shifts of B. This yields a sequence of degree-wise split short exact sequences of dg B-modules

$$\begin{array}{ccccccccc} 0 = L_{-1} & \longrightarrow & L_0 & \longrightarrow & L_1 & \longrightarrow \cdots & \quad L_{n-1} & \longrightarrow & L_n & \quad \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 = M_{-1} & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow \cdots & M_{n-1} & \longrightarrow & M_n & \quad \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 = N_{-1} & \longrightarrow & N_0 & \longrightarrow & N_1 & \longrightarrow \cdots & N_{n-1} & \longrightarrow & N_n & \quad \cdots, \end{array}$$

where for each $n \geq 0$ there is a morphisms of degree-wise split short exact sequences of dg B-modules

$$\begin{array}{ccccc} M_{n-1} & \longrightarrow & M_n & \longrightarrow & M_n/M_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ N_{n-1} & \longrightarrow & N_n & \longrightarrow & N_n/N_{n-1} \end{array}$$

where the vertical arrows are degree-wise split surjections and N_n/N_{n-1} is the direct summand of M_n/M_{n-1} formed by the positive shifts of B. Write $L = \mathsf{colim}_{n\geq 0} L_n$ and $N = \mathsf{colim}_{n\geq 0} N_n$. The short exact sequence of dg B-modules

$$0 \to L \to M \to N \to 0$$

induces a triangle

$$L \to M \to N \to \Sigma L$$

in $\mathcal{D}B$. Note that $L = \mathsf{Mcolim}_{n\geq 0} L_n$ and $N = \mathsf{Mcolim}_n N_n$. Since $\mathcal{D}^{w<0}$ is closed under small coproducts, positive shifts and extensions, then $N \in \mathcal{D}^{w<0}$. On the other hand, if $Y \in \mathcal{D}^{w<0}$ then

$$\mathsf{Hom}(L, Y) = \mathsf{lim}_{n\geq 0} \mathsf{Hom}(L_n, Y) = 0,$$

which proves that $L \in \mathcal{D}^{w\geq 0}$.

3) We can construct a commutative diagram as follows:

$$\begin{array}{ccccccc}
\sigma_{\geq 0}X & \longrightarrow & \sigma_{\geq -1}X & \longrightarrow & \sigma_{\geq -2}X & \longrightarrow & \cdots \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_{-2}} & & \\
X & = & X & = & X & = & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\sigma_{<0}X & \longrightarrow & \sigma_{<-1}X & \longrightarrow & \sigma_{>-2}X & \longrightarrow & \cdots
\end{array}$$

which induces a morphism

$$f : \mathsf{Mcolim}_{p\geq 0} \sigma_{\geq -p}X \to X.$$

For each $n \in \mathbf{Z}$ this yields a morphism

$$H^n(f) : \mathsf{colim}_{p\geq 0} H^n(\sigma_{\geq -p}X) \to H^n X$$

induced by the commutative diagram

$$\begin{array}{ccccccc}
H^n(\sigma_{\geq 0}X) & \longrightarrow & H^n(\sigma_{\geq -1}X) & \longrightarrow & H^n(\sigma_{\geq -2}X) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle H^n(f_0)} & & \downarrow{\scriptstyle H^n(f_{-1})} & & \downarrow{\scriptstyle H^n(f_{-2})} & & \\
H^n X & = & H^n X & = & H^n X & = & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H^n(\sigma_{<0}X) & \longrightarrow & H^n(\sigma_{<-1}X) & \longrightarrow & H^n(\sigma_{>-2}X) & \longrightarrow & \cdots
\end{array}$$

We deduce that $H^n(f)$ is an isomorphism from the fact that almost every map $H^n(f_{-p})$, $p \geq 0$, is an isomorphism.

4) Note that we have

$$\mathsf{Hom}(X, Y) = \mathsf{Hom}(\mathsf{Mcolim}_{q\geq 0} \sigma_{\geq -q}X, Y) = \mathsf{lim}_{q\geq 0} \mathsf{Hom}(\sigma_{\geq -q}X, Y).$$

Now for a fix $q \geq 0$, we apply $\mathsf{Hom}(\sigma_{\geq -q}X, ?)$ to the diagram

$$\begin{array}{ccccccc}
\sigma_{\geq 0}Y & \longrightarrow & \sigma_{\geq -1}Y & \longrightarrow & \sigma_{\geq -2}Y & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
Y & = & Y & = & Y & = & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\sigma_{<0}Y & \longrightarrow & \sigma_{<-1}Y & \longrightarrow & \sigma_{>-2}Y & \longrightarrow & \cdots
\end{array}$$

to get the diagram

$$\begin{array}{ccccccc}
\mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{\geq 0}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{\geq -1}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{\geq -2}Y) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathsf{Hom}(\sigma_{\geq -q}X, Y) & = & \mathsf{Hom}(\sigma_{\geq -q}X, Y) & = & \mathsf{Hom}(\sigma_{\geq -q}X, Y) & = & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{<0}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{<-1}Y) & \longrightarrow & \mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{>-2}Y) & \longrightarrow & \cdots
\end{array}$$

in which $\mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{<-p}Y) = 0$ for $p \gg 0$. Thus the induced morphism

$$\mathsf{colim}_{p\geq 0}\, \mathsf{Hom}(\sigma_{\geq -q}X, \sigma_{\geq -p}Y) \to \mathsf{Hom}(\sigma_{\geq -q}X, Y)$$

is an isomorphism. √

11. Appendix 2: Milnor colimits versus homotopy colimits

Let $\mathbb{D}$ be a triangulated derivator defined on the 2-category of small categories (see [11] and the notation therein). Let us denote by e the 1-point category. For any small category I, we will write $p : I \to e$ to refer to the unique possible functor. We have an adjoint pair of triangle functors

$$\begin{array}{c} \mathbb{D}(e) \\ {\scriptstyle p_!}\uparrow \;\downarrow{\scriptstyle p^*} \\ \mathbb{D}(I) \end{array}$$

and, by definition, if $F \in \mathbb{D}(I)$ we say that $p_!F$ is the *homotopy colimit* of F. Sometimes this will be denoted by $\mathsf{hocolim}\, F$ or $\Gamma_!(F, I)$.

In this Appendix, we will show that if a triangulated category $\mathcal{D}$ is at the base of a triangulated derivator, then Milnor colimits of sequences of morphisms of $\mathcal{D}$ are isomorphic to homotopy colimits.

The key tool will be the diagram functor (see [11, §1.10]):

$$d_I : \mathbb{D}(I) \to \underline{\mathsf{Hom}}(I^{\mathrm{op}}, \mathbb{D}(e))$$

(sometimes we shall omit the subscript I). If F is an object of $\mathbb{D}(I)$, we say that $d_I F$ is the *diagram* or *presheaf* associated to F. Given a presheaf $F \in \underline{\mathsf{Hom}}(I^{\mathrm{op}}, \mathbb{D}(e))$, we say that an object $G \in \mathbb{D}(I)$ is *lifts* F if $d_I(G)$ is isomorphic to F in $\underline{\mathsf{Hom}}(I^{\mathrm{op}}, \mathbb{D}(e))$.

For each $i \in I$, we denote by $? \otimes i : \mathbb{D}(e) \to \underline{\mathsf{Hom}}(I^{\mathrm{op}}, \mathbb{D}(e))$ the left adjoint of the functor $(?)_i$ *evaluation at* i:

$$\begin{array}{c} \underline{\mathsf{Hom}}(I^{\mathrm{op}}, \mathbb{D}(e)) \\ {\scriptstyle ?\otimes i}\uparrow \;\downarrow{\scriptstyle (?)_i} \\ \mathbb{D}(e) \end{array}$$

For $j \in I$ and X in $\mathbb{D}(e)$, we have the canonical isomorphism

$$(X \otimes i)_j = \coprod_{\mathsf{Hom}(j,i)} X.$$

Lemma 11.1. *For each i in I, the triangle*

$$\begin{array}{ccc} \mathbb{D}(I) & \xrightarrow{d_I} & \underline{\mathsf{Hom}}(I^{op}, \mathbb{D}(e)) \\ {\scriptstyle i_!}\uparrow & \nearrow_{?\otimes i} & \\ \mathbb{D}(e) & & \end{array}$$

commutes up to a canonical isomorphism.

Proof. Recall that by axiom Der4d, for each functor $u : J \to I$ and each object j of I, we have a canonical isomorphism

$$j^* u_! = p_! \, l^* \, ,$$

where the functors are those of the square

$$\begin{array}{ccc} j\backslash J & \xrightarrow{l} & J \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u} \\ e & \xrightarrow[j]{} & I \end{array}$$

and $j\backslash J$ is the comma-category of pairs $(j', u(j') \to j)$. Let us specialize J to e and u to the inclusion determined by the object i of I. Then we get a canonical isomorphism

$$j^* i_! = p_! \, p^* \, ,$$

where now $i\backslash J = i\backslash e$ is the discrete category $\mathsf{Hom}(j, i)$ and p the unique functor $\mathsf{Hom}(j, i) \to e$. By axiom Der1, the composition $p_! \, p^*$ is the coproduct composed with the diagonal functor. So for each object X of $\mathbb{D}(e)$, we get a canonical isomorphism

$$(i_! X)_j = \coprod_{\mathsf{Hom}(j,i)} X = (X \otimes i)_j .$$

One checks that these isomorphisms yield a canonical isomorphism as claimed. $\surd$

Remark 11.2. For $I^{\mathrm{op}} = \mathbf{N}$ and $n \in \mathbf{N}$, the object $X \otimes n$ is the presheaf

$$0 \to \ldots \to 0 \to X \xrightarrow{1} X \xrightarrow{1} X \to \ldots ,$$

where the first X appears in position n and by the lemma, the triangle

$$\begin{array}{ccc} \mathbb{D}(\mathbf{N}^{\mathrm{op}}) & \xrightarrow{d_{\mathbf{N}^{\mathrm{op}}}} & \underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e)) \\ {\scriptstyle n_!}\uparrow & \nearrow_{?\otimes n} & \\ \mathbb{D}(e) & & \end{array}$$

commutes up to isomorphism.

Proposition 11.3. 1) *Given an object X of $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$, there exists an object of $\mathbb{D}(\mathbf{N}^{op})$ which lifts X.*

2) *Given a morphism $f : X \to X'$ in $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ there exists a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{X}'$ in $\mathbb{D}(\mathbf{N}^{op})$ such that $d_{\mathbf{N}^{op}}(\widetilde{f})$ is isomorphic to f.*

3) *The homotopy colimit of an object X of $\mathbb{D}(\mathbf{N}^{op})$ is isomorphic to the Milnor colimit of its associated diagram $d_{\mathbf{N}^{op}} X$.*

Proof. 1) *Step 1: an exact category with global dimension* 1. Every additive category can be endowed with an exact structure by taking as conflations the split exact pairs (see [19] and the terminology therein). Let us consider $\mathbb{D}(e)$ as an exact category in this way, and let us regard $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ as an exact category with the pointwise split exact structure. Let us calculate a projective resolution of an arbitrary object of this category. Given an object X of $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$, *i.e.* a sequence of morphisms in $\mathbb{D}(e)$

$$X_0 \xrightarrow{x_0} X_1 \xrightarrow{x_1} X_2 \xrightarrow{x_2} \ldots ,$$

we can start the projective resolution by considering the deflation

$$P_0 = \coprod_{n \in N} X_n \otimes n \to X$$

defined by using the counit of the adjunctions $(? \otimes n, (?)_n)$. It turns out that $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ has global dimension 1. Indeed, in the kernel $P_1 \xrightarrow{u} P_0$ of the former deflation we can take

$$P_1 = \coprod_{n \in \mathbf{N}} X_n \otimes (n+1),$$

which is a projective object. An explicit diagram might help

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_0 & \longrightarrow & X_0 \oplus X_1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \begin{bmatrix} \mathbf{1} \\ -x_0 \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} \mathbf{1} & 0 \\ -x_0 & \mathbf{1} \\ 0 & -x_1 \end{bmatrix}} & & \\
X_0 & \longrightarrow & X_0 \oplus X_1 & \longrightarrow & X_0 \oplus X_1 \oplus X_2 & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \begin{bmatrix} x_0 & \mathbf{1} \end{bmatrix}} & & \downarrow{\scriptstyle \begin{bmatrix} x_1 x_0 & x_1 & \mathbf{1} \end{bmatrix}} & & \\
X_0 & \xrightarrow{x_0} & X_1 & \xrightarrow{x_1} & X_2 & \longrightarrow & \cdots
\end{array}$$

Step 2: lifting a projective resolution along the diagram functor. Put

$$\widetilde{P}_1 = \coprod_{n \in \mathbf{N}} (n+1)_!(X_n)$$

and

$$\widetilde{P}_0 = \coprod_{n \in \mathbf{N}} n_!(X_n).$$

For each $n \in \mathbf{N}$, let $a_n \in \mathsf{Hom}_{\mathbb{D}(\mathbf{N}^{\mathrm{op}})}((n+1)_! X, n_! X)$ be the image of the identity $\mathbf{1}_{n_!(X_n)}$ by the composition of the morphisms

$$\begin{aligned}
\mathsf{Hom}_{\mathbb{D}(\mathbf{N}^{\mathrm{op}})}(n_!(X_n), n_!(X_n)) &\xrightarrow{\sim} \mathsf{Hom}_{\mathbb{D}(e)}(X_n, n^* n_!(X_n)) \\
&\to \mathsf{Hom}_{\mathbb{D}(e)}(X_n, (n+1)^* n_!(X_n)) \\
&\xrightarrow{\sim} \mathsf{Hom}_{\mathbb{D}(\mathbf{N}^{\mathrm{op}})}((n+1)_!(X_n), n_!(X_n))
\end{aligned}$$

induced by the adjoint pairs $(n_!, n^*)$ and $((n+1)_!, (n+1)^*)$ and the 2-arrow

$$\mathbb{D}(e) \underset{n^*}{\overset{(n+1)^*}{\rightleftarrows}} \mathbb{D}(\mathbf{N}^{\mathrm{op}}) \quad \Uparrow (\alpha_n^{n+1})^*$$

coming from the only possible 2-arrow

$$e \underset{n}{\overset{n+1}{\rightrightarrows}} \mathbf{N}^{\mathrm{op}}. \quad \Downarrow \alpha_n^{n+1}$$

Consider now the morphism

$$\widetilde{P}_1 \xrightarrow{\widetilde{u}} \widetilde{P}_0$$

in $\mathbb{D}(\mathbf{N}^{\mathrm{op}})$ determined by

$$\begin{array}{ccc}
\widetilde{P}_1 & \xrightarrow{\widetilde{u}} & \widetilde{P}_0 \\
\uparrow & & \uparrow \\
(n+1)_!(X_n) & \xrightarrow{\begin{bmatrix} a_n & -(n+1)_!(x_n) \end{bmatrix}^t} & n_!(X_n) \oplus (n+1)_!(X_{n+1})
\end{array}$$

Remark 11.2 tells us that the diagram functor $d_{\mathbf{N}^{\mathrm{op}}}$ sends $\widetilde{u}$ to $u : P_1 \to P_0$.

Step 3: a triangle over the lifted morphism. Now consider a triangle

$$\widetilde{P}_1 \xrightarrow{\widetilde{u}} \widetilde{P}_0 \to \widetilde{X} \to \Sigma \widetilde{P}_1$$

in $\mathbb{D}(\mathbf{N}^{\mathrm{op}})$. For each $m \in \mathbf{N}$, after applying the triangle functor $m^* : \mathbb{D}(\mathbf{N}^{\mathrm{op}}) \to \mathbb{D}(e)$ we get a triangle

$$\textstyle\bigoplus_{n=0}^{m-1} X_n \xrightarrow{u_m} \bigoplus_{n=0}^{m} X_n \longrightarrow d_{\mathbf{N}^{\mathrm{op}}}(\widetilde{X})_m \longrightarrow \Sigma \bigoplus_{n=0}^{m-1} X_n$$

in $\mathbb{D}(e)$. Since u_m is a section, $d_{\mathbf{N}^{\mathrm{op}}}(\widetilde{X})_m$ is the cokernel of u_m and so $d_{\mathbf{N}^{\mathrm{op}}}(\widetilde{X})_m \cong X_m$.

2) Given a morphism $f : X \to X'$ in $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$, we can consider as before the projective resolutions

$$P_1 \overset{u}{\to} P_0 \to X$$

and

$$P_1' \overset{u'}{\to} P_0' \to X'.$$

By using $f : X \to X'$ we can define a morphism $g : P_0 \to P_0'$ making commutative the square

$$\begin{array}{ccc} P_0 & \longrightarrow & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ P_0' & \longrightarrow & X \end{array}$$

and the universal property of the cokernel guarantees the existence of a morphism of conflations

$$\begin{array}{ccccc} P_1 & \xrightarrow{u} & P_0 & \longrightarrow & X \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ P_1' & \xrightarrow{u'} & P_0' & \longrightarrow & X \end{array}$$

Thanks to Remark 11.2, we can prove that there exists a commutative square

$$\begin{array}{ccc} \widetilde{P}_1 & \xrightarrow{\widetilde{u}} & \widetilde{P}_0 \\ \downarrow{\scriptstyle \widetilde{h}} & & \downarrow{\scriptstyle \widetilde{g}} \\ \widetilde{P}_1' & \xrightarrow{\widetilde{u}'} & \widetilde{P}_0' \end{array}$$

in $\mathbb{D}(\mathbf{N}^{\mathrm{op}})$ which is mapped to

$$\begin{array}{ccc} P_1 & \xrightarrow{u} & P_0 \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ P_1' & \xrightarrow{u'} & P_0' \end{array}$$

by $d_{\mathbf{N}^{\mathrm{op}}}$. The commutative square in $\mathbb{D}(\mathbf{N}^{\mathrm{op}})$ can be completed to a morphism of triangles

$$\begin{array}{ccccccc} \widetilde{P}_1 & \xrightarrow{\widetilde{u}} & \widetilde{P}_0 & \longrightarrow & \widetilde{X} & \longrightarrow & \Sigma\widetilde{P}_1 \\ \downarrow{\scriptstyle \widetilde{h}} & & \downarrow{\scriptstyle \widetilde{g}} & & \downarrow{\scriptstyle \widetilde{f}} & & \downarrow \\ \widetilde{P}_1' & \xrightarrow{\widetilde{u}'} & \widetilde{P}_0' & \longrightarrow & \widetilde{X}' & \longrightarrow & \Sigma\widetilde{P}_1' \end{array}$$

For each $m \in \mathbf{N}$, we apply the triangle functor m^* and obtain a morphism of triangles

$$\begin{array}{ccccccc} \bigoplus_{n=0}^{m-1} X_n & \xrightarrow{u_m} & \bigoplus_{n=0}^{m} X_n & \longrightarrow & (d_{\mathbf{N}^{\mathrm{op}}}\widetilde{X})_m & \longrightarrow & \Sigma \bigoplus_{n=0}^{m-1} X_n \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{n=0}^{m-1} X_n' & \xrightarrow{u_m'} & \bigoplus_{n=0}^{m} X_n' & \longrightarrow & (d_{\mathbf{N}^{\mathrm{op}}}\widetilde{X}')_m & \longrightarrow & \Sigma \bigoplus_{n=0}^{m-1} X_n' \end{array}$$

Since both u_m and u_m' are sections, $(d_{\mathbf{N}^{\mathrm{op}}}\widetilde{X})_m$ is the cokernel of u_m and $(d_{\mathbf{N}^{\mathrm{op}}}\widetilde{X}')_m$ is the cokernel of u_m'. Thus, $d_{\mathbf{N}^{\mathrm{op}}}(\widetilde{f})_m$ is isomorphic to f_m.

3) Given an object $X \in \mathbb{D}(\mathbf{N}^{\mathrm{op}})$ we consider a triangle

$$Y \to \coprod_{n\in\mathbf{N}} n_! n^* X \xrightarrow{\varepsilon} X \to \Sigma Y$$

where ε is defined by using the counit of the adjunctions $(n_!, n^*)$. For each $n \in \mathbf{N}$, let $a_n \in \mathsf{Hom}_{\mathbb{D}(\mathbf{N}^{\mathrm{op}})}((n+1)_! n^* X, n_! n^* X)$ be the image of the identity $\mathbf{1}_{n_! n^* X}$ by the composition of the morphisms

$$\begin{aligned}\mathsf{Hom}_{\mathbb{D}(\mathbf{N}^{\mathrm{op}})}(n_! n^* X, n_! n^* X) &\xrightarrow{\sim} \mathsf{Hom}_{\mathbb{D}(e)}(n^* X, n^* n_! n^* X)\\ &\to \mathsf{Hom}_{\mathbb{D}(e)}(n^* X, (n+1)^* n_! n^* X)\\ &\xrightarrow{\sim} \mathsf{Hom}_{\mathbb{D}(\mathbf{N}^{\mathrm{op}})}((n+1)_! n^* X, n_! n^* X)\end{aligned}$$

induced by the adjoint pairs $(n_!, n^*)$ and $((n+1)_!, (n+1)^*)$ and the 2-arrow

$$\mathbb{D}(e) \;\underset{n^*}{\overset{(n+1)^*}{\Uparrow (\alpha_n^{n+1})^*}}\; \mathbb{D}(\mathbf{N}^{\mathrm{op}})$$

coming from the only possible 2-arrow

$$e \;\underset{n}{\overset{n+1}{\Downarrow \alpha_n^{n+1}}}\; \mathbf{N}^{\mathrm{op}}.$$

Consider the morphism

$$\coprod_{n\in\mathbf{N}} (n+1)_! n^* X \xrightarrow{u} \coprod_{n\in\mathbf{N}} n_! n^* X$$

described by

$$\begin{array}{ccc} \coprod_{n\in\mathbf{N}} (n+1)_! n^* X & \xrightarrow{\quad u \quad} & \coprod_{n\in\mathbf{N}} n_! n^* X \\ \uparrow & & \uparrow \\ (n+1)_! n^* X & \xrightarrow{\begin{bmatrix} a_n & -x_n \end{bmatrix}^t} & n_! n^* X \oplus (n+1)_! (n+1)^* X \end{array}$$

Since the composition εu vanishes, there exists a morphism φ making commutative the diagram

$$\begin{array}{ccccccc} & & \coprod_{n\in\mathbf{N}} (n+1)_! n^* X & & & & \\ & \overset{\varphi}{\swarrow} & \downarrow u & \overset{0}{\searrow} & & & \\ Y & \longrightarrow & \coprod_{n\in\mathbf{N}} n_! n^* X & \xrightarrow{\varepsilon} & X & \longrightarrow & \Sigma Y \end{array}$$

For each $m \in \mathbf{N}$, after applying the triangle functor m^* we get a triangle

$$m^* Y \longrightarrow \coprod_{n\in\mathbf{N}} m^* n_! n^* X \xrightarrow{m^*\varepsilon} m^* X \longrightarrow \Sigma m^* Y$$

By using Remark 11.2 we know that

$$\coprod_{n\in\mathbf{N}} m^* n_! n^* X = \bigoplus_{n=0}^{m} n^* X,$$

and it is easy to check that the nth composite of the morphism $m^*\varepsilon$ is the morphism

$$(\alpha_n^m)^* : n^* X \to m^* X$$

given by the unique 2-arrow

$$e \;\underset{n}{\overset{m}{\Downarrow \alpha_n^m}}\; \mathbf{N}^{\mathrm{op}}.$$

Thus, $m^*\varepsilon$ is a section, with retraction given by

$$[\ 0\ \ \ldots\ \ 0\ \ \mathbf{1}\ ]^t : m^*X \to \bigoplus_{n=0}^{m} n^*X.$$

¿From this, we deduce that the morphism

$$m^*Y \to \bigoplus_{n=0}^{m} n^*X$$

is the kernel of $m^*\varepsilon$. On the other hand, it is easy to check that the kernel of $m^*\varepsilon$ is m^*u. Therefore, $m^*\varphi$ is an isomorphism for each $m \in \mathbf{N}$, and the conservative axiom of derivators (see [11, Definition 1.11]) says that φ is an isomorphism. Finally, if we apply the triangle functor $\mathsf{hocolim}$ to the triangle

$$\coprod_{n\in\mathbf{N}}(n+1)_!n^*X \xrightarrow{u} \coprod_{n\in\mathbf{N}} n_!n^*X \xrightarrow{\varepsilon} X \longrightarrow \Sigma\coprod_{n\in\mathbf{N}}(n+1)_!n^*X$$

we get the triangle

$$\coprod_{n\in\mathbf{N}} n^*X \xrightarrow{\mathbf{1}-\sigma} \coprod_{n\in\mathbf{N}} n^*X \longrightarrow \mathsf{hocolim}\, X \longrightarrow \Sigma\coprod_{n\in\mathbf{N}} n^*X.$$

The nth composite of σ is the composition

$$n^*X \xrightarrow{(\alpha_n^{n+1})^*} (n+1)^*X \to \coprod_{n\in\mathbf{N}} n^*X,$$

where α_n^{n+1} is the only possible 2-arrow

$$e \;\overset{n+1}{\underset{n}{\Downarrow \alpha_n^{n+1}}}\; \mathbf{N}^{\mathrm{op}}.$$

Therefore,

$$\mathsf{hocolim}\, X \cong \mathsf{Mcolim}\, d_{\mathbf{N}^{\mathrm{op}}}X.$$

$\surd$

If X is an object of $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ given by

$$X_0 \overset{x_0}{\to} X_1 \overset{x_1}{\to} X_2 \to \ldots,$$

we denote by ΣX the object $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ given by

$$\Sigma X_0 \overset{\Sigma x_0}{\to} \Sigma X_1 \overset{\Sigma x_1}{\to} \Sigma X_2 \to \ldots$$

If $\mathcal{D}$ is a triangulated category and $f : X \to Y$ is a morphism in the category $\underline{\mathsf{Hom}}(\mathbf{N}, \mathcal{D})$:

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots, \end{array}$$

we write $\mathsf{Mcolim}\, f$ to refere to a morphism which fits in a morphism of triangles

$$\begin{array}{ccccccc} \coprod_{n\in\mathbf{N}} X_n & \xrightarrow{\mathbf{1}-\sigma} & \coprod_{n\in\mathbf{N}} X_n & \longrightarrow & \mathsf{Mcolim}\, X_n & \longrightarrow & \Sigma\coprod_{n\in\mathbf{N}} X_n \\ \downarrow{\scriptstyle \coprod_{n\in\mathbf{N}} f_n} & & \downarrow{\scriptstyle \coprod_{n\in\mathbf{N}} f_n} & & \downarrow{\scriptstyle \mathsf{Mcolim}\, f} & & \downarrow \\ \coprod_{n\in\mathbf{N}} Y_n & \xrightarrow{\mathbf{1}-\sigma} & \coprod_{n\in\mathbf{N}} Y_n & \longrightarrow & \mathsf{Mcolim}\, Y_n & \longrightarrow & \Sigma\coprod_{n\in\mathbf{N}} Y_n \end{array}$$

Corollary 11.4. *Let*

$$X \xrightarrow{f} Y \to Z \to \Sigma X$$

be a diagram in $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ *such that for each* $n \in \mathbf{N}$ *the corresponding diagram*

$$X_n \xrightarrow{f_n} Y_n \to Z_n \to \Sigma X_n$$

is a triangle in $\mathbb{D}(e)$*. There exists a triangle*

$$\mathsf{Mcolim}\, X \xrightarrow{\mathsf{Mcolim}\, f} \mathsf{Mcolim}\, Y \longrightarrow \mathsf{Mcolim}\, Z' \longrightarrow \Sigma\, \mathsf{Mcolim}\, X$$

in $\mathbb{D}(e)$*, where* Z' *is an object of* $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ *such that* $Z'_n \cong Z_n$ *for each* $n \in \mathbf{N}$*.*

Proof. Part 2) of Proposition 11.3 tells us that there exists a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in $\mathbb{D}(\mathbf{N}^{\mathrm{op}})$ such that $d_{\mathbf{N}^{\mathrm{op}}}(\widetilde{f}) = f$. Let us complete this morphism to a triangle

$$\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y} \to \widetilde{Z} \to \Sigma\widetilde{X}$$

in $\mathbb{D}(\mathbf{N}^{\mathrm{op}})$. For a natural number $n \in \mathbf{N}$ the triangle functor n^* sends this triangle to a triangle

$$X_n \xrightarrow{f_n} Y_n \to n^*\widetilde{Z} \to \Sigma X_n,$$

which proves that $n^*\widetilde{Z} \cong Z_n$. On the other hand, by using part 1) of Proposition 11.3 we get that the triangle functor $\mathsf{hocolim}$ sends the triangle in $\mathbb{D}(\mathbf{N}^{\mathrm{op}})$ to a triangle

$$\mathsf{Mcolim}\, X \xrightarrow{\mathsf{Mcolim}\, f} \mathsf{Mcolim}\, Y \longrightarrow \mathsf{hocolim}\, \widetilde{Z} \longrightarrow \Sigma\, \mathsf{Mcolim}\, X.$$

Finally, part 3) of Proposition 11.3 tells us that

$$\mathsf{hocolim}\, \widetilde{Z} \cong \mathsf{Mcolim}\, d_{\mathbf{N}^{\mathrm{op}}}(\widetilde{Z}).$$

√

12. Appendix 3: From compact objects to t-structures

It is well known that from a set $\mathcal{S}$ of compact objects of a triangulated category $\mathcal{D}$ with small coproducts one can produce in a natural way an interesting t-structure $t_{\mathcal{S}}$. For example, in [4, Theorem III.2.3], it is proved that if $\mathcal{Y}_{\mathcal{S}}$ is the full subcategory of $\mathcal{D}$ formed by those objects Y such that $\mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, Y) = 0$ for each $n \geq 0$ and each $S \in \mathcal{S}$, then $\mathcal{Y}_{\mathcal{S}}$ is the right aisle of a t-structure. In fact, this can be deduced from [1, Theorem A.1]. For the convenience of the reader we will include here the statement and the proof of that theorem:

Theorem 12.1. *Let* $\mathcal{D}$ *be a triangulated category with small coproducts, and let* $\mathcal{S}$ *be a set of compact objects of* $\mathcal{D}$*. Then:*

1) *the smallest full subcategory* $\mathsf{Susp}_{\mathcal{D}}(\mathcal{S})$ *of* $\mathcal{D}$ *containing* $\mathcal{S}$ *and closed under extensions, positive shifts and small coproducts is a left aisle,*
2) *every object* X *of* $\mathsf{Susp}_{\mathcal{D}}(\mathcal{S})$ *fits in a triangle*

$$\coprod_{i\geq 0} X_i \to X \to \coprod_{i\geq 0} \Sigma X_i \to \coprod_{i\geq 0} \Sigma X_i$$

where X_i *is an* i*-fold extension of small coproducts of non negative shifts of objects of* $\mathcal{S}$*.*

Proof. Let M be an object of $\mathcal{D}$, and let us consider an approximation

$$P_0 \to M$$

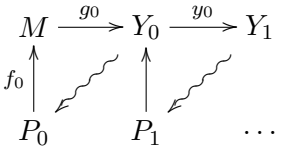
of M with respect to the full subcategory of $\mathcal{D}$ formed by the small coproducts of non negative shifts of objects of $\mathcal{S}$. Let us consider a triangle

$$P_0 \xrightarrow{f_0} M \xrightarrow{g_0} Y_0 \to \Sigma P_0$$

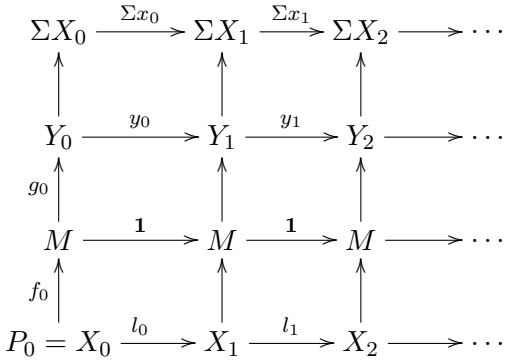
and a new approximation

$$P_1 \to Y_0$$

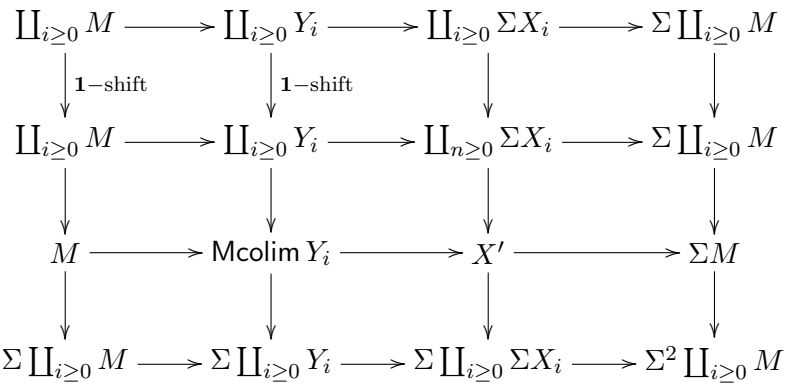
with respect to the same subcategory. By iterating this procedure we get a diagram of the form



This diagram yields a diagram



in which every column is a triangle. The octahedron axiom implies that each X_i is an i-fold extension of small coproducts of non negative shifts of objects of $\mathcal{S}$. Now, by using [3, Proposition 1.1.11] (*i.e.* , Verdier's 3×3 lemma) we get a diagram



where the columns and rows are triangles. It is clear that $\Sigma^{-1}X' \in \mathsf{Susp}_{\mathcal{D}}(\mathcal{S})$. On the other hand, for each $S \in \mathcal{S}$ and each $n \geq 0$ we have

$$\mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, \mathsf{Mcolim}\, Y_i) \cong \mathsf{colim}_{i\in\mathbf{N}}\, \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, Y_i) = 0$$

because the induced morphisms

$$\mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, Y_i) \to \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, Y_{i+1})$$

vanish. $\surd$

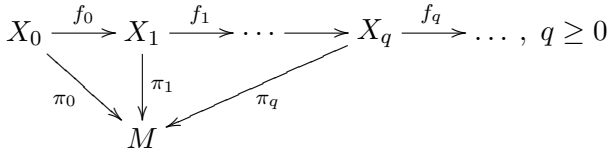
Of course, one would like to express the objects of the left aisle of $t_{\mathcal{S}}$ in terms of the objects of $\mathcal{S}$, for instance as a kind of colimit. In [4, Proposition III.2.6] it is proved that this is the case when $\mathcal{S}$ satisfies a certain vanishing condition. Here we give an alternative proof of this result:

Theorem 12.2. *Let $\mathcal{D}$ be a triangulated category with small coproducts, and let $\mathcal{S}$ be a set of compact objects in $\mathcal{D}$ such that*

$$\mathsf{Hom}_{\mathcal{D}}(S, \Sigma^n S') = 0$$

for all $S, S' \in \mathcal{S}$ and each $n \geq 1$. Then every object of $\mathsf{Susp}_{\mathcal{D}}(\mathcal{S})$ is the Milnor colimit of a sequence $X_0 \to X_1 \to X_2 \to \dots$ where X_i is an i-fold extension of small coproducts of non negative shifts of objects of $\mathcal{S}$.

Proof. Given $M \in \mathcal{D}$ we will inductively construct a commutative diagram



such that:

a) X_i is an i-fold extension of small coproducts of non negative shifts of objects of $\mathcal{S}$,

b) π_i induces a surjection

$$\pi_i^{\wedge} : \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, X_i) \to \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, M)$$

for each $S \in \mathcal{S}$, $n \geq 0$.

For $i = 0$ we take $X_0 = \coprod_{S \in \mathcal{S}} \coprod_{n \geq 0} \coprod_{\mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, M)} \Sigma^n S$ and the obvious morphism

$$\pi_0 : X_0 \to M.$$

Suppose for some $i \geq 0$ we have constructed X_i and π_i. Consider the triangle

$$C_i \stackrel{\alpha_i}{\to} X_i \stackrel{\pi_i}{\to} M \to \Sigma C_i$$

induced by π_i. Consider $Z_i = \coprod_{S \in \mathcal{S}} \coprod_{n \geq 0} \coprod_{\mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, C_i)} \Sigma^n S$ and the obvious morphism

$$\beta_i : Z_i \to C_i.$$

The triangle

$$Z_i \stackrel{\alpha_i \beta_i}{\to} X_i \to X_{i+1} \to \Sigma Z_i$$

defines X_{i+1} up to non unique isomorphism. Note that the surjectivity required for $\pi_{i+1}^{\wedge}$ comes from the surjectivity of $\pi_i^{\wedge}$.

Define X_∞ to be the Milnor colimit of the sequence f_i , $i \geq 0$:

$$\coprod_{i \geq 0} X_i \xrightarrow{\mathbf{1} - \sigma} \coprod_{i \geq 0} X_i \xrightarrow{\psi} X_\infty \longrightarrow \Sigma \coprod_{i \geq 0} X_i.$$

Consider the morphism

$$\theta = \left[\begin{array}{ccc} \pi_0 & \pi_1 & \ldots \end{array} \right] : \coprod_{i \geq 0} X_i \to M.$$

Since $\pi_{i+1} f_i = \pi_i$ for every $i \geq 0$, we have $\theta(\mathbf{1} - \sigma) = 0$, and so we obtain a morphism $\pi_\infty : X_\infty \to M$ such that $\pi_\infty \psi = \theta$. If we prove that π_∞ induces an isomorphism

$$\pi_\infty^{\wedge} : \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, X_\infty) \stackrel{\sim}{\to} \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, M)$$

for every $S \in \mathcal{S}$, $n \geq 0$, then we have

$$\mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, \mathsf{Cone}(\pi_\infty)) = 0$$

for every $S \in \mathcal{S}$, $n \geq 1$. For the case $n = 0$, let us consider the exact sequence

$$\mathsf{Hom}_{\mathcal{D}}(S, X_\infty) \stackrel{\sim}{\to} \mathsf{Hom}_{\mathcal{D}}(S, M) \to \mathsf{Hom}_{\mathcal{D}}(S, \mathsf{Cone}(\pi_\infty)) \to \mathsf{Hom}_{\mathcal{D}}(S, \Sigma X_\infty)$$

Since S is compact, there exists a short exact sequence

$$\coprod_{i \geq 0} \mathsf{Hom}_{\mathcal{D}}(S, \Sigma X_i) \to \mathsf{Hom}_{\mathcal{D}}(S, \Sigma X_\infty) \to \coprod_{i \geq 0} \mathsf{Hom}_{\mathcal{D}}(S, \Sigma^2 X_i)$$

¿From the hypothesis on the set $\mathcal{S}$ and the construction of the objects X_i we can deduce that both the left and the right hand side of the former sequence vanish, and so $\mathsf{Hom}_{\mathcal{D}}(S, \Sigma X_\infty) = 0$. Therefore, then we would have

$$\mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, \mathsf{Cone}(\pi_\infty)) = 0$$

for every $S \in \mathcal{S}$, $n \geq 0$. This, by infinite dévissage, implies that

$$\mathsf{Hom}_{\mathcal{D}}(N, \mathsf{Cone}(\pi_\infty)) = 0$$

for each $N \in \mathsf{Susp}(\mathcal{S})$. Hence, we have proved that $\mathsf{Susp}_{\mathcal{D}}(\mathcal{S})$ is an aisle in $\mathcal{D}$.

Let us prove the required bijectivity for $\pi_\infty^\wedge$. The surjectivity follows from the identity $\pi_\infty^\wedge \psi^\wedge = \theta^\wedge$ and the fact that $\theta^\wedge$ is surjective (thanks to the surjectivity of the $\pi_i^\wedge$, $i \geq 0$ and the compactness of the $S \in \mathcal{S}$). Now consider the commutative diagram

$$\begin{array}{ccccccc} \coprod_{i\geq 0} \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, X_i) & \xrightarrow{(\mathbf{1}-\sigma)^\wedge} & \coprod_{i\geq 0} \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, X_i) & \xrightarrow{\psi^\wedge} & \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, X_\infty) & \longrightarrow & 0 \\ & & & \searrow^{\theta^\wedge} & \downarrow{\scriptstyle \pi_\infty^\wedge} & & \\ & & & & \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, M) & & \end{array}$$

The map $\psi^\wedge$ is surjective since the map

$$(\Sigma(\mathbf{1}-\sigma))^\wedge : \coprod_{i\geq 0} \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, \Sigma X_i) \to \coprod_{i\geq 0} \mathsf{Hom}_{\mathcal{D}}(\Sigma^n S, \Sigma X_i)$$

is injective. If we prove that the kernel of $\theta^\wedge$ is contained in the image of $(\mathbf{1}-\sigma)^\wedge$, then we obtain the injectivity of $\pi_\infty^\wedge$ by an easy diagram chase. Let

$$g = \left[\begin{array}{ccc} g_0 & g_1 & \dots \end{array} \right]^t : \Sigma^n S \to \coprod_{i\geq 0} X_i$$

be such that

$$\left[\begin{array}{ccc} \pi_0 & \pi_1 & \dots \end{array} \right] \left[\begin{array}{ccc} g_0 & g_1 & \dots \end{array} \right]^t = \pi_0 g_0 + \pi_1 g_1 + \dots = 0.$$

Notice that there exists an $s \geq 0$ such that $g_{s+1} = g_{s+2} = \dots = 0$. Then

$$\pi_0 g_0 + \dots + \pi_s g_s = 0$$

implies

$$\pi_s(f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s) = 0$$

and so the morphism

$$f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s$$

factors through α_s:

$$f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s = \alpha_s \gamma_s : \Sigma^n S \to C_s \to X_s.$$

By construction of Z_s we have that γ_s factors through β_s, and so

$$f_{s-1} \dots f_0 g_0 + f_{s-1} \dots f_1 g_1 + \dots + g_s = \alpha_s \beta_s \xi_s.$$

This implies

$$f_s \dots f_0 g_0 + f_s \dots f_1 g_1 + \dots + f_s g_s = f_s \alpha_s \beta_s \xi_s = 0,$$

since $f_s \alpha_s \beta_s = 0$ by construction of f_s. Therefore, the morphism

$$h : \Sigma^n S \to \coprod_{i\geq 0} X_i$$

with non-vanishing components

$$\Sigma^n S \to X_r \to \coprod_{i\geq 0} X_i$$

induced by

$$g_r + \dots + f_{r-1} \dots f_1 g_1 + f_{r-1} \dots f_0 g_0 : \Sigma^n S \to X_r$$

with $0 \leq r \leq s$, satisfies $\varphi^\wedge(h) = g$. $\surd$

In practice, every triangulated category is at the basis of a triangulated derivator (see [10]). If we assume that our triangulated category $\mathcal{D}$ satisfies this property, we can use Appendix 1 to get rid of the extra hypothesis on the set $\mathcal{S}$ of compact objects, to simplify the proof of Theorem 12.2 and to enhance the proof of Theorem 12.1.

Theorem 12.3. *Let $\mathbb{D}$ be a triangulated derivator, and let $\mathcal{S}$ be a set of compact objects of $\mathbb{D}(e)$. Then:*

1) *the smallest full subcategory $\mathsf{Susp}_{\mathbb{D}(e)}(\mathcal{S})$ of $\mathbb{D}(e)$ containing $\mathcal{S}$ and closed under extensions, positive shifts and small coproducts is a left aisle,*

2) *every object of* $\mathsf{Susp}_{\mathbb{D}(e)}(\mathcal{S})$ *is the Milnor colimit of a sequence* $X_0 \to X_1 \to X_2 \to \dots$ *where* X_i *is an* i*-fold extension of small coproducts of non negative shifts of objects of* $\mathcal{S}$*.*

Proof. The proof starts as the one of Theorem12.1. Thus, starting from an object M of $\mathbb{D}(e)$ we produce a diagram of the form

$$\begin{array}{ccccccccc}
\Sigma X_0 & \xrightarrow{\Sigma x_0} & \Sigma X_1 & \xrightarrow{\Sigma x_1} & \Sigma X_2 & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
Y_0 & \xrightarrow{y_0} & Y_1 & \xrightarrow{y_1} & Y_2 & \longrightarrow & \cdots \\
\uparrow{\scriptstyle g_0} & & \uparrow & & \uparrow & & \\
M & \xrightarrow{\mathbf{1}} & M & \xrightarrow{\mathbf{1}} & M & \longrightarrow & \cdots \\
\uparrow{\scriptstyle f_0} & & \uparrow & & \uparrow & & \\
P_0 = X_0 & \xrightarrow{l_0} & X_1 & \xrightarrow{l_1} & X_2 & \longrightarrow & \cdots
\end{array}$$

in which every column is a triangle, each X_i is an i-fold extension of small coproducts of non negative shifts of objects of $\mathcal{S}$ and

$$\mathsf{Hom}_{\mathbb{D}(e)}(\Sigma^n S, \mathsf{Mcolim}\, Y_i) = 0$$

for each $S \in \mathcal{S}$ and each $n \geq 0$. Let us regard the rows of this diagram as objects X , M and Y of the category $\underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ of presheaves. Thanks to Corollary 11.4 we know that there exists a triangle

$$\mathsf{Mcolim}\, X' \to M \to \mathsf{Mcolim}\, Y \to \Sigma\, \mathsf{Mcolim}\, X',$$

where $X' \in \underline{\mathsf{Hom}}(\mathbf{N}, \mathbb{D}(e))$ is such that $X'_i \cong X_i$ for each $i \geq 0$. In particular, $X'_i \in \mathsf{Susp}_{\mathbb{D}(e)}(\mathcal{S})$ for all $i \geq 0$, which implies that $\mathsf{Mcolim}\, X' \in \mathsf{Susp}_{\mathbb{D}(e)}(\mathcal{S})$. $\surd$

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