On the classifying space of a linear algebraic group

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1. BG in algebraic topology

G topological group, X space

$$H^1(X,G) := \{\text{isomorphism classes of } G\text{-torsors} \\ (= \text{principal homogeneous spaces}) \text{ with base } X\}$$

Theorem 1.1. $X \mapsto H^1(X,G)$ *is representable in homotopy category* $\mathcal{H}_\bullet$.

Representing object denoted by BG.

1.1. Construction of BG.

(1) **General**: take "nice" contractible space EG where G acts freely and properly: $BG = EG/G$.

(2) **Specific**: $(E_nG = G^{n+1})$ simplicial space (faces = projections, degeneracies = diagonals), $(B_nG = E_nG/G \simeq G^n)$ other simplicial space: $EG = |E_\bullet G|$, $BG = |B_\bullet G|$.

(3) **Special**: if G compact, take faithful unitary representation $G \hookrightarrow U(N)$ $(N \gg 0)$: $EG = \varinjlim_r U(N+r)/U(r)$.

2. BG IN ALGEBRAIC GEOMETRY

2.1. Classical. (cf. Hodge II) G group scheme $/S$, $B_n G = G^n$, usual faces and degeneracies defines simplicial scheme $B_\bullet G$.

2.2. If G linear over a field. (Serre-Rost, 90'es, for the Serre-Rost invariant) $\rho : G \hookrightarrow GL_N$, $B_\rho G = \varinjlim GL_{N+r}/(G \times GL_r)$.

Related to 2.2:

Morel-Voevodsky: in homotopy category of schemes

Totaro: Chow ring of BG.

2.3. Morel-Voevodsky. S Noetherian of finite Krull dimension

$$\begin{array}{ll} \mathcal{H}_s(S) & \text{“simplicial” hom. cat. of schemes} \\ \downarrow & \quad\text{(for Nisnevich topology)} \\ \mathcal{H}(S) & \mathbf{A}^1-\text{homotopy category of schemes} \end{array}$$

G S-group scheme: $B_\bullet G$ defines object $BG \in \mathcal{H}_s(S)$.

Theorem 2.1. *X smooth S-scheme of finite type: natural bijection*

$$H^1_{\mathrm{Nis}}(X, G) \simeq \mathrm{Hom}_{\mathcal{H}_s(S)}(X, BG).$$

$\alpha : (Sm/S)_{\text{ét}} \to (Sm/)_{\text{Nis}}$: $BG \to B_{\text{ét}}G := R\alpha_* \alpha^* BG$.

Theorem 2.2. *X smooth S-scheme of finite type: natural bijection*

$$H^1_{\text{ét}}(X, G) \simeq \operatorname{Hom}_{\mathcal{H}_s(S)}(X, B_{\text{ét}}G).$$

Theorem 2.3. *If* G/S *linear, i.e.* $\rho : G \hookrightarrow GL_N/S$, $N \gg 0$, *then* $B_{\text{ét}}G \simeq B_\rho G$ *in* $\mathcal{H}(S)$.

2.4. Parallel (independent): Totaro. k field, G linear algebraic group.

Theorem 2.4 (Totaro)**.** *$\forall c \geq 1$ $\exists \rho : G \hookrightarrow GL(E)$, E vector space and $j : U \subset E$ G-stable open subset such that*

(i) *G acts freely on U, geometric quotient U/G exists, is quasi-projective, and $U \to U/G$ G-torsor.*

(ii) *$\delta(j) := \operatorname{codim}_E(E - U) \geq c$.*

Definition 2.5 (Totaro)**.** $CH^n(BG) := CH^n(U/G)$ for $c \gg 0$ ($c > n$ is enough).

Does not depend on choice of ρ.

Generalised by Edidin-Graham (Borel constructions) $\mapsto$ Brosnan's construction of Steenrod operations on CH^*/p.

2.5. This project. Totaro defines $CH^*(BG)$, but what is "his" BG?

Answer: different and more elementary than Morel-Voevodsky.

Applications:

- new invariants associated to G (dimensions, sets of torsors)
- functoriality in Totaro's construction
- "good" spectral sequences for motivic and étale motivic cohomology of BG
- lots of open questions!

3. The double fibration construction, functorially

3.1. Review of no name lemma. Why does $CH^n(U/G)$ not depend on choice of U?

Answer: the double fibration construction (aka no name lemma: Bogomolov-Katsylo et al)

$G \hookrightarrow GL(E), G \hookrightarrow GL(E')$ two "good" representations, $E \supset U$, $E' \supset U'$

$$\begin{array}{ccccc} (U \times E')/G & \overset{j}{\supset} & (U \times U')/G & \overset{j'}{\subset} & (E \times U')/G \\ p\downarrow & & & & \downarrow p' \\ U/G & & & & U'/G \end{array}$$

j, j' "small" open immersions, p, p' vector bundles (by faithfully flat descent).

3.2. Conceptualisation.

$$\mathrm{Sm}_{\mathrm{dom}} := \{\text{smooth quasi-projective } k\text{-schemes} \mid \text{morphisms} = \text{dominant morphisms}\}$$

Definition 3.1. $c \geq 1$, $F : \mathrm{Sm}_{\mathrm{dom}} \to \mathcal{C}$ functor: F is *homotopic of coniveau* $\geq c$ if

(i) F inverts $p : V \to X$, vector bundle projections

(ii) F inverts open immersions j with $\delta(j) \geq c$.

G linear algebraic group, $c \geq 1$.

Definition 3.2. $U, U' \in \mathrm{Sm}_{\mathrm{dom}}$ with G-action: (U, U') is an *admissible pair of coniveau* $\geq c$ if

(i) U is the total space of a G-torsor

(ii) U': nonempty G-stable open subset $j : U' \hookrightarrow E'$, E' linear representation of G, with $\delta(j') \geq c$, geometric quotient U'/G exists and is quasi-projective.

Construction 3.3 (with Nguyen T. K. Ngan). *F homotopic of coniveau $\geq c$: canonical morphisms*

$$(U,U') \text{ adm. of coniveau} \geq c \mapsto \varphi_{U,U'} : F(U/G) \to F(U'/G)$$

with

Reflexivity: (U,U) *admissible* $\Rightarrow \varphi_{U,U} = 1$.

Symmetry: (U,U') *and* (U',U) *admissible* $\Rightarrow \varphi_{U,U'} \circ \varphi_{U',U} = 1$ *($\Rightarrow \varphi_{U,U'}$ iso).*

Transitivity: (U,U'), (U',U'') *and* (U,U'') *admissible* $\Rightarrow \varphi_{U,U''} = \varphi_{U',U''} \circ \varphi_{U,U'}$.

Sketch. $U' \subset E'$

$$\begin{array}{ccc} (U \times E')/G & \overset{j}{\supset} & (U \times U')/G \\ p \downarrow & & p' \downarrow \\ U/G & & U'/G \end{array}$$

$\varphi_{U,U'} = F(p')F(j)^{-1}F(p)^{-1}$. □

Definition 3.4.

$$F(BG) = \varinjlim_{\varphi_{U,U'},(U,U),(U',U') \text{ adm}} F(U/G)$$

($\varinjlim$ of isos over trivial groupoid!)

Proposition 3.5. *$G \mapsto F(BG)$ functorial for group scheme homomorphisms.*

3.3. Universal case(s).

$$\mathrm{Sm}_{\mathrm{dom}} \supset S^{o}_{c} = \{ j \mid \text{open imm.}, \delta(j) \geq c \}$$

$$S_h = \{ p \mid \text{vector bundle projections} \}$$

$$S_c = S^{o}_{c} \cup S_h$$

$$\begin{array}{ccccccccc}
 & & & \swarrow^{F_{c+1}} & \mathrm{Sm}_{\mathrm{dom}} & \searrow^{F_{c-1}} & & & \\
 & & & & \downarrow F_c & & & & \\
\cdots \longrightarrow & S_{c+1}^{-1}\mathrm{Sm}_{\mathrm{dom}} & \longrightarrow & & S_{c}^{-1}\mathrm{Sm}_{\mathrm{dom}} & & \longrightarrow & S_{c-1}^{-1}\mathrm{Sm}_{\mathrm{dom}} & \longrightarrow \cdots
\end{array}$$

$$F_{c+1}(BG) \overset{\sim}{\mapsto} \qquad F_{c}(BG) \overset{\sim}{\mapsto} \qquad F_{c-1}(BG) \overset{\sim}{\mapsto}$$

We write $F_c(BG) =: B_c G$.

Similarly, Borel constructions: $\mathrm{Sm}^G_{\mathrm{dom}} = \{G - \text{objects of } \mathrm{Sm}_{\mathrm{dom}}\}$

$$\mathrm{Bor}^G_c : S_c^{-1}(\mathrm{Sm}^G_{\mathrm{dom}}) \to S_c^{-1}\,\mathrm{Sm}_{\mathrm{dom}}$$

$$X \mapsto E_c G \times^G X.$$

E.g.: $c = 1$:

$$\begin{array}{ccc} (S_1^o)^{-1}\,\mathrm{Sm}_{\mathrm{dom}} & \simeq & \mathrm{field}^{op} \\ \downarrow & & \downarrow \\ S_1^{-1}\,\mathrm{Sm}_{\mathrm{dom}} & \simeq & S_r^{-1}\,\mathrm{field}^{op} \\ B_1G & \leftrightarrow & k(BG) \end{array}$$

field = category of function field extensions of k (with smooth model), $S_r =$

$\{K(t_1, \ldots, t_n)/K\}$.

(Challenge: compute Homs in $S_r^{-1}\,\mathrm{field}$!)

Can push $\mathrm{Sm}_{\mathrm{dom}}$ to Sm (all morphisms): get other (weaker) BGs.

3.4. Representability.

$$\begin{array}{ccccccc} & \mathrm{Sm}_{\mathrm{dom}} & \xrightarrow{y} & \widehat{\mathrm{Sm}_{\mathrm{dom}}} & \ni & H^1(-,G) \\ & F_c\Big\downarrow & & (F_c)_!\Big\downarrow & & \Big\downarrow \\ B_cG \in & S_c^{-1}\,\mathrm{Sm}_{\mathrm{dom}} & \xrightarrow{\bar{y}} & \widehat{S_c^{-1}\,\mathrm{Sm}_{\mathrm{dom}}} & \ni & H^1(-,G)_c \end{array}$$

(Recall:

$$H^1(X,G)_c = \varinjlim_{Y \in X \setminus F_c} H^1(Y,G).)$$

Theorem 3.6. *$H^1(-,G)_c$ is representable by B_cG.*

3.5. Comparison with Morel-Voevodsky.

$$\begin{array}{ccccc} & \mathrm{Sm}_{\mathrm{dom}} & \longrightarrow & \mathcal{H} & \ni B_{\text{ét}}G \\ & \downarrow & & \downarrow & \\ B_cG \in & S_c^{-1}\,\mathrm{Sm}_{\mathrm{dom}} & \longrightarrow & S_c^{-1}\mathcal{H} & \end{array}$$

same image.

4. Computations

4.1. Direct products.

Warning 4.1. $\mathrm{Sm}_{\mathrm{dom}}$ has no (or very few) products! (when is $\Delta_X : X \to X \times X$ dominant?)

Yet $(X, Y) \mapsto X \times Y$ defines a bifunctor $* : \mathrm{Sm}_{\mathrm{dom}} \times \mathrm{Sm}_{\mathrm{dom}} \to \mathrm{Sm}_{\mathrm{dom}}$ (becoming product in Sm).

Proposition 4.2. *a) $*$ induces bifunctor $* : S_c^{-1}\mathrm{Sm}_{\mathrm{dom}} \times S_c^{-1}\mathrm{Sm}_{\mathrm{dom}} \to S_c^{-1}\mathrm{Sm}_{\mathrm{dom}}$; $\times$ induces product on $S_c^{-1}\mathrm{Sm}$.*
b) $B_c(G \times H) \simeq B_cG * B_cH$ in $S_c^{-1}\mathrm{Sm}_{\mathrm{dom}}$ (hence $\simeq B_cG \times B_cH$ in $S_c^{-1}\mathrm{Sm}$).

4.2. Semi-direct products.

Proposition 4.3. $B_C(G \ltimes H) \simeq \mathrm{Bor}_C^G(B_C H)$

(To be taken with a pinch of salt. . .)

4.3. Unipotent subgroups.

$$\mathrm{Sm}_{\mathrm{dom}} \supset S_J := \{p \mid \text{ projection from torsor under vector group}\}$$

(J is for Jouanolou)

$$\bar{S}_c := S_c \cup S_J.$$

Theorem 4.4. *k perfect; G linear, N normal connected unipotent subgroup. Suppose the extension*

$$1 \to N \to G \to G/N \to 1$$

is split and N has a composition series by characteristic subgroups with vector groups as successive quotients. Then, $B_c G \xrightarrow{\sim} B_c(G/N)$ in $\bar{S}_c^{-1}\mathrm{Sm}_{\mathrm{dom}}$.

Splitting hypothesis true if

- $N = G$
- G connected, G/N reductive and
  - $\operatorname{char} k = 0$ or
  - $G =$ parabolic subgroup of a reductive group.

  (Then composition series also true.)

Remark 4.5. Can also show $B\mathbf{Z}/p \sim *$ in char. p. But does not seem to extend to Borel constructions (may $\exists$ nontrivial Artin-Schreier extensions of $k\ldots$)

4.4. Selected examples.

$$B_c\mathbf{G}_m = \mathbf{P}^{c-1}$$

$$B_c\mu_m = \mathcal{O}_{\mathbf{P}^{c-1}}(-m) - 0$$

$$B_cGL_n = \mathrm{Grass}_n(\mathbf{A}^{c+n-1}).$$

Application: "Fischer's theorem" over any field: G of split multiplicative type, E faithful representation $\Rightarrow k(E)^G/k$ stably rational.

5. Examples of pure homotopic functors

5.1.

Definition 5.1. $F : \mathrm{Sm}_{\mathrm{dom}} \to \mathcal{C}$:

$$\nu(F) = \inf\{c \mid F \text{ is pure of coniveau } > c\}.$$

This is the *coniveau* of F.

Examples 5.2.

- $H^i_{\text{ét}}(-, \mu_m^{\otimes n})$: $\nu = [i/2]$.
- $H^i(-, \mathbf{Z}(n))$: $\nu = n$.
- $H^i_{\text{ét}}(-, \mathbf{Z}(n))$: $\nu = \sup(n, [i/2])$.
- Rost's Chow groups with coefficients $A^i(-, M_j)$:

$$\nu \leq \inf(j - \delta(M), i+1)$$

where $\delta(M) = \inf\{j \mid M_j \neq 0\}$ (the *connectivity* of M).

Non-example: K_0 (need to replace by $K_0/F^p K_0$: cf. Atiyah-Segal, also in Totaro).

Theorem 5.3. *k separably closed, G finite group: canonical isomorphisms*

$$H^i_{\text{ét}}(BG, \mathbf{Z}(n)) \simeq H^i_{\text{ét}}(k, \mathbf{Z}(n)) \oplus H^i(G, \mathbf{Z})\{p'\}(n) \quad (i \in \mathbf{Z})$$

p exponential characteristic, $A\{p'\} = p'$-primary component of torsion of abelian group A.

Other examples in DM_{-}^{eff}:

- $X \mapsto \nu_{\leq n} M(X)$ (slice filtration): $\nu = n$.
- $\Rightarrow X \mapsto c_n(M(X)) \in DM_{-}^{o}$: $\nu = n$.

(Recall:

$$\nu_{\leq n} M := \mathrm{cone}\,(\underline{\mathrm{Hom}}(\mathbf{Z}(n), M)(n) \to M)$$

$$c_n(M) := \mathrm{cone}\left(\nu_{\leq n+1} M \to \nu_{\leq n} M\right)(-n)[-2n].)$$

5.2. Spectral sequences. X smooth variety:

$$\begin{array}{ccccc} I_2^{p,q} = A^p(X, \mathcal{H}^q_{\mathrm{Nis}}(\mathbf{Z}(n))) & \Rightarrow H^{p+q}(X, \mathbf{Z}(n)) & \Leftarrow H^{p-q}(c_q(X), \mathbf{Z}(n-q)) = II_2^{p,q} \\ \downarrow & \downarrow & \downarrow \\ {}_{\text{ét}}I_2^{p,q} = A^p(X, \mathcal{H}^q_{\text{ét}}(\mathbf{Z}(n))) & \Rightarrow H^{p+q}_{\text{ét}}(X, \mathbf{Z}(n)) & \Leftarrow H^{p-q}_{\text{ét}}(c_q(X), \mathbf{Z}(n-q)) = {}_{\text{ét}}II_2^{p,q} \end{array}$$

$I_r^{p,q}$ coniveau, $II_r^{p,q}$ slice.

How about BG?

Consider $F : \mathrm{Sm}_{\mathrm{dom}} \to \{\text{spectral sequences}\}$: if lucky, F homotopic and pure of some coniveau.

OK in Nisnevich case $\nu = n$, but *not* in étale case.

For ${}_{\text{ét}}II$, can write it as $\varinjlim$ of sequences for $\nu_{\leq n} M(X)$, hence OK.

For ${}_{\text{ét}}I$, still works but more messy (need to talk of "chunks of spectral sequences"...)

6. Dimensions

6.1. Review of essential dimension.

Definition 6.1 (Buhler-Reichstein)**.** G algebraic group over k.

a) K/k extension, $\alpha \in H^1(K, G)$ G-torsor:

$$\mathrm{ed}(\alpha) = \inf\{\mathrm{trdeg}(L/k) \mid k \subset L \subset K, \alpha \text{ is defined over } L\}.$$

b) $\mathrm{ed}(G) = \sup_{K,\alpha} \mathrm{ed}(\alpha)$.

Merkurjev: generalised to any functor : field $\longrightarrow$ $Sets$.

6.2. Generalisation of Merkurjev's generalisation. $\mathcal{C}$ category, $\dim = Ob(\mathcal{C}) \to \mathbf{N}$ dimension function (respecting isomorphisms), $y : \mathcal{C} \to \widehat{\mathcal{C}}$ Yoneda embedding.

Definition 6.2. $F \in \hat{\mathcal{C}}$.

a) $X \in \mathcal{C}$, $\alpha \in F(X)$:

$$\mathrm{ed}(\alpha) = \inf\{\dim Y \mid Y \in \mathcal{C}, \exists f : X \to Y, \beta \in F(Y) : \alpha = f^*\beta\}.$$

b) $\mathrm{ed}(F) = \sup_{X,\alpha} \mathrm{ed}(\alpha)$.

Question 6.3. $X \in \mathcal{C}$: compare $\dim X$ and $\mathrm{ed}\, y(X)$.

Answer:

Proposition 6.4.

$$\mathrm{ed}(y(X)) = \mathrm{ed}(1_X) = \inf\{\dim Y \mid X \textit{ is a retract of } Y.\}$$

For "usual" categories, $\dim X = \mathrm{ed}\, y(X)$ but not necessarily in general.

Leads to *normalise dimensions*:

Definition 6.5. $\mathcal{C}$, dim as above, $X \in \mathcal{C}$:

$$\dim^{\nu}(X) := \mathrm{ed}\, y(X).$$

Remark 6.6.

- $\dim^{\nu}(X) = \mathrm{ed}^{\nu}(y(X))$
- $(\dim^{\nu})^{\nu} = \dim^{\nu}$.

6.3. Dimensions and localisations. E set, $\dim : E \to \mathbf{N}$ dimension function, $\sim$ equivalence relation on E:

$$\dim_\sim : E/\sim \to \mathbf{N}$$
$$\bar{x} \mapsto \inf\{\dim x \mid x \in \bar{x}\}.$$

Examples 6.7. $\mathcal{C}$ essentially small category, $E = \langle \mathcal{C} \rangle$ set of iso classes of objects, S class of arrows of $\mathcal{C}$: $\langle \mathcal{C} \rangle \twoheadrightarrow \langle S^{-1}\mathcal{C} \rangle$, so any dimension function dim on $\mathcal{C}$ (respecting isomorphisms) induces one, $\dim_S$, on $S^{-1}\mathcal{C}$.

Is it the right one? Not necessarily because of Proposition 6.4 (if dim normalised, maybe $\dim_S$ is not normalised). Thus replace $\dim_S$ by $\dim_S^\nu$.

6.4. Dimensions and coniveaux.

Definition 6.8. a) $X \in S_c^{-1}\,\mathrm{Sm}_{\mathrm{dom}}$: $\dim_c(X) := \dim^{\nu}_{S_c}(X)$.

b) G linear algebraic group: $d_c(G) := \dim_c(B_c G)$.

$X \in \mathrm{Sm}_{\mathrm{dom}}$:

$$\dim X \geq \dots \geq \dim_c X \geq \dim_{c-1} X \geq \dots$$

G linear:

$$\dots \geq d_c(G) \geq d_{c-1}(G) \geq \dots$$

Variants with $\bar{S}_c$, Sm…

Proposition 6.9. *X smooth projective of dimension n:* $\dim_{n+1}(X) = n$.

Sketch. Use $Y \mapsto H^{2n}_{\text{ét}}(\bar{Y}, \mathbf{Z}/l)$. □

Corollary 6.10. $d_c(\mathbf{G}_m) = c - 1 \to \infty$.

(Should be true for any non unipotent linear algebraic group.)

Question 6.11. Relationship between $(d_c(G))_{c \geq 1}$ and $\mathrm{ed}(G)$?

I don't know!

The end.