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MOTIVES OF AZUMAYA ALGEBRAS

BRUNO KAHN[1] AND MARC LEVINE[2]

[1] *Institut de Mathématiques de Jussieu, 175–179 rue du Chevaleret, 75013 Paris, France* (kahn@math.jussieu.fr)
[2] *Department of Mathematics, Northeastern University, Boston, MA 02115, USA* (marc@neu.edu)

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Abstract We study the slice filtration for the K-theory of a sheaf of Azumaya algebras A, and for the motive of a Severi–Brauer variety, the latter in the case of a central simple algebra of prime degree over a field. Using the Beilinson–Lichtenbaum conjecture, we apply our results to show the vanishing of $SK_2(A)$ for a central simple algebra A of square-free index (prime to the characteristic). This proves a conjecture of Merkurjev.

Keywords: Bloch–Lichtenbaum spectral sequence; algebraic cycles; Morel–Voevodsky stable homotopy category; slice filtration; Azumaya algebras; Severi–Brauer schemes

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Contents

Introduction

Voevodsky [**59**] has defined an analog of the classical Postnikov tower in the setting of motivic stable homotopy theory by replacing the classical suspension $\Sigma := - \wedge S^1$ with t-suspension $\Sigma_T := - \wedge \mathbb{P}^1$; we call this construction the *motivic Postnikov tower*. In this paper, we use this idea to associate invariants to a central simple algebra A over a field k, and to study them.

For this, we consider the motivic Postnikov tower in the category of S^1-spectra, $\mathcal{SH}_{S^1}(k)$, and its analog in the category of effective motives, $DM^{\mathrm{eff}}(k)$. In the setting of S^1-spectra, we look at the presheaf of the K-theory spectra K^A:

$$Y \mapsto K^A(Y) := K(Y; A),$$

where $K(Y; A)$ is the K-theory spectrum of the category of $\mathcal{O}_Y \otimes_k A$-modules which are locally free as $\mathcal{O}_Y$-modules. In the motivic setting, we study the motive $M(X) \in DM^{\mathrm{eff}}(k)$, where X is the Severi–Brauer variety of A.

Of course, K^A is a twisted form of the presheaf K of K-theory spectra $Y \mapsto K(Y)$ and X is a twisted form of a projective space over k, so one would expect the layers in the respective Postnikov towers of K^A and $M(X)$ to be twisted forms of the layers for K and $M(\mathbb{P}^n)$. The second author has shown in [**33**] that the nth layer for K is the Eilenberg–Mac Lane spectrum for the Tate motive $\mathbb{Z}(n)[2n]$; similarly, the direct sum decomposition

$$M(\mathbb{P}^N) = \bigoplus_{n=0}^{N} \mathbb{Z}(n)[2n]$$

shows that nth layer for $M(\mathbb{P}^N)$ is $\mathbb{Z}(n)[2n]$ for $0 \leqslant n \leqslant N$, and is 0 for n outside this range. The twisted version of $\mathbb{Z}(n)$ turns out to be $\mathbb{Z}_A(n)$, where $\mathbb{Z}_A \in DM^{\mathrm{eff}}(k)$ is the subsheaf of the constant sheaf with transfers $\mathbb{Z}$ whose sections $\mathbb{Z}_A(Y)$ on a smooth irreducible k-scheme Y are the subgroup of $\mathbb{Z}(Y) = \mathbb{Z}$ equal to the image of the reduced norm map

$$\mathrm{Nrd} : K_0(A \otimes_k k(Y)) \to K_0(k(Y)) = \mathbb{Z}.$$

We like to call $\mathbb{Z}_A$ the *motive of A*.

Letting s_n and s_n^{mot} denote the nth layer of the motivic Postnikov tower in $\mathcal{SH}_{S^1}(k)$ and $DM^{\mathrm{eff}}(k)$, respectively, and letting $\mathrm{EM}_{\mathbb{A}^1} : DM^{\mathrm{eff}}(k) \to \mathcal{SH}_{S^1}(k)$ denote the Eilenberg–Mac Lane functor [**44**], our main results are the following theorems.

Theorem 1. *Let A be a central simple algebra over a field k. Then*

$$s_n(K^A) = \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_A(n)[2n])$$

for all $n \geqslant 0$.

Theorem 2. *Let A be a central simple algebra over a field k of prime degree $\ell \neq \operatorname{char} k$, $X := \mathrm{SB}(A)$ the associated Severi–Brauer variety. Then*

$$s_n^{\mathrm{mot}}(M(X)) = \mathbb{Z}_{A^{\otimes n+1}}(n)[2n]$$

for $0 \leqslant n \leqslant \ell - 1$, 0 otherwise.

See Theorems 6.5.5 and 7.4.2, respectively, in the body of the paper.

Remark 1. Let A be a quaternion algebra over k. Then $\mathbb{Z}_A$ is in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$. Indeed, by Theorem 2, we have the distinguished triangle

$$\mathbb{Z}(1)[2] \to M(\mathrm{SB}(A)) \to \mathbb{Z}_A \to \mathbb{Z}(1)[3].$$

We do not know if $\mathbb{Z}_A$ is in $DM_{\mathrm{gm}}^{\mathrm{eff}}(k)$ for A of larger degree.

Since $s_n K^A$ and $s_n^{\mathrm{mot}} M(X)$ are the layers in the respective motivic Postnikov towers

$$\cdots \to f_{n+1}K^A \to f_n K^A \to \cdots \to f_0 K^A = K^A,$$
$$0 = f_\ell^{\mathrm{mot}} M(X) \to f_{\ell-1}^{\mathrm{mot}} M(X) \to \cdots \to f_0^{\mathrm{mot}} M(X) = M(X),$$

our computation of the layers gives us some handle on the spectral sequences

$$E_2^{p,q} := \pi_{-p-q}(s_{-q}K^A(Y)) \Longrightarrow \pi_{-p-q}K^A(Y)$$

and

$$E_2^{p,q} := \mathbb{H}^{p+q}(Y, s_{-q}^{\mathrm{mot}} M(X)(n)) \Longrightarrow \mathbb{H}^{p+q}(Y, M(X)(n))$$

arising from the towers. In fact, we use a version of the first sequence to help compute the layers of $M(X)$. Putting our computation of the layers into the K^A-spectral sequence gives us the spectral sequence

$$E_2^{p,q} := H^{p-q}(Y, \mathbb{Z}_A(-q)) \Longrightarrow K_{-p-q}(Y; A)$$

generalizing the Bloch–Lichtenbaum/Friedlander–Suslin spectral sequence from motivic cohomology to K-theory [**8**, **16**]. In particular, taking $Y = \operatorname{Spec} k$, we get

$$K_1(A) = H^1(k, \mathbb{Z}_A(1))$$

and for A of square-free index, prime to the characteristic,

$$K_2(A) = H^2(k, \mathbb{Z}_A(2)).$$

See Theorem 6.7.1 and Theorem 6.8.2.

To go further, we must use the Beilinson–Lichtenbaum conjecture. Recall that this conjecture is equivalent to the Milnor–Bloch–Kato conjecture relating Milnor's K-theory with Galois cohomology [**18**, **56**]. It seems to be now a theorem (see [**66**]), thanks to work of Rost and Voevodsky; accepted proofs are certainly that of Merkurjev and Suslin in the special case of weight 2 [**38**] and that of Voevodsky at the prime 2 (in all weights) [**61**].

Since this seems important to some people, we shall specify in what weights we need the Beilinson–Lichtenbaum (or Milnor–Bloch–Kato) conjecture for our statements.

We use our knowledge of the layers of $M(X)$, together with the Beilinson–Lichtenbaum conjecture, to deduce a result comparing $H^p(k, \mathbb{Z}_A(q))$ and $H^p(k, \mathbb{Z}(q))$ via the *reduced norm map*

$$\mathrm{Nrd} : H^p(k, \mathbb{Z}_A(q)) \to H^p(k, \mathbb{Z}(q)),$$

this just being the map induced by the inclusion $\mathbb{Z}_A \subset \mathbb{Z}$. By identifying Nrd with the change of topologies map from the Nisnevich to the étale topology (using the fact that $\mathbb{Z}_A(n)^{\text{ét}} = \mathbb{Z}(n)^{\text{ét}}$), a duality argument leads to the following corollary.

Corollary 1. *Let A be a central simple algebra of square-free index e over k, with $(e, \operatorname{char} k) = 1$. Let $n \geqslant 0$ and assume the Beilinson–Lichtenbaum conjecture in weights $w \leqslant n + 1$ at all primes dividing the index of A. Then*

$$\mathrm{Nrd} : H^p(k, \mathbb{Z}_A(n)) \to H^p(k, \mathbb{Z}(n))$$

is an isomorphism for $p < n$, and we have an exact sequence

$$0 \to H^n(k, \mathbb{Z}_A(n)) \xrightarrow{\mathrm{Nrd}} H^n(k, \mathbb{Z}(n)) \simeq K_n^M(k)$$
$$\xrightarrow{\cup[A]} H^{n+2}_{\text{ét}}(k, \mathbb{Z}/e(n+1)) \to H^{n+2}_{\text{ét}}(k(X), \mathbb{Z}/e(n+1)).$$

Here $[A] \in H^3_{\text{ét}}(k, \mathbb{Z}(1)) = H^2_{\text{ét}}(k, \mathbb{G}_m)$ is the class of A in the Brauer group of k, and the map $\cup[A]$ is shorthand for the composition

$$H^n(k, \mathbb{Z}(n)) \xrightarrow{\sim} H^n_{\text{ét}}(k, \mathbb{Z}(n)) \xrightarrow{\cup[A]} H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1))$$

(note that this cup-product map lands into ${}_eH^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)) \simeq H^{n+2}_{\text{ét}}(k, \mathbb{Z}/e(n+1))$, the latter isomorphism being a consequence of the Beilinson–Lichtenbaum conjecture in weight $n + 1$).

See Theorem 8.2.2 in the body of the paper for this result.

Combining this result with our identification above of $K_1(A)$ and $K_2(A)$ as 'twisted Milnor K-theory' of k, we have the following corollary (see Theorem 8.2.2).

Corollary 2. *Let A be a central simple algebra over k of square-free index e, with $(e, \operatorname{char} k) = 1$. Then the reduced norm maps on $K_0(A)$, $K_1(A)$ and $K_2(A)$*

$$\mathrm{Nrd} : K_n(A) \to K_n(k), \quad n = 0, 1, 2,$$

are injective; in fact, we have an exact sequence

$$0 \to K_n(A) \xrightarrow{\mathrm{Nrd}} K_n(k) = H^n(k, \mathbb{Z}(n))$$
$$\xrightarrow{\cup[A]} H^{n+2}_{\text{ét}}(k, \mathbb{Z}/e(n+1)) \to H^{n+2}_{\text{ét}}(k(X), \mathbb{Z}/e(n+1))$$

for $n = 0, 1, 2$. (For $n = 2$ we need the Beilinson–Lichtenbaum conjecture in weight 3.)

For $n = 2$, this proves a conjecture of Merkurjev [**36**, p. 81].

The injectivity of Nrd on $K_1(A)$ is Wang's theorem [**64**], and it was proved for $K_2(A)$ and A a quaternion algebra by Rost [**51**] and Merkurjev [**35**]. They used it as a step towards the proof of the Milnor conjecture in degree 3; conversely, the Milnor conjecture in degree 3 was used in [**26**, proof of Theorem 9.3] to give a simple proof of the injectivity in this case. This proof was one of the starting points of the present paper.

For $n = 0$, the exact sequence reduces to Amitsur's theorem that $\ker(\mathrm{Br}(k) \to \mathrm{Br}(k(X)))$ is generated by the class of A [**1**]. For $n = 1$, the exactness at $K_1(k)$ is due to Merkurjev–Suslin [**38**, Theorem 12.2] and the exactness at $H^3_{\text{ét}}(k, \mathbb{Z}/e(2))$ could be extracted from Suslin [**53**]. For $n = 1$ and A a quaternion algebra, the exactness at $H^3_{\text{ét}}(k, \mathbb{Z}/2)$ is due to Arason [**2**, Satz 5.4]. For $n = 2$ and a quaternion algebra, it is due to Merkurjev [**37**, Proposition 3.15].

The injectivity for $K_2(A)$ with A of square-free index has also been proven by Merkurjev and Suslin [**39**, Theorem 2.4]; their method also relies on the Beilinson–Lichtenbaum conjecture, using it to give a computation of the motivic cohomology of the 'Čech cosimplicial scheme' $\check{\mathrm{C}}(X)$.

This paper is divided into three parts. Part I is foundational material concerning the slice filtration in both the homotopical and the motivic context, and their comparisons: it may be skipped at first reading by those readers primarily interested in the applications to central simple algebras, which can be found in Part II. Part III contains three appendices.

We begin in § 1 with a quick review of the motivic Postnikov tower in $\mathcal{SH}_{S^1}(k)$ and $DM^{\text{eff}}(k)$, recalling the basic constructions and properties. In § 2, we recall from [**33**] the *homotopy coniveau tower* and its relation to the motivic Postnikov tower in $\mathcal{SH}_{S^1}(k)$; we also explain how to modify this theory to give an analogous homotopy coniveau tower for motives. We discuss *well-connected* spectra in § 3, showing how the slices for these spectra can be expressed using a generalization of Bloch's cycle complexes. In § 4 we recall some of the first author's theory of *birational motives** as well as pointing out the role these motives play as the Tate twists of slices of an arbitrary T-spectrum.

We proceed in § 5 to define and study the special case of the birational motive $\mathbb{Z}_A$ arising from a central simple algebra A over k; we actually work in the more general setting of a sheaf of Azumaya algebras on a scheme. In § 6 we prove our first main result: we compute the slices of the 'homotopy coniveau tower' for the G-theory spectrum $G(X; \mathcal{A})$, where $\mathcal{A}$ is a sheaf of Azumaya algebras on a scheme X. This result relies on some regularity properties of the functors $K_p(-, A)$ which in turn rely on results due to Vorst and generalized by van der Kallen; we collect and prove what we need in this direction in Appendix B. We also recall some basic results on Azumaya algebras in Appendix A. Specializing to the case in which X is smooth over a field k and $\mathcal{A}$ is the pullback to X of a central simple algebra A over k, the results of [**33**] translate our computation of the slices of the homotopy coniveau tower to give Theorem 1.

We also give in § 6.9 a construction of homomorphisms from SK_1 and SK_2 of a central simple algebra A to quotients of étale cohomology groups of k, in the spirit of an idea

* Work done jointly with Sujatha.

of Suslin [**54**, **55**], albeit with a very different technique (for SK_2 we need the Beilinson–Lichtenbaum conjecture in weight 3).

We turn to our study of the motive of a Severi–Brauer variety in § 7, proving Theorem 2 there. We conclude in § 8 with a discussion of the reduced norm map and the proofs of Corollaries 1 and 2. In Appendix C, we recall the construction and basic properties of the category of motives $DM^{\mathrm{eff}}(S)$ over a regular base S, as well as the version for the étale topology $DM^{\mathrm{eff}}(S)^{\text{ét}}$.

Notation

For a scheme B, let $\mathbf{Sch}_B$ denote the category of finite type B-schemes, and $\mathbf{Sm}/B$ the full subcategory of smooth quasi-projective B-schemes. For $B = \operatorname{Spec} R$, we often write $\mathbf{Sch}_R$ and $\mathbf{Sm}/R$ for $\mathbf{Sch}_B$ and $\mathbf{Sm}/B$. We let $\mathbf{Ord}$ denote the usual indexing category for (co)simplicial objects, that is, $\mathbf{Ord}$ has objects the sets $[n] := \{0, 1, \ldots, n\}$ and morphisms $[n] \to [m]$ the non-decreasing maps of sets. We write $\Delta[n]$ for the representable simplicial set $\mathrm{Hom}_{\mathbf{Ord}}(-, [n])$. For a set S, $\mathbb{Z}[S]$ denotes the free abelian group on S; for a simplicial set S, $\mathbb{Z}[S]$ is the corresponding simplicial abelian group $n \mapsto \mathbb{Z}[S_n]$.

For categories $\mathcal{A}$ and $\mathcal{C}$, with $\mathcal{C}$ essentially small, we let $\mathrm{PS}_{\mathcal{A}}(\mathcal{C})$ denote the category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$; in case $\mathcal{A}$ is the category of sets, we just write $\mathrm{PS}(\mathcal{C})$, and for the category of pointed sets we write $\mathrm{PS}_\bullet(\mathcal{C})$. Since an $\mathcal{A}$-valued presheaf on $\mathbf{Ord}$ is just a simplicial object of $\mathcal{A}$, we write $s\mathcal{A}$ for $\mathrm{PS}_{\mathcal{A}}(\mathbf{Ord})$.

For an additive category $\mathcal{A}$, we let $C(\mathcal{A})$ denote the category of complexes over $\mathcal{A}$, with differential of degree $+1$. We let $K(\mathcal{A})$ denote the homotopy category of complexes, with the standard structure of a triangulated category. If $\mathcal{A}$ is an abelian category, we denote the derived category by $D(\mathcal{A})$. We have as well the bounded versions $C^?(\mathcal{A})$, $K^?(\mathcal{A})$, $D^?(\mathcal{A})$, with $? = \emptyset, +, -, b$. We let $C^{\leqslant 0}(\mathcal{A})$ denote the category of complexes supported in non-positive degrees. We will systematically use the cohomological translation functor: $(E[1])^n := E^{n+1}$. On the occasion that we use a homological complex C_*, we will always consider C_* as a cohomological complex by setting $C^n := C_{-n}$, and the translation functor will be applied to C^*. As homological complexes, we thus have $(C_*[1])_n = C_{n-1}$.

Part I. Slice filtrations and birational motives

1. The motivic Postnikov tower in $\mathcal{SH}_{S^1}(k)$ and $DM^{\mathrm{eff}}(k)$

In this section, we assume that k is a perfect field. We review Voevodsky's construction of the motivic Postnikov tower in $\mathcal{SH}_{S^1}(k)$, as well as the analog of the tower in $DM^{\mathrm{eff}}(k)$. We also give the description of these towers in terms of the homotopy coniveau tower, following [**33**].

1.1. Constructions in $\mathbb{A}^1$ stable homotopy theory

We start with the unstable $\mathbb{A}^1$ homotopy category over k, $\mathcal{H}_\bullet(k)$, which is the homotopy category of the category $\mathbf{Spc}_\bullet(k)$ of pointed presheaves of simplicial sets on $\mathbf{Sm}/k$, with respect to the Nisnevich- and $\mathbb{A}^1$-local model structure defined in [**15**, § 2] (in [**15**] $\mathbf{Spc}_\bullet(k)$ is denoted $\mathcal{M}$ and the model structure $\mathcal{M}_{\mathrm{mo}}$ is called the *motivic* model structure). We recall that the cofibrations in $\mathbf{Spc}_\bullet(k)$ are generated by maps of the form

$$h_X \wedge \partial\Delta[n] \to h_X \wedge \Delta[n], \quad n = 0, 1, \dots,$$

where h_X is the pointed representable presheaf $h_X(U) := \mathrm{Hom}_{\mathbf{Sm}/k}(U, X)_+$.

$\mathbf{Spc}_\bullet(k)$ contains the category of simplicial sets by taking the constant presheaf; in particular, we have the suspension operation

$$\Sigma_s : \mathbf{Spc}_\bullet(k) \to \mathbf{Spc}_\bullet(k)$$

defined by $\Sigma_s X := X \wedge S^1$. For $S \in \mathbf{Spc}_\bullet(k)$, we have the associated $\mathbb{A}^1$-*homotopy sheaf* $\pi_n^{\mathbb{A}^1}(S)$, this being the Nisnevich sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(\Sigma_s^n h_U, S).$$

We note that the weak equivalences in $\mathbf{Spc}_\bullet(k)$ are the maps inducing an isomorphism on $\pi_n^{\mathbb{A}^1}$ for all $n \geqslant 0$.* Below, we simplify the notation $\pi_n^{\mathbb{A}^1}$ into π_n.

We let $\mathbf{Spt}_{S^1}(k)$ denote the category of Σ_s-spectra in $\mathbf{Spc}_\bullet(k)$, i.e. the category with objects sequences $(E_0, E_1, \dots)$ in $\mathbf{Spc}_\bullet(k)$ together with bonding maps $\epsilon_n : \Sigma_s E_n \to E_{n+1}$; morphisms are sequences of morphisms in $\mathbf{Spc}_\bullet(k)$ commuting with the bonding maps. Thus, $\mathbf{Spt}_{S^1}(k)$ is just the category of presheaves of classical spectra on $\mathbf{Sm}/k$.

For $E = (E_0, E_1, \dots) \in \mathbf{Spt}_{S^1}(k)$, one has the *stable homotopy sheaf*

$$\pi_n^s(E) := \varinjlim_N \pi_{n+N} E_N.$$

A map $f : E \to F$ in $\mathbf{Spt}_{S^1}(k)$ is a *stable weak equivalence* if $f_* : \pi_n^s(E) \to \pi_n^s(F)$ is an isomorphism for all n.

Hovey [**21**, § 3] defines the *stable* model structure on $\mathbf{Spt}_{S^1}(k)$. It follows from [**21**, Theorem 4.12] that the weak equivalences are the stable weak equivalences. We denote the homotopy category of $\mathbf{Spt}_{S^1}(k)$ by $\mathcal{SH}_{S^1}(k)$.

Remark 1.1.1. There is a natural functor

$$\mathcal{SH}^{S^1}(k) \to \mathcal{SH}_{S^1}(k),$$

where $\mathcal{SH}^{S^1}(k)$ is the stable $\mathbb{A}^1$-homotopy category defined by Morel in [**40**, § 3.2]. This functor is in fact an equivalence of categories.

To see this, we use the Nisnevich-local model structure $\mathcal{M}_s$ on $\mathcal{M} := \mathbf{Spc}_\bullet(k)$ defined in [**15**]. The results of Hovey [**21**, Theorems 3.4, 4.9 and 4.12] tell us that the fibrant objects in $\mathbf{Spt}_{S^1}(\mathcal{M}_s)$ are (up to weak equivalence) the S^1-spectra $E = (E_0, E_1, \dots)$ such that E_n is fibrant in $\mathcal{M}_s$ and $E_n \to \Omega_s E_{n+1}$ is a weak equivalence in $\mathcal{M}_s$. Changing $\mathcal{M}_s$ to $\mathcal{M}_{\mathrm{mo}}$ gives us a similar description of the fibrant objects in $\mathbf{Spt}_{S^1}(\mathcal{M}_{\mathrm{mo}}) =: \mathbf{Spt}_{S^1}(k)$.

* To see this, note that, for a map f between fibrant objects, this implies that f induces an isomorphism on the homotopy presheaves $U \mapsto \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(\Sigma_s^n h_U, -)$, and the $\Sigma_s^n h_U$ generate.

As $\mathcal{M}_{\mathrm{mo}}$ is the Bousfield localization of $\mathcal{M}_s$ with respect to $\mathbb{A}^1$-homotopy, it is then easy to see that the Bousfield localization of $\mathbf{Spt}_{S^1}(\mathcal{M}_s)$ with respect to $\mathbb{A}^1$-homotopy has the same fibrant objects as $\mathbf{Spt}_{S^1}(\mathcal{M}_{\mathrm{mo}})$, from which it follows that the respective homotopy categories are equal.

We shall not however use this identification of the category $\mathcal{SH}^{S^1}(k)$ of [**40**] with $\mathcal{SH}_{S^1}(k)$ in this paper.

The infinite suspension functor

$$\Sigma_s^\infty : \mathbf{Spc}_\bullet(k) \to \mathbf{Spt}_{S^1}(k), \qquad \Sigma^\infty(X) := (X, \Sigma_s X, \Sigma_s^2 X, \dots)$$

admits as right adjoint the 0-space functor $(E_0, E_1, \dots) \mapsto E_0$, giving the Quillen adjoint pair $(\Sigma_s^\infty, \Omega_s^\infty)$ and inducing the pair of adjoint functors on the homotopy categories

$$\Sigma_s^\infty : \mathbf{Spc}_\bullet(k) \rightleftarrows \mathcal{SH}_{S^1}(k) : \Omega_s^\infty.$$

Let $\mathbb{G}_m$ be the pointed space $(\mathbb{A}^1 \setminus \{0\}, 1)$. Let T denote the pointed presheaf $S^1 \wedge \mathbb{G}_m$, and Σ_T the operation $- \wedge T$. The functor Σ_T on $\mathbf{Spt}_{S^1}(k)$ has as right adjoint the T-loops functor $\Omega_T := \mathcal{H}\mathrm{om}(T, -)$. These functors form a Quillen pair of adjoint functors on the model category $\mathbf{Spt}_{S^1}(k)$ and thus define an adjoint pair of functors

$$\Sigma_T : \mathcal{SH}_{S^1}(k) \rightleftarrows \mathcal{SH}_{S^1}(k) : \Omega_T$$

on the homotopy category $\mathcal{SH}_{S^1}(k)$.

We have the *pointwise* model structure on $\mathbf{Spt}_{S^1}(k)$, with the same cofibrations as above, and with the weak equivalences the maps $E \to F$ for which $E(Y) \to F(Y)$ is a weak equivalence of spectra for each $Y \in \mathbf{Sm}/k$. We write $\mathcal{H}\,\mathbf{Spt}_{S^1}(k)$ for the homotopy category of $\mathbf{Spt}_{S^1}(k)$ with respect to the pointwise model structure.

Remark 1.1.2. For $E \in \mathbf{Spt}_{S^1}(k)$, define $\Omega_{\mathbb{P}^1} E(X)$ as the homotopy fibre

$$\Omega_{\mathbb{P}^1} E(X) := \mathrm{fib}(E(X \times \mathbb{P}^1) \to E(X \times \infty)).$$

As $T \cong (\mathbb{P}^1, \infty)$ in $\mathcal{H}_\bullet$, the adjoint functors Σ_T, Ω_T on $\mathcal{SH}_{S^1}(k)$ are isomorphic to $\Sigma_{\mathbb{P}^1}$, $\Omega_{\mathbb{P}^1}$; we often use the model $\Omega_{\mathbb{P}^1} E$ for $\Omega_T E$.

We let $\mathbf{Spc}_\bullet$ denote the category of pointed simplicial sets, $\mathbf{Spt}$ the category of spectra (i.e. $- \wedge S^1$ spectra in $\mathbf{Spc}_\bullet$) and $\mathcal{SH}$ the homotopy category of $\mathbf{Spt}$, i.e. the classical stable homotopy category. For each $Y \in \mathbf{Sm}/k$, the evaluation functor at Y defines as usual an exact functor

$$R\Gamma(Y, -) : \mathcal{SH}_{S^1}(k) \to \mathcal{SH}$$

with $R\Gamma(Y, E) := E^{\mathrm{fib}}(Y)$, where $E \to E^{\mathrm{fib}}$ is a fibrant model. As we will usually apply $R\Gamma(Y, -)$ to presheaves E for which $E(Y) \to E^{\mathrm{fib}}(Y)$ is a weak equivalence for all Y, we usually will write $E(Y)$ for $R\Gamma(Y, E)$.

Remark 1.1.3. There are other model structures on $\mathbf{Spc}_\bullet(k)$ and $\mathbf{Spt}_{S^1}(k)$ with the same weak equivalences, and thus yielding the same homotopy categories as above; see for instance [**24**, **41**, **49**].

1.2. Postnikov towers for S^1-spectra

Voevodsky [**59**] has defined a canonical tower on the motivic stable homotopy category $\mathcal{SH}_{S^1}(k)$, which we call the *motivic Postnikov tower*.

Recall from [**42**, Definition 3.2.6] that a thick subcategory $\mathcal{A}$ of a triangulated category $\mathcal{T}$ is a *localizing* subcategory if each (not necessarily finite) coproduct of objects of $\mathcal{A}$ that exists in $\mathcal{T}$ is in $\mathcal{A}$. Let $\Sigma_T^n \mathcal{SH}_{S^1}(k)$ be the localizing subcategory of $\mathcal{SH}_{S^1}(k)$ generated by objects of the form $\Sigma_T^n E$, $E \in \mathcal{SH}_{S^1}(k)$. This gives us the tower of localizing subcategories

$$\cdots \subset \Sigma_T^{n+1} \mathcal{SH}_{S^1}(k) \subset \Sigma_T^n \mathcal{SH}_{S^1}(k) \subset \cdots \subset \mathcal{SH}_{S^1}(k).$$

Take $E \in \mathcal{SH}_{S^1}(k)$ and consider the cohomological functor

$$\mathrm{Hom}_{\Sigma_T^n \mathcal{SH}_{S^1}(k)}(-, E) : \Sigma_T^n \mathcal{SH}_{S^1}(k) \to \mathbf{Ab}\,.$$

By Neeman's version [**42**, Theorem 8.3.3] of Brown representability, this functor is represented by an object $r_n E$ of $\Sigma_T^n \mathcal{SH}_{S^1}(k)$; sending E to $r_n E$ defines a right adjoint $r_n : \mathcal{SH}_{S^1}(k) \to \Sigma_T^n \mathcal{SH}_{S^1}(k)$ to the inclusion $i_n : \Sigma_T^n \mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k)$. Let $f_n := i_n \circ r_n$ with counit $f_n \to \mathrm{id}$. Thus, for each $E \in \mathcal{SH}_{S^1}(k)$, there is a canonical tower in $\mathcal{SH}_{S^1}(k)$

$$\cdots \to f_{n+1}E \to f_n E \to \cdots \to f_0 E = E, \tag{1.1}$$

the *motivic Postnikov tower* for S^1-spectra. We write $f_{n/n+r}E$ for the cofibre of $f_{n+r}E \to f_n E$; we use the notation $s_n := f_{n/n+1}$ to denote the nth slice in the Postnikov tower.

By [**33**, Theorem 7.4.2], the T-loops functor Ω_T is compatible with the truncation functors f_n up to canonical isomorphism

$$\Omega_T \circ f_{n+1} \cong f_n \circ \Omega_T. \tag{1.2}$$

1.3. The motivic Postnikov tower for motives

There is an analogous Postnikov tower for motives, where the corresponding category of motives is the enlargement $DM^{\mathrm{eff}}(k)$ of the category $DM_-^{\mathrm{eff}}(k)$. For details on the construction and basic properties of $DM^{\mathrm{eff}}(k)$, we refer the reader to Appendix C.

Let $DM^{\mathrm{eff}}(k)(n)$ be the localizing subcategory of $DM^{\mathrm{eff}}(k)$ generated by objects $M(X)(n)[2n]$, $X \in \mathbf{Sm}/k$, giving the tower of localizing subcategories (for $n \geqslant 0$)

$$\cdots \subset DM^{\mathrm{eff}}(k)(n+1) \subset DM^{\mathrm{eff}}(k)(n) \subset \cdots \subset DM^{\mathrm{eff}}(k)(0) = DM^{\mathrm{eff}}(k).$$

Just as for $\mathcal{SH}_{S^1}(k)$, we have the right adjoint $r_n^{\mathrm{mot}} : DM^{\mathrm{eff}}(k) \to DM^{\mathrm{eff}}(k)(n)$ to the inclusion i_n^{mot}. Thus, for E in $DM^{\mathrm{eff}}(k)$, we have the *motivic Postnikov tower* in $DM^{\mathrm{eff}}(k)$

$$\cdots \to f_{n+1}^{\mathrm{mot}}E \to f_n^{\mathrm{mot}}E \to \cdots \to f_0^{\mathrm{mot}}E = E \tag{1.3}$$

with $f_n^{\mathrm{mot}} := i_n^{\mathrm{mot}} \circ r_n^{\mathrm{mot}}$.

Remark 1.3.1. We lift the functors s_n, f_n to operations on $\mathbf{Spt}_{S^1}(k)$ by taking the fibrant model of the corresponding object in $\mathcal{SH}_{S^1}(k)$; we make a similar lifting to $C(\mathrm{PST}(k))$ for the functors f_n^{mot}, s_n^{mot}.

1.4. Comparing Postnikov towers

We use the motivic Eilenberg–Mac Lane functor to compare the Postnikov towers in $\mathcal{SH}_{S^1}(k)$ and $DM^{\mathrm{eff}}(k)$; we begin by recalling the construction of the Eilenberg–Mac Lane functor from [**45**, § 1].

We recall the *Dold–Kan correspondence* [**14, 28**]. Sending a simplicial abelian group $n \mapsto C_n$ to the *normalized chain complex* (NC, d):

$$NC^{-n} := \bigcap_{i=1}^{n} \ker(d_i : C_n \to C_{n-1}), \quad d = d_0,$$

defines an equivalence of categories

$$N : s\,\mathbf{Ab} \to C^{\leqslant 0}(\mathbf{Ab}).$$

The inverse is the *Dold–Kan functor*

$$\mathrm{DK} : C^{\leqslant 0}(\mathbf{Ab}) \to s\,\mathbf{Ab},$$

where $\mathrm{DK}(C)$ is the simplicial object

$$q \mapsto \mathrm{Hom}_{C^{\leqslant 0}(\mathbf{Ab})}(N\mathbb{Z}[\Delta[q]], C).$$

If $\mathcal{C}$ is a category, applying the functors N and DK pointwise gives an equivalence of presheaf categories $C^{\leqslant 0}(\mathrm{PS}_{\mathbf{Ab}}(\mathcal{C})) \sim s\,\mathrm{PS}_{\mathbf{Ab}}(\mathcal{C})$.

We have the forgetful functor

$$\mathcal{U} : \mathrm{PST}(k) \to \mathrm{PS}_{\bullet}(\mathbf{Sm}/k)$$

sending a presheaf with transfers P to the associated presheaf of sets (pointed by 0). $\mathcal{U}$ induces the functor

$$s\mathcal{U} : s\,\mathrm{PST}(k) \to \mathbf{Spc}_{\bullet}(k)$$

on the associated categories of simplicial objects. Sending $h_X \wedge \Delta[n]$ to $\mathbb{Z}^{\mathrm{tr}}(X) \otimes \mathbb{Z}[\Delta[n]]$ extends, by taking the left Kan extension, to a functor

$$\mathbb{Z}^{\mathrm{tr}} : \mathbf{Spc}_{\bullet}(k) \to s\,\mathrm{PST}(k)$$

left adjoint to $s\mathcal{U}$.

Composing with the Dold–Kan functor $\mathrm{DK} : C^{\leqslant 0}(\mathrm{PST}(k)) \to s\,\mathrm{PST}(k)$ gives

$$\mathrm{DK} \circ s\mathcal{U} : C^{\leqslant 0}(\mathrm{PST}(k)) \to \mathbf{Spc}_{\bullet}(k),$$

with left adjoint

$$N \circ \mathbb{Z}^{\mathrm{tr}} : \mathbf{Spc}_{\bullet}(k) \to C^{\leqslant 0}(\mathrm{PST}(k)).$$

One defines a model structure $C^{\leqslant 0}(\mathrm{PST}(k))_{\mathbb{A}^1}$ on $C^{\leqslant 0}(\mathrm{PST}(k))$ with the cofibrations generated by maps of the form

$$\mathbb{Z}^{\mathrm{tr}}(X)[n-1] \to D^{\mathrm{tr}}(X)[n], \quad n \geqslant 1, \quad \text{and} \quad 0 \to \mathbb{Z}^{\mathrm{tr}}(X), \quad X \in \mathbf{Sm}/k,$$

where $D^{\mathrm{tr}}(X)$ is the complex $\mathbb{Z}^{\mathrm{tr}}(X) \xrightarrow{\mathrm{id}} \mathbb{Z}^{\mathrm{tr}}(X)$, concentrated in degrees 0, 1, the weak equivalences the maps in $C^{\leqslant 0}(\mathrm{PST}(k))$ which are weak equivalences in $C(\mathrm{PST}(k))_{\mathbb{A}^1}$, and the fibrations are the maps having the right lifting property with respect to acyclic cofibrations. It is easy to show that $N \circ \mathbb{Z}^{\mathrm{tr}}$ defines a left Quillen functor with right adjoint $\mathrm{DK} \circ s\mathcal{U}$ (see [**45**, § 2] for details).

Let $\mathbf{Spt}(C^{\leqslant 0}(\mathrm{PST}(k)))$ be the category of spectrum objects in $C^{\leqslant 0}(\mathrm{PST}(k))$ with respect to the suspension operator $\Sigma C := C[1]$. As $C[1] = (\mathbb{Z}[1]) \otimes C$, Hovey's methods apply to give a stable model category structure $\mathbf{Spt}(C^{\leqslant 0}(\mathrm{PST}(k)))_{\mathbb{A}^1}$ to $\mathbf{Spt}(C^{\leqslant 0}(\mathrm{PST}(k)))$. $(N \circ \mathbb{Z}^{\mathrm{tr}}, \mathrm{DK} \circ s\mathcal{U})$ extends to a Quillen adjoint pair $(\mathbf{Spt}(N \circ \mathbb{Z}^{\mathrm{tr}}), \mathbf{Spt}(\mathrm{DK} \circ s\mathcal{U}))$ on the spectrum categories.

Sending $(C_0, C_1, \dots)$ to $\varinjlim_n C_n[-n]$ defines a left Quillen equivalence

$$\mathbf{Spt}(C^{\leqslant 0}(\mathrm{PST}(k))_{\mathbb{A}^1} \to C(\mathrm{PST}(k))_{\mathbb{A}^1},$$

with inverse the functor

$$[C \in C(\mathrm{PST}(k))] \mapsto (\tau_{\leqslant 0} C, \tau_{\leqslant 0}(C[1]), \dots, \tau_{\leqslant 0}(C[n]), \dots).$$

Thus, on the homotopy categories, $(\mathbf{Spt}(N \circ \mathbb{Z}^{\mathrm{tr}}), \mathbf{Spt}(\mathrm{DK} \circ s\mathcal{U}))$ induces the pair of adjoint functors $(\mathrm{Mot}, \mathrm{EM}_{\mathbb{A}^1})$:

$$\mathrm{Mot} : \mathcal{SH}_{S^1}(k) \rightleftarrows DM^{\mathrm{eff}}(k) : \mathrm{EM}_{\mathbb{A}^1}.$$

Remark 1.4.1. Actually, Østvær–Röndigs define the adjoint pair $(\mathrm{Mot}, \mathrm{EM}_{\mathbb{A}^1})$ between the category of T-spectra $\mathcal{SH}(k)$, and the category of $\mathbb{Z}(1)[2]$-spectra $DM(k)$. The constructions of [**44**,**45**] work in the (somewhat simpler) setting described above, by replacing the T-suspension functor used in [**44**, **45**] with the S^1-suspension Σ_s.

Remark 1.4.2. Replacing $\mathrm{PST}(k)$ with $\mathbf{Ab}$ and $\mathbf{Spc}_\bullet(k)$ with $\mathbf{Spc}_\bullet$, exactly the same construction gives the classical Eilenberg–Mac Lane functor

$$\mathrm{EM} : D(\mathbf{Ab}) \to \mathcal{SH}.$$

For $Y \in \mathbf{Sm}/k$, $\mathcal{F} \in DM^{\mathrm{eff}}(k)$, we have a canonical isomorphism in $\mathcal{SH}$,

$$\mathrm{EM}(\mathcal{F}(Y)) \cong (\mathrm{EM}_{\mathbb{A}^1} \mathcal{F})(Y),$$

as follows from the adjunction computation for a general $E \in \mathcal{SH}$:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{SH}}(E, \mathrm{EM}(\mathcal{F}(Y))) &\simeq \mathrm{Hom}_{D(\mathbf{Ab})}(C_* E, \mathcal{F}(Y)) \\
&\simeq \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(C_* E \otimes M(Y), \mathcal{F}) \\
&\simeq \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathrm{Mot}(E \wedge Y), \mathcal{F}) \\
&\simeq \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(E \wedge Y, \mathrm{EM}_{\mathbb{A}^1} \mathcal{F}) \\
&\simeq \mathrm{Hom}_{\mathcal{SH}}(E, (\mathrm{EM}_{\mathbb{A}^1} \mathcal{F})(Y)),
\end{aligned}$$

where C_* is the left adjoint of EM and the third isomorphism uses the fact that Mot is a strict monoidal functor.

Lemma 1.4.3. *For every $n \geqslant 0$, we have* $\mathrm{Mot}(\Sigma_T^n \mathcal{SH}_{S^1}(k)) \subset DM^{\mathrm{eff}}(k)(n)$ *and* $\mathrm{EM}_{\mathbb{A}^1}(DM^{\mathrm{eff}}(k)(n)) \subset \Sigma_T^n \mathcal{SH}_{S^1}(k)$.

Proof. Note that, as the infinite suspension spectra $\Sigma_s^\infty h_{X+}$ are generators for $\mathcal{SH}_{S^1}(k)$, the $\Sigma_T^n \Sigma_s^\infty h_{X+}$ generate $\Sigma_T^n \mathcal{SH}_{S^1}(k)$ as a localizing subcategory of $\mathcal{SH}_{S^1}(k)$. Since Mot is exact and commutes with colimits, we need only show that $\mathrm{Mot}(\Sigma_T^n \Sigma_s^\infty h_{X+})$ is in $DM^{\mathrm{eff}}(k)(n)$ for each $X \in \mathbf{Sm}/k$.

Since

$$\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{P}^1) = \mathbb{Z}^{\mathrm{tr}}(X) \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^1) \cong \mathbb{Z}^{\mathrm{tr}}(X) \oplus \mathbb{Z}^{\mathrm{tr}}(X)(1)[2],$$

we have

$$\mathrm{Mot}(\Sigma_T(\Sigma^\infty h_{X+})) \cong \mathrm{Mot}(\Sigma^\infty h_{X \times \mathbb{P}^1}/h_X) \cong M(X)(1)[2],$$

and similarly, $\mathrm{Mot}(\Sigma_T^n(\Sigma^\infty h_{X+})) \cong M(X)(n)[2n]$. This verifies the first inclusion.

The second inclusion is more subtle; we will postpone the proof until we introduce the homotopy coniveau construction in § 2.2 (see Remark 2.2.4). □

Proposition 1.4.4. *We have canonical isomorphisms for all $n \geqslant 0$,*

$$\mathrm{EM}_{\mathbb{A}^1} \circ f_n^{\mathrm{mot}} \cong f_n \circ \mathrm{EM}_{\mathbb{A}^1}, \qquad \mathrm{EM}_{\mathbb{A}^1} \circ s_n^{\mathrm{mot}} \cong s_n \circ \mathrm{EM}_{\mathbb{A}^1},$$

inducing an isomorphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{EM}_{\mathbb{A}^1} \circ f_{n+1}^{\mathrm{mot}} \arrow[r] \arrow[d, "\wr"] & \mathrm{EM}_{\mathbb{A}^1} \circ f_n^{\mathrm{mot}} \arrow[r] \arrow[d, "\wr"] & \mathrm{EM}_{\mathbb{A}^1} \circ s_n^{\mathrm{mot}} \arrow[r] \arrow[d, "\wr"] & \mathrm{EM}_{\mathbb{A}^1} \circ f_{n+1}^{\mathrm{mot}}[1] \arrow[d, "\wr"] \\
f_{n+1} \circ \mathrm{EM}_{\mathbb{A}^1} \arrow[r] & f_n \circ \mathrm{EM}_{\mathbb{A}^1} \arrow[r] & s_n \circ \mathrm{EM}_{\mathbb{A}^1} \arrow[r] & f_{n+1} \circ \mathrm{EM}_{\mathbb{A}^1}[1]
\end{tikzcd}$$

Proof. By Lemma 1.4.3 and the fact that $\mathrm{EM}_{\mathbb{A}^1}$ is a right adjoint, we see that $\mathrm{EM}_{\mathbb{A}^1}(f_n^{\mathrm{mot}}\mathcal{F}) \to \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})$ satisfies the universal property of $f_n \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F}) \to \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})$, giving the canonical isomorphism

$$\mathrm{EM}_{\mathbb{A}^1} \circ f_n^{\mathrm{mot}} \cong f_n \circ \mathrm{EM}_{\mathbb{A}^1}.$$

Let $\Sigma_T^{n+1} \mathcal{SH}_{S^1}(k)^\perp \subset \Sigma_T^n \mathcal{SH}_{S^1}$ denote the right perpendicular of $\Sigma_T^{n+1} \mathcal{SH}_{S^1}(k)$ in $\Sigma_T^n \mathcal{SH}_{S^1}$, and similarly let $DM^{\mathrm{eff}}(k)(n+1)^\perp \subset DM^{\mathrm{eff}}(k)(n)$ be the right perpendicular of $DM^{\mathrm{eff}}(k)(n+1)$ in $DM^{\mathrm{eff}}(k)(n)$. For $E \in \Sigma_T^n \mathcal{SH}_{S^1}$, the distinguished triangle

$$f_{n+1}E \to E \to s_n E \to f_{n+1}E[1]$$

is characterized as the unique distinguished triangle $A \to E \to B \to A[1]$ with $A \in \Sigma_T^{n+1} \mathcal{SH}_{S^1}(k)$ and $B \in \Sigma_T^{n+1} \mathcal{SH}_{S^1}(k)^\perp$. We have an analogous characterization of the distinguished triangle

$$f_{n+1}^{\mathrm{mot}}\mathcal{F} \to \mathcal{F} \to s_n^{\mathrm{mot}}\mathcal{F} \to f_{n+1}^{\mathrm{mot}}\mathcal{F}[1]$$

for $\mathcal{F} \in DM^{\mathrm{eff}}(k)(n)$. Since

$$\mathrm{Mot}(\Sigma_T^{n+1} \mathcal{SH}_{S^1}(k)) \subset DM^{\mathrm{eff}}(k)(n+1),$$

the right adjoint $\mathrm{EM}_{\mathbb{A}^1}$ satisfies

$$\mathrm{EM}_{\mathbb{A}^1}(DM^{\mathrm{eff}}(k)(n+1)^{\perp}) \subset \Sigma_T^{n+1}\,\mathcal{SH}_{S^1}(k)^{\perp}.$$

Thus the isomorphisms

$$\begin{aligned}\mathrm{EM}_{\mathbb{A}^1}\circ f_n^{\mathrm{mot}} &\cong f_n\circ \mathrm{EM}_{\mathbb{A}^1},\\ \mathrm{EM}_{\mathbb{A}^1}\circ f_{n+1}^{\mathrm{mot}} &\cong f_{n+1}\circ \mathrm{EM}_{\mathbb{A}^1}\end{aligned}$$

extend to an isomorphism of distinguished triangles

$$\begin{tikzcd}
\mathrm{EM}_{\mathbb{A}^1}\circ f_{n+1}^{\mathrm{mot}} \arrow[r] \arrow[d, "\wr"] & \mathrm{EM}_{\mathbb{A}^1}\circ f_n^{\mathrm{mot}} \arrow[r] \arrow[d, "\wr"] & \mathrm{EM}_{\mathbb{A}^1}\circ s_n^{\mathrm{mot}} \arrow[r] \arrow[d, "\wr"] & \mathrm{EM}_{\mathbb{A}^1}\circ f_{n+1}^{\mathrm{mot}}[1] \arrow[d, "\wr"] \\
f_{n+1}\circ \mathrm{EM}_{\mathbb{A}^1} \arrow[r] & f_n\circ \mathrm{EM}_{\mathbb{A}^1} \arrow[r] & s_n\circ \mathrm{EM}_{\mathbb{A}^1} \arrow[r] & f_{n+1}\circ \mathrm{EM}_{\mathbb{A}^1}[1]
\end{tikzcd}$$

completing the proof. □

2. The homotopy coniveau tower

The *homotopy coniveau tower* gives a fairly explicit construction of the motivic Postnikov towers in $\mathcal{SH}_{S^1}(k)$ and $DM^{\mathrm{eff}}(k)$. We review the main results of [**33**] on the homotopy coniveau tower for $\mathbf{Spt}_{S^1}(k)$, and show how these can be modified to give analogous results for $C(\mathrm{PST}(k))$.

2.1. Purity

Let E be in $\mathcal{SH}_{S^1}(k)$, $Y \in \mathbf{Sm}/k$ and $W \subset Y$ a closed subset. We let $E^W(Y)$ denote the homotopy fibre of

$$\tilde{E}(Y) \to \tilde{E}(Y\setminus W),$$

where $\tilde{E}$ is a fibrant model of E in $\mathbf{Spt}_{S^1}(k)$. We make a similar definition for $\mathcal{F} \in C(\mathrm{PST}(k))$. If E is homotopy invariant and satisfies Nisnevich excision, then the map of the homotopy fibre of $E(Y) \to E(Y\setminus W)$ to $E^W(Y)$ is a weak equivalence [**43**]; in this setting, we will sometimes use the homotopy fibre spectrum for $E^W(Y)$ without explicit mention.

Let $i : W \to Y$ be a closed immersion in $\mathbf{Sm}/k$ such that the normal bundle $\nu := N_{W/Y}$ admits a trivialization $\varphi : \mathcal{O}_W^q \to \nu$. This gives us the Morel–Voevodsky purity isomorphism [**41**, Theorem 2.23] in $\mathcal{SH}$

$$\theta_{\varphi,E} : E^W(Y) \to (\Omega_T^q E)(W) \tag{2.1}$$

and the isomorphism on homotopy groups

$$\theta_{\varphi,n,E} : \pi_n(E^W(Y)) \to \pi_n((\Omega_T^q E)(W)). \tag{2.2}$$

In general, the $\theta_{\varphi,n,E}$ depend on the choice of φ.

2.2. The tower

The construction of the homotopy coniveau tower relies on the cosimplicial scheme of algebraic n-simplices

$$n \mapsto \Delta^n := \operatorname{Spec} k[t_0, \ldots, t_n]/\sum_i t_i - 1,$$

with coface and codegeneracy maps defined as in the topological setting (see, for example, [**6**]). We recall that a *face* of Δ^n is a subscheme defined by equations of the form $t_{i_1} = \cdots = t_{i_r} = 0$.

Definition 2.2.1.

(1) For $X \in \mathbf{Sch}_k$, locally equidimensional over k, and $q, n \geqslant 0$ integers, set

$$\mathcal{S}_X^{(q)}(n) := \{W \subset X \times \Delta^n \mid W \text{ is closed, and } \operatorname{codim}_{X\times F} W \cap X \times F \geqslant q \text{ for all faces } F \subset \Delta^n\}.$$

Set

$$X^{(q)}(n) := \{w \in X \times \Delta^n \mid w \text{ is the generic point of some irreducible } W \in \mathcal{S}_X^{(q)}(n)\}.$$

(2) For $E \in \mathbf{Spt}_{S^1}(k)$, $X \in \mathbf{Sm}/k$ and integer $q \geqslant 0$, define

$$f^q(X, n; E) = \varinjlim_{W \in \mathcal{S}_X^{(q)}(n)} E^W(X \times \Delta^n).$$

(3) For $E \in \mathbf{Spt}_{S^1}(k)$, $X \in \mathbf{Sm}/k$ and integer $q \geqslant 0$, define

$$s^q(X, n; E) = \varinjlim_{\substack{W \in \mathcal{S}_X^{(q)}(n) \\ W' \in \mathcal{S}_X^{(q+1)}(n)}} E^{W \setminus W'}(X \times \Delta^n \setminus W').$$

For fixed q, the cosimplicial structure on $n \mapsto \Delta^n$ makes $n \mapsto \mathcal{S}_X^{(q)}(n)$ a simplicial set, and $n \mapsto f^q(X, n; E)$, $n \mapsto s^q(X, n; E)$ similarly form simplicial spectra. We let $f^q(X, -; E)$ and $s^q(X, -; E)$ denote the respective total spectra.

For $\mathcal{F} \in C(\mathrm{PST}(k))$, we make the analogous definition yielding the simplicial complexes $n \mapsto f^q_{\mathrm{mot}}(X, n; \mathcal{F})$ and $n \mapsto s^q_{\mathrm{mot}}(X, n; \mathcal{F})$; we let $f^q_{\mathrm{mot}}(X, *; \mathcal{F})$ and $s^q_{\mathrm{mot}}(X, *; \mathcal{F})$ be the associated total complexes. It follows from Remark 1.4.2 that we have isomorphisms in $\mathcal{SH}$:

$$\left.\begin{aligned} \mathrm{EM}(f^q_{\mathrm{mot}}(X, *; \mathcal{F})) &\cong f^q(X, -; \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})), \\ \mathrm{EM}(s^q_{\mathrm{mot}}(X, *; \mathcal{F})) &\cong s^q(X, -; \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})). \end{aligned}\right\} \tag{2.3}$$

The following result relates the homotopy coniveau construction to the motivic Postnikov tower.

Proposition 2.2.2 (Levine [33, Theorem 7.1.1]). *Take $X \in \mathbf{Sm}/k$ and $q \geqslant 0$ an integer. Let $E \in \mathbf{Spt}_{S^1}(k)$ be homotopy invariant and satisfy Nisnevich excision. Then there are isomorphisms in $\mathcal{SH}$*

$$\alpha_{X,q;E} : f^q(X, -; E) \xrightarrow{\sim} f_q(E)(X).$$

The maps $\alpha_{X,q;E}$ define an isomorphism of towers in $\mathcal{SH}$

$$\alpha_{X,-;E} : f^{(-)}(X, -; E) \xrightarrow{\sim} f_{(-)}(E)(X)$$

and induce isomorphisms in $\mathcal{SH}$

$$\beta_{X,q;E} : s^q(X, -; E) \xrightarrow{\sim} s_q(E)(X).$$

All these transformations are natural in X (with respect to smooth maps in $\mathbf{Sm}/k$) and in E.

We have as well a version in $C(\mathrm{PST}(k))$.

Proposition 2.2.3. *Take $X \in \mathbf{Sm}/k$ and $q \geqslant 0$ an integer. Let $\mathcal{F} \in C(\mathrm{PST}(k))$ be homotopy invariant and satisfy Nisnevich excision. Then there are isomorphisms in $D(\mathbf{Ab})$*

$$\alpha^{\mathrm{mot}}_{X,q;\mathcal{F}} : f^q_{\mathrm{mot}}(X, -; \mathcal{F}) \xrightarrow{\sim} f^{\mathrm{mot}}_q(\mathcal{F})(X).$$

The isomorphisms $\alpha^{\mathrm{mot}}_{X,q;\mathcal{F}}$ define an isomorphism of towers in $D(\mathbf{Ab})$

$$\alpha^{\mathrm{mot}}_{X,-;\mathcal{F}} : f^{(-)}_{\mathrm{mot}}(X, -; \mathcal{F}) \xrightarrow{\sim} f^{\mathrm{mot}}_{(-)}(\mathcal{F})(X)$$

and induce isomorphisms in $D(\mathbf{Ab})$

$$\beta^{\mathrm{mot}}_{X,q;\mathcal{F}} : s^q_{\mathrm{mot}}(X, -; \mathcal{F}) \xrightarrow{\sim} s^{\mathrm{mot}}_q(\mathcal{F})(X).$$

All these transformations are natural in X (with respect to smooth maps in $\mathbf{Sm}/k$) and in $\mathcal{F}$.

Proof. The proof of Proposition 2.2.3 goes by constructing a functorial model for the $f^q_{\mathrm{mot}}(X, -; E)$, as in [**32**, Theorems 2.6.2 and 7.4.1], and then following the proof of [**33**, Theorem 7.1.1], changing presheaves of spectra to complexes of presheaves with transfer throughout. We give an outline of the proof, referring to the relevant results in [**32**, **33**] as needed.

Step 1. Take $\mathcal{F} \in C(\mathrm{PST}(k))$ which is homotopy invariant and satisfies Nisnevich excision. We apply the method of [**32**] to define $\tilde{f}^q_{\mathrm{mot}}(\mathcal{F}) \in C(\mathrm{PST}(k))$, forming a tower in $C(\mathrm{PST}(k))$

$$\cdots \to \tilde{f}^{q+1}_{\mathrm{mot}}(\mathcal{F}) \to \tilde{f}^q_{\mathrm{mot}}(\mathcal{F}) \to \cdots \to \tilde{f}^0_{\mathrm{mot}}(\mathcal{F}),$$

and, for each $X \in \mathbf{Sm}/k$, an isomorphism of towers in $D(\mathbf{Ab})$

$$\gamma_{(-),X,\mathcal{F}} : \tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})(X) \xrightarrow{\sim} f^{(-)}_{\mathrm{mot}}(X, *; \mathcal{F}),$$

natural in $\mathcal{F}$ and natural in X for smooth maps.

To form the tower $\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})$, we apply the functoriality results [**32**, Theorems 2.6.2 and 7.4.1], replacing the presheaf of spectra in that result with the complex of presheaves $\mathcal{F}$, throughout. As the results of [**32**] construct a presheaf on $\mathbf{Sm}/k$, rather than on $\mathrm{SmCor}(k)$, we need to make a few modifications to achieve this extension. Fortunately, the main technical result [**32**, Theorem 2.6.2] does not need to be modified at all, we need only modify the application to functoriality given in [**32**, Theorem 7.4.1] as follows (we use the notation of [**32**, § 7]).

(1) We replace the category $\mathcal{L}(\mathbf{Sm}/k)$ (see [**32**, § 7.4]) with a version adapted to finite correspondences and defined as follows: $\mathcal{L}(\mathrm{SmCor}(k))$ has objects the morphisms $f : X' \to X$ in $\mathrm{SmCor}(k)$ such that

(i) f is an effective (finite) cycle on $X' \times X$,

(ii) X' can be written as a disjoint union, $X' = X'_0 \amalg X'_1$ such that the restriction of f to $f_0 : X'_0 \to X$ is the graph of an isomorphism $f_0 : X'_0 \to X$ in $\mathbf{Sm}/k$.

The choice of the decomposition of X' as a disjoint union is not part of the data. We identify $f : X' \to X$ with $f \amalg p : X' \amalg X'' \to X$ if $p = \sum_j m_j p_j : X'' \to X$ with each p_j the graph of a smooth morphism in $\mathbf{Sm}/k$, and we identify $f : X' \to X$ with $f \circ g : X'' \to X$ for $g : X'' \to X'$ the graph of an isomorphism in $\mathbf{Sm}/k$.

$\mathrm{Hom}_{\mathcal{L}(\mathrm{SmCor}(k))}(f_X : X' \to X, f_Y : Y' \to Y)$ is by definition the subgroup of $\mathrm{Hom}_{\mathrm{SmCor}(k)}(X, Y)$ generated by effective $g \in \mathrm{Hom}_{\mathrm{SmCor}(k)}(X, Y)$ such that there is a morphism $q = \sum_i n_i q_i : X' \to Y'$ in $\mathrm{SmCor}(k)$, with $n_i > 0$ and with each q_i the graph of a smooth morphism in $\mathbf{Sm}/k$, such that $g \circ f_X = f_Y \circ q$. The choice of q is not part of the data; composition is the composition in $\mathrm{SmCor}(k)$.

(2) Let $f : Y \to X$ be a morphism in $\mathrm{SmCor}(k)$, with $f = \sum_i n_i Z_i$ and the $Z_i \subset Y \times X$ integral. Let $p_i : Z_i \to X$ be the projection, giving us the subset

$$\mathcal{S}^{(q)}_i(X)(p) = \{W \in \mathcal{S}^{(q)}(X)(p) \mid (p_i \times \mathrm{id}_{\Delta^p})^{-1}(W) \in \mathcal{S}^{(q)}(Z_i)(p)\}.$$

Define

$$\mathcal{S}^{(q)}_f(X)(p) := \cap_i \mathcal{S}^{(q)}_i(X)(p).$$

Given $\mathcal{F} \in \mathrm{PST}(k)$, define

$$f^q_{\mathrm{mot}}(X, p, \mathcal{F})_f := \varinjlim_{W \in \mathcal{S}^{(q)}_f(X)(p)} \mathcal{F}^W(X \times \Delta^p)$$

giving us the associated simplicial complex $p \mapsto f^q_{\mathrm{mot}}(X, p, \mathcal{F})_f$ and total complex $f^q_{\mathrm{mot}}(X, *, \mathcal{F})_f$.

If we take $W \in \mathcal{S}^{(q)}_f(X)(p)$, then, as each Z_i is finite over Y, there is a unique minimal closed subset $W' \in \mathcal{S}^{(q)}(Y)(p)$ such that

$$p_{Y \times \Delta^p}(Z_i \otimes \delta_{\Delta^p} \cap p^{-1}_{X \times \Delta^p}(W)) \subset W',$$

where δ_{Δ^p} is the diagonal correspondence. Thus, the correspondence $Z \otimes \delta_{\Delta^p}$ gives a well-defined map of complexes

$$Z^* : f^q_{\mathrm{mot}}(X, *, \mathcal{F})_f \to f^q_{\mathrm{mot}}(Y, *, \mathcal{F}).$$

More generally, if $g : (f_X : X' \to X) \to (f_Y : Y' \to Y)$ is a morphism in $\mathcal{L}(\mathrm{SmCor}(k))$, we have a well-defined map of complexes

$$g^* : f^q_{\mathrm{mot}}(X, *, \mathcal{F})_{f_X} \to f^q_{\mathrm{mot}}(Y, *, \mathcal{F})_{f_Y}$$

(cf. [**32**, Lemma 7.4.3]). Thus, sending $f_X : X' \to X$ to $f^q_{\mathrm{mot}}(X, *, \mathcal{F})_{f_X}$ defines a presheaf of complexes on $\mathcal{L}(\mathrm{SmCor}(k))$.

Noting this, and making the two changes described above, the proof of [**32**, Theorem 7.4.1] goes through word for word as in [**32**, § 7.4] to give us the tower of presheaves $\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})_{\mathrm{Nis}}$ and an isomorphism of towers in $D(\mathrm{Sh}^{\mathbf{Ab}}(X_{\mathrm{Nis}}))$:

$$\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})_{\mathrm{Nis}}|_{X_{\mathrm{Nis}}} \cong f^{(-)}_{\mathrm{mot}}(X_{\mathrm{Nis}}, *; \mathcal{F}).$$

Letting $\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})$ be a fibrant model of the Nisnevich sheafification of $\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})_{\mathrm{Nis}}$ in $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ and recalling that $f^{(-)}_{\mathrm{mot}}(X_{\mathrm{Nis}}, *; \mathcal{F})$ has the Brown–Gersten property on X_{Nis}, we have the tower $\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})$ in $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ with value $\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})(X)$ isomorphic to $f^{(-)}_{\mathrm{mot}}(X, *; \mathcal{F})$ in $D(\mathbf{Ab})$, naturally in X for smooth morphisms.

We conclude by defining $\tilde{s}^q_{\mathrm{mot}}(\mathcal{F})$ as the cone of $\tilde{f}^{q+1}_{\mathrm{mot}}(\mathcal{F}) \to \tilde{f}^q_{\mathrm{mot}}(\mathcal{F})$; the isomorphisms $\tilde{f}^{(-)}_{\mathrm{mot}}(\mathcal{F})(X) \cong f^{(-)}_{\mathrm{mot}}(X, *; \mathcal{F})$ extend to an isomorphism in $D(\mathbf{Ab})$

$$\tilde{s}^q_{\mathrm{mot}}(\mathcal{F})(X) \cong s^q_{\mathrm{mot}}(X, *; \mathcal{F})$$

with the same naturality as above.

Step 2. We now just repeat the proof of [**33**, Theorem 7.1.1], replacing presheaves of spectra on $\mathbf{Sm}/k$ with complexes of presheaves $\mathcal{F}$ on $\mathrm{SmCor}(k)$. Making this change, the proofs of the preliminary results [**33**, Theorem 5.3.1 and Lemmas 7.3.1, 7.3.2, 7.3.3 and 7.3.4] as well as the concluding argument following [**33**, Lemma 7.3.4] finish the proof of Proposition 2.2.3. □

Remark 2.2.4. We can now finish the proof of Lemma 1.4.3.

Take $\mathcal{F} \in DM^{\mathrm{eff}}(k)(n)$; we may assume that $\mathcal{F}$ is fibrant in $C(\mathrm{PST}(k))_{\mathbb{A}^1}$. For each $Y \in \mathbf{Sm}/k$, we have isomorphisms in $\mathcal{SH}$:

$$\begin{aligned} s_m(\mathrm{EM}_{\mathbb{A}^1}(\mathcal{F}))(Y) &\cong s^m(Y, -, \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})) \\ &\cong \mathrm{EM}(s^m_{\mathrm{mot}}(Y, *, \mathcal{F})) \\ &\cong \mathrm{EM}(s^{\mathrm{mot}}_m(\mathcal{F})(Y)), \end{aligned}$$

using Proposition 2.2.2, Proposition 2.2.3 and (2.3). But $s^{\mathrm{mot}}_m(\mathcal{F}) = 0$ for $0 \leqslant m < n$, hence $s_m(\mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})) = 0$ for $0 \leqslant m < n$, so $\mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})$ is in $\Sigma^n_T \mathcal{SH}_{S^1}(k)$.

The identity (1.2) is also valid for the truncation functors f^{mot}_n.

Proposition 2.2.5. *For each $n \geqslant 0$, we have a natural isomorphism*

$$\Omega_T \circ f_{n+1}^{\mathrm{mot}} \cong f_n^{\mathrm{mot}} \circ \Omega_T. \tag{2.4}$$

Proof. One repeats the argument for (1.2) given in [**33**, Theorem 7.4.2], changing $\mathbf{Spt}_{S^1}(k)$ to $C(\mathrm{PST}(k))$ throughout, as in the proof of Proposition 2.2.3. □

Remark 2.2.6. As a particular case, Proposition 2.2.2 gives an explicit description of the 0th slice of $E \in \mathbf{Spt}_{S^1}(k)$, assuming E is $\mathbb{A}^1$-homotopy invariant and satisfies Nisnevich excision, as follows. For $Y \in \mathbf{Sm}/k$, $(s_0E)(Y)$ can be described using the cosimplicial scheme of *semi-local ℓ-simplices* $\hat{\Delta}^\ell$ (denoted Δ_0^ℓ in [**33**]). In fact, for $Y \in \mathbf{Sm}/k$, let $\mathcal{O}(\ell)_{k(Y),v}$ be the semi-local ring of the set v of vertices of $\Delta^\ell_{k(Y)}$ and set

$$\hat{\Delta}^\ell_{k(Y)} := \operatorname{Spec} \mathcal{O}(\ell)_{k(Y),v}.$$

Clearly, $\ell \mapsto \hat{\Delta}^\ell_{k(Y)}$ forms a cosimplicial subscheme of $\Delta^*_{k(Y)}$. It follows from Proposition 2.2.2 that $(s_0E)(Y)$ weakly equivalent to total spectrum $E(\hat{\Delta}^*_{k(Y)})$ of the simplicial spectrum

$$\ell \mapsto E(\hat{\Delta}^\ell_{k(Y)}).$$

Proposition 2.2.3 yields a similar description of $s_0^{\mathrm{mot}}\mathcal{F}(Y)$, for $\mathcal{F} \in C(\mathrm{PST}(k))$ which is $\mathbb{A}^1$-homotopy invariant and satisfies Nisnevich excision: $s_0^{\mathrm{mot}}\mathcal{F}(Y)$ is represented by the total complex $\mathcal{F}(\hat{\Delta}^*_{k(Y)})$ associated to the simplicial object of $C(\mathbf{Ab})$

$$\ell \mapsto \mathcal{F}(\hat{\Delta}^\ell_{k(Y)}).$$

The construction in [**25**, Definition 2.14] is closely related to this.

2.3. Miscellaneous results

We conclude this section recalling a few results from [**33**] that will be useful later.

Lemma 2.3.1. *Let $W \subset Y$ be a closed subset, $Y \in \mathbf{Sm}/k$, such that* $\mathrm{codim}_Y W \geqslant q$ *for some integer $q \geqslant 0$. For $E \in \mathcal{SH}_{S^1}(k)$, the canonical map $f_qE \to E$ induces a weak equivalence*

$$(f_qE)^W(Y) \to E^W(Y).$$

Proof. This is [**33**, Lemma 7.3.2]. □

Lemma 2.3.2. *Let E be in $\mathcal{SH}_{S^1}(k)$. Let $W \subset Y$ be a closed subset, $Y \in \mathbf{Sm}/k$.*

(1) *Suppose* $\mathrm{codim}_Y W > q$. *Then $s_q(E)^W(Y) \cong 0$ in $\mathcal{SH}$.*

(2) *Suppose that* $\mathrm{codim}_Y W \geqslant q$. *Let $Y_W^{(q)}$ be the set of points generic points w of W with* $\mathrm{codim}_Y w = q$. *For $y \in Y$, let $Y_y := \operatorname{Spec} \mathcal{O}_{Y,y}$. Then the restriction map*

$$s_q(E)^W(Y) \to \bigoplus_{w \in Y_W^{(q)}} s_q(E)^w(Y_w)$$

is an isomorphism in $\mathcal{SH}$.

Proof. It follows from Lemma 2.3.1 that the canonical map

$$f_{q+1}(E)^W(Y) \to f_q(E)^W(Y)$$

is an isomorphism in $\mathcal{SH}$, hence the cofibre $s_q(E)^W(Y)$ is zero, proving (1). For (2), if $W^0 \subset W$ is a closed subset with $\operatorname{codim}_Y W^0 > q$, then we have the homotopy fibre sequence

$$s_q(E)^{W^0}(Y) \to s_q(E)^W(Y) \to s_q(E)^{W\setminus W^0}(Y \setminus W^0),$$

hence by (1), the restriction map $s_q(E)^W(Y) \to s_q(E)^{W\setminus W^0}(Y \setminus W^0)$ is an isomorphism in $\mathcal{SH}$. Now (2) follows by taking the limit over $W^0 \subset W$. $\square$

For $E \in \mathcal{SH}_{S^1}(k)$, we have the diagram

$$E \xleftarrow{\tau_q} f_qE \xrightarrow{\pi_q} s_qE.$$

Lemma 2.3.3. *Take $E \in \mathcal{SH}_{S^1}(k)$, $X \in \mathbf{Sm}/k$ and integers $q, n \geqslant 0$. For all $p \geqslant q$ the map $\tau_q : f_qE \to E$ induces weak equivalences*

$$f^p(X, n; f_qE) \xrightarrow{\tau_q} f^p(X, n; E),$$
$$s^p(X, n; f_qE) \xrightarrow{\tau_q} s^p(X, n; E).$$

Proof. That $\tau_q : f^p(X, n; f_qE) \to f^p(X, n; E)$ is a weak equivalence follows from Lemma 2.3.1. We have the map of distinguished triangles

$$\begin{tikzcd}
f^{p+1}(X, n; f_qE) \arrow[r] \arrow[d, "\tau_q"] & f^p(X, n; f_qE) \arrow[r] \arrow[d, "\tau_q"] & s^p(X, n; f_qE) \arrow[d, "\tau_q"] \\
f^{p+1}(X, n; E) \arrow[r] & f^p(X, n; E) \arrow[r] & s^p(X, n; E)
\end{tikzcd}$$

hence $\tau_q : s^p(X, n; f_qE) \to s^p(X, n; E)$ is also a weak equivalence. $\square$

Proposition 2.3.4. *Take $E \in \mathcal{SH}_{S^1}(k)$, $X \in \mathbf{Sm}/k$ and an integer $q \geqslant 0$.*

(1) *For all $p \geqslant q$, the map $\tau_q : f_qE \to E$ induces weak equivalences*

$$f^p(X, -; f_qE) \xrightarrow{\tau_q} f^p(X, -; E),$$
$$s^p(X, -; f_qE) \xrightarrow{\tau_q} s^p(X, -; E).$$

(2) *The map $\pi_q : f_q \to s_q$ induces a weak equivalence*

$$s^q(X, -; f_qE) \xrightarrow{\pi_q} s^q(X, -; s_qE).$$

Proof. Item (1) follows from Lemma 2.3.3. For (2), we have the commutative diagram in $\mathcal{SH}$

$$\begin{tikzcd} s^q(X,-;f_qE) \arrow[r, "\pi_q"] \arrow[d, "\beta_{X,q;f_qE}"'] & s^q(X,-;s_qE) \arrow[d, "\beta_{X,q;s_qE}"] \\ s_q(f_qE)(X) \arrow[r, "s_q(\pi_q)"'] & s_q(s_qE)(X) \end{tikzcd}$$

with vertical arrows isomorphisms. The bottom horizontal diagram extends to the distinguished triangle

$$s_q(f_{q+1}E) \to s_q(f_qE) \xrightarrow{s_q(\pi_q)} s_q(s_qE) \to s_q(f_{q+1}E)[1]$$

and we have the defining distinguished triangle for s_q:

$$f_{q+1}(f_{q+1}E) \to f_q(f_{q+1}E) \to s_q(f_{q+1}E) \to f_{q+1}(f_{q+1}E)[1].$$

Since $f_{q+1}E$ is in $\Sigma_T^{q+1}\mathcal{SH}_{S^1}(k) \subset \Sigma_T^q\mathcal{SH}_{S^1}(k)$, the canonical maps

$$f_{q+1}(f_{q+1}E) \to f_{q+1}E, f_q(f_{q+1}E) \to f_{q+1}E$$

are isomorphisms, hence $s_q(f_{q+1}E) \cong 0$ and $s_q(\pi_q)$ is an isomorphism. □

Remark 2.3.5. Making the evident changes, the analogues of Lemma 2.3.1, Lemma 2.3.3 and Proposition 2.3.4 hold for $\mathcal{F} \in DM^{\mathrm{eff}}(k)$.

3. Slices and cycles

We show how, for special objects in $\mathbf{Spt}_{S^1}(k)$, the *well-connected* spectra, the slices in the motivic Postnikov tower have a cycle-theoretic description via a generalization of Bloch's cycle complex. This material is taken largely from [**33**, §§ 5, 6].

3.1. Connected spectra

We continue to assume the field k is perfect.

Definition 3.1.1. Call $E \in \mathcal{SH}_{S^1}(k)$ *connected* if for each $X \in \mathbf{Sm}/k$, the spectrum $\tilde{E}(X)$ is -1 connected, where $\tilde{E} \in \mathbf{Spt}_{S^1}(k)$ is a fibrant model for E.

Note that this is a global, quite strong notion.

Lemma 3.1.2. *Let $E \in \mathcal{SH}_{S^1}(k)$ be connected. Then*

(1) *For each $q \geqslant 0$, $\Omega_T^q E$ is connected.*

(2) *For $X \in \mathbf{Sm}/k$ and $W \subset X$ a closed subset, the spectrum with supports $E^W(X)$ is -1 connected.*

(3) *Let $j : U \to X$ be an open immersion in* $\mathbf{Sm}/k$, *$W \subset X$ a closed subset. Then*

$$j^* : \pi_0(E^W(X)) \to \pi_0(E^{W\cap U}(U))$$

is surjective.

Proof. For (1) it suffices to prove the case $q = 1$. Take $X \in \mathbf{Sm}/k$. Since $\infty \hookrightarrow \mathbb{P}^1$ is split by $\mathbb{P}^1 \to \operatorname{Spec} k$, $(\Omega_T E)(X)$ is a retract of $E(X \times \mathbb{P}^1)$. Since $E(X \times \mathbb{P}^1)$ is -1 connected by assumption, it follows that $(\Omega_T E)(X)$ is also -1 connected, hence $\Omega_T E$ is connected.

For (2), suppose first that $i : W \to X$ is a closed immersion in $\mathbf{Sm}/k$ and that the normal bundle ν of W in X admits a trivialization, $\nu \cong \mathcal{O}_W^q$. We have the Morel–Voevodsky purity isomorphism (2.1)

$$E^W(X) \cong (\Omega_T^q E)(W).$$

By (1) $(\Omega_T^q E)(W)$ is -1 connected, verifying (2) in this case.

In general, we proceed by descending induction on $\operatorname{codim}_X W$, starting with the trivial case $\operatorname{codim}_X W = \dim_k X + 1$, i.e. $W = \emptyset$. In general, suppose that $\operatorname{codim}_X W \geqslant q$ for some integer $q \leqslant \dim_k X$. Then there is a closed subset $W' \subset W$ with $\operatorname{codim}_X W' > q$ such that $W \setminus W'$ is smooth and has trivial normal bundle in $X \setminus W'$. We have the homotopy fibre sequence

$$E^{W'}(X) \to E^W(X) \to E^{W\setminus W'}(X \setminus W')$$

thus the induction hypothesis, and the -1 connectedness of $E^{W\setminus W'}(X \setminus W')$ implies that $E^W(X)$ is -1 connected.

Item (3) follows from the homotopy fibre sequence (note that $\tilde{E}$ satisfies Zariski excision)

$$E^{W\setminus U}(X) \to E^W(X) \to E^{W\cap U}(U)$$

and the -1 connectedness of $E^{W\setminus U}(X)$. □

Lemma 3.1.3. *Suppose $E \in \mathcal{SH}_{S^1}(k)$ is connected. Then for $X \in \mathbf{Sm}/k$ and every $q, n \geqslant 0$, $f^q(X, n; E)$ and $s^q(X, n; E)$ are -1 connected.*

Proof. This follows from Lemma 3.1.2 (2), noting that $f^q(X, n; E)$ and $s^q(X, n; E)$ are colimits over spectra with supports $E^W(X \times \Delta^n)$, $E^{W\setminus W'}(X \times \Delta^n \setminus W')$. □

Proposition 3.1.4. *Suppose $E \in \mathcal{SH}_{S^1}(k)$ is connected. Then for every $q \geqslant 0$, $f_q E$ and $s_q E$ are connected.*

Proof. Take $X \in \mathbf{Sm}/k$. By Proposition 2.2.2, we have isomorphism in $\mathcal{SH}$:

$$f_q E(X) \cong f^q(X, -; E), \qquad s_q E(X) \cong s^q(X, -; E).$$

By Lemma 3.1.3, the total spectra $f^q(X, -; E)$ and $s^q(X, -; E)$ are -1 connected, whence the result. □

Definition 3.1.5. Fix an integer $q \geqslant 0$ and let $W \subset Y$ be a closed subset with $Y \in \mathbf{Sm}/k$ and $\operatorname{codim}_Y W \geqslant q$. For $E \in \mathcal{SH}_{S^1}(k)$, define the *comparison map*

$$\psi^E_W(Y) : \pi_0(E^W(Y)) \to \pi_0(s_q(E)^W(Y))$$

as the composition

$$\pi_0(E^W(Y)) \xleftarrow{\sim} \pi_0((f_qE)^W(Y)) \to \pi_0(s_q(E)^W(Y)),$$

noting that $\pi_0((f_qE)^W(Y)) \to \pi_0(E^W(Y))$ is an isomorphism by Lemma 2.3.1.

Lemma 3.1.6. *Let $w \in Y^{(q)}$ be a codimension q point of $Y \in \mathbf{Sm}/k$ and let $Y_w := \operatorname{Spec} \mathcal{O}_{Y,w}$. Take $E \in \mathcal{SH}_{S^1}(k)$ and suppose that E is connected. Then the comparison map*

$$\psi^E_w(Y_w) : \pi_0(E^w(Y_w)) \to \pi_0(s_q(E)^w(Y_w))$$

is an isomorphism.

Proof. Recall from Remark 2.2.6 the cosimplicial subscheme $\hat{\Delta}^*_{k(Y)}$ of $\Delta^*_{k(Y)}$.

Since $\hat{\Delta}^0_{k(Y)} = \operatorname{Spec} k(Y)$, we have the natural map

$$\pi_0((\Omega^q_T E)(k(Y))) \to \pi_0((\Omega^q_T E)(\hat{\Delta}^*_{k(Y)})),$$

which is an isomorphism. Indeed, by Lemma 3.1.2 (1), $\Omega^q_T E$ is connected for all $q \geqslant 0$. In particular, $(\Omega^q_T E)(\hat{\Delta}^n_{k(Y)})$ is -1 connected for all Y and all $n \geqslant 0$. Thus we have the presentation of $\pi_0((\Omega^q_T E)(\hat{\Delta}^*_{k(Y)}))$:

$$\pi_0((\Omega^q_T E)(\hat{\Delta}^1_{k(Y)})) \xrightarrow{i_0^* - i_1^*} \pi_0((\Omega^q_T E)(k(Y))) \to \pi_0((\Omega^q_T E)(\hat{\Delta}^*_{k(Y)})) \to 0.$$

By Lemma 3.1.2 (3) and a limit argument, the map

$$\pi_0((\Omega^q_T E)(\Delta^1_{k(Y)})) \to \pi_0((\Omega^q_T E)(\hat{\Delta}^1_{k(Y)}))$$

is surjective; since $\Delta^1_{k(Y)} = \mathbb{A}^1_{k(Y)}$ and $\Omega^q_T E$ is homotopy invariant, the map $i_0^* - i_1^*$ is the zero map.

Choose a trivialization of the normal bundle ν of $w \in Y_w$, $k(w)^q \cong \nu$. This gives us the purity isomorphisms $E^w(Y_w) \cong (\Omega^q_T E)(w)$, $(s_qE)^w(Y_w) \cong s_0(\Omega^q_T E)(w)$; from Remark 2.2.6 we have the isomorphism $s_0(\Omega^q_T E)(w) \cong (\Omega^q_T E)(\hat{\Delta}^*_{k(w)})$. This gives us the commutative diagram

$$\begin{tikzcd}
\pi_0(E^w(Y_w)) \arrow[r, "\psi^E_w(Y_w)"] \arrow[d] & \pi_0(s_q(E)^w(Y_w)) \arrow[d] \\
\pi_0(\Omega^q_T E(w)) \arrow[r] & \pi_0((\Omega^q_T E)(\hat{\Delta}^*_{k(w)}))
\end{tikzcd}$$

with the two vertical arrows and the bottom horizontal arrow isomorphisms. Thus $\psi^E_w(Y_w)$ is an isomorphism. $\square$

Lemma 3.1.7. *Suppose $E \in \mathcal{SH}_{S^1}(k)$ is connected. Fix an integer $q \geqslant 0$ and let $W \subset Y$ be a closed subset, with $Y \in \mathbf{Sm}/k$ and* $\operatorname{codim}_Y W \geqslant q$. *Then the comparison map*

$$\psi^E_W(Y) : \pi_0(E^W(Y)) \to \pi_0(s_q(E)^W(Y))$$

is surjective.

Proof. Recall that $Y^{(q)}_W$ denotes the set of generic points w of W with $\operatorname{codim}_Y w = q$. Let $Y_W := \operatorname{Spec} \mathcal{O}_{Y,Y^{(q)}_W}$. By Lemma 2.3.2, the restriction map

$$s_q(E)^W(Y) \to \coprod_{w \in Y^{(q)}_W} s_q(E)^w(Y_W)$$

is a weak equivalence. By Lemma 3.1.6,

$$\psi^E_w(Y_W) : \pi_0(E^w(Y_W)) \to \pi_0(s_q(E)^w(Y_W))$$

is an isomorphism for all $w \in Y^{(q)}_W$. Thus we have the commutative diagram

$$\begin{tikzcd}
\pi_0(E^W(Y)) \arrow[r, "\psi^E_W(Y)"] \arrow[d] & \pi_0(s_q(E)^W(Y)) \arrow[d] \\
\bigoplus_{w \in Y^{(q)}_W} \pi_0(E^w(Y_W)) \arrow[r, "\Sigma_w \psi^E_w(Y_W)"'] & \bigoplus_{w \in Y^{(q)}_W} \pi_0(s_q(E)^w(Y_W))
\end{tikzcd}$$

By Lemma 2.3.2, the right-hand vertical arrow is an isomorphism; the bottom horizontal arrow is an isomorphism by Lemma 3.1.6. It follows from Lemma 3.1.2 (3) that the left-hand vertical arrow is surjective, hence $\psi^E_W(Y)$ is surjective as well. □

Lemma 3.1.8. *Suppose that $E \in \mathcal{SH}_{S^1}(k)$ is connected. Take $Y \in \mathbf{Sm}/k$, $w \in Y^{(q)}$ and let $Y_w := \operatorname{Spec} \mathcal{O}_{Y,w}$. Then the purity isomorphism*

$$\theta_{\varphi,0,E} : \pi_0(E^w(Y_w)) \to \pi_0(\Omega^q_T E(w))$$

is independent of the choice of trivialization φ.

Proof. We have the commutative diagram of isomorphisms

$$\begin{tikzcd}
\pi_0(E^w(Y_w)) \arrow[r, "\psi^E_w(Y_w)"] \arrow[d, "\theta_{\varphi,0E}"'] & \pi_0(s_q(E)^w(Y_w)) \arrow[d, "\theta_{\varphi,0,s_qE}"] \\
\pi_0(\Omega^q_T E(w)) \arrow[r] & \pi_0((\Omega^q_T E)(\hat{\Delta}^*_{k(w)}))
\end{tikzcd}$$

By [**33**, Corollary 4.2.4], $\theta_{\varphi,0,s_qE}$ is independent of the choice of φ, whence the result. □

Take $E \in \mathcal{SH}_{S^1}(k)$ connected. For each closed subset $W \subset Y$, $Y \in \mathbf{Sm}/k$, $E^W(Y)$ is -1 connected, giving us the canonical map

$$\rho_{E,Y,W} : E^W(Y) \to \mathrm{EM}(\pi_0(E^W(Y))).$$

Definition 3.1.9. Let $E \in \mathcal{SH}_{S^1}(k)$ be connected. Let Y be in $\mathbf{Sm}/k$ and let $W \subset Y$ be a closed subset of codimension $\geqslant q$. The *cycle map*

$$\mathrm{cyc}_E^W(Y) : E^W(Y) \to \mathrm{EM}\left(\bigoplus_{w \in Y_W^{(q)}} \pi_0((\Omega_T^q E)(w))\right)$$

is the composition

$$\begin{aligned} E^W(Y) &\xrightarrow{\rho_{E,Y,W}} \mathrm{EM}(\pi_0(E^W(Y))) \\ &\xrightarrow{\mathrm{res}} \mathrm{EM}\left(\bigoplus_{w \in Y_W^{(q)}} \pi_0(E^w(Y_w))\right) \\ &\xrightarrow{\theta_{\varphi,0,E}} \mathrm{EM}\left(\bigoplus_{w \in Y_W^{(q)}} \pi_0((\Omega_T^q E)(w))\right). \end{aligned}$$

We let

$$\pi_0(\mathrm{cyc}_E^W(Y)) : \pi_0(E^W(Y)) \to \bigoplus_{w \in Y_W^{(q)}} \pi_0((\Omega_T^q E)(w))$$

be the map on π_0 induced by $\mathrm{cyc}_E^W(Y)$.

Definition 3.1.10. Let $E \in \mathcal{SH}_{S^1}(k)$ be connected. For $X \in \mathbf{Sm}/k$ and integers $q, n \geqslant 0$ define

$$z^q(X, n; E) := \bigoplus_{w \in X^{(q)}(n)} \pi_0((\Omega_T^q E)(w)).$$

Taking the limit of the maps $\mathrm{cyc}_E^{W \setminus W'}(X \times \Delta^n \setminus W')$ for $E \in \mathcal{SH}_{S^1}(k)$ connected, $W \in \mathcal{S}_X^{(q)}(n)$, $W' \in \mathcal{S}_X^{(q+1)}(n)$ we have the maps of spectra

$$\mathrm{cyc}_E(X, n) : s^q(X, n; E) \to \mathrm{EM}(z^q(X, n; E)) \tag{3.1}$$

and the maps of abelian groups

$$\pi_0(\mathrm{cyc}_E(X, n)) : \pi_0(s^q(X, n; E)) \to z^q(X, n; E).$$

Lemma 3.1.11. *Let $E \in \mathcal{SH}_{S^1}(k)$ be connected and let X be in $\mathbf{Sm}/k$. Then*

$$\pi_0(\mathrm{cyc}_{s_q E}(X, n)) : \pi_0(s^q(X, n; s_q E)) \to z^q(X, n; s_q E)$$

is an isomorphism.

Proof. First note that, by Proposition 3.1.4, s_qE is connected, hence all terms in the statement are defined. By Lemma 2.3.2, the restriction map

$$\pi_0((s_qE)^W(Y)) \to \bigoplus_{w\in Y_W^{(q)}} \pi_0((s_qE)^w(Y_W))$$

is an isomorphism; since $\pi_0(\mathrm{cyc}_{s_qE}(X,n))$ is constructed by composing restriction maps with purity isomorphisms, this proves the result. □

Proposition 3.1.12. *Let $E \in \mathcal{SH}_{S^1}(k)$ be connected and let X be in $\mathbf{Sm}/k$. There is a unique structure of a simplicial abelian group*

$$n \mapsto z^q(X,n;E)$$

such that the maps $\pi_0(\mathrm{cyc}_E(X,n))$ define a map of simplicial abelian groups

$$[n \mapsto \pi_0(s^q(X,n;E))] \xrightarrow{\pi_0(\mathrm{cyc}_E(X,-))} [n \mapsto z^q(X,n;E)].$$

Proof. Since E is connected, the cycle maps

$$\pi_0(E^W(Y)) \xrightarrow{\mathrm{res}} \bigoplus_{w\in Y_W^{(q)}} \pi_0(E^w(Y_W)) \cong \bigoplus_{w\in Y_W^{(q)}} \pi_0(\Omega_T^q E(w))$$

are surjective. Thus $\pi_0(\mathrm{cyc}_E(X,n))$ is surjective, which proves the uniqueness.

For existence, the map $\pi_0(\mathrm{cyc}_E(X,n))$ is natural with respect to E. In addition, by Proposition 3.1.4, both f_qE and s_qE are connected; applying $\pi_0(\mathrm{cyc}_?(X,n))$ to the diagram

$$E \leftarrow f_qE \to s_qE$$

gives the commutative diagram

$$\begin{tikzcd}
\pi_0(s^q(X,n;E)) \arrow[d, "\pi_0(\mathrm{cyc}_E(X,n))"'] & \pi_0(s^q(X,n;f_qE)) \arrow[l] \arrow[r] \arrow[d, "\pi_0(\mathrm{cyc}_{f_qE}(X,n))"'] & \pi_0(s^q(X,n;s_qE)) \arrow[d, "\pi_0(\mathrm{cyc}_{s_qE}(X,n))"'] \\
z^q(X,n;E) & z^q(X,n;f_qE) \arrow[l] \arrow[r] & z^q(X,n;s_qE)
\end{tikzcd}$$

By Lemma 2.3.3, the left-hand map in the top row is an isomorphism. The maps in the bottom row are induced by maps

$$\pi_0((\Omega_T^q E)(w)) \leftarrow \pi_0((\Omega_T^q f_qE)(w)) \to \pi_0((\Omega_T^q s_qE)(w)).$$

By (1.2), $\Omega_T^q f_qE = f_0(\Omega_T^q f_qE) = \Omega_T^q E$ and similarly $\Omega_T^q s_qE = s_0(\Omega_T^q E)$. Thus the bottom row is a sum of isomorphisms (see Lemma 3.1.6)

$$\pi_0((\Omega_T^q E)(w)) \to \pi_0(s_0(\Omega_T^q E)(w)).$$

Finally, the right-hand vertical map is an isomorphism by Lemma 3.1.11. As the top row is the degree n part of a diagram of maps of simplicial abelian groups, the isomorphisms

$$\pi_0(s^q(X,n;s_qE)) \to z^q(X,n;s_qE) \leftarrow z^q(X,n;E)$$

induce the structure of a simplicial abelian group from $[n \mapsto \pi_0(s^q(X, n; s_q E))]$ to $[n \mapsto z^q(X, n; E)]$, so that the maps $\pi_0(\mathrm{cyc}_E(X, n))$ define a map of simplicial abelian groups. $\square$

Remark 3.1.13. For $\mathcal{F} \in C(\mathrm{PST}(k))$, we call $\mathcal{F}$ connected if $\mathbb{H}^n(X_{\mathrm{Nis}}, \mathcal{F}) = 0$ for $n > 0$, $X \in \mathbf{Sm}/k$. Making the obvious modifications, all the results of this section carry over from $\mathcal{SH}_{S^1}(k)$ to $DM^{\mathrm{eff}}(k)$.

We use the above results to give a generalization of the higher cycle complexes of Bloch.

Definition 3.1.14. Let $E \in \mathbf{Spt}_{S^1}(k)$ be connected, homotopy invariant and satisfy Nisnevich excision. For $X \in \mathbf{Sm}/k$, and $q, n \geqslant 0$ integers, let $z^q(X, *; E)$ be the complex associated to the simplicial abelian group $n \mapsto z^q(X, n; E)$. Similarly, for $\mathcal{F} \in C(\mathrm{PST}(k))$ which is connected, homotopy invariant and satisfies Nisnevich excision, we set

$$z^q(X, n; \mathcal{F}) = \bigoplus_{w \in X^{(q)}(n)} H^0((\Omega_T^q \mathcal{F})(w)),$$

giving the simplicial abelian group $n \mapsto z^q(X, n; \mathcal{F})$. We denote the associated complex by $z^q(X, *; \mathcal{F})$.

For integers $q, n \geqslant 0$, set

$$\mathrm{CH}^q(X, n; E) := H_n(z^q(X, *; E))$$

and

$$\mathrm{CH}^q(X, n; \mathcal{F}) := H_n(z^q(X, *; \mathcal{F})).$$

For $\mathcal{F} \in C(\mathrm{PST}(k))$ as above, we note that $z^q(X, *, \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F}))$ is naturally isomorphic to $z^q(X, *, \mathcal{F})$, via the canonical isomorphisms

$$\begin{aligned} H^0((\Omega_T^q \mathcal{F})(w)) &\cong \pi_0(EM((\Omega_T^q \mathcal{F})(w))) \\ &\cong \pi_0((\mathrm{EM}_{\mathbb{A}^1}\, \Omega_T^q \mathcal{F})(w)) \\ &\cong \pi_0((\Omega_T^q\, \mathrm{EM}_{\mathbb{A}^1}\, \mathcal{F})(w)). \end{aligned}$$

Definition 3.1.15. Take $X \in \mathbf{Sm}/k$. For connected $E \in \mathcal{SH}_{S^1}(k)$, let

$$\mathrm{cyc}_E(X) : s^q(X, -; E) \to \mathrm{EM}(z^q(X, -; E))$$

be the map of spectra induced by the maps (3.1); this is well-defined by Proposition 3.1.12. Similarly, for connected $\mathcal{F} \in DM^{\mathrm{eff}}(k)$, let

$$\mathrm{cyc}_{\mathcal{F}}(X) : s^q_{\mathrm{mot}}(X, *; \mathcal{F}) \to z^q(X, *; \mathcal{F})$$

be the map of complexes induced by the maps $\mathrm{cyc}_{\mathcal{F}}(X, n)$ analogous to the maps (3.1).

3.2. Well-connected spectra

Following [**33**] we have the following definition.

Definition 3.2.1. $E \in \mathcal{SH}_{S^1}(k)$ is *well-connected* if

(1) E is connected;

(2) for each $Y \in \mathbf{Sm}/k$, and each $q \geqslant 0$, the total spectrum $(\Omega_T^q E)(\hat{\Delta}^*_{k(Y)})$ has

$$\pi_n((\Omega_T^q E)(\hat{\Delta}^*_{k(Y)})) = 0$$

for $n \neq 0$.

Remark 3.2.2. The corresponding notion in $DM^{\mathrm{eff}}(k)$ is Let $\mathcal{F} \in C(\mathrm{PST}(k))$ be $\mathbb{A}^1$ homotopy invariant and satisfy Nisnevich excision. Call $\mathcal{F}$ well-connected if

(1) $\mathcal{F}$ is connected;

(2) for each $Y \in \mathbf{Sm}/k$, the total complex $(\Omega_T^q \mathcal{F})(\hat{\Delta}^*_{k(Y)})$ satisfies

$$H^n((\Omega_T^q \mathcal{F})(\hat{\Delta}^*_{k(Y)})) = 0$$

for $n \neq 0$.

Remark 3.2.3. We gave a slightly different definition of well-connectedness in [**33**, Definition 6.1.1], replacing the connectedness condition (1) with: $E^W(Y)$ is -1 connected for all closed subsets $W \subset Y$, $Y \in \mathbf{Sm}/k$. By Lemma 3.1.2, this condition is equivalent with the connectedness of E.

The main result on well-connected spectra is the following theorem.

Theorem 3.2.4.

(1) *Suppose $E \in \mathcal{SH}_{S^1}(k)$ is well-connected. Then*

$$\mathrm{cyc}_E(X) : s^q(X, -; E) \to \mathrm{EM}(z^q(X, -; E))$$

is a weak equivalence for each $X \in \mathbf{Sm}/k$. In particular, there is a natural isomorphism

$$\mathrm{CH}^q(X, n; E) \cong \pi_n((s_q E)(X)) \cong \mathrm{Hom}_{\mathcal{SH}_{S^1}(k)}(\Sigma_T^\infty X_+, \Sigma_s^{-n} s_q(E)).$$

(2) *Suppose $\mathcal{F} \in C(\mathrm{PST}(k))$ is well-connected. Then*

$$\mathrm{cyc}^{\mathrm{mot}}_{\mathcal{F}}(X) : s^q_{\mathrm{mot}}(X, *; \mathcal{F}) \to z^q(X, *; \mathcal{F})$$

is a quasi-isomorphism for each $X \in \mathbf{Sm}/k$. In particular, there is a natural isomorphism

$$\mathrm{CH}^q(X, n; \mathcal{F}) \cong \mathbb{H}^{-n}_{\mathrm{Nis}}(X, s_q^{\mathrm{mot}} \mathcal{F}) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), s_q^{\mathrm{mot}}(\mathcal{F})[-n]).$$

Proof. We prove (1), the proof of (2) is the same. We have the commutative diagram in $\mathcal{SH}$:

$$\begin{tikzcd}
s^q(X,-;E) \arrow[d, "\mathrm{cyc}_E(X)"'] & s^q(X,-;f_qE) \arrow[l, "\tau_q"'] \arrow[r, "\pi_q"] \arrow[d, "\mathrm{cyc}_{f_qE}(X)"'] & s^q(X,-;s_qE) \arrow[d, "\mathrm{cyc}_{s_qE}(X)"] \\
\mathrm{EM}(z^q(X,-;E)) & \mathrm{EM}(z^q(X,-;f_qE)) \arrow[l, "\tau_q"] \arrow[r, "\pi_q"'] & \mathrm{EM}(z^q(X,-;s_qE))
\end{tikzcd}$$

By Proposition 2.3.4, the arrows in the top row are isomorphisms. As we have seen in the proof of Proposition 3.1.12 the arrows in the bottom row are also isomorphisms. Thus, it suffices to prove the result with E replaced by s_qE.

The map $\mathrm{cyc}_{s_qE}(X)$ is just the map on total spectra induced by the map on n-simplices

$$\mathrm{cyc}_{s_qE}(X,n) : s^q(X,n;s_qE) \to \mathrm{EM}(z^q(X,n;s_qE)).$$

By Lemma 3.1.11, the map on π_0,

$$\pi_0(\mathrm{cyc}_{s_qE}(X,n)) : \pi_0(s^q(X,n;s_qE)) \to z^q(X,n;s_qE),$$

is an isomorphism. However, since E is well-connected, and since

$$s^q(X,n;s_qE) \cong \coprod_{w\in X^{(q)}(n)} (\Omega^q_T s_qE)(k(w)) \cong \coprod_{w\in X^{(q)}(n)} s_0(\Omega^q_T E)(k(w)),$$

it follows that

$$s^q(X,n;s_qE) = \mathrm{EM}(\pi_0(s^q(X,n;s_qE))),$$

and $\mathrm{cyc}_{s_qE}(X,n)$ is the map induced by $\pi_0(\mathrm{cyc}_{s_qE}(X,n))$. Thus $\mathrm{cyc}_{s_qE}(X,n)$ is a weak equivalence for every n, hence $\mathrm{cyc}_{s_qE}(X)$ is an isomorphism in $\mathcal{SH}$, as desired. □

We recall that the functoriality results of [**32**, Theorems 2.6.2 and 7.4.1] applied to the spectra $s^q(X,-;E)$ gives a presheaf of spectra $\tilde{s}^q(E)$ on $\mathbf{Sm}/k$, together with isomorphisms

$$\gamma_{q,X,E} : s^q(X,-;E) \to \tilde{s}^q(E)(X)$$

in $\mathcal{SH}$, natural in X for smooth maps in $\mathbf{Sm}/k$. In addition, by [**33**, Theorem 7.1.1], there is an isomorphism

$$\varphi_{q,E} : \tilde{s}^q(E) \to s_q(E)$$

in $\mathcal{H}\,\mathbf{Spt}_{S^1}(k)$ and the isomorphism $\beta_{X,q;E}$ of Proposition 2.2.2 is the composition $\varphi_{q,E}(X) \circ \gamma_{q,X,E}$.

Using the same methods, we extend the assignment $X \mapsto z^q(X,-;E)$ to a presheaf $X \mapsto z^q(E)(X)$ of simplicial abelian groups on $\mathbf{Sm}/k$, together with isomorphisms

$$\delta_{q,X,E} : z^q(X,-;E) \to \tilde{z}^q(E)(X)$$

in the homotopy category of $s\,\mathbf{Ab}$, natural in X for smooth maps in $\mathbf{Sm}/k$. It follows from the naturality of the functorial models of [**32**, Theorems 2.6.2 and 7.4.1] that we have the canonical maps of presheaves on $\mathbf{Sm}/k$

$$\mathrm{cyc}_E : \tilde{s}^q(E) \to \mathrm{EM}(\tilde{z}^q(E)),$$

giving for each $X \in \mathbf{Sm}/k$ the commutative diagram

$$\begin{tikzcd} s^q(X,-;E) \arrow[r, "\gamma_{q,X,E}"] \arrow[d, "\mathrm{cyc}_E(X)"'] & \tilde{s}^q(E)(X) \arrow[d, "\mathrm{cyc}_E(X)"] \\ \mathrm{EM}(z^q(X,-;E)) \arrow[r, "\mathrm{EM}(\delta_{q,X,E})"'] & \mathrm{EM}(\tilde{z}^q(E))(X) \end{tikzcd}$$

Similarly, using the functoriality machinery of [**32**, Theorems 2.6.2 and 7.4.1], and the comparison results of [**33**, Theorem 7.1.1], extended as explained in the proof of Proposition 2.2.3, we can extend the assignments $X \mapsto s^q_{\mathrm{mot}}(X,*;\mathcal{F})$, $X \mapsto z^q(X,*;\mathcal{F})$ to objects of $C(\mathrm{PST}(k))$, $\tilde{s}^q_{\mathrm{mot}}(\mathcal{F})$, $\tilde{z}^q(\mathcal{F})$, together with isomorphisms

$$\gamma^{\mathrm{mot}}_{q,X,E} : s^q_{\mathrm{mot}}(X,*;\mathcal{F}) \to \tilde{s}^q_{\mathrm{mot}}(\mathcal{F})(X),$$
$$\delta^{\mathrm{mot}}_{q,X,\mathcal{F}} : z^q(X,*;\mathcal{F}) \to \tilde{z}^q(\mathcal{F})(X)$$

in $D(\mathbf{Ab})$, natural in X for smooth maps in $\mathbf{Sm}/k$. We also have an isomorphism

$$\varphi^{\mathrm{mot}}_{q,\mathcal{F}} : \tilde{s}^q_{\mathrm{mot}}(\mathcal{F}) \to s^{\mathrm{mot}}_q(\mathcal{F})$$

in the derived category $D(\mathrm{PST}(k))$, such that the isomorphism $\beta^{\mathrm{mot}}_{X,q;\mathcal{F}}$ of Proposition 2.2.3 is the composition $\varphi^{\mathrm{mot}}_{q,\mathcal{F}}(X) \circ \gamma^{\mathrm{mot}}_{q,X,\mathcal{F}}$.

In addition, the maps $\mathrm{cyc}^{\mathrm{mot}}_{\mathcal{F}}(X)$ extend to maps in $C(\mathrm{PST}(k))$

$$\mathrm{cyc}^{\mathrm{mot}}_{\mathcal{F}} : \tilde{s}^q_{\mathrm{mot}}(\mathcal{F}) \to \tilde{z}^q(\mathcal{F})$$

compatible with the maps $\mathrm{cyc}^{\mathrm{mot}}_{\mathcal{F}}(X) : s^q_{\mathrm{mot}}(X,*;\mathcal{F}) \to z^q(X,*;\mathcal{F})$ via the isomorphisms γ^{mot}, δ^{mot}. Theorem 3.2.4 thus yields the following corollary.

Corollary 3.2.5.

(1) *Suppose $E \in \mathcal{SH}_{S^1}(k)$ is well-connected. Then*

$$\mathrm{cyc}_E \circ \varphi^{-1}_{q,E} : s_q(E) \to \mathrm{EM}(\tilde{z}^q(E))$$

is an isomorphism in $\mathcal{SH}_{S^1}(k)$.

(2) *Suppose $\mathcal{F} \in C(\mathrm{PST}(k))$ is well-connected. Then*

$$\mathrm{cyc}^{\mathrm{mot}}_{\mathcal{F}} \circ (\varphi^{\mathrm{mot}}_{q,\mathcal{F}})^{-1} : s^{\mathrm{mot}}_q(\mathcal{F}) \to \tilde{z}^q(\mathcal{F})$$

is an isomorphism in $DM^{\mathrm{eff}}(k)$.

4. Birational motives and higher Chow groups

Birational motives have been introduced and studied by Kahn and Sujatha in [**27**] and by Huber and Kahn in [**22**]. In this section we reexamine their theory, emphasizing the relation to the slices in the motivic Postnikov tower. We also extend Bloch's construction of cycle complexes and higher Chow groups: Bloch's construction may be considered as the case of the cycle complex with constant coefficients $\mathbb{Z}$ whereas our generalization allows the coefficients to be in a *birational motivic sheaf*. Finally, we extend the identification of Bloch's higher Chow groups with motivic cohomology [**17**, **60**] to the setting of birational motivic sheaves.

4.1. Birational motives

Definition 4.1.1. A motive $\mathcal{F} \in DM^{\mathrm{eff}}(k)$ is called *birational* if for every dense open immersion $j : U \to X$ in $\mathbf{Sm}/k$ and every integer n, the map

$$j^* : \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) \to \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(U), \mathcal{F}[n])$$

is an isomorphism. If $\mathcal{F}$ is a sheaf, i.e. $\mathcal{F} \cong \mathcal{H}^0(\mathcal{F})$ in $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$, we call $\mathcal{F}$ a *birational motivic sheaf.*

Remark 4.1.2. For $X \in \mathbf{Sm}/k$ and $\mathcal{F} \in DM^{\mathrm{eff}}(k) \subset D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$, there is a natural isomorphism

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) \cong \mathbb{H}^n_{\mathrm{Nis}}(X, \mathcal{F}).$$

Thus a motive $\mathcal{F} \in DM^{\mathrm{eff}}(k)$ is birational if and only if the hypercohomology presheaf

$$U \mapsto \mathbb{H}^n_{\mathrm{Nis}}(U, \mathcal{F})$$

on X_{Zar} is the constant presheaf on each connected component of X, for all $X \in \mathbf{Sm}/k$.

Lemma 4.1.3. *Let $\mathcal{F}$ be a presheaf with transfers that is birational and homotopy invariant. Then $\mathcal{F}$ is a birational motivic sheaf.*

Proof. Any presheaf of sets $\mathcal{G}$ on $\mathbf{Sm}/k$ which transforms coproducts into products and is birationally invariant in the sense that $\mathcal{G}(X) \xrightarrow{\sim} \mathcal{G}(U)$ for any open immersion $U \hookrightarrow X$ is a sheaf for the Nisnevich topology: this follows from the fact that the Nisnevich topology is generated by elementary Nisnevich covers, see [**41**, Proposition 1.4, p. 96]. This shows that $\mathcal{F}$ is a Nisnevich sheaf with transfers. Then we have

$$\mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))}(\mathbb{Z}^{\mathrm{tr}}(X), \mathcal{F}[n]) = H^n_{\mathrm{Nis}}(X, \mathcal{F});$$

the Nisnevich cohomology $H^n_{\mathrm{Nis}}(X, \mathcal{F})$ is zero for $n > 0$ by Lemma 4.1.4 below. In particular, $\mathcal{F}$ is strictly homotopy invariant and thus an object of $DM^{\mathrm{eff}}_{-}(k) \subset DM^{\mathrm{eff}}(k)$. Finally,

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) = \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))}(\mathbb{Z}^{\mathrm{tr}}(X), \mathcal{F}[n]) = \begin{cases} \mathcal{F}(X) & \text{for } n = 0, \\ 0 & \text{for } n \neq 0, \end{cases}$$

hence $\mathcal{F}$ is a birational motive. □

Lemma 4.1.4 (Riou). *Let X be a Noetherian scheme, and let $\mathcal{F}$ be a Nisnevich sheaf of abelian groups over X. Assume that $\mathcal{F}$ is flasque viewed as a Zariski sheaf, i.e. for any open immersion $V \hookrightarrow U$ in the small Nisnevich site of X, the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Then $H^i_{\mathrm{Nis}}(X, \mathcal{F}) = 0$ for all $i > 0$.*

The proof is an elaboration of Godement's proof for the Zariski topology: see [**50**, Lemma 1.40].

4.2. The Postnikov tower for birational motives

In this section, we give a treatment of the slices of a birational motive. These results are obtained in [**27**]; here we develop part of the theory of [**27**] in a slightly different and independent way.

Let $\mathcal{F}$ be in $DM^{\mathrm{eff}}(k)$. Since $f_0^{\mathrm{mot}}\mathcal{F} \to \mathcal{F}$ is an isomorphism, we have the canonical map

$$\pi_0 : \mathcal{F} \to s_0^{\mathrm{mot}}\mathcal{F}.$$

The following result is taken from [**27**] in slightly modified form.

Theorem 4.2.1. *For $\mathcal{F}$ in $DM^{\mathrm{eff}}(k)$, $\pi_0 : \mathcal{F} \to s_0^{\mathrm{mot}}\mathcal{F}$ is an isomorphism if and only if $\mathcal{F}$ is a birational motive. In particular, since $s_0^{\mathrm{mot}}\mathcal{F} = s_0^{\mathrm{mot}}(s_0^{\mathrm{mot}}\mathcal{F})$, $s_0^{\mathrm{mot}}\mathcal{F}$ is a birational motive.*

Proof. Since we have the distinguished triangle

$$f_1^{\mathrm{mot}}\mathcal{F} \to \mathcal{F} \xrightarrow{\pi_0} s_0^{\mathrm{mot}}\mathcal{F} \to f_1^{\mathrm{mot}}\mathcal{F}[1],$$

π_0 is an isomorphism if and only if $f_1^{\mathrm{mot}}\mathcal{F} \cong 0$.

Suppose that π_0 is an isomorphism. Let $j : U \to X$ be a dense open immersion in $\mathbf{Sm}/k$ and let $W = X \setminus U$. We show that

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) \xrightarrow{j^*} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(U), \mathcal{F}[n])$$

is an isomorphism by induction on $\mathrm{codim}_X W$, starting with $W = \emptyset$. We may assume that X is irreducible.

By induction we may assume that W is smooth of codimension $d \geqslant 1$, giving us the Gysin distinguished triangle

$$M(U) \xrightarrow{j} M(X) \to M(W)(d)[2d] \to M(U)[1].$$

But as $d \geqslant 1$, we have

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(W)(d)[2d], \mathcal{F}[n]) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(W)(d)[2d], f_1^{\mathrm{mot}}\mathcal{F}[n]) = 0,$$

hence, by adjunction, j^* is an isomorphism.

Now suppose that $\mathcal{F}$ is birational. We may assume that $\mathcal{F}$ is fibrant as a complex of Nisnevich sheaves, so that

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) = H^n(\mathcal{F}(X))$$

for all $X \in \mathbf{Sm}/k$.

Take an irreducible $X \in \mathbf{Sm}/k$. By Remark 2.2.6, we have a natural isomorphism

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), s_0^{\mathrm{mot}}\mathcal{F}[n]) \cong H^n(\mathcal{F}(\hat{\Delta}^*_{k(X)})).$$

Also, as $\mathcal{F}$ is birational, the restriction to the generic point gives an isomorphism

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) \cong H^n(\mathcal{F}(k(X))),$$

and the map

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) \xrightarrow{\pi_0} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), s_0^{\mathrm{mot}}\mathcal{F}[n])$$

is given by the map on H^n induced by the canonical map

$$\mathcal{F}(k(X)) = \mathcal{F}(\hat{\Delta}^0_{k(X)}) \to \mathcal{F}(\hat{\Delta}^*_{k(X)}).$$

On the other hand, since $\mathcal{F}$ is birational, the map

$$\mathcal{F}(\Delta^n_{k(X)}) \to \mathcal{F}(\hat{\Delta}^n_{k(X)})$$

is a quasi-isomorphism for all n, and hence the map of total complexes

$$\mathcal{F}(\Delta^*_{k(X)}) \to \mathcal{F}(\hat{\Delta}^*_{k(X)})$$

is a quasi-isomorphism. Since $\mathcal{F}$ is homotopy invariant, the map

$$\mathcal{F}(k(X)) = \mathcal{F}(\Delta^0_{k(X)}) \to \mathcal{F}(\Delta^*_{k(X)})$$

is a quasi-isomorphism; thus the composition

$$\mathcal{F}(k(X)) \to \mathcal{F}(\Delta^*_{k(X)}) \to \mathcal{F}(\hat{\Delta}^*_{k(X)})$$

is a quasi-isomorphism as well. Taking H^n, we see that

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}[n]) \xrightarrow{\pi_0} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), s_0^{\mathrm{mot}}\mathcal{F}[n])$$

is an isomorphism for all $X \in \mathbf{Sm}/k$. Since the localizing subcategory of $DM^{\mathrm{eff}}(k)$ generated by the $M(X)$ for $X \in \mathbf{Sm}/k$ is all of $DM^{\mathrm{eff}}(k)$, it follows that π_0 is an isomorphism. □

Corollary 4.2.2. *Let $\mathcal{F}$ be a birational motive. Then*

$$f_m^{\mathrm{mot}}(\mathcal{F}(n)) = \begin{cases} 0 & \text{for } m > n, \\ \mathcal{F}(n) & \text{for } m \leqslant n. \end{cases}$$

Proof. Suppose $n \geqslant m \geqslant 0$. As $\mathcal{F}(n)$ is in $DM^{\mathrm{eff}}(k)(m)$, we have $f_m^{\mathrm{mot}}(\mathcal{F}(n)) = \mathcal{F}(n)$.

Now take $m > n$. As a localizing subcategory of $DM^{\mathrm{eff}}(k)$, $DM^{\mathrm{eff}}(k)(m)$ is generated by objects $M(X)(m)$, $X \in \mathbf{Sm}/k$. Thus it suffices to show that

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X)(m), \mathcal{F}(n)[p]) = 0$$

for all $X \in \mathbf{Sm}/k$ and all p. By Voevodsky's cancellation theorem [**62**], we have

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X)(m), \mathcal{F}(n)[p]) = \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X)(m-n), \mathcal{F}[p]).$$

But since $m - n \geqslant 1$, we have

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X)(m-n), \mathcal{F}[p]) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X)(m-n), f_1^{\mathrm{mot}}\mathcal{F}[p]),$$

which is zero by Theorem 4.2.1. □

Remark 4.2.3. Let $\mathcal{F}$ be a birational motive. Then $\mathcal{F}(n) = s_n^{\mathrm{mot}}(\mathcal{F}(n))$ for all $n \geqslant 0$. Indeed, $f_n^{\mathrm{mot}}(\mathcal{F}(n)) = \mathcal{F}(n)$ and $f_{n+1}^{\mathrm{mot}}(\mathcal{F}(n)) = 0$.

Remark 4.2.4. Let $\mathcal{F}$ be a birational motive. Then for all $\mathcal{G}$ in $DM^{\mathrm{eff}}(k)$ and all integers $m > n \geqslant 0$, and all p, we have

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}(m), \mathcal{F}(n)[p]) = 0.$$

Indeed, the universal property of $f_m^{\mathrm{mot}}(\mathcal{F}(n)) \to \mathcal{F}(n)$ gives the isomorphism

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}(m), f_m^{\mathrm{mot}}(\mathcal{F}(n))[p]) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}(m), \mathcal{F}(n)[p])$$

but $f_m^{\mathrm{mot}}(\mathcal{F}(n)) = 0$ by Corollary 4.2.2.

4.3. Birational motivic sheaves

If F/k is a finitely generated field extension, we define the motive $M(F)$ in $DM^{\mathrm{eff}}(k)$ as the homotopy limit of the motives $M(Y)$ as $Y \in \mathbf{Sm}/k$ runs over all smooth models of F. Since we will really only be using the functor $\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(F), -)$, the reader can, if she prefers, view this as a notational shorthand for the functor on $DM^{\mathrm{eff}}(k)$:

$$M \mapsto \varinjlim_{\substack{Y \\ k(Y)=F}} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(Y), M).$$

This limit is just

$$\varinjlim_{\substack{Y \\ k(Y)=F}} \mathbb{H}^0_{\mathrm{Zar}}(Y, M),$$

in other words, just the stalk of the 0th hypercohomology sheaf of M at the generic point of Y.

Lemma 4.3.1. *Let $\mathcal{F} \in DM^{\mathrm{eff}}(k)$ be such that $\mathcal{H}^i(\mathcal{F}) = 0$ for all $i > 0$. Then*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(k(Y)), \mathcal{F}(n)[2n+r])) = 0$$

for $r > 0$, $n \geqslant 0$ and for all $Y \in \mathbf{Sm}/k$.

Proof. Let $F = k(Y)$. $\mathcal{F}(n)[2n]$ is a summand of $\mathcal{F} \otimes M(\mathbb{P}^n)$, so it suffices to show that

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(F), \mathcal{F} \otimes M(\mathbb{P}^n)[r]) = 0$$

for $r > 0$. We can represent $\mathcal{F} \otimes M(\mathbb{P}^n)$ by $C_*(\mathcal{F} \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^n))$. For each $n \in \mathbb{Z}$, let $\mathcal{F}_n$ be the nth term of $\mathcal{F}$ (homological notation). Replacing $\mathcal{F}$ with the canonical truncation $\tau_{\leqslant 0}\mathcal{F}$, we may assume that $\mathcal{F}_n = 0$ for $n < 0$. We have the functorial left resolutions

$$\mathcal{L}(\mathcal{F}_n) \to \mathcal{F}_n$$

of $\mathcal{F}_n$ (as Nisnevich sheaves with transfers), where the terms in $\mathcal{L}(\mathcal{F}_n)$ are direct sums of representable sheaves; let $\mathcal{L}(\mathcal{F})$ denote the total complex of the double complex $\mathcal{L}(\mathcal{F}_p)_q$. Then we can replace $C_*(\mathcal{F} \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^n))$ with the total complex of

$$\cdots \to C_*(\mathcal{L}(\mathcal{F})_n \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^n)) \to \cdots \to C_*(\mathcal{L}(\mathcal{F})_0 \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^n)).$$

This in turn is a complex supported in (cohomological) degree $\leqslant 0$ with all terms direct sums of representable sheaves $\mathbb{Z}^{\mathrm{tr}}(Y)$, $Y \in \mathbf{Sm}/k$. But for any $X \in \mathbf{Sm}/k$, we have

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), M(Y)[r]) \cong \mathbb{H}^r_{\mathrm{Zar}}(X, C_*(Y)).$$

Thus

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(F), M(Y)[r]) \cong H^r(C_*(Y)(F)),$$

which is zero for $r > 0$, and thus

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(F), \mathcal{F}(n)[2n + r]) \subset H^r(C_*(\mathcal{L}(\mathcal{F}) \otimes^{\mathrm{tr}} \mathbb{Z}^{\mathrm{tr}}(\mathbb{P}^n))) = 0$$

for $r > 0$. $\square$

Proposition 4.3.2. *Let $\mathcal{F}$ be a birational motivic sheaf. Then for all $n \geqslant 0$, $\mathcal{F}(n)[2n]$ is well-connected.*

Proof. We first show that $\mathcal{F}(n)[2n]$ is connected, i.e. that

$$\mathbb{H}^r_{\mathrm{Zar}}(X, \mathcal{F}(n)[2n]) = \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}(n)[2n + r]) = 0$$

for all $r > 0$ and all $X \in \mathbf{Sm}/k$. We have the Gersten–Quillen spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(k(x))(p)[2p], \mathcal{F}(n)[2n + p + q]) \Longrightarrow \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}(n)[2n + p + q]).$$

For $p > n$, $E_1^{p,q} = 0$ by Remark 4.2.4. Using Lemma 4.3.1 and Voevodsky's cancellation theorem [**62**], we see that $E_1^{p,q} = 0$ for $p + q > 0$, $p \leqslant n$, whence the claim.

Next, note that

$$\Omega_T^m(\mathcal{F}(n)[2n]) = \begin{cases} \mathcal{F}(n-m)[2n-2m] & \text{for } 0 \leqslant m \leqslant n, \\ 0 & \text{for } m > n. \end{cases}$$

Indeed, note that, for $\mathcal{G} \in DM^{\mathrm{eff}}(k)$,

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}, \Omega_T^m(\mathcal{F}(n)[2n])) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}(m)[2m], \mathcal{F}(n)[2n]).$$

For $m \leqslant n$, we have the canonical evaluation map $\mathrm{ev} : \mathcal{F}(n-m)[2n-2m] \to \Omega_T^m(\mathcal{F}(n)[2n])$; the above identity says that ev induces the Tate twist map

$$\begin{aligned} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}, \mathcal{F}(n-m)[2n-2m]) \to \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}(m)[2m], \mathcal{F}(n)[2n]),\\ f \mapsto f \otimes \mathrm{id}_{\mathbb{Z}(m)[2m]}\,. \end{aligned}$$

Voevodsky's cancellation theorem [**62**] implies that the Tate twist map is an isomorphism; as $\mathcal{G}$ was arbitrary, it follows that ev is an isomorphism. For the case $m > n$, the right-hand side $\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{G}(m)[2m], \mathcal{F}(n)[2n])$ is zero by Remark 4.2.4.

Thus

$$s_0^{\mathrm{mot}}(\Omega_T^m(\mathcal{F}(n)[2n])) = \begin{cases} 0 & \text{for } m \geqslant 0,\ m \neq n, \\ \mathcal{F} & \text{for } m = n. \end{cases}$$

In fact, we need only check for $0 \leqslant m \leqslant n$. If $0 \leqslant m < n$, then $\Omega_T^m(\mathcal{F}(n)[2n])$ is in $DM^{\mathrm{eff}}(k)(1)$, hence the $s_0^{\mathrm{mot}}(\Omega_T^m(\mathcal{F}(n)[2n])) = 0$. Finally, $\Omega_T^n(\mathcal{F}(n)[2n]) = \mathcal{F}$, and thus $s_0\Omega_T^n(\mathcal{F}(n)[2n]) = s_0^{\mathrm{mot}}(\mathcal{F}) = \mathcal{F}$ by Remark 4.2.3.

As $\mathcal{F}$ is a sheaf, $s_0^{\mathrm{mot}}(\Omega_T^m(\mathcal{F}(n)[2n]))$ is concentrated in cohomological degree 0 for all m, which shows that $\mathcal{F}(n)[2n]$ is well-connected. □

Theorem 4.3.3. *Let $\mathcal{F}$ be a birational motivic sheaf. Then for $q \geqslant 0$, there is a natural isomorphism*

$$H^{2q-p}(X, \mathcal{F}(q)) := \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathcal{F}(q)[2q-p]) \cong \mathrm{CH}^q(X, p; \mathcal{F}(q)[2q]).$$

Proof. Since $\mathcal{F}(q)[2q]$ is well-connected (Proposition 4.3.2), it follows from Theorem 3.2.4 that the slices $s_q^{\mathrm{mot}}(\mathcal{F}(q)[2q])$ are computed by the cycle complexes, i.e. there is a natural isomorphism

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), s_q^{\mathrm{mot}}(\mathcal{F}(q)[2q])[-p]) \cong \mathrm{CH}^q(X, p; \mathcal{F}(q)[2q]).$$

But $s_q^{\mathrm{mot}}(\mathcal{F}(q)[2q]) = \mathcal{F}(q)[2q]$ by Remark 4.2.3. □

Remark 4.3.4. Let $\mathcal{F}$ be a birational sheaf. For $Y \in \mathbf{Sm}/k$, we can define the group of codimension q cycles on Y with values in $\mathcal{F}$ as

$$z^q(Y)_{\mathcal{F}} := \bigoplus_{w \in Y^{(q)}} \mathcal{F}(k(w)),$$

that is, an $\mathcal{F}$-valued cycle on Y is a formal finite sum $\sum_i a_i W_i$ with each W_i a codimension q integral closed subscheme of Y and $a_i \in \mathcal{F}(k(W_i))$. The canonical identification

$$\mathcal{F}(k(w)) \cong H^0((\mathcal{F}(q)[2q])^W(Y))$$

for $W \subset Y$ a codimension q integral closed subscheme gives the $\mathcal{F}$-valued cycle groups the usual properties of algebraic cycles, including proper pushforward, and partially defined pullback. In particular, for $\mathcal{F} = \mathbb{Z}$, we have the identification

$$z^q(Y)_{\mathbb{Z}} = z^q(Y);$$

we will show in the next subsection that this identification is compatible with the operations of proper pushforward, and pullback (when defined).

In addition, we have

$$\begin{aligned} s_0^{\mathrm{mot}}(\Omega_T^q(\mathcal{F}(q)[2q])) &\cong s_0^{\mathrm{mot}}(\mathcal{F}) \\ &\cong \mathcal{F}, \end{aligned}$$

hence

$$z^q(X, n; \mathcal{F}(q)[2q]) = \bigoplus_{w \in X^{(q)}(n)} \mathcal{F}(k(w)).$$

Thus we can think of $z^q(X, *; \mathcal{F}(q)[2q])$ as the cycle complex of codimension q $\mathcal{F}$-valued cycles in good position on $X \times \Delta^*$.

4.4. The sheaf $\mathbb{Z}$

The most basic example of a birational motivic sheaf is the constant sheaf $\mathbb{Z}$. Here we show that the constructions of the previous subsection are compatible with the classical operations on algebraic cycles.

Let $W \subset Y$ be a closed subset with $Y \in \mathbf{Sm}/k$. We let $z^q_W(Y)$ be the subgroup of $z^q(Y)$ consisting of cycles with support contained in W.

Definition 4.4.1. The category of closed immersions Imm_k has objects (Y, W) with $Y \in \mathbf{Sm}/k$ and $W \subset Y$ a closed subset. A morphism $f : (Y, W) \to (Y', W')$ is a morphism $f : Y \to Y'$ in $\mathbf{Sm}/k$ such that $f^{-1}(W')_{\mathrm{red}} \subset W$. Let $\mathrm{Imm}_k(q) \subset \mathrm{Imm}_k$ be the full subcategory of closed subsets $W \subset Y$ such that each component of W has codimension at least q.

Note that for each morphism $f : (W \subset Y) \to (W' \subset Y')$ in $\mathrm{Imm}_k(q)$, the pullback of cycles gives a well-defined map $f^* : z^q_{W'}(Y') \to z^q_W(Y)$.

Definition 4.4.2. Let $f : Y' \to Y$ be a morphism in $\mathbf{Sch}_k$, with Y and Y' equidimensional over k. We let $z^q(Y, *)_f \subset z^q(Y, *)$ be the subcomplex defined by letting $z^q(Y, n)_f$ be the subgroup of $z^q(Y, n)$ generated by irreducible $W \subset Y \times \Delta^n$, $W \in z^q(Y, n)$, such that for each face $F \subset \Delta^n$, each irreducible component of $(f \times \mathrm{id}_F)^{-1}(W \cap X \times F)$ has codimension q on $Y' \times F$.

Assuming that $f(Y')$ is contained in the smooth locus of Y, the maps $(f \times \mathrm{id}_{\Delta^n})^*$ thus define the morphism of complexes

$$f^* : z^q(Y, *)_f \to z^q(Y', *).$$

We recall Chow's moving lemma in the following form.

Theorem 4.4.3 (Bloch [7]). *Suppose that Y is a quasi-projective k-scheme, and that $f: Y' \to Y$ has image contained in the smooth locus of Y. Then the inclusion $z^q(Y,*)_f \to: z^q(Y,*)$ is a quasi-isomorphism.*

Lemma 4.4.4. *Take $Y \in \mathbf{Sm}/k$, $W \subset Y$ a closed subset. Suppose that each irreducible component of $W \subset Y$ has codimension at least q. Then there is an isomorphism*

$$\rho_{Y,W,q} : H^{2q}_W(Y, \mathbb{Z}(q)) \to z^q_W(Y)$$

such that the $\rho_{Y,W,q}$ define a natural isomorphism of functors from $\mathrm{Imm}_k(q)^{\mathrm{op}}$ to **Ab**. *In addition, the maps $\rho_{Y,W,q}$ are natural with respect to proper pushforward.*

Proof. For $U \in \mathbf{Sm}/k$, we have the sheaf $z_{\mathrm{q.fin}}(U) \in \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$, where for $X \in \mathbf{Sm}/k$, $z_{\mathrm{q.fin}}(U)(X)$ is the free abelian group on the integral subschemes $W \subset X \times_k U$ with $W \to X$ quasi-finite and dominant over some component of X.

Let $f : (Y', W') \to (Y, W)$ be a map in $\mathrm{Imm}_k(q)$. By definition, $\mathbb{Z}(1)[2]$ is the reduced motive of $\mathbb{P}^1$,

$$\mathbb{Z}(1)[2] = \tilde{M}(\mathbb{P}^1) \cong \mathrm{cone}(M(k) \xrightarrow{i_{\infty *}} M(\mathbb{P}^1)),$$

and $\mathbb{Z}(q)[2q]$ is the qth tensor power of $\mathbb{Z}(1)[2]$. Via the localization functor

$$RC^{\mathrm{Sus}}_* : D^-(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(\mathbf{Sm}/k)) \to DM^{\mathrm{eff}}_-(k)$$

and using [**58**, Corollary 4.1.8], we have the isomorphism

$$\mathbb{Z}(q)[2q] \cong C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))$$

and the natural identification

$$H^{2q+p}(Y, \mathbb{Z}(q)) \cong \mathbb{H}^p_{\mathrm{Nis}}(Y, C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))) \cong H^p(C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y)).$$

In particular, we have the natural identification of the motivic cohomology with supports

$$H^{2q}_W(Y, \mathbb{Z}(q)) \cong H_0(\mathrm{cone}(C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y) \to C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y \setminus W))[-1]).$$

Set

$$C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))^W(Y) := \mathrm{cone}(C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y) \to C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y \setminus W))[-1].$$

In addition, from the definition of the Suslin complex, we have the evident inclusion of complexes

$$C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y) \subset z^q(Y \times \mathbb{A}^q, *)_{f \times \mathrm{id}} \subset z^q(Y \times \mathbb{A}^q, *).$$

It follows from [**17**, VI, Theorem 3.2; V, Theorem 4.2.2] that the inclusion

$$C^{\mathrm{Sus}}_*(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y) \subset z^q(Y \times \mathbb{A}^q, *)$$

is a quasi-isomorphism; by Theorem 4.4.3, the inclusion

$$C_*^{\mathrm{Sus}}(z_{\mathrm{q.fin}}(\mathbb{A}^q))(Y) \subset z^q(Y \times \mathbb{A}^q, *)_{f\times\mathrm{id}}$$

is a quasi-isomorphism as well.

Let $U = Y \setminus W$, $U' := Y' \setminus W'$ and let $f_U : U' \to U$ be the restriction of f. Setting

$$z^q_W(Y,*)_f = \mathrm{cone}(z^q(Y,*)_f \to z^q(U,*)_{f_U})[-1],$$

we thus have the quasi-isomorphism

$$C_*^{\mathrm{Sus}}(z_{\mathrm{q.fin}}(\mathbb{A}^q))^W(Y) \to z^q_{W\times\mathbb{A}^q}(Y \times \mathbb{A}^q, *)_{f\times\mathrm{id}}.$$

We have the commutative diagram

$$\begin{tikzcd}
C_*^{\mathrm{Sus}}(z_{\mathrm{q.fin}}(\mathbb{A}^q))^W(Y) \arrow[r] \arrow[d, "(f^*, f_U^*)"'] & z^q_{W\times\mathbb{A}^q}(Y \times \mathbb{A}^q, *)_{f\times\mathrm{id}} \arrow[d, "(f\times\mathrm{id}^*, f_U\times\mathrm{id}^*)"] \\
C_*^{\mathrm{Sus}}(z_{\mathrm{q.fin}}(\mathbb{A}^q))^{W'}(Y') \arrow[r] & z^q_{W'\times\mathbb{A}^q}(Y' \times \mathbb{A}^q, *)
\end{tikzcd}$$

Since the horizontal maps are quasi-isomorphisms, we can use the right-hand side to compute $f^* : H^{2q}_W(Y, \mathbb{Z}(q)) \to H^{2q}_{W'}(Y', \mathbb{Z}(q))$.

By the homotopy property for the higher Chow groups, and using the moving lemma again, the pullback maps

$$p_1^* : z^q_W(Y,*)_f \to z^q_{W\times\mathbb{A}^q}(Y \times \mathbb{A}^q, *)_{f\times\mathrm{id}},$$
$$p_1^* : z^q_{W'}(Y',*) \to z^q_{W'\times\mathbb{A}^q}(Y \times \mathbb{A}^q, *)$$

are quasi-isomorphisms. Thus we can use

$$f^* : z^q_W(Y,*)_f \to z^q_{W'}(Y',*)$$

to compute $f^* : H^{2q}_W(Y, \mathbb{Z}(q)) \to H^{2q}_{W'}(Y', \mathbb{Z}(q))$.

Let $d = \dim_k Y$. Chow's moving lemma together with the localization distinguished triangle

$$z_{d-q}(W,*) \to z_{d-q}(Y,*) \to z_{d-q}(U,*)$$

shows that the inclusion $z_{d-q}(W,*) \subset z_{d-q}(Y,*)_f$ induces a quasi-isomorphism

$$z_{d-q}(W,*) \to z^q_W(Y,*)_f.$$

Similarly, the inclusion $z_{d'-q}(W',*) \subset z_{d'-q}(Y',*)$, $d' := \dim_k Y'$, induces a quasi-isomorphism

$$z_{d'-q}(W',*) \to z^q_{W'}(Y',*).$$

Since each component of W has codimension at least q on Y, it follows that the inclusion

$$z_{d-q}(W) = z_{d-q}(W,0) \to z_{d-q}(W,*)$$

is a quasi-isomorphism. As $z_{d-q}(W) = z^q_W(Y)$, we thus have the isomorphism

$$\rho_{Y,W,q} : z^q_W(Y) \to H^{2q}_W(Y, \mathbb{Z}(q)).$$

In addition, the following diagram commutes:

$$\begin{tikzcd}
z_{d-q}(W) \arrow[r, equal] & z^q_W(Y) \arrow[r] \arrow[d, "f^*"'] & z^q_W(Y,*)_f \arrow[d, "f^*"] \\
z_{d'-q}(W') \arrow[r, equal] & z^q_{W'}(Y') \arrow[r] & z^q_{W'}(Y',*)
\end{tikzcd}$$

Combining this with our previous identification of $H^{2q}_W(Y, \mathbb{Z}(q))$ with $H_0(z^q_W(Y,*)_f)$ and $H^{2q}_{W'}(Y', \mathbb{Z}(q))$ with $H_0(z^q_{W'}(Y',*))$ shows that the isomorphisms $\rho_{Y,W,q}$ are natural with respect to pullback.

The compatibility of the $\rho_{Y,W,q}$ with proper pushforward is similar, but easier, as one does not need to introduce the complexes $z^q(Y \times \mathbb{A}^q, *)_{f\times\mathrm{id}}$, etc., or use Chow's moving lemma. We leave the details to the reader. $\square$

Now take $X \in \mathbf{Sm}/k$, $W \in \mathcal{S}^{(q)}_X(n)$. By Lemma 4.4.4, we have the isomorphism

$$\rho_{X\times\Delta^n,W,q} : H^{2q}_W(X \times \Delta^n, \mathbb{Z}(q)) \to z^q_W(X \times \Delta^n).$$

In addition, if $W' \subset W$ is a closed subset of codimension greater than q on $X \times \Delta^n$, then the restriction map

$$H^{2q}_W(X \times \Delta^n, \mathbb{Z}(q)) \to H^{2q}_{W\setminus W'}(X \times \Delta^n \setminus W', \mathbb{Z}(q))$$

is an isomorphism. Noting that

$$H^0((\mathbb{Z}(q)[2q])^W(X \times \Delta^n)) = H^{2q}_W(X \times \Delta^n, \mathbb{Z}(q)),$$

it follows from the definition of $z^q(X, n; \mathbb{Z}(q)[2q])$ that we have

$$z^q(X, n; \mathbb{Z}(q)[2q]) = \varinjlim_{\substack{W\subset X\times\Delta^n \\ W\in\mathcal{S}^{(q)}_X(n)}} H^{2q}_W(X \times \Delta^n, \mathbb{Z}(q)).$$

Thus taking the limit of the isomorphisms $\rho_{X\times\Delta^n,W,q}$ over $W \in \mathcal{S}^{(q)}_X(n)$ gives the isomorphism

$$\rho_{X,n} : z^q(X, n; \mathbb{Z}(q)[2q]) \to z^q(X, n).$$

Proposition 4.4.5. *For $X \in \mathbf{Sm}/k$, the maps $\rho_{X,n}$ define an isomorphism of complexes*

$$z^q(X, *; \mathbb{Z}(q)[2q]) \xrightarrow{\rho_X} z^q(X, *)$$

natural with respect to flat pullback.

Proof. It follows from Lemma 4.4.4 that the isomorphisms $\rho_{X,W,n}$ are natural with respect to the pullback maps in $\mathrm{Imm}_k(q)$; in particular, with respect to flat pullback and with respect to the face maps $X \times \Delta^{n-1} \to X \times \Delta^n$. Passing to the limit over $W \in \mathcal{S}^{(q)}_X(n)$ proves the result. $\square$

Part II. Motivic cohomology of Azumaya algebras

5. The sheaves $\mathcal{K}_0^{\mathcal{A}}$ and $\mathbb{Z}_{\mathcal{A}}$

In this section we develop a theory of '$\mathcal{K}_0^{\mathcal{A}}$-valued cycles' leading to a generalization of Bloch's cycle complex and higher Chow groups. In the next section, we show how one extends the Bloch–Lichtenbaum spectral sequence (as generalized in [**31**]) to the case of the G-theory of an Azumaya algebra $\mathcal{A}$ over some scheme X, with the higher Chow groups being replaced by our modified version.

The general theory developed in the previous sections is restricted to presheaves of spectra on $\mathbf{Sm}/k$; as we would like to have our spectral sequence for an arbitrary sheaf of Azumaya algebras over some base-scheme X, rather than just a central simple algebra over k, we are forced to repeat some of the constructions of the previous sections in this more general setting. However, the proof of our main result (Theorem 6.1.3) in the next section will be accomplished by using localization properties to reduce to the case $X = \operatorname{Spec} k$, allowing us to apply the results of the previous sections.

Returning to the case of a central simple algebra over k, we use Theorem 6.1.3 to prove a more precise result, identifying the slice $s_q K^{\mathcal{A}}$ with the Eilenberg–Mac Lane spectra of the motive $\mathbb{Z}_{\mathcal{A}}(q)[2q]$ (Theorem 6.5.5). This is the main result we will need for our applications to Severi–Brauer varieties and the K-theory of central simple algebras.

5.1. $\mathcal{K}_0^{\mathcal{A}}$: definition and first properties

Let R be a noetherian ring and fix a sheaf of Azumaya algebras $\mathcal{A}$ on an R-scheme of finite type X. For $p : Y \to X \in \mathbf{Sch}_X$, we have the sheaf $p^*\mathcal{A}$ of Azumaya algebras on Y. We may sheafify the K-groups of $p^*\mathcal{A}$ for the Zariski topology on Y, giving us the Zariski sheaves $\mathcal{K}_n^{\mathcal{A}}$ on $\mathbf{Sch}_X$.

Lemma 5.1.1. *Suppose that X is regular. Then*

(1) $\mathcal{K}_0^{\mathcal{A}}$ *is an $\mathbb{A}^1$ homotopy invariant presheaf on $\mathbf{Sm}/X$;*

(2) $\mathcal{K}_0^{\mathcal{A}}$ *is a birational presheaf on $\mathbf{Sm}/X$, i.e. for $Y \in \mathbf{Sm}/X$, $j : U \to Y$ a dense open subscheme, the restriction map*

$$j^* : \mathcal{K}_0^{\mathcal{A}}(Y) \to \mathcal{K}_0^{\mathcal{A}}(U)$$

is an isomorphism; equivalently, $\mathcal{K}_0^{\mathcal{A}}$ is locally constant for the Zariski topology on $\mathbf{Sm}/X$, hence is a sheaf for the Nisnevich topology on $\mathbf{Sm}/X$.

Proof. The homotopy invariance follows from the fact that $Y \mapsto K_0(Y; \mathcal{A})$ is homotopy invariant, and that the restriction map $K_0(Y, \mathcal{A}) \to K_0(U, \mathcal{A})$ is surjective for each open immersion $U \to Y$ in $\mathbf{Sm}/X$.

For the birationality property, we may assume that Y is irreducible. By Corollary A.4, any object in the category $\mathcal{P}_{X;\mathcal{A}}$ is locally $\mathcal{A}$-projective, hence it suffices to show that for each $y \in Y$, the map

$$K_0(\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{O}_{Y,y}) \to K_0(\mathcal{A} \otimes_{\mathcal{O}_B} k(Y))$$

is an isomorphism.

Since Y is regular, surjectivity follows easily from Corollary A.5. On the other hand, since $\mathcal{O}_{Y,y}$ is local, the category of finitely generated projective $\mathcal{A} \otimes_{\mathcal{O}_B} \mathcal{O}_{Y,y}$-modules has a unique indecomposable generator (see [**12**] and [**29**, III.5.2.2]) and similarly, the category of finitely generated projective $\mathcal{A} \otimes_{\mathcal{O}_B} k(Y)$-modules has a unique simple generator. Thus the map is also injective, completing the proof that $\mathcal{K}_0^{\mathcal{A}}$ is birational.

To see that $\mathcal{K}_0^{\mathcal{A}}$ is a sheaf for the Nisnevich topology, it suffices to check the sheaf condition on elementary Nisnevich squares (compare proof of Lemma 4.1.3); this follows directly from the birationality property. □

5.2. The reduced norm map

For $Y \in \mathbf{Sm}/X$, let $\operatorname{Spec} F \to Y$ be a point. We define a map

$$\mathrm{Nrd}_F : \mathbb{Z} \simeq K_0(\mathcal{A}_F) \to K_0(F) = \mathbb{Z}$$

by mapping the positive generator of $K_0(\mathcal{A}_F)$ to $e_F[F]$, where e_F is the *index* of $\mathcal{A}_F$. Recall that, by definition, $e_F^2 = [D : F]$ where D is the unique division F-algebra similar to $\mathcal{A}_F$.

Lemma 5.2.1. *The assignment $F \mapsto \mathrm{Nrd}_F$ defines a morphism of sheaves on $\mathbf{Sm}/X_{\mathrm{Nis}}$*

$$\mathrm{Nrd} : \mathcal{K}_0^{\mathcal{A}} \to \mathbb{Z}$$

which realizes $\mathcal{K}_0^{\mathcal{A}}$ as a subsheaf of the constant sheaf $\mathbb{Z}$. This is the reduced norm map attached to $\mathcal{A}$.

Proof. In view of Lemma 5.1.1, it suffices to check that if L is a separable extension of F, the diagram

$$\begin{tikzcd} K_0(\mathcal{A}_L) \arrow[r, "\mathrm{Nrd}_L"] & K_0(L) \\ K_0(\mathcal{A}_F) \arrow[r, "\mathrm{Nrd}_K"] \arrow[u] & K_0(F) \arrow[u] \end{tikzcd}$$

commutes. This is classical: by Morita invariance, we may replace $\mathcal{A}_F$ by a similar division algebra D. Choose a maximal commutative subfield $E \subset D$ which is separable over F. First assume that $L = E$: then D_L is split and Nrd_L is an isomorphism by Morita invariance; on the other hand, the generator $[D]$ of $K_0(D)$ maps to e times the generator of $K_0(D_L)$, which proves the claim in this special case. The general case reduces to the special case by considering a commutative cube involving the extension LE. □

5.3. The presheaf with transfers $\mathbb{Z}_{\mathcal{A}}$

For a scheme X we let $\mathcal{M}_X$ denote the category of coherent sheaves of $\mathcal{O}_X$-modules on X. Given a sheaf of Azumaya algebras $\mathcal{A}$ on X, we let $\mathcal{M}_X(\mathcal{A})$ denote the category of sheaves of $\mathcal{A}$-modules $\mathcal{F}$ which are coherent as $\mathcal{O}_X$-modules, using the structure map $\mathcal{O}_X \to \mathcal{A}$ to define the $\mathcal{O}_X$-module structure on $\mathcal{F}$. We let $G(X; \mathcal{A})$ denote the K-theory spectrum of the abelian category $\mathcal{M}_X(\mathcal{A})$. If $f : Y \to X$ is a morphism, we often write

$G(Y;\mathcal{A})$ for $G(Y;f^*\mathcal{A})$. For $Y \in \mathbf{Sm}/X$, we let $\mathcal{G}_n(Y,\mathcal{A})$ denote the Zariski sheaf on Y associated to the presheaf $U \mapsto G_n(U,\mathcal{A})$.

Suppose that X is regular. Let $f: Z \to Y$ be a finite morphism in $\mathbf{Sch}_X$ with Y in $\mathbf{Sm}/X$. Restriction of scalars defines a map of sheaves

$$f_*: f_*\mathcal{K}_0(Z;\mathcal{A}) \to \mathcal{G}_0(Y;\mathcal{A}).$$

Using Corollary A.5, we see that the natural map

$$\mathcal{K}_0(Y;\mathcal{A}) \to \mathcal{G}_0(Y;\mathcal{A})$$

is an isomorphism, giving us the pushforward map

$$f_*: \mathcal{K}_0^{\mathcal{A}}(Z) \to \mathcal{K}_0^{\mathcal{A}}(Y).$$

Now take $Y, Y' \in \mathbf{Sm}/X$ and let $Z \subset Y \times_X Y'$ be an integral subscheme which is finite over Y and surjective onto a component of Y; let $p: Z \to Y$, $p': Z \to Y'$ be the maps induced by the projections. Define

$$Z^*: \mathcal{K}_0^{\mathcal{A}}(Y') \to \mathcal{K}_0^{\mathcal{A}}(Y)$$

by $Z^* := p_* \circ p'^*$. For X regular, this operation extends to $\mathrm{Cor}_X(Y,Y')$ by linearity.

Lemma 5.3.1. *Suppose X regular. For $Z_1 \in \mathrm{Cor}_X(Y,Y')$, $Z_2 \in \mathrm{Cor}_X(Y',Y'')$ we have*

$$(Z_2 \circ Z_1)^* = Z_1^* \circ Z_2^*.$$

Proof. We already have a canonical operation of $\mathrm{Cor}_X(-,-)$ on the constant sheaf $\mathbb{Z}$ making $\mathbb{Z}$ a sheaf with transfers; one easily checks that this action agrees with the action we have defined above for $\mathcal{A} = \mathcal{O}_X$. It is similarly easy to check that, for Z integral and $f: Z \to Y$ finite and surjective with Y smooth, f_* commutes with Nrd. Since Nrd is injective, this implies that $\mathcal{K}_0^{\mathcal{A}}$ is also a sheaf with transfers, as desired. □

Definition 5.3.2. Let X be a regular R-scheme of finite type, $\mathcal{A}$ a sheaf of Azumaya algebras on X. We let $\mathbb{Z}_{\mathcal{A}}$ denote the Nisnevich sheaf with transfers on $\mathbf{Sm}/X$ defined by $\mathcal{K}_0^{\mathcal{A}}$.

Remark 5.3.3. The reduced norm map $\mathrm{Nrd}: \mathcal{K}_0^{\mathcal{A}} \to \mathbb{Z}$ defines a monomorphism of Nisnevich sheaves with transfers $\mathrm{Nrd}: \mathbb{Z}_{\mathcal{A}} \to \mathbb{Z}$.

Lemma 5.3.4. *The subsheaf with transfers $(\mathbb{Z}_{\mathcal{A}}, \mathrm{Nrd})$ of the constant sheaf (with transfers) $\mathbb{Z}$ only depends on the subgroup of $\mathrm{Br}(X)$ generated by $\mathcal{A}$. In particular, it is Morita-invariant.*

Proof. Indeed, if $\mathcal{B}$ generates the same subgroup of $\mathrm{Br}(X)$ as $\mathcal{A}$, there exist integers r, s such that $\mathcal{A}^{\otimes_X r}$ is similar to $\mathcal{B}$ and $\mathcal{B}^{\otimes_X s}$ is similar to $\mathcal{A}$. This implies readily that $\mathcal{A}$ and $\mathcal{B}$ have the same splitting fields (say, over a point $\operatorname{Spec} F$ of X), hence have the same index (say, over any extension of F). □

Remark 5.3.5. The maps $K_0(F) \to K_0(\mathcal{A}_F)$ given by extension of scalars also define a morphism of sheaves $\mathbb{Z} \to \mathbb{Z}_{\mathcal{A}}$. But this morphism is not Morita-invariant.

In case X is the spectrum of a field, Lemma 5.1.1 yields the following proposition.

Proposition 5.3.6. *Take $X = \operatorname{Spec} k$, k a field, and let A be a central simple algebra over k. Then the sheaf with transfers $\mathbb{Z}_A$ on $\mathbf{Sm}/k$ is a birational motivic sheaf.*

5.4. Severi–Brauer schemes

Let $p : \mathrm{SB}(\mathcal{A}) \to X$ be the Severi–Brauer scheme associated to $\mathcal{A}$.

Lemma 5.4.1. *Suppose $X = \operatorname{Spec} k$, k a field. Then the subgroup $\mathrm{Nrd}(K_0(\mathcal{A})) \subset K_0(k) = \mathbb{Z}$ is the same as the image*

$$p_*(\mathrm{CH}_0(\mathrm{SB}(\mathcal{A}))) \subset \mathrm{CH}_0(B) = \mathbb{Z}.$$

Moreover, $p_* : \mathrm{CH}_0(\mathrm{SB}(\mathcal{A})) \to \mathbb{Z}$ is injective.

Proof. This is a theorem of Panin [**46**], see also [**10**, Corollary 7.3]. We recall the proof of the first statement. Let $x = \operatorname{Spec} K$ be a closed point of $\mathrm{SB}(\mathcal{A})$. Then K is a finite extension of F which is a splitting field of $\mathcal{A}$. It is classical that K is a maximal commutative subfield of some algebra similar to $\mathcal{A}$; in particular, $[K : F]$ is divisible by the index of A. Conversely, replacing A by a similar division algebra D, for any maximal commutative subfield $L \subset D$, $[L : F]$ equals the index of A. □

Now let us come back to the case where X is regular. Let us denote by $\mathcal{CH}_0(\mathrm{SB}(\mathcal{A})/X)$ the sheafification (for the Zariski topology) of the presheaf on $\mathbf{Sm}/X$

$$U \mapsto \mathrm{CH}_{\dim_k U}(\mathrm{SB}(\mathcal{A}) \times_X U).$$

The pushforward

$$p_{U*} : \mathrm{CH}_{\dim_k U}(\mathrm{SB}(\mathcal{A}) \times_X U) \to \mathrm{CH}_{\dim_k U}(U) = \mathbb{Z}$$

defines the map

$$\deg : \mathcal{CH}_0(\mathrm{SB}(\mathcal{A})/X) \to \mathbb{Z},$$

where $\mathbb{Z}$ is viewed as a constant sheaf on $(\mathbf{Sm}/X)_{\mathrm{Zar}}$.

Lemma 5.4.2. *The map* $\deg$ *identifies $\mathcal{CH}_0(\mathrm{SB}(\mathcal{A})/X)$ with the locally constant subsheaf* $\mathrm{Nrd}(\mathbb{Z}_{\mathcal{A}}) \subset \mathbb{Z}$. *In other words, there is a canonical isomorphism of subsheaves of* $\mathbb{Z}$

$$(\mathbb{Z}_{\mathcal{A}}, \mathrm{Nrd}) \simeq (\mathcal{CH}_0(\mathrm{SB}(\mathcal{A})/X), \deg).$$

Proof. By Lemma 5.4.1, the result is true at $\operatorname{Spec} F$, F a field. For Y local, the restriction map

$$j^* : \mathrm{CH}_{\dim X}(\mathrm{SB}(\mathcal{A}) \times_X Y) \to \mathrm{CH}_0(\mathrm{SB}(\mathcal{A} \otimes_{\mathcal{O}_B} k(Y)))$$

($\dim X :=$ the Krull dimension) is surjective, from which the result easily follows. □

Remark 5.4.3. It is evident that the transfer structure of Lemma 5.3.1 on $\mathbb{Z}_{\mathcal{A}}$ coincides with the natural transfer structure on $\mathcal{CH}_0(\mathrm{SB}(\mathcal{A})/X)$.

5.5. $\mathcal{K}_0^{\mathcal{A}}$ for embedded schemes

Let k be a field. We fix a sheaf of Azumaya algebras $\mathcal{A}$ on some finite type k-scheme X; we do not assume that X is regular.

As a technical tool, we extend the definition of the category Imm_k (Definition 4.4.1) as follows.

Definition 5.5.1. The category of closed immersions $\mathrm{Imm}_{X,k}$ has objects (Y, W) with $Y \in \mathbf{Sm}/k$ and $W \subset X \times_k Y$ a closed subset. A morphism $f : (Y, W) \to (Y', W')$ is a morphism $f : Y \to Y'$ in $\mathbf{Sm}/k$ such that $(\mathrm{id} \times f)^{-1}(W')_{\mathrm{red}} \subset W$.

Let Y be a smooth k-scheme, let $i : W \to X \times_k Y$ be a reduced closed subscheme of pure codimension. Letting $W_{\mathrm{reg}} \subset W$ be the regular locus, we have the (constant) Zariski sheaf $\mathcal{K}_0^{\mathcal{A}}$ defined on W_{reg}. We describe how to extend $\mathcal{K}_0^{\mathcal{A}}$ to $W \subset X \times_k Y$ so that

$$(Y, W) \mapsto \mathcal{K}_0^{\mathcal{A}}(W \subset X \times_k Y)$$

defines a presheaf $\mathcal{K}_0^{\mathcal{A}}$ on $\mathrm{Imm}_{X,k}$.

For this, we define $\mathcal{K}_0^{\mathcal{A}}$ on $i : W \to X \times_k Y$ to be $\mathcal{K}_0^{\mathcal{A}}(W_{\mathrm{reg}})$, where $j : W_{\mathrm{reg}} \to W$ is the regular locus of W. The trick is to define the pullback maps.

We let $G^W(X \times_k Y; \mathcal{A})$ denote the homotopy fibre of the restriction map

$$G(X \times_k Y; \mathcal{A}) \to G(X \times_k Y \setminus W; \mathcal{A}).$$

Lemma 5.5.2. *Suppose that X is local, with closed point x. Let $i : Y' \to Y$ be a closed embedding in $\mathbf{Sm}^{\mathrm{ess}}/k$, with Y local having closed point y. Let $W \subset X \times Y$ be a closed subset such that $X \times Y' \cap W = (x, y)$ (as a closed subset). If $\mathrm{codim}_{X\times Y} W > \mathrm{codim}_{X\times Y'}(x, y)$, then the restriction map*

$$i^* : G_0^W(X \times Y; \mathcal{A}) \to G_0^{(x,y)}(X \times Y'; \mathcal{A})$$

is the zero map.

Proof. The proof is a modification of Quillen's proof of Gersten's conjecture. Making a base-change to $k(x, y)$, and noting that $G_0^{(x,y)}(X \times Y; \mathcal{A}) = G_0((x, y); \mathcal{A})$, we may assume that $k(y) = k(x) = k$. Since K-theory commutes with direct limits (of rings) we may replace Y and Y' with finite type, smooth affine k-schemes, and we are free to shrink to a smaller neighbourhood of y in Y as needed.

Let $\bar{W} \subset Y$ be the closure of $p_2(W)$. Note that the condition $\mathrm{codim}_{X\times Y} W > \mathrm{codim}_{X\times Y'}(x, y)$ implies that $\dim_k W < \dim_k Y$, hence $\bar{W}$ is a proper closed subset of Y. Take a divisor $D \subset Y$ containing $\bar{W}$. Then there is a morphism

$$\pi : Y \to \mathbb{A}^n,$$

$n = \dim_k Y - 1$, such that π is smooth in a neighbourhood of y and $\pi : D \to \mathbb{A}^n$ is finite. Let

$$W' := \pi^{-1}(\pi(W)).$$

Choosing π general enough, and noting that

$$\operatorname{codim}_{X\times Y} W' = \operatorname{codim}_{X\times Y} W - 1 \geqslant \operatorname{codim}_{X\times Y'}(x,y) = \dim_k X\times Y',$$

we may assume that $W'\cap X\times Y'$ is a finite set of closed points, say T. Let $S\subset D$ be the finite set of closed points $\pi^{-1}(\pi(y))\cap D$.

The inclusion $D\to Y$ induces a section $s: D\to Y\times_{\mathbb{A}^n} D$ to $p_2: Y\times_{\mathbb{A}^n} D\to D$; since π is smooth at y', $s(D)$ is contained in the regular locus of $Y\times_{\mathbb{A}^n} D$ and is hence a Cartier divisor on $Y\times_{\mathbb{A}^n} D$. Noting that $p_1: Y\times_{\mathbb{A}^n} D\to Y$ is finite, there is an open neighbourhood U of S in Y such that $s(D)\cap Y\times_{\mathbb{A}^n} U$ is a principal divisor; let t be a defining equation. Let $D_U := D\cap U$.

This gives us the commutative diagram

$$\begin{tikzcd} Y\times_{\mathbb{A}^n} D_U \arrow[r, "q"] \arrow[d, shift left, "p"] & U \\ D_U \arrow[u, shift left, "s"] \arrow[ur, "i"'] & \end{tikzcd}$$

with q finite. Applying $X\times_k -$, this gives us the commutative diagram

$$\begin{tikzcd} X\times_k Y\times_{\mathbb{A}^n} D_U \arrow[r, "\hat{q}"] \arrow[d, shift left, "\hat{p}"] & X\times_k U \\ X\times_k D_U \arrow[u, shift left, "\hat{s}"] \arrow[ur, "\hat{i}"'] & \end{tikzcd}$$

with $\hat{q}$ finite.

Thus we have, for $M\in\mathcal{M}_{X\times_k D_U;\mathcal{A}}$, the exact sequence

$$0\to \hat{q}_*(\hat{p}^*M)\xrightarrow{\hat{q}_*(\times t)}\hat{q}_*(\hat{p}^*M)\to \hat{i}_*M\to 0$$

natural in M.

Note that, if M is supported in W, then $\hat{q}_*(\hat{p}^*M)$ is supported in W'. Letting $i': W\to W'$ be the inclusion, our exact sequence gives us the identity

$$[i'_*M]=0 \quad \text{in } G_0^{W'}(Y;\mathcal{A}),$$

hence

$$i^*([i'_*M])=0 \quad \text{in } G_0^{W'\cap X\times_k Y'}(Y';\mathcal{A}).$$

Let $\bar{i}:(x,y)\to T:=W'\cap X\times_k Y'$ be the inclusion. We have the commutative diagram

$$\begin{tikzcd} G_0^W(X\times Y;\mathcal{A}) \arrow[r, "i'_*"] \arrow[d, "i^*"'] & G_0^{W'}(X\times Y';\mathcal{A}) \arrow[d, "i^*"] \\ G_0^{(x,y)}(X\times Y';\mathcal{A}) \arrow[r, "\bar{i}_*"] & G_0^T(X\times Y';\mathcal{A}) \end{tikzcd}$$

Since T is a finite set of points containing (x, y),

$$G_0^T(X \times Y'; \mathcal{A}) = G_0^{(x,y)}(X \times Y'; \mathcal{A}) \oplus G_0^{T \setminus \{(x,y)\}}(X \times Y'; \mathcal{A}),$$

with $\bar{i}_*$ the inclusion of the summand $G_0^{(x,y)}(X \times Y'; \mathcal{A})$, from which the result follows directly. $\square$

For a closed immersion $i : W \to X \times Y$, restricting to the generic points of W and using the canonical weak equivalence

$$G(W; \mathcal{A}) \to G^W(X \times Y; \mathcal{A})$$

gives the map

$$\varphi_W : G_0^W(X \times Y; \mathcal{A}) \to \mathcal{K}_0^{\mathcal{A}}(W).$$

Each map of pairs $f : (i' : W' \to X \times Y') \to (i : W \to X \times Y)$ induces a commutative diagram of inclusions

$$\begin{tikzcd} X \times Y' \setminus W' \arrow[r] \arrow[d] & X \times Y' \arrow[d] \\ X \times Y \setminus W \arrow[r] & X \times Y \end{tikzcd}$$

Noting that $\mathrm{id} \times f : X \times Y' \to X \times Y$ is a local complete intersection morphism, we may apply $G(-)$ to this diagram, giving us the induced map on the homotopy fibres

$$f^* : G_0^W(X \times Y; \mathcal{A}) \to G_0^{W'}(X'; \mathcal{A}).$$

Thus, we have the diagram

$$\begin{tikzcd} G_0^W(X \times Y; \mathcal{A}) \arrow[r, "f^*"] \arrow[d, "\varphi_W"'] & G_0^{W'}(X \times Y'; \mathcal{A}) \arrow[d, "\varphi_{W'}"] \\ \mathcal{K}_0^{\mathcal{A}}(W) & \mathcal{K}_0^{\mathcal{A}}(W') \end{tikzcd}$$

In order that f^* descend to a map

$$f^* : \mathcal{K}_0^{\mathcal{A}}(W) \to \mathcal{K}_0^{\mathcal{A}}(W'),$$

it therefore suffices to prove the following lemma.

Lemma 5.5.3.

(1) *For each $i : W \to X \times Y$, the map φ_W is surjective.*

(2) $\varphi_{W'}(f^*(\ker \varphi_W)) = 0.$

Proof. The surjectivity of φ_W follows from Quillen's localization theorem, which first of all identifies $K_0^W(X \times Y; \mathcal{A})$ with $G_0(W; \mathcal{A})$ and secondly implies that the restriction map

$$j^* : G_0(W; \mathcal{A}) \to G_0(k(W); \mathcal{A}) = K_0(k(W); \mathcal{A})$$

is surjective.

For (2), we can factor f as a composition of a closed immersion followed by a smooth morphism. In the second case, $f^{-1}(W \setminus \operatorname{Spec} k(W))$ contains no generic point of W', hence classes supported in $W \setminus \operatorname{Spec} k(W)$ die when pulled back by f and restricted to $k(W')$. Thus we may assume f is a closed immersion.

Fix a generic point $w' = (x, y)$ of W'. We may replace X with $\operatorname{Spec} \mathcal{O}_{X,x}$ and replace Y with $\operatorname{Spec} \mathcal{O}_{Y,y}$. Making a base-change, we may assume that $k(x, y)$ is finite over k. Since $X \times_k Y$ is smooth, it follows that

$$\operatorname{codim}_{X\times Y} W \geqslant \operatorname{codim}_{X\times Y'}(x, y).$$

Let $W'' \subset W$ is a closed subset of W containing no generic point of W. Then

$$\operatorname{codim}_{X\times Y} W'' > \operatorname{codim}_{X\times Y'}(x, y),$$

hence by Lemma 5.5.2 the restriction map

$$G_0^{W''}(X \times Y; \mathcal{A}) \to G_0^{(x,y)}(X \times Y'; \mathcal{A})$$

is the zero map. By Quillen's localization theorem we have

$$\ker \varphi_W = \varinjlim G_0^{W''}(X \times Y; \mathcal{A})$$

over such W'', which proves the lemma. □

5.6. The cycle complex

Let T be a finite type k-scheme. We let $\dim_k T$ denote the Krull dimension of T; we sometimes write d_T for $\dim_k T$.

We fix as above a finite type k-scheme X and a sheaf of Azumaya algebras $\mathcal{A}$ on X. We let $\mathcal{S}^X_{(r)}(n)$ be the set of closed subsets $W \subset X \times \Delta^n$ with

$$\dim_k W \cap X \times F \leqslant r + \dim_k F$$

for all faces $F \subset \Delta^n$ (compare with Definition 2.2.1, where we index by codimension instead of dimension). We order $\mathcal{S}^X_{(r)}(n)$ by inclusion. If $g : \Delta^m \to \Delta^n$ is the map corresponding to a map $g : [m] \to [n]$ in **Ord**, and W is in $\mathcal{S}^X_{(r)}(n)$, then $g^{-1}(W)$ is in $\mathcal{S}^X_{(r)}(m)$, so $n \mapsto \mathcal{S}^X_{(r)}(n)$ defines a simplicial set. We let $X_r(n) \subset \mathcal{S}^X_{(r)}(n)$ denote the set of irreducible $W \in \mathcal{S}^X_{(r)}(n)$ with $\dim_k W = r + n$.

Definition 5.6.1.

$$z_r(X, n; \mathcal{A}) := \bigoplus_{W\in X_r(n)} K_0(k(W); \mathcal{A}).$$

Remark 5.6.2. Let $W \subset X \times \Delta^n$ be a closed subset. Then restriction to the generic points of W gives the isomorphism

$$\mathcal{K}_0^{\mathcal{A}}(W \subset X \times \Delta^n) \cong \bigoplus_{w \in W^{(0)}} K_0(k(w); \mathcal{A}).$$

Thus, we can identify $z_r(X, n; \mathcal{A})$ with the quotient:

$$z_r(X, n; \mathcal{A}) \cong \frac{\varinjlim_{W \in \mathcal{S}^X_{(r)}(n)} \mathcal{K}_0^{\mathcal{A}}(W \subset X \times \Delta^n)}{\varinjlim_{W' \in \mathcal{S}^X_{(r-1)}(n)} \mathcal{K}_0^{\mathcal{A}}(W' \subset X \times \Delta^n)}.$$

Suppose each irreducible $W' \in \mathcal{S}^X_{(r-1)}(n)$ is contained in some irreducible $W \in \mathcal{S}^X_{(r)}(n)$ with $\dim_k W = r + n$; as the map

$$\mathcal{K}_0^{\mathcal{A}}(W' \subset X \times \Delta^n) \to \mathcal{K}_0^{\mathcal{A}}(W \subset X \times \Delta^n)$$

is in this case the zero-map, it follows that

$$z_r(X, n; \mathcal{A}) \cong \varinjlim_{W \in \mathcal{S}^X_{(r)}(n)} \mathcal{K}_0^{\mathcal{A}}(W \subset X \times \Delta^n)$$

if this condition is satisfied, e.g. for X quasi-projective over k.

Let $g : \Delta^m \to \Delta^n$ be the map corresponding to $g : [m] \to [n]$ in **Ord**. By Lemma 5.5.3 and the above remark, we have a well-defined pullback map

$$\mathrm{id} \times g^* : z_r(X, n; \mathcal{A}) \to z_r(X, m; \mathcal{A}),$$

giving us the simplicial abelian group $n \mapsto z_r(X, n; \mathcal{A})$. We let $(z_r(X, *; \mathcal{A}), d)$ denote the associated complex, i.e.

$$d_n := \sum_{i=0}^{n} (-1)^i (\mathrm{id} \times \delta_i^{n-1})^* : z_r(X, n; \mathcal{A}) \to z_r(X, n-1; \mathcal{A}).$$

Definition 5.6.3. We define the higher Chow groups of dimension r with coefficients in $\mathcal{A}$ as

$$\mathrm{CH}_r(X, n; \mathcal{A}) := H_n(z_r(X, *; \mathcal{A})).$$

5.7. Elementary properties

The standard elementary properties of the cycle complexes are also valid with coefficients in $\mathcal{A}$, if properly interpreted.

Proper pushforward

Let $f : X' \to X$ be a proper morphism. For $Y \in \mathbf{Sm}/k$ and $W \subset X' \times Y$, we have the pushforward map

$$f \times \mathrm{id}_* : G_0^W(X' \times Y, f^*\mathcal{A}) \to G_0^{f \times \mathrm{id}(W)}(X \times Y; \mathcal{A})$$

commuting with pullback by morphisms $\mathrm{id} \times g$, for $g : Y' \to Y$ in $\mathbf{Sm}/k$. Thus, the maps $(f \times \mathrm{id}_{\Delta^n})_*$ induce a map of complexes

$$f_* : z_r(X', *; f^*\mathcal{A}) \to z_r(X, *; \mathcal{A})$$

with the evident functoriality.

Flat pullback

Let $f : X' \to X$ be a flat morphism. For $Y \in \mathbf{Sm}/k$ and $W \subset X \times_k Y$, we have the pullback map

$$f \times \mathrm{id}^* : G_0^W(X \times Y, \mathcal{A}) \to G_0^{(f\times\mathrm{id})^{-1}(W)}(X' \times Y, f^*\mathcal{A})$$

commuting with the pullback maps $\mathrm{id} \times g^*$ for $g : Y' \to Y$ a map in $\mathbf{Sm}/k$. Since f is flat, the codimension of W is preserved, hence the pullback maps $f \times \mathrm{id}^*_{\Delta^n}$ induce a map of complexes

$$f^* : z_r(X, *; \mathcal{A}) \to z_r(X', *; f^*\mathcal{A})$$

functorially in f.

Elementary moving lemmas and the homotopy property

Definition 5.7.1. Fix a $Y \in \mathbf{Sm}/k$ and let $\mathcal{C}$ be a finite set of locally closed subsets of Y. Let $X \times Y_r^{\mathcal{C}}(n)$ be the set of irreducible dimension $r+n$ closed subsets W of $X \times Y \times \Delta^n$ such that W is in $X \times Y_r(n)$ and for each $C \in \mathcal{C}$

$$W \cap X \times C \times \Delta^n \text{ is in } \mathcal{S}_{(r)}^{X\times C}(n).$$

We have the subcomplex $z_r(X \times Y, *; \mathcal{F})_{\mathcal{C}}$ of $z_r(X \times Y, *; \mathcal{F})$, with

$$z_r(X \times Y, n; \mathcal{F})_{\mathcal{C}} = \bigoplus_{W \in X \times Y_r^{\mathcal{C}}(n)} \mathcal{K}_0^{\mathcal{A}}(W).$$

Exactly the same proof as for [**6**, Lemma 2.2], using translation by GL_n, gives the following.

Lemma 5.7.2. *Let $\mathcal{C}$ be a finite set of locally closed subsets of Y, with $Y = \mathbb{A}^n$ or $Y = \mathbb{P}^{n-1}$. Then the inclusion*

$$z_r(X \times Y, *; \mathcal{A})_{\mathcal{C}} \to z_r(X \times Y, *; \mathcal{A})$$

is a quasi-isomorphism.

Similarly, we have the following lemma.

Lemma 5.7.3. *The pullback map*

$$z_r(X, *; \mathcal{A}) \to z_{r+1}(X \times \mathbb{A}^1; \mathcal{A})$$

is a quasi-isomorphism.

5.8. Localization

Let $j : U \to X$ be an open immersion with closed complement $i : Z \to X$. Let Y be in $\mathbf{Sm}/k$. If $W \subset X \times Y$ is an irreducible closed subset supported in $Z \times Y$, then $i \times \mathrm{id}$ induces an isomorphism

$$i \times \mathrm{id}_* : G_0^W(Z \times Y, \mathcal{A}) \to G_0^W(X \times Y; \mathcal{A}),$$

which in turn induces the isomorphism

$$i_* : \mathcal{K}_0^{i^*\mathcal{A}}(W) \to \mathcal{K}_0^{\mathcal{A}}(W).$$

Similarly, if the generic point of W lives over $U \times Y$, then we have the surjection

$$j \times \mathrm{id}^* : G_0^W(X \times Y; \mathcal{A}) \to G_0^{W \cap U \times Y}(U \times Y, \mathcal{A})$$

inducing an isomorphism

$$j^* : \mathcal{K}_0^{\mathcal{A}}(W) \to \mathcal{K}_0^{\mathcal{A}}(W \cap U \times Y).$$

This yields the termwise exact sequence of complexes

$$0 \to z_r(Z, *, \mathcal{A}) \xrightarrow{i_*} z_r(X, *; \mathcal{A}) \xrightarrow{j^*} z_r(U, *, \mathcal{A}). \tag{5.1}$$

The lemma below follows from [**31**, § 7, Theorem 8.2].

Lemma 5.8.1. *The inclusion*

$$j^*(z_r(X, *, \mathcal{A})) \subset z_r(U, *, \mathcal{A})$$

is a quasi-isomorphism.

Therefore, we have the following corollary.

Corollary 5.8.2. *The sequence (5.1) determines a canonical distinguished triangle in $D^-(\mathbf{Ab})$, and we have the long exact localization sequence*

$$\cdots \to \mathrm{CH}_r(Z, n; \mathcal{A}) \xrightarrow{i_*} \mathrm{CH}_r(X, n; \mathcal{A}) \xrightarrow{j^*} \mathrm{CH}_r(U, n; \mathcal{A}) \to \mathrm{CH}_r(Z, n-1; \mathcal{A}) \to \cdots .$$

This in turn yields the Mayer–Vietoris distinguished triangle for $X = U \cup V$, $U, V \subset X$ Zariski open subschemes

$$z_r(X, *; \mathcal{A}) \to z_r(U, *; \mathcal{A}) \oplus z_r(V, *; \mathcal{A}) \to z_r(U \cap V, *; \mathcal{A}) \to z_r(X, * - 1; \mathcal{A}). \tag{5.2}$$

5.9. Reduced norm

For $X \in \mathbf{Sch}_k$, $\mathcal{A} = k$, the complex $z_r(X, *; k)$ is just Bloch's cycle complex $z_r(X, *)$. Indeed, for a field F, we have the canonical identification of $K_0(F)$ with $\mathbb{Z}$ by the dimension function, giving the isomorphism

$$z_r(X, n; k) = \bigoplus_{w \in X_{(r)}(n)} K_0(k(w)) \cong \bigoplus_{w \in X_{(r)}(n)} \mathbb{Z} = z_r(X, n).$$

In addition, if $W \subset X \times \Delta^n$ is an integral closed subscheme of dimension d, $i : \Delta^{n-1} \to \Delta^n$ is a codimension one face and if W is not contained in $X \times i(\Delta^{n-1})$, then it follows directly from Serre's intersection multiplicity formula that the image of $(\mathrm{id} \times i)^*([\mathcal{O}_W])$ in $\bigoplus_{w \in (X \times \Delta^{n-1})_{(d-1)}} K_0(k(w))$ goes to the pullback cycle $(\mathrm{id} \times i)^*([W])$ under the isomorphism

$$\bigoplus_{w \in (X \times \Delta^{n-1})_{(d-1)}} K_0(k(w)) \cong z_{d-1}(X \times \Delta^{n-1}).$$

Now take $\mathcal{A}$ to be a sheaf of Azumaya algebras on X. The collection of reduced norm maps

$$\mathrm{Nrd}_{\mathcal{A}_{k(w)}} : K_0(k(w); \mathcal{A}) \to K_0(k(w))$$

defines the homomorphism

$$\mathrm{Nrd}_{X,n;\mathcal{A}} : z_r(X, n; \mathcal{A}) \to z_r(X, n).$$

Lemma 5.9.1. *The maps* $\mathrm{Nrd}_{X,n;\mathcal{A}}$ *define a map of simplicial abelian groups*

$$n \mapsto [\mathrm{Nrd}_{X,n;\mathcal{A}} : z_r(X, n; \mathcal{A}) \to z_r(X, n)].$$

Proof. We note that the maps $\mathrm{Nrd}_{X',n;\mathcal{A}}$ for $X' \to X$ étale define a map of presheaves on $X_{\text{ét}}$. Both $z_r(X, n; \mathcal{A})$ and $z_r(X, n)$ are sheaves for the Zariski topology on X and $\mathrm{Nrd}_{X,n;\mathcal{A}}$ defines a map of sheaves, so we may assume that X is local. If $X' \to X$ is an étale cover, then $z_r(X, n; \mathcal{A}) \to z_r(X', n; \mathcal{A})$ and $z_r(X, n) \to z_r(X', n)$ are injective, so we may replace X with any étale cover. Since $\mathcal{A}$ is locally a sheaf of matrix algebras on $X_{\text{ét}}$, we may assume that $\mathcal{A} = M_n(\mathcal{O}_X)$. In this case, $\mathrm{Nrd}_{X,n;\mathcal{A}}$ is just the Morita isomorphism; we thus may extend $\mathrm{Nrd}_{X,n;\mathcal{A}}$ to the Morita isomorphism

$$\mathrm{Nrd}^W : G_0^W(X \times \Delta^n; \mathcal{A}) \to G_0^W(X \times \Delta^n)$$

for every $W \in \mathcal{S}^X_{(r)}(n)$. But the pullback maps $g^* : z_r(X, n; \mathcal{A}) \to z_r(X, m; \mathcal{A})$ and $g^* : z_r(X, n) \to z_r(X, m)$ for $g : [m] \to [n]$ in **Ord** are defined by lifting elements in $z_r(X, n; \mathcal{A})$ (respectively $z_r(X, n)$) to $G_0^W(X \times \Delta^n; \mathcal{A})$ (respectively $G_0^W(X \times \Delta^n)$) for some W, applying $(\mathrm{id} \times g)^*$ and mapping to $z_r(X, m; \mathcal{A})$ (respectively $z_r(X, m)$). Thus the maps $\mathrm{Nrd}_{X,n;\mathcal{A}}$ define an isomorphism of simplicial abelian groups, completing the proof. □

Thus we have maps

$$\mathrm{Nrd}_{X,\mathcal{A}} : z_r(X, *; \mathcal{A}) \to z_r(X, *),$$
$$\mathrm{Nrd}_{X;\mathcal{A}} : \mathrm{CH}_r(X, n; \mathcal{A}) \to \mathrm{CH}_r(X, n).$$

The naturality properties of Nrd show that the maps $\mathrm{Nrd}_{X,\mathcal{A}}$ are natural with respect to flat pullback and proper pushforward (on the level of complexes).

6. The spectral sequence

We are now ready for the first of our main constructions and results. We begin by discussing the *homotopy coniveau tower* associated to the G-theory of a sheaf of Azumaya algebras $\mathcal{A}$ on a scheme X. Our main result (Theorem 6.1.3) is the identification of the layers in the homotopy coniveau tower with the Eilenberg–Mac Lane spectra associated to the twisted cycle complex $z_p(X,*;\mathcal{A})$. The proof is exactly the same as for standard K-theory $K(X)$ (see [**33**, **34**]), except that at one point we need to use an extension of some regularity results from $K(-)$ to $K(-;\mathcal{A})$; this extension is given in Appendix B.

We then turn to the case $X = \operatorname{Spec} k$, where we have the motivic Postnikov tower for the presheaf $K^{\mathcal{A}}$. We show how our computation of the layers in the homotopy coniveau tower for $K^A(X) = K(X; A \otimes_k \mathcal{O}_X)$, for each $X \in \mathbf{Sm}/k$, lead to a computation of the layers in the motivic Postnikov tower for K^A. This completes the proof of our first main Theorem 1 (see Theorem 6.5.5). We conclude this section with a comparison of the reduced norm maps in motivic cohomology and K-theory, and some computations of the Atiyah–Hirzebruch spectral sequence in low degrees.

6.1. The homotopy coniveau filtration

Following [**33**] we define

$$G_{(p)}(X,n;\mathcal{A}) := \varinjlim_{W\in\mathcal{S}^X_{(p)}(n)} G^W(X\times\Delta^n;\mathcal{A})$$

giving the simplicial spectrum $n \mapsto G_{(p)}(X,n;\mathcal{A})$, with associated total spectrum denoted $G_{(p)}(X,-;\mathcal{A})$. Note that, for all $p \geqslant d_X$, the 'forget supports' map

$$G_{(p)}(X,-;\mathcal{A}) \to G(X\times\Delta^*;\mathcal{A})$$

is an isomorphism.

Remark 6.1.1. In order that $n \mapsto G_{(p)}(X,n;\mathcal{A})$ form a simplicial spectrum, one needs to make the G-theory with support strictly functorial. This is done by first replacing the categories $\mathcal{M}_{X\times\Delta^n}(\mathcal{A})$ with the full subcategory $\mathcal{M}_{X\times\Delta^n}(\mathcal{A})'$ of $\mathcal{A}$-modules which are coherent sheaves on $X\times\Delta^n$ and are *flat* with respect to all inclusions $X\times F \to X\times\Delta^n$, $F \subset \Delta^n$ a face. Quillen's resolution theorem shows that

$$K(\mathcal{M}_{X\times\Delta^n}(\mathcal{A})') \to K(\mathcal{M}_{X\times\Delta^n}(\mathcal{A}))$$

is a weak equivalence. One then uses the usual trick of replacing $\mathcal{M}_{X\times\Delta^n}(\mathcal{A})'$ with sequences of objects together with isomorphisms (indexed by the morphisms in **Ord**) to make the pullbacks strictly functorial.

A similar construction makes $Y \mapsto G(X\times_k Y,\mathcal{A})$ strictly functorial on $\mathbf{Sm}/k$; we will use this modification from now on without further mention.

Since $G(X\times -;\mathcal{A})$ is homotopy invariant, the canonical map

$$G(X;\mathcal{A}) \to G_{(d_X)}(X,-;\mathcal{A})$$

is a weak equivalence. This gives us the *homotopy coniveau tower*

$$\cdots \to G_{(p-1)}(X,-;\mathcal{A}) \to G_{(p)}(X,-;\mathcal{A}) \to \cdots \to G_{(d_X)}(X,-;\mathcal{A}) \sim G(X;\mathcal{A}). \quad (6.1)$$

Setting $G_{(p/p-r)}(X,-;\mathcal{A})$ equal to the homotopy cofibre of $G_{(p-r)}(X,-;\mathcal{A}) \to G_{(p)}(X,-;\mathcal{A})$, the tower (6.1) yields the spectral sequence

$$E_2^{p,q} = \pi_{-p-q}(G_{(q/q-1)}(X,-;\mathcal{A})) \Longrightarrow G_{-p-q}(X;\mathcal{A}). \quad (6.2)$$

Remarks 6.1.2.

(1) Let T be a finite type k-scheme, $W \subset T$ a closed subscheme with open complement $j : U \to T$ and $\mathcal{A}$ a sheaf of Azumaya algebras on T. We have the homotopy fibre sequence

$$G^W(T;\mathcal{A}) \to G(T;\mathcal{A}) \to G(U;j^*\mathcal{A}).$$

In addition, the spectra $G(T;\mathcal{A})$ and $G(U;j^*\mathcal{A})$ are -1 connected, and the restriction map

$$j^* : G_0(T;\mathcal{A}) \to G_0(U;j^*\mathcal{A})$$

is surjective. Thus $G^W(T;\mathcal{A})$ is -1 connected, hence the spectra $G_{(p)}(X,n;\mathcal{A})$ are -1 connected for all n and p.

(2) Noting that $\mathcal{S}^X_{(p)}(n) = \emptyset$ for $p+n < 0$, the -1 connectedness of $G_{(p)}(X,n;\mathcal{A})$ implies that

$$\pi_N(G_{(p)}(X,-;\mathcal{A})) = 0$$

for $N < -p$, i.e. that $G_{(p)}(X,-;\mathcal{A})$ is $-p-1$ connected. This in turn implies that $G_{(p/p-r)}(X,-;\mathcal{A})$ is $-p-1$ connected for all $r \geqslant 0$, that the natural map

$$G(X;\mathcal{A}) \to \operatorname{holim}_n G_{(d_X/-n)}(X;\mathcal{A})$$

is a weak equivalence and that the spectral sequence (6.2) is strongly convergent.

Our main result in this section is the following theorem.

Theorem 6.1.3. *There is a natural isomorphism*

$$\pi_n(G_{(p/p-1)}(X,-;\mathcal{A})) \cong \mathrm{CH}_p(X,n;\mathcal{A}).$$

Corollary 6.1.4. *There is a strongly convergent spectral sequence*

$$E_2^{p,q} = \mathrm{CH}_q(X,-p-q;\mathcal{A}) \Longrightarrow G_{-p-q}(X;\mathcal{A}).$$

The proof is in three steps: we first define a natural 'cycle map'

$$\mathrm{cyc} : \pi_n(G_{(p/p-1)}(X,-;\mathcal{A})) \to \mathrm{CH}_p(X,n;\mathcal{A}),$$

which will define the isomorphism. We then go on to use the localization properties of $G_{(p/p-1)}(X,-;\mathcal{A})$ and $\mathrm{CH}_p(X,*;\mathcal{A})$ to reduce to the case $X = \operatorname{Spec} F$, F a field, and finally we apply Theorem 3.2.4 to complete the proof.

6.2. The cycle map

We have already seen in Remark 6.1.2 that the spectra $G_{(p)}(X, n; \mathcal{A})$ are all -1 connected. A similar argument shows that the spectra $G_{(p/p-r)}(X, n; \mathcal{A})$ are all -1 connected.

Let $\mathrm{EM}(\pi_0 G_{(p/p-1)}(X, n; \mathcal{A}))$ denote the Eilenberg–Mac Lane spectrum with $\pi_0 = \pi_0 G_{(p/p-1)}(X, n; \mathcal{A})$ and all other homotopy groups 0. Since $G_{(p/p-1)}(X, n; \mathcal{A})$ is -1 connected, we have the map of spectra

$$\varphi_n : G_{(p/p-1)}(X, n; \mathcal{A}) \to \mathrm{EM}(\pi_0 G_{(p/p-1)}(X, n; \mathcal{A}))$$

natural in n. Letting $\mathrm{EM}(\pi_0 G_{(p/p-1)}(X, -; \mathcal{A}))$ denote the total spectrum of the simplicial spectrum $n \mapsto \mathrm{EM}(\pi_0 G_{(p/p-1)}(X, n; \mathcal{A}))$, this gives us the natural map of spectra

$$\varphi_X : G_{(p/p-1)}(X, -; \mathcal{A}) \to \mathrm{EM}(\pi_0 G_{(p/p-1)}(X, -; \mathcal{A})).$$

Lemma 6.2.1. *There is a natural map*

$$\psi_n : \pi_0(G_{(p/p-1)}(X, n; \mathcal{A})) \to z_p(X, n; \mathcal{A}),$$

which is an isomorphism if $X = \operatorname{Spec} F$, F a field.

Proof. Let $W \subset X \times \Delta^n$ be a closed subset with generic points $w_1, \ldots, w_r$. We have the evident restriction map

$$G_0^W(X \times \Delta^n; \mathcal{A}) = G_0(W; \mathcal{A}) \to \bigoplus_i G_0(k(w_i); \mathcal{A}).$$

Since $\mathbb{Z}_{\mathcal{A}}(W) = \bigoplus_i G_0(k(w_i); \mathcal{A})$, we may define

$$\psi_n : \pi_0(G_{(p/p-1)}(X, n; \mathcal{A})) \to z_p(X, n; \mathcal{A})$$

by projecting $\bigoplus_i G_0(k(w_i); \mathcal{A})$ on the factors coming from the generic points of $W \in \mathcal{S}^X_{(p)}(n)$ having dimension $n + r$ over k. By Lemma 5.5.3, ψ_n is natural in n.

Suppose now that $X = \operatorname{Spec} F$, F a field; making a base-change and replacing p with $p - \dim_k X$, we may assume that $F = k$ (note that in this case we may assume $p \leqslant 0$). This implies that $X \times \Delta^n \cong \mathbb{A}^n_k$. It is easy to see that, for each $W \in \mathcal{S}^X_{(p)}(n)$, the intersection of $-p$ hypersurfaces of sufficiently high degree, containing W, is in $\mathcal{S}^X_{(p)}(n)$ and has pure dimension $p + n$. Thus the closed subsets $W \in \mathcal{S}^X_{(p)}(n)$ of pure dimension $p + n$ are cofinal in $\mathcal{S}^X_{(p)}(n)$.

Identify $z_p(X, n; \mathcal{A})$ with the direct sum $\bigoplus_w G_0(k(w); \mathcal{A})$ as w runs over the generic points of $\mathcal{S}^X_{(p)}(n)$ of dimension exactly n. From the localization sequence, we see that the map

$$\varinjlim_{W \in \mathcal{S}^X_{(p)}(n)} G_0(W; \mathcal{A}) \to \bigoplus_w G_0(k(w); \mathcal{A})$$

is surjective, with kernel the subgroup generated by the image of groups $G_0(W'; \mathcal{A})$ with $\dim W' < p + n$ and $W' \subset W$ for some $W \in \mathcal{S}^X_{(p)}(n)$. The result thus follows from Lemma 6.2.2 below. $\square$

Lemma 6.2.2. *Suppose that $X = \operatorname{Spec} k$. Let $W' \subset \Delta^n_k$ be a closed subset with $W' \in \mathcal{S}^{(q)}_X(n)$ and $\operatorname{codim}_{\Delta^n} W' > q$. Then the natural map*

$$G_0(W'; \mathcal{A}) \to \varinjlim_{W \in \mathcal{S}^{(q)}_X(n)} G_0(W; \mathcal{A})$$

is the zero-map.

Proof. This is a modification of the proof of Sherman [**52**] that the Gersten complex for $\mathbb{A}^n$ is exact. We may assume that k is infinite. Take a general linear projection

$$\pi : \Delta^n_k = \mathbb{A}^n_k \to \mathbb{A}^{n-1}_k$$

and let $W = \pi^{-1}(\pi(W'))$. Then

$$\pi : W' \to \mathbb{A}^{n-1}_k$$

is finite and W is in $\mathcal{S}^{(q)}_X(n)$. In addition, π makes $\mathbb{A}^n$ into a trivial $\mathbb{A}^1$-bundle over $\mathbb{A}^{n-1}$. Thus the canonical section $s : W' \to W' \times_{\mathbb{A}^{n-1}} \mathbb{A}^n$ makes $W' \times_{\mathbb{A}^{n-1}} \mathbb{A}^n \to W'$ into a trivial line bundle over W', hence $s(W') \subset W' \times_{\mathbb{A}^{n-1}} \mathbb{A}^n$ is a principal Cartier divisor. Letting t be a defining equation, we have the functorial exact sequence

$$0 \to p_{2*}p_1^*(M) \xrightarrow{\times t} p_{2*}p_1^*(M) \to i_*(M) \to 0, \quad M \in \mathcal{M}_{W'}(\mathcal{A}),$$

where $p_1 : W' \times_{\mathbb{A}^{n-1}} \mathbb{A}^n \to W'$, $p_2 : W' \times_{\mathbb{A}^{n-1}} \mathbb{A}^n \to W \subset \mathbb{A}^n$ are the projections and $i : W' \to W$ is the inclusion. Thus

$$i_* : G_0(W'; \mathcal{A}) \to G_0(W; \mathcal{A})$$

is the zero-map, completing the proof. □

We denote the composition $\mathrm{EM}(\psi_n) \circ \varphi_n$ by

$$\mathrm{cyc}_n(X) : G_{(p)}(X, n; \mathcal{A}) \to \mathrm{EM}(z_p(X, n; \mathcal{A}))$$

and the map on the associated total spectra by

$$\mathrm{cyc}(X) : G_{(p)}(X, -; \mathcal{A}) \to \mathrm{EM}(z_p(X, -; \mathcal{A})).$$

6.3. Localization

Consider an open subscheme $j : U \to X$ with closed complement $i : Z \to X$. We let $\mathcal{S}^{U_X}_{(r)}(n) \subset \mathcal{S}^U_{(r)}(n)$ denote the set of closed subsets $W \subset U \times \Delta^n$ such that

(1) W is in $\mathcal{S}^U_{(r)}(n)$,

(2) the closure $\bar{W}$ of W in $X \times \Delta^n$ is in $\mathcal{S}^X_{(r)}(n)$.

Define the spectrum $G_{(r)}(U_X, n; \mathcal{A})$ by

$$G_{(r)}(U_X, n; \mathcal{A}) := \varinjlim_{W \in \mathcal{S}^{U_X}_{(r)}(n)} G^W(U \times \Delta^n; \mathcal{A})$$

giving us the simplicial spectrum $n \mapsto G_{(r)}(U_X, n; \mathcal{A})$ and the associated total spectrum $G_{(r)}(U_X, -; \mathcal{A})$. The restriction map

$$j^* : G_{(r)}(X, n; \mathcal{A}) \to G_{(r)}(U, n; \mathcal{A})$$

factors through $G_{(r)}(U_X, n; \mathcal{A})$, giving us the commutative diagram

$$\begin{tikzcd} G_{(r)}(X, -; \mathcal{A}) \arrow[r, "\hat{j}^*"] \arrow[dr, "j^*"'] & G_{(r)}(U_X, -; \mathcal{A}) \arrow[d, "\psi"] \\ & G_{(r)}(U, -; \mathcal{A}) \end{tikzcd}$$

Lemma 6.3.1. *The sequence*

$$G_{(r)}(Z, -; i^*\mathcal{A}) \xrightarrow{i_*} G_{(r)}(X, -; \mathcal{A}) \xrightarrow{\hat{j}^*} G_{(r)}(U_X, -; \mathcal{A})$$

is a homotopy fibre sequence.

Proof. In fact, it follows from Quillen's localization theorem for $G(-; \mathcal{A})$ that the sequence

$$G_{(r)}(Z, n; i^*\mathcal{A}) \xrightarrow{i_*} G_{(r)}(X, n; \mathcal{A}) \xrightarrow{\hat{j}^*} G_{(r)}(U_X, n; \mathcal{A})$$

is a homotopy fibre sequence for each n, whence the result. $\square$

The localization techniques of [**31**, § 7, Theorem 8.2] yield the following result.

Theorem 6.3.2. *The map*

$$\psi : G_{(r)}(U_X, -; \mathcal{A}) \to G_{(r)}(U, -; \mathcal{A})$$

is a weak equivalence.

Thus, we have the following corollary.

Corollary 6.3.3. *The sequences*

$$G_{(r)}(Z, -; \mathcal{A}) \xrightarrow{i_*} G_{(r)}(X, -; \mathcal{A}) \xrightarrow{j^*} G_{(r)}(U, -; \mathcal{A})$$

and

$$G_{(r/r-s)}(Z, -; \mathcal{A}) \xrightarrow{i_*} G_{(r/r-s)}(X, -; \mathcal{A}) \xrightarrow{j^*} G_{(r/r-s)}(U, -; \mathcal{A})$$

are homotopy fibre sequences.

In addition, we have the following lemma.

Lemma 6.3.4. *The diagram*

$$\begin{tikzcd}
G_{(r/r-1)}(Z,-;\mathcal{A}) \arrow[r, "i_*"] \arrow[d, "\mathrm{cyc}"'] & G_{(r/r-1)}(X,-;\mathcal{A}) \arrow[r, "j^*"] \arrow[d, "\mathrm{cyc}"'] & G_{(r/r-1)}(U,-;\mathcal{A}) \arrow[d, "\mathrm{cyc}"'] \\
\mathrm{EM}(z_r(Z,-;\mathcal{A})) \arrow[r, "i_*"'] & \mathrm{EM}(z_r(X,-;\mathcal{A})) \arrow[r, "j^*"'] & \mathrm{EM}(z_r(U,-;\mathcal{A}))
\end{tikzcd}$$

defines a map of distinguished triangles in $\mathcal{SH}$.

Proof. It is clear the maps cyc_n are functorial with respect to the closed immersion i and the open immersion j, hence the diagram

$$\begin{tikzcd}
\pi_0 G_{(r/r-1)}(Z,n;\mathcal{A}) \arrow[r, "i_*"] \arrow[d, "\mathrm{cyc}_n"'] & \pi_0 G_{(r/r-1)}(X,n;\mathcal{A}) \arrow[r, "\hat{\jmath}^*"] \arrow[d, "\mathrm{cyc}_n"'] & \pi_0 G_{(r/r-1)}(U_X,n;\mathcal{A}) \arrow[d, "\mathrm{cyc}_n"'] \\
z_r(Z,n;\mathcal{A}) \arrow[r, "i_*"'] & z_r(X,n;\mathcal{A}) \arrow[r, "j^*"'] & z_r(U_X,n;\mathcal{A})
\end{tikzcd}$$

commutes for each n. Similarly, the diagram

$$\begin{tikzcd}
\pi_0 G_{(r/r-1)}(U_X,n;\mathcal{A}) \arrow[r, "\psi"] \arrow[d, "\mathrm{cyc}_n"'] & \pi_0 G_{(r/r-1)}(U,n;\mathcal{A}) \arrow[d, "\mathrm{cyc}_n"'] \\
z_r(U_X,n;\mathcal{A}) \arrow[r, "\psi"'] & z_r(U,n;\mathcal{A})
\end{tikzcd}$$

commutes for each n. The result follows directly from this. $\square$

Proposition 6.3.5. *Suppose that the map*

$$\mathrm{cyc}(X) : G_{(r/r-1)}(X,-;\mathcal{A}) \to \mathrm{EM}(z_r(X,-;\mathcal{A}))$$

is a weak equivalence for all X of the form $X = \mathrm{Spec}\, F$, F a finitely generated field extension of k. Then $\mathrm{cyc}(X)$ is a weak equivalence for all X essentially of finite type over k.

Proof. This follows from Corollary 5.8.2, Corollary 6.3.3, Lemma 6.3.4 and noetherian induction. $\square$

6.4. The case of a field

We have reduced the proof of Theorem 6.1.3 to the case $X = \mathrm{Spec}\, k$, where we may apply the method of [**33**, § 6.4], as explained in § 3.2.

Let $K^{\mathcal{A}} \in \mathbf{Spt}_{S^1}(k)$ be the presheaf of spectra $X \mapsto K(X;\mathcal{A})$. We note the following lemma.

Lemma 6.4.1.

(1) *$K^{\mathcal{A}}$ is homotopy invariant and satisfies Nisnevich excision.*

(2) *$K^{\mathcal{A}}$ is connected.*

(3) *$K^{\mathcal{A}} \cong \Omega_T(K^{\mathcal{A}})$.*

We have already seen (1); (2) follows from the weak equivalence $K(-;\mathcal{A}) \to G(-;\mathcal{A})$ on $\mathbf{Sm}/k$; and (3) follows from the projective bundle formula (which in turn is a direct consequence of localization and homotopy invariance)

$$\Omega_T(K^{\mathcal{A}})(Y) \cong \mathrm{fib}[K(Y \times \mathbb{P}^1, \mathcal{A}) \xrightarrow{i_\infty^*} K(Y, \mathcal{A})] \cong K(Y, \mathcal{A}).$$

In particular, for Y in $\mathbf{Sm}/k$ and integer $q \geqslant 0$, we have the simplicial abelian group $z^q(Y, -; K^{\mathcal{A}})$ and the cycle map (see Definition 3.1.15)

$$\mathrm{cyc}_{K^{\mathcal{A}}}(Y) : s^q(Y, -; K^{\mathcal{A}}) \to \mathrm{EM}(z^q(Y, -; K^{\mathcal{A}})).$$

Lemma 6.4.2. *Let Y be in $\mathbf{Sm}/k$, $d = \dim_k Y$. Fix an integer $q \geqslant 0$ and let $r = d - q$. There is a weak equivalence of simplicial spectra*

$$n \mapsto \varphi_n : s^q(Y, n; K^{\mathcal{A}}) \to G_{(r/r-1)}(Y, n; \mathcal{A})$$

and an isomorphism of simplicial abelian groups

$$n \mapsto \psi_n : z^q(Y, n; K^{\mathcal{A}}) \to z_r(Y, n; \mathcal{A})$$

such that the diagram of total spectra

$$\begin{tikzcd}
s^q(Y, -; K^{\mathcal{A}}) \arrow[r, "\varphi"] \arrow[d, "\mathrm{cyc}_{K^{\mathcal{A}}}(Y)"'] & G_{(r/r-1)}(Y, -; \mathcal{A}) \arrow[d, "\mathrm{cyc}(Y)"] \\
\mathrm{EM}(z^q(Y, -; K^{\mathcal{A}})) \arrow[r, "\mathrm{EM}(\psi)"'] & \mathrm{EM}(z_r(Y, -; \mathcal{A}))
\end{tikzcd}$$

commutes in $\mathcal{SH}$.

Proof. We have the natural transformation of functors on $\mathbf{Sm}/k$

$$K(-;\mathcal{A}) \to G(-;\mathcal{A}).$$

In particular, for $T \in \mathbf{Sm}/k$ and $W \subset T$ a closed subset, we have the map

$$\varphi_{T,W} : K^W(T;\mathcal{A}) \to G^W(T;\mathcal{A})$$

defining a natural transformation of presheaves of spectra on Imm_k. Applying $\varphi_{-,-}$ to the colimit of spectra with supports forming the definition of $s^q(Y, n; K^{\mathcal{A}})$ and $G_{(r/r-1)}(Y, n; \mathcal{A})$ gives φ_n. The map ψ_n is defined similarly, using the maps $\pi_0(\varphi_{T,W})$. The compatibility with the cycle maps follows directly from the definitions. $\square$

Thus, to prove that $\mathrm{cyc}(Y) : G_{(r/r-1)}(Y, -; \mathcal{A}) \to \mathrm{EM}(z_r(Y, -; \mathcal{A}))$ is an isomorphism in $\mathcal{SH}$ for all r and all $Y \in \mathbf{Sm}/k$ (in particular, for $Y = \operatorname{Spec} k$), it suffices to show the following lemma.

Lemma 6.4.3. *The object $K^{\mathcal{A}} \in \mathbf{Spt}_{S^1}(k)$ is well-connected. The map*

$$\mathrm{cyc}_{K^{\mathcal{A}}}(Y) : s^q(Y, -; K^{\mathcal{A}}) \to \mathrm{EM}(z^q(Y, -; K^{\mathcal{A}}))$$

is an isomorphism in $\mathcal{SH}$ for all q and all $Y \in \mathbf{Sm}/k$.

Proof. By Theorem 3.2.4, the first statement implies the second.

We have already seen that $K^{\mathcal{A}}$ is connected (Lemma 6.4.1 (2)). By Lemma 6.4.1 (3) we need only show that

$$\pi_n(K(\hat{\Delta}^*_{k(Y)}; \mathcal{A})) = 0$$

for $n \neq 0$.

We have shown in [**33**, Theorem 6.4.1] that the theory $Y \mapsto K(Y)$ is well-connected, in particular, that $\pi_n(K(\hat{\Delta}^*_{k(Y)}; \mathcal{A})) = 0$ for $n \neq 0$ and for $\mathcal{A} = k$. Using the results of Appendix B, especially Proposition B.5, the same argument shows $\pi_n(K(\hat{\Delta}^*_{k(Y)}; \mathcal{A})) = 0$ for $n \neq 0$ for arbitrary $\mathcal{A}$. □

This completes the proof of Theorem 6.1.3.

6.5. The slice filtration for an Azumaya algebra

By Proposition 5.3.6, $\mathbb{Z}_{\mathcal{A}}$ is a birational motivic sheaf, hence the cycle complex $z^q(X, *; \mathbb{Z}_{\mathcal{A}}(q)[2q])$ is defined.

Proposition 6.5.1. *Let $\mathcal{A}$ be a central simple algebra over a field k. For $X \in \mathbf{Sm}/k$, there is an isomorphism of complexes*

$$z^q(X, *; \mathcal{A}) \xrightarrow{\varphi_{X,\mathcal{A}}} z^q(X, *; \mathbb{Z}_{\mathcal{A}}(q)[2q]),$$

natural with respect to proper pushforward and flat pullback.

Proof. We first define for each $n, q \geqslant 0$ an isomorphism

$$\varphi_{X,\mathcal{A},n} : z^q(X, n; \mathcal{A}) \cong z^q(X, n; \mathbb{Z}_{\mathcal{A}}(q)[2q]).$$

Indeed, by definition,

$$z^q(X, n; \mathcal{A}) = \bigoplus_{w \in X^{(q)}(n)} \mathcal{K}_0^{\mathcal{A}}(k(w)).$$

By Remark 4.3.4, we have

$$z^q(X, n; \mathbb{Z}_{\mathcal{A}}(q)[2q]) = \bigoplus_{w \in X^{(q)}(n)} \mathbb{Z}_{\mathcal{A}}(k(w)).$$

But $\mathbb{Z}_{\mathcal{A}}$ is just $\mathcal{K}_0^{\mathcal{A}}$ considered as a sheaf with transfers, giving us the desired isomorphism.

This isomorphism $\varphi_{X,\mathcal{A},n}$ is clearly compatible with proper pushforward and flat pullback. It thus suffices to show that the $\varphi_{X,\mathcal{A},n}$ are compatible with the face maps $X \times \Delta^{n-1} \to X \times \Delta^n$.

Let $k \to k'$ be an extension of fields. As the base-change maps

$$z^q(X, n; \mathcal{A}) \to z^q(X_{k'}, n; \mathcal{A}(q)[2q]),$$
$$z^q(X, n; \mathbb{Z}_{\mathcal{A}}(q)[2q]) \to z^q(X_{k'}, n; \mathbb{Z}_{\mathcal{A}}(q)[2q])$$

are injective, it suffices to check in case $\mathcal{A}$ is a matrix algebra. By Morita equivalence, it suffices to check for $\mathcal{A} = k$.

Recall from Proposition 4.4.5 the isomorphism of simplicial abelian groups

$$n \mapsto [\rho_{X,n} : z^q(X, n; \mathbb{Z}(q)[2q]) \to z^q(X, n)]$$

and from § 5.9 and Lemma 5.9.1 the reduced norm map (of simplicial abelian groups)

$$n \mapsto [\mathrm{Nrd}_{X,n;\mathcal{A}} : z^q(X, n; \mathcal{A}) \to z^q(X, n)].$$

In case $\mathcal{A} = k$, the maps $\mathrm{Nrd}_{X,n;\mathcal{A}}$ are isomorphisms. It is easy to check that (for $\mathcal{A} = k$) the diagram of isomorphisms

$$\begin{tikzcd} z^q(X, n; \mathcal{A}) \arrow[rr, "\varphi_{X,\mathcal{A},n}"] \arrow[dr, "\mathrm{Nrd}_{X,n;\mathcal{A}}"'] & & z^q(X, n; \mathbb{Z}(q)[2q]) \arrow[dl, "\rho_{X,n}"] \\ & z^q(X, n) & \end{tikzcd}$$

commutes. Since both the $\mathrm{Nrd}_{X,n;\mathcal{A}}$ and $\rho_{X,n}$ define maps of simplicial abelian groups, it follows that the $\varphi_{X,\mathcal{A},n}$ define maps of simplicial abelian groups as well. □

Remark 6.5.2. We have the reduced norm map $\mathrm{Nrd}_{\mathcal{A}} : \mathbb{Z}_{\mathcal{A}} \to \mathbb{Z}$ (as a map of Nisnevich sheaves with transfers) inducing a reduced norm map $\mathrm{Nrd}_{\mathcal{A}}(q) : \mathbb{Z}_{\mathcal{A}}(q)[2q] \to \mathbb{Z}(q)[2q]$ and thus a map of complexes

$$\mathrm{Nrd}_{\mathcal{A}}(q)_X : z^q(X, *; \mathbb{Z}_{\mathcal{A}}(q)[2q]) \to z^q(X, *; \mathbb{Z}(q)[2q]).$$

We have as well the reduced norm map of § 5.9:

$$\mathrm{Nrd}_{X;\mathcal{A}} : z^q(X, *; \mathcal{A}) \to z^q(X, *).$$

We claim that the diagram

$$\begin{tikzcd} z^q(X, *; \mathcal{A}) \arrow[r, "\mathrm{Nrd}_{X;\mathcal{A}}"] \arrow[d, "\varphi_{X,\mathcal{A}}"'] & z^q(X, *) \arrow[d, "\varphi_{X,k}"] \\ z^q(X, *; \mathbb{Z}_{\mathcal{A}}(q)[2q]) \arrow[r, "\mathrm{Nrd}_{\mathcal{A}}(q)_X"'] & z^q(X, *; \mathbb{Z}(q)[2q]) \end{tikzcd}$$

commutes. Indeed, on $z^q(X, n; \mathcal{A}) = \bigoplus_w \mathcal{K}_0^{\mathcal{A}}(k(w))$, both compositions are just sums of the reduced norm maps

$$\mathrm{Nrd} : K_0(\mathcal{A}_{k(w)}) \to K_0(k(w)) = \mathbb{Z}.$$

Theorem 6.5.3. *Let A be a central simple algebra over a perfect field k, $Y \in \mathbf{Sm}/k$. Then there is an isomorphism*

$$\psi_{p,q;\mathcal{A}} : \mathrm{CH}^q(Y, 2q-p; \mathcal{A}) \to H^p(Y, \mathbb{Z}_{\mathcal{A}}(q))$$

natural with respect to flat pullback and proper pushforward, and compatible with the respective reduced norm maps.

Proof. This follows from Theorem 4.3.3 and Proposition 6.5.1. □

Corollary 6.5.4. *Let A be a central simple algebra over a perfect field k, $Y \in \mathbf{Sm}/k$. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H^{p-q}(Y, \mathbb{Z}_{\mathcal{A}}(-q)) \Longrightarrow K_{-p-q}(Y; \mathcal{A}).$$

Proof. By Corollary 6.1.4, we have the strongly convergent E_2 spectral sequence

$$E_2^{p,q} = \mathrm{CH}^{-q}(Y, -p-q; \mathcal{A}) \Longrightarrow K_{-p-q}(Y; \mathcal{A}).$$

By Theorem 6.5.3 we have the isomorphism

$$\mathrm{CH}^{-q}(Y, -p-q; \mathcal{A}) \cong H^{p-q}(Y, \mathbb{Z}_{\mathcal{A}}(-q)),$$

yielding the result. □

In fact, we have the following theorem.

Theorem 6.5.5. *Let $\mathcal{A}$ be a central simple algebra over a perfect field k. Then for each $q \geqslant 0$, there is an isomorphism*

$$s_q(K^{\mathcal{A}}) \cong \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_{\mathcal{A}}(q)[2q]).$$

Proof. By Proposition 6.5.1, we have an isomorphism of complexes

$$z^q(X, *, \mathbb{Z}_{\mathcal{A}}(q)[2q]) \cong z^q(X, *; \mathcal{A});$$

as $z^q(X, *; \mathcal{A}) \cong z^q(X, *; K^{\mathcal{A}})$ (Lemma 6.4.3), this gives us an isomorphism of complexes

$$\tau_X : z^q(X, *; K^{\mathcal{A}}) \to z^q(X, *, \mathbb{Z}_{\mathcal{A}}(q)[2q]).$$

Referring to the construction in § 3.2 of functorial models $\tilde{z}^q(K^{\mathcal{A}})$, $\tilde{z}^q(\mathbb{Z}_{\mathcal{A}}(q)[2q])$ for the complexes $z^q(X, *, \mathbb{Z}_{\mathcal{A}}(q)[2q])$, $z^q(X, *; K^{\mathcal{A}})$, the isomorphisms τ_X give rise to an isomorphism in $\mathbf{Spt}_{S^1}(k)$

$$\tau : \mathrm{EM}(\tilde{z}^q(K^{\mathcal{A}})) \to \mathrm{EM}(\tilde{z}^q(\mathbb{Z}_{\mathcal{A}}(q)[2q])).$$

By Proposition 5.3.6, $\mathbb{Z}_{\mathcal{A}}$ is a birational motivic sheaf, hence by Proposition 4.3.2, $\mathbb{Z}_{\mathcal{A}}(q)[2q]$ is well-connected. $K^{\mathcal{A}}$ is well-connected by Lemma 6.4.3. Thus, Corollary 3.2.5 yields isomorphisms (in $\mathcal{H}\,\mathbf{Spt}_{S^1}(k)$, $D(\mathrm{PS}_{\mathbf{Ab}}(\mathbf{Sm}/k))$, respectively)

$$\mathrm{cyc}_{K^{\mathcal{A}}} \circ \varphi_{q,K^{\mathcal{A}}} : s_q(K^{\mathcal{A}}) \to \mathrm{EM}(\tilde{z}^q(K^{\mathcal{A}})),$$
$$\mathrm{cyc}^{\mathrm{mot}}_{\mathbb{Z}_{\mathcal{A}}(q)[2q]} \circ \varphi^{\mathrm{mot}}_{q,\mathbb{Z}_{\mathcal{A}}(q)[2q]} : s_q^{\mathrm{mot}}(\mathbb{Z}_{\mathcal{A}}(q)[2q]) \to \tilde{z}^q(\mathbb{Z}_{\mathcal{A}}(q)[2q]),$$

and therefore we have an isomorphism

$$s_q(K^{\mathcal{A}}) \cong \mathrm{EM}_{\mathbb{A}^1}(s_q^{\mathrm{mot}}(\mathbb{Z}_{\mathcal{A}}(q)[2q]))$$

in $\mathcal{SH}_{S^1}(k)$.

Finally, as $\mathbb{Z}_{\mathcal{A}}$ is a birational motivic sheaf, it follows from Remark 4.2.3 that $s_q^{\mathrm{mot}}(\mathbb{Z}_{\mathcal{A}}(q)[2q]) \cong \mathbb{Z}_{\mathcal{A}}(q)[2q]$, giving us the desired isomorphism

$$s_q(K^{\mathcal{A}}) \cong \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_{\mathcal{A}}(q)[2q]).$$

□

6.6. The reduced norm map

Let A be a central simple algebra over k. We have already mentioned the reduced norm map

$$\mathrm{Nrd} : K_0(A) \to K_0(k)$$

in § 5.2; there are in fact reduced norm maps

$$\mathrm{Nrd} : K_n(A) \to K_n(k)$$

for $n = 0, 1, 2$. For $n = 0, 1$, these may be defined with the help of a splitting field $L \supset k$ for A and Morita equivalence. Use the composition $A \subset A \otimes_k L \cong M_d(L)$ to define maps on the K-groups

$$K_n(A) \to K_n(A_L) \cong K_n(M_d(L)) \cong K_n(L).$$

For $n = 0$, the map $K_0(k) \to K_0(L)$ is an isomorphism; one checks that the resulting map $K_0(A) \to K_0(k)$ is the reduced norm we have already defined. For $n = 1$, one can take L to be Galois over k (with group say G) and use that fact that there is a 1-cocycle $\{\bar{g}_\sigma\} \in Z^1(G; \mathrm{PGL}_d(L))$ such that $A \subset M_d(L)$ is the invariant subalgebra under the G action

$$(\sigma, m) \mapsto \bar{g}_\sigma \cdot {}^\sigma m \cdot \bar{g}_\sigma^{-1}.$$

As $\det : K_1(M_d(L)) \to K_1(L) = L^\times$ is the isomorphism given by Morita equivalence, one sees that the image of $K_1(A)$ in $L^\times$ lands in the G-invariants, i.e. in $k^\times = K_1(k)$.

For $n = 2$, the definition of the reduced norm map (due to Merkurjev–Suslin in the square-free degree case [**38**, Theorem 7.3] and to Suslin in general [**53**, Corollary 5.7]) is more complicated; however, we do have the following result. Let Spl_A be the set of field extensions L/k that split A.

Proposition 6.6.1. *Let $L \supset k$ be an extension field.*

(1) *For $n = 0, 1, 2$, the diagram*

$$\begin{tikzcd} K_n(A_L) \arrow[r, "\mathrm{Nrd}"] \arrow[d, "\mathrm{Nm}_{A_L/A}"'] & K_n(L) \arrow[d, "\mathrm{Nm}_{L/k}"] \\ K_n(A) \arrow[r, "\mathrm{Nrd}"'] & K_n(k) \end{tikzcd}$$

commutes. Here $\mathrm{Nm}_{A_L/A} : K_n(A_L) \to K_n(A)$ is the map on the K-groups induced by the restriction of scalars functor, and similarly for $\mathrm{Nm}_{L/k}$.

(2) *For $n = 0, 1$, the map*

$$\sum \mathrm{Nm}_{A_L/A} : \bigoplus_{L \in \mathrm{Spl}_A} K_n(A_L) \to K_n(A)$$

is surjective. If A has square free index, $\sum \mathrm{Nm}_{A_L/A}$ is surjective for $n = 2$ as well.

For a proof of the last statement, see [**38**, Theorem 5.2].

Let $L \supset k$ be a field. Since $\mathrm{CH}^m(L, n; \mathcal{A}) = 0$ for $m > n$, due to reasons of dimension, we have the edge homomorphism

$$p_{n,L;A} : \mathrm{CH}^n(L, n; A) \to K_n(A_L)$$

coming from the spectral sequence of Corollary 6.1.4.

Let L/k be a finite field extension. We let

$$\mathrm{Nm}_{L/k} : \mathrm{CH}^q(Y_L, p; A) \to \mathrm{CH}^q(Y, p; A)$$

denote the pushforward map for the finite morphism $Y_L \to Y$.

Lemma 6.6.2. *Let L/k be a finite field extension, $f : \mathrm{Spec}\, L \to \mathrm{Spec}\, k$ the corresponding morphism. Then the diagram*

$$\begin{tikzcd} \mathrm{CH}^n(L, n; A) \arrow[r, "p_{n,L;A}"] \arrow[d, "\mathrm{Nm}_{L/k}"'] & K_n(A_L) \arrow[d, "\mathrm{Nm}_{A_L/A}"] \\ \mathrm{CH}^n(k, n; A) \arrow[r, "p_{n,k;A}"'] & K_n(A) \end{tikzcd}$$

commutes.

Proof. Let w be a closed point of Δ^n_L, not contained in any face. We have the composition

$$K_0(L(w); A) \cong K_0^w(\Delta^n_L; A) \cong K_0^w(\Delta^n_L, \partial\Delta^n_L; A) \xrightarrow{\alpha} K_0(\Delta^n_L, \partial\Delta^n_L; A) \cong K_n(A_L)$$

defined as follows. The first isomorphism is obtained via the localization sequence for $K(-; A)$. We have the canonical map

$$K^w(\Delta^n_L, \partial\Delta^n_L; A) \to K^w(\Delta^n_L; A),$$

which is a weak equivalence since $w \cap \partial\Delta^n_L = \emptyset$, giving us the second isomorphism. The map α is 'forget supports' and the last isomorphism follows from the homotopy property of $K(-; A)$. Denote this composition by

$$\beta^w_{n,L;A} : K_0(k(w); A) \to K_n(A_L).$$

Since $z^n(L, n; A) = \bigoplus_w K_0(k(w); A)$, where the sum is over all closed points $w \in \Delta^n_L \setminus \partial\Delta^n_L$, the maps $\beta^w_{n,L;A}$ induce

$$\beta_{n,L;A} : z^n(L, n; A) \to K_n(A_L);$$

we have as well the canonical surjection

$$\gamma_{n,L;A} : z^n(L,n;A) \to \mathrm{CH}^n(L,n;A).$$

It follows easily from the definitions that the diagram

$$\begin{tikzcd} z^n(L,n;A) \arrow[r, "\gamma_{n,L;A}"] \arrow[dr, "\beta_{n,L;A}"'] & \mathrm{CH}^n(L,n;A) \arrow[d, "p_{n,L;A}"] \\ & K_n(A_L) \end{tikzcd}$$

commutes.

On the other hand, it is also a direct consequence of the definitions that, for $x \in \Delta^n_k$ the image of w under $\Delta^n_L \to \Delta^n_k$, we have

$$\mathrm{Nm}_{L/k} \circ \gamma_{n,L;A} = \gamma_{n,k;A} \circ \mathrm{Nm}_{A_{L(w)}/A_{k(x)}},$$
$$\mathrm{Nm}_{A_L/A_k} \circ \beta_{n,L;A} = \beta_{n,k;A} \circ \mathrm{Nm}_{A_{L(w)}/A_{k(x)}},$$

whence the result. □

Lemma 6.6.3. *For all $n \geqslant 0$, the map*

$$\sum_L \mathrm{Nm}_{L/k} : \bigoplus_{L \in \mathrm{Spl}_A} \mathrm{CH}^n(L,n;A) \to \mathrm{CH}^n(k,n;A)$$

is surjective.

Proof. In fact, the map

$$\sum_L \mathrm{Nm}_{L/k} : \bigoplus_{L \in \mathrm{Spl}_A} z^n(L,n;A) \to z^n(k,n;A)$$

is surjective. Indeed, let x be a closed point of $\Delta^n_k \setminus \partial\Delta^n_k$. Then

$$A_{k(x)} = M_n(D)$$

for some division algebra D over $k(x)$. Letting $L \subset D$ be a maximal subfield of D containing $k(x)$, L splits D, hence L/k splits A. Since $L \supset k(x)$, there is a closed point $w \in \Delta^n_L \setminus \partial\Delta^n_L$ lying over x with $L(w) = L$, i.e. w is an L-point.

Since L is a maximal subfield of D, the degree of L over $k(x)$ is exactly the index of $\mathrm{Nrd}(K_0(D)) \subset K_0(k(x))$. Thus the norm map

$$\mathrm{Nm}_{L/k(x)} : K_0(A_L) \to K_0(A_{k(x)})$$

is surjective, i.e. $K_0(A_{k(x)}) \cdot x$ is contained in the image of $\mathrm{Nm}_{L/k}(z^n(L,n;A))$. As

$$z^n(k,n;A) = \bigoplus_x K_0(A_{k(x)})$$

with the sum over all closed points $x \in \Delta^n_k \setminus \partial\Delta^n_k$, this proves the lemma. □

Recall from § 5.9 the reduced norm map

$$\mathrm{Nrd}_{Y,A} : z^q(Y,*;A) \to z^q(Y,*).$$

Lemma 6.6.4. *Let $j : k \hookrightarrow L$ be a finite extension field, $Y \in \mathbf{Sm}/k$. Then the diagram*

$$\begin{tikzcd}
z^q(Y_L,-;A) \arrow[r, "\mathrm{Nrd}_{Y_L,A}"] \arrow[d, "\mathrm{Nm}_{L/k}"'] & z^q(Y_L,-) \arrow[d, "\mathrm{Nm}_{L/k}"] \\
z^q(Y,-;A) \arrow[r, "\mathrm{Nrd}_{Y,A}"'] & z^q(Y,-)
\end{tikzcd}$$

commutes.

Proof. Take $w \in Y_L^{(q)}(n)$ and let $x \in Y^{(q)}(n)$ be the image of w under $Y_L \times \Delta^n \to Y \times \Delta^n$. It is easy to check that the diagram

$$\begin{tikzcd}
K_0(A_{k(w)}) \arrow[r, "\mathrm{Nrd}"] \arrow[d, "\mathrm{Nm}_{A_{k(w)}/A_{k(x)}}"'] & K_0(k(w)) \arrow[d, "\mathrm{Nm}_{k(w)/k(x)}"] \\
K_0(A_{k(x)}) \arrow[r, "\mathrm{Nrd}"'] & K_0(k(x))
\end{tikzcd}$$

commutes, from which the lemma follows. □

Proposition 6.6.5. *For $n = 0, 1, 2$ the diagram*

$$\begin{tikzcd}
\mathrm{CH}^n(k,n;A) \arrow[r, "p_{n,k;A}"] \arrow[d, "\mathrm{Nrd}"'] & K_n(A) \arrow[d, "\mathrm{Nrd}"] \\
\mathrm{CH}^n(k,n) \arrow[r, "p_{n,k}"'] & K_n(k)
\end{tikzcd}$$

commutes.

Proof. Let $j : k \hookrightarrow L$ be a finite extension field of k. We have the diagram

$$\begin{tikzcd}
\mathrm{CH}^n(L,n;A_L) \arrow[rr, "p_{n,L;A}"] \arrow[dd, "\mathrm{Nrd}"'] \arrow[dr, "\mathrm{Nm}_{L/k}"'] & & K_n(A_L) \arrow[dd, "\mathrm{Nrd}"' near start] \arrow[dr, "\mathrm{Nm}_{L/k}"] & \\
& \mathrm{CH}^n(k,n;A) \arrow[rr, "p_{n,k;A}" near start] \arrow[dd, "\mathrm{Nrd}" near start] & & K_n(A) \arrow[dd, "\mathrm{Nrd}"] \\
\mathrm{CH}^n(L,n) \arrow[rr, "p_{n,L}"' near end] \arrow[dr, "\mathrm{Nm}_{L/k}"'] & & K_n(L) \arrow[dr, "\mathrm{Nm}_{L/k}"'] & \\
& \mathrm{CH}^n(k,n) \arrow[rr, "p_{n,k}"'] & & K_n(k)
\end{tikzcd}$$

The left-hand square commutes by Lemma 6.6.4, the right-hand square commutes by Proposition 6.6.1, and the top and bottom squares commute by Lemma 6.6.2.

Now suppose that L splits A. Then, after using the Morita equivalence, the maps Nrd are identity maps, hence the back square commutes. Thus for $b \in \mathrm{CH}^n(L, n; A_L)$, $a = \mathrm{Nm}_{L/k}(b) \in \mathrm{CH}^n(k, n; A)$, we have

$$\mathrm{Nrd}(p_{n,k,A}(a)) = p_{n,k}(\mathrm{Nrd}(a)).$$

But by Lemma 6.6.3, $\mathrm{CH}^n(k, n; A)$ is generated by elements a of this form, as L runs over all splitting fields of A, proving the result. □

6.7. Computations

Theorem 6.7.1 (see also Theorem 6.8.2). *Let A be a central simple algebra over k.*

(1) *For $n = 0, 1$, the edge homomorphism*

$$\mathrm{CH}^n(k, n; A) \xrightarrow{p_{n,k;A}} K_n(A)$$

is an isomorphism.

(2) *The sequence*

$$0 \to \mathrm{CH}^1(k, 3; A) \xrightarrow{d_2^{-2,-1}} \mathrm{CH}^2(k, 2; A) \xrightarrow{p_{2,k;A}} K_2(A) \to \mathrm{CH}^1(k, 2; A) \to 0$$

is exact.

Proof. We first note that $\mathrm{CH}^m(k, n; A) = 0$ for $m > n$ for dimensional reasons. In addition $z^0(k, -; A)$ is the constant simplicial abelian group $n \mapsto K_0(A)$, hence $\mathrm{CH}^0(k, n; A) = 0$ for $n \neq 0$. Item (1) follows thus from the spectral sequence of Corollary 6.1.4.

For (2), the same argument gives the exact sequence

$$0 \to \mathrm{CH}^1(k, 3; A) \xrightarrow{d_2^{-2,-1}} \mathrm{CH}^2(k, 2; A) \xrightarrow{p_{2,k;A}} K_2(A) \to \mathrm{CH}^1(k, 2; A) \to 0.$$

□

6.8. Codimension one

We recall the computation of the codimension one higher Chow groups due to Bloch.

Proposition 6.8.1 (Bloch [6, Theorem 6.1]). *Let F be a field. Then*

$$\mathrm{CH}^1(F, n) = \begin{cases} F^\times & \text{for } n = 1, \\ 0 & \text{for } n \neq 1. \end{cases}$$

Note that $\mathrm{CH}^1(F,0) = 0$ for dimensional reasons. To show that $\mathrm{CH}^1(F,n) = 0$ for $n > 1$, let $D = \sum_i n_i D_i$ be a divisor on Δ^n_F, intersecting each face properly, i.e. containing no vertex of Δ^n_F in its support. Suppose that D represents an element $[D] \in \mathrm{CH}^1(F,n)$, that is, $d_n(D) = 0$. Using the degeneracy maps to add 'trivial' components, we may assume that $D \cdot \Delta^{n-1}_j = 0$ for all j, where Δ^{n-1}_j is the face $t_j = 0$.

As $\Delta^n_F \cong \mathbb{A}^n_F$, the divisor D is the divisor of a rational function f on Δ^n_F. Since D intersects each Δ^{n-1}_j properly, the restriction f_j of f to Δ^{n-1}_j is a well-defined rational function on Δ^{n-1}_j; as $D \cdot \Delta^{n-1}_j = 0$, $\mathrm{Div}(f_j) = 0$, so f_j is a unit on Δ^{n-1}_j, that is, $f_j = a_j$ for some $a_j \in k^\times$. Since $\Delta^{n-1}_j \cap \Delta^{n-1}_l \neq \emptyset$ for all j, l,* all the a_j are equal, thus $f_j = a \in k^\times$ for all j. Dividing f by a we may assume that $f_j \equiv 1$ for all j.

Now let $\mathcal{D}$ be the divisor of $g := tf - (1-t)$ on $\Delta^n_F \times \mathbb{A}^1_F$, where $\mathbb{A}^1_F := \operatorname{Spec} F[t]$. As the restriction of g to $\Delta^{n-1}_j \times \mathbb{A}^1$ is 1, $\mathcal{D}$ defines an element $[\mathcal{D}] \in \mathrm{CH}^1(\mathbb{A}^1_F, n)$ with $i_0^*([\mathcal{D}]) = [D]$, $i_1^*([\mathcal{D}]) = 0$. By the homotopy property, $[D] = 0$.

We use essentially the same argument plus Wang's theorem [**64**] to complete Theorem 6.7.1 as follows.

Theorem 6.8.2. *Let A be a central simple algebra over a field F. Suppose A has square-free index e, with $(e, \operatorname{char} k) = 1$. Then $\mathrm{CH}^1(F,n;A) = 0$ for $n \neq 1$, and the edge homomorphism*

$$\mathrm{CH}^2(k,2;A) \xrightarrow{p_{2,k;A}} K_2(A)$$

is an isomorphism.

Proof. We reduce as usual to the case where $\deg A = p$ is prime (with $(p, \operatorname{char} k) = 1$). As above, the case $n = 0$ is trivially true. We mimic the proof for $\mathrm{CH}^1(F,n)$ in case $n > 1$.

If $A = M_p(k)$, then $\mathrm{CH}^1(F,n;A) = \mathrm{CH}^1(F,n)$, so there is nothing to prove; we therefore assume that A is a degree p division algebra over k. Then A admits a splitting field k' of degree p over k; since $\mathrm{CH}^1(F \otimes_k k', n; A) = \mathrm{CH}^1(F \otimes_k k', n) = 0$ for $n > 1$, a norm argument shows that $\mathrm{CH}^1(F,n;A)$ is p-torsion.

We have seen in Lemma 6.2.2 that the argument of Sherman [**52**, Theorem 2.4] for the degeneration of the Quillen spectral sequence for $K(\mathbb{A}^n_F)$ goes through word for word to give the degeneration of the Quillen spectral sequence for $K(\mathbb{A}^n_F; A)$. We will use this fact throughout the remainder of the proof.

Let $\mathcal{M}^{(1)}_v(\Delta^n_F; A)$ denote the category of $A \otimes_k \mathcal{O}_{\Delta^n_F}$-modules M which are coherent as $\mathcal{O}_{\Delta^n_F}$-modules, and such that the support of M has codimension at least one on Δ^n_F and contains no vertex of Δ^n_F. Take a 'divisor' D representing a class $[D] \in \mathrm{CH}^1(F,n;A)$, that is, represent $[D]$ by an element

$$\tilde{D} := \sum_j \alpha_j \cdot D_j$$

with each D_j an integral codimension one closed subscheme of Δ^n_F, containing no vertex of Δ^n_F, $\alpha_j \in K_0(A \otimes_F F(D_j))$ and extend $\bigoplus_j \alpha_j$ to an element $D \in K_0(\mathcal{M}^{(1)}_v(\Delta^n_F; A))$.

* This is where we use the hypothesis $n > 1$.

As above, we may assume that the restriction $\tilde{D}\cdot\Delta^{n-1}_j$ of $\tilde{D}$ to Δ^{n-1}_j is zero for each $j=0,\dots,n$.

Let v_n denote the set of vertices of Δ^n, $\mathcal{O}_{\Delta^n,v_n}$ the semi-local ring of v_n in Δ^n. Since $K_0(\Delta^n_F;A)=K_0(A)$ by the homotopy property, the localization sequence

$$K_1(A\otimes\mathcal{O}_{\Delta^n,v_n})\xrightarrow{\partial}K_0(\mathcal{M}^{(1)}_v(\Delta^n;A))\to K_0(\Delta^n_F;A)\to K_0(A\otimes\mathcal{O}_{\Delta^n,v_n})$$

for $K(\Delta^n_F;A)$ gives us an element $f\in K_1(A\otimes\mathcal{O}_{\Delta^n,v_n})$ with

$$\partial f=D.$$

Since $\mathcal{O}_{\Delta^n,v_n}$ is semi-local, we have a surjection

$$(A\otimes\mathcal{O}_{\Delta^n,v_n})^\times\to K_1(A\otimes\mathcal{O}_{\Delta^n,v_n});$$

we lift f to an element $\tilde{f}$ of $(A\otimes\mathcal{O}_{\Delta^n,v_n})^\times$, and let $\tilde{f}_j\in(A\otimes\mathcal{O}_{\Delta^{n-1}_j,v_{n-1}})^\times$ denote the restriction of $\tilde{f}$ to Δ^{n-1}_j.

We have the localization sequence

$$0\to K_1(\Delta^{n-1}_j;A)\to K_1(A\otimes\mathcal{O}_{\Delta^{n-1}_j,v_{n-1}})\xrightarrow{\partial}K_0(\mathcal{M}^{(1)}_v(\Delta^{n-1}_j;A))\to .$$

By the degeneration of the Quillen spectral sequence on Δ^{n-1}_j, it follows that

$$K_0(\mathcal{M}^{(1)}_v(\Delta^{n-1}_j;A))=\bigoplus_{w\in(\Delta^{n-1}_j,v_{n-1})^{(1)}}K_0(A\otimes_k k(w)),$$

where $(\Delta^{n-1}_j,v_{n-1})^{(1)}$ is the set of codimension one points of Δ^{n-1}_j whose closure contains no vertex. Thus, the fact that $\tilde{D}\cdot\Delta^{n-1}_j=0$ implies that restriction of f to $f_j\in K_1(A\otimes\mathcal{O}_{\Delta^{n-1}_j,v_{n-1}})$ lifts uniquely to $K_1(\Delta^{n-1}_j;A)=K_1(A)$.

The degeneracy maps give compatible splittings to the inclusions $\Delta^{n-1}_j\to\Delta^n$ for $j=1,\dots,n$, so we can modify f and $\tilde{f}$ so that $\tilde{f}_j=1\in(A\otimes\mathcal{O}_{\Delta^{n-1}_j,v_{n-1}})^\times$ for $j=1,\dots,n$.

Now let $L:=k(\Delta^{n-1}_0)$ and consider $\tilde{f}_0\in(A_L)^\times$. As $n>1$, $\Delta^{n-1}_0\cap\Delta^{n-1}_1\neq\emptyset$; restricting to $\Delta^{n-1}_0\cap\Delta^{n-1}_1$ shows that $f_0=1\in K_1(A_L)$. Furthermore, the reduced norm map

$$\mathrm{Nrd}:K_1(A_L)\to K_1(L)=L^\times$$

is injective [**64**], and finally, for $a\in(A_L)^\times$, we have

$$\mathrm{Nrd}(a)=\begin{cases}a^p & \text{for } a\in L^\times,\\ \mathrm{Nm}_{L(a)/L}(a) & \text{for } a\in A_L^\times\setminus L^\times.\end{cases}$$

Now, $L(\tilde{f}_0)$ is a subfield of A_L of degree at most p over L. But since A is a division algebra and L is a pure transcendental extension of k, A_L is still a division algebra, and hence either $L(\tilde{f}_0)=L$ or $L(\tilde{f}_0)$ has degree exactly p over L. In the former case, $1=\mathrm{Nrd}(\tilde{f}_0)=\tilde{f}_0^p$, and since $\tilde{f}_j=1$ for $j>0$, it follows that $\tilde{f}_0=1$ as well.

In case $L(\tilde{f}_0)$ has degree exactly p over L, then $\mathrm{Nm}_{L(\tilde{f}_0)/L}(\tilde{f}_0) = 1$. Let $M \supset L(\tilde{f}_0)$ be the Galois closure of $L(\tilde{f}_0)$ over L, let $M_0 \subset M$ be the unique subfield of index p, and let $\sigma \in \mathrm{Gal}(M/L)$ be the generator for $\mathrm{Gal}(M/M_0)$. Then $\mathrm{Nm}_{M/M_0}(\tilde{f}_0) = 1$, so by Hilbert's theorem 90, there is a $\tilde{g} \in M^\times$ with

$$\tilde{f}_0 = \frac{g^\sigma}{g}.$$

Looking at the proof of Hilbert's theorem 90, we see that we may take g in the integral closure of $\mathcal{O}_{\Delta_0^{n-1}, v_{n-1}}$, with $g \equiv 1$ over all generic points of $\partial\Delta_0^{n-1}$.

By the Skolem–Noether theorem, there is an element $a_g \in A_{M_0}^\times$ with $g^\sigma = a_g^{-1} g a_g$, i.e.

$$\tilde{f}_0 = a_g^{-1} g a_g g^{-1}.$$

As above, we may take a_g to be a unit in the integral closure of $A \otimes \mathcal{O}_{\Delta_0^{n-1}, v_{n-1}}$.

Let $\hat{L} := k(\Delta^n)$, and let $\hat{M} \supset \hat{L}(\tilde{f})$ be the Galois closure of $\hat{L}(\tilde{f})$ over $\hat{L}$. Lift g to $\hat{g} \in \hat{M}$ (or rather, in the integral closure $\hat{R}$ of $\mathcal{O}_{\Delta^n, v_n}$ in $\hat{M}$), with $\hat{g} \equiv 1$ over the generic point of Δ_j^{n-1}, for each $j > 0$. Lift a_g similarly to $\hat{a}_g$. Let $d = [\hat{M}_0 : \hat{L}]$. We may replace $\tilde{f}^d$ with

$$\hat{f} := \mathrm{Nm}_{A_{\hat{M}_0}/A_{\hat{L}}}(\tilde{f}\hat{a}_g^{-1}\hat{g}\hat{a}_g\hat{g}^{-1}).$$

Then $\hat{f}$ restricts to 1 in $A \otimes \mathcal{O}_{\Delta_0^{n-1}, v_{n-1}}$ for all j, giving a trivialization of $d \cdot [D]$ in $\mathrm{CH}^1(F, n)$. Since d is prime to p, it follows that $[D] = 0$ in $\mathrm{CH}^1(F, n; A)$. ☐

Remark 6.8.3. We shall give in Corollary 8.1.5 below a second proof of Theorem 6.8.2, relying on the Merkurjev–Suslin theorem, by proving that $H^p(k; \mathbb{Z}_A(1)) = 0$ for $p \neq 1$, if A has square-free index e over a perfect field k, $(e, \mathrm{char}\, k) = 1$. Via the isomorphism of Theorem 6.5.3

$$\mathrm{CH}^1(k, n; A) \cong H^{2-n}(k, \mathbb{Z}_A(1))$$

this shows a second time that $\mathrm{CH}^1(k, n; A) = 0$ for $n \neq 1$ in the square-free index case. We do not know if this holds for A of arbitrary index.

6.9. A map from $SK_i(A)$ to étale cohomology

In this section, we use the étale version of the spectral sequence in the previous section to construct homomorphisms from $SK_i(A)$ to quotients of $H^{i+2}_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z}(i+1))$ for $i = 1, 2$. In what follows, we invert the exponential characteristic of k throughout, but we do not write this explicitly, to simplify the notation. We refer to Appendix C, especially § C.4, for details on the category of étale motives and the change of topology functor.

The motivic Postnikov tower for K^A

$$\cdots \to f_{n+1}K^A \to f_nK^A \to \cdots \to f_0K^A = K^A$$

induces by the étale sheafification functor α^* the étale version

$$\cdots \to [f_{n+1}K^A]^{\text{ét}} \to [f_nK^A]^{\text{ét}} \to \cdots \to [f_0K^A]^{\text{ét}} = [K^A]^{\text{ét}}$$

with layers the étale sheafifications $s_n^{\text{ét}} K^A$ of the layers $s_n K^A$ of the original tower. Since $s_n K^A = \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_A(n)[2n])$ (Theorem 6.5.5), and $\mathbb{Z}_A(n)^{\text{ét}} = \mathbb{Z}(n)^{\text{ét}}$, we have

$$s_n^{\text{ét}} K^A = \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}(n)^{\text{ét}}[2n]).$$

Evaluating at $\operatorname{Spec} k$ and taking the spectral sequence of this tower gives the *étale motivic Atiyah–Hirzebruch spectral sequence for* A, with Bloch–Lichtenbaum numbering:

$$E_2^{p,q} = H_{\text{ét}}^{p-q}(k, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}^{\text{ét}}(A).$$

Here is part of the corresponding E_2-plane:

-2	-1	0	1	2	3
0	0	$H_{\text{ét}}^0(k, \mathbb{Z})$	0	$H_{\text{ét}}^2(k, \mathbb{Z})$	
0	0	$H_{\text{ét}}^1(k, \mathbb{Z}(1))$	0	$H_{\text{ét}}^3(k, \mathbb{Z}(1))$	
$H_{\text{ét}}^0(k, \mathbb{Z}(2))$	$H_{\text{ét}}^1(k, \mathbb{Z}(2))$	$H_{\text{ét}}^2(k, \mathbb{Z}(2))$	0	$H_{\text{ét}}^4(k, \mathbb{Z}(2))$	
$H_{\text{ét}}^1(k, \mathbb{Z}(3))$	$H_{\text{ét}}^2(k, \mathbb{Z}(3))$	$H_{\text{ét}}^3(k, \mathbb{Z}(3))$	0	$H_{\text{ét}}^5(k, \mathbb{Z}(3))$	
$H_{\text{ét}}^2(k, \mathbb{Z}(4))$	$H_{\text{ét}}^3(k, \mathbb{Z}(4))$	$H_{\text{ét}}^4(k, \mathbb{Z}(4))$	0	$H_{\text{ét}}^6(k, \mathbb{Z}(4))$	$H_{\text{ét}}^7(k, \mathbb{Z}(4))$

For $i = 1, 2$, the composition

$$K_i(A) \to K_i^{\text{ét}}(A) \xrightarrow{\varepsilon} H_{\text{ét}}^i(k, \mathbb{Z}(i)) = K_i(k)$$

coincides with the reduced norm, where ε is the edge homomorphism of the spectral sequence and the isomorphism follows from the Beilinson–Lichtenbaum conjecture in weight i (that is, Kummer theory for $i = 1$ and the Merkurjev–Suslin theorem for $i = 2$). Hence we get an induced map

$$SK_1(A) \to \operatorname{coker}(K_2^M(k) \simeq H_{\text{ét}}^2(k, \mathbb{Z}(2)) \xrightarrow{d_2^A} H_{\text{ét}}^5(k, \mathbb{Z}(3))).$$

Note that the map $H_{\text{ét}}^4(k, \mathbb{Q}/\mathbb{Z}(3)) \to H_{\text{ét}}^5(k, \mathbb{Z}(3))$ is an isomorphism, independent of the Beilinson–Lichtenbaum conjecture. The spectral sequence shows that there is a map from the kernel of this homomorphism to a quotient of $H_{\text{ét}}^7(k, \mathbb{Z}(4)) \simeq H_{\text{ét}}^6(k, \mathbb{Q}/\mathbb{Z}(4))$.

For $SK_2(A)$, we get *a priori* a map to the quotient of

$$\operatorname{coker}(K_3^M(k) \simeq H_{\text{ét}}^3(k, \mathbb{Z}(3)) \xrightarrow{d_2^A} H_{\text{ét}}^6(k, \mathbb{Z}(4)))$$

by the image of a d_3 differential starting from $H_{\text{ét}}^1(k, \mathbb{Z}(2)) \simeq K_3(k)_{\text{ind}}$. If k contains a separably closed field, this group is divisible, hence its image by the torsion differential d_3 is 0. Note that we also have an isomorphism

$$H_{\text{ét}}^5(k, \mathbb{Q}/\mathbb{Z}(4)) \xrightarrow{\sim} H_{\text{ét}}^6(k, \mathbb{Z}(4)).$$

Here, the isomorphism $K_3^M(k) \simeq H_{\text{ét}}^3(k, \mathbb{Z}(3))$ follows from the Beilinson–Lichtenbaum conjecture in weight 3; if one does not want to assume it, one gets a slightly more obscure quotient.

To compute d_2^A, we use the fact that this spectral sequence is a module on the corresponding spectral sequence for $K^{\text{ét}}F$ [**47**]. The latter is multiplicative [**47**] and d_2 is obviously 0 on $K_0(F)$ and $K_1(F)$, hence on all $K_i^M(F)$. For d_2^A, we then have

$$d_2^A(x) = x \cdot d_2^A(1), \quad x \in E_2^{0,-2}, E_2^{0,-3},$$

where $d_2^A(1)$ is the image of $1 \in K_0(F)$ in $\mathrm{Br}(F)$.

When we pass to the function field K of the Severi–Brauer variety of A, A gets split so $d_2^A(1)_K = 0$. By Amitsur's theorem, $d_2^A(1)$ is a multiple $\delta[A]$ of $[A]$.

In fact, we have $\delta = 1$. The computation is very similar to our computation of a related boundary map for the motive of a Severi–Brauer variety (see Proposition 8.2.1) so we will be a little sketchy in our discussion here.

Proposition 6.9.1. $d_2^A(1) = [A]$.

Proof. We begin by noting that by naturality, it suffices to restrict the presheaf $Y \mapsto K(Y; A)$ to the small étale site over k. Fix a Galois splitting field L over k of A with group G. As the field extensions of L are cofinal in $k_{\text{ét}}$, it suffices to consider the functor

$$F \mapsto K(F; A)$$

on finite extensions F of k containing L; denote this subcategory of $k_{\text{ét}}$ by $k_{\text{ét}}(L)$.

For such an F, A_F is isomorphic to a matrix algebra, say $A_F \cong M_n(F)$, so by Morita equivalence, $K(F; A)$ is weakly equivalent to $K(F)$. Similarly, $\mathbb{Z}_A = \mathbb{Z}$ on $k_{\text{ét}}(L)$. Since

$$H^p(F, \mathbb{Z}(n)) = 0$$

for $p > n$, and since $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$, it follows from our identification of the slices (Theorem 6.5.5)

$$s_n K^A \cong \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_A(n)[2n]),$$

that the cofibre $f_{0/2}K^A$ of $f_2 K^A \to f_0 K^A$ is the same as the presheaf of cofibres of K^A by its 1-connected cover

$$\tau_{\leqslant 1} K^A := \mathrm{cofib}[\tau_{\geqslant 2} K^A \to K^A].$$

Thus, to compute $d_2^A(1)$, we just need to apply the usual machinery of G-cohomology to the fibre sequence

$$\Sigma\,\mathrm{EM}(\mathcal{K}_1^A) \to \tau_{\leqslant 1} K^A \to \mathrm{EM}(\mathcal{K}_0^A)$$

(see the proof of Proposition 8.2.1 below for more details).

Let us choose a cocycle $\sigma \mapsto \bar{g}_\sigma \in \mathrm{PGL}_n(L)$ representing the class of A in $H^1(G, \mathrm{PGL}_n(L))$. Thus, if $g_\sigma \in \mathrm{GL}_n(L)$ is a lifting of $\bar{g}_\sigma$, we have the action of G on $M_n(L)$

$$\varphi_\sigma(m) := g_\sigma \cdot {}^\sigma m \cdot g_\sigma^{-1},$$

where ${}^\sigma m$ is the usual action of G by conjugation of the matrix coefficients. A is isomorphic to the G-invariant k-subalgebra of $M_n(L)$. Also, the coboundary in $H^2_{\text{ét}}(k, \mathbb{G}_m)$ of the class of A in $H^1(k, \mathrm{PGL}_n)$ is represented by the 2-cocycle $\{c_{\tau,\sigma}\} \in Z^2(G, L^\times)$ defined by

$$c_{\tau,\sigma} \cdot \mathrm{id}_{L^n} = g_\tau \cdot {}^\tau g_\sigma \cdot g_{\tau\sigma}^{-1}.$$

The ring homomorphism $\varphi_\sigma : M_n(L) \to M_n(L)$ induces an exact functor

$$\varphi_{\sigma *} : \mathrm{Mod}_{M_n(L)} \to \mathrm{Mod}_{M_n(L)}$$

sending projectives to projectives, hence a natural map $\varphi_{\sigma_*} : K(L; A) \to K(L; A)$ and thereby a map $\varphi_{\sigma_*} : \tau_{\leqslant 1} K(L; A) \to \tau_{\leqslant 1} K(L; A)$. To compute $d_2^A(1)$, we apply the following procedure: lift $1 \in K_0(L; A)$ to a representing $M_n(L)$-module F. For each $\sigma \in G$, choose an isomorphism $\psi_\sigma : \varphi_{\sigma *}(F) \to F$, which gives us a path γ_σ in the 0-space of $K(L; A)$. The path

$$\gamma(\tau, \sigma) := \gamma_\tau \cdot \varphi_{\tau_*}[\gamma_\sigma] \cdot \gamma_{\tau\sigma}^{-1}$$

is a loop in $K(L; A)$, giving an element $c'_{\sigma,\tau} \in K_1(L; A) = L^\times$. This gives us a cocycle $\{c'_{\tau,\sigma}\} \in Z^2(G; L^\times)$, which represents $d_2^A(1) \in H^3_{\text{ét}}(k, \mathbb{Z}(1)) = H^2_{\text{ét}}(k, \mathbb{G}_m)$.

To make the computation concrete, let F be a left $M_n(L)$-module. Then the isomorphism of abelian groups $F \to M_n(L) \otimes_{M_n(L)} F$ sending v to $1 \otimes v$ identifies $\varphi_{\sigma *}(F)$ with the $M_n(L)$-module with underlying abelian group F, and with multiplication

$$m \cdot_\sigma v := {}^{\sigma^{-1}}[g_\sigma^{-1} m g_\sigma] \cdot v.$$

Under this identification, $\varphi_{\sigma *}$ acts by the identity on morphisms.

Take $F = L^n$ with the standard $M_n(L)$-module structure. One sees immediately that sending v to $g_\sigma \cdot {}^\sigma v$ gives an $M_n(L)$-module isomorphism $\psi_\sigma : \varphi_{\sigma_*}(F) \to F$. The loop $\gamma(\tau, \sigma)$ is thus represented by the automorphism $\psi_\tau \circ \varphi_{\tau *}(\psi_\sigma) \circ \psi_{\tau\sigma}^{-1}$:

$$\begin{aligned} \psi_\tau \circ \varphi_{\tau *}(\psi_\sigma) \circ \psi_{\tau\sigma}^{-1}(v) &= \psi_\tau \circ \varphi_{\sigma *}(\psi_\sigma)({}^{(\tau\sigma)^{-1}}[g_{\tau\sigma}^{-1} v]) \\ &= \psi_\tau(g_\sigma \cdot {}^{\tau^{-1}}[g_{\tau\sigma}^{-1} \cdot v]) \\ &= (g_\tau \cdot {}^\tau g_\sigma \cdot g_{\tau\sigma}^{-1})(v). \end{aligned}$$

Since the Morita equivalence $\mathrm{Mod}_{M_n(L)} \to \mathrm{Mod}_L$ sends multiplication by $c \in L$ on F to multiplication by c on L, we have the explicit representation of $d_2^A(1)$ by the cocycle $\{c_{\tau,\sigma}\}$, completing the computation. $\square$

7. The motivic Postnikov tower for a Severi–Brauer variety

Results of Huber and Kahn [**22**] give a computation of the sheaf H^0 of the delooped slices of $M(X)$ for X any smooth projective variety and show that H^n vanishes for $n > 0$. For the motive of a Severi–Brauer variety $X = \mathrm{SB}(A)$, we are able to show (in case A has prime degree ℓ over k) that the negative cohomology vanishes as well. We do this by comparing with the slices of the K-theory of X and using Adams operations to split the appropriate spectral sequence, proving our second main result Theorem 2 (see Theorem 7.4.2).

7.1. The motivic Postnikov tower for a smooth variety

Take $X \in \mathbf{Sm}/k$, $n \geqslant 0$ an integer. We recall that the sheaf $z_{\mathrm{equi}}(X,n) \in \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$ has sections $z_{\mathrm{equi}}(X,n)(Y)$ over $Y \in \mathbf{Sm}/k$ the free abelian group on integral subschemes $W \subset Y \times_k X$ such that $W \to Y$ is dominant and equidimensional of relative dimension n over a component of Y.

Lemma 7.1.1. *Let X be a smooth projective variety, $M(X) \in DM^{\mathrm{eff}}(k)$ the motive of X.*

(1) $f_n^{\mathrm{mot}} M(X) = 0$ *for* $n > \dim_k X$.

(2) *For* $0 \leqslant n \leqslant \dim_k X$, $\Omega_T^n f_n^{\mathrm{mot}} X$ *is represented by* $C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X,n))$.

Proof. (1) Since the collection of objects $\{M(Z)[p] \mid Z \in \mathbf{Sm}/k, p \in \mathbb{Z}\}$ are dense in $DM^{\mathrm{eff}}(k)$, it suffices to show that

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(Z)(n)[p], M(X)) = 0$$

for all Z, p and all $n > \dim_k X$. Since $RC_*^{\mathrm{Sus}} \circ K^b(\mathbb{Z}^{\mathrm{tr}}) : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{eff}}(k)$ is fully faithful (see Remark C.6.3), it suffices to show the same vanishing for the morphisms in $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$; since $DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM_{\mathrm{gm}}(k)$ is fully faithful, it suffices to show the vanishing for the morphisms in $DM_{\mathrm{gm}}(k)$.

As X is smooth and projective, we have

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M(Z)(n)[p], M(X)) = \mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M(Z \times X), \mathbb{Z}(d-n)[2d-p]),$$

where $d = \dim_k X$. But

$$\mathrm{Hom}_{DM_{\mathrm{gm}}(k)}(M(Z \times X), \mathbb{Z}(d-n)[2d-p]) = H^{2d-p}(Z \times X, \mathbb{Z}(d-n)),$$

which is zero for $d - n < 0$.

For (2), it follows from (2.4) that

$$\Omega_T^n f_n^{\mathrm{mot}} M(X) = f_0^{\mathrm{mot}} \Omega_T^n M(X) = \Omega_T^n M(X).$$

By [**58**, Theorem 4.2.2], the inclusion

$$\mathbb{Z}^{\mathrm{tr}}(X)(Y \times \mathbb{P}^n) = z_{\mathrm{equi}}(X,0)(Y \times \mathbb{P}^n) \subset z_{\mathrm{equi}}(X \times \mathbb{P}^n, n)(Y)$$

induces a natural isomorphism

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_-(k)}(M(Y \times \mathbb{P}^n), M(X)[m]) \cong H^m(C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X \times \mathbb{P}^n, n))(Y)).$$

One checks that the projection

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_-(k)}(M(Y \times \mathbb{P}^n), M(X)[m]) \to \mathrm{Hom}_{DM^{\mathrm{eff}}_-(k)}(M(Y)(n)[2n], M(X)[m])$$

corresponding to the summand $M(Y)(n)[2n] \subset M(Y \times \mathbb{P}^n)$ corresponds to the map

$$z_{\mathrm{equi}}(X \times \mathbb{P}^n, n) \to z_{\mathrm{equi}}(X, n)$$

induced by the projection $X \times \mathbb{P}^n \to X$. This gives us the isomorphism

$$\Omega_T^n M(X) = R\mathcal{H}\mathrm{om}(\mathbb{Z}(n)[2n], M(X)) \cong C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, n)).$$

□

For later use, we make the following explicit computation.

Lemma 7.1.2. *Let Y be in $\mathbf{Sm}/k$. Let X be smooth, irreducible and projective of dimension d over k. The canonical map $f_d^{\mathrm{mot}} M(X) \to f_{d-1}^{\mathrm{mot}} M(X)$ induces the map (in $D(\mathbf{Ab})$)*

$$[\Omega_T^{d-1} f_d^{\mathrm{mot}} M(X)](Y) \xrightarrow{\alpha} [\Omega_T^{d-1} f_{d-1}^{\mathrm{mot}} M(X)](Y).$$

Then α is isomorphic to the map on Bloch's cycle complexes

$$p_2^* : z^1(Y, *) \to z^1(X \times Y, *)$$

induced by the projection $p_2 : X \times Y \to Y$.

Proof. By (2.4), we have

$$\Omega_T^{d-1} f_d^{\mathrm{mot}} M(X) = f_1^{\mathrm{mot}} \Omega_T^{d-1} M(X) = f_1^{\mathrm{mot}} (\Omega_T^{d-1} f_{d-1}^{\mathrm{mot}} M(X)).$$

By Lemma 7.1.1 (2), we have

$$\Omega_T^{d-1} f_{d-1}^{\mathrm{mot}} M(X) = C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1)),$$

hence

$$\Omega_T^{d-1} f_d^{\mathrm{mot}} M(X) \cong f_1^{\mathrm{mot}} C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))$$

and the map $\Omega_T^{d-1} f_d^{\mathrm{mot}} M(X) \to \Omega_T^{d-1} f_{d-1}^{\mathrm{mot}} M(X)$ is just the canonical map

$$f_1^{\mathrm{mot}} C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1)) \to C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1)).$$

Applying Proposition 2.2.3, we have isomorphisms in $D(\mathbf{Ab})$

$$f_1^{\mathrm{mot}} C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))(Y) \cong f_{\mathrm{mot}}^1(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))),$$

and the canonical map

$$f_1^{\mathrm{mot}} C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))(Y) \to C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))(Y)$$

is isomorphic to

$$\begin{tikzcd} f_{\mathrm{mot}}^1(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))) \arrow[r] & f_{\mathrm{mot}}^0(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))) \arrow[d, equal] \\ & C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))(Y \times \Delta^*) \end{tikzcd}$$

Next, for any $T \in \mathbf{Sm}/k$, the inclusion

$$C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))(T) \subset z^1(T \times X, *)$$

is a quasi-isomorphism [**60**]. Thus, if $W \subset T$ is a closed subset, we have the quasi-isomorphism

$$C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))^W(T) \to \mathrm{cone}(z^1(T \times X, *) \to z^1(T \times X \setminus W \times X, *))[-1].$$

Now suppose that W has pure codimension 1. By Bloch's localization theorem, we have the quasi-isomorphism

$$z^0(W, *) \to \mathrm{cone}(z^1(T \times X, *) \to z^1(T \times X \setminus W \times X, *))[-1];$$

$z^0(W) = z^0(W, 0) \to z^0(W, *)$ is also a quasi-isomorphism. If $\mathrm{codim}_X W > 1$, a similar computation shows that $C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))^W(T)$ is acyclic. Applying this to the computation of $f^1_{\mathrm{mot}}(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1)))$, we have the isomorphism in $D(\mathbf{Ab})$

$$\varphi : z^1(Y, *) \to f^1_{\mathrm{mot}}(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))).$$

Furthermore, the composition

$$z^1(Y, *) \xrightarrow{\varphi} f^1_{\mathrm{mot}}(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))) \to f^1_{\mathrm{mot}}(Y, *; z^1(X \times -, *))$$

is the map

$$W \subset Y \times \Delta^n \mapsto X \times W \times \Delta^0 \subset X \times Y \times \Delta^n \times \Delta^0.$$

It is then easy to see that the composition

$$z^1(Y, *) \xrightarrow{\varphi} f^1_{\mathrm{mot}}(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))) \to f^0_{\mathrm{mot}}(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1)))$$

combined with the isomorphism in $D(\mathbf{Ab})$

$$f^0_{\mathrm{mot}}(Y, *; C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))) \cong C_*^{\mathrm{Sus}}(z_{\mathrm{equi}}(X, d-1))(Y) \cong z^1(X \times Y, *)$$

is just the pullback

$$p_2^* : z^1(Y, *) \to z^1(X \times Y, *).$$

□

Let X be in $\mathbf{Sm}/k$. For a presheaf of spectra E on $\mathbf{Sm}/k$, we have the associated presheaf $\mathcal{H}\mathrm{om}(X, E)$, defined by

$$\mathcal{H}\mathrm{om}(X, E)(Y) := E(X \times Y).$$

Applying $\mathcal{H}\mathrm{om}(X, -)$ to a fibrant model defines the functor

$$R\,\mathcal{H}\mathrm{om}(X, -) : \mathcal{SH}_{S^1}(k) \to \mathcal{SH}_{S^1}(k).$$

We use the notation $\mathcal{H}\mathrm{om}^{\mathrm{mot}}$ and $R\,\mathcal{H}\mathrm{om}^{\mathrm{mot}}$ for the analogous operations on $C(\mathrm{PST}(k))$ and on $DM^{\mathrm{eff}}(k)$. We note that

$$R\,\mathcal{H}\mathrm{om}(X,-) \cong \mathcal{H}\mathrm{om}_{\mathcal{SH}_{S^1}(k)}(\Sigma_s^\infty h_X,-)$$

and, similarly,

$$R\,\mathcal{H}\mathrm{om}^{\mathrm{mot}}(X,-) \cong \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(M(X),-).$$

The operation $R\,\mathcal{H}\mathrm{om}(X,-)$ does not in general commute with the truncation functors f_n. However, we do have the following lemma.

Lemma 7.1.3. *Take $m > \dim_k X$. Then for all $E \in \mathcal{SH}_{S^1}(k)$,*

$$s_0 R\,\mathcal{H}\mathrm{om}(X, f_m E) \cong 0.$$

Proof. Let F be a presheaf of spectra on $\mathbf{Sm}/k$ which is $\mathbb{A}^1$-homotopy invariant and satisfies Nisnevich excision. By Remark 2.2.6, we have a natural isomorphism in $\mathcal{SH}$

$$(s_0F)(X) \cong F(\hat{\Delta}^*_{k(Y)}).$$

Similarly, for E homotopy invariant and satisfying Nisnevich excision, the spectrum $\mathcal{H}\mathrm{om}(X, f_mE)(Y) := f_mE(X \times Y)$ is weakly equivalent to the simplicial spectrum $q \mapsto f_mE(X \times Y)(q)$ with

$$f_mE(X\times Y)(q) = \varinjlim_{W \in \mathcal{S}^{(m)}_{X\times Y}(q)} E^W(X \times Y \times \Delta^q).$$

The moving lemma [**32**, Theorem 2.6.2] gives us the natural weak equivalence

$$f_mE(X \times \hat{\Delta}^p_{k(Y)})(q) \cong \varinjlim_{W\in \mathcal{S}^{(m)}_{X\times\hat{\Delta}^p_{k(Y)}}(q)_{\mathcal{C}(p)}} E^W(X \times \hat{\Delta}^p_{k(Y)} \times \Delta^q),$$

where $\mathcal{C}(p)$ is the set $X \times F$, with F a face of $\hat{\Delta}^p_{k(Y)}$.

Thus $s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)$ is weakly equivalent to the total spectrum of the bi-simplicial spectrum

$$(p,q) \mapsto s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(p,q) = \varinjlim_{W\in \mathcal{S}^{(m)}_{X\times\hat{\Delta}^p_{k(Y)}}(q)_{\mathcal{C}(p)}} E^W(X \times \hat{\Delta}^p_{k(Y)} \times \Delta^q).$$

We denote the total spectrum by $s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(-,-)$.

Let $s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(-,q)$ be the total spectrum of the simplicial spectrum

$$p \mapsto s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(p,q).$$

By [**33**, Claim, Lemma 5.2.1], the face maps

$$\delta_i^{q*} : s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(-,q) \to s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(-,q-1)$$

are weak equivalences for all $i = 0,\dots,q$, $q \geqslant 1$, and therefore the canonical map

$$s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(-,0) \to s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(-,-)$$

is a weak equivalence.

Take

$$W \in \mathcal{S}^{(m)}_{X\times\hat{\Delta}^p_{k(Y)}}(0)_{\mathcal{C}(p)},$$

so W is a closed subset of $X \times \hat{\Delta}^p_{k(Y)}$ of codimension greater than or equal to $m > \dim_k X$, and $W \cap X \times F$ has codimension greater than or equal to m on $X \times F$ for all faces F of $\hat{\Delta}^p_{k(Y)}$. In particular, for each vertex v of $\hat{\Delta}^p_{k(Y)}$,

$$\mathrm{codim}_{X\times v}\, W \cap X \times v > \dim_k X.$$

Thus $W \cap X \times v = \emptyset$. Since X is proper, the projection of W, $p_2(W) \subset \hat{\Delta}^p_{k(Y)}$, is a closed subset disjoint from all vertices v. Since $\hat{\Delta}^p_{k(Y)}$ is semi-local with closed points the set of vertices, this implies that $p_2(W) = \emptyset$. Thus, $W = \emptyset$, that is,

$$\mathcal{S}^{(m)}_{X\times\hat{\Delta}^p_{k(Y)}}(0)_{\mathcal{C}(p)} = \{\emptyset\},$$

and therefore $s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)(-,0) \sim 0$. The description we have given of $s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y)$ as a simplicial spectrum thus yields

$$s_0\,\mathcal{H}\mathrm{om}(X, f_mE)(Y) \sim 0$$

for all $Y \in \mathbf{Sm}/k$, completing the proof. □

Thus, for $X \in \mathbf{Sm}/k$, smooth and projective of dimension d over k, and for $E \in \mathcal{SH}_{S^1}(k)$, we have the tower in $\mathcal{SH}_{S^1}(k)$

$$\begin{aligned} 0 = s_0R\,\mathcal{H}\mathrm{om}(X, f_{d+1}E) \to s_0R\,\mathcal{H}\mathrm{om}(X, f_dE) \to \cdots \\ \to s_0R\,\mathcal{H}\mathrm{om}(X, f_0E) = s_0R\,\mathcal{H}\mathrm{om}(X, E) \end{aligned} \tag{7.1}$$

gotten by applying $s_0R\,\mathcal{H}\mathrm{om}(X,-)$ to the T-Postnikov tower of E. Since the functors s_0 and $R\,\mathcal{H}\mathrm{om}(X,-)$ are exact, the mth layer in the tower (7.1) is isomorphic to $s_0R\,\mathcal{H}\mathrm{om}(X, s_mE)$, $m = 0,\dots,\dim_k X$. Evaluating at some $Y \in \mathbf{Sm}/k$, we have the strongly convergent spectral sequence

$$E^1_{a,b} = \pi_{a+b}(s_0R\,\mathcal{H}\mathrm{om}(X, s_{-a}E)(Y)) \Longrightarrow \pi_{a+b}(s_0R\,\mathcal{H}\mathrm{om}(X, E)(Y)). \tag{7.2}$$

7.2. The case of K-theory

We take $E = K$, where $K(Y)$ is the Quillen K-theory spectrum of the smooth k-scheme Y. By [**33**, Theorem 6.4.2] we have the natural isomorphism

$$(s_m K)(Y) \cong \mathrm{EM}(z^m(Y,*)) \cong \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}(m)[2m])(Y).$$

In addition, we have natural Adams operations ψ_k, $k = 2, 3, \ldots$ acting on K and on the T-Postnikov tower of K, with ψ_k acting on $\pi_*(s_m K)(Y)$ by multiplication by k^m for all $Y \in \mathbf{Sm}/k$ (see [**34**, § 12, Theorem 12.1]).

Thus we have the following lemma.

Lemma 7.2.1. *Suppose X has dimension $p-1$ over k for some prime p. Then the spectral sequence (7.2) degenerates at E_1 after localizing at p.*

Proof. We have to show that all differentials are killed by some integer prime to p. The Adams operations act on the spectral sequence and ψ_k acts by multiplication by k^a on $E^r_{-a,b}$. Thus the differential $d^r_{-a,b} : E^r_{-a,b} \to E^r_{-a-r,b+r-1}$ is killed by $k^a(k^r-1)$. We have $E^1_{-a,b} = 0$ if $a > p$ or $a < 0$, so $d^r_{-a,b} = 0$ unless $0 \leqslant a \leqslant p-2$ and $1 \leqslant r \leqslant p-a-1$. Thus, if $a \geqslant 1$, then we need only consider r with $1 \leqslant r \leqslant p-2$, and we need to find an integer $k \geqslant 2$ such that k and $k^r - 1$ are prime to p. This is possible since $(\mathbb{Z}/p)^\times$ is cyclic of order $p-1$. If $a = 0$, we can take $k = p$. □

7.3. The Chow sheaf

For a smooth projective variety X, we have the Nisnevich sheaf with transfers $\mathcal{CH}^n(X)$ on $\mathbf{Sm}/k$, this being the sheaf associated to the presheaf

$$Y \mapsto \mathrm{CH}^n(X \times Y).$$

It is shown in [**22**, Remark 2.3] that $\mathcal{CH}^n(X)$ is a birational motivic sheaf. We can also label with the relative dimension, defining

$$\mathcal{CH}_n(X) := \mathcal{CH}^{\dim_k X - n}(X).$$

For our next computation, we need the following lemma.

Lemma 7.3.1. *Take*

$$\mathcal{F} \in C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)),$$

which is homotopy invariant and satisfies Nisnevich excision. Suppose in addition that $\mathcal{F}$ is connected. Then the sheaf $\mathcal{H}^{\mathrm{Nis}}_0(s^{\mathrm{mot}}_0 R\,\mathcal{H}\mathrm{om}(X, s^{\mathrm{mot}}_n \mathcal{F}))$ is the Nisnevich sheaf associated to the presheaf $H_0(s^{\mathrm{mot}}_0 R\,\mathcal{H}\mathrm{om}(X, s^{\mathrm{mot}}_n \mathcal{F}))$ with value at $Y \in \mathbf{Sm}/k$ given by the *exactness of*

$$\begin{aligned}\varinjlim_{\substack{W'\in\mathcal{S}^{(n+1)}_{X\times Y}(1)\\ W\in\mathcal{S}^{(n)}_{X\times Y}(1)}} & H_0(\mathcal{F}^{W\setminus W'}(X\times Y\times\Delta^1\setminus W'))\\ &\xrightarrow{i_1^*-i_0^*}\varinjlim_{\substack{W'\in\mathcal{S}^{(n+1)}_{X\times Y}(0)\\ W\in\mathcal{S}^{(n)}_{X\times Y}(0)}} H_0(\mathcal{F}^{W\setminus W'}(X\times Y\setminus W'))\\ &\to H_0(s_0^{\mathrm{mot}}R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F}))(Y)\\ &\to 0.\end{aligned}$$

Proof. From Proposition 2.2.3, $R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})(Y)=(s_n^{\mathrm{mot}}\mathcal{F})(X\times Y)$ is isomorphic in $D(\mathbf{Ab})$ to $s^n_{\mathrm{mot}}(X\times Y,-;\mathcal{F})$, the total complex of the simplicial complex

$$m\mapsto s^n_{\mathrm{mot}}(X\times Y,m;\mathcal{F}):=\varinjlim_{\substack{W\in\mathcal{S}^{(n)}_{X\times Y}(m)\\ W'\in\mathcal{S}^{(n+1)}_{X\times Y}(m)}}\mathcal{F}^{W\setminus W'}(X\times Y\times\Delta^m\setminus W').$$

By Lemma 3.1.3, the spectra $s^n_{\mathrm{mot}}(X\times Y,m;\mathcal{F})$ are all -1-connected. Thus we have the exact sequence

$$H_0(s^n(X\times Y,1;\mathcal{F}))\xrightarrow{i_0^*-i_1^*}H_0(s^n_{\mathrm{mot}}(X\times Y,0;\mathcal{F}))\to H_0(s^n_{\mathrm{mot}}(X\times Y,-;\mathcal{F})).$$

In any case $R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})$ is in $DM^{\mathrm{eff}}(k)$, hence the homology presheaf

$$Y\mapsto H_0(R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})(Y))=H_0(s_n^{\mathrm{mot}}(X\times Y,-;\mathcal{F}))$$

is a homotopy invariant presheaf with transfers. Thus, by [**17**, III, Corollary 4.18], if Y is local, the restriction map

$$H_0(s_n^{\mathrm{mot}}(X\times Y,-;\mathcal{F}))\to H_0(s_n^{\mathrm{mot}}(X_{k(Y)},-;\mathcal{F}))\tag{7.3}$$

is injective. In addition, $R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})$ is connected. Indeed, $s_n^{\mathrm{mot}}\mathcal{F}$ is connected by Proposition 3.1.4, and this implies that $R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})$ is connected. Thus the restriction map (7.3) is also surjective, hence an isomorphism.

By Theorem 4.2.1, $s_0^{\mathrm{mot}}R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})$ is also birational, and is connected by Proposition 3.1.4, hence the same argument shows that

$$H_0(s_0^{\mathrm{mot}}R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})(Y))\to H_0(s_0^{\mathrm{mot}}R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})(k(Y)))$$

is an isomorphism.

We now return to the situation $Y\in\mathbf{Sm}/k$. As in the proof of Lemma 7.1.3, $s_0^{\mathrm{mot}}R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})(Y)$ is given by evaluating $R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})$ on $\hat\Delta^*_{k(Y)}$. Since $R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})$ is connected by Proposition 3.1.4, it follows that we have the exact sequence

$$\begin{aligned}H_0(R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F}))(\hat\Delta^1_{k(Y)})&\xrightarrow{i_0^*-i_{1*}}H_0(R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F}))(\hat\Delta^0_{k(Y)})\\ &\to H_0(s_0^{\mathrm{mot}}R\,\mathcal{H}\mathrm{om}(X,s_n^{\mathrm{mot}}\mathcal{F})(Y))\to 0.\end{aligned}$$

But since $R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})$ is connected, the restriction map

$$H_0(R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})(\Delta^1_{k(Y)})) \to H_0(R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})(\hat{\Delta}^1_{k(Y)}))$$

is surjective, which shows that

$$H_0(R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})(k(Y))) \cong H_0(s_0^{\mathrm{mot}}R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})(Y)).$$

Since the restriction map (7.3) is an isomorphism for Y local, it follows that the canonical map

$$H_0(R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})(Y)) \to H_0(s_0^{\mathrm{mot}}R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})(Y))$$

is an isomorphism for Y local.

Putting this together with our description above of $H_0(R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}\mathcal{F})(Y))$ proves the result. □

Lemma 7.3.2. *Let X be a smooth projective variety of dimension d. There is a natural isomorphism*

$$\mathcal{H}_0^{\mathrm{Nis}}(s_0^{\mathrm{mot}}R\mathcal{H}\mathrm{om}(X, \mathbb{Z}(n)[2n])) \cong \mathcal{CH}^n(X).$$

Proof. Since $\mathbb{Z}$ is a birational motive, we have (Remark 4.2.3)

$$\mathbb{Z}(n)[2n] \cong s_n^{\mathrm{mot}}(\mathbb{Z}(n)[2n]).$$

We can now use Lemma 7.3.1 to compute $H_0^{\mathrm{Nis}}(s_0^{\mathrm{mot}}R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}(\mathbb{Z}(n)[2n])))$.

By Lemma 4.4.4, for $W \subset Y$ a closed subvariety of codimension n, $Y \in \mathbf{Sm}/k$, there is a natural isomorphism

$$H_0((\mathbb{Z}(n)[2n])^W(T)) = H_W^{2n}(Y, \mathbb{Z}(n)) \xrightarrow{\rho_{Y,W,n}} z_W^n(Y).$$

From this, it follows from Lemma 7.3.1 that $H_0^{\mathrm{Nis}}(s_0^{\mathrm{mot}}R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}(\mathbb{Z}(n)[2n])))$ is just the sheafification of

$$Y \mapsto \mathrm{CH}^n(X \times Y),$$

i.e.

$$\mathcal{H}_0^{\mathrm{Nis}}(s_0^{\mathrm{mot}}R\mathcal{H}\mathrm{om}(X, s_n^{\mathrm{mot}}(\mathbb{Z}(n)[2n]))) \cong \mathcal{CH}^n(X).$$

□

7.4. The slices of $M(X)$

To prove our main theorem on the slices of the motive of a Severi–Brauer variety, we use duality to shift the computation of the nth slice to a 0th slice of a related motive. 0th slices are easier to handle, because their cohomology sheaves are birational sheaves.

Lemma 7.4.1. *Let X be smooth and projective of dimension d over k. Then for $0 \leqslant n \leqslant d$ there is a natural isomorphism*

$$s_n^{\mathrm{mot}}M(X) \cong s_0^{\mathrm{mot}}(R\mathcal{H}\mathrm{om}(X, \mathbb{Z}(d-n)))(n)[2d].$$

Proof. By [**22**],

$$\begin{aligned} f_n^{\mathrm{mot}} M(X) &= \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(n), M(X))(n) \\ &= \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(d)[2d], M(X)(d-n)[2d])(n) \\ &= \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(M(X), \mathbb{Z}(d-n))(n)[2d]. \end{aligned}$$

In addition, using the isomorphism (2.4), we have

$$f_{n-1}^{\mathrm{mot}} \circ \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}}(\mathbb{Z}(1), -) = \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}}(\mathbb{Z}(1), -) \circ f_n^{\mathrm{mot}}. \tag{7.4}$$

This plus Voevodsky's cancellation theorem [**62**] implies

$$f_n^{\mathrm{mot}}(F(1)) \cong f_{n-1}^{\mathrm{mot}}(F)(1).$$

Indeed

$$\begin{aligned} f_n^{\mathrm{mot}}(F(1)) &\cong \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(n), F(1))(n) \\ &\cong \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(n-1), F)(n) \\ &\cong f_{n-1}^{\mathrm{mot}}(F)(1). \end{aligned}$$

Thus

$$\begin{aligned} s_n^{\mathrm{mot}} M(X) &= s_n^{\mathrm{mot}}(f_n^{\mathrm{mot}}(M(X))) \\ &= s_n^{\mathrm{mot}}(\mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(M(X), \mathbb{Z}(d-n))(n)[2d]) \\ &= s_0^{\mathrm{mot}}(\mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(k)}(M(X), \mathbb{Z}(d-n)))(n)[2d] \\ &= s_0^{\mathrm{mot}}(R\,\mathcal{H}\mathrm{om}(X, \mathbb{Z}(d-n)))(n)[2d]. \end{aligned}$$

□

Theorem 7.4.2. *Let X be a Severi–Brauer variety of dimension $p-1$, p a prime, associated to a central simple algebra $\mathcal{A}$ of degree p over k. Then*

(1)
$$s_n^{\mathrm{mot}} M(X) \cong \mathcal{CH}_n(X)(n)[2n]$$
for $n = 0, \ldots, p-1$, $s_n^{\mathrm{mot}} M(X) = 0$ for $n \geqslant p$.

(2) *There is a canonical isomorphism*
$$\bigoplus_{n=0}^{p-1} \mathcal{CH}^n(X) \cong \bigoplus_{n=0}^{p-1} \mathbb{Z}_{\mathcal{A}^{\otimes n}}.$$

(3) *For $n = 0, \ldots, p-1$, we have*
$$\mathcal{CH}^n(X) \cong \mathbb{Z}_{\mathcal{A}^{\otimes n}} \cong \begin{cases} \mathbb{Z}_{\mathcal{A}} & \text{for } n = 1, \ldots, p-1, \\ \mathbb{Z} & \text{for } n = 0. \end{cases}$$

Proof. We first note that the spectral sequence (7.2) has

$$p \cdot d_r^{a,b} = 0$$

for all a, b, r. Indeed, if $X = \mathbb{P}^{p-1}$, then the projective bundle formula gives the weak equivalence

$$R\mathcal{H}\mathrm{om}(\mathbb{P}^{p-1}, f_m K) \cong \bigoplus_{i=0}^{p-1} f_{m-i} K$$

from which the degeneration of the spectral sequence at E_1 for all $Y \in \mathbf{Sm}/k$ easily follows. In general, there is a splitting field L for $\mathcal{A}$ of degree p over k, so $X_L \cong \mathbb{P}_L^{p-1}$, and thus the differentials are all killed by $\times p$. But now by Lemma 7.2.1, it follows that the spectral sequence (7.2) actually degenerates at E_1.

We recall that $s_n K \cong \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}(n)[2n])$ [**33**, Theorem 6.4.2]. By Quillen's computation [**48**] of the K-theory of Severi–Brauer varieties,

$$R\mathcal{H}\mathrm{om}(X, K) \cong \bigoplus_{n=0}^{p-1} K(-; \mathcal{A}^{\otimes n}).$$

Finally, the fact that $K(-; \mathcal{A}^{\otimes n})$ is well-connected (Lemma 6.4.3) implies

$$s_0(K(-; \mathcal{A}^{\otimes n})) = \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_{\mathcal{A}^{\otimes n}}).$$

Since our spectral sequence degenerates at E^1, we therefore have the isomorphism

$$\bigoplus_{n=0}^{p-1} \pi_*^{\mathrm{Nis}} s_0(R\mathcal{H}\mathrm{om}(X, \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}(n)[2n]))) \cong \bigoplus_{n=0}^{p-1} \pi_*^{\mathrm{Nis}} \mathrm{EM}_{\mathbb{A}^1}(\mathbb{Z}_{\mathcal{A}^{\otimes n}}).$$

Also, by Proposition 1.4.4, we have $s_0 \circ \mathrm{EM}_{\mathbb{A}^1} = \mathrm{EM}_{\mathbb{A}^1} \circ s_0^{\mathrm{mot}}$. In addition,

$$R\mathcal{H}\mathrm{om}(X, \mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})) = \mathrm{EM}_{\mathbb{A}^1}(R\mathcal{H}\mathrm{om}(X, \mathcal{F})),$$
$$\pi_m^{\mathrm{Nis}}(\mathrm{EM}_{\mathbb{A}^1}(\mathcal{F})) = \mathcal{H}_{\mathrm{Nis}}^{-m}(\mathcal{F})$$

for $\mathcal{F} \in DM^{\mathrm{eff}}(k)$. Thus we see that

$$\mathcal{H}_{\mathrm{Nis}}^{m}(s_0^{\mathrm{mot}}(R\mathcal{H}\mathrm{om}(X, \mathbb{Z}(n)[2n]))) = 0$$

for $m \neq 0$ and

$$\bigoplus_{n=0}^{p-1} \mathcal{H}_{\mathrm{Nis}}^{0}(s_0^{\mathrm{mot}}(R\mathcal{H}\mathrm{om}(X, \mathbb{Z}(n)[2n]))) \cong \bigoplus_{n=0}^{p-1} \mathbb{Z}_{\mathcal{A}^{\otimes n}}. \tag{7.5}$$

In particular, $s_0^{\mathrm{mot}}(R\mathcal{H}\mathrm{om}(X, \mathbb{Z}(n)[2n]))$ is concentrated in degree 0. Thus, it follows from Lemma 7.3.2 that

$$s_0^{\mathrm{mot}}(R\mathcal{H}\mathrm{om}(X, \mathbb{Z}(n)[2n])) \cong \mathcal{CH}^n(X)$$

for $n = 0, \ldots, p-1$, which together with (7.5) proves (2).

Together with Lemma 7.4.1, this gives

$$\begin{aligned} s_n^{\mathrm{mot}} M(X) &\cong s_0^{\mathrm{mot}}(R\,\mathcal{H}\mathrm{om}^{\mathrm{mot}}(X, \mathbb{Z}(p-1-n)))(n)[2p-2] \\ &\cong \mathcal{CH}_n(X)(n)[2n], \end{aligned}$$

proving (1).

For (3), take a finite Galois splitting field L/k for $\mathcal{A}$ with Galois group G. We have the natural map

$$\pi^* : \mathcal{CH}^n(X) \to \mathcal{CH}^n(X_L)^G \cong \mathbb{Z}$$

with kernel and cokernel killed by p. By (2), $\mathcal{CH}^n(X)$ is torsion-free. Similarly, we have the inclusion

$$\pi^* : \mathbb{Z}_{\mathcal{A}^{\otimes n}} \to (\mathbb{Z}_{\mathcal{A}_L^{\otimes n}})^G \cong \mathbb{Z}.$$

We thus have compatible inclusions

$$\begin{tikzcd} \bigoplus_{n=0}^{p-1} \mathcal{CH}^n(X) \arrow[r, phantom, "\cong"] \arrow[d, hook] & \bigoplus_{n=0}^{p-1} \mathbb{Z}_{\mathcal{A}^{\otimes n}} \arrow[d, hook] \\ \bigoplus_{n=0}^{p-1} \mathcal{CH}^n(X_L)^G \arrow[r, phantom, "\cong"] & \bigoplus_{n=0}^{p-1} (\mathbb{Z}_{\mathcal{A}_L^{\otimes n}})^G \end{tikzcd}$$

Clearly, $\mathcal{CH}^0(X) \cong \mathbb{Z}$. For $y \in Y \in \mathbf{Sm}/k$ the quotient

$$\left(\bigoplus_{n=0}^{p-1} (\mathbb{Z}_{\mathcal{A}_L^{\otimes n}})^G\right)_y \Big/ \left(\bigoplus_{n=0}^{p-1} \mathbb{Z}_{\mathcal{A}^{\otimes n}}\right)_y$$

has order p^{p-1} if $\mathcal{A}_y$ is not split, and order 1 otherwise. Thus, for $n = 1, \ldots, p-1$, $\mathcal{CH}^n(X)_y \subset \mathcal{CH}^n(X_L)_y^G = \mathbb{Z}$ has index p if $\mathcal{A}_y$ is not split and index 1 if $\mathcal{A}_y$ is split. Thus we can write

$$\mathcal{CH}^n(X) \cong \mathbb{Z}_{\mathcal{A}^{\otimes n}}$$

for $n = 0, \ldots, p-1$, completing the proof. □

8. Applications

In this section, we let X be the Severi–Brauer variety $\mathrm{SB}(A)$ associated to a central simple algebra A of prime degree ℓ over k. We use our computations of the layers for $M(X)$, together with a duality argument and the Beilinson–Lichtenbaum conjecture, to study the reduced norm map

$$\mathrm{Nrd} : H^p(k, \mathbb{Z}_A(q)) \to H^p(k, \mathbb{Z}(q))$$

and prove the first of our main applications: Corollary 1 (see Theorem 8.1.4). Combining these results with our identification of the low-degree K-theory of A with the twisted Milnor K-theory of k gives us our main result on the vanishing of $SK_2(A)$ for A of square-free index (Corollary 2; see also Theorem 8.2.2).

8.1. A spectral sequence for motivic homology

Throughout this section, we invert the exponential characteristic of k. We omit writing this explicitly, to simplify the notation.

We have the motivic Postnikov tower for $M(X)$

$$0 = f_\ell^{\mathrm{mot}} M(X) \to f_{\ell-1}^{\mathrm{mot}} M(X) \to \cdots \to f_1^{\mathrm{mot}} M(X) \to f_0^{\mathrm{mot}} M(X) = M(X) \tag{8.1}$$

with slices

$$s_b^{\mathrm{mot}} M(X) \cong \mathbb{Z}_{A^{\otimes b+1}}(b)[2b], \quad b = 0, \ldots, \ell - 1.$$

Let $\alpha^* : DM^{\mathrm{eff}}(k) \to DM^{\mathrm{eff}}(k)^{\text{ét}}$ be the change of topologies functor, with right adjoint $\alpha_* : DM^{\mathrm{eff}}(k)^{\text{ét}} \to DM^{\mathrm{eff}}(k)$ (see § C.4). The functors α^* and α_* are exact, and applying α^* to the morphism $\mathbb{Z}_A(n) \to \mathbb{Z}(n)$ gives an isomorphism $\alpha^*\mathbb{Z}_A(n) \xrightarrow{\sim} \alpha^*\mathbb{Z}(n)$. Thus, we have the tower

$$\begin{aligned} 0 = \alpha_*\alpha^* f_\ell^{\mathrm{mot}} M(X) &\to \alpha_*\alpha^* f_{p-1}^{\mathrm{mot}} M(X) \to \cdots \\ &\to \alpha_*\alpha^* f_1^{\mathrm{mot}} M(X) \to \alpha_*\alpha^* f_0^{\mathrm{mot}} M(X) = \alpha_*\alpha^* M(X) \end{aligned} \tag{8.2}$$

with slices

$$\alpha_*\alpha^* s_b^{\mathrm{mot}} M(X) \cong \alpha_*\alpha^*\mathbb{Z}(b)[2b], \quad b = 0, \ldots, \ell - 1.$$

Since α_* is right adjoint to α^*, the unit η of the adjunction gives the natural transformation of towers $\eta : (8.1) \to (8.2)$. Defining $\bar{M}(X)$, $\bar{M}(X)^{(n)}$ and $\bar{\mathbb{Z}}_{A^{\otimes b+1}}(a)$ by the distinguished triangles

$$\begin{gathered} M(X) \to \alpha_*\alpha^* M(X) \to \bar{M}(X) \to M(X)[1], \\ f_n^{\mathrm{mot}} M(X) \to \alpha_*\alpha^* f_n^{\mathrm{mot}} M(X) \to \bar{M}(X)^{(n)} \to f_n^{\mathrm{mot}} M(X)[1], \\ \mathbb{Z}_{A^{\otimes b+1}}(a) \to \alpha_*\alpha^*\mathbb{Z}(a) \to \bar{\mathbb{Z}}_{A^{\otimes b+1}}(a) \to \mathbb{Z}_{A^{\otimes b+1}}(a)[1], \end{gathered}$$

we have the tower

$$0 = \bar{M}(X)^{(p)} \to \bar{M}(X)^{(p-1)} \to \cdots \to \bar{M}(X)^{(1)} \to \bar{M}(X)^{(0)} = \bar{M}(X) \tag{8.3}$$

with slices

$$\bar{M}(X)^{[p]} \cong \bar{\mathbb{Z}}_{A^{\otimes b+1}}(b)[2b], \quad b = 0, \ldots, p - 1.$$

Note that there are many non-canonical choices leading to these isomorphisms, but they are not important for the sequel.

This last tower thus gives rise to the strongly convergent spectral sequence

$$E_2^{p,q} \Longrightarrow \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(a)[b], \bar{M}(X)(a')[p+q]) \tag{8.4}$$

with

$$E_2^{p,q} = \begin{cases} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(a)[b], \bar{\mathbb{Z}}_{A^{\otimes -q+1}}(a'-q)[p-q]) & \text{for } 0 \leqslant -q \leqslant \ell - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 8.1.1. *For $U \in \mathbf{Sm}/k$, $\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(U)(r'), \mathbb{Z}_A(r)[q]) = 0$ for*

(1) *$r' > r$ and all q,*

(2) *$r' = r$ and $q \neq 0$,*

(3) *$1 \leqslant r - r' < q$ if $U = \operatorname{Spec} k$.*

In addition, $\mathrm{Hom}_{DM^{\mathrm{eff}}(k)^{\text{ét}}}(M(U)^{\text{ét}}(r'), \mathbb{Z}^{\text{ét}}(r)[q]) = 0$ for

$(1)^{\text{ét}}$ *$r' > r$ and $q \leqslant 2(r - r')$,*

$(2)^{\text{ét}}$ *$r' = r$ and $q < 0$.*

Proof. By cancellation (see Theorem C.7.1 and Corollary C.7.2), it suffices to prove (1), $(1)^{\text{ét}}$, (2) and $(2)^{\text{ét}}$ with $r = 0$, and (3) with $r' = 0$.

We first prove (1) and (2). For this, $\mathbb{Z}_A$ is a homotopy invariant Nisnevich sheaf with transfers, so

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(U)(r'), \mathbb{Z}_A[q - 2r']) = \ker[H^q_{\mathrm{Zar}}(U \times \mathbb{P}^{r'}, \mathbb{Z}_A) \to H^q_{\mathrm{Zar}}(U \times \mathbb{P}^{r'-1}, \mathbb{Z}_A)].$$

We may assume U irreducible. Since $\mathbb{Z}_A$ is a constant sheaf in the Zariski topology and is homotopy invariant,

$$H^q_{\mathrm{Zar}}(U \times \mathbb{P}^{r'}, \mathbb{Z}_A) = \begin{cases} 0 & \text{for } q \neq 0, \\ \mathbb{Z}_A(k(U)) & \text{for } q = 0. \end{cases}$$

The proof of $(1)^{\text{ét}}$ and $(2)^{\text{ét}}$ is similar: $(2)^{\text{ét}}$ follows from the vanishing of $H^q_{\text{ét}}(U, \mathbb{Z}^{\text{ét}})$ for $q < 0$. For $(1)^{\text{ét}}$, we use the argument for (1), noting that

$$H^q_{\text{ét}}(U \times \mathbb{P}^{r'}, \mathbb{Z}^{\text{ét}}) = \begin{cases} 0 & \text{for } q < 0, \\ \mathbb{Z} & \text{for } q = 0. \end{cases}$$

For (3), Theorem 6.5.3 gives us isomorphisms

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X), \mathbb{Z}_A(r)[2r + n]) \cong \mathrm{CH}^r(X, n; A)$$

for all n. Taking $X = \operatorname{Spec} k$, (3) follows from the fact that $z^r(\operatorname{Spec} k; A, n) = 0$ for $n < r$ by reason of dimension. □

For the rest of the paper we use the convention that, for $\mathcal{F}, \mathcal{G} \in DM^{\mathrm{eff}}(k)$, $a, b \in \mathbb{Z}$,

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{F}(a), \mathcal{G}(b)[m]) := \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathcal{F}(a + N), \mathcal{G}(b + N)[m]),$$

where N is chosen so that $a + N \geqslant 0$ and $b + N \geqslant 0$; we use a similar convention in $DM^{\mathrm{eff}}(k)^{\text{ét}}$. We define motivic cohomology with twisted coefficients $\mathcal{F}(-q)$, $q > 0$, by

$$H^p(X, \mathcal{F}(-q)) := \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(X)(q), \mathcal{F}[p])$$

and similarly for the étale version. By the cancellation theorems (Theorem C.7.1 and Corollary C.7.2), the convention is well-defined.

Remark 8.1.2. Define as before $\bar{\mathbb{Z}}(n)$ by the distinguished triangle

$$\mathbb{Z}(n) \to \alpha_*\alpha^*\mathbb{Z}(n) \to \bar{\mathbb{Z}}(n) \to \mathbb{Z}(n)[1].$$

The Bloch–Kato conjecture in weight n may be formulated as the statement that the cohomology sheaves of $\bar{\mathbb{Z}}(n)$ are zero in degree $d \leqslant n+1$. We note that the case $n=0$, although not often considered, is in fact true: this comes down to the statement that $\mathcal{H}^1_{\text{ét}}(\mathbb{Z}^{\text{ét}}) = 0$. This in turn follows from the exact sheaf sequence

$$0 \to \mathcal{H}^0_{\text{ét}}(\mathbb{Z}) \to \mathcal{H}^0_{\text{ét}}(\mathbb{Q}) \to \mathcal{H}^0_{\text{ét}}(\mathbb{Q}/\mathbb{Z}) \to \mathcal{H}^1_{\text{ét}}(\mathbb{Z}) \to 0$$

and the surjectivity of $\mathcal{H}^0_{\text{ét}}(\mathbb{Q}) \to \mathcal{H}^0_{\text{ét}}(\mathbb{Q}/\mathbb{Z})$.

Lemma 8.1.3. *For $n+1 \geqslant 0$, the Beilinson–Lichtenbaum conjecture for weight $n+1$ implies that*

$$\operatorname{Hom}_{DM^{\text{eff}}(k)}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[m]) = 0 \quad \text{for } m \leqslant n+2$$

and the sequence

$$0 \to H^{n+3}(X, \mathbb{Z}(n+1)) \to H^{n+3}_{\text{ét}}(X, \mathbb{Z}^{\text{ét}}(n+1)) \to H^{n+3}_{\text{ét}}(k(X), \mathbb{Z}^{\text{ét}}(n+1))$$

is exact.

Proof. The Beilinson–Lichtenbaum conjecture for weight $n+1 \geqslant 0$ says that the cohomology sheaves of $\bar{\mathbb{Z}}(n+1)$ are 0 in degree $d \leqslant n+2$, hence the natural map

$$H^m(X, \mathbb{Z}(n+1)) \to H^m_{\text{ét}}(X, \mathbb{Z}^{\text{ét}}(n+1))$$

is an isomorphism for $m \leqslant n+2$ and there is an exact sequence

$$0 \to H^{n+3}(X, \mathbb{Z}(n+1)) \to H^{n+3}_{\text{ét}}(X, \mathbb{Z}^{\text{ét}}(n+1)) \to H^0_{\text{Zar}}(X, \mathcal{H}^{n+3}_{\text{ét}}(\mathbb{Z}(n+1)))$$

since the cohomology sheaves of $\mathbb{Z}(n+1)$ vanish in degree $d \geqslant n+1$. By the Gersten conjecture for $\mathcal{H}^{n+3}_{\text{ét}}(\mathbb{Z}(n+1))$, the map

$$\mathcal{H}^{n+3}_{\text{ét}}(\mathbb{Z}(n+1)) \to H^{n+3}_{\text{ét}}(k(X), \mathbb{Z}(n+1))$$

is injective, which gives the exact sequence in the statement of the lemma.

In terms of morphisms in $DM^{\text{eff}}(k)$ and $DM^{\text{eff}}(k)^{\text{ét}}$, this says that the change of topologies map

$$\operatorname{Hom}_{DM^{\text{eff}}(k)}(M(X), \mathbb{Z}(n+1)[m]) \to \operatorname{Hom}_{DM^{\text{eff}}(k)^{\text{ét}}}(\alpha^* M(X), \alpha^*\mathbb{Z}(n+1)[m])$$

is an isomorphism for $m \leqslant n+2$ and an injection for $m = n+3$.

By Corollary C.7.3, we have natural isomorphisms

$$\operatorname{Hom}_{DM^{\text{eff}}(k)}(M(X), \mathbb{Z}(n+1)[m]) \cong \operatorname{Hom}_{DM^{\text{eff}}(k)}(\mathbb{Z}(d)[2d], M(X)(n+1)[m])$$

and

$$\begin{aligned}\mathrm{Hom}_{DM^{\mathrm{eff}}(k)^{\text{ét}}}&(\alpha^*M(X),\alpha^*\mathbb{Z}(n+1)[m])\\ &\cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)^{\text{ét}}}(\alpha^*\mathbb{Z}(d)[2d],\alpha^*M(X)(n+1)[m])\\ &\cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(d)[2d],\alpha_*\alpha^*M(X)(n+1)[m]).\end{aligned}$$

Thus, the natural map $M(X)\to\alpha_*\alpha^*M(X)$ induces an isomorphism

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(d)[2d],M(X)(n+1)[m])\to \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(d)[2d],\alpha_*\alpha^*M(X)(n+1)[m])$$

for $m\leqslant n+2$ and an injection for $m=n+3$, hence the lemma. □

Theorem 8.1.4. *Let A be a central simple algebra over k of prime degree ℓ, with $(\ell,\operatorname{char}k)=1$. Let $n\geqslant -1$ be an integer, and assume that the Beilinson–Lichtenbaum conjecture holds in weights $w\leqslant n+1$, and for the prime ℓ.*

(1) *For $m<n$, the reduced norm*

$$\mathrm{Nrd}: H^m(k,\mathbb{Z}_A(n))\to H^m(k,\mathbb{Z}(n))$$

is an isomorphism.

(2) *There is an exact sequence*

$$\begin{aligned}0\to H^n(k,\mathbb{Z}_A(n))\xrightarrow{\mathrm{Nrd}} H^n(k,\mathbb{Z}(n))&\xrightarrow{\partial_n}\\ &H^{n+3}_{\text{ét}}(k,\mathbb{Z}(n+1))\xrightarrow{\gamma} H^{n+3}_{\text{ét}}(k(X),\mathbb{Z}(n+1)),\end{aligned}$$

where X is the Severi–Brauer variety of A, γ is given by extension of scalars, and ∂_n is induced by the spectral sequence (8.4).

Proof. For $n=-1$, $H^m(k,\mathbb{Z}_A(n))=H^m(k,\mathbb{Z}(n))=0$ for all m by Lemma 8.1.1, and so the assertion is just that $H^2_{\text{ét}}(k,\mathbb{Z})\to H^2_{\text{ét}}(k(X),\mathbb{Z})$ is injective, As $H^2_{\text{ét}}(-,\mathbb{Z})\cong H^1_{\text{ét}}(-,\mathbb{Q}/\mathbb{Z})$, this is the assertion that the base-change map

$$H^1_{\text{ét}}(k,\mathbb{Q}/\mathbb{Z})\to H^1_{\text{ét}}(k(X),\mathbb{Q}/\mathbb{Z})$$

is injective. As $H^1_{\text{ét}}(-,\mathbb{Q}/\mathbb{Z})$ classifies cyclic étale covers and k is algebraically closed in $k(X)$, the injectivity is clear.

For $n\geqslant 0$, we proceed by induction on n: assume the result for all $n'<n$, $n'\geqslant -1$. By the Beilinson–Lichtenbaum conjecture in weight n',

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z},\bar{\mathbb{Z}}(n')[m])=0\quad\text{for } m\leqslant n'+1,\ n'\geqslant 0.$$

Similarly, applying (1) and (2) to the distinguished triangle defining $\bar{\mathbb{Z}}_A$, our induction assumption gives

$$\mathrm{Hom}_{DM^{\mathrm{eff}}}(\mathbb{Z},\bar{\mathbb{Z}}_A(n')[m])=0\quad\text{for } m<n',\ n'\geqslant -1. \tag{8.5}$$

Finally, by Lemma 8.1.3, the Beilinson–Lichtenbaum conjecture for weight $n+1$ gives

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[m]) = 0 \quad \text{for } m \leqslant n+2. \tag{8.6}$$

Now consider our spectral sequence (8.4) with $a = d$, $b = 2d - n - 2$ and $a' = n+1$, where $d = \dim_k X = \ell - 1$. We have

$$\begin{aligned}\mathrm{Hom}(&\mathbb{Z}(d)[2d-n-2], \bar{M}(X)(n+1)[p+q]) \\ &= \mathrm{Hom}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[n+2+p+q]),\end{aligned}$$

so by (8.6) the spectral sequence converges to 0 for $p+q \leqslant 0$.

The $E_2^{p,q}$ term is

$$E_2^{p,q} = \mathrm{Hom}(\mathbb{Z}(d)[2d], \bar{\mathbb{Z}}_{A^{\otimes -q+1}}(n+1-q)[n+2+p-q])$$

for $0 \leqslant -q \leqslant d$ and 0 otherwise. For $0 \leqslant -q < d-1$ and $p+q \leqslant 0$, we have

$$\begin{aligned} n' &:= n+1-d-q < n, \\ n+2-2d&+p-q < n'. \end{aligned}$$

For $-q = d$, $A^{\otimes -q+1}$ is a matrix algebra, hence $\bar{\mathbb{Z}}_{A^{\otimes -q+1}}(N) = \bar{\mathbb{Z}}(N)$. Thus

$$E_2^{p,-d} = \mathrm{Hom}(\mathbb{Z}, \bar{\mathbb{Z}}(n+1)[n+2-d+p]).$$

We claim that

$$E_2^{p,q} = 0 \quad \text{for } 0 \leqslant -q \leqslant d, \ \ -q \neq d-1, \ p+q \leqslant 0. \tag{8.7}$$

Indeed, if $p+q \leqslant 0$, then $p \leqslant d$, so $n+2-d+p \leqslant n+2$. Thus $E_2^{p,-d} = 0$ by Hilbert's theorem 90 in weight $n+1$. Next, suppose that $n+1-d-q < 0$. We have

$$n+2-2d+p-q \leqslant 2(n+1-q-d),$$

so $E_2^{p,q} = 0$ by Lemma 8.1.1 $(1)^{\text{ét}}$. Finally, in case $n+1-d-q \geqslant 0$, we use our induction hypothesis for $n' = n+1-d-q$ to conclude that $E_2^{p,q} = 0$ for $0 \leqslant -q < d-1$, finishing the proof of (8.7).

Thus, in the range $0 \leqslant -q \leqslant d$, $p+q \leqslant 0$, there is for each p exactly one E_2 term that is possibly non-zero, namely

$$E_2^{p,1-d} = \mathrm{Hom}(\mathbb{Z}, \bar{\mathbb{Z}}_{A^{\otimes d}}(n)[n+1-d+p]);$$

the d_2 differential is

$$E_2^{p,1-d} \xrightarrow{d_2} E_2^{p+2,-d}.$$

Suppose $p+q < 0$. Since $p+2-d \leqslant 0$, $E_2^{p+2,-d} = 0$. Since $E_2^{*,q} = 0$ for $q < -d$, there are no higher differentials coming out of $E_2^{p,1-d}$. Similarly, there are no d_r differentials going to $E_r^{p,1-d}$. Thus $E_2^{p,1-d} = E_\infty^{p,1-d} = 0$.

Now take $p+q=0$. The abutment of the spectral sequence is still 0 and there is still only one possibly non-zero E_2 term,

$$E_2^{d-1,1-d} = \mathrm{Hom}(\mathbb{Z}, \bar{\mathbb{Z}}_A(n)[n]).$$

The d_2 differential maps to

$$E_2^{d+1,-d} = \mathrm{Hom}(\mathbb{Z}, \bar{\mathbb{Z}}(n+1)[n+3]).$$

Since $E_2^{p,q} = 0$ for $-q > d$, $d_r^{d-1,1-d} = 0$ for $r > 2$, hence

$$d_2^{d-1,1-d} : E_2^{d-1,1-d} \to E_2^{d+1,-d}$$

is an injection. Moreover, for $r > 2$, all d_r differentials mapping to $E_r^{d+1,-d}$ have a source equal to 0, hence $E_3^{d+1,-d} = E_\infty^{d+1,-d}$.

Let us collect the information obtained so far.

- $E_2^{p,q} = 0$ for $p+q \leqslant 0$, except possibly $(p,q) = (d-1, 1-d)$.
- The differential $d_2^{d-1,1-d}$ induces an exact sequence

$$0 \to E_2^{d-1,1-d} \to E_2^{d+1,-d} \to \mathrm{Hom}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[n+3]). \tag{8.8}$$

Since $E_2^{p,1-d} = 0$ for $p < d-1$, we find that the map

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_{A^{\otimes d}}(n)[n+1-d+p]) \to \mathrm{Hom}(\mathbb{Z}, \alpha_*\alpha^*\mathbb{Z}_{A^{\otimes d}}(n)[n+1-d+p])$$

is an isomorphism for $p < d-1$ and an injection for $p = d-1$. Since $\mathbb{Z}_A \cong \mathbb{Z}_{A^{\otimes \ell-1}}$, we have

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_{A^{\otimes d}}(n)[n+1-d+p]) \cong H^{n+1+p-d}(k, \mathbb{Z}_A(n)),$$
$$\mathrm{Hom}(\mathbb{Z}, \alpha_*\alpha^*\mathbb{Z}_{A^{\otimes d}}(n)[n+1-d+p]) \cong H_{\text{ét}}^{n+1+p-d}(k, \mathbb{Z}(n)),$$

hence the canonical map

$$\alpha_A : H^m(k, \mathbb{Z}_A(n)) \to H_{\text{ét}}^m(k, \mathbb{Z}(n))$$

is an isomorphism for $m < n$ and an injection for $m = n$. Since α_A factors as

$$\begin{tikzcd} H^m(k, \mathbb{Z}_A(n)) \arrow[r, "\alpha_A"] \arrow[d, "\mathrm{Nrd}"'] & H_{\text{ét}}^m(k, \mathbb{Z}(n)) \\ H^m(k, \mathbb{Z}(n)) \arrow[ur, "\alpha"'] & \end{tikzcd}$$

and $\alpha : H^m(k, \mathbb{Z}(n)) \to H_{\text{ét}}^m(k, \mathbb{Z}(n)^{\text{ét}})$ is an isomorphism for $m \leqslant n$ by the Beilinson–Lichtenbaum conjecture in weight n, it follows that Nrd is an isomorphism for $m < n$ and an injection for $m = n$, proving (1) and the injectivity of Nrd in (2).

From the distinguished triangles defining $\bar{\mathbb{Z}}_A(n)$ and $\bar{\mathbb{Z}}(n)$, we have exact sequences

$$\cdots \to H^{n-1}(k, \mathbb{Z}_A(n)) \to H^{n-1}_{\text{ét}}(k, \mathbb{Z}(n)) \to E_2^{d-2,1-d} \\ \to H^n(k, \mathbb{Z}_A(n)) \to H^n_{\text{ét}}(k, \mathbb{Z}(n)) \to E_2^{d-1,1-d} \to H^{n+1}(k, \mathbb{Z}_A(n)) \to \cdots$$

and

$$\cdots \to H^{n+3}(k, \mathbb{Z}(n+1)) \to H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)) \to E_2^{d+1,-d} \to H^{n+4}(k, \mathbb{Z}(n+1)) \to \cdots .$$

But we have already shown $E_2^{d-2,1-d} = 0$. Also, using Theorem 6.5.3, we have $H^{n+1}(k, \mathbb{Z}_A(n)) = \mathrm{CH}^n(k, n-1; A)$. Thus

$$H^{n+1}(k, \mathbb{Z}_A(n)) = H^{n+3}(k, \mathbb{Z}(n+1)) = H^{n+4}(k, \mathbb{Z}(n+1)) = 0$$

for dimensional reasons; additionally, $H^n(k, \mathbb{Z}(n)) = H^n_{\text{ét}}(k, \mathbb{Z}(n))$ by Bloch–Kato in weight n. Thus we get an exact sequence

$$0 \to H^n(k, \mathbb{Z}_A(n)) \overset{\mathrm{Nrd}}{\to} H^n(k, \mathbb{Z}(n)) \to E_2^{d-1,1-d} \to 0$$

and an isomorphism

$$H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)) \xrightarrow{\sim} E_2^{d+1,-d}.$$

Putting this together with (8.8), we get the exact sequence

$$0 \to H^n(k, \mathbb{Z}_A(n)) \xrightarrow{\mathrm{Nrd}} H^n(k, \mathbb{Z}(n)) \xrightarrow{\partial_n} H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)) \\ \to \mathrm{Hom}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[n+3]),$$

where ∂_n is the map induced by $d_2^{d-1,1-d}$. By comparing the spectral sequence for

$$\mathrm{Hom}(\mathbb{Z}(d)[2d], M(X)(n+1)[*]), \qquad \mathrm{Hom}(\mathbb{Z}(d)[2d], \alpha_*\alpha^* M(X)(n+1)[*])$$

and

$$\mathrm{Hom}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[*]),$$

we see that $H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)) \to \mathrm{Hom}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[n+3])$ factors through the image of

$$\mathrm{Hom}(\mathbb{Z}(d)[2d], \alpha_*\alpha^* M(X)(n+1)[n+3]) \to \mathrm{Hom}(\mathbb{Z}(d)[2d], \bar{M}(X)(n+1)[n+3]).$$

By the exact sequence of Lemma 8.1.3 and the duality result Corollary C.7.3, we thus have the exact sequence

$$0 \to H^n(k, \mathbb{Z}_A(n)) \xrightarrow{\mathrm{Nrd}} H^n(k, \mathbb{Z}(n)) \xrightarrow{\partial_n} H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)) \to H^{n+3}_{\text{ét}}(k(X), \mathbb{Z}(n+1)).$$

The resulting map

$$H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)) \to H^{n+3}_{\text{ét}}(k(X), \mathbb{Z}(n+1))$$

is induced by an edge homomorphism of our spectral sequence, hence equals the extension of scalars map. This completes the proof. □

Corollary 8.1.5. *Let A be a central simple algebra of square-free index e over k, with $(e, \operatorname{char} k) = 1$. For $n \neq 1$, $H^n(k, \mathbb{Z}_A(1)) = 0$.*

Of course, we have already proved this by a direct argument (Theorem 6.8.2). This second argument uses our main result on the reduced norm, Theorem 8.1.4, which, in the weight one case, relies on the Merkurjev–Suslin theorem to prove Beilinson–Lichtenbaum in weight two (using in turn [**18**] or [**56**]).

Proof. We first reduce to the case of A of prime degree ℓ. Write

$$\deg(A) = \prod \ell_i = d,$$

where the ℓ_i are distinct primes. Write $A = M_n(D)$ for some division algebra D of degree d over k, and let $F \subset D$ be a maximal subfield. Then F has degree d over k and splits D. Let $\ell = \ell_i$ for some i, let $k(\ell) \supset k$ be the maximal prime to ℓ extension of k and let $F(\ell) := Fk(\ell)$. Then clearly $F(\ell)$ has degree ℓ over $k(\ell)$ and splits $A_{k(\ell)}$; since $k(\ell)$ has no prime to ℓ extensions, $F(\ell)$ is Galois over $k(\ell)$. Passing from k to the $\operatorname{Gal}(k(\ell)/k)$ invariants alters only the prime to ℓ torsion. Thus we may replace k with $k(\ell)$ and assume that A is split by a degree ℓ Galois extension of k. But then A is Morita equivalent to an algebra of degree ℓ, which achieves the reduction.

It follows from [**6**, Theorem 6.1] that

$$0 = \mathrm{CH}^1(k, 2-n) \cong H^n(k, \mathbb{Z}(1))$$

for $n \neq 1$. By Theorem 8.1.4 (1), this implies that $H^n(k, \mathbb{Z}_A(1)) = 0$ for $n < 1$. Additionally, we have

$$H^n(k, \mathbb{Z}_A(1)) \cong \mathrm{CH}^1(k, 2-n; A)$$

by Theorem 6.5.3. Since $\mathrm{CH}^1(k, m; A) = 0$ for $m < 0$ and $\mathrm{CH}^1(k, 0; A) = 0$ for dimensional reasons, the proof is complete. □

Corollary 8.1.6. *Let A be a central simple algebra of square-free index e over k, with $(e, \operatorname{char} k) = 1$. Then the edge homomorphism*

$$p_{2,k;A} : \mathrm{CH}^2(k, 2; A) \to K_2(A)$$

is an isomorphism.

Proof. From Corollary 8.1.5, $\mathrm{CH}^1(k, n; A) = 0$ for $n \neq 1$. From Theorem 6.7.1 (2), we have the exact sequence

$$0 \to \mathrm{CH}^1(k, 3; A) \xrightarrow{d_2^{-2,-1}} \mathrm{CH}^2(k, 2; A) \xrightarrow{p_{2,k;A}} K_2(A) \to \mathrm{CH}^1(k, 2; A) \to 0,$$

hence the edge-homomorphism $p_{2,k;A} : \mathrm{CH}^2(k, 2; A) \to K_2(A)$ is an isomorphism. □

Finally, here is a global version of Theorem 8.1.4.

Corollary 8.1.7. *Let $\tilde{\mathbb{Z}}_A$ denote the cokernel of the reduced norm map* $\mathrm{Nrd} : \mathbb{Z}_A \to \mathbb{Z}$. *Suppose that A has square-free index e, with $(e, \operatorname{char} k) = 1$, and assume the Beilinson–Lichtenbaum conjecture. Then,*

(1) *for all $n \geqslant 0$, the complex $\tilde{\mathbb{Z}}_A(n) = \tilde{\mathbb{Z}}_A \otimes \mathbb{Z}(n) \in DM^{\mathrm{eff}}(k)$ is concentrated in degree n;*

(2) *let $\mathcal{F}_n = \mathcal{H}^n(\tilde{\mathbb{Z}}_A(n))$. Then the stalk of $\mathcal{F}_n$ at a function field K is isomorphic to*

$$\ker(H^{n+3}_{\text{ét}}(K, \mathbb{Z}(n+1)) \to H^{n+3}_{\text{ét}}(K(X), \mathbb{Z}(n+1))),$$

where X is the Severi–Brauer variety of A;

- *for any smooth scheme U we have a Gersten resolution*

$$0 \to \mathcal{F}_n \to \bigoplus_{x \in U^{(0)}} (i_x)_*(\mathcal{F}_n) \\ \to \bigoplus_{x \in U^{(1)}} (i_x)_*(\mathcal{F}_{n-1}) \to \cdots \to \bigoplus_{x \in U^{(p)}} (i_x)_*(\mathcal{F}_{n-p}) \to \cdots .$$

Proof. As in the proof of Corollary 8.1.5, it suffices to handle the case of A of prime degree over k.

Clearly, $\mathbb{Z}_A(n)$ has no cohomology in degrees greater than n; by Voevodsky's form of Gersten's conjecture [**57**, Corollary 4.19, Theorem 4.27], the vanishing of $\mathcal{H}^i(\tilde{\mathbb{Z}}_A(n))$ for $i < n$ reduces to Theorem 8.1.4. The computation of the stalks of $\mathcal{H}^n(\tilde{\mathbb{Z}}_A(n))$ also follows from Theorem 8.1.4.

For (3), we first show (with the notation of [**57**, § 3.1]) that the Zariski sheaf associated to the presheaf $(\mathcal{F}_n)_{-1}$ is $\mathcal{F}_{n-1}$. This follows immediately from Voevodsky's cancellation theorem [**62**]: by definition,

$$\begin{aligned}(\mathcal{F}_n)_{-1}(U) &= \operatorname{coker}(\mathcal{F}_n(U \times \mathbb{A}^1) \to \mathcal{F}_n(U \times (\mathbb{A}^1 - \{0\}))) \\ &= \operatorname{coker}(H^n(U \times \mathbb{A}^1, \tilde{\mathbb{Z}}_A(n)) \to H^n(U \times (\mathbb{A}^1 - \{0\}), \tilde{\mathbb{Z}}_A(n))).\end{aligned}$$

By purity, the localization sequence for $U \times (\mathbb{A}^1 - \{0\}) \subset U \times \mathbb{A}^1$, and part (1) of the corollary, the latter cokernel is isomorphic to

$$\ker(H^{n-1}(U, \tilde{\mathbb{Z}}_A(n-1)) \to H^{n+1}(U, \tilde{\mathbb{Z}}_A(n))) \simeq H^1_{\mathrm{Zar}}(U, \mathcal{F}_n),$$

hence the Zariski sheaf associated to $(\mathcal{F}_n)_{-1}$ is the sheaf associated to

$$U \mapsto H^{n-1}(U, \tilde{\mathbb{Z}}_A(n-1)) \simeq \mathcal{F}_{n-1}(U).$$

The statement on the Gersten complex follows from this and [**57**, Theorem 4.37]. □

8.2. Computing the boundary map

To finish our study of $H^n(k, \mathbb{Z}_A(n))$, we need to compute the boundary map ∂_n in Theorem 8.1.4. As above, we fix a central simple algebra A over k of prime degree ℓ, let $d = \ell - 1$ and let X be the Severi–Brauer variety $\mathrm{SB}(A)$. We let $[A] \in H^2_{\text{ét}}(k, \mathbb{G}_m)$ denote the class of A in the (cohomological) Brauer group of k. As in the previous section, we invert the exponential characteristic of k.

Concentrating on $f^{\mathrm{mot}}_{d-1}M(X)$ gives us the distinguished triangle

$$s^{\mathrm{mot}}_d M(X) \to f^{\mathrm{mot}}_{d-1}M(X) \to s^{\mathrm{mot}}_{d-1}M(X) \to s^{\mathrm{mot}}_d M(X)[1],$$

which by Theorem 7.4.2 is

$$\mathbb{Z}(d)[2d] \to f^{\mathrm{mot}}_{d-1}M(X) \to \mathbb{Z}_A(d-1)[2d-2] \to \mathbb{Z}(d)[2d+1].$$

Applying Ω^{d-1}_T gives

$$\mathbb{Z}(1)[2] \to \Omega^{d-1}_T f^{\mathrm{mot}}_{d-1}M(X) \to \mathbb{Z}_A \to \mathbb{Z}(1)[3].$$

Applying the étale sheafification α^* and noting that $\mathbb{Z}^{\text{ét}}_A \cong \mathbb{Z}^{\text{ét}}$ gives the distinguished triangle

$$\mathbb{Z}(1)^{\text{ét}}[2] \to \alpha^*\Omega^{d-1}_T f^{\mathrm{mot}}_{d-1}M(X) \to \mathbb{Z}^{\text{ét}} \xrightarrow{\partial} \mathbb{Z}(1)^{\text{ét}}[3]. \tag{8.9}$$

Thus $\partial : \mathbb{Z}^{\text{ét}} \to \mathbb{Z}(1)^{\text{ét}}[3]$ gives us the element

$$\beta_A \in H^3_{\text{ét}}(k, \mathbb{Z}(1)^{\text{ét}}) = H^2_{\text{ét}}(k, \mathbb{G}_m).$$

Proposition 8.2.1. $\beta_A = [A]$.

Proof. To calculate β_A, it suffices to restrict (8.9) to the small étale site on k. By Lemma 7.1.2, (8.9) on $k_{\text{ét}}$ is isomorphic (in $D(\mathrm{Sh}_{\text{ét}}(k))$) to the sheafification of the sequence of presheaves

$$L \mapsto (z^1(L,*) \xrightarrow{p^*} z^1(X_L,*) \to \mathrm{cone}(p^*) \to z^1(L,*)[1]). \tag{8.10}$$

Here, and in the remainder of this proof, we consider the cycle complexes as cohomological complexes:

$$z^1(Y,*)^n := z^1(Y,-n).$$

We recall that $z^1(X_L,*)$ has non-zero cohomology only in degrees 0 and -1, and that

$$H^{-1}(z^1(X_L,*)) = \Gamma(X_L, \mathcal{O}^\times_{X_L}),$$
$$H^0(z^1(X_L,*)) = \mathrm{CH}^1(X_L).$$

Similarly, $H^{-1}(z^1(L,*)) = L^\times$ and all other cohomology of $z^1(L,*)$ vanishes. Since X is geometrically irreducible and projective,

$$p^* : L^\times \to \Gamma(X_L, \mathcal{O}^\times_{X_L})$$

is an isomorphism, and thus the cone of $z^1(L,*) \xrightarrow{p^*} z^1(X_L,*)$ has only cohomology in degree 0, namely

$$H^0(\mathrm{cone}(p^*)) = \mathrm{CH}^1(X_L).$$

Thus the sequence (8.10) is naturally isomorphic (in $D(\mathbf{Spc}_{\bullet_{\text{ét}}}(k))$) to the canonical sequence

$$L \mapsto (H^{-1}(z^1(X_L,*))[1] \to \tau_{\geqslant -1} z^1(X_L,*) \to H^0(z^1(X_L,*)) \to H^{-1}(z^1(X_L,*))[2]). \tag{8.11}$$

We can explicitly calculate a cocycle representing β_A as follows. Take L/k to be a Galois extension with group G such that A_L is a matrix algebra over L. Then (8.11) gives a distinguished triangle in the derived category of G-modules, so we have in particular the connecting homomorphism

$$\partial_L : H^0(G, H^0(z^1(X_L,*))) \to H^2(G; H^{-1}(z^1(X_L,*))) = H^2(G; L^\times).$$

Also $X_L \cong \mathbb{P}^d_L$. As $H^0(z^1(X_L,*)) = \mathrm{CH}^1(X_L)$, $H^0(z^1(X_L,*))$ has a canonical G-invariant generator 1, namely the element corresponding to $c_1(\mathcal{O}(1))$. We can apply ∂_L to 1, giving the element $\partial_L(1) \in H^2(G; L^\times)$ which maps to β_A under the canonical map

$$H^2(G, L^\times) \to H^2_{\text{ét}}(k, \mathbb{G}_m).$$

Since A_L is a matrix algebra over L, A is given by a 1-cocycle

$$\{\bar{g}_\sigma \mid \sigma \in G\} \in Z^1(G, \mathrm{PGL}_\ell(L))$$

and X is the form of $\mathbb{P}^d$ defined by $\{\bar{g}_\sigma\}$. This mean that there is an L isomorphism $\psi : X_L \to \mathbb{P}^d_L$ such that, for each $\sigma \in G$, we have

$$\bar{g}_\sigma := \psi \circ {}^\sigma\psi^{-1},$$

under the usual identification $\mathrm{Aut}_L(\mathbb{P}^d_L) = \mathrm{PGL}_{d+1}(L)$.

Lifting $\bar{g}_\sigma$ to $g_\sigma \in \mathrm{GL}_\ell(L)$ and defining $c_{\tau,\sigma} \in L^\times$ by

$$c_{\tau,\sigma}\,\mathrm{id} := g_\tau\, {}^\tau g_\sigma g_{\tau\sigma}^{-1},$$

we have the cocycle $\{c_{\tau,\sigma}\} \in Z^2(G, L^\times)$ representing $[A]$.

For a G-module M, let $(C^*(G; M), \hat{d})$ denote the standard co-chain complex computing $H^*(G; M)$, i.e. $C^n(G; M)$ is a group of n co-chains of G with values in M. We will show that $\partial_L(1) = \{c_{\tau,\sigma}\}$ in $H^2(G, L^\times)$ by applying $C^*(G; -)$ to the sequence (8.11) and making an explicit computation of the boundary map.

Fix a hyperplane $H \subset \mathbb{P}^d_k$. Then $D := \psi^*(H_L) \in z^1(X_L,*)^0$ represents the positive generator $1 \in \mathrm{CH}^1(X_L) \cong \mathbb{Z}$. As the class of D in $\mathrm{CH}^1(X_L)$ is G-invariant, there is for each $\sigma \in G$ a rational function f_σ on X_L such that

$$\mathrm{Div}(f_\sigma) = {}^\sigma D - D.$$

Given $\tau, \sigma \in G$, we thus have

$$\mathrm{Div}(f_\sigma^\tau f_{\tau\sigma}^{-1} f_\tau) = {}^{\tau\sigma}D - {}^\tau D - ({}^{\tau\sigma}D - D) + {}^\tau D - D = 0,$$

so there is a $c'_{\tau,\sigma} \in \Gamma(X_L, \mathcal{O}_{X_L}^\times) = L^\times$ with

$$c'_{\tau,\sigma} = f_\sigma^\tau f_{\tau\sigma}^{-1} f_\tau.$$

Using the fact that

$${}^\sigma D = \psi^*(\bar{g}_\sigma(H_L)),$$

one can easily calculate that

$$c'_{\tau,\sigma} = c_{\tau,\sigma}.$$

Indeed, take a k-linear form L_0 so that H is the hyperplane defined by $L_0 = 0$. Let

$$F_\sigma := \frac{L_0 \circ g_\sigma^{-1}}{L_0},$$

so $\mathrm{Div}(F_\sigma) = \bar{g}_\sigma(H) - H$. Letting $f_\sigma := \psi^* F_\sigma$, we have

$$\mathrm{Div}(f_\sigma) = \psi^*(\mathrm{Div}(F_\sigma)) = \psi^*(\bar{g}_\sigma(H) - H) = {}^\sigma D - D,$$

and

$${}^\tau f_\sigma = \psi^*\left(\frac{L_0 \circ {}^\tau g_\sigma^{-1} \circ g_\tau^{-1}}{L_0 \circ g_\tau^{-1}}\right).$$

Thus

$$\begin{aligned} c'_{\tau,\sigma} &= \psi^*\left(\frac{L_0 \circ {}^\tau g_\sigma^{-1} \circ g_\tau^{-1}}{L_0 \circ g_\tau^{-1}}\right) \cdot \psi^*\left(\frac{L_0 \circ g_{\tau\sigma}^{-1}}{L_0}\right)^{-1} \cdot \psi^*\left(\frac{L_0 \circ g_\tau^{-1}}{L_0}\right) \\ &= \psi^*\left(\frac{L_0 \circ {}^\tau g_\sigma^{-1} g_\tau^{-1}}{L_0 \circ g_{\tau\sigma}^{-1}}\right) \\ &= c_{\tau,\sigma}. \end{aligned}$$

On the other hand, we can calculate the boundary $\partial_L(1)$ by lifting the generator $1 = [D] \in \mathrm{CH}^1(X_L)^G$ to the element $D \in z^1(X_L, *)^0$ and taking Čech co-boundaries. Explicitly, let $\Gamma_\sigma \subset X_L \times \Delta^1$ be the closure of graph of f_σ, after identifying $(\Delta^1, 0, 1)$ with $(\mathbb{P}^1 \setminus \{1\}, 0, \infty)$. Define $\Gamma_{c_{\sigma,\tau}} \in z^1(L, *)^{-1}$ similarly as the point of Δ^1_L corresponding to $c_{\tau,\sigma} \in \mathbb{A}^1(k) \subset \mathbb{P}^1(k)$, and let δ denote the boundary in the complex $z^1(X_L, *)$. For $\sigma \in G$, we have

$$\delta^{-1}(\Gamma_\sigma) = {}^\sigma D - D = \hat{d}^0(D)_\sigma.$$

Since $H^{-1}(z^1(X_L, *)) = \Gamma(X_L, \mathcal{O}_{X_L}^\times) = L^\times$, there is for each $\sigma, \tau \in G$, an element $B_{\sigma,\tau} \in z^1(X_L, 2)$ with

$$\begin{aligned} p^*\Gamma_{c_{\sigma,\tau}} &= {}^\tau\Gamma_\sigma - \Gamma_{\tau\sigma} + \Gamma_\tau + \delta^{-2}(B_{\sigma,\tau}) \\ &= \hat{d}^1(\sigma \mapsto \Gamma_\sigma)_{\tau,\sigma} \in \tau_{\geqslant -1} z^1(X_L, *)^{-1}. \end{aligned}$$

Thus

$$\partial_L([D]) = \{c_{\sigma,\tau}\} \in H^2(G, H^{-1}(z^1(L, *))) = H^2(G, L^\times).$$

This completes the computation of $\partial_L(1)$ and the proof of the proposition. $\square$

Theorem 8.2.2. *Let A be a central simple algebra over k of square-free index e with $(e, \operatorname{char} k) = 1$. Let $n \geqslant 0$, and assume that the Beilinson–Lichtenbaum conjecture holds in weights $w \leqslant n+1$ at all primes dividing e.*

(1) *For $m < n$, the reduced norm*

$$\mathrm{Nrd} : H^m(k, \mathbb{Z}_A(n)) \to H^m(k, \mathbb{Z}(n))$$

is an isomorphism.

(2) *We have an exact sequence*

$$\begin{aligned} 0 \to H^n(k, \mathbb{Z}_A(n)) \xrightarrow{\mathrm{Nrd}} H^n(k, \mathbb{Z}(n)) \simeq K_n^M(k) \\ \xrightarrow{\cup[A]} H^{n+2}_{\text{ét}}(k, \mathbb{Z}/e(n+1)) \to H^{n+2}_{\text{ét}}(k(X), \mathbb{Z}/e(n+1)). \end{aligned}$$

(3) $(n = 1)$ $SK_1(A) = 0$. *More precisely, we have an exact sequence*

$$0 \to K_1(A) \xrightarrow{\mathrm{Nrd}} K_1(k) \xrightarrow{\cup[A]} H^3_{\text{ét}}(k, \mathbb{Z}/e(2)) \to H^3_{\text{ét}}(k(X), \mathbb{Z}/e(2)).$$

(4) $(n = 2)$ $SK_2(A) = 0$. *More precisely, we have an exact sequence*

$$0 \to K_2(A) \xrightarrow{\mathrm{Nrd}} K_2(k) \xrightarrow{\cup[A]} H^4_{\text{ét}}(k, \mathbb{Z}/e(3)) \to H^4_{\text{ét}}(k(X), \mathbb{Z}/e(3)).$$

To explain the map $\cup[A]$ in (2), (3) and (4), we have isomorphisms

$$\begin{aligned} K_1(k) = k^\times &\cong H^1(k, \mathbb{Z}(1)), \\ K_2(k) &\cong H^2(k, \mathbb{Z}(2)), \\ H^n_{\text{ét}}(k, \mathbb{G}_m) &\cong H^{n+1}_{\text{ét}}(k, \mathbb{Z}(1)^{\text{ét}}). \end{aligned}$$

Thus we have $[A] \in H^3_{\text{ét}}(k, \mathbb{Z}(1)^{\text{ét}})$ and cup product maps

$$H^n(k, \mathbb{Z}(n)) \to H^n_{\text{ét}}(k, \mathbb{Z}(n)^{\text{ét}}) \xrightarrow{\cup[A]} H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)^{\text{ét}}),$$

which obviously land in ${}_eH^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)^{\text{ét}})$. On the other hand, the exact triangle

$$\mathbb{Z}(n+1)^{\text{ét}} \xrightarrow{e} \mathbb{Z}(n+1)^{\text{ét}} \to \mathbb{Z}/e(n+1) \xrightarrow{+1}$$

and the Beilinson–Lichtenbaum conjecture in weight $n+1$ give an isomorphism

$$H^{n+2}_{\text{ét}}(k, \mathbb{Z}/e(n+1)) \xrightarrow{\sim} {}_eH^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)^{\text{ét}}).$$

Proof. As in the proof of Corollary 8.1.5, it suffices to handle the case of A of prime degree over k. Thus, (1) follows from Theorem 8.1.4 (1).

For (2), applying α^* to the distinguished triangle

$$\mathbb{Z}(1)[2] \to \Omega_T^{d-1} f_{d-1}^{\text{mot}} M(X) \to \mathbb{Z}_A \to \mathbb{Z}(1)[2]$$

we have

$$\mathbb{Z}(1)^{\text{ét}}[2] \to \alpha^* \Omega_T^{d-1} f_{d-1}^{\text{mot}} M(X) \to \mathbb{Z}^{\text{ét}} \xrightarrow{\partial} \mathbb{Z}(1)^{\text{ét}}[3].$$

It follows from Proposition 8.2.1 that the ∂ is given by cup product with $[A] \in H^3_{\text{ét}}(k, \mathbb{Z}(1)^{\text{ét}})$. Since the map ∂_n in Theorem 8.1.4 is just the map induced by ∂ after tensoring with $\mathbb{Z}(n)^{\text{ét}}[n]$, (2) is proven in the form of an exact sequence

$$0 \to H^n(k, \mathbb{Z}_A(n)) \xrightarrow{\text{Nrd}} H^n(k, \mathbb{Z}(n)) \\ \xrightarrow{\cup[A]} H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)^{\text{ét}}) \to H^{n+3}_{\text{ét}}(k(X), \mathbb{Z}(n+1)^{\text{ét}}).$$

But the Beilinson–Lichtenbaum conjecture in weight $n+1$, applied both to k and $k(X)$, shows that in the commutative diagram

$$\begin{tikzcd} H^{n+2}_{\text{ét}}(k, \mathbb{Z}/e(n+1)) \arrow[r] \arrow[d, "\partial"'] & H^{n+2}_{\text{ét}}(k(X), \mathbb{Z}/e(n+1)) \arrow[d, "\partial"'] \\ H^{n+3}_{\text{ét}}(k, \mathbb{Z}(n+1)^{\text{ét}}) \arrow[r] & H^{n+3}_{\text{ét}}(k(X), \mathbb{Z}(n+1)^{\text{ét}}) \end{tikzcd}$$

both horizontal maps have isomorphic kernels, hence the form of (2) appearing in Theorem 8.2.2.

For (3) and (4), we have the isomorphism (Theorem 6.5.3)

$$\psi_{p,q;A} : H^p(k, \mathbb{Z}_A(q)) \to \text{CH}^q(k, 2q-p; A)$$

compatible with the respective reduced norm maps. From Corollary 8.1.6, the edge-homomorphism $p_{2,k;A} : \text{CH}^2(k, 2; A) \to K_2(A)$ is an isomorphism. It follows from Theorem 6.7.1 (1) that the edge homomorphism $p_{1,k;A} : \text{CH}^1(k, 1; A) \to K_1(A)$ is an isomorphism as well. Together with Proposition 6.6.5, this gives us the commutative diagram for $n = 1, 2$:

$$\begin{tikzcd} H^n(k, \mathbb{Z}_A(n)) \arrow[r, "\psi_{n,n;A}"] \arrow[d, "\text{Nrd}"] & \text{CH}^n(k, n; A) \arrow[r, "p_{n,k;A}"] \arrow[d, "\text{Nrd}"] & K_n(A) \arrow[d, "\text{Nrd}"] \\ H^n(k, \mathbb{Z}(n)) \arrow[r, "\psi_{n,n;k}"'] & \text{CH}^n(k, n) \arrow[r, "p_{n,k;k}"'] & K_n(k) \end{tikzcd}$$

with all horizontal maps isomorphisms. Thus, in the sequence (1), we may replace $H^n(k, \mathbb{Z}_A(n))$ with $K_n(A)$ and $H^n(k, \mathbb{Z}(n))$ with $K_n(k)$ for $n = 1, 2$, proving (3) and (4). □

Remark 8.2.3. Taking $n = 0$ in Theorem 8.2.2, we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times e} \mathbb{Z} \xrightarrow{\cup[A]} H^2_{\text{ét}}(k, \mu_e) \to H^2_{\text{ét}}(k(X), \mu_e),$$

i.e. the kernel of $H^2_{\text{ét}}(k, \mathbb{G}_m) \to H^2_{\text{ét}}(k(X), \mathbb{G}_m)$ is generated by $[A]$. This relies only on the Bloch–Kato conjecture in weight 1, i.e. the classical Hilbert theorem 90, and recovers Amitsur's result in this special case [**1**].

Part III. Appendices

Appendix A. Modules over Azumaya algebras

We collect some basic results for use throughout the paper.

Let R be a commutative ring and A an Azumaya R-algebra.

Lemma A.1. *If R is Noetherian, A is left and right Noetherian.*

Proof. Indeed, A is a Noetherian R-module, hence a Noetherian A-module (on the left and on the right). $\square$

Lemma A.2. *For an A–A-bimodule M, let*

$$M^A = \{m \in M \mid am = ma\}.$$

Then the functor $M \mapsto M^A$ is exact and sends injective A–A-bimodules to injective R-modules.

Proof. Let $A^{\mathrm{e}} = A \otimes_R A^{\mathrm{op}}$ be the enveloping algebra of A. We may view M as a left A^{e}-module. A special A–A-bimodule is A itself, and we clearly have

$$M^A = \mathrm{Hom}_{A^{\mathrm{e}}}(A, M).$$

Since A is an Azumaya algebra, the map $A^{\mathrm{e}} \to \mathrm{End}_R(A)$ is an isomorphism of R-algebras; via this isomorphism, $\mathrm{Hom}_{A^{\mathrm{e}}}(A, M)$ may be canonically identified with $A^* \otimes_{\mathrm{End}_R(A)} M$, where $A^* = \mathrm{Hom}_R(A, R)$. Hence M^A is the transform of M under the Morita functor from $\mathrm{End}_R(A)$-modules to R-modules; since this functor is an equivalence of categories, it is exact and preserves injectives. $\square$

Proposition A.3. *For any two left A-modules M, N and any $q \geqslant 0$, we have*

$$\mathrm{Ext}^q_A(M, N) \simeq \mathrm{Ext}^q_R(M, N)^A.$$

(Note that $\mathrm{Ext}^q_R(M, N)$ is naturally an A–A-bimodule, which gives a meaning to the statement.)

Proof. The bifunctor $(M, N) \mapsto \mathrm{Hom}_A(M, N)$ is clearly the composition of the two functors

$$(M, N) \mapsto \mathrm{Hom}_R(M, N)$$

(from left A-modules to A–A-bimodules) and

$$Q \mapsto Q^A$$

(from A–A-bimodules to R-modules). Note also that, if P is A-projective and I is A-injective, then $\mathrm{Hom}_R(P, I)$ is an injective A–A-bimodule. The conclusion therefore follows from Lemma A.1. $\square$

Corollary A.4. *Let M be a left A-module. Then M is A-projective if and only if it is R-projective.*

Proof. If M is A-projective, it is R-projective since A is a projective R-module. The converse follows from Proposition A.3. □

Corollary A.5. *Suppose R regular of dimension d. Then any finitely generated left A-module M has a left resolution of length at most d by finitely generated projective A-modules. In particular, A is regular.*

Proof. Since R is regular, it is Noetherian and so is A by Lemma A.1. Proposition A.3 also shows that $\mathrm{Ext}_A^{d+1}(M,N) = 0$ for any N. The conclusion is now classical [**9**, VI, Proposition 2.1; V, Proposition 1.3]. □

Appendix B. Regularity

We prove the main result on the regularity properties of the functor $K(-;A)$ that we need to compute the layers in the homotopy coniveau tower for $G(X;\mathcal{A})$ in § 6.

Fix a noetherian commutative ring R. We let $\boldsymbol{R}$**-alg** denote the category of commutative R-algebras which are localizations of finitely generated commutative R-algebras.

Following Bass [**4**, XII, § 7, pp. 657–658], for an additive functor $F : \boldsymbol{R}\text{-}\mathbf{alg} \to \mathbf{Ab}$, we let $NF : \boldsymbol{R}\text{-}\mathbf{alg} \to \mathbf{Ab}$ be the functor

$$NF(A) := \ker(F(A[t]) \to F(A[t]/(t))),$$

where $A[t]$ is the polynomial algebra over A. We set $N^qF := N(N^{q-1}F)$.

For $a \in A$ the morphism $A[X] \to A[X]$, $X \mapsto a \cdot X$ induces a group endomorphism $NF(A) \to NF(A)$. So $NF(A)$ becomes a $\mathbb{Z}[T]$-module. We denote by $NF(A)_{[a]}$ the $\mathbb{Z}[T,T^{-1}]$-module $\mathbb{Z}[T,T^{-1}] \otimes_{\mathbb{Z}[T]} NF(A)$. With this notation Vorst proves the following theorem in [**63**].

Theorem B.1. *Let $A \in \boldsymbol{R}$**-alg** and let $a_1, \dots, a_n$ be elements of A which generate the unit ideal. Suppose further that the map*

$$NF(R[T]_{a_{i_0},\dots,\widehat{a_{i_j}},\dots,a_{i_p}})_{[a_{i_j}]} \to NF(A[T]_{a_{i_0},\dots,a_{i_p}})$$

is an isomorphism, for each set of indexes $1 \leqslant i_0 < \cdots < i_p \leqslant n$. Then the canonical morphism

$$\epsilon : NF(A) \to \bigoplus_{j=1}^{n} NF(A_{a_j})$$

is injective.

Proof. Compare [**63**, Theorem 1.2] or [**30**, Lemma 1.1]. □

This is extended by van der Kallen, in the case of the functor $A \mapsto K_n(A)$, to prove an étale descent result: namely, the following theorem.

Theorem B.2. *Let A be a noetherian commutative ring such that each zero divisor of A is contained in a minimal prime ideal of A. Let $A \to B$ be an étale and faithfully flat extension of A. Then the Amitsur complex*

$$0 \to N^q K_n(A) \to N^q K_n(B) \to N^q K_n(B \otimes_A B) \to \cdots$$

is exact for each q and n.

In fact, one can abstract van der Kallen's argument to give conditions on a functor $F : \boldsymbol{R}\text{-}\mathbf{alg} \to \mathbf{Ab}$ as above so that the conclusion of Theorem B.2 holds for the Amitsur complex for NF. For this, we recall the *big Witt vectors* $W(A)$ of a commutative ring A, with the canonical surjection $W(A) \to A$ and the multiplicative *Teichmüller lifting* $A \to W(A)$ sending $a \in A$ to $[a] \in W(A)$. We have as well the Witt vectors of length n, with surjection $W(A) \to W_n(A)$; we let $F^n W(A) \subset W(A)$ be the kernel. If M is a $W(A)$-module, we say M is a *continuous* $W(A)$ module if M is a union of the submodules M_n killed by $F^n W(A)$. Then one has the following theorem.

Theorem B.3. *Let $F : \boldsymbol{R}\text{-}\mathbf{alg} \to \mathbf{Ab}$ be a functor. Suppose that F satisfies the following conditions.*

1. *Given $a \in A \in \boldsymbol{R}\text{-}\mathbf{alg}$, the natural map $F(A_a) \to F(A)_{[a]}$ is an isomorphism.*
2. *Sending $a \in A$ to the endomorphism $[a] : NF(A) \to NF(A)$ extends to a continuous $W(A)$-module structure on $NF(A)$, natural in A, with the Teichmüller lifting $[a] \in W(A)$ acting by $[a] : NF(A) \to NF(A)$.*
3. *F commutes with filtered direct limits.*

Let $A \in \boldsymbol{R}\text{-}\mathbf{alg}$ be such that each zero-divisor of A is contained in a minimal prime ideal of A. Let $A \to B$ be an étale and faithfully flat extension of A. Then the Amitsur complex

$$0 \to NF(A) \to NF(B) \to NF(B \otimes_A B) \to NF(B \otimes_A B \otimes_A B) \to \cdots$$

is exact.

The main example of interest for us is the following. Let $\mathcal{A}$ be a noetherian central R-algebra, and let $K_n(\mathcal{A})$ be the nth K-group of the category of finitely generated projective (left) $\mathcal{A}$-modules.

Corollary B.4. *Let $F : \boldsymbol{R}\text{-}\mathbf{alg} \to \mathbf{Ab}$ be the functor*

$$F(A) := N^q K_n(\mathcal{A} \otimes_R A).$$

Then F satisfies the conditions of Theorem B.3, hence (assuming A satisfies the hypothesis on zero-divisors) if $A \to B$ is an étale and faithfully flat extension of A, then the Amitsur complex

$$0 \to N^q K_n(\mathcal{A} \otimes_R A) \to N^q K_n(\mathcal{A} \otimes_R B) \to N^q K_n(\mathcal{A} \otimes_R B \otimes_A B) \to \cdots$$

is exact.

Proof. Weibel [**65**] has shown that $N^qK_n(\mathcal{A})$ admits a $W(A)$-module structure, satisfying the conditions (1) and (2) of Theorem B.3. Since K-theory commutes with filtered direct limits, this proves that the given F satisfies the conditions of Theorem B.3, whence the result. $\square$

Now let X be an R-scheme and let $\mathcal{A}$ be a sheaf of Azumaya algebras over $\mathcal{O}_X$. We have the category $\mathcal{P}_{X;\mathcal{A}}$ of left $\mathcal{A}$-modules $\mathcal{E}$ which are locally free as $\mathcal{O}_X$-modules. We let $K(X;\mathcal{A})$ denote the K-theory spectrum of $\mathcal{P}_{X;\mathcal{A}}$. We extend $K(X;\mathcal{A})$ to a spectrum which is (possibly) not (-1)-connected by taking the Bass delooping, and denote this spectrum by $KB(X;\mathcal{A})$. For $f : Y \to X$ an X-scheme, we write $K(Y;\mathcal{A})$ for $K(Y;f^*\mathcal{A})$, and similarly for KB.

The spectra $KB(X;\mathcal{A})$ have the following properties.

(1) There is a canonical map $K(X;\mathcal{A}) \to KB(X;\mathcal{A})$, identifying $K(X;\mathcal{A})$ with is the -1-connected cover of $KB(X;\mathcal{A})$.

(2) There is the natural exact sequence

$$\begin{aligned} 0 \to KB_p(X;\mathcal{A}) \to KB_p(X \times \mathbb{A}^1;\mathcal{A}) \oplus KB_p(X \times \mathbb{A}^1;\mathcal{A}) \\ \to KB_p(X \times \mathbb{G}_m;\mathcal{A}) \to KB_{p-1}(X;\mathcal{A}) \to 0 \end{aligned}$$

called the *fundamental exact sequence.*

(3) If X is regular, then $K(X;\mathcal{A}) \to KB(X;\mathcal{A})$ is a weak equivalence.

From now on, we will drop the notation $KB(X;\mathcal{A})$ and write $K(X;\mathcal{A})$ for the (possibly) non-connected version.

Proposition B.5. *Let X be a noetherian affine R-scheme such that $\mathcal{O}_X$ has no nilpotent elements, and let $p : Y \to X$ be an étale cover. Let $\tilde{\mathcal{A}}$ be a sheaf of Azumaya algebras over $\mathcal{O}_X$. For each point $y \in Y$, let $Y_y := \operatorname{Spec}\mathcal{O}_{Y,y}$ and let $p_y : Y_y \to X$ be the map induced by p. Fix an integer $q \geqslant 1$. Suppose there is an M such that, for each smooth affine k-scheme T, $N^qK_n(T \times_k Y_y, (p_y \circ p_2)^*\mathcal{A}) = 0$ for each $y \in Y$ and each $n \leqslant M$. Then $N^qK_n(T \times_k X;\mathcal{A}) = 0$ for each smooth affine T and each $n \leqslant M$.*

Proof. Write $X = \operatorname{Spec} A$. Then $\prod p_y^* : A \to B := \prod_y \mathcal{O}_{Y,y}$ is faithfully flat and étale. Since X is affine, $\tilde{\mathcal{A}}$ is the sheaf associated to a central A-algebra $\mathcal{A}$ and since $\tilde{\mathcal{A}}$ is a sheaf of Azumaya algebras, each finitely generated projective left $\mathcal{A}$ module is finitely generated and projective as an A-module. Thus $N^qK_n(X,\tilde{\mathcal{A}}) = N^qK_n(\mathcal{A})$. Similarly, $N^qK_n(Y_y, p_y^*\tilde{\mathcal{A}}) = N^qK_n(p_y^*\mathcal{A})$. By Corollary B.4, $N^qK_n(\mathcal{A}) = 0$ for $n \geqslant 0$. The same argument, with $T \times X$ replacing X and $T \times Y_y$ replacing Y_y, proves the result for $M \geqslant n \geqslant 0$ and all T. To handle the cases $n < 0$, use the Bass fundamental sequence and descending induction starting with $n = 0$. $\square$

Appendix C. Categories of motives

Categories of motives have been defined by Ivorra [**23**] over a Noetherian separated base and by Cisinski and Déglise [**11**] over a regular base. In this appendix, we recall the construction of the category $DM^{\mathrm{eff}}(S)$, and various adjoint pairs of functors involving this category. For the construction of adjoint pairs, it is useful to invoke the general theory of model categories, applied to the various model structures on complexes over a Grothendieck abelian category discussed in [**11**]. This theory also gives a tensor structure and internal Hom functors for $DM^{\mathrm{eff}}(S)$. We will need as well the étale version $DM^{\mathrm{eff}}(S)^{\text{ét}}$; for lack of a suitable reference in the literature, we apply the methods of [**11**] in the étale setting and use the model structure to give a tensor structure with internal Hom functors, as well as an adjoint pair for change of topology. We conclude with the special case $S = \operatorname{Spec} k$, k a field, where one can apply Voevodsky's cancellation theorem to give a twisted duality result.

C.1. Categories of correspondences

We begin by recalling the construction; for details, we refer the reader to [**5**,**11**] and [**23**, Chapter 4].

We work at first in a fairly general setting. Let S be a regular scheme. The starting point is the category $\mathrm{SmCor}(S)$, with objects the smooth quasi-projective S-schemes $\mathbf{Sm}/S$, and morphisms given by the *finite correspondences* $\mathrm{Cor}_S(X,Y)$, this latter being the group of cycles on $X \times_S Y$ generated by the integral closed subschemes $W \subset X \times_S Y$ such that $W \to X$ is finite and surjective over some component of X. Composition is by the usual formula for composition of correspondences:

$$W' \circ W := p_{XZ*}(p_{XY}^*(W) \cdot p_{YZ}^*(W')).$$

Sending $f : X \to Y$ to the graph $\Gamma_f \subset X \times_S Y$ defines a functor $m : \mathbf{Sm}/S \to \mathrm{SmCor}(S)$.

Next, one has the category $\mathrm{PST}(S)$ of *presheaves with transfer*, this being simply the category of additive presheaves of abelian groups on $\mathrm{SmCor}(S)$. Restriction to $\mathbf{Sm}/S$ gives the functor to the category of presheaves on $\mathbf{Sm}/S$

$$i^* : \mathrm{PST}(S) \to \mathrm{PS}(\mathbf{Sm}/S);$$

we let $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S) \subset \mathrm{PST}(S)$ be the full subcategory with objects P such that $i^*(P)$ is a Nisnevich sheaf on $\mathbf{Sm}/S$. Such a P is a *Nisnevich sheaf with transfers* on $\mathbf{Sm}/S$. We have as well the subcategory $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S) \subset \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ of *étale sheaves with transfer*, that is, presheaves P such that i^*P is an étale sheaf on $\mathbf{Sm}/S$.

We record the following facts about the categories $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$. For $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$, these are proven in [**13**, §§ 4.2.4, 4.2.5]; the analogous facts for $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ follow by exactly the same arguments.

The inclusion $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S) \to \mathrm{PST}(S)$ has as left adjoint: the *sheafification functor*. $\mathrm{PST}(S)$ is a Grothendieck abelian category with kernel and cokernel defined pointwise and generators the representable presheaves; as usual, $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ is a Grothendieck abelian category with kernel the presheaf kernel, cokernel the sheafification of the presheaf cokernel and generators the representable sheaves. The corresponding statements for $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ hold as well.

We write $\mathbb{Z}^{\mathrm{tr}}_S(X)$ for the presheaf with transfers represented by $X \in \mathbf{Sm}/S$; this is in fact an étale sheaf with transfers. $\mathrm{PST}(S)$, $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ are all Grothendieck abelian categories, with generators $\mathbb{Z}^{\mathrm{tr}}_S(X)$, $X \in \mathbf{Sm}/S$. For $\mathcal{F} \in \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $n \in \mathbb{Z}$, we have a natural isomorphism

$$\mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))}(\mathbb{Z}^{\mathrm{tr}}_S(X), \mathcal{F}[n]) \cong H^n(X_{\mathrm{Nis}}, \mathcal{F}|_{X_{\mathrm{Nis}}}).$$

Similarly, for $\mathcal{F} \in \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ and $n \in \mathbb{Z}$, we have a natural isomorphism

$$\mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))}(\mathbb{Z}^{\mathrm{tr}}_S(X), \mathcal{F}[n]) \cong H^n(X_{\text{ét}}, \mathcal{F}|_{X_{\text{ét}}}).$$

The category $\mathrm{PST}(S)$ is a tensor category, with tensor operation $\otimes^{\mathrm{tr}}_S$ satisfying

$$\mathbb{Z}^{\mathrm{tr}}_S(X) \otimes^{\mathrm{tr}}_S \mathbb{Z}^{\mathrm{tr}}(Y) = \mathbb{Z}^{\mathrm{tr}}(X \times_S Y)$$

for $X, Y \in \mathbf{Sm}/S$. Taking as usual the sheaf associated to the presheaf tensor product gives $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ the structure of tensor categories. As the functor

$$- \otimes^{\mathrm{tr}}_S M : \mathrm{PST}(S) \to \mathrm{PST}(S)$$

preserves colimits, there is a right adjoint

$$\mathcal{H}om_{\mathrm{PST}}(M, -) : \mathrm{PST}(S) \to \mathrm{PST}(S);$$

for N a sheaf, $\mathcal{H}om_{\mathrm{PST}}(M, N)$ is automatically a sheaf, so we have internal Hom functors $\mathcal{H}om_{\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}}(-,-)$ and $\mathcal{H}om_{\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}}(-,-)$ in $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ as well. In other words, the categories $\mathrm{PST}(S)$, $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ are closed symmetric monoidal categories.

C.2. Model structures

We can now apply the machinery of [**11**] to define the *motivic model structure* on the categories of unbounded complexes $C(\mathrm{PST}(S))$, $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ and $C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))$. We first recall the general set-up. Let $\mathcal{A}$ be a Grothendieck abelian category. A *descent structure* for $\mathcal{A}$ is a pair $(\mathcal{G}, \mathcal{H})$ of subsets of $C(\mathcal{A})$ such that, for $C \in C(\mathcal{A})$,

$$\begin{aligned}&\mathrm{Hom}_{K(\mathcal{A})}(H, C[n]) = 0 \quad \text{for all } H \in \mathcal{H},\ n \in \mathbb{Z}\\ &\qquad \Longrightarrow \mathrm{Hom}_{K(\mathcal{A})}(G, C[n]) \cong \mathrm{Hom}_{D(\mathcal{A})}(G, C[n]) \quad \text{for all } G \in \mathcal{G},\ n \in \mathbb{Z}.\end{aligned}$$

If $(\mathcal{G}, \mathcal{H})$ is a descent structure for $\mathcal{A}$, then by [**11**, Theorem 1.7], the following defines a proper cellular model category $C(\mathcal{A})_{\mathcal{G}}$ with underlying category $C(\mathcal{A})$.

(1) Cofibrations. For $E \in \mathcal{G}$, let $D(E)$ be the complex $E \xrightarrow{\mathrm{id}} E$, concentrated in degrees 0 and 1, and let

$$\iota_E : E[-1] \to D(E)$$

be the map given by the identity in degree 1. The cofibrations are generated (by push-out, transfinite compositions and retracts) by the morphisms $\iota_E[n]$, $E \in \mathcal{G}$, $n \in \mathbb{Z}$.

(2) Weak equivalences. The weak equivalences are the quasi-isomorphisms.

(3) Fibrations. The fibrations are the maps having the having the right lifting property with respect to acyclic cofibrations.

In particular, the homotopy category $\mathcal{H}C(\mathcal{A})_{\mathcal{G}}$ is the derived category $D(\mathcal{A})$.

If $\mathcal{A}$ is a presheaf category $\mathrm{PS}_{\mathbf{Ab}}(\mathcal{C})$, for $\mathcal{C}$ an essentially small category, then $\mathcal{A}$ is a Grothendieck abelian category with set of generators the representable presheaves $\mathbb{Z}(X)$, $X \in \mathcal{C}$ (more correctly, X running through a set of representatives of isomorphism classes of $\mathcal{C}$). One can take $\mathcal{G} = \{\mathbb{Z}(X) \mid X \in \mathcal{C}\}$ and $\mathcal{H} = \{0\}$:

$$\mathrm{Hom}_{K(\mathcal{A})}(\mathbb{Z}(X), C[n]) \cong H^n(C(X)),$$

hence $\mathrm{Hom}_{K(\mathcal{A})}(\mathbb{Z}(X), C[n]) \cong \mathrm{Hom}_{D(\mathcal{A})}(\mathbb{Z}(X), C[n])$ for all n, C. We denote the resulting model category by $C(\mathrm{PS}_{\mathbf{Ab}}(\mathcal{C}))_{\mathrm{proj}}$.

In particular, we have the proper cellular model category $C(\mathrm{PST}(S))_{\mathrm{proj}}$ with homotopy category $D(\mathrm{PST}(S))$. For the sheaf categories $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$, we also let $\mathcal{G}$ be the set of representable (pre)sheaves. Let $\mathcal{H}_{\mathrm{Nis}}$ be the set of complexes of the form $\mathbb{Z}^{\mathrm{tr}}_S(\mathcal{X}) \to \mathbb{Z}^{\mathrm{tr}}_S(X)$, with $\mathcal{X} \to X$ a Nisnevich hypercover of $X \in \mathbf{Sm}/S$. By [**11**, Example 1.5], $(\mathcal{G}, \mathcal{H}_{\mathrm{Nis}})$ defines a descent structure on $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$, giving us the proper cellular model category $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{proj}}$ with homotopy category $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$. Replacing Nisnevich hypercovers with étale hypercovers defines the set $\mathcal{H}_{\text{ét}}$; the same argument as in [**11**, Example 1.5], $(\mathcal{G}, \mathcal{H}_{\mathrm{Nis}})$ shows that $(\mathcal{G}, \mathcal{H}_{\text{ét}})$ defines a descent structure on $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$. Thus, we have the proper cellular model category $C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{\mathrm{proj}}$ with homotopy category $D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))$.

Returning to the general situation, the fact that $C(\mathcal{A})_{\mathcal{G}}$ is a proper cellular model category allows one to apply the localization machinery of Hirschhorn [**19**, Theorem 4.1.1]. Specifically, let $\mathcal{T}$ be a set of objects of $C(\mathcal{A})$, and suppose we have a descent structure $(\mathcal{G}, \mathcal{H})$ for $\mathcal{A}$. By [**11**, Proposition 3.5], the left Bousfield localization $C(\mathcal{A})_{\mathcal{T}}$ of $C(\mathcal{A})_{\mathcal{G}}$ exists, $C(\mathcal{A})_{\mathcal{T}}$ is again proper and cellular, and the homotopy category is the localization of $D(\mathcal{A})$ with respect to the localizing subcategory $\mathcal{T}(\mathcal{A})$ generated by $\mathcal{T}$. In addition, the general theory of Bousfield localization tells us that the quotient functor

$$Q_{\mathcal{T}} : D(\mathcal{A}) \to D(\mathcal{A})_{\mathcal{T}} := D(\mathcal{A})/\mathcal{T}(\mathcal{A})$$

admits a right adjoint $r_{\mathcal{T}}$, which in turn defines an equivalence of $D(\mathcal{A})_{\mathcal{T}}$ with the full subcategory $D(\mathcal{A})^{\mathcal{T}\text{-loc}}$ of $D(\mathcal{A})$ of $\mathcal{T}$-local objects, that is, objects C of $D(\mathcal{A})$ such that

$$\mathrm{Hom}_{D(\mathcal{A})}(T, C[n]) = 0$$

for all $T \in \mathcal{T}$ and all $n \in \mathbb{Z}$. In particular, $D(\mathcal{A})^{\mathcal{T}\text{-loc}}$ is a triangulated subcategory of $D(\mathcal{A})$ and is equal to the essential image of $r_{\mathcal{T}}$. In addition, letting $i_{\mathcal{T}} : D(\mathcal{A})^{\mathcal{T}\text{-loc}} \to D(\mathcal{A})$ be the inclusion, we have the functor

$$L_{\mathcal{T}} := r_{\mathcal{T}} \circ Q_{\mathcal{T}} : D(\mathcal{A}) \to D(\mathcal{A})^{\mathcal{T}\text{-loc}},$$

which is left adjoint to $i_{\mathcal{T}}$.

Example C.2.1. Let $\mathcal{T}_{\mathrm{Nis}}$ be the set of complexes of the form

$$\mathbb{Z}^{\mathrm{tr}}(V) \to \mathbb{Z}^{\mathrm{tr}}(U) \oplus \mathbb{Z}^{\mathrm{tr}}(Y) \to \mathbb{Z}^{\mathrm{tr}}(X)$$

for each elementary Nisnevich square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

in $\mathbf{Sm}/S$. By an argument analogous to that of [**41**, Proposition 3.1.16], a complex $C \in C(\mathrm{PST}(S))$ is $\mathcal{T}$-local if and only if

$$\mathrm{Hom}_{K(\mathrm{PST}(S))}(H, C[n]) = 0$$

for all $H \in \mathcal{H}_{\mathrm{Nis}}$. From this, it follows that $D(\mathrm{PST}(S))_{\mathcal{T}_{\mathrm{Nis}}}$ is equivalent to $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$.

We can now define the motivic model structures. Let $\mathcal{T}_{\mathbb{A}^1}$ be the set of complexes of the form $\mathbb{Z}^{\mathrm{tr}}(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}(X)$ for $X \in \mathbf{Sm}/S$; when we need to explicitly indicate the ambient category, we write $\mathcal{T}^{\mathrm{Nis}}_{\mathbb{A}^1}$ or $\mathcal{T}^{\text{ét}}_{\mathbb{A}^1}$. We set

$$\begin{aligned} C(\mathrm{PST}(S))_{\mathrm{mot}} &:= C(\mathrm{PST}(S))_{T_{\mathrm{Nis}} \cup T_{\mathbb{A}^1}}, \\ C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{mot}} &:= C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{T^{\mathrm{Nis}}_{\mathbb{A}^1}}, \\ C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{\mathrm{mot}} &:= C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{T^{\text{ét}}_{\mathbb{A}^1}}. \end{aligned}$$

Definition C.2.2. Define the triangulated category of *effective motives over* S, $DM^{\mathrm{eff}}(S)$, by

$$DM^{\mathrm{eff}}(S) := D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))^{\mathbb{A}^1\text{-loc}}.$$

The category of *effective étale motives over* S, $DM^{\mathrm{eff}}(S)^{\text{ét}}$, is

$$DM^{\mathrm{eff}}(S)^{\text{ét}} := D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))^{\mathbb{A}^1\text{-loc}}.$$

Since $D(\mathrm{PST}(S))_{\mathcal{T}_{\mathrm{Nis}}}$ is equivalent to $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$, it follows that the localization $D(\mathrm{PST}(S))^{\mathcal{T}_{\mathrm{Nis}} \cup T_{\mathbb{A}^1} - loc}$ is equivalent to $DM^{\mathrm{eff}}(S)$.

The general theory, as explained above, gives us the left adjoints

$$\begin{aligned} L_{\mathbb{A}^1} &: D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to DM^{\mathrm{eff}}(S), \\ L^{\text{ét}}_{\mathbb{A}^1} &: D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)) \to DM^{\mathrm{eff}}(S)^{\text{ét}} \end{aligned}$$

to the respective inclusions $i : DM^{\mathrm{eff}}(S) \to D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$, $i^{\text{ét}} : DM^{\mathrm{eff}}(S)^{\text{ét}} \to D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))$. We let

$$\begin{aligned} M_S &: \mathbf{Sm}/S \to DM^{\mathrm{eff}}(S), \\ M^{\text{ét}}_S &: \mathbf{Sm}/S \to DM^{\mathrm{eff}}(S)^{\text{ét}} \end{aligned}$$

denote the functors $M_S(X) := L_{\mathbb{A}^1}(\mathbb{Z}^{\mathrm{tr}}_S(X))$, $M^{\text{ét}}_S(X) := L^{\text{ét}}_{\mathbb{A}^1}(\mathbb{Z}^{\mathrm{tr}}_S(X))$.

C.3. Tensor and internal Hom

We have seen that the categories $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ and $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S)$ are tensor categories with internal Homs. The descent structures $(\mathcal{G}, \mathcal{H}_{\mathrm{Nis}})$ and $(\mathcal{G}, \mathcal{H}_{\mathrm{\acute{e}t}})$ are *weakly flat* [**11**, § 2.1], that is, for each $X, Y \in \mathbf{Sm}/S$ and each Nisnevich (respectively étale) hypercover $\mathcal{X} \to X$, the complex

$$\mathbb{Z}^{\mathrm{tr}}_S(Y) \otimes^{\mathrm{tr}} (\mathbb{Z}^{\mathrm{tr}}_S(\mathcal{X}) \to \mathbb{Z}^{\mathrm{tr}}(X))$$

is in $\mathcal{H}_{\mathrm{Nis}}$ (respectively $\mathcal{H}_{\mathrm{\acute{e}t}}$). Indeed, $Y \times_S \mathcal{X} \to Y \times_S X$ is clearly a Nisnevich (respectively étale) hypercover of $Y \times_S X$ and the tensor product is just $\mathbb{Z}^{\mathrm{tr}}_S(Y \times_S \mathcal{X}) \to \mathbb{Z}^{\mathrm{tr}}(Y \times_S X)$. In addition $\mathcal{T}_{\mathbb{A}^1}$ is *$\mathcal{G}$-flat*, that is

$$\mathbb{Z}^{\mathrm{tr}}_S(Y) \otimes^{\mathrm{tr}} (\mathbb{Z}^{\mathrm{tr}}_S(X \times \mathbb{A}^1) \to \mathbb{Z}^{\mathrm{tr}}_S(X))$$

is in $\mathcal{T}_{\mathbb{A}^1}$ for each $X, Y \in \mathbf{Sm}/S$. Thus, by [**11**, Corollary 3.14], we have the following proposition.

Proposition C.3.1. *The tensor product on $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$, respectively $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S))$, makes $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{proj}}$ and $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1}$, respectively $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S))_{\mathrm{proj}}$ and $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S))_{\mathbb{A}^1}$, symmetric monoidal model categories.*

The general theory of symmetric monoidal model categories (see [**20**, Theorem 4.3.2]) yields the following theorem.

Theorem C.3.2.

(1) *The categories $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$, $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S))$, $DM^{\mathrm{eff}}(S)$ and $DM^{\mathrm{eff}}(S)^{\mathrm{\acute{e}t}}$ are triangulated tensor categories with internal Hom functors.*

(2) *The localization functors $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to DM^{\mathrm{eff}}(S)$ and $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(S)) \to DM^{\mathrm{eff}}(S)^{\mathrm{\acute{e}t}}$ are tensor functors.*

(3) *The adjunction $\mathrm{Hom}_{DM^{\mathrm{eff}}}(L_{\mathbb{A}^1}X, Y) \cong \mathrm{Hom}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})}(X, iY)$ induces the isomorphism*

$$i\, \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}}(L_{\mathbb{A}^1}X, Y) \cong \mathcal{H}\mathrm{om}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})}(X, iY).$$

Similarly, we have the natural isomorphism

$$i^{\mathrm{\acute{e}t}}\, \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff\acute{e}t}}}(L^{\mathrm{\acute{e}t}}_{\mathbb{A}^1}X, Y) \cong \mathcal{H}\mathrm{om}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}})}(X, iY).$$

Remark C.3.3. Take $X \in \mathbf{Sm}/S$. Then $\mathbb{Z}^{\mathrm{tr}}_S(X)$ is cofibrant, hence the internal Hom $\mathcal{H}\mathrm{om}_{C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})}(\mathbb{Z}^{\mathrm{tr}}_S(X), C)$ represents $\mathcal{H}\mathrm{om}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})}(\mathbb{Z}^{\mathrm{tr}}_S(X), C)$ for all fibrant $C \in C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{proj}}$. But $\mathcal{H}\mathrm{om}_{C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})}(\mathbb{Z}^{\mathrm{tr}}_S(X), C)$ is the sheafification of the presheaf

$$Y \mapsto C(X \times_S Y),$$

so we have an explicit description of $\mathcal{H}\mathrm{om}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})}(\mathbb{Z}^{\mathrm{tr}}_S(X), C)$. Similarly, the internal Hom $\mathcal{H}\mathrm{om}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}})}(\mathbb{Z}^{\mathrm{tr}}_S(X), C)$ is the étale sheafification of the same presheaf as above.

Using the adjunction of Theorem C.3.2, we have a similar description of the internal Homs $\mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}}(M_S(X), -)$ and $\mathcal{H}\mathrm{om}_{DM^{\mathrm{eff\acute{e}t}}}(M^{\mathrm{\acute{e}t}}_S(X), -)$.

C.4. Change of topology

Let $\alpha^* : \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S) \to \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)$ be the sheafification functor, with right adjoint $\alpha_* : \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S) \to \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)$ the inclusion. As α^* is exact, we have the canonical extension to the derived categories

$$\alpha^* : D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S));$$

as $\alpha^*(\mathcal{T}^{\mathrm{Nis}}_{\mathbb{A}^1}) = \mathcal{T}^{\text{ét}}_{\mathbb{A}^1}$, α^* descends to an exact functor

$$\alpha^*_{\mathrm{mot}} : D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathcal{T}^{\mathrm{Nis}}_{\mathbb{A}^1}} \to D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{\mathcal{T}^{\text{ét}}_{\mathbb{A}^1}}.$$

Via the equivalences

$$DM^{\mathrm{eff}}(S) \sim D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathcal{T}^{\mathrm{Nis}}_{\mathbb{A}^1}}, \qquad DM^{\mathrm{eff}}(S)^{\text{ét}} \sim D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{\mathcal{T}^{\text{ét}}_{\mathbb{A}^1}},$$

a^*_{mot} induces the exact functor

$$\alpha^*_{\mathrm{mot}} : DM^{\mathrm{eff}}(S) \to DM^{\mathrm{eff}}(S)^{\text{ét}}.$$

On the other hand, the sheaf-level functor α^* clearly sends $\mathcal{G}_{\mathrm{Nis}}$ to $\mathcal{G}_{\text{ét}}$ and as a left-adjoint, α^* preserves colimits. Thus, the extension $C(\alpha^*) : C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))$ maps cofibrations in $C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{proj}}$ to cofibrations in $C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{\mathrm{proj}}$. Noting that

$$\text{cof-}C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1} = \text{cof-}C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{proj}}$$

and similarly for $C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))$, the fact that α^* is exact, respectively that α^* descends to α^*_{mot}, says $C(\alpha^*)$ preserves acyclic cofibrations, for both the projective as well as the motivic model structures. Thus, we have Quillen adjoint functors

$$C(\alpha^*) : C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathrm{proj}} \rightleftarrows C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{\mathrm{proj}} : C(\alpha_*),$$
$$C(\alpha^*) : C(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))_{\mathbb{A}^1} \rightleftarrows C(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))_{\mathbb{A}^1} : C(\alpha_*).$$

The general theory of model categories thus gives us right adjoints

$$R\alpha_* : D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S)) \to D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)),$$
$$R\alpha_{\mathrm{mot}*} : DM^{\mathrm{eff}}(S)^{\text{ét}} \to DM^{\mathrm{eff}}(S)$$

to α^*, α^*_{mot}. As $a^*_{\mathrm{mot}} \circ Q_{\mathrm{Nis}} = Q_{\text{ét}} \circ \alpha^*$, we have

$$\alpha^*_{\mathrm{mot}} \circ L_{\mathbb{A}^1} \cong L^{\text{ét}}_{\mathbb{A}^1} \circ a^*,$$
$$R\alpha_* \circ i_{\text{ét}} \cong i_{\mathrm{Nis}} \circ R\alpha_{\mathrm{mot}*},$$

where $i_{\mathrm{Nis}} : DM^{\mathrm{eff}}(S) \to D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$, $i_{\text{ét}} : DM^{\mathrm{eff}}(S)^{\text{ét}} \to D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(S))$ are the respective inclusions. We sometimes write α_*, α^{mot}_* for $R\alpha_*$, $R\alpha^{\mathrm{mot}}_*$ when the context makes the meaning clear.

The sheaf-level functor α^* is a tensor functor, and thus $C(\alpha^*)$ is a functor of symmetric monoidal model categories, for both model structures ${}_{\mathrm{proj}}$ and ${}_{\mathbb{A}^1}$. Thus, the derived functors α^* and α^*_{mot} are tensor functors, and we have the projection formulae

$$R\alpha_* \,\mathcal{H}\mathrm{om}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}}(\alpha^* B, C) \cong \mathcal{H}\mathrm{om}_{D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}})}(B, R\alpha_* C),$$
$$R\alpha_{\mathrm{mot}*} \,\mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(S)^{\text{ét}}}(\alpha^* B, C) \cong \mathcal{H}\mathrm{om}_{DM^{\mathrm{eff}}(S))}(B, R\alpha_{\mathrm{mot}*} C).$$

Remark C.4.1. As for $\mathcal{SH}_{S^1}(k)$, we have the evaluation functor for $Y \in \mathbf{Sm}/S$,

$$R\Gamma(Y, -) : DM^{\mathrm{eff}}(S) \to D(\mathbf{Ab}), \qquad R\Gamma(Y, \mathcal{F}) := \mathcal{F}^{\mathrm{fib}}(Y).$$

We use the notation $\mathcal{F}(Y)$ for $R\Gamma(Y, \mathcal{F})$ in case $\mathcal{F} \to \mathcal{F}^{\mathrm{fib}}$ is a pointwise quasi-isomorphism.

C.5. The case of a field

We now specialize to $S = \operatorname{Spec} k$, k a perfect field; we drop the subscript $\operatorname{Spec} k$ from the notation for e.g. $\mathbb{Z}_S^{\mathrm{tr}}(X)$. We let $\mathrm{PST}(k)[1/p] \subset \mathrm{PST}(k)$ denote the subcategory of presheaves of $\mathbb{Z}[1/p]$-modules, and use a similar notation for $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)$, $\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(k)$, etc. We recall the following fundamental result of Voevodsky.

Theorem C.5.1 (Voevodsky [58, Theorem 3.1.12]).

(1) *Let $\mathcal{F} \in \mathrm{PST}(k)$ be an $\mathbb{A}^1$-homotopy invariant presheaf. Then for every $n \geqslant 0$, the cohomology presheaf $X \mapsto H^n_{\mathrm{Nis}}(X, \mathcal{F}_{\mathrm{Nis}})$ has a natural structure of a presheaf with transfers, and is homotopy invariant.*

(2) *Let $\mathcal{F} \in \mathrm{PST}(k)[1/p]$ be an $\mathbb{A}^1$-homotopy invariant presheaf, where p is the exponential characteristic of k. Then for every $n \geqslant 0$ the cohomology presheaf $X \mapsto H^n_{\mathrm{\acute{e}t}}(X, \mathcal{F}_{\mathrm{\acute{e}t}})$ has a natural structure of a presheaf with transfers, and is homotopy invariant.*

(For the homotopy invariance in (2), see [**3**, Lemma D.1.3]; the existence of transfers follows by using the same argument as for the Nisnevich topology, as given in the proof of [**57**, Theorem 5.3].)

Corollary C.5.2.

(1) *$DM^{\mathrm{eff}}(k) \subset D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ is the full subcategory of complexes X such that the cohomology sheaves (for the Nisnevich topology) $\mathcal{H}^n_{\mathrm{Nis}}(X)$ are $\mathbb{A}^1$-homotopy invariant for all n.*

(2) *$DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p] \subset D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(k))[1/p]$ is the full subcategory of complexes X such that the cohomology sheaves (for the étale topology) $\mathcal{H}^n_{\mathrm{\acute{e}t}}(X)$ are $\mathbb{A}^1$-homotopy invariant for all n.*

(3) *For $\mathcal{F} \in DM^{\mathrm{eff}}(k)^{\mathrm{Nis}}[1/p]$ we have*

$$\alpha^*(\mathcal{F}) \in DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p]$$

and

$$\alpha^*_{\mathrm{mot}}(\mathcal{F}) = \alpha^*(\mathcal{F}) \in DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p].$$

Proof. We first prove (2); the proof of (1) is similar, but a bit easier. If C is in $DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p] \subset D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(k))$, then as $\operatorname{Hom}_{DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p]}(\mathbb{Z}^{\mathrm{tr}}(X), C) = \mathbb{H}^n_{\mathrm{\acute{e}t}}(X, C)$, the pullback map

$$\mathbb{H}^n_{\mathrm{\acute{e}t}}(X, C) \to \mathbb{H}^n_{\mathrm{\acute{e}t}}(X \times \mathbb{A}^1, C)$$

is an isomorphism for all $X \in \mathbf{Sm}/k$, in other words, the presheaf $X \mapsto \mathbb{H}^n_{\text{ét}}(X, C)$ is homotopy invariant. By Theorem C.5.1, the associated étale sheaf $\mathcal{H}^n_{\text{ét}}(C)$ is homotopy invariant.

Now suppose that $\mathcal{H}^n_{\text{ét}}(C)$ is homotopy invariant for each n. Take $X \in \mathbf{Sm}/k$, and let $p : X \times \mathbb{A}^1 \to X$ be the projection. Then

$$\mathbb{H}^n_{\text{ét}}(X \times \mathbb{A}^1, C) \cong \mathbb{H}^n_{\text{ét}}(X, Rp_*(C|_{X \times \mathbb{A}^1_{\text{ét}}})).$$

Extending the canonical map $p^* : C|_{X_{\text{ét}}} \to Rp_*(C|_{X \times \mathbb{A}^1_{\text{ét}}})$ to a distinguished triangle

$$C|_{X_{\text{ét}}} \to Rp_*(C|_{X \times \mathbb{A}^1_{\text{ét}}}) \to \bar{C} \to C|_{X_{\text{ét}}}[1],$$

we need to show that $\mathbb{H}^n_{\text{ét}}(X, \bar{C}) = 0$ for all n. For this, it suffices to show that $\bar{C} \cong 0$ in $D(\mathrm{Sh}_{\text{ét}}(X))$, that is, it suffices to show that $\mathcal{H}^n_{\text{ét}}(\bar{C}) = 0$ for all n.

Take $x \in X$, let $\mathcal{O}^{\text{sh}}_{X,x}$ be the strict Henselization of $\mathcal{O}_{X,x}$ and let $X^{\text{ét}}_x = \operatorname{Spec} \mathcal{O}^{\text{sh}}_{X,x}$. Letting $p_x : X^{\text{ét}}_x \times \mathbb{A}^1 \to X^{\text{ét}}_x$ be the projection, we have the long exact sequence

$$\cdots \to \mathcal{H}^n_{\text{ét}}(C)_x \xrightarrow{p_x^*} R^n p_*(C)_x \to \mathcal{H}^n_{\text{ét}}(\bar{C}) \to \cdots$$

so it suffices to show that $p_x^* : \mathcal{H}^n_{\text{ét}}(C)_x \to R^n p_*(C)_x$ is an isomorphism for all n. But this is just the map

$$p_x^* : \mathbb{H}^n_{\text{ét}}(X^{\text{ét}}_x, C) \to \mathbb{H}^n(X^{\text{ét}}_x \times \mathbb{A}^1, C).$$

As $X^{\text{ét}}_x$ is strictly Hensel local, $X^{\text{ét}}_x \times \mathbb{A}^1$ has finite étale cohomological dimension, so we have the strongly convergent spectral sequence

$$E_1^{p,q} = H^p_{\text{ét}}(X^{\text{ét}}_x \times \mathbb{A}^1, \mathcal{H}^q_{\text{ét}}(C)) \Longrightarrow \mathbb{H}^{p+q}_{\text{ét}}(X^{\text{ét}}_x \times \mathbb{A}^1, C).$$

Since the sheaf $\mathcal{H}^q_{\text{ét}}(C)$ is homotopy invariant, the cohomology presheaves

$$Y \mapsto H^p_{\text{ét}}(Y, \mathcal{H}^q_{\text{ét}}(C))$$

are homotopy invariant (Theorem C.5.1 (2) again). Since $X^{\text{ét}}_x$ is strictly Hensel local, we have $E_1^{p,q} = 0$ except for $p = 0$, and $E_1^{0,q} = \mathcal{H}^q_{\text{ét}}(C)_x$. Thus $\mathbb{H}^n_{\text{ét}}(X^{\text{ét}}_x, C) \cong \mathbb{H}^n(X^{\text{ét}}_x \times \mathbb{A}^1, C)$, completing the proof.

For (3), take $\mathcal{F} \in DM^{\text{eff}}(k)[1/p]$. As the presheaf with transfers

$$X \mapsto \mathcal{H}^n_{\text{Nis}}(\mathcal{F})(X)$$

is homotopy invariant, Theorem C.5.1 (2) tells us that $\mathcal{H}^n_{\text{ét}}(\alpha^*\mathcal{F}) = a^*(\mathcal{H}^n_{\text{Nis}}(\mathcal{F}))$ is homotopy invariant. By (2), $\alpha^*\mathcal{F}$ is in $DM^{\text{eff}}(k)^{\text{ét}}[1/p]$. Thus, the natural maps $\alpha^*\mathcal{F} \to i^{\text{ét}} L^{\text{ét}}_{\mathbb{A}^1} \alpha^* \mathcal{F}$ and $\mathcal{F} \to L^{\text{Nis}}_{\mathbb{A}^1}\mathcal{F}$ are isomorphisms; as $L^{\text{ét}}_{\mathbb{A}^1}\alpha^* \cong \alpha^*_{\text{mot}} L^{\text{Nis}}_{\mathbb{A}^1}$ and $i^{\text{ét}}$ is an embedding, we have $\alpha^*\mathcal{F} \cong \alpha^*_{\text{mot}}\mathcal{F}$. □

Remark C.5.3. We have presented here the approach of Cisinski–Déglise to the construction of $DM^{\text{eff}}(S)$; Ivorra has defined a category of effective motives over S in the setting of triangulated categories, without using a model structure on the category of complexes (see [**23**, Definition 4.1.2]). Ivorra defines the triangulated category $DM^{\mathrm{eff}}(S)$ for a general base-scheme S as a localization

$$Q_S : D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S)) \to DM^{\mathrm{eff}}(S)$$

of the derived category $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$. By [**23**, Corollary 4.1.16], the localization functor Q_S admits a right adjoint; this gives an identification of $DM^{\mathrm{eff}}(S)$ with the full triangulated subcategory of $\mathbb{A}^1$-local objects in $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ (in the sense of [**23**]). By [**23**, Proposition 4.1.12], an object $\mathcal{F} \in D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(S))$ is $\mathbb{A}^1$-local if and only if the presheaf, $X \mapsto \mathbb{H}^n_{\mathrm{Nis}}(X, \mathcal{F})$, is $\mathbb{A}^1$-homotopy invariant, that is, the natural map

$$\mathbb{H}^n_{\mathrm{Nis}}(X, \mathcal{F}) \to \mathbb{H}^n_{\mathrm{Nis}}(X \times \mathbb{A}^1, \mathcal{F})$$

is an isomorphism for all $X \in \mathbf{Sm}/S$; by [**11**, Example 3.15], this agrees with the notion of $\mathbb{A}^1$-local object defined in § C.2. Thus the definition given of $DM^{\mathrm{eff}}(S)$ given here is equivalent to that given in [**23**].

Beilinson and Vologodsky [**5**] define the triangulated tensor category $DM^{\mathrm{eff}}(k)$, as the homotopy category of a DG tensor category (see [**5**, § 2.3]), and give a description of $DM^{\mathrm{eff}}(k)$ as both a localization of $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ and as a triangulated subcategory of $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$, equivalent to the descriptions found in [**11**] and [**23**].

Recall [**58**, § 3.1] that the category $DM^{\mathrm{eff}}_{-}(k)$ is the full subcategory of the bounded above derived category $D^{-}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ with objects the complexes C^* for which the cohomology sheaves $\mathcal{H}^n_{\mathrm{Nis}}(C^*)$ are $\mathbb{A}^1$ homotopy invariant for all n. For bounded above complexes, this condition is equivalent to $\mathbb{A}^1$-homotopy invariance, as defined above.

Noting that the bounded above category $D^{-}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ is a full triangulated subcategory of $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$, we therefore have a commutative diagram of full embeddings

$$\begin{tikzcd}
DM^{\mathrm{eff}}_{-}(k) \arrow[r] \arrow[d] & D^{-}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)) \arrow[d] \\
DM^{\mathrm{eff}}(k) \arrow[r] & D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))
\end{tikzcd}$$

Voevodsky has also shown [**58**, Proposition 3.2.3] that the inclusion

$$i_{-} : DM^{\mathrm{eff}}_{-}(k) \to D^{-}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$$

admits a left adjoint $L^{-}_{\mathbb{A}^1} : D^{-}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k)) \to DM^{\mathrm{eff}}_{-}(k)$; the uniqueness of adjoints shows that $L^{-}_{\mathbb{A}^1}$ is the restriction of $L^{\mathrm{Nis}}_{\mathbb{A}^1}$.

C.6. Geometric motives

We recall the category of *effective geometric motives* $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, defined in [**58**, Definition 2.1.1] as a localization of $K^b(\mathrm{SmCor}(k))$.

Remarks C.6.1 (the Suslin complex [58, § 3.2]).

(1) We have the cosimplicial scheme $n \mapsto \Delta^n$, with

$$\Delta^n := \operatorname{Spec} k[t_0, \ldots, n] \Big/ \sum_i t_i - 1$$

and with coface and codegeneracy maps defined as in the topological setting.

For $\mathcal{F} \in \mathrm{PST}(k)$, let $C_n^{\mathrm{Sus}}(\mathcal{F})$ be the presheaf $C_n^{\mathrm{Sus}}(\mathcal{F})(X) := \mathcal{F}(X \times \Delta^n)$, giving us the simplicial object $n \mapsto C_n^{\mathrm{Sus}}(\mathcal{F})$ of $\mathrm{PST}(k)$ and the associated complex $C_*^{\mathrm{Sus}}(\mathcal{F}) \in C^-(\mathrm{PST}(k))$. It is easy to show that

$$p^* : C_*^{\mathrm{Sus}}(\mathcal{F})(X) \to C_*^{\mathrm{Sus}}(\mathcal{F})(X \times \mathbb{A}^1)$$

is a homotopy equivalence for every $X \in \mathbf{Sm}/k$; by Voevodsky's theorem [**58**, Theorem 3.1.12], this implies that $C_*^{\mathrm{Sus}}(\mathcal{F})$ is in fact in $DM_-^{\mathrm{eff}}(k)$. We extend C_*^{Sus} to

$$C_*^{\mathrm{Sus}} : C^-(\mathrm{PST}(k)) \to DM_-^{\mathrm{eff}}(k)$$

by taking the total complex of the evident double complex.

(2) Sending $X \in \mathbf{Sm}/k$ to $\mathbb{Z}^{\mathrm{tr}}(X)$ extends to a functor

$$\mathbb{Z}^{\mathrm{tr}} : \mathrm{SmCor}(k) \to C^-(\mathrm{PST}(k));$$

we extend $\mathbb{Z}^{\mathrm{tr}}$ to $C^b(\mathrm{SmCor}(k))$ by taking the evident total complex. This gives us the exact functor

$$K^b(\mathbb{Z}^{\mathrm{tr}}) : K^b(\mathrm{SmCor}(k)) \to D^-(\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{tr}}(k)).$$

Similarly, composing $C^b(\mathbb{Z}^{\mathrm{tr}})$ with C_*^{Sus} defines the exact functor

$$C_*^{\mathrm{Sus}} : K^b(\mathrm{SmCor}(k)) \to DM^{\mathrm{eff}}(k).$$

We recall Voevodsky's embedding theorem.

Theorem C.6.2 (Voevodsky [58, Theorem 3.2.6]).

(1) *The Suslin complex functor*

$$C_*^{\mathrm{Sus}} \circ K^b(\mathbb{Z}^{\mathrm{tr}}) : K^b(\mathrm{SmCor}(k)) \to DM_-^{\mathrm{eff}}(k)$$

descends to a full embedding $i_{\mathrm{gm}}^{\mathrm{eff}} : DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \to DM_-^{\mathrm{eff}}(k)$.

(2) *There is a natural isomorphism of functors* $K^b(\mathrm{SmCor}(k)) \to DM_-^{\mathrm{eff}}(k)$

$$C_*^{\mathrm{Sus}} \circ K^b(\mathbb{Z}^{\mathrm{tr}}) \cong L_{\mathbb{A}^1}^- \circ K^b(\mathbb{Z}^{\mathrm{tr}}).$$

Remark C.6.3. As the inclusion functor $D^-(\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{tr}}(k)) \to D(\mathrm{Sh}_{\mathrm{Nis}}^{\mathrm{tr}}(k))$ is a full embedding, the embedding theorem together with the results of [**11**, Example 3.15] yields the full embedding

$$C_*^{\mathrm{Sus}} \circ K^b(\mathbb{Z}^{\mathrm{tr}}) : DM_{\mathrm{gm}}^{\mathrm{eff}}(k) \to DM^{\mathrm{eff}}(k).$$

C.7. Cancellation theorems

We have as well the category of *geometric motives* $DM_{\mathrm{gm}}(k)$, formed by inverting the functor $-\otimes\mathbb{Z}(1)$ on $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$. We recall Voevodsky's cancellation theorem.

Theorem C.7.1 (see [58, Theorem 4.3.1] and [62]). *For $M, N \in DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$, the canonical map*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M,N) \to \mathrm{Hom}_{DM^{\mathrm{eff}}_{\mathrm{gm}}(k)}(M(1),N(1))$$

is an isomorphism. In consequence, the canonical functor $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)\to DM_{\mathrm{gm}}(k)$ is a full embedding.

Huber and Kahn [**22**, Appendix A] have extended this result to $DM^{\mathrm{eff}}_{-}(k)$ and in case k has finite étale cohomological dimension, they extend the result to a bounded above version $DM^{\mathrm{eff}}_{-}(k)^{\mathrm{\acute{e}t}}$ of $DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p]$; the same proof extends the cancellation theorem to $DM^{\mathrm{eff}}(k)$ and $DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p]$. A direct proof for $DM^{\mathrm{eff}}(k)$ can be found in [**5**, proposition in 6.1].

Corollary C.7.2.

(1) *For $M, N\in DM^{\mathrm{eff}}(k)$, the canonical map*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M,N)\to \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(1),N(1))$$

is an isomorphism.

(2) *For $M, N\in DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}[1/p]$, the canonical map*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}}(M,N)\to \mathrm{Hom}_{DM^{\mathrm{eff}}(k)^{\mathrm{\acute{e}t}}}(M(1),N(1))$$

is an isomorphism.

Proof. (1) We have the adjunction

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M(1),N(1))\cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(M,\mathcal{H}\mathrm{om}(\mathbb{Z}(1),N(1))),$$

so to prove (1), it suffices to show that the canonical map

$$\varphi_N : N\to \mathcal{H}\mathrm{om}(\mathbb{Z}(1),N(1))$$

is an isomorphism for all N. As $\mathbb{Z}(1)\cong\mathbb{G}_m[-1]$ is compact, the category $\mathcal{B}$ of N such that φ_N is an isomorphism is a localizing subcategory of $DM^{\mathrm{eff}}(k)$. As $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)\to DM^{\mathrm{eff}}(k)$ is a full embedding, the cancellation theorem for $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ (Theorem C.7.1 or [**5**, Theorem 3.3]) shows that $\mathcal{B}$ contains $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$; by [**58**, Theorem 3.2.6], $DM^{\mathrm{eff}}_{\mathrm{gm}}(k)$ is dense in $DM^{\mathrm{eff}}_{-}(k)$, hence $\mathcal{B}$ contains $DM^{\mathrm{eff}}_{-}(k)$. But $DM^{\mathrm{eff}}_{-}(k)$ is the essential image of $D^{-}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ under $L^{\mathrm{Nis}}_{\mathbb{A}^1}$; as $D^{-}(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ is dense in $D(\mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k))$ and the left adjoint $L^{\mathrm{Nis}}_{\mathbb{A}^1}$ preserves colimits, $\mathcal{B}=DM^{\mathrm{eff}}(k)$, proving (1).

For (2), we need to show as above that

$$\varphi_N : N\to \mathcal{H}\mathrm{om}_{\mathrm{\acute{e}t}}(\mathbb{Z}(1)^{\mathrm{\acute{e}t}},N(1))$$

is an isomorphism in $DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]$ for all $N \in DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]$. Suppose we know that φ_N is an isomorphism in $DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]$ for each homotopy invariant $N \in \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(k)[1/p]$. Take N to be an arbitrary object of $DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]$ and take $x \in X \in \mathbf{Sm}/k$. To show that φ_N is an isomorphism, it suffices to show that the map on the stalk

$$\varphi_{N,x} : N_x \to \mathcal{H}om_{\text{ét}}(\mathbb{Z}(1)^{\text{ét}}, N(1))_x$$

is an isomorphism in $D(\mathbf{Ab})$. But $\mathcal{H}om_{\text{ét}}(\mathbb{Z}(1)^{\text{ét}}, N(1))_x$ fits into the split exact sequence

$$0 \to \mathcal{H}om_{\text{ét}}(\mathbb{Z}(1)^{\text{ét}}, N(1))_x \to N(1)(\mathbb{P}^1 \times X_x^{\text{ét}})[2] \to N(1)(X_x^{\text{ét}})[2] \to 0.$$

As $\mathbb{P}^1 \times X_x^{\text{ét}}$ has finite cohomological dimension, we have the strongly convergent spectral sequence

$$E_1^{p,q} = H^p_{\text{ét}}(\mathbb{P}^1 \times X_x^{\text{ét}}, \mathcal{H}^q_{\text{ét}}(N)(1)) \Longrightarrow \mathbb{H}^{p+q}_{\text{ét}}(\mathbb{P}^1 \times X_x^{\text{ét}}, N(1)).$$

The assumption that each $\varphi_{\mathcal{H}^q_{\text{ét}}(N)}$ is an isomorphism implies that $E_1^{2,q} \cong \mathcal{H}^q_{\text{ét}}(N)$, $E_1^{0,q} \cong \mathcal{H}^q_{\text{ét}}(N)(1)$ and all other terms are zero; as the above sequence is split, the d_2 differential is also zero, and thus $\varphi_{N,x}$ is an isomorphism.

Suppose then that N is a sheaf. Suppose first that N is a sheaf of $\mathbb{Q}$-vector spaces. By [**17**, Chapter III, Proposition 5.24, 5.27], the canonical map $N \to a^*Ra_*N$ is a quasi-isomorphism. As we thus have the isomorphism $N(1) \to a^*((Ra_*N)(1))$, (1) implies (2) for N a complex of sheaves of $\mathbb{Q}$-vector spaces.

Next, suppose that N is a torsion sheaf. By [**17**, Chapter III, Theorem 5.25], N is a locally constant sheaf. Since we need only check φ_N on stalks, we may replace k with k^{sep}, reducing us to the case $N = \mathbb{Z}/m$ for some m prime to the characteristic. Thus, φ_N is

$$\varphi_{\mathbb{Z}/m} : \mathbb{Z}/m \to \mathcal{H}om_{\text{ét}}(\mathbb{Z}(1)^{\text{ét}}, \mathbb{Z}/m(1)^{\text{ét}}).$$

As above, $\mathcal{H}om_{\text{ét}}(\mathbb{Z}(1)^{\text{ét}}, \mathbb{Z}/m(1)^{\text{ét}})$ can be computed from the étale cohomology of $\mathbb{P}^1 \times X_x$ with $\mathbb{Z}/m(1)$-coefficients; as $\mathbb{Z}/m(1)^{\text{ét}} \cong \mu_m$, and we have the proper base-change isomorphism

$$H^n_{\text{ét}}(\mathbb{P}^1 \times X_x, \mu_m) \cong H^n_{\text{ét}}(\mathbb{P}^1_x, \mu_m),$$

the result follows from the known étale cohomology of $\mathbb{P}^1$.

In general, we use the exact sequence

$$0 \to N_{\mathrm{tor}} \to N \to N \otimes \mathbb{Q} \to N_{\mathrm{cotor}} \to 0$$

to reduce to the case of torsion sheaves and sheaves of $\mathbb{Q}$-vector spaces. □

Via the cancellation theorem, we have a twisted version of duality in the categories $DM^{\mathrm{eff}}(k)$ and $DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]$.

Corollary C.7.3. *Let $X \in \mathbf{Sm}/k$ be smooth and projective of dimension d over k. Then there are natural isomorphisms*

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A \otimes M(X), B) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A(d)[2d], B \otimes M(X)),$$
$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]}(A \otimes M^{\text{ét}}(X), B) \cong \mathrm{Hom}_{DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]}(A(d)[2d], B \otimes M^{\text{ét}}(X)),$$

where in the first isomorphism, A and B are arbitrary objects of $DM^{\mathrm{eff}}(k)$ and in the second, A and B are arbitrary objects of $DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]$.

Proof. In the category $DM_{\mathrm{gm}}(k)$, the object $M(X)$ has dual $M(-d)[-2d]$ (see [**58**]), thus there are morphisms

$$\begin{aligned} \delta_X &: \mathbb{Z} \to M_{\mathrm{gm}}(X) \otimes M_{\mathrm{gm}}(X)(-d)[-2d], \\ \epsilon_X &: M_{\mathrm{gm}}(X)(-d)[-2d] \otimes M_{\mathrm{gm}}(X) \to \mathbb{Z} \end{aligned}$$

with

$$(\mathrm{id}_{M_{\mathrm{gm}}(X)} \otimes \epsilon_X) \circ (\delta_X \otimes \mathrm{id}_{M_{\mathrm{gm}}(X)}) = \mathrm{id}_{M_{\mathrm{gm}}(X)} .$$

Twisting by $\mathbb{Z}(d)[2d]$ and applying the embedding $i^{\mathrm{eff}}_{\mathrm{gm}} : DM^{\mathrm{eff}}_{\mathrm{gm}}(k) \to DM^{\mathrm{eff}}(k)$ gives the maps

$$\begin{aligned} \delta^{\mathrm{eff}}_X &: \mathbb{Z}(d)[2d] \to M(X) \otimes M(X), \\ \epsilon^{\mathrm{eff}}_X &: M(X) \otimes M(X) \to \mathbb{Z}(d)[2d] \end{aligned}$$

in $DM^{\mathrm{eff}}(k)$ with

$$(\mathrm{id}_{M(X)} \otimes \epsilon^{\mathrm{eff}}_X) \circ (\delta^{\mathrm{eff}}_X \otimes \mathrm{id}_{M(X)}) = \mathrm{id}_{M(X)(d)[2d]} . \tag{C.1}$$

Now take $A, B \in DM^{\mathrm{eff}}(k)$. We have the natural transformation

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A \otimes M(X), B) \xrightarrow{\varphi_{A,B}} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A(d)[2d], B \otimes M(X))$$

sending $f : M(X) \otimes A \to B$ to the composition

$$A(d)[2d] = A \otimes \mathbb{Z}(d)[2d] \xrightarrow{\mathrm{id}_A \otimes \delta^{\mathrm{eff}}_X} A \otimes M(X) \otimes M(X) \xrightarrow{f \otimes \mathrm{id}_{M(X)}} B \otimes M(X).$$

We have as well the natural transformation

$$\mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A(d)[2d], B \otimes M(X)) \xrightarrow{\psi_{A,B}} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A \otimes M(X)(d)[2d], B(d)[2d])$$

sending $g : A(d)[2d] \to B \otimes M(X)$ to the composition

$$\begin{aligned} A \otimes M(X)(d)[2d] \cong A(d)[2d] \otimes M(X) &\xrightarrow{g \otimes \mathrm{id}_{M(X)}} B \otimes M(X) \otimes M(X) \\ &\xrightarrow{\mathrm{id}_B \otimes \epsilon^{\mathrm{eff}}_X} B \otimes \mathbb{Z}(d)[2d] = B(d)[2d]. \end{aligned}$$

It follows from (C.1) that $\psi_{A,B} \circ \varphi_{A,B}$ and $\psi_{A(d)[2d],B(d)[2d]} \circ \psi_{A,B}$ are the respective twists by $\mathbb{Z}(d)[2d]$:

$$\begin{aligned} \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A \otimes M(X), B) &\to \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A \otimes M(X)(d)[2d], B(d)[2d]), \\ \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A(d)[2d], B \otimes M(X)) &\to \mathrm{Hom}_{DM^{\mathrm{eff}}(k)}(A(2d)[4d], B \otimes M(X)(d)[2d]), \end{aligned}$$

which are isomorphisms by the cancellation theorem Corollary C.7.2 (1). In particular, $\varphi_{A,B}$ gives us the desired natural isomorphism.

The proof for $DM^{\mathrm{eff}}(k)^{\text{ét}}[1/p]$ is the same, noting α^* is a tensor functor, that applying α^* to the identity (C.1) yields the identity

$$(\mathrm{id}_{M^{\text{ét}}(X)} \otimes \alpha^* \epsilon_X^{\mathrm{eff}}) \circ (\alpha^* \delta_X^{\mathrm{eff}} \otimes \mathrm{id}_{M^{\text{ét}}(X)}) = \mathrm{id}_{M^{\text{ét}}(X)(d)[2d]},$$

and using Corollary C.7.2 (2) instead of Corollary C.7.2 (1). $\square$

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