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BOTT ELEMENTS IN ALGEBRAIC K THEORY

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CONTENTS

1. INTRODUCTION

Let K be a number field, R its ring of integers and l a rational prime number. A classical result of Harris and Segal [12] asserts that, for a well-chosen prime $\mathfrak{p} \subseteq R$, with residue field E, the natural map

$$K_{2i-1}(R) \otimes_{\mathbf{Z}} \mathbf{Z}_{(l)} \to K_{2i-1}(E) \otimes_{\mathbf{Z}} \mathbf{Z}_{(l)} \tag{1}$$

is split surjective for any $i \geqslant 1$, at least when $l > 2$ or K is not "exceptional". Here, as in the sequel, $\mathbf{Z}_{(l)}$ denotes the ring of rational l-adic integers, that is, the localization of $\mathbf{Z}$ at l. For this choice of $\mathfrak{p}$, $K_{2i-1}(E) \otimes_{\mathbf{Z}} \mathbf{Z}_{(l)}$ is isomorphic to $H^0(K, \mathbf{Q}_l/\mathbf{Z}_l(i))$, the Galois fixed points of the l-primary roots of unity Tate-twisted $i - 1$ times (this depends on Quillen's computation of the algebraic K-theory of finite fields [27]). So, in particular, $K_{2i-1}(R)$ contains a cyclic direct summand isomorphic to $H^0(K, \mathbf{Q}_l/\mathbf{Z}_l(i))$.

Harris and Segal's proof is not constructive. Let l^n be a power of l. Let A be a ring in which l is invertible and containing a primitive l^nth root of unity. If μ denotes the group of l^nth roots of unity of A, there are natural maps

$$\pi_j^S(\mathbf{B}\mu_+) \to K_j(A), \quad j > 0. \tag{2}$$

When A is a finite field and contains exactly l^n l-primary roots of unity, Harris and Segal show that these maps are split on the l-primary components. Applying this to $A = R$ and to $A = E$ as above, the composite

$$K_{2i-1}(E) \otimes_{\mathbf{Z}} \mathbf{Z}_{(l)} \to \pi_{2i-1}(\mathbf{B}\mu_+) \otimes_{\mathbf{Z}} \mathbf{Z}_{(l)} \xrightarrow{(\Phi_R)_*} K_{2i-1}(R) \otimes_{\mathbf{Z}} \mathbf{Z}_{(l)}$$

is a splitting of (1), where the first map is a Harris–Segal section and the second one is induced by (2). But there are *a priori* lots of Harris–Segal sections, which induce *a priori* lots of splittings of (1). It is not clear at all, at that stage, how many different cyclic summands of $K_{2i-1}(R)$ the Harris–Segal theorem provides.

It turns out, however, that all these cyclic summands are the same. This is a consequence of a result of Dwyer, Friedlander and Mitchell [7] implying that the map $(\Phi_R)_*$ factors through $K_{2i-1}(E)\otimes_{\mathbf{Z}}\mathbf{Z}_{(l)}$. It follows that the Harris–Segal cyclic summands all coincide with $\mathrm{Im}(\Phi_R)_*$; in particular, they do not even depend on the choice of $\mathfrak{p}$.[†]

In this paper, we do two things. First, we show how one can get a homotopy-theoretic canonical, conceptual formulation of the results described above. Second, we use these results and this formulation to solve most of [15, Conjecture 1] in the affirmative. This is a key step in the construction of higher 'anti-Chern classes' $\beta_R^{ij}: H^j(R,\mathbf{Z}/l^\nu(i)) \to K_{2i-j}(R,\mathbf{Z}/l^\nu)$ for R a semi-local ring [16].

To be more precise, assume l odd for simplicity. To any $\mathbf{Z}[1/l]$-scheme X we associate in Section 7 a "cyclotomic" ring spectrum $\mathrm{j}_l(X)$, such that

$$\pi_n(\mathrm{j}_l(X)) = \begin{cases} H^0(X_{\text{ét}},\ \mathbf{Q}_l/\mathbf{Z}_l(i)) & \text{if } n = 2i-1>0 \\ \mathbf{Z}_{(l)}^{\pi_0(X)} & \text{if } n=0 \\ 0 & \text{otherwise.} \end{cases}$$

$\mathrm{j}_l(X)$ generalises the connective cover of the "Im **J**" spectrum, that one gets for $X = \operatorname{Spec}\mathbf{Z}[1/l]$; it is equivalent to the algebraic K-theory of a suitable finite field, localised at l. If $f: Y \to X$ is a morphism of schemes (resp. a finite flat morphism), there is an "extension of scalars" morphism $f^*: \mathrm{j}_l(X) \to \mathrm{j}_l(Y)$ (resp. a "transfer" morphism $f_*: \mathrm{j}_l(Y) \to \mathrm{j}_l(X)$), satisfying standard properties. In Section 9, we define a homotopy class of map

$$\Omega^\infty \mathrm{j}_l(X) \xrightarrow{\bar\beta_X} \Omega^\infty \mathbf{L}KX.$$

Here KX denotes the algebraic K-theory spectrum of X and, for any space or spectrum S, $\mathbf{L}S$ denotes its Bousfield localisation at l [5]. We show that $\bar\beta_X$ commutes with extension of scalars, transfer in the semi-local case and with product if X is essentially of finite type over $\mathbf{Z}$. It induces homomorphisms to K-theory with finite coefficients

$$H^0(X_{\text{ét}}, \mathbf{Z}/l^\nu(i)) \xrightarrow{\beta_X^i} K_{2i}(X, \mathbf{Z}/l^\nu), \tag{3}$$

which inherit the corresponding compatibility properties, and split natural homomorphisms going the other way.

If we specialise to $X = \operatorname{Spec} R$, where R is the ring of integers of a number field, we get the following picture:

$$\Omega^\infty \mathrm{j}_l(R) \xrightarrow{\bar\beta_R} \Omega^\infty \mathbf{L}KR \to \Omega^\infty \mathbf{L}K(R/\mathfrak{p}).$$

For $\mathfrak{p}$ an appropriate prime, the composition of these two maps in a homotopy equivalence.

Harris and Segal start from $\mathfrak{p}$ and want to split the map $\Omega^\infty\mathbf{L}KR \to \Omega^\infty\mathbf{L}K(R/\mathfrak{p})$. To do this they introduce a "monomial" space mapping both to $\Omega^\infty\mathbf{L}KR$ and $\Omega^\infty\mathbf{L}K(R/\mathfrak{p})$, and

[†] Let me take this opportunity to point out a gap in a proof of [12]. It is claimed there that, if K is an exceptional number field and $l = 2$, then $K_{2i-1}(R)$ contains a cyclic direct summand of order w_i, $2w_i$ or $w_i/2$, where $w_i = |H^0(K, \mathbf{Q}/\mathbf{Z}(i))|$. The proof is a reduction to the non-exceptional case, by using transfer from the K-theory of $R(\sqrt{-1})$ to that of R. One point in the argument is to study the Galois action on the cyclic direct summand of $K_{2i-1}(R(\sqrt{-1}))$ (compare [12, p. 30, Section 2]). But this presupposes that the direct summand is stable under Galois action (compare *loc. cit.*, lines-9/-6, where the argument is incorrect). This gap is now filled, thanks to the Dwyer–Friedlander–Mitchell theorem.

show that the map to $\Omega^\infty LK(R/\mathfrak{p})$ is split. Our $\Omega^\infty \mathrm{j}_l(R)$ is precisely an appropriate localisation of this space. So we have swung the focus the other side: the canonical map is now $\bar{\beta}_R$, which is retracted by the choice of a suitable $\mathfrak{p}$.

The following analogy may inspire the reader. If X is a curve over a field k, for any closed point $\mathfrak{p} \xhookrightarrow{\iota_\mathfrak{p}} X$ we have a reduction map $(\iota_\mathfrak{p})^*: KX \to Kk(\mathfrak{p})$, where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$. When $\mathfrak{p}$ is rational, $(\iota_\mathfrak{p})^*$ has a section π^* given by the structural morphism $X \xrightarrow{\pi} \operatorname{Spec} k$, which evidently does not depend on the choice of $\mathfrak{p}$.

The formalism described above is reminiscent of this situation: suitable $\mathfrak{p}$'s correspond to rational points, and the map $\bar{\beta}_R$ corresponds to that coming from the structural morphism in the example above. But in the present case there is no base and the construction of $\bar{\beta}_R$ is far from trivial. To complete the analogy, one might think of an imaginary base, perhaps consisting of roots of unity. This fits nicely with Iwasawa's philosophy according to which cyclotomic extensions in the number field case play the role of base field extensions in the function field case.

We now describe the contents of this paper. In Section 3 we introduce auxiliary spectra, corresponding to the Harris–Segal "monomial" spaces. The main piece of work is the construction of $\mathrm{j}_l(X)$ in the case of rings of cyclotomic integers: this is done in Section 4, except in certain cases for $l = 2$. In Section 5 we construct $\bar{\beta}_X$, still for cyclotomic integers, using the results indicated above, and in Section 6 we prove its functorial properties. Sections 7–9 are devoted to the (essentially trivial) extension to general schemes and the proof of (most of) [15, Conjecture 1] (Corollary 9.5 and Theorem 9.8). The latter rests partially on the Suslin–Gillet–Thomason–Gabber rigidity theorem for the K-theory of strict Hensel local rings.

There are five appendices. Appendix A is a rather careful review of facts on symmetric (bi)monoidal categories. Appendix B is a detailed construction of the localisation *à la* Dwyer–Friedlander–Snaith–Thomason [8] of a ring spectrum (or modules over it) relative to a mod m homotopy class. Special attention has been devoted to the cases $m = 3$ and 4; we hope this section will be useful to some readers (it certainly was to the author!) In Appendix C we start developing the theory in the remaining $l = 2$ cases.

A conjecture of [7] implies that $\bar{\beta}_X$ is an infinite loop map. It is shown in Appendix D (Theorem D.2) that this conjecture follows from the validity of Lichtenbaum–Quillen conjecture for rings of l-cyclotomic integers (this was observed before by Mitchell). We also show in Appendix E (Theorem E.1) that $\bar{\beta}_X$ has retractions when X is of finite type over $\mathbf{Z}$; this is a higher-dimensional generalisation of the Harris–Segal theorem.

2. NOTATION

2.1. We fix a prime number l. $\mathbf{Z}_l$ denotes the l-adic integers and $\mathbf{Z}_{(l)} = \mathbf{Z}_l \cap \mathbf{Q}$ the rational l-adic integers. If M is a $\mathbf{Z}_l$-module on which $\mathbf{Z}_l^*$ acts by a homomorphism $\mu: \mathbf{Z}_l^* \to \operatorname{Aut}(M)$ and $i \in \mathbf{Z}$, we denote by $M(i)$ the "ith Tate twist" of M, i.e. the same module on which $\mathbf{Z}_l^*$ acts by the new action $\mu'(u) = u^i\mu(u)$. In particular, we have the $\mathbf{Z}_l[\mathbf{Z}_l^*]$-modules $\mathbf{Z}/l^\nu(i)$ and $\mathbf{Q}_l/\mathbf{Z}_l(i)$, twisted from the trivial modules $\mathbf{Z}/l^\nu$ and $\mathbf{Q}_l/\mathbf{Z}_l$.

2.2. We note

$$l^* = \begin{cases} l & \text{if } l > 2 \\ 4 & \text{if } l = 2. \end{cases}$$

2.3. We shall often use the notational shortcut

$$1 + 2l\mathbf{Z}_l = \begin{cases} 1 + l\mathbf{Z}_l & \text{if } l > 2 \\ 1 + 4\mathbf{Z}_2 & \text{if } l = 2. \end{cases}$$

2.4. Let Δ be a closed subgroup of $\mathbf{Z}_l^*$. We write $\Delta_1 = \Delta \cap (1 + l\mathbf{Z}_l)$ and $\bar{\Delta} = \Delta/\Delta_1$. Then $\bar{\Delta}$ is a finite group of order dividing $l-1$ if $l > 2$ and dividing 2 if $l = 2$, hence prime to l if $l > 2$.

2.5. If F is a field, we denote by G_F its absolute Galois group: so $G_F = \mathrm{Gal}(F_s/F)$ for some separable closure F_s of F. If l is invertible in F, we denote by $\kappa_l: G_F \to \mathbf{Z}_l^*$ the *cyclotomic character* at l: this is the continuous homomorphism given by the action of G_F on the l-primary roots of unity of F_s. This way any $\mathbf{Z}_l^*$-module becomes a Galois module; $\mathbf{Z}/l^n(1)$ and $\mathbf{Q}_l/\mathbf{Z}_l(1)$ become isomorphic to the modules μ_{l^n} and μ_{l^∞} of l^nth and l-primary roots of unity. More generally, we can do the same for any connected scheme X over $\mathbf{Z}[1/l]$, using its algebraic fundamental group $\pi_1(X, \bar{\eta})$ relative to some geometric point $\bar{\eta}$. The image

$$\Delta_X = \kappa_l(\pi_1(X, \bar{\eta})) \subseteq \mathbf{Z}_l^*$$

does not depend on the choice of $\bar{\eta}$.

2.6. For any ring A, we denote by $\mathscr{P}(A)$ the category of finitely generated projective A-modules.

2.7. For any unpointed space X, we denote by X_+ the union of X and a disjoint base point. If X is a pointed space and N is a perfect normal subgroup of $\pi_1(X)$, we denote by X^+ Quillen's$^+$ construction on X with respect to N [18]. Usually, N will be clear from the context. If $X = \mathrm{BGL}(R)$ for some ring R, we have $N = E(R)$ and $K_0(R) \times \mathrm{BGL}(R)^+ \approx \Omega^\infty KR$, where KR is the K-theory spectrum of R.

2.8. $\mathbb{S}$ is the sphere spectrum. For an abelian group D, $H(D)$ denotes its Eilenberg–MacLane spectrum and $M(D)$ its Moore spectrum. In particular, for an integer $m \geqslant 1$ we simply write $M(m)$ for $M(\mathbf{Z}/m)$, so that there is a fibration $\mathbb{S} \xrightarrow{m} \mathbb{S} \to M(m)$ and a corresponding fibration $S \xrightarrow{m} S \to S \wedge M(m)$ for any spectrum S. We denote by $M(l^\infty)$ the spectrum $M(\mathbf{Q}_l/\mathbf{Z}_l) = \mathrm{hocolim}\, M(l^\nu)$. For any space X, $\Sigma^\infty X$ is the suspension spectrum of X and, for any spectrum S, $\Omega^\infty S$ (resp. $\Omega_0^\infty S$) is the zero space of S (resp. its connected component at 0). For a spectrum S and an integer n, we denote by $S_{\geqslant n}$ the truncation of S above n (denoted by $S > n-1 <$ in [41]). If a group G acts on S, we write S^{hG} for the spectrum of homotopy fixed points ($S^{hG} = \mathrm{holim}_{g \in G}(S \xrightarrow{g} S)$). Finally, for a spectrum S, we denote by $\mathbf{L}_S$ the functor "localisation at S" of [5]. In fact, we shall only use this localisation in simple "arithmetic" cases. If $S = M(\mathbf{Z}_{(l)})$, we write $\mathbf{L}$ for $\mathbf{L}_{M(\mathbf{Z}_{(l)})}$ (when l is unambiguous): this is localisation at the prime l, so that $\pi_*(\mathbf{L}S) \simeq \pi_*(S) \otimes \mathbf{Z}_{(l)}$. Similarly, $S[1/l]$ (resp. $S_\mathbf{Q}$) denotes localisation away from l (resp. at $\mathbf{Q}$): $\pi_*(S[1/l]) = \pi_*(S) \otimes \mathbf{Z}[1/l]$, $\pi_*(S_\mathbf{Q}) = \pi_*(S) \otimes \mathbf{Q}$. If $S = M(\mathbf{Z}/l)$, we write $X^\wedge$ for $L_S X$: this is the "completion at l" of X ($\approx \mathrm{holim}_\nu\, X \wedge M(l^\nu)$).

3. THE SPECTRA $\Sigma(l, \Delta)$

3.1. Recall [31, 20, 40, 13] that to a (small) symmetric monoidal category $\mathscr{S}$ one can associate a spectrum $\mathrm{Spt}(\mathscr{S})$. The assignment $\mathscr{S} \mapsto \mathrm{Spt}(\mathscr{S})$ defines a functor from the category of symmetric monoidal categories (with morphisms functors commuting to the monoidal structure) to the category of spectra (and strict morphisms). A symmetric bimonoidal category yields a ring spectrum, and a multiplicative functor yields a morphism of ring spectra.

3.2. *Examples.* (1) Let $\mathscr{S} = \mathscr{P}(R)$ be the category of projective modules over the ring R. Then $\mathrm{Spt}(\mathscr{S}) = K(R)$ is the K-theory spectrum of R. Its 0-space $\Omega^\infty K(R)$ is homotopy equivalent to $K_0(R) \times \mathrm{BGL}(R)^+$, where $^+$ is Quillen's$^+$ construction.

(2) Let $\mathscr{S} = \mathscr{S}(A)$, where A is a group (see A.2). Then $\mathrm{Spt}(\mathscr{S}) = \Sigma^\infty(BA_+)$, the spectrum of suspensions of BA union a disjoint base point. By the Barratt–Priddy–Quillen–Segal theorem [31, Proposition 3.6], its 0-space $Q(BA_+)$ is homotopy equivalent to $\mathbf{Z} \times B(\mathfrak{S}_\infty \wr A)^+$, where $\mathfrak{S}_\infty$ is the infinite symmetric group and $\wr$ is wreath product.

3.3. The construction Spt composed with a pseudo-functor from a category $\mathscr{C}$ to the 2-category of symmetric monoidal categories (compare A.1) does not yield a functor from $\mathscr{C}$ to spectra, but of course a pseudo-functor, if one considers spectra as forming a 2-category by using homotopies. A rectification of this pseudo-functor, however, yields a genuine functor from $\mathscr{C}$ to spectra; compare, e.g. [13] for details. We shall keep quiet on this question and assume here that pseudo-functors have been rectified, following the practice of [41] (see bottom of p. 440).

3.4. *Definition.* (a) For $\Delta \subseteq 1 + 2l\mathbf{Z}_l^*$ we set $\Sigma(l, \Delta) = \mathbf{L}\,\mathrm{Spt}(\mathscr{S}^\Delta)$, where $\mathscr{S}^\Delta$ is the symmetrical bimonoidal category defined in A.8.

(b) Suppose $l > 2$. For $\Delta \subseteq \mathbf{Z}_l^*$, we set $\Sigma(l, \Delta) = \Sigma(l, \Delta)^{h\bar{\Delta}}$ (see 1.4).

3.5. *Remarks.* (1) By Example 3.2(2), $\Sigma(l, \Delta) = \mathbf{L}\Sigma^\infty B(\mu_{l^\infty}^\Delta)^+$ if $\Delta \subseteq 1 + 2l\mathbf{Z}_l$.

(2) We shall define $\mathscr{S}^\Delta$ and $\Sigma(2, \Delta)$ for $\Delta \not\subseteq 1 + 4\mathbf{Z}_2$ in Appendix C.

3.6. Lemma. *Let Δ be a closed subgroup of $\mathbf{Z}_l^*$. Then the map*

$$\operatorname*{hocolim}_{U \supseteq \Delta} \Sigma(l, U) \to \Sigma(l, \Delta)$$

is an equivalence, where U runs through the open subgroups of $\mathbf{Z}_l^*$ containing Δ.

Proof. If $\Delta \subseteq 1 + 2l\mathbf{Z}_l$, this is immediate from the description of $\Sigma(l, \Delta)$ (Example 3.2(2)). In general, observe that the groups $\bar{U}$ are finite, hence the inverse system $(\bar{U})$ stabilises, and homotopy fixed points under the action of a finite group commute with filtering colimits. □

Let $\iota: KR^\Delta \to KR^{\Delta_1}$ be the map coming from extension of scalars $\mathscr{P}(R^\Delta) \to \mathscr{P}(R^{\Delta_1})$. For any $g \in \bar{\Delta}$, $g \circ \iota = \iota$, hence ι induces a map

$$KR^\Delta \xrightarrow{\iota'} (KR^{\Delta_1})^{h\bar{\Delta}}.$$

3.7. Proposition. *If $l > 2$, $\mathbf{L}KR^\Delta \xrightarrow{\approx} (\mathbf{L}KR^{\Delta_1})^{h\bar{\Delta}}$.*

Proof. It suffices to show that for all i, $K_i(R^\Delta) \xrightarrow{\sim} \pi_i(\mathbf{L}KR^{\Delta_1})^{h\bar{\Delta}}$. Since l is odd, $\bar{\Delta}$ has order prime to l and the target group is just $K_i(R^{\Delta_1})^{\bar{\Delta}}$. Let $\tau: KR^{\Delta_1} \to KR^\Delta$ be the K-theoretic transfer coming from the restriction of scalars $\tau: \mathscr{P}(RT^{\Delta_1}) \to \mathscr{P}(R^\Delta)$. By the projection formula, we have

$$\tau_* \circ \iota_* = \text{multiplication by } [R^{\Delta_1}]$$

and the class $[R^{\Delta_1}] \in K_0(R^\Delta) \otimes \mathbf{Z}_{(l)}$ is invertible, since its rank is prime to l and $\mathrm{Ker}(K_0(R^\Delta) \xrightarrow{rk} \mathbf{Z})$ is nilpotent. On the other hand,

$$\iota_* \circ \tau_* = \sum_{g \in \bar{\Delta}} g,$$

hence $\iota_* \circ \tau_*$ restricted to $K_*(R^{\Delta_1})^{\bar{\Delta}}$ induces multiplication by $|\bar{\Delta}|$. □

3.8. For any closed subgroup Δ of $1 + 2l\mathbf{Z}_l^*$, we still denote by

$$\Phi^\Delta : \Sigma(l, \Delta) \to \mathbf{L}KR^\Delta$$

the morphism of localized spectra induced by the functor Φ^Δ of equation (A6) via the Spt-construction (without abusing notation, one should write this morphism $\mathbf{L}\,\mathrm{Spt}(\Phi^\Delta)$.) Similarly, we simply denote by $\iota_{\Delta'/\Delta}$, $\tau_{\Delta'/\Delta}$, $\iota_{R^{\Delta'}/R^\Delta}$ and $\tau_{R^{\Delta'}/R^\Delta}$ the morphisms we should denote by $\mathrm{L}\,\mathrm{Spt}(\iota_{\Delta'/\Delta})$, etc. The morphisms Φ^Δ, $\iota_{\Delta'/\Delta}$ and $\iota_{R^{\Delta'}/R^\Delta}$ are multiplicative, unlike $\tau_{\Delta'/\Delta}$ (resp. $\tau_{R^{\Delta'}/R^\Delta}$) which satisfy the projection formula.

3.9. *Definition.* Suppose $l > 2$. For Δ an arbitrary closed subgroup of $\mathbf{Z}_l^*$, we define

$$\Phi^\Delta : \Sigma(l, \Delta) \to \mathbf{L}KR^\Delta$$

as the composite of $(\Phi^{\Delta_1})^{h\bar{\Delta}}$ (see 3.8) with the inverse of the equivalence of Proposition 3.7.

3.10. PROPOSITION. *The map Φ^Δ is multiplicative. With notation as in* 3.8, *the following diagrams are commutative*:

$$\begin{array}{ccc} \Sigma(l,\Delta') & \xrightarrow{\Phi^{\Delta'}} & \mathbf{L}KR^{\Delta'} \\ {\scriptstyle \iota_{\Delta'/\Delta}}\uparrow & & \uparrow{\scriptstyle \iota_{R^{\Delta}/R^{\Delta'}}} \\ \Sigma(l,\Delta) & \xrightarrow{\Phi^{\Delta}} & \mathbf{L}KR^{\Delta} \end{array} \qquad \begin{array}{ccc} \Sigma(l,\Delta') & \xrightarrow{\Phi^{\Delta'}} & \mathbf{L}KR^{\Delta'} \\ {\scriptstyle \tau_{\Delta'/\Delta}}\downarrow & & \downarrow{\scriptstyle \tau_{R^{\Delta'}/R^{\Delta}}} \\ \Sigma(l,\Delta) & \xrightarrow{\Phi^{\Delta}} & \mathbf{L}KR^{\Delta}. \end{array}$$

Proof. Multiplicativity in the case $\Delta \in 1 + 2l\mathbf{Z}_l$ follows from Eqs (A4); in general (for $l > 2$) it follows from this and the fact that the equivalence of Proposition 3.7 is multiplicative. For Δ, $\Delta' \subseteq 1 + 2l\mathbf{Z}_l$, the commutativity of the two diagrams in Proposition 3.10 follows from Propositions A.11 and A.13. If $l = 2$ the proof is finished, since Φ^Δ is only defined for such subgroups. If $l > 2$, consider the intermediate group $\Delta'' = \Delta'\Delta_1$. We have

$$\Delta_1'' = \Delta_1$$

$$\overline{\Delta''} = \overline{\Delta'}.$$

This allows us to reduce the proof to the following two special cases:

(i) $\overline{\Delta'} = \overline{\Delta}$;
(ii) $\Delta_1' = \Delta_1$.

Case (i) reduces to the case $\overline{\Delta'} = \overline{\Delta} = 1$ by taking holims relatively to $\overline{\Delta}$. Case (ii) is obvious in view of the definition of Φ^Δ and $\Phi^{\Delta'}$. □

4. THE SPECTRA $\mathrm{j}(l, \Delta)$ AND $\mathrm{J}(l, \Delta)$

4.1. *Bott elements.* Let $F_1 = \Sigma(l, 1 + 2l\mathbf{Z}_l^*)$. We define an element β in $\pi_2(F_1, \mathbf{Z}/l^*)$ as follows. Pick a generator ζ of $\mathbf{Q}_l/\mathbf{Z}_l(1)^{1+2l\mathbf{Z}_l} \simeq \mathbf{Z}/l^*$. Then β is the image of ζ by the composite

$$\mathbf{Z}/l^* =_{l^*} \pi_1(\mathbf{BZ}/l^*) = \pi_2(\mathbf{BZ}/l^*, \mathbf{Z}/l^*) \xrightarrow{\text{stabilisation}} \pi_2(F_1, \mathbf{Z}/l^*).$$

Suppose $l > 2$. Then $\bar{\Delta} = \mathbf{Z}_l^*/(1 + l\mathbf{Z}_l) \simeq (\mathbf{Z}/l)^*$ acts on β by scalar multiplication. Hence β^{l-1} is invariant under this action. By Proposition 3.7, it defines an element of $\pi_{2(l-1)}(\Sigma(l, \mathbf{Z}_l^*), \mathbf{Z}/l)$, which we still denote by β.

4.2. Set

$$F = \begin{cases} \Sigma(l, \mathbf{Z}_l^*) & \text{if } l > 2 \\ \Sigma(2, 1 + 4\mathbf{Z}_2) & \text{if } l = 2. \end{cases}$$

For $\beta \in \pi_*(F, \mathbf{Z}/l^*)$ as in 4.1, we can define a localised spectrum $\beta^{-1}X$ with an F-linear map $X \xrightarrow{l_\beta} \beta^{-1}X$ for any F-module X, as in B.22. This assignment is strictly natural for F-linear maps. Moreover, if one wishes, one can get a canonical β^{-1} functor depending only on the "l-adic isogeny class" of β; see B.26. By Proposition B.23 and Theorem B.24, it has the following properties:

(1) $X_{\mathbf{Q}} \xrightarrow{\sim} \beta^{-1}X_{\mathbf{Q}}$;

(2) $X \mapsto \beta^{-1}X$ commutes with arbitrary homotopy colimits;

(3) *Universal property*: (i) β acts as an equivalence on $\beta^{-1}X \wedge M(l^*)$. (ii) Let Y be an F-module and $f: X \to Y$ an F-linear morphism such that β acts as an equivalence on $Y \wedge M(l^*)$. Then there is a unique (up to homotopy) factorisation

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\ell_\beta"'] & Y \\ & \beta^{-1}X. \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

(4) $\beta^{-1}(X \wedge M(l^*))$ can be computed by (equation (B1)) mapping telescope.

4.3. In particular, $\mathbf{L}K\mathbf{Z}[1/l]$ if $l > 2$ and $\mathbf{L}K\mathbf{Z}[1/2, i]$ if $l = 2$ are F-modules by the map Φ^Δ of 3.8. Therefore, for any scheme $\mathscr{X}$ (over $\mathbf{Z}[i]$ if $l = 2$), $\mathbf{L}K\mathscr{X}$ is an F-module, and the localised spectrum $\beta^{-1}\mathbf{L}K\mathscr{X}$ makes sense. By 4.2(2) and [28, Proposition 7.2.2], $\mathscr{X} \mapsto \beta^{-1}\mathbf{L}K\mathscr{X}$ commutes with arbitrary filtered inverse limits of schemes with affine transition maps.

4.4. By the main result of [41], $\beta^{-1}\mathbf{L}K\mathscr{X} \wedge M(l^\nu)$ satisfies étale cohomological descent for reasonable $\mathscr{X}$.

The following lemma is quite classical; we include it for the convenience of the reader.

4.5. Lemma. *Let Δ be an open subgroup of $\mathbf{Z}_l^*$. If $l = 2$, suppose that Δ is procyclic, i.e. $-1 \notin \Delta$. Then there exists a finite field E of characteristic $\neq l$ such that $\Delta_E = \Delta$ (cf. 2.5).*

Proof. Notice that the hypothesis in the lemma is necessary, since $G_E \simeq \hat{\mathbf{Z}}$ is procyclic for any finite field E. Let u be a generator of Δ and let $\Delta_1 = 1 + l^n\mathbf{Z}_l$. By Dirichlet's arithmetic progression theorem, there exists a prime number p such that $p \equiv u \pmod{l^{n+1}}$. Let $E = \mathbf{F}_p$ and $\phi_E \in G_E$ be the Frobenius automorphism. Then $\kappa_l(\phi_E) = p \in \mathbf{Z}_l^*$. By assumption, u and p coincide modulo Δ_1^l. It follows that p also generates Δ. (This proof shows that there are infinitely many such E's, which can be chosen as prime fields.) □

4.6. Proposition. *Let Δ be an open subgroup of $\mathbf{Z}_l^*$ (contained in $1 + 4\mathbf{Z}_2$ if $l = 2$) and E be a finite field such that $\kappa_l(G_E) = \Delta$. Denote by Φ_E the composite*

$$\Sigma(l, \Delta) \xrightarrow{\Phi^\Delta} \mathbf{L}KR^\Delta \to \mathbf{L}KE$$

where the latter map is induced by the natural ring homomorphism $R^{\Delta} \to E$. Then the map

$$\beta^{-1}(\Sigma(l, \Delta) \wedge M(l^*)) \to \beta^{-1}(KE \wedge M(l^*))$$

induced by Φ_E is an equivalence.

Proof. Suppose first that $l = 2$ or $\Delta \subseteq 1 + l\mathbf{Z}_l$ (i.e. $\mu_l \subseteq E$). In this case, Snaith proves this in [32, II.1.10] by a calculation. The general case follows from this one by an analogue of Proposition 3.7.

4.7. Corollary. *$\beta^{-1}\Phi_E : \beta^{-1}\Sigma(l, \Delta) \to \beta^{-1}\mathbf{L}KE$ is an equivalence.*

This follows from Proposition 4.6 and the universal property of β^{-1}, noting that $\Sigma(l, \Delta)_{\mathbf{Q}} \xrightarrow{\approx} KE_{\mathbf{Q}} \approx H\mathbf{Q}$ by Quillen's computation of K_*E [27].

4.8. Lemma. *For any finite field E (containing $\sqrt{-1}$ if $l = 2$), $\mathbf{L}KE \to (\beta^{-1}\mathbf{L}KE)_{\geqslant 0}$ is an equivalence.*

Proof. It is enough to see this equivalence at $\mathbf{Q}$ and after smashing by $M(l^*)$. The first one is trivial, and the second follows from [3, Theorem 2.6], at least for $l > 2$. For $l = 2$, Browder only computes $K_*(E, \mathbf{Z}/2)$, but a similar computation shows that $\pi_*(E, \mathbf{Z}/4) = \Lambda(x) \otimes \mathbf{Z}/4[\beta]$, where β is the image of the Bott element $\beta \in Z(2, 1 + 4\mathbf{Z}_2)$ above. □

Corollary 4.7 and Lemma 4.8 motivate

4.9. *Definition.* Let Δ be an closed subgroup of $\mathbf{Z}_l^*$ (of $1 + 4\mathbf{Z}_2$ if $l = 2$). We define spectra $\mathsf{j}(l, \Delta)$ and $\mathsf{J}(l, \Delta)$ by

$$\mathsf{J}(l, \Delta) = \beta^{-1}\Sigma(l, \Delta), \qquad \mathsf{j}(l, \Delta) = \mathsf{J}(l, \Delta)_{\geqslant 0}.$$

They imply

4.10. Proposition. *With notation as in Proposition 4.6, the map Φ_E induces equivalences of ring spectra*

$$\beta_E : \mathsf{j}(l, \Delta) \xrightarrow{\approx} \mathbf{L}KE$$
$$\beta_E : \mathsf{J}(l, \Delta) \xrightarrow{\approx} \beta^{-1}\mathbf{L}KE.$$

As an immediate consequence we have the following, no doubt well-known to experts (compare [1, Ch. IX, Theorem 3.2]):

4.11. Corollary. *For a finite field E of characteristic $\neq l$, the ring spectrum $\mathbf{L}KE$ only depends on Δ_E, at least if $l > 2$ or $l = 2$ and $\Delta_E \subseteq 1 + 4\mathbf{Z}_2$.*

4.12. *Remark.* See Propositions C.13, Corollary C.14 and Proposition C.15 for an extension of these results to the case $l = 2$, $\Delta \not\subseteq 1 + 4\mathbf{Z}_2$.

4.13. By construction, $\mathsf{j}(l, \Delta)$ and $\mathsf{J}(l, \Delta)$ are homotopy commutative and associative ring spectra. If $\Delta' \subseteq \Delta$ is another closed subgroup of $\mathbf{Z}_l^*$, there are "restriction" maps $\iota : \mathsf{j}(l, \Delta) \to \mathsf{j}(l, \Delta')$, $\iota : \mathsf{J}(l, \Delta) \to \mathsf{J}(l, \Delta')$; if Δ' has finite index in Δ, there are transfer maps

$\tau:\mathrm{j}(l,\Delta')\to\mathrm{j}(l,\Delta)$, $\tau:\mathrm{J}(l,\Delta')\to\mathrm{J}(l,\Delta)$; all are included by the corresponding maps on $\Sigma(l,\Delta)$. By Propositions 3.10 and 4.10, they correspond to extension of scalars and transfer for the K-theory of finite fields.

4.14. LEMMA. *Let Δ be a closed subgroup of $\mathbf{Z}_l^*$ ($\Delta = 1$ if $l = 2$). Then*

$$\mathrm{j}(l,\Delta) = \operatorname*{hocolim}_U \mathrm{j}(l,U)$$
$$\mathrm{J}(l,\Delta) = \operatorname*{hocolim}_U \mathrm{J}(l,U)$$

where U runs through the open subgroups containing Δ and the homotopy colimits are taken relatively to the restriction maps of 4.13.

Proof. For $\mathrm{J}(l,\Delta)$, this follows from 4.2(2) and the same property for $\Sigma(l,\Delta)$ (Lemma 3.6); then it follows for $\mathrm{j}(l,\Delta)$ by truncation. □

The following theorem gives some insight into the spectra $\mathrm{j}(l,\Delta)$ and $\mathrm{J}(l,\Delta)$:

4.15. THEOREM. (a) *The homotopy groups of* $\mathrm{j}(l,\Delta)$ *and* $\mathrm{J}(l,\Delta)$ *are the following*:

$$\pi_{2i}(\mathrm{j}(l,\Delta)) = \pi_{2i}(\mathrm{J}(l,\Delta)) = 0 \quad \textit{for all } i\in\mathbf{Z}\textit{, except } \pi_0(\mathrm{j}(l,\Delta)) = \pi_0(\mathrm{J}(l,\Delta)) = \mathbf{Z}_{(l)}$$

$$\pi_{2i-1}(\mathrm{J}(l,\Delta)) = H^0(\Delta,\mathbf{Q}_l/\mathbf{Z}_l(i)) \textit{ for all } i\in\mathbf{Z}$$

$$\pi_{2i-1}(\mathrm{j}(l,\Delta)) = \begin{cases} H^0(\Delta,\mathbf{Q}_l/\mathbf{Z}_l(i) & \textit{if } i>0 \\ 0 & \textit{if } i\leqslant 0. \end{cases}$$

(b) *For l odd,* $\mathrm{j}(l,\mathbf{Z}_l^*)$ *is homotopy equivalent to the convective cover of the* "Im J" *spectrum.*

(c) *The natural morphism* $\mathrm{j}(l,\Delta)\to\mathbb{H}(\Delta,\mathrm{j}(l,1))_{\geqslant 0}$ *is an equivalence.*

In (c), $\mathbb{H}(\Delta,\mathrm{j}(l,1))$ is the hypercohomology spectrum of the profinite group Δ with values in the sheaf of spectra associated to the presheaf $U\mapsto\mathrm{j}(l,U)$ on the site associated to the category of open subgroups of Δ, as in [41, 1.33]. It should be thought of as homotopy fixed points of Δ on $\mathrm{j}(l,1)$, taking the topology of Δ into account.

Proof. For $\mathrm{j}(l,\Delta)$, (a) follows from Proposition 4.10 and [27]; for $\mathrm{J}(l,\Delta)$, it follows similarly from Proposition 4.10 and [23, (2.4)] (applied with $X = \operatorname{Spec} E$, with E as in Proposition 4.10). (b) also follows from Proposition 4.10. □

Finally, to prove (c), we use the spectral sequence [41, 1.36]

$$E_2^{p,q} = H^p(\Delta,\pi_q(\mathrm{j}(l,1))) \Rightarrow \pi_{q-p}(\mathbb{H}(\Delta,\mathrm{j}(l,1))).$$

Since $\Delta\simeq T$ or $\mathbf{Z}_l\times T$, where T is a finite group of order prime to l and $\pi_q(\mathrm{j}(l,1))$ is l-primary torsion for $q>0$, we have $E_2^{p,q}=0$ for $p>1$ and $q>0$ because the profinite group $\mathbf{Z}_l$ has cohomological dimension 1. By Tate's lemma [39, Lemma], one also has $E_2^{1,q}=0$ for all $q>0$. Finally, $E_2^{0,0}=\mathbf{Z}_{(l)}$ (we also have $E_2^{1,0}=H^1(\Delta,\mathbf{Z}_{(l)})=0$ and $E_2^{2,0}=H^2(\Delta,\mathbf{Z}_{(l)})=H^1(\Delta,\mathbf{Q}_l/\mathbf{Z}_l)$, but do not actually need this). The claim is now clear by comparing homotopy groups.

4.16. COROLLARY. *There is a canonical equivalence*

$$\mathrm{j}(l,1)\hat{} \approx \mathrm{bu}\hat{}$$

and homotopy fibre sequences

$$\mathrm{j}(l, 1) \to \mathrm{L\,bu} \to (\mathrm{bu}_{\mathbf{Q}})_{\geq 1}$$

$$\mathrm{j}(l, 1) \to \mathrm{bu}^{\wedge} \to (\mathrm{bu}^{\wedge}_{\mathbf{Q}})_{\geq 1}$$

where bu *is the connective cover of* KU.

Proof. Consider the natural homomorphism of spectra

$$\Sigma(l, 1) \xrightarrow{\theta} \mathrm{L\,bu}$$

induced by the inclusion $\mu_{l^\infty} \hookrightarrow U(1) \hookrightarrow U(\infty)$. Observing that $\beta^{-1}\mathrm{Lbu} = \mathrm{LKU}$ we get a morphism

$$\mathrm{J}(l, 1) \to \mathrm{LKU}$$

hence, by truncating, a morphism factoring θ

$$\mathrm{j}(l, 1) \xrightarrow{\bar{\theta}} \mathrm{Lbu}.$$

The composite $\mathrm{j}(l, 1) \xrightarrow{\bar{\theta}} \mathrm{L\,bu} \to (\mathrm{bu}_{\mathbf{Q}})_{\geq 1}$ is obviously null-homotopic, hence $\mathrm{j}(l, 1)$ maps to the homotopy fibre of $\mathrm{L\,bu} \to (\mathrm{bu}_{\mathbf{Q}})_{\geq 1}$; Theorem 4.15(a) implies that this map is a homotopy equivalence, hence Corollary 4.16 follows. □

4.17. *Definition.* For $\Delta \subseteq \mathbf{Z}_l^*$ ($\subseteq 1 + 4\mathbf{Z}_2$ for $l = 2$), we denote by ℓ^Δ the morphism of spectra making the diagram

$$\begin{tikzcd}
\Sigma(l, \Delta) \arrow[r, "\ell_\beta"] \arrow[d, "\ell^\Delta"'] & \mathrm{J}(l, \Delta) \\
\mathrm{j}(l, \Delta) \arrow[ur] &
\end{tikzcd}$$

commutative.

This collection of morphisms commutes with product, restriction and transfer in both theories. Moreover, for E as in Proposition 4.6, the diagram

$$\begin{tikzcd}
\Sigma(l, \Delta) \arrow[r, "\Phi_E"] \arrow[d, "\ell^\Delta"'] & LKE \\
\mathrm{j}(l, \Delta) \arrow[ur, "\beta_E"] &
\end{tikzcd}$$

is commutative.

4.18. Localising Φ^Δ, we get a morphism of ring spectra

$$\mathrm{J}(l, \Delta) \xrightarrow{\beta^\Delta} \beta^{-1}\mathrm{L}KR^\Delta$$

such that the diagram

$$\begin{tikzcd} \Sigma(l,\Delta) \arrow[r, "\Phi^\Delta"] \arrow[d, "\ell^\Delta"'] & \mathbf{L}KR^\Delta \arrow[d] \\ \mathrm{J}(l,\Delta) \arrow[r, "\beta^\Delta"] & \beta^{-1}\mathbf{L}KR^\Delta. \end{tikzcd}$$

is commutative. The collection of β^Δ commutes with product (by Proposition B.25(b)), scalar extension and transfer.

4.19. *Remark.* By a theorem of Snaith (and Zaldívar for $l = 2$), $\beta^{-1}\mathbf{L}K\mathscr{X} \approx A^{-1}\mathbf{L}K\mathscr{X}$ for any $\mathscr{X}$, where A is the Adams map, hence $\beta^{-1}\mathbf{L}K\mathscr{X} \approx L_1K\mathscr{X}$ by Example B.27. So one may wonder why deal with Bott localisation at all. However, $A^{-1}\Sigma(l,\Delta)$ is bigger than $\beta^{-1}\Sigma(l,\Delta)$, so applying the functor L_1 to the map Φ^Δ will not give the construction I want (for all these remarks, compare [23]).

The main expectation is the maps β^Δ of 4.18 are Bott localisations of morphisms of ring spectra

$$\mathrm{j}(l,\Delta) \xrightarrow{\beta^\Delta} \mathbf{L}KR^\Delta.$$

In the next section, we shall construct homotopy classes of such morphisms at the zero-space level.

5. THE HARRIS–SEGAL THEOREM AND THE DWYER–FRIEDLANDER–MITCHELL THEOREM

5.1. THEOREM. *Let ℓ^Δ be the morphism of Definition* 4.17. *If Δ is open, $\Omega^\infty\ell^\Delta$ has homotopy sections.*

Proof. By Proposition 4.10, ℓ^Δ is equivalent to the natural map $\Sigma(l,\Delta) \to \mathbf{L}KE$ for any finite field E of characteristic $\neq l$ such that $\Delta_E = \Delta$. On the connected components of 0 in $\Omega^\infty\Sigma(l,\Delta)$ and $\Omega^\infty\mathrm{j}(l,\Delta)$, the theorem follows from the Harris–Segal theorem [12; 7, Section 4]). On the factors $\mathbf{Z}_{(l)}$, we simply take the identity. □

5.2. THEOREM. *Suppose Δ open and $\subseteq 1 + 2l\mathbf{Z}_l$, and let $s:\Omega^\infty\mathrm{j}(l,\Delta) \to \Omega^\infty\Sigma(l,\Delta)$ be a section of $\Omega^\infty\ell^\Delta$. Then the maps $\Omega^\infty\Phi^\Delta$ and $\Omega^\infty\Phi^\Delta \circ s \circ \Omega^\infty\ell^\Delta$ are homotopic:*

$$\begin{tikzcd} \Omega^\infty\Sigma(l,\Delta) \arrow[r, "\Omega^\infty\Phi^\Delta"] \arrow[d, "\Omega^\infty l^\Delta"', shift right] & \Omega^\infty(\mathbf{L}KR^\Delta) \\ \Omega^\infty\mathrm{j}(l,\Delta). \arrow[u, bend right, "s"'] & \end{tikzcd}$$

Proof. Choose a finite field E of characteristic $\neq l$ such that $\Delta_E = \Delta$ (Lemma 4.5). If E is a prime field, it is quotient of R^Δ. As above, we reduce to the diagram

$$\begin{tikzcd} \Omega^\infty\Sigma(l,\Delta) \arrow[r, "\Omega^\infty\Phi^\Delta"] \arrow[d, "\Omega^\infty\Phi_E"', shift right] & \Omega^\infty(\mathbf{L}KR^\Delta) \\ \Omega^\infty(\mathbf{L}KE). \arrow[u, bend right, "s"'] & \end{tikzcd}$$

The claim follows this time from the Dwyer–Friedlander–Mitchell theorem [7, Theorem 4.1]. □

The following theorem is the main result of this section.

5.3. THEOREM. *Let Δ be as in Theorem 5.2 and s be a section of $\Omega^\infty\ell^\Delta$. Then the homotopy class of*

$$\Omega^\infty\Phi^\Delta \circ s : \Omega^\infty \mathrm{j}(l, \Delta) \to \Omega^\infty(\mathbf{L}KR^\Delta)$$

does not depend on s.

Proof. Let s' be another section. Then

$$\Omega^\infty\Phi^\Delta \circ s' \approx \Omega^\infty\Phi^\Delta \circ s \circ \Omega^\infty\ell^\Delta \circ s' \approx \Omega^\infty\Phi^\Delta \circ s. \qquad \square$$

5.4. *Definition.* We denote the (homotopy class of) map $\Omega^\infty\Phi^\Delta \circ s$ by $\bar\beta^\Delta$.

5.5. PROPOSITION. *Let $\Delta \subseteq \mathbf{Z}_l^*$ be an open subgroup and Δ_1, $\bar\Delta$ be as in 2.4. Then $\bar\beta^{\Delta_1}$ is $\bar\Delta$-equivariant (up to homotopy).*

Proof. Write $\bar\beta^\Delta = \Omega^\infty\Phi^\Delta \circ s$ for some s. Let $\sigma \in \bar\Delta$, considered as acting on $\Sigma(l, \Delta_1)$, KR^{Δ_1} and $\mathrm{j}(l, \Delta_1)$. Then σ commutes with Φ^{Δ_1} and ℓ^{Δ_1}. In particular, $\sigma^{-1} \circ s \circ \sigma$ is another section of $\Omega^\infty\ell^{\Delta_1}$. Therefore,

$$\beta^{\Delta_1} \circ \sigma = \Omega^\infty\Phi^\Delta \circ s \circ \sigma = \Omega^\infty\Phi^\Delta \circ \sigma \circ \sigma^{-1} \circ s \circ \sigma \approx \sigma \circ \Omega^\infty\Phi^\Delta \circ \sigma^{-1} \circ s \circ \sigma \approx \sigma \circ \beta^{\Delta_1}. \qquad \square$$

5.6. *Definition.* Suppose $l > 2$ or $\Delta \subseteq 1 + 4\mathbf{Z}_2$. We define $\bar\beta^\Delta \in [\Omega^\infty \mathrm{j}(l, \Delta), \Omega^\infty(\mathbf{L}KR^\Delta)]$ as the composite of

$$\Omega^\infty \mathrm{j}(l, \Delta) \xrightarrow{\approx} \Omega^\infty \mathrm{j}(l, \Delta_1)^{h\bar\Delta} \xrightarrow{\bar\beta^{\Delta_1}} \Omega^\infty(\mathbf{L}KR^{\Delta_1})^{h\bar\Delta}$$

and the inverse of the homotopy equivalence $\mathbf{L}KR^\Delta \to (\mathbf{L}KR^{\Delta_1})^{h\bar\Delta}$ of Proposition 3.7.

Note that $\bar\beta^\Delta$ is (contrary to β^Δ in 4.18) merely a homotopy class of maps. When writing diagrams involving it, we shall mean some representative of this class.

5.7. PROPOSITION. *In all cases, the diagram*

$$\begin{tikzcd}
\Omega^\infty\Sigma(l, \Delta) \arrow[r, "\Omega^\infty\Phi^\Delta"] \arrow[dr, "\Omega^\infty l^\Delta"'] & \Omega^\infty(\mathbf{L}KR^\Delta) \\
 & \Omega^\infty \mathrm{j}(l, \Delta) \arrow[u, "\bar\beta^\Delta"']
\end{tikzcd}$$

is homotopy commutative.

Proof. In the case $\Delta \subseteq 1 + 2l\mathbf{Z}_l$, this follows from Theorem 5.2 and the definition of $\bar\beta^\Delta$. In general, we observe that, by the definition of $\bar\beta^\Delta$ in Definition 5.6, we have to

$\iota_{R^{\Delta_1}/R^\Delta} \circ \bar{\beta}^\Delta \approx \bar{\beta}^{\Delta_1} \iota_{\Delta_1/\Delta}$; on the other hand, we have seen in Section 3 that Φ and ℓ also commute with $\iota_{\Delta_1/\Delta}$. Therefore,

$$\iota_{R^{\Delta_1}/R^\Delta}\bar{\beta}^\Delta \Omega^\infty \ell^\Delta \approx \bar{\beta}^{\Delta_1} \iota_{\Delta_1/\Delta} \Omega^\infty \ell^\Delta \approx \bar{\beta}^{\Delta_1} \Omega^\infty \ell^{\Delta_1} \iota_{\Delta_1/\Delta} \approx \Omega^\infty \Phi^{\Delta_1} \iota_{\Delta_1/\Delta} \approx \iota_{\Delta_1/\Delta} \Omega^\infty \Phi^\Delta.$$

Since $\iota_{R^{\Delta_1}/R^\Delta}$ is a retraction, this yields the result. □

5.8. Corollary. *For $l > 2$, Theorem 5.2 holds for any Δ open in $\mathbf{Z}_l^*$; we have*

$$\bar{\beta}^\Delta \approx \Omega^\infty \Phi^\Delta \circ s$$

for any section s of $\Omega^\infty \ell^\Delta$.

Proof. Let s be such a section. For the first claim,

$$\Omega^\infty \Phi^\Delta \circ s \circ \Omega^\infty \ell^\Delta \approx \bar{\beta}^\Delta \circ \Omega^\infty \ell^\Delta \circ s \circ \Omega^\infty \ell^\Delta \approx \bar{\beta}^\Delta \circ \Omega^\infty \ell^\Delta \approx \Omega^\infty \Phi^\Delta.$$

The second claim follows trivially from Proposition 5.7. □

5.9. Proposition. *For all open Δ, the diagram*

$$\begin{tikzcd} \Omega^\infty \mathrm{j}(l, \Delta) \arrow[r, "\bar{\beta}^\Delta"] \arrow[d] & \Omega^\infty \mathbf{L}KR^\Delta \arrow[d] \\ \Omega^\infty \mathrm{J}(l, \Delta) \arrow[r, "\Omega^\infty \beta^\Delta"] & \Omega^\infty \beta^{-1} \mathbf{L}KR^\Delta \end{tikzcd}$$

is homotopy commutative.

Proof. In the diagram

$$\begin{tikzcd} \Omega^\infty \Sigma(l, \Delta) \arrow[d, "\Omega^\infty l^\Delta"'] \arrow[dr, "\Omega^\infty \Phi^\Delta"] & \\ \Omega^\infty \mathrm{j}(l, \Delta) \arrow[r, "\bar{\beta}^\Delta"] \arrow[d] & \Omega^\infty \mathbf{L}KR^\Delta \arrow[d] \\ \Omega^\infty \mathrm{j}(l, \Delta) \arrow[r, "\Omega^\infty \beta^\Delta"] & \Omega^\infty \beta^{-1} \mathbf{L}KR^\Delta \end{tikzcd}$$

the big pentagon is commutative, since $\beta^\Delta = \beta^{-1}\Phi^\Delta$. We deduce Proposition 5.9 from this fact by using a section of $\Omega^\infty \ell^\Delta$ and Theorem 5.2, as before. □

5.10. Let Δ be a closed subgroup of $\mathbf{Z}_l^*$ of *infinite* index; if $l = 2$ we restrict to $\Delta = 1$. Let U run through the open subgroups of $\mathbf{Z}_l^*$ containing Δ ($U \subseteq 1 + 4\mathbf{Z}_2$ if $l = 2$). Taking the homotopy colimit of the homotopy classes

$$\Omega^\infty \mathrm{j}(l, U) \xrightarrow{\bar{\beta}^U} \Omega^\infty \mathbf{L}KR^U$$

and taking Lemma 4.14 and [28, Proposition 7.2.2] into account, we get a homotopy class

$$\Omega^\infty \mathrm{j}(l, \Delta) \xrightarrow{\bar{\beta}^\Delta} \Omega^\infty \mathbf{L}KR^\Delta.$$

5.11. PROPOSITION (compare [7, Theorem 4.13]). *For any U as in 5.10, the diagram*

$$\begin{tikzcd}
\Omega^\infty\Sigma(l, U) \arrow[r] & \Omega^\infty\Sigma(l, \Delta) \arrow[r, "\Omega^\infty\Phi^\Delta"] \arrow[dr, "\Omega^\infty l^\Delta"'] & \Omega^\infty \mathbf{L}KR^\Delta \\
& & \Omega^\infty \mathrm{j}(l, \Delta) \arrow[u, "\bar\beta^\Delta"']
\end{tikzcd}$$

is homotopy commutative.

Proof. This follows from the definition of $\bar\beta^\Delta$ and Proposition 5.7. □

5.12 *Remark.* We do not know if the diagram in Proposition 5.11 commutes if one removes the term $\Omega^\infty\Sigma(l, U)$, and similarly we do not know if Proposition 5.9 holds for arbitrary Δ (cf. [7, end of proof of Theorem 4.13]). The problem is that, in the Milnor exact sequences

$$\begin{aligned}0 &\to \varprojlim{}^1 [\Omega^\infty\Sigma(l, U), \Omega^{\infty+1}\mathrm{j}(l, \Delta)] \to [\Omega^\infty\Sigma(l, \Delta), \Omega^\infty\mathrm{j}(l, \Delta)]\\ &\to \varprojlim [\Omega^\infty\Sigma(l, U), \Omega^\infty\mathrm{j}(l, \Delta)] \to 0\end{aligned}$$

and

$$\begin{aligned}0 &\to \varprojlim{}^1 [\Omega^\infty\Sigma(l, U), \Omega^{\infty+1}\beta^{-1}\mathbf{L}KR^\Delta]_1 \to [\Omega^\infty\Sigma(l, \Delta), \Omega^\infty\beta^{-1}\mathbf{L}KR^\Delta]\\ &\to \varprojlim [\Omega^\infty\Sigma(l, U), \Omega^\infty\beta^{-1}\mathbf{L}KR^\Delta] \to 0\end{aligned}$$

the $\varprojlim^1$ term may be nonzero.

6. MAIN PROPERTIES OF $\bar\beta^\Delta$

In [7], it is conjectured that a map λ_p equivalent to $\bar\beta^\Delta$ is an infinite loop map (we shall come back to this conjecture in Appendix D) and proved as a token that it commutes with the Dyer–Lashof operations (*op. cit.*, Proposition 4.9). In the same spirit, we show that $\bar\beta^\Delta$ has a number of very nice properties.

6.1. THEOREM. *If Δ is open, $\bar\beta^\Delta$ commutes with products.*

Proof. Here, as in further proofs of this section, we sometimes drop the symbols Ω^∞ and $\mathbf{L}$ for simplicity. Moreover, we write $=$ rather than $\approx$, working in the homotopy category. For any Δ, denote by π^Δ the product on $\mathrm{j}(l, \Delta)$ and, for any ring A, denote by π_A the product on the algebraic K-theory of A. We have to prove that the diagram

$$\begin{tikzcd}
\Omega^\infty\mathrm{j}(l, \Delta) \wedge \Omega^\infty\mathrm{j}(l, \Delta) \arrow[r, "\bar\beta^\Delta \wedge \bar\beta^\Delta"] \arrow[d, "\pi^\Delta"'] & \Omega^\infty\mathbf{L}KR^\Delta \wedge \Omega^\infty\mathbf{L}KR^\Delta \arrow[d, "\pi_{R^\Delta}"'] \\
\Omega^\infty\mathrm{j}(l, \Delta) \arrow[r, "\bar\beta^\Delta"] & \Omega^\infty\mathbf{L}KR^\Delta
\end{tikzcd}$$

commutes up to homotopy.

Let s be a section of $\Omega^\infty\ell^\Delta$. Observing that Φ^Δ commutes with product, we get

$$\begin{aligned}\pi_{R^\Delta}(\bar\beta^\Delta \wedge \bar\beta^\Delta) &= \pi_{R^\Delta}(\Phi^\Delta s \wedge \Phi^\Delta s) = \pi_{R^\Delta}(\Phi^\Delta \wedge \Phi^\Delta)(s \wedge s)\\ &= \Phi^\Delta\pi^\Delta(s \wedge s) = \bar\beta^\Delta\ell^\Delta\pi^\Delta(s \wedge s) = \bar\beta^\Delta\pi^\Delta(\ell^\Delta \wedge l^\Delta)(s \wedge s)\\ &= \bar\beta^\Delta\pi^\Delta(\ell^\Delta s \wedge \ell^\Delta s) = \bar\beta^\Delta\pi^\Delta.\end{aligned}$$

(We used Corollary 5.8.) □

6.2. THEOREM. *Let $\Delta' \subseteq \Delta$ be two closed subgroups of $\mathbf{Z}_l^*$ (contained in $1 + 4\mathbf{Z}_2$ if $l = 2$). Then the diagram*

$$\begin{array}{ccc} \Omega^\infty \mathrm{j}(l, \Delta') & \xrightarrow{\bar\beta^{\Delta'}} & \Omega^\infty \mathbf{L}KR^{\Delta'} \\ {\scriptstyle \iota_{\Delta'/\Delta}}\uparrow & & {\scriptstyle \iota_{R^{\Delta'}/R^\Delta}}\uparrow \\ \Omega^\infty \mathrm{j}(l, \Delta) & \xrightarrow{\bar\beta^{\Delta}} & \Omega^\infty \mathbf{L}KR^{\Delta} \end{array}$$

commutes up to homotopy. If Δ' is open in Δ the same holds for the diagram

$$\begin{array}{ccc} \Omega^\infty \mathrm{j}(l, \Delta') & \xrightarrow{\bar\beta^{\Delta'}} & \Omega^\infty \mathbf{L}KR^{\Delta'} \\ {\scriptstyle \tau_{\Delta'/\Delta}}\downarrow & & {\scriptstyle \tau_{R^{\Delta'}/R^\Delta}}\downarrow \\ \Omega^\infty \mathrm{j}(l, \Delta) & \xrightarrow{\bar\beta^{\Delta}} & \Omega^\infty \mathbf{L}KR^{\Delta}. \end{array}$$

Proof. First assume that both Δ and Δ' are open in $\mathbf{Z}_l^*$. In the case of restriction, let s be a section of $\Omega^\infty \ell^\Delta$. Then

$$\iota_{R^{\Delta'}/R^\Delta}\bar\beta^\Delta = \iota_{R^{\Delta'}/R^\Delta}\Phi^\Delta s = \Phi^{\Delta'}\iota s = \bar\beta^{\Delta'}\ell^{\Delta'}\iota s = \bar\beta^{\Delta'}\iota_{\Delta'/\Delta}\ell^\Delta s = \bar\beta^{\Delta'}\iota_{\Delta'/\Delta}.$$

In the case of the transfer, let s' be a section of $\Omega^\infty \ell^{\Delta'}$. Then

$$\tau_{R^{\Delta'}/R^\Delta}\bar\beta^{\Delta'} = \tau_{R^{\Delta'}/R^\Delta}\Phi^{\Delta'} s' = \Phi^\Delta \tau s' = \bar\beta^\Delta \ell^\Delta \tau s' = \bar\beta^\Delta \tau_{\Delta'/\Delta}\ell^{\Delta'} s' = \bar\beta^\Delta \tau_{\Delta'/\Delta}.$$

For the first diagram, the case where Δ is open and Δ' has infinite index follows from the definition of $\bar\beta^\Delta$ given at the end of the last section. When both Δ and Δ' have infinite index, we note that either they are equal and the theorem is trivial for both diagrams, or $l > 2$ and they are both finite, contained in $\mu_{l-1} \subseteq \mathbf{Z}_l^*$. Since this group has order prime to l, an easy argument shows that they are the homotopy fixed points of $\bar\beta^{\{1\}}$ under Δ and Δ', via the analogue of Proposition 3.7. The commutativity of both diagrams then follows trivially. □

The next proposition is a trivial consequence of the formula $\Omega^\infty \mathbf{L}\Phi^\Delta = \bar\beta^\Delta \Omega^\infty \ell^\Delta$.

6.3. PROPOSITION. *Let $B^\Delta : \mu_{l^\nu} \to K_2(R^\Delta, \mathbf{Z}/l^\nu)$ be the composite*

$$\mu_{l^\nu} =_{l^\nu}\pi_1(\mathbf{B}\mu) = \pi_2(\mathbf{B}\mu, \mathbf{Z}/l^\nu) \to \pi_2(Q_0(\mathbf{B}\mu_+), \mathbf{Z}/l^\nu) \xrightarrow{\pi_2(\Phi^\Delta, \mathbf{Z}/l^\nu)} K_2(R^\Delta, \mathbf{Z}/l^\nu).$$

Then, one has $B^\Delta = \pi_2(\bar\beta^\Delta, \mathbf{Z}/l^\nu)$, with the identification $\pi_2(\mathrm{j}(l, \Delta), \mathbf{Z}/l^\nu) = H^0(\Delta, \mathbf{Z}/l^\nu(1))$ from Theorem 4.15(a).

7. THE SPECTRA $\Sigma(l, X)$, $\mathrm{j}_l(X)$ AND $\mathrm{J}_l(X)$

Throughout this section, X denotes a scheme over $\operatorname{Spec} \mathbf{Z}[1/l]$ (over $\operatorname{Spec} \mathbf{Z}[1/l, i]$ if $l = 2$).

7.1. Let X be connected and Δ_X be as in 2.5. There is a tautological morphism

$$X \to \operatorname{Spec} R^{\Delta_X},$$

hence a morphism of ring spectra $\Sigma(l, \Delta_X) \xrightarrow{\Phi_X} \mathbf{L}KX$, obtained by composing Φ^{Δ_X} with the natural morphism $\mathbf{L}KR^{\Delta_X} \to \mathbf{L}KX$.

7.2. *Definition.* (a) Let $f\colon Y \to X$ be a morphism of connected schemes over $\operatorname{Spec} \mathbf{Z}[1/l]$. f is *l-cyclotomic* if the diagram

$$\begin{array}{ccc} Y & \longrightarrow & \operatorname{Spec} R^{\Delta_Y} \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & \operatorname{Spec} R^{\Delta_X} \end{array}$$

is cartesian.

(b) A morphism of (arbitrary) schemes over $\operatorname{Spec} \mathbf{Z}[1/l]$ is *l-cyclotomic* if all its components are.

7.3. *Definition.* Let X be an arbitrary scheme over $\operatorname{Spec} \mathbf{Z}[1/l]$, and $(X_i)_{i\in I}$ its connected components. We define three ring spectra $\Sigma(l, X)$, $\mathsf{j}_l(X)$ and $\mathsf{J}_l(X)$ as follows:

$$\Sigma(l, X) = \bigvee_{i\in I} \Sigma(l, \Delta_{X_i})$$

$$\mathsf{j}_l(X) = \bigvee_{i\in I} \mathsf{j}(l, \Delta_{X_i})$$

$$\mathsf{J}_l(X) = \bigvee_{i\in I} \mathsf{J}(l, \Delta_{X_i}).$$

We define a morphism of ring spectra Φ_X:

$$\Sigma(l, X) \xrightarrow{\Phi_X} \mathbf{L}KX$$

by taking the wedge over I of the morphisms of 7.1.

By Theorem 4.15, the homotopy groups of $\mathsf{j}_l(X)$ and $\mathsf{J}_l(X)$ are as follows:

$$\begin{aligned} &\pi_{2i}(\mathsf{j}_l(X)) = \pi_{2i}(\mathsf{J}_l(X)) = 0 \quad \text{for all } i\in\mathbf{Z}, \text{ except } \pi_0(\mathsf{J}_l(l,\Delta)) = \mathbf{Z}_{(l)} \\ &\pi_{2i-1}(\mathsf{J}_l(X)) = H^0(X_{\text{ét}}, \mathbf{Q}_l/\mathbf{Z}_l(i)) \quad \text{for all } i\in\mathbf{Z} \\ &\pi_{2i-1}(\mathsf{j}_l(X)) = \begin{cases} H^0(X_{\text{ét}}, \mathbf{Q}_l/\mathbf{Z}_l(i)) & \text{if } i>0 \\ 0 & \text{if } i\leqslant 0. \end{cases} \end{aligned} \tag{4}$$

7.4. *Definition.* Let $Y \xrightarrow{f} X$ be a morphism.

(a) We define a morphism of ring spectra $\Sigma(l, X) \xrightarrow{f^*} \Sigma(l, Y)$ as follows:

- (i) If X and Y are connected and f is l-cyclotomic, $f^*\colon \Sigma(l, \Delta_X) \to \Sigma(l, \Delta_Y)$ is given by the morphism $\iota_{\Delta_Y/\Delta_X}$ of 3.8.
- (ii) If X and Y are connected and $\Delta_X = \Delta_Y$, f^* is the identity.
- (iii) If X and Y are connected, there is a unique Z such that f factors as $Y \xrightarrow{h} Z \xrightarrow{g} X$, with g as in (i) and h as in (ii). We define $f^* = h^* \circ g^*$.
- (iv) In general, f^* is given on every connected component of X by (iii).

(b) Suppose f is finite and flat. We define a morphism of spectra $\Sigma(l, Y) \xrightarrow{f_*} \Sigma(l, X)$ as follows:

- (i) If X and Y are connected and f is l-cyclotomic, $f_*\colon \Sigma(l, \Delta_Y) \to \Sigma(l, \Delta_X)$ is given by the morphism τ_{Δ_Y/Δ_X} of 3.8.
- (ii) If X and Y are connected and $\Delta_X = \Delta_Y$, f_* is multiplication by $\deg f$.
- (iii) If X and Y are connected, there is a unique Z such that f factors as $Y \xrightarrow{h} Z \xrightarrow{g} X$, with g as in (i) and h as in (ii). We define $f_* = g_* \circ h_*$.
- (iv) In general, f_* is given on every connected component of Y by (iii).

There are analogous definitions for j_l and J_l.

7.5. PROPOSITION. *Let $Y \xrightarrow{f} X$ be a morphism.*

(a) *The diagram*

$$\begin{array}{ccc} \Sigma(l, Y) & \xrightarrow{\Phi_Y} & \mathbf{L}KY \\ {\scriptstyle f^*}\uparrow & & {\scriptstyle f^*}\uparrow \\ \Sigma(l, X) & \xrightarrow{\Phi_X} & \mathbf{L}KX \end{array}$$

is commutative.

(b) *Assume f is finite and flat. Then the diagram*

$$\begin{array}{ccc} \Sigma(l, Y) & \xrightarrow{\Phi_Y} & \mathbf{L}KY \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ \Sigma(l, X) & \xrightarrow{\Phi_X} & \mathbf{L}KX \end{array}$$

commutes in the following cases: X is semi-local or f is l-cyclotomic.

Proof. The claim for f^* in general and f_* in the cyclotomic case follow from Proposition 3.10. In the semi-local case, we reduce to X, Y connected and $\Delta_X = \Delta_Y$. Let $n = \deg f$. It is enough to see that the diagram

$$\begin{array}{ccc} KR^{\Delta_X} & \longrightarrow & KY \\ {\scriptstyle n}\downarrow & & {\scriptstyle f_*}\downarrow \\ KR^{\Delta_X} & \longrightarrow & KX \end{array}$$

is commutative. But this follows from the projection formula in algebraic K-theory for proper morphisms with finite Tor dimension [28, Proposition 7.2.10]:

$$f_*f^* = \text{smash product by the class of } f_*\mathcal{O}_Y$$

since, X being semi-local, $f_*\mathcal{O}_Y$ is a free $\mathcal{O}_X$-module of rank n.

7.6. *Definition.* Let $(X_i)_{i \in I}$ be the connected component of the scheme X, and $\Delta_i = \Delta_{X_i}$ for all i. We define a morphism $\beta_X : \mathsf{J}_l(X) \to \beta^{-1}\mathbf{L}KX$ as the composite

$$\mathsf{J}_l(X) = \bigvee \mathsf{J}_l(l, \Delta_i) \xrightarrow{\vee \beta^{\Delta_i}} \bigvee \beta^{-1}\mathbf{L}KR^{\Delta_i} \to \bigvee \beta^{-1}\mathbf{L}KX_i = \beta^{-1}\mathbf{L}KX.$$

7.7. PROPOSITION. *The collection of β_X commutes with products, pull-backs and with transfer in the same conditions as in Proposition* 7.5(b). *Moreover the diagram*

$$\begin{array}{ccc} \Sigma(l, X) & \xrightarrow{\Phi_X} & \mathbf{L}KX \\ \downarrow & & \downarrow \\ \mathsf{J}_l(X) & \xrightarrow{\beta_X} & \beta^{-1}\mathbf{L}KX \end{array}$$

is commutative.

Proof. This follows from 4.18 and (for transfer) the argument in the proof of Proposition 7.5.

8. RIGIDITY

Let R a strict Hensel local ring and l a prime number invertible in l. The aim of this section is to describe well-determined equivalences

$$KR\hat{} \xrightarrow{\approx} \mathrm{bu}\hat{} \qquad (5)$$

following the theorems of Quillen, Suslin and Gabber.

(1) *The case* $R = \mathbf{C}$. By Suslin's theorem [37], the natural morphism $K\mathbf{C} \to \mathrm{bu}$ in an equivalence after l-completion. This is (5).

(2) *The case where R is an algebraically closed field of characteristic* 0. By Suslin's theorem [36], the two maps

$$\begin{tikzcd} & KR \\ K\bar{\mathbf{Q}} \arrow[ur] \arrow[dr] & \\ & K\mathbf{C} \end{tikzcd}$$

are equivalences after l-completion. The equivalence (5) is obtained from these two equivalences and (1).

(3) *The case* $R = \bar{\mathbf{F}}_p$ $(p \neq l)$. Let q run through the powers of p. By Quillen's theorem [27], Brauer lifting induces equivalences of spectra

$$K\mathbf{F}_q \to F\Psi^q[1/p]$$

where $F\Psi^q[1/p]$ is the homotopy fibre of $\Psi^q - 1 : \mathrm{bu}[1/p] \to \mathrm{bu}[1/p]$, ($\Psi^q$ being the qth Adams operation). Passing to the limit, these equivalences induce the desired equivalence (5).

(4) *The case where R is a separably closed field of characteristic p.* As in (2), using (3) and applying [36] to the inclusion $\bar{\mathbf{F}}_p \to R$.

(5) *The general case.* Let F be the residue field of R. By Gabber's theorem [10], the map $KR \to KF$ is an equivalence after l-completion. (5) is obtained from this and (2) or (4), according to the characteristic of F.

Let X be a scheme over $\mathbf{Z}[1/l]$ and x a geometric point of X. To x is associated a morphism

$$KX\hat{} \xrightarrow{e_x} \mathrm{bu}\hat{}$$

by composing $KX\hat{} \to (K\mathcal{O}^{\mathrm{sh}}_{X,x})\hat{}$ with (5) applied to $R = \mathcal{O}^{\mathrm{sh}}_{X,x}$.

8.1. Proposition. *If X is connected and essentially of finite type over a field or a Dedekind domain, the morphism* e_x *does not depend on the choice of x up to homotopy.*

Proof. Since X is noetherian and catenary [19, Corollary 2 to Theorem 31.7], it suffices to show that, if y is a specialisation of x of codimension 1, we have $e_x = e_y$. Without loss of generality, we may assume that X is strictly local at y, or in other words, $X = \operatorname{Spec} R$ with R a strict Hensel local ring with maximal ideal y. We may even assume that R is a domain with $x = (0)$, and (up to passing to its integral closure in its quotient field) that it is integrally closed. Then R is a henselian discrete valuation ring. Let E be its field of fractions and F its residue field. We want to show that the diagram

$$\begin{tikzcd} & KE^\wedge \arrow[dr] & \\ KR^\wedge \arrow[ur] \arrow[dr] & & \mathrm{bu}^\wedge \\ & KF^\wedge \arrow[ur] & \end{tikzcd}$$

is homotopy commutative. First suppose that R has equal characteristic, hence contains a separably closed field k. Then, by definition of the equivalences, the diagram

$$\begin{tikzcd} & & KE^\wedge \arrow[dr] & \\ Kk^\wedge \arrow[r] & KR^\wedge \arrow[ur] \arrow[dr] & & \mathrm{bu}^\wedge \\ & & KF^\wedge \arrow[ur] & \end{tikzcd}$$

is homotopy commutative, and so is the former one since $Kk^\wedge \to KR^\wedge$ is an equivalence. Suppose now that R has unequal characteristic. Let $p = \mathrm{char}(F)$. Then the strict henselisation R_0 of $\mathbf{Z}$ at p is contained in R. We have a diagram

$$\begin{tikzcd} & K\bar{\mathbf{Q}}^\wedge \arrow[r] & KE^\wedge \arrow[dr] & \\ KR_0^\wedge \arrow[ur] \arrow[r] \arrow[dr] & & KR^\wedge \arrow[u] \arrow[d] & \mathrm{bu}^\wedge \\ & K\bar{\mathbf{F}}_p^\wedge \arrow[r] & KF^\wedge \arrow[ur] & \end{tikzcd}$$

in which all maps are equivalences, and the part excluding $\mathrm{bu}^\wedge$ commutes; this reduces us to the case $R = R_0$. But then the commutativity follows from the definition of the equivalence $k\bar{\mathbf{F}}_p^\wedge \to \mathrm{bu}^\wedge$ given above. □

9. EXTENSIONS OF β TO GENERAL SCHEMES

9.1. Let X be a scheme over $\mathbf{Z}[1/l]$ (over $\mathbf{Z}[\frac{1}{2}, i]$ if $l = 2$). We define a homotopy class of map

$$\Omega^\infty \mathrm{j}_l(X) \xrightarrow{\beta_X} \Omega^\infty \mathbf{L}KX$$

as the composite

$$\Omega^\infty \mathrm{j}_l(X) = \bigvee \Omega^\infty \mathrm{j}_l(l, \Delta_i) \xrightarrow{\vee \bar\beta^{\Delta_i}} \bigvee \Omega^\infty \mathbf{L}KR^{\Delta_i} \to \Omega^\infty \mathbf{L}KX$$

where the X_i are the connected components of X, as in the former section. We have the following generalisation of Proposition 5.9:

9.2. Proposition. *If X is essentially of finite type over $\mathbf{Z}$, the diagram*

$$\begin{array}{ccc} \Omega^\infty \mathrm{j}_l(X) & \xrightarrow{\bar\beta_X} & \Omega^\infty \mathbf{L}KX \\ \downarrow & & \downarrow \\ \Omega^\infty \mathrm{J}_l(X) & \xrightarrow{\Omega^\infty \beta_X} & \Omega^\infty \beta^{-1}\mathbf{L}KX \end{array}$$

is homotopy commutative.

Proof. The hypothesis on X implies that Δ_{X_i} is open in $\mathbf{Z}_l^*$ for all connected components X_i of X, and the result follows immediately from Proposition 5.9. □

The following theorem is the main result of this paper.

9.3. Theorem. *For l, X as in* 9.1, *the map $\bar\beta_X$ has the following properties:*

(o) *$\bar\beta_X$ commutes with base change;*
(i) *if X is essentially of finite type over $\mathbf{Z}$, $\bar\beta_X$ commutes with products;*
(ii) *$\pi_0(\bar\beta_X)$ maps 1 to the class of $\mathcal{O}_X$; $\pi_1(\bar\beta_X)$ is the classical Bott element construction.*
(iv) *let $f: Y \to X$ be a finite flat morphism. Then $\bar\beta_X$ and $\bar\beta_Y$ commute with f_* in both theories in the following two cases: f is l-cyclotomic or X is semi-local.*
(v) *If X is strictly local, there is a map of spectra $\beta_X: \mathrm{j}_l(X) \to \mathbf{L}KX$ such that the diagram*

$$\begin{array}{ccc} \mathrm{j}_l(X) & \xrightarrow{\beta_X} & \mathbf{L}KX \\ \downarrow & & \downarrow \\ \mathrm{J}_l(X) & \xrightarrow{\beta_X} & \beta^{-1}\mathbf{L}KX \end{array}$$

commutes, and $\Omega^\infty \beta_X \wedge 1_{M(l^\nu)}$ coincides with $\bar\beta_X \wedge 1_{M(l^\nu)}$ for all ν. These maps are equivalences. In particular, β_X is the inverse of (5) *after l-completion.*

Proof. Properties (0), (i) and (iv) follow from Theorems 6.1 and 6.2. Property (ii) for $\pi_0(\bar\beta_X)$ follows from the definition of $\bar\beta^{\Delta_{X_i}}$, and for $\pi_1(\bar\beta_X)$ it follows from Proposition 6.3.

It remains to prove (v). To define $\beta_X: \mathrm{j}_l(X) \to \mathbf{L}KX$, we simply truncate $\mathrm{J}_l(X) \xrightarrow{\beta_X} \beta^{-1}\mathbf{L}KX$ by observing that $\mathbf{L}KX \xrightarrow{\approx} (\beta^{-1}\mathbf{L}KX)_{\geq 0}$ by Suslin's and Gabber's theorems [37, 10]. The same references show that β_X is an equivalence after completion. Finally, for the fact that $\Omega^\infty \beta_X = \bar\beta_X$, we need the following lemma:

9.4. Lemma. *Let (S_α) be a filtering direct system of spaces and T a spectrum. Assume that $\pi_i(T)$ is finite for all i. Then the natural map* $[\operatorname{hocolim} S_\alpha, T] \to \varprojlim [S_\alpha, T]$ *is bijective.*

Proof. (a) By Sullivan's theory ([35], see also [24]), the sets $[S_\alpha, \Omega T]$ have a natural profinite structure and the transition maps are continuous. It follows that $\varprojlim^1 [S_\alpha, \Omega T] = 0$. The lemma then follows from the Milnor exact sequence

$$0 \to \varprojlim{}^1 [S_\alpha, \Omega T] \to [\operatorname{hocolim} S_\alpha, T] \to \varprojlim [S_\alpha, T] \to 0.$$ □

Let $X = \operatorname{Spec} A$; write A as a union $\bigcup A_\alpha$ of its finitely generated subrings. By Lemma 9.4, the map

$$[\Omega^\infty \mathbf{j}_l(A) \wedge M(l^\nu), \Omega^\infty \mathbf{L}KA \wedge M(l^\nu)] \to \varprojlim [\Omega^\infty \mathbf{j}_l(A_\alpha) \wedge M(l^\nu), \Omega^\infty \mathbf{L}KA \wedge M(l^\nu)]$$

is bijective. The equality now follows from Proposition 9.2. □

9.5. Corollary. (Compare [15, Conjecture 1].) *Let l, X be as in 9.1. There exist homomorphisms*

$$\beta_X^i : H^0(X_{\text{ét}}, \mathbf{Z}/l^\nu(i)) \to K_{2i}(X, \mathbf{Z}/l^\nu), \quad i, \nu \geqslant 0,$$

with the following properties:

(o) *β_X^i commutes with base change;*
(i) *β_X^i commutes with products;*
(ii) *For $i = 0$, β_X^i maps 1 to the class of $\mathcal{O}_X$; for $i = 1$, β_X^i is the classical Bott element construction;*
(iii) *β_X^i commutes with change of coefficients;*
(iv) *Let $f: Y \to X$ be a finite morphism. Then β_X^i and β_Y^i commute with direct image in cohomology and transfer in K-theory in the following to cases: f is l-cyclotomic or X is semi-local;*
(v) *β_X^i is a section of the natural map $\mathrm{ch}_{i,0} : K_{2i}(X, \mathbf{Z}/l^\nu) \to H^0(X_{\text{ét}}, \mathcal{K}_{2i}(\mathbf{Z}/l^\nu)) = H^0(X_{\text{ét}}, \mathbf{Z}/l^\nu(i))$ (denoted by α_X^i in [15, Section 1]) given by (5).*

Proof. Define β_X^i as $\pi_i(\bar{\beta}_X, \mathbf{Z}/l^\nu)$, taking (4) into account. Everything follows immediately from Theorem 9.3, except that we get (i) *a priori* only for X essentially of finite type over $\mathbf{Z}$. In the general case, it is enough to prove (i) when $X = \operatorname{Spec} R^\Delta$ for Δ a closed subgroup of $\mathbf{Z}_l^*$. But then, the commutation follows from the open subgroup case by taking direct limits.

9.6. Corollary. *With the hypothesis of Theorem 9.5, $H^0(X_{\text{ét}}, \mathbf{Z}/l^\nu(i))$ is a natural direct summand of $K_{2i}(X, \mathbf{Z}/l^\nu)$, and $H^0(X_{\text{ét}}, \mathbf{Q}_l/\mathbf{Z}_l(i))$ is a natural direct summand of $K_{2i-1}(X)\{l\}$.*

Proof. The first claim follows from Theorem 9.5(o) and (v); the second one follows from the first and [15, Lemma 1.1]. □

We shall see in Appendix E that $H^0(X, \mathbf{Q}_l/\mathbf{Z}_l(i))$ is even a direct summand of $K_{2i-1}(X)$, provided X is of finite type over $\mathbf{Z}$.

9.7. Proposition. (Compare [15, Proposition 1.3].) *Assume that X has positive characteristic p and is connected. Let E be the algebraic closure of $\mathbf{F}_p$ in X (the absolute field of constants of X). Then $\bar{\beta}_X$ can be identified to the composite $\Omega^\infty \mathbf{L}KE \to \Omega^\infty \mathbf{L}KX$.*

Proof. Follows from Proposition 4.10. □

Proposition 9.7 allows us to refine Theorem 9.3 and Corollary 9.5 in nonzero characteristic, removing in particular the hypothesis that -1 is a square when $l = 2$.

9.8. Theorem. *Let X be a scheme over $\mathbf{F}_p$, where p is a prime number $\neq l$. Then the map $\bar{\beta}_X$ of 9.1 extends to a morphism of spectra β_X, which always commutes with products. Moreover, if $l = 2$, we can extend the definition of β_X to the case when -1 is not necessarily* *a square on* X, *using Definition* C.3 *for* $\mathfrak{j}_2(X)$ *on the model above. Theorem* 9.3 *and Corollary* 9.5 *extend to this case.*

Proof. Use Propositions 4.10 and C.13(a). □

APPENDIX A. SYMMETRIC MONOIDAL CATEGORIES

In this section, we recall some well-known constructions on symmetric monoidal and bimonoidal categories. A symmetric monoidal category is a category $\mathscr{S}$ equipped with a functor $\oplus : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ (sum) which is coherently associative and commutative. A symmetric bimonoidal category is a symmetric monoidal category equipped with a further functor $\otimes : \mathscr{S} \times \mathscr{S} \to \mathscr{S}$ (product) which is coherently associative and distributive with respect to $\oplus$. Symmetric monoidal categories form a (2-)category, in which morphisms are functors respecting the sum, and symmetric bimonoidal categories form a (2-)-category in which morphisms are functors respecting the sum and the product. We shall have to consider functors between symmetric bimonoidal categories which respect the sum but not necessarily the product; in case they do respect the product, we shall sometimes stress it by calling them *multiplicative*.

We say that a diagram of functors is *naturally commutative* if it is commutative up to natural equivalences.

A.1. Recall that a *pseudo-functor* T from a category $\mathscr{C}$ to a 2-category $\mathscr{D}$ is an assignment $c \mapsto T(c)$ from objects of $\mathscr{C}$ to objects of $\mathscr{D}$ together with an assignment $f \mapsto T(f)$ from morphisms of $\mathscr{C}$ to functors between objects of $\mathscr{D}$ and a set of natural isomorphisms $T(gf) \xrightarrow{\sim} T(g)T(f)$ for all composable morphisms in $\mathscr{C}$, satisfying some coherence conditions. Two typical examples of pseudo-functors are $A \mapsto \mathscr{P}(A)$ from rings to symmetric monoidal categories and $A \mapsto \mathscr{S}(A)$ from abelian groups to symmetric bimonoidal categories (see A.2 and A.5). Pseudo-functors can be "rectified" into true functors. These questions have been carefully considered, for example, by Jardine [13; 14, Ch. 5].

A.2. Let A be a group. An *A-set* is a set provided with an action of A. An A-set X is *finitely generated* if X/A is finite, free if the action of A is free.

Let $\mathscr{S}(A)$ be the category of finitely generated free A-sets with morphisms the equivariant maps. Disjoint union makes $\mathscr{S}(A)$ a symmetric monoidal category. Suppose A is abelian. If X, $Y \in \mathscr{S}(A)$, define $X \times_A Y$ as $X \times Y/\sim$ where $(x, y) \sim (x', y')$ if there exists $a \in A$ such that $x' = ax$, $y' = a^{-1}y$. Then $X \times_A Y \in \mathscr{S}(A)$ and $\times_A$ gives $\mathscr{S}(A)$ a structure of a symmetric bimonoidal category with unit $\mathbf{1}_A = A$ with the translation action. We denote by $(x, y)_A$ the image of $(x, y) \in X \times Y$ in $X \times_A Y$.

A.3. For A an abelian group, define a function

$$\# : \mathscr{S}(A) \to \mathrm{N}$$

by

$$\# X = |X/A|.$$

This is an "Euler characteristic" in the following sense:

(1) $X \simeq Y \Rightarrow \# X = \# Y$;
(2) $\#(X \coprod Y) = \# X + \# Y$;
(3) $\#(X \times_A Y) = \# X \# Y$.

Moreover:

A.4. LEMMA. *Let $\alpha: X \to Y$ be a morphism in $\mathscr{S}(A)$. Then any two of the following conditions imply the third*:

(i) *α is a monomorphism.*

(ii) *α is an epimorphism.*

(iii) $\# X = \# Y$.

If these conditions are verified, then α is an isomorphism.

A.5. Let $f: A \to B$ be a homomorphism of abelian groups. Define a functor

$$f^*: \mathscr{S}(A) \to \mathscr{S}(B)$$

as follows: For any A-set X, we let B act on $B \times_A X$ by $b(b', x)_A = (bb', x)_A$. This is a well-defined action, and the B-set $B \times_A X$ is finitely generated (resp. free) if X is. Clearly, f^* is a multiplicative functor of symmetric bimonoidal categories. For any $X \in \mathscr{S}(A)$, we have

$$\#(f^*X) = \# X. \tag{A1}$$

Moreover, there is a natural isomorphism of functors

$$(g \circ f)^* \simeq g^* \circ f^* \tag{A2}$$

if f and g are composable. The assignment $f \mapsto f^*$ defines a covariant pseudofunctor from $\mathscr{C}$ to the category of symmetric bimonoidal categories.

A.6. PROPOSITION. *With the same setting as in* A.5, *suppose that f is injective and $f(A)$ is of finite index in B. Then f^* has a right adjoint f_* given by*

$$f_*X = X.$$

We have

$$\#(f_*X) = (A : f(B)) \# X$$

for any $X \in \mathscr{S}(B)$. Moreover, for any $(X, Y) \in \mathscr{S}(A) \times \mathscr{S}(B)$, there is a canonical isomorphism (projection formula)

$$X \times_A f_* Y \simeq f_*(f^*X \times_B Y).$$

In particular,

$$f_* f^* X \simeq X \times_A f_*(1_B).$$

Proof. The action is A-free because f is injective; if X has r B-orbits, then f_*X has $r(A : f(B))$ A-orbits. This shows that $f_*X \in \mathscr{S}(A)$ and also that the claim on $\#$ holds. The adjointness property of f_* is easily checked. Finally, let (X, Y) be as in the proposition. The counit map

$$f^* f_* Y \to Y$$

yields a map

$$f^*X \times_B f^*f_*Y \to f^*X \times_B Y.$$

Composing with the isomorphism $f^*(X \times_A f_* Y) \xrightarrow{\sim} f^*X \times_B f^*f_*Y$, this gives a map

$$f^*(X \times_A f_* Y) \to f^*X \times_B Y$$

hence by adjunction map

$$X \times_A f_* Y \to f_*(f^*X \times_B Y).$$

To check that this map is an isomorphism, it is enough to check that the map of A-orbits is bijective (this is because the action of A is free), which is obvious. □

Note that (A2) yields by adjunction an isomorphism of functors

$$(g \circ f)_* \simeq f_* \circ g_* \tag{A3}$$

if f, g are composable and both satisfy the hypotheses of Proposition A.6 (hence a contravariant pseudofunctor...). Note however that f_* is not multiplicative unless f is an isomorphism.

A.7. Let l be a prime number. Let $[\mathbf{Z}_l^*]$ be the category whose objects are the closed subgroups of the l-adic units $\mathbf{Z}_l^*$ and morphisms are inclusion maps, and let $[\mathbf{Z}_l^*]_1$ be the full subcategory of $[\mathbf{Z}_l^*]$ consisting of torsion-free subgroups, i.e. the closed subgroups of $1 + 2l\mathbf{Z}_l$.

The profinite group $\mathbf{Z}_l^*$ acts continuously by multiplication on $\mathbf{Z}_l$, $\mathbf{Q}_l$, hence on $\mathbf{Q}_l/\mathbf{Z}_l$: this last $\mathbf{Z}_l^*$-module is denoted by $\mathbf{Q}_l/\mathbf{Z}_l(1)$. We define a (contravariant) functor

$$\mu : [\mathbf{Z}_l^*]_1 \to \mathscr{AB}$$

by

$$\mu(\Delta) = \mathbf{Q}_l/\mathbf{Z}_l(1)^\Delta$$

where $\mathscr{AB}$ is the category of abelian groups.

A.8. Let $\Delta \in [\mathbf{Z}_l^*]_1$. We set

$$\mathscr{S}^\Delta = \mathscr{S}(\mu(\Delta))$$

where μ is the functor of A.7. Given $\alpha : \Delta' \hookrightarrow \Delta$, we have an induced functor

$$\iota_{\Delta'/\Delta} = \mu(\alpha)^* : \mathscr{S}^{\Delta'} \to \mathscr{S}^\Delta$$

This is a multiplicative functor. If $(\Delta' : \Delta)$ is finite, the conditions of Proposition A.6 are satisfied and $\iota_{\Delta'/\Delta}$ has a right adjoint

$$\tau_{\Delta'/\Delta} = \mu(\alpha)_* : \mathscr{S}^{\Delta'} \to \mathscr{S}^\Delta$$

and the pair $(\iota_{\Delta'/\Delta}, \tau_{\Delta'/\Delta})$ satisfies the "projection formula" of *loc. cit.*

A.9. Let R be a commutative ring, $\mathscr{P}(R)$ the category of finitely generated projective R-modules and R^* the group of invertible elements of R. We define a functor

$$\mathscr{S}(R^*) \xrightarrow{L} \mathscr{P}(R)$$

by

$$X \mapsto L(X) = (RX/\approx)$$

where RX is the free R-module with basis X and $\approx$ is the R-linear equivalence relation generated by $[rx] \approx r[x]$ $((x,r) \in X \times R^*)$. Note that $RX/\approx$ is a finitely generated free R-module, so that indeed $L(X) \in \mathscr{P}(R)$.

One has the following formulas:

$$\begin{aligned} L(X \textstyle\coprod Y) &\simeq L(X) \oplus L(Y) \\ L(X \times_{R^*} Y) &\simeq L(X) \otimes_R L(Y) \\ \dim_R L(X) &= \# X \end{aligned} \tag{A4}$$

where $\dim_R M$ denotes the rank of a projective R-module M (supposed of constant rank).

Only the second and third ones require some explanation. It is clear that the S-homomorphisms

$$R(X \times_{R^*} Y) \to LX \otimes_R LY$$
$$RX \otimes_R RY \to L(X \times_{R^*} Y)$$

defined by

$$[(x,y)_{R^*}] \mapsto [x] \otimes [y]$$
$$[x] \otimes [y] \mapsto [(x,y)_{R^*}]$$

yield inverse isomorphisms between $L(X \times_{R^*} Y)$ and $L(X) \otimes_R L(Y)$. To show (A1), we reduce by additivity to the special case $X = 1_A$; then $L(X) = R$. □

A.10. In general, let A be an abelian group and $f: A \to R^*$ be a homomorphism. By composing L with f^* we get a multiplicative functor $\mathscr{S}(A) \to \mathscr{P}(R)$.

Let $R \to R'$ be a homomorphism of commutative rings. We have:

A.11. Proposition. *The diagram of functors*

$$\begin{array}{ccc} \mathscr{S}(R'^*) & \xrightarrow{L} & \mathscr{P}(R') \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle \iota_{R'/R}} \\ \mathscr{S}(R^*) & \xrightarrow{L} & \mathscr{P}(R) \end{array}$$

is naturally commutative, where $\iota_{R'/R}$ is extension of scalars.

Proof. For $X \in S(1, R^*)$, map $R'(R'^* \times_{R^*} X)$ to $R' \otimes_R (RX/\approx)$ by $[(r', x)_{R'^*}] \mapsto r' \otimes [x]$ and RX to $(R' \otimes_R (RX/\approx))_{(R)}$ (hence $R' \otimes_R (RX)$ to $R' \otimes_R (RX/\approx)$) by $[x] \mapsto [(1, x)_{R'^*}]$ and check that this induces inverse natural isomorphism between $L \circ f^*$ and $\iota_{R'/R} \circ L$. □

A.12. Let l be a prime number and $R = \mathbf{Z}[\mu_{l^\infty}]$ the subring of $\mathbf{C}$ generated by all the l-primary roots of unity. The profinite group $\mathbf{Z}_l^*$ acts on μ_{l^∞} as on $\mathbf{Q}_l/\mathbf{Z}_l$ (1) (cf. A.7); we choose an isomorphism of $\mathbf{Z}_l^*$-modules

$$\mathbf{Q}_l/\mathbf{Z}_l(1) \xrightarrow{\sim} \mu_{l^\infty}. \tag{A5}$$

For any closed subgroup $\Delta \subseteq \mathbf{Z}_l^*$, let R^Δ be the fixed ring of the restriction of this action to Δ. For $\Delta \in [\mathbf{Z}_l^*]_1$, we define a functor

$$\Phi^\Delta : \mathscr{S}^\Delta \to \mathscr{P}(R^\Delta) \tag{A6}$$

where $\mathscr{S}^\Delta$ was defined in A.8, by

$$\Phi^\Delta = L \circ \rho^*$$

where $\rho : \mu_{l^\infty}^{\Delta_1} \hookrightarrow S^*$ is the inclusion and L is as in A.9 (compare A.10). This is a multiplicative functor of symmetric bimonoidal categories. We have:

A.13. PROPOSITION. *Let $\Delta' \subseteq \Delta$ be two closed subgroups of $1 + 2l\mathbf{Z}_l$. Then*

(a) *The diagram of functors*

$$\begin{array}{ccc} \mathscr{S}^{\Delta'} & \xrightarrow{\Phi^{\Delta'}} & \mathscr{P}(R^{\Delta'}) \\ {\scriptstyle \iota_{\Delta'/\Delta}}\uparrow & & \uparrow{\scriptstyle \iota_{R^{\Delta'}/R^{\Delta}}} \\ \mathscr{S}^{\Delta} & \xrightarrow{\Phi^{\Delta}} & \mathscr{P}(R^{\Delta}) \end{array}$$

is naturally commutative, where $\iota_{R^{\Delta'}/R^{\Delta}}$ is the extension of scalars $M \mapsto S \otimes_R M$.

(b) *If $(\Delta : \Delta')$ is finite, the diagram of functors*

$$\begin{array}{ccc} \mathscr{S}^{\Delta'} & \xrightarrow{\Phi^{\Delta'}} & \mathscr{P}(R^{\Delta'}) \\ {\scriptstyle \tau_{\Delta'/\Delta}}\downarrow & & \downarrow{\scriptstyle \tau_{R^{\Delta'}/R^{\Delta}}} \\ \mathscr{S}^{\Delta} & \xrightarrow{\Phi^{\Delta}} & \mathscr{P}(R^{\Delta}) \end{array}$$

is naturally commutative, where $\tau_{R^{\Delta'}/R^{\Delta}}$ is the restriction of scalars.

Proof. (a) follows from Proposition A.11 and (A2). We now proceed to prove (b). We first define a natural transformation (base change)

$$\Phi^{\Delta} \circ \tau_{\Delta'/\Delta} \to \tau_{R^{\Delta'}/R^{\Delta}} \circ \Phi^{\Delta'} \tag{A7}$$

by abstract nonsense. Start from the counit

$$\iota_{\Delta'/\Delta} \circ \tau_{\Delta'/\Delta} \to Id_{\mathscr{S}^{\Delta'}}.$$

Compose it with $\Phi^{\Delta'}$:

$$\Phi^{\Delta'} \circ \iota_{\Delta'/\Delta} \circ \tau_{\Delta'/\Delta} \to \Phi^{\Delta'}$$

use (a) to transform this into

$$\iota_{R^{\Delta'}/R^{\Delta}} \circ \Phi^{\Delta} \circ \tau_{\Delta'/\Delta} \to \Phi^{\Delta'}$$

and use adjunction again to get the desired natural transformation. Note that $[R^{\Delta'} : R^{\Delta}] = (\Delta : \Delta')$ (by Gauss' theorem), hence, by (A1), (A4) and Proposition A.6,

$$\# \Phi^{\Delta} \circ \tau_{\Delta'/\Delta}(X) = \# \tau_{R^{\Delta'}/R^{\Delta}} \circ \Phi^{\Delta'}(X).$$

Therefore, to prove that (A7) is an isomorphism, it suffices to show that, for any X, the homomorphism $\Phi^{\Delta} \circ \tau_{\Delta'/\Delta}(X) \to \tau_{R^{\Delta'}/R^{\Delta}} \circ \Phi^{\Delta'}(X)$ is surjective. It suffices to do it when $X = \mathbf{1}_\mu$, where $\mu = \mu_{l^\infty}^{\Delta'}$. This is clear, since $R^{\Delta'} = \sum_{\zeta \in \mu} R^{\Delta}\zeta$. □

APPENDIX B. LOCALISATION AND MAPPING TELESCOPES

B.1. We begin by reviewing the localisation of modules over a ring spectrum by means of mapping telescopes [32, 8, 23; 41, Appendix]. Let E be a commutative, associative and unital ring spectrum and X an associative, unital E-module spectrum. This means that X is provided with an E-action $\mu_X : E \wedge X \to X$ which is homotopy associative and strictly unital. Let $d \in \mathbf{Z}$ and $a \in \pi_d(E)$. Then a defines an endomorphism of X:

$$a_X : \Sigma^d X \xrightarrow{a \wedge \mathrm{Id}_X} E \wedge X \xrightarrow{\mu_X} X,$$

hence a "mapping telescope"

$$a^{-1}X = \operatorname{hocolim}(X \xrightarrow{\Sigma^{-d}a_X} \Sigma^{-d}X \xrightarrow{\Sigma^{-2d}a_X} \Sigma^{-2d}X \to \cdots). \tag{B1}$$

By associativity and commutativity, the pairings

$$\Sigma^{-md}E \wedge \Sigma^{-nd}X \to \Sigma^{-(m+n)d}X$$

deduced from μ_X commute up to homotopy with the transition maps induced by a_E and a_X, hence yield at the limit a pairing

$$a^{-1}E \wedge a^{-1}X \to a^{-1}X.$$

In particular, $a^{-1}E$ is a (commutative, associative, unital) ring spectrum and $a^{-1}X$ is a(n associative, unital) $a^{-1}E$-module. This operation is called *localisation with respect to* (or *away from*) a. Clearly, if a, $b \in \pi_*(E)$ verify $a^m = b^n$ for some m, n, they yield the same localisations. In the stable homotopy category, the natural map $X \xrightarrow{\ell_a} a^{-1}X$ is universal for maps from X to associative unital E-modules Y such that a_Y is an equivalence.

The following result says that $x \mapsto a^{-1}X$ is an "extension of scalars" from E-modules to $a^{-1}E$-modules:

B.2. PROPOSITION. *For any associative, unital E-module X, denote by μ'_X the composite*

$$a^{-1}E \wedge X \xrightarrow{1 \wedge \ell_a} a^{-1}E \wedge a^{-1}X \xrightarrow{a^{-1}\mu_X} a^{-1}X.$$

Then there is a homotopy cocartesian square

$$\begin{array}{ccc} a^{-1}E \wedge E \wedge X & \xrightarrow{\mu'_E \wedge 1} & a^{-1}E \wedge X \\ {\scriptstyle 1 \wedge \mu'_X}\downarrow & & \downarrow{\scriptstyle \mu'_X} \\ a^{-1}E \wedge X & \xrightarrow{\mu'_X} & a^{-1}X. \end{array}$$

Proof. Let $\widetilde{a^{-1}X}$ be the homotopy pushout of the two maps $\mu'_E \wedge 1$ and $1 \wedge \mu'_X$ in the square. Since their domain and ranges are $a^{-1}E$-modules, $\widetilde{a^{-1}X}$ inherits the same structure. It remains to see that it enjoys the universal property of $a^{-1}X$, which follows from associativity and unitality. □

B.3. Let F be another commutative, associative and unital ring spectrum and $F \xrightarrow{f} E$ a morphism of ring spectra. Let $a' \in \pi_d(F)$ and $a = f_*(a')$. View X as an F-module via f. Then $a'_X = a_X$, hence $a'^{-1}X = a^{-1}X$. If $X = E$, then the natural map $a^{-1}F \to a^{-1}E$ is a morphism of ring spectra.

B.4. If E is a ring spectrum which is not necessarily unital, commutative and associative and $a \in \pi_d(E)$, then the construction of B.1 still makes sense; however, $a^{-1}E$ is not a ring spectrum in general. It is if a is central and associates with every pair of elements of $\pi_*(E)$ ($a(xy) = (ax)y$ for every x, $y \in \pi_*(E)$). Then $a^{-1}X$ inherits a structure of $a^{-1}E$-module, provided the similar associativity condition holds for it.

B.5. The assignment $X \mapsto a^{-1}X$ is strictly functorial for E-linear maps. The following remarks are occasionally useful. Let b be a power of a. Then $b^{-1}X$ and $a^{-1}X$ are equivalent naturally in X. This holds more generally if $a^m = b^n$ for suitable integers m, n. Following [23, p. 826], call the set of those b's the *isogeny class* of a. If one wishes, one can define a canonical localisation which does not depend on the choice of an element in this isogeny class cl(a). To do this, one takes a hocolim over all suspensions of products of elements of cl(a), rather than just over suspensions of a (compare [41, p. 545]).

B.6. Let l be a prime number and $N = l^\nu$ a prime power. Suppose $N \neq 2$. We recall that, by Barratt [2], $N\mathrm{Id}_{M(N)} = 0$ in this case.

By [26, Theorem 2], $M(N)$ has a structure of unital ring spectrum $M(N) \wedge M(N) \xrightarrow{\mu_N} M(N)$, for the unit $\rho_N : \mathbb{S} \to M(N)$ coming from the defining the fibre sequence

$$\mathbb{S} \xrightarrow{N} \mathbb{S} \xrightarrow{\rho_N} M(N),$$

such that the mod N Bockstein is a derivation.

If $l > 2$, there is (up to homotopy) a unique such structure, which is commutative; it is associative except for $N = 3$ (*ibid.* and [42, Theorem 6]). If $l = 2$, there are two such structures; they are both associative, and are commutative except for $N = 4$ (*ibid.*). More precisely, for $l = 3$ one has

$$\mu_N \circ (\mu_N \wedge 1) = \mu_N \circ (1 \wedge \mu_N) + \tfrac{1}{3} N \rho_N \circ \alpha \circ (\delta_N \wedge \delta_N \wedge \delta_N) \tag{B2}$$

and for $l = 2$,

$$\mu_N \circ T = \mu_N + \tfrac{1}{4} N \rho_N \circ \eta^2 \circ (\delta_N \wedge \delta_N) \tag{B3}$$

where α is a generator of (the 3-primary component of) π_3^S, η the generator of π_1^S (the Hopf map), $\delta_N : M(N) \to \Sigma\mathbb{S}$ the integral Bockstein and $T : M(N) \wedge M(N) \to M(N) \wedge M(N)$ is the map switching factors (*ibid.*).

Finally, given $N'|N$ and such a multiplication over $M(N')$, there is one (unique up to homotopy) on $M(N)$ which is compatible with the former via the reduction map $M(N) \to M(N')$ [26, Lemma 5 and Remark, p. 266].

In what follows, if $l = 2$ we choose a good multiplication on $M(4)$ once and for all, and take the compatible one on $M(N)$ for all $N \geqslant 4$.

B.7. Let E, X be as in B.1 and l, N as in B.6. Then $E \wedge M(N)$ inherits the structure of a ring spectrum by

$$E \wedge M(N) \wedge E \wedge M(N) \xrightarrow{1 \wedge T \wedge 1} E \wedge E \wedge M(N) \wedge M(N) \xrightarrow{\mu \wedge \mu_N} E \wedge M(N)$$

with regularity properties as in B.6. Similarly, $X \wedge M(N)$ becomes an $E \wedge M(N)$-module. In particular, formulas (B2) and (B3) imply, for $x, y, z \in \pi_*(E, \mathbf{Z}/3)$,

$$(xy)z - x(yz) = (\rho_N)_*(\alpha\delta_N(x)\delta_N(y)\delta_N(z)) \tag{B4}$$

and, for $x, y \in \pi_*(E, \mathbf{Z}/4)$

$$yx - (-1)^{\deg(x)\deg(y)} xy = (\rho_N)_*(\eta^2 \delta_N(x)\delta_N(y)). \tag{B5}$$

This shows that $\pi_*(E, Z/3)$ (resp. $\pi_*(E, \mathbf{Z}/4)$) is associative (resp. commutative) if $(\rho_N)_*(\alpha) = 0 \in \pi_3(E, \mathbf{Z}/3)$ (resp. if $(\rho_N)_*(\eta^2) = 0 \in \pi_2(E, \mathbf{Z}/4)$). Moreover, for any E, every element of $\pi_*(E, \mathbf{Z}/3)$ coming from $\pi_*(E)$ associates with any other two elements, and every element of $\pi_*(E, \mathbf{Z}/4)$ coming from $\pi_*(E)$ is central (in the graded sense). This applies in particular to Bocksteins. Moreover, for $x, y \in \pi_*(E, \mathbf{Z}/3)$, $x^2 y = x(xy)$ if x has even degree, since then $\delta(x)^2 = 0$. It follows from a well-known argument that, for $x, y \in \pi_*(E, \mathbf{Z}/3)$ with even degree, the subalgebra generated by x and y is associative (and commutative) (compare [23, Proposition 0.3]).

B.8. LEMMA. *Let $a \in \pi_d(E, \mathbf{Z}/N)$, $N'|N$ and $\bar{a}$ the image of a in $\pi_*(E, \mathbf{Z}/N')$. If $l = 2$, suppose $N' \geqslant 4$. Let X be an associative, unital E-module. Then, if $a_{X \wedge M(N)}$ is an equivalence, so is $\bar{a}_{X \wedge M(N')}$.*

Proof. Smashing the equivalence $\Sigma^d X \wedge M(N) \xrightarrow{a_{X \wedge M(N)}} X \wedge M(N)$ by $M(N')$, we get a new equivalence

$$\Sigma^d X \wedge M(N) \wedge M(N') \xrightarrow{a_{X \wedge M(N)} \wedge 1} X \wedge M(N) \wedge M(N').$$

The unitality of $\mu_{N'}$ shows that the composition $M(N) \wedge M(N') \xrightarrow{\rho \wedge 1} M(N') \wedge M(N) \xrightarrow{\mu_{N'}} M(N')$ is split by $\rho_N \wedge 1$, where $\rho: M(N) \to M(N')$ is the projection. It follows that $\mu_{N'} \circ (\rho \wedge 1)$ provides an equivalence

$$\Sigma^d X \wedge M(N') \to X \wedge M(N')$$

as a wedge summand of the former one. It now suffices to check that this new equivalence coincides with $\bar{a}_{X \wedge M(N')}$. □

B.9. Let N', a, $\bar{a}$ be as in Lemma B.8. If $E \wedge M(N)$ is homotopy commutative and associative, we can apply B.1 to the $E \wedge M(N)$-module $X \wedge M(N)$ and get a localised spectrum $a^{-1}(X \wedge M(N))$. Let $N'|N$ be such that $E \wedge M(N')$ is still associative and commutative. There is a natural map from the telescope of $X \wedge M(N)$ to that of $X \wedge M(N')$, hence from $a^{-1}(X \wedge M(N))$ to $\bar{a}^{-1}(X \wedge M(N'))$.

B.10. Let E, l, N be as above, and let $N'|N$. If $l = 2$, we assume that N' and N/N' are divisible by 4. Denote by $\rho_N: \mathbb{S} \to M(N)$, $\rho_{N'}: \mathbb{S} \to M(N')$ the unit maps and by $\rho: M(N) \to M(N')$, $i: M(N') \to M(N)$ the maps, respectively, induced by the projection $\mathbf{Z}/N \to \mathbf{Z}/N'$ and the inclusion $\mathbf{Z}/N' \to \mathbf{Z}/N$ (sending, respectively, 1 to 1 and 1 to N/N'). Denote, respectively, by μ_N and $\mu_{N'}$ the multiplications on $M(N)$ and $M(N')$. We could not find the following compatibility in the literature:

B.11. PROPOSITION. (a) *The diagram*

$$\begin{tikzcd}
 & M(N') \wedge M(N') \arrow[r, "\mu_{N'}"] & M(N') \arrow[dd, "i"] \\
M(N') \wedge M(N) \arrow[ur, "1 \wedge \rho"] \arrow[dr, "i \wedge 1"'] & & \\
 & M(N) \wedge M(N) \arrow[r, "\mu_N"] & M(N)
\end{tikzcd}$$

is homotopy commutative if $l > 2$. If $l = 2$, the difference between the two paths of the diagram equals

$$ci \circ b \circ (\delta_{N'} \wedge \delta_N)$$

where $c \in \mathbf{Z}$, δ_N, $\delta_{N'}$ are the integral Bocksteins ($\delta_{N'} = \delta_N \circ i$), and $b \in \pi_2(\mathbb{S}, \mathbf{Z}/N')$ is an element such that $\delta_{N'}(b) = \eta$, the Hopf map generating π_1^S.

(b) *Let E be a ring spectrum. For $(x, y) \in \pi_*(E, \mathbf{Z}/N') \times \pi_*(E, \mathbf{Z}/N)$, we have the formula*

$$i_*x \cdot y = i_*(x \cdot \rho_* y)$$

if $l > 2$. If $l = 2$, we have

$$i_*x \cdot y = i_*(x \cdot \rho_* y) + c i_* b \cdot \delta_{N'}(x) \cdot \delta_N(y)$$

for c as in (a).

Remark. Presumably one has $c = 1$.

Proof. Clearly, (a)$\Rightarrow$(b). Smashing the fibre sequence $\mathbb{S} \xrightarrow{N} \mathbb{S} \xrightarrow{\rho_N} M(N) \xrightarrow{\delta_N} \Sigma\mathbb{S} \xrightarrow{N} \Sigma\mathbb{S}$ by $M(N')$, we get another fibre sequence, with somewhat fastidious notation:

$$M(N') \wedge \mathbb{S} \xrightarrow{N} M(N') \wedge \mathbb{S} \xrightarrow{1 \wedge \rho_N} M(N') \wedge M(N) \xrightarrow{1 \wedge \delta_N} M(N') \wedge \Sigma\mathbb{S} \xrightarrow{N} M(N') \wedge \Sigma\mathbb{S}.$$

As noted above, $N = 0$ on $M(N')$. If l is odd, $[M(N') \wedge \Sigma\mathbb{S}, M(N)] = 0$ [26, Lemma 7]. Hence, the map $[M(N') \wedge M(N), M(N)] \xrightarrow{(1 \wedge \rho_N)^*} [M(N') \wedge \mathbb{S}, M(N)]$ is bijective. Therefore, it suffices to check (a) after composing on the right by $1 \wedge \rho_N$.

Let $\pi : \mathbb{S} \wedge M(N) \xrightarrow{\approx} M(N)$, $\pi' : \mathbb{S} \wedge M(N') \xrightarrow{\approx} M(N')$ be the tautological equivalences. On the one hand,

$$\mu_{N'} \circ (i \wedge \rho) \circ (1 \wedge \rho_N) = \mu_{N'} \circ (1 \wedge \rho_{N'}) = \pi'$$

and on the other hand

$$\mu_N \circ (i \wedge 1) \circ (1 \wedge \rho_N) = \mu_N \circ (1 \wedge \rho_N) \circ (i \wedge 1) = \pi \circ (i \wedge 1)$$

by unitality of μ_N and $\mu_{N'}$. The claim now follows from the obvious identity

$$i \circ \pi' = \pi \circ (i \wedge 1).$$

When $l = 2$, the above computation remains valid, but only proves that the difference between the two compositions factors through $1 \wedge \delta_N$. To go further, we need:

B.12. Lemma. *With notation as in Proposition* B.11, *compositions on the left with ρ of the two paths of the diagram coincide.*

Indeed, we have

$$\rho \circ i \circ \mu_{N'} \circ (1 \wedge \rho) = N/N' \mu_{N'} \circ (1 \wedge \rho)$$

and

$$\rho \circ \mu_N \circ (i \wedge 1) = \mu_{N'} \circ (\rho \wedge \rho) \circ (i \wedge 1) = \mu_{N'} \circ (\rho i \wedge \rho) = N/N' \mu_{N'} \circ (1 \wedge \rho)$$

by the compatibility between μ_N and $\mu_{N'}$. $\square$

To finish the proof of Proposition B.11(a), consider the diagram

$$\begin{tikzcd}
M(N') \wedge M(N) \arrow[r, "\alpha"] \arrow[d, "1 \wedge \delta_N"'] & M(N) \arrow[r, "\rho"] & M(N') \\
M(N') \wedge \Sigma\mathbb{S} \arrow[urr, "\bar{\alpha}"] & &
\end{tikzcd}$$

in which α is the difference between the two paths of the diagram in Proposition B.11 and $\bar{\alpha}$ is the map induced by the above reasoning. Since the vertical map is split, the composition $\rho\circ\bar{\alpha}$ is 0 by Lemma B.12. Hence,

$$\bar{\alpha}\in \operatorname{Ker}([M(N')\wedge\Sigma\mathbb{S}, M(N)] \xrightarrow{\rho_*} [M(N')\wedge\Sigma\mathbb{S}, M(N')]).$$

Considering the commutative diagram with exact columns

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\pi_2(M(N))/N' \arrow[r, "\rho_*"] \arrow[d] & \pi_2(M(N'))/N' \arrow[d] \\
{[M(N')\wedge\Sigma\mathbb{S}, M(N)]} \arrow[r, "\rho_*"] \arrow[d] & {[M(N')\wedge\Sigma\mathbb{S}, M(N')]} \arrow[d] \\
\pi_1(M(N)) \arrow[r, "\rho_*"] & \pi_1(M(N'))
\end{tikzcd}$$

and using e.g. [26, Lemma 7], we see that this kernel is generated by $i\circ b\circ(\delta_{N'}\wedge 1)$. This concludes the proof.

B.13. Corollary. *Suppose $N'^2|N$. With the above notation,*

(a) *For $l>2$, we have $i_*x\cdot i_*y=0$ for all $x, y\in\pi_*(E,\mathbf{Z}/N')$.*
(b) *For $l=2$, we have $2i_*x\cdot i_*y=0$ and $i_*x\cdot i_*y\cdot i_*z\cdot i_*t=0$ for all $x,y,z,t\in\pi_*(E,\mathbf{Z}/N')$.*
(c) *For $l=2$, $i_*x\cdot y^2=i_*(x\cdot\rho_*y^2)$ for any $(x,y)\in\pi_*(E,\mathbf{Z}/N')\times\pi_*(E,\mathbf{Z}/N)$.*

Proof. If $l>2$,

$$i_*x\cdot i_*y=i_*(x\cdot\rho_*i_*y)=i_*(N/N'x\cdot y)=0.$$

Suppose $l=2$. For simplicity, set $\delta_{N'}=\delta$. This time, we have

$$i_*x\cdot i_*y=i_*(x\cdot\rho_*i_*y)+ci_*b\cdot\delta(x)\cdot\delta(y)=i_*(N/N'x\cdot y)+ci_*b\cdot\delta(x)\cdot\delta(y)=ci_*b\cdot\delta(x)\cdot\delta(y)$$

since N/N' is divisible by 4. Since b is killed by 2, this gives $2i_*x\cdot i_*y=0$. Next

$$i_*x\cdot i_*y\cdot i_*z=ci_*x\cdot i_*b\cdot\delta(y)\cdot\delta(z)=c^2i_*b\cdot\delta(x)\cdot\delta(b)\cdot\delta(y)\cdot\delta(z)=c^2i_*b\cdot\eta\cdot\delta(x)\cdot\delta(y)\cdot\delta(z).$$

Finally,

$$\begin{aligned}i_*x\cdot i_*y\cdot i_*z\cdot i_*t&=c^2i_*x\cdot i_*b\cdot\eta\cdot\delta(x)\cdot\delta(y)\cdot\delta(z)\\&=c^3i_*b\cdot\delta(x)\cdot\delta(b)\cdot\eta\cdot\delta(y)\cdot\delta(z)\cdot\delta(t)=c^3i_*b\cdot\eta^2\cdot\delta(x)\cdot\delta(y)\cdot\delta(z)\cdot\delta(t)=0\end{aligned}$$

because $i_*b\cdot\eta^2=0\in\pi_4(\mathbb{S},\mathbf{Z}/N)$. (Indeed, since $\pi_4^{\mathbb{S}}=0$, the Bockstein $\pi_4(\mathbb{S},\mathbf{Z}/N)\to\pi_3^{\mathbb{S}}$ is injective. But $\delta(i_*b\cdot\eta^2)=\delta(b)\cdot\eta^2=\eta^3=0$.)

Note that in this computation, we freely used commutativity because all elements but one come from $\pi_*(E)$ and all products are killed by 2.

To prove (c), we compute

$$i_*x\cdot y^2=i_*(x\cdot\rho_*y^2)+ci_*b\cdot\delta(x)\cdot\delta(y^2)=i_*(x\cdot\rho_*y^2)+ci_*b\cdot\partial(x)\cdot\partial(y^2)$$

where ∂ is the Bockstein modulo N. By (17) and the subsequent remarks, y commutes with its Bockstein, hence $\partial(y^2)=2y\cdot\partial(y)$. The claim now follows from the fact that $2i_*b=0$. □

B.14. Let E, X, l, ν, a be as above. Suppose $l > 2$. There are fibre sequences

$$M(l) \to M(lN) \to M(N)$$

$$E \wedge M(l) \to E \wedge M(lN) \to E \wedge M(N)$$

hence a long exact sequence of homotopy groups

$$\cdots \to \pi_i(E, \mathbf{Z}/l) \to \pi_i(E, \mathbf{Z}/lN) \to \pi_i(E, \mathbf{Z}/N) \xrightarrow{\partial} \pi_{i-1}(E, \mathbf{Z}/l) \to \cdots .$$

We have

$$\partial(a^l) = la^{l-1}\partial(a) = 0 \in \pi_{dl-1}(E, \mathbf{Z}/l).$$

This is clear if $l > 3$. If $l = 3$, it follows from (B4) and the subsequent remarks plus the commutativity of $\pi_*(E, \mathbf{Z}/3)$ that

$$\partial(a^3) = a^2\partial(a) + (\partial(a)a + a\partial(a))a = a^2\partial(a) + 2a(a\partial(a)) = 3a^2\partial(a) = 0.$$

Hence, in all cases, a^l is the reduction of an element $\tilde{a} \in \pi_{dl}(E, \mathbf{Z}/lN)$. For another such lift $\tilde{a}'$, we have $\tilde{a}' - \tilde{a} = i_* x$ with $x \in \pi_{dl}(E, \mathbf{Z}/l)$. Using Corollary B.13, this gives

$$\tilde{a}'^l = \tilde{a}^l + \sum_{k=1}^{l} \binom{l}{k} (i_* x)^k a^{l-k} = \tilde{a}^l$$

even if $l = 3$ by the remarks following (B4). It follows that the mapping telescopes defined on $X \wedge M(N)$ by $\tilde{a}$ and $\tilde{a}'$ are homotopy equivalent.

B.15. In the case $l = 2$, we have to be a little more careful. This time we use the fibre sequences

$$M(4) \to M(4N) \to M(N)$$

$$E \wedge M(4) \to E \wedge M(4N) \to E \wedge M(N).$$

Let N be divisible by 4. Denote by ∂ the Bockstein $\pi_*(E, \mathbf{Z}/N) \to \pi_{*-1}(E, \mathbf{Z}/4)$. Then, even if $N = 4$, we have for $a \in \pi_*(E, \mathbf{Z}/N)$:

$$\partial(a^4) = 4a^3\partial(a) = 0$$

by (B5) and the subsequent remarks. Hence, a comes from an element $\tilde{a} \in \pi_*(E, \mathbf{Z}/4N)$. For another lift $\tilde{a}'$, we have $\tilde{a}' - \tilde{a} = i_* x$ with $x \in \pi_{dl}(E, \mathbf{Z}/4)$. Using Corollary B.13 again, we have $2(i_* x)^2 = (i_* x)^4 = 0$, hence

$$\tilde{a}'^4 = \tilde{a}^4 + 4i_* x\tilde{a}^3 + 6(i_* x)^2\tilde{a}^2 + 4(i_* x)^3\tilde{a} + (i_* x)^4 = \tilde{a}^4.$$

B.16. Let $a \in \pi_{2d}(E, \mathbf{Z}/l^*)$. Iterating the above process, we get a sequence of elements

$$a_\nu \in \pi_*(E, \mathbf{Z}/l^{*\nu})$$

such that $a_1 = a$ and, for any ν, the reduction of a_ν modulo $l^{*\nu-1}$ is $a_{\nu-1}^{l^*}$. This gives rise to a well-defined homotopy inverse system

$$(a_\nu^{-1}(X \wedge M(l^{*\nu})))_{\nu \geqslant 1}.$$

By B.14 and B.15, another choice of (a_ν) gives an equivalent homotopy inverse system.

We have:

B.17. PROPOSITION. *The following conditions are equivalent:*

(a) $(a_\nu)_{X \wedge M(l^{*\nu})}$ *is an equivalence for all* ν;
(b) $(a_\nu)_{X \wedge M(l^{*\nu})}$ *is an equivalence for some* ν.

Proof. (a) $\Rightarrow$ (b) is obvious. To see (b) $\Rightarrow$ (a), note that if $\mu < \nu$ then the image of a_ν in $\pi_*(E, \mathbf{Z}/l^{*\mu})$ is a power of a_μ, so if (b) is true for ν it is true for μ by Lemma B.8. Conversely, let us show that if it is true for $\nu - 1$, then it is true for ν. By the above, a defines an equivalence on $X \wedge M(l^*)$. Consider the diagram

$$\begin{tikzcd}
\pi_{n+k+1}(X, \mathbf{Z}/l^{*\nu-1}) \arrow[r] & \pi_{n+k}(X, \mathbf{Z}/l^*) \arrow[r] & \pi_{n+k}(X, \mathbf{Z}/l^{*\nu}) \arrow[r] & \pi_{n+k}(X, \mathbf{Z}/l^{*\nu-1}) \arrow[r] & \pi_{n+k-1}(X, \mathbf{Z}/l^*) \\
\pi_{n+1}(X, \mathbf{Z}/l^{*\nu-1}) \arrow[r] \arrow[u, "\cdot a_{\nu-1}^{l^*}"] & \pi_n(X, \mathbf{Z}/l^*) \arrow[r] \arrow[u, "\cdot a^{l^{*\nu}}"] & \pi_n(X, \mathbf{Z}/l^{*\nu}) \arrow[r] \arrow[u, "\cdot a_\nu"] & \pi_n(X, \mathbf{Z}/l^{*\nu-1}) \arrow[r] \arrow[u, "\cdot a_{\nu-1}^{l^*}"] & \pi_{n-1}(X, \mathbf{Z}/l^*) \arrow[u, "\cdot a^{l^{*\nu}}"]
\end{tikzcd}$$

where k is an appropriate integer. The left and right squares commute because the mod l^* Bockstein ∂ acts like a derivation and $\partial(a_{\nu-1}^{l^*}) = 0$ as seen above; the second square from the right also commutes because the reduction modulo $l^{*\nu-1}$ is multiplicative. If $l > 2$, the second square from the left also commutes, thanks to Proposition B.11. This may not hold if $l = 2$. However, in this case, the diagram

$$\begin{tikzcd}
\pi_{n+2k+1}(X, \mathbf{Z}/4^{\nu-1}) \arrow[r] & \pi_{n+2k}(X, \mathbf{Z}/4) \arrow[r] & \pi_{n+2k}(X, \mathbf{Z}/4^\nu) \arrow[r] & \pi_{n+2k}(X, \mathbf{Z}/4^{\nu-1}) \arrow[r] & \pi_{n+2k-1}(X, \mathbf{Z}/4) \\
\pi_{n+1}(X, \mathbf{Z}/4^{\nu-1}) \arrow[r] \arrow[u, "\cdot a_{\nu-1}^8"] & \pi_n(X, \mathbf{Z}/4) \arrow[r] \arrow[u, "\cdot a^{2^{2\nu+1}}"] & \pi_n(X, \mathbf{Z}/4^\nu) \arrow[r] \arrow[u, "\cdot a_\nu^2"] & \pi_n(X, \mathbf{Z}/4^{\nu-1}) \arrow[r] \arrow[u, "\cdot a_{\nu-1}^8"] & \pi_{n-1}(X, \mathbf{Z}/4) \arrow[u, "\cdot a^{2^{2\nu+1}}"]
\end{tikzcd}$$

does commute by Corollary B.13(c). In all cases, the claim now follows from the five lemma applied to either diagram. $\square$

B.18. If $l = 2$, we redefine

$$a_{2\nu}(\text{new}) = a_\nu(\text{old})$$

$$a_{2\nu+1} = \text{image of } a_{2\nu+2} \text{ in } \pi_*(E, \mathbf{Z}/l^{2\nu+1})$$

for $\nu \geq 1$. This way, $a_\nu \in \pi_*(E, \mathbf{Z}/2^\nu)$ and hence $a_\nu^{-1}(E \wedge M(2^\nu))$, $a_\nu^{-1}(X \wedge M(2^\nu))$ are defined for all $\nu \geq 2$.

B.19. PROPOSITION. *Let* $N = l^\nu$, $N' = l^{\nu'}$, $N'' = l^{\nu''}$ *with* $\nu < \nu' < \nu''$. *If* $l = 2$, *suppose* $\nu > 1$. *Then there is a homotopy commutative diagram of fibre sequences*

$$\begin{tikzcd}
\Sigma a_\nu^{-1}(X \wedge M(N)) \arrow[r] \arrow[d, "N''/N'"'] & a_{\nu''}^{-1}(X \wedge M(N'')) \wedge M(N) \arrow[r] \arrow[d, "1 \wedge \rho_{N'' \to N'} \wedge 1"'] & a_\nu^{-1}(X \wedge M(N)) \arrow[d, equal] \\
\Sigma a_\nu^{-1}(X \wedge M(N)) \arrow[r] & a_{\nu'}^{-1}(X \wedge M(N')) \wedge M(N) \arrow[r] & a_\nu^{-1}(X \wedge M(N)).
\end{tikzcd}$$

where $\rho_{N'' \to N'}$ *is the reduction map* $M(N'') \to M(N')$.

Proof. Using unitality and associativity of μ_N, the derivation property of $\partial_{N'}$ and the compatibility between μ_N and $\mu_{N'}$, we get by inspection a homotopy commutative diagram of spectra

$$\begin{tikzcd}
M(N') \wedge M(N') \wedge M(N) \arrow[r, "\mu_{N'} \wedge 1"] \arrow[d, "\begin{pmatrix} 1 \wedge (\mu_N \circ (\rho \wedge 1)) \\ 1 \wedge \delta_{N'} \wedge 1 \end{pmatrix}"'] & M(N') \wedge M(N) \arrow[d, "\begin{pmatrix} \mu_N \circ (\rho \wedge 1) \\ \delta_{N'} \wedge 1 \end{pmatrix}"'] \\
M(N') \wedge M(N) \vee M(N') \wedge \Sigma M(N) \arrow[r, "\lambda"] & M(N) \vee \Sigma M(N)
\end{tikzcd}$$

where

$$\lambda = \begin{pmatrix} \mu_{N'} \circ (\rho \wedge 1) & 0 \\ \delta_{N'} \wedge 1 & \mu_N \circ (\rho \wedge 1) \end{pmatrix}$$

and the two vertical morphisms are equivalences; similarly for N''. (The reader who wants to check out our computation carefully should notice, for $N = 3$, the relation $\delta_3 \wedge \delta_3 \wedge \delta_3 \circ \rho \wedge \rho \wedge 1 = (N'/3)^2 \delta_{N'} \wedge \delta_{N'} \wedge \delta_3$, which kills the defect of associativity (14) of μ_3.) Therefore, we get a homotopy commutative diagram

$$\begin{tikzcd} \Sigma^d X \wedge M(N') \wedge M(N) \arrow[r, "a_{v'} \wedge 1"] \arrow[d, "\begin{pmatrix} 1 \wedge (\mu_N \circ (\rho \wedge 1)) \\ 1 \wedge \delta_{N'} \wedge 1 \end{pmatrix}"'] & X \wedge M(N') \wedge M(N) \arrow[d, "\begin{pmatrix} 1 \wedge \mu_N \circ (\rho \wedge 1) \\ 1 \wedge \delta_{N'} \wedge 1 \end{pmatrix}"'] \\ \Sigma^d X \wedge M(N) \vee \Sigma^{d+1} X \wedge M(N) \arrow[r, "\lambda'"] & X \wedge M(N) \vee \Sigma X \wedge M(N) \end{tikzcd}$$

in which

$$\lambda' = \begin{pmatrix} \overline{a_{v'}}_{X \wedge M(N)} & 0 \\ \delta_{N'}(a)_X \wedge 1 & \overline{a_{v'}}_{X \wedge M(N)} \end{pmatrix}$$

and the vertical morphisms are equivalences; similarly for N''. Here $\overline{a_{v'}}$ is the image of $a_{v'}$ in $\pi_*(E, \mathbf{Z}/N)$. The two rows in the diagram of Proposition B.19 follow, noting that hocolim commutes with $\wedge M(N)$ [41, Lemma 5.20]. Finally, the value of the left vertical map follows from the equality $\delta_{N'} \circ \rho_{N'' \to N'} = N''/N' \delta_{N''}$. □

B.20. Define

$$\hat{a}^{-1} X = \operatorname{holim}_v a_v^{-1}(X \wedge M(l^{*v}))$$

where the transition morphisms are as in B.9. Up to homotopy, this is independent of the choice of the sequence (a_v). The maps

$$X \wedge M(l^{*v}) \to a_v^{-1}(X \wedge M(l^{*v}))$$

are compatible with each other, hence induce a canonical map

$$X^{\wedge} \to \hat{a}^{-1} X.$$

Moreover, the actions of $a_v^{-1}(E \wedge M(l^{*v}))$ onto $a_v^{-1}(X \wedge M(l^{*v}))$ fit together when v varies; hence, $\hat{a}^{-1} E$ inherits the structure of a (homotopy commutative, associative, unital) ring spectrum and $\hat{a}^{-1} X$ that of an $\hat{a}^{-1} E$-module spectrum. As a consequence of Proposition B.19, we have:

B.21. Corollary. *There is a natural homotopy equivalence*

$$\hat{a}^{-1} X \wedge M(l^v) \approx a_v^{-1}(X \wedge M(l^v))$$

for all $v \geqslant 1$ ($v \geqslant 2$ *if* $l = 2$).

Proof. From Proposition B.19, we get a homotopy equivalence

$$\operatorname{holim}_{v' \geqslant v} (a_{v'}^{-1}(X \wedge M(l^{v'})) \wedge M(l^v)) \xrightarrow{\approx} a_v^{-1}(X \wedge M(l^v)).$$

Now, by S-duality, there is a natural equivalence for any spectrum F

$$F \wedge M(l^v) \approx \operatorname{Map}_*(M(l^v), F),$$

and the natural map

$$\mathrm{Map}_*(M(l^\nu), \operatorname{holim} a_{\nu'}^{-1}(X \wedge M(l^{\nu'}))) \to \operatorname{holim} \mathrm{Map}_*(M(l^\nu), a_{\nu'}^{-1}(X \wedge M(l^{\nu'})))$$

is an equivalence [33, Lemma 5.11]. □

B.22. With notation as above, define $a^{-1}X$ as the homotopy pull-back of the diagram

$$\begin{tikzcd} & \hat{a}^{-1}X \arrow[d] \\ X[1/l] \arrow[r] & \hat{a}^{-1}X[1/l]. \end{tikzcd}$$

In particular, $(a^{-1}X)^{\wedge} \xrightarrow{\approx} \hat{a}^{-1}X$. There is a natural (homotopy class of) map

$$X \xrightarrow{\iota_a} a^{-1}X$$

given by the definition of $a^{-1}X$. As above, $a^{-1}E$ is a ring spectrum and $a^{-1}X$ an $a^{-1}E$-module.

B.23. Proposition. (a) *We have*

$$X[1/l] \xrightarrow{\approx} a^{-1}X[1/l].$$

(b) *For all* $\nu \geqslant 1$ ($\nu \geqslant 2$ *if* $l = 2$),

$$(a^{-1}X) \wedge M(l^\nu) \xrightarrow{\approx} a_\nu^{-1}(X \wedge M(l^\nu)).$$

Proof. (a) is obvious; to see (b), we need only show it for $\hat{a}^{-1}X$ instead of $a^{-1}X$. This is Corollary B.21.

B.24. Theorem. (a) (*Universal property of* $a^{-1}X$). *Let* Y *be an* E-*module and* $f: X \to Y$ *an* E-*linear morphism such that* $a_{Y \wedge M(l^*)}$ *is a homotopy equivalence. Then there is a unique (up to homotopy) factorisation*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\iota_a"'] & Y \\ & a^{-1}X. \arrow[u, "\tilde{f}"'] \end{tikzcd}$$

(b) *The functor* a^{-1} *commutes with arbitrary homotopy colimits.*

Proof. (a) Recall the homotopy cartesian diagram (cf. [5, Proposition 1.9])

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y^{\wedge} \arrow[d] \\ Y[1/l] \arrow[r] & Y^{\wedge}[1/l]. \end{tikzcd}$$

In view of this diagram, we have to construct two compatible maps $a^{-1}X \to Y[1/l]$, $a^{-1}X \to Y^{\wedge}$. The first one is obtained from the equivalence $X[1/l] \xrightarrow{\approx} a^{-1}X[1/l]$. To get the second, we observe that, by the property of Y and Proposition B.17, a_ν acts invertibly on $Y \wedge M(l^\nu)$ for all $\nu \geqslant 1$ ($\geqslant 2$ if $l = 2$). Therefore $f \wedge 1_{M(l^\nu)}$ factors into a (unique) map

$$\tilde{f}_\nu: a_\nu^{-1}(X \wedge M(l^\nu)) \to Y \wedge M(l^\nu)$$

by the universal property of localisation away from $a_\nu \in \pi_*(E \wedge \mathbf{Z}/l^\nu)$. The $\tilde{f}_\nu$ are compatible with each other (to show compatibility between $\tilde{f}_\nu$ and $\tilde{f}_{\nu+1}$, consider $Y \wedge M(l^\nu)$ as an $E \wedge M(l^{\nu+1})$-module via the map $E \wedge M(l^{\nu+1}) \to E \wedge M(l^\nu)$, observe that $a_{\nu+1}$ acts invertibly on it and apply the universal property to $a_{\nu+1}^{-1}(X \wedge M(l^{\nu+1}))$. Hence, they induce a map

$$\hat{a}^{-1}X \to Y^\wedge$$

which gives the required map by composition with $a^{-1}X \to \hat{a}^{-1}X$. The compatibility with $X[1/l] \to Y[1/l]$ is obvious. Uniqueness is proven similarly, using the same homotopy cartesian diagram.

(b) Let (X_α) be a (homotopy) direct system of E-modules. There is a natural map $a^{-1}\operatorname{hocolim} X_\alpha \xrightarrow{\varphi} \operatorname{hocolim}(a^{-1}X_\alpha)$. To see that φ is an equivalence it suffices to see that $\varphi[1/l]$ and $\varphi \wedge M(l^*)$ are. The first is obvious in view of the equivalence $Y[1/l] \xrightarrow{\approx} a^{-1}Y[1/l]$ for all Y. But the $(a^{-1}X_\alpha) \wedge M(l^*) \approx a^{-1}(X_\alpha \wedge M(l^*))$ (Proposition B.23) are defined by a homotopy colimit, so the second is obvious too. □

By the same method, we have:

B.25. Proposition. (a) (*Compare Proposition* B.2.): *For any associative, unital E-module X, denote by μ'_X the composite*

$$a^{-1}E \wedge X \xrightarrow{1 \wedge \ell_a} a^{-1}E \wedge a^{-1}X \xrightarrow{a^{-1}\mu_X} a^{-1}X.$$

Then there is a homotopy cocartesian square

$$\begin{array}{ccc} a^{-1}E \wedge E \wedge X & \xrightarrow{\mu'_E \wedge 1} & a^{-1}E \wedge X \\ {\scriptstyle 1 \wedge \mu'_X}\downarrow & & \downarrow{\scriptstyle \mu'_X} \\ a^{-1}E \wedge X & \xrightarrow{\mu'_X} & a^{-1}X. \end{array}$$

(b) (*Compare* B.3.): *If F is another ring spectrum and $f: E \to F$ is a morphism of ring spectra, then $a^{-1}F$ is a ring spectrum and $a^{-1}F$ is a morphism of ring spectra.*

B.26. As in B.5, $X \mapsto a^{-1}X$ is strictly natural for E-linear morphisms after choosing a collection of a_ν once and for all. If one wishes, one can avoid such a choice by defining a more canonical version, only depending on the l-adic isogeny class of a in the sense of [23, p. 826] by applying the construction outlined in B.5 at all stages of the l-adic tower.

B.27. *Example.* Let $E = \mathbb{S}$ and $A \in \pi_*(E, \mathbf{Z}/l^*)$ be an Adams map. It follows from a theorem of Bousfield that $A^{-1}\mathbf{L}X \approx L_1X$ for any spectrum X, where L_1 is Bousfield localisation with respect to $K(0) \wedge K(1)$ (compare e.g. [23]). Theorem B.24(b) corresponds to [30, Corollary 8.2] and Proposition B.25(a) to [30, Theorem 8.1].

B.28. *Exercise.* Show that the universal property of $a^{-1}X$ (Proposition B.23(b) + Theorem B.24(a)) formally implies Proposition B.23(a) and Theorem B.24(b).

APPENDIX C. THE CASE $l = 2$

C.1. Let $\Delta \subseteq \mathbf{Z}_2^*$ be such that $\bar{\Delta} = \mathbf{Z}/2$. We consider the quadratic extension $R^\Delta \subseteq R^{\Delta_1}$. We have $R^{\Delta_1} = \mathbf{Z}[\mu]$, where $\mu = \mu_{2^\infty}^{\Delta_1}$. If $\Delta_1 = 1 + 2^n\mathbf{Z}_2$, then $|\mu| = 2^n$. There are two cases:

The nonexceptional case: Δ is procyclic; then Δ is topologically generated by $-1 + 2^{n-1}$. In this case, $\bar{\Delta}$ acts on μ by $\zeta \mapsto \zeta^{-1+2^{n-1}}$ and $R^\Delta = \mathbf{Z}[2i\sin(2\pi/2^n)]$.

The exceptional case: $\Delta = \Delta_1 \times \{\pm 1\}$. In this case, $\bar{\Delta}$ acts on μ by $\zeta \mapsto \zeta^{-1}$ and $R^{\Delta} = \mathbf{Z}[2\cos(2\pi/2^n)]$. This is a totally real cyclotomic ring.

C.2. For $\Delta \subseteq \mathbf{Z}_2^*$ as in C.1, we define

$$\mathscr{S}^{\Delta} = \mathscr{S}(\bar{\Delta} \ltimes \mu)$$

$$\Sigma(2, \Delta) = \mathbf{L}\,\mathrm{Spt}(\mathscr{S}^{\Delta})$$

where $\mu = \mu_{2^\infty}^{\Delta_1}$. $\mathscr{S}^{\Delta}$ is a symmetric monoidal category. What is less obvious is that it has a bimonoidal structure. For X, $Y \in \mathscr{S}^{\Delta}$, define

$$X \cdot Y = X \times_{\mu} Y \quad \text{(cf. A.2)}$$

and define a $\bar{\Delta} \ltimes \mu$-action on it by letting μ act as usual and $\bar{\Delta}$ by

$$g(x, y)_{\mu} = (gx, gy)_{\mu}.$$

This law makes $\mathscr{S}^{\Delta}$ a *nonunital* symmetric bimonoidal category, hence $\Sigma(2, \Delta)$ is a nonunital commutative and associative ring spectrum. If $\Delta' \subseteq \Delta$, there is a multiplicative functor

$$\mathscr{S}^{\Delta} \xrightarrow{\iota_{\Delta'/\Delta}} \mathscr{S}^{\Delta'}$$

defined as in Appendix A if $\bar{\Delta}' = \bar{\Delta}$ and by first forgetting the $\bar{\Delta}$-action if $\bar{\Delta}' = 1$. There is a corresponding morphism $\Sigma(2, \Delta) \xrightarrow{\iota_{\Delta'/\Delta}} \Sigma(2, \Delta')$. There is a natural action of Δ/Δ' over $\Sigma(2, \Delta')$ and it is clear that $\iota_{\Delta'/\Delta}$ is equivariant for this action. (One could also define a transfer extending that of Section 3.)

Note that $\mathbf{Z}_2^* = \mathrm{Aut}(\mathbf{Q}_2/\mathbf{Z}_2)$ acts continuously on $\mathrm{j}(2, 1)$. Motivated by Theorem 4.15(c), we give the following definition:

C.3. *Definition.* Let Δ be an *arbitrary* closed subgroup of $\mathbf{Z}_2^*$. We define

$$\mathrm{j}(2, \Delta) = \mathbb{H}^{\cdot}(\Delta, \mathrm{j}(2, 1))_{\geq 0}.$$

We define a morphism of ring spectra

$$\ell^{\Delta}\colon \Sigma(2, \Delta) \to \mathrm{j}(2, \Delta)$$

by truncating the composite

$$\Sigma(2, \Delta) \to \mathbb{H}^{\cdot}(\Delta, \Sigma(2, 1)) \xrightarrow{\mathbb{H}^{\cdot}(\Delta, \ell^1)} \mathbb{H}^{\cdot}(\Delta, \mathrm{j}(2, 1)).$$

C.4. Let Δ, μ be as in C.1 and $X \in \mathscr{S}^{\Delta}$. The free R^{Δ_1}-module $R^{\Delta_1}X$ is endowed with an action of $\bar{\Delta}$ given by $\overline{s[x]} = \bar{s}[\bar{x}]$ for $(s, x) \in R^{\Delta_1} \times X$, where $^-$ denotes the $\bar{\Delta}$-action on R^{Δ_1} and X. Define an R^{Δ_1}-linear equivalence relation $\approx$ on $R^{\Delta_1}X$, as in A.9, by $\zeta[x] \approx [\zeta x]$ for $(\zeta, x) \in \mu \times X$. This relation commutes with the action of $\bar{\Delta}$; the invariants

$$(R^{\Delta_1}X/\approx)^{\bar{\Delta}}$$

form a projective R^{Δ}-module of rank $|X/\mu|$. Actually, $(R^{\Delta_1}X/\approx)^{\bar{\Delta}}$ is generated by the elements of the form $r[x] + \bar{r}[\bar{x}]$ for $r \in R^{\Delta_1}$, $x \in X$. If $X = \bar{\Delta} \ltimes \mu$ with left-translation action, the map $r \mapsto r[1] + \bar{r}[-1]$ is an isomorphism of the R^{Δ}-module R^{Δ_1} onto $(R^{\Delta_1}X/\approx)^{\bar{\Delta}}$.

C.5. In the non-exceptional case, define a functor

$$\mathscr{S}^{\Delta} \xrightarrow{\Phi^{\Delta}} \mathscr{P}(R^{\Delta})$$

by

$$X \mapsto (R^{\Delta_1}X/\approx)^{\bar{\Delta}}.$$

This is clearly a symmetric monoidal functor. Moreover,

C.6. PROPOSITION. *Φ^{Δ} is multiplicative.*

Proof. Define a natural transformation

$$\Phi^{\Delta}(X) \otimes_{R^{\Delta}} \Phi^{\Delta}(Y) \to \Phi^{\Delta}(X \times_{\mu} Y) \tag{C1}$$

via the multiplicativity of Φ^{Δ_1} and the obvious map

$$M^{\bar{\Delta}} \otimes_{R^{\Delta}} N^{\bar{\Delta}} \to (M \otimes_{R^{\Delta_1}} N)^{\bar{\Delta}}$$

for two $\bar{\Delta}$-equivariant R^{Δ_1}-modules M, N. To check that (C1) is bijective, we may assume by distributivity that $X = Y = X_0$, where $X_0 = \bar{\Delta} \rtimes \mu$ provided with the left-translation action. Let $\bar{\Delta} = \{1, -1\}$. Then $\Phi^{\Delta_1}(X_0)$ has R^{Δ_1}-basis $[1], [-1]$, hence $\Phi^{\Delta}(X_0)$ has R^{Δ}-basis $[1]$, $i[-1]$ and $\Phi^{\Delta}(X_0) \otimes_{R^{\Delta}} \Phi^{\Delta}(X_0)$ has R^{Δ}-basis $[1] \otimes [1]$, $[1] \otimes i[-1]$, $i[-1] \otimes [1]$, $i[-1] \otimes i[-1]$ (where $i = \sqrt{-1}$). On the other hand, $\Phi^{\Delta_1}(X_0 \times_{\mu} X_0)$ also has R^{Δ_1}-basis $[1] \otimes [1]$, $[1] \otimes i[-1]$, $i[-1] \otimes [1]$, $i[-1] \otimes i[-1]$, hence $\Phi^{\Delta}(X_0 \cdot X_0)$ has the same R^{Δ}-basis. □

C.7. For a commutative ring R, recall the L-theory spectrum

$$LR = \mathrm{Spt}(\mathcal{Q}(R)),$$

where $\mathcal{Q}(R)$ is the symmetric bimonoidal category of finitely generated projective R-modules provided with unimodular symmetric bilinear form. There is a forgetful functor $\mathcal{Q}(R) \to \mathcal{P}(R)$, which is equivariant relatively to the $\mathbf{Z}/2$-action on $\mathcal{P}(R)$ given by

$$P \mapsto \mathrm{Hom}(P, R), \qquad f \mapsto {}^t f.$$

We have $\Omega^{\infty} LR = L_0(R) \times BO(R)^+$, where $O(R)$ is the infinite orthogonal group of R.

For the sake of smoothness of the exposition, let $\mathcal{Q}'(R)$ denote temporarily the symmetric bimonoidal category of finitely generated projective R-modules provided with a non-necessarily unimodular symmetric bilinear form.

C.8. In the exceptional case, define a functor

$$\mathcal{S}^{\Delta} \xrightarrow{\Phi^{\Delta}} \mathcal{Q}'(R^{\Delta})$$

by

$$X \mapsto ((R^{\Delta_1}X/\approx)^{\bar{\Delta}}, b)$$

where the symmetric bilinear form b is given by

$$b(rx + \bar{r}[\bar{x}], sy + \bar{s}[\bar{y}]) = \begin{cases} 0 & \text{if } x \notin \{y, \bar{y}\} \\ \mathrm{Tr}_{R^{\Delta_1}/R^{\Delta}}(r\bar{s}) & \text{if } x = y \end{cases}$$

for $x, y \in X$ and $r, s \in R^{\Delta_1}$.

Note that this assignment really defines a (symmetrical monoidal) functor because of the relation $\bar{\zeta} = \zeta^{-1}$, which ensures that elements of μ define isometries of the symmetric bilinear form. This would not hold in the non-exceptional case.

C.9. LEMMA. *The quadratic form* $b_0:(r,s)\mapsto Tr_{R^{\Delta_1}/R^\Delta}(r\bar{s})$ *on the* R^Δ*-module* R^{Δ_1} *has* $(1,\sqrt{-1})$ *as an orthogonal basis, and* $b_0(1,1)=b_0(\sqrt{-1},\sqrt{-1})=2$.

C.10. PROPOSITION. Φ^Δ *is multiplicative.*

Proof. Use the natural transformation (18) and check that it respects the bilinear structures. □

C.11. Note that b_0 in Lemma C.9 is not unimodular, but becomes so when we invert 2 in R^Δ. So Φ^Δ in the exceptional case lands into $\mathscr{Q}(R^\Delta[\frac{1}{2}])$ when composed with the functor $\mathscr{Q}'(R^\Delta)\to\mathscr{Q}'(R^\Delta[\frac{1}{2}])$. If we took $\frac{1}{2}\mathrm{Tr}_{R^{\Delta_1}/R^\Delta}(r\bar{s})$ instead of $\mathrm{Tr}_{R^{\Delta_1}/R^\Delta}(r\bar{s})$ in C.8, we would get a unimodular form, but would lose multiplicativity.

C.12. The map $\Phi^\Delta:\Sigma(2,\Delta)\to \mathbf{L}KR^\Delta$ or $\mathbf{L}LR^\Delta[\frac{1}{2}]$ defined by the functor Φ^Δ above can be described on the 0-spaces as follows. We have

$$\Omega^\infty\mathrm{Spt}(\mathscr{S}^\Delta)\approx \mathbf{Z}\times B(\bar{\Delta}\ltimes\mu)^+$$

$$\Omega^\infty KR^\Delta\approx K_0(R^\Delta)\times BGL(R^\Delta)^+$$

$$\Omega^\infty LR^\Delta\approx L_0(R^\Delta)\times BO(R^\Delta)^+.$$

In the non-exceptional case, $\Omega^\infty\Phi^\Delta$ maps the generator of $\mathbf{Z}=\pi_0(\mathrm{Spt}(\mathscr{S}^\Delta))$ to $[R^{\Delta_1}]\in K_0(R^\Delta)$. The map on the $+$-constructions is induced by the group homomorphism

$$\bar{\Delta}\ltimes\mu\to \mathrm{GL}(R^{\Delta_1}_{(R^\Delta)})$$

given by the action of μ (resp. $\bar{\Delta}$) on R^{Δ_1} by homotheties (resp. by Galois action). In the exceptional case, $\Omega^\infty\Phi^\Delta$ maps the generator of $\mathbf{Z}=\pi_0(\mathrm{Spt}(\mathscr{S}^\Delta))$ to $\langle 2,2\rangle\in L_0(R^\Delta[\frac{1}{2}])$ and the map on the $+$-constructions is induced by the same group homomorphism as above, which this time lands into $\mathrm{O}(\mathrm{Tr}_{R^{\Delta_1}/R^\Delta}(x\bar{x}))$.

We now want to recognise the spectra $\mathrm{j}(2,\Delta)$ as more familiar objects. The method is to start from the equivalence

$$\mathrm{j}(2,\{1\})\xrightarrow{\approx}\mathbf{L}KE'$$

deduced from Propositions 4.10, Lemma 4.14 and a passage to the limit, where $E'=E(\mu_{2^\infty})$ for a finite field E of characteristic $\neq 2$, hence an equivalence

$$\mathrm{j}(2,\Delta)\xrightarrow{\approx}\mathbb{H}^\cdot(\Delta,\mathbf{L}KE')$$

and to recognise the latter hypercohomology spectrum. For our effort we also get a splitting result. First the non-exceptional case:

C.13. PROPOSITION. *Let* Δ, *as in* C.1, *be procyclic.*

(a) *Let E be a finite field of characteristic* $\neq 2$ *such that* $\kappa_2(G_E)=\Delta$ (*Lemma* 4.5). *Then there is a canonical equivalence*

$$\mathrm{j}(2,\Delta)\xrightarrow{\approx}\mathbf{L}KE.$$

(b) $\Omega_0^\infty\ell^\Delta$ *has homotopy sections.*

Proof. (a) Let $E'=E(\mu_{2^\infty})$. We have $\mathrm{Gal}(E'/E)=\Delta$. Using the computation of K_*E in [27] and a computation of $\pi_*\mathbb{H}^\cdot(\Delta,\mathbf{L}KE')_{\geq 0}$ similar to that in the proof of Theorem 4.15(c), we get that $\mathbf{L}KE\to\mathbb{H}^\cdot(\Delta,\mathbf{L}KE')_{\geq 0}$ is an equivalence (note for this computation that

$\Delta \simeq \mathbf{Z}_2$, hence exactly the same arguments apply). More specifically, on the E side, we get by a computation similar to [3]

$$K_*(E, \mathbf{Z}/4) = \Lambda(x) \otimes \mathbf{Z}/4[b, B]$$

with $\deg x = 1$, $\deg b = 2$, $\deg B = 4$, subject to the only relations $2x = 2b = 0$, $b^2 = 2B$, $\partial b = x$. If $E'' = E(\sqrt{-1})$, the image of b (resp. B) in $K_*(E'', \mathbf{Z}/4)$ is 2β (resp. β^2), where β is a Bott element as in Section 3.

(b) The claim will follow from the Harris–Segal theorem that $B(\mathfrak{S}_\infty \wr (\bar{\Delta} \ltimes \mu)_+) \to \mathrm{BGL}(E)^+$ has homotopy sections, as in the proof of Theorem 5.1, provided we show that the diagram

$$\begin{array}{ccc} \Sigma(2, \Delta) & \xrightarrow{\Phi^\Delta} & \mathbf{L}KR^\Delta \\ {\scriptstyle \ell^\Delta}\downarrow & & \downarrow \\ \mathrm{j}(l, \Delta) & \xrightarrow{\approx} & \mathbf{L}KE \end{array} \tag{C2}$$

is commutative. To see this, we simply note that the corresponding diagram replacing Δ by $\{1\}$ and E by E' commutes (by definition of the maps in this case!) and that (C2) maps to its homotopy fixed points, the map being an equivalence on the bottom rows. □

C.14. Corollary. *For $l = 2$, Corollary 4.11 extends to the case where -1 is not a square in E.*

Now the exceptional case. Let $\Delta = \{\pm 1\} \times \Delta_1$. If $\Delta_1 \neq \{1\}$, i.e. $\Delta_1 = 1 + 2^n\mathbf{Z}_2$, we set $\Delta_1' = (-1 + 2^n)\mathbf{Z}_2$; otherwise we set $\Delta_1' = \{1\}$. Note that we can equally well write

$$\Delta = \{\pm 1\} \times \Delta_1'.$$

For simplicity, say that a field is *ind-finite* if it is a union of finite fields (i.e. is of positive characteristic and algebraic over its prime subfield). We want to outline a proof of:

C.15. Proposition. *Let Δ be as above.*

(a) *Let E be an ind-finite field of characteristic $\neq 2$ such that $\kappa(G_E) = \Delta_1$ or Δ_1'. Then there is a canonical equivalence*

$$\mathrm{j}(2, \Delta) \xrightarrow{\approx} \mathbf{L}LE.$$

(b) *If Δ is infinite, $\Omega_0^\infty \ell^\Delta$ has homotopy sections.*

(c) *If $\Delta = \{\pm 1\}$, there is a canonical equivalence*

$$\mathrm{j}(2, \{\pm 1\})\hat{} \approx \mathrm{bo}\hat{}.$$

Recall:

C.16. Proposition. (a) (cf. Suslin [38]). *The natural map*

$$\mathrm{KO} \to \mathrm{KU}^{h\mathbf{Z}/2}$$

is an equivalence, where $\mathbf{Z}/2$ acts on U by complex conjugation.

(b) [9, Theorem III 3.1 d] *Let $\mathbf{F}_q$ be a finite field with q elements, where 1 is odd. Then the "Brauer lifting"*

$$L\mathbf{F}_q \to F\Psi_O^q[1/q]$$

is an equivalence of spectra, where $F\Psi^q_O[1/q]$ is the homotopy fixed point spectrum of Ψ^q on $\mathrm{bo}[1/q]$. □

Proposition C.15(c) immediately follows from the Proposition C.16(a), by comezlpleting, truncating and applying Corollary 4.16. To prove Proposition C.15(a), let Γ denote either Δ_1 or Δ'_1. We need only prove that the map

$$\mathbf{L}LE \to \mathbb{H}^\cdot(\Delta, \mathbf{L}KE')_{\geqslant 0}$$

is an equivalence, where $E' = E(\mu_{2^\infty})$ and the action of Δ on $\mathbf{L}KE'$ is defined as follows: Γ acts by Galois action and $\{\pm 1\}$ acts by inverse-transpose matrices. We have

$$\mathbb{H}^\cdot(\Delta, \mathbf{L}KE')_{\geqslant 0} \approx \mathbb{H}^\cdot(\{\pm 1\}, \mathbb{H}^\cdot(\Delta, \mathbf{L}KE')_{\geqslant 0})_{\geqslant 0} \approx \mathbb{H}^\cdot(\{\pm 1\}, \mathbf{L}KE))_{\geqslant 0} = (\mathbf{L}KE)^{h\{\pm 1\}}_{\geqslant 0}.$$

It follows from Proposition C.16(a) that the map

$$\Psi^q_O[1/q] \to \Psi^q[1/q]^{h\mathbf{Z}/2}$$

is an equivalence. Proposition C.15(a) follows from this fact and Proposition C.16(b). Finally, Proposition C.15(b) follows from (a) and an analogue of the Harris–Segal theorem (see [11]) by the same argument as for Proposition C.13. □

C.17. *Example.* For $\Delta = \mathbf{Z}_2^*$ and $E = \mathbf{F}_3$, we get back a well-known description of $\operatorname{Im} J$ at 2. (This is not quite true: the usual $\operatorname{Im} J$ at 2 has $\pi_0(J) = \mathbf{Z}$, while $\pi_0(\mathrm{j}(2, \mathbf{Z}_2^*)) \simeq L_0(\mathbf{F}_3) \simeq \mathbf{Z} \oplus Z/2$.) Taking $E = \mathbf{F}_5$ gives another, less usual one.

C.18. From Propositions C.13 and C.15, we get retracts:

$$\Omega_0^\infty \mathrm{j}(2, \Delta) \xrightarrow{\beta^\Delta} \Omega_0^\infty \mathbf{L}KR^\Delta$$

in the non-exceptional case

$$\Omega_0^\infty \mathrm{j}(2, \Delta) \xrightarrow{\beta^\Delta} \Omega_0^\infty \mathbf{L}KR^\Delta[\tfrac{1}{2}]$$

in the exceptional case. Note that they were defined differently from those of Section 5: we do not have an immediately obvious Bott element to localise when -1 is not a square. These maps will be investigated in [11].

APPENDIX D. THE LICHTENBAUM–QUILLEN CONJECTURE AND A CONJECTURE OF MITCHELL

Recall that the Lichtenbaum–Quillen conjecture predicts isomorphisms:

$$\mathrm{ch}_{i,1}: K_{2i-1}(R_K) \otimes \mathbf{Z}_l \to H^1(R_K, \mathbf{Z}_l(i))$$

$$\mathrm{ch}_{i,2}: K_{2i-2}(R_K) \otimes \mathbf{Z}_l \to H^2(R_K, \mathbf{Z}_l(i))$$

for any number field K (containing $\sqrt{-1}$ if $l = 2$). Here the cohomology groups are l-adic étale cohomology.

Soulé defined the homomorphisms $\mathrm{ch}_{i,1}$ and $\mathrm{ch}_{i,2}$ for $i < l$ in [33], proved that $\mathrm{ch}_{i,2}$ is surjective [33] and that $\mathrm{ch}_{i,1}$ is surjective with finite kernel [34]. Dwyer and Friedlander introduced étale K-theory, that they used to define $\mathrm{ch}_{i,1}$ and $\mathrm{ch}_{i,2}$ and prove their surjectivity for all i [6]. Finally, Thomason proved that étale K-theory coincides with Bott-localised algebraic K-theory for a large class of schemes including rings of S-integers in number fields

[41]. Following these developments, the original Lichtenbaum–Quillen conjecture has been reformulated in terms of Bott-localised algebraic K-theory:

D.1. CONJECTURE. *For any number field K (containing $\sqrt{-1}$ if $l = 2$), any $i > 0$ and any $v \geqslant 1$, the natural map $K_i(R_K, \mathbf{Z}/l^v) \to \beta^{-1}K_i(R_K, \mathbf{Z}/l^v)$ is an isomorphism.*

On the other hand, it is conjectured in [7] that maps λ_p equivalent to β^Δ are infinite loop maps. The aim of this section is to point out that this conjecture follows from Conjecture D.1 applied to the l-cyclotomic extensions of $\mathbf{Q}$. This was observed before by Mitchell [22], and actually motivated [21] (personal communication). More precisely, we have:

D.2. THEOREM. *Assume that Conjecture* D.1 *holds for R^Δ, where Δ is open in $\mathbf{Z}_l^*$ (in $1 + 4\mathbf{Z}_l$ if $l = 2$). Then the map β^Δ of Definition 5.6 extends to a morphism of spectra (still denoted by) $\beta^\Delta : \mathrm{j}(l, \Delta) \to \mathbf{L}KR^\Delta$.*

Proof. Truncate the map β^Δ of Proposition 4.18 above 0 to get

$$(\beta^\Delta)_{\geqslant 0} : \mathrm{j}(l, \Delta) \to (\beta^{-1}\mathbf{L}KR^\Delta)_{\geqslant 0}.$$

Theorem D.2 now follows from:

D.3. LEMMA. *Under the hypothesis of Theorem* D.2, *the natural map $\mathbf{L}KR^\Delta \to (\beta^{-1}\mathbf{L}KR^\Delta)_{\geqslant 0}$ is a homotopy equivalence.*

Proof. Since, by Quillen's theorem [29], the K-groups of R^Δ are finitely generated, it suffices to see this after smashing by $M(l^v)$. Conjecture D.1 asserts that $\pi_i(\mathbf{L}KR^\Delta \wedge M(l^v)) \to \pi_i((\beta^{-1}(\mathbf{L}KR^\Delta \wedge M(l^v)))_{\geqslant 0})$ is an isomorphism for $i > 0$. It remains to look at $i = 0$. The left-hand side is $K_0(R^\Delta, \mathbf{Z}/l^v) = K_0(R^\Delta)/l^v = \mathbf{Z}/l^v \oplus \mathrm{Pic}(R^\Delta)/l^v$. By the descent spectral sequence for Bott-localised algebraic K-theory [41], the right-hand side is $\beta^{-1}K_0(R^\Delta, \mathbf{Z}/l^v) = \mathbf{Z}/l^v \oplus H^2(R^\Delta, \mathbf{Z}/l^v(1))$. The map $K_0(R^\Delta, \mathbf{Z}/l^v) \to \beta^{-1}K_0(R^\Delta, \mathbf{Z}/l^v)$ induces the natural injection $\mathrm{Pic}(R^\Delta)/l^v \hookrightarrow H^2(R^\Delta, \mathbf{Z}/l^v(1))$ coming from the isomorphism $\mathrm{Pic}(R^\Delta) = H^1(R^\Delta, \mathbb{G}_m)$ and the Kummer exact sequence $1 \to \mu_{l^v} \to \mathbb{G}_m \xrightarrow{l^v} \mathbb{G}_m \to 1$. The next term in the corresponding long exact sequence is $H^2(R^\Delta, \mathbb{G}_m) = \mathrm{Br}(R^\Delta)$. But since there is only one place above l in R^Δ, one has $\mathrm{Br}(R^\Delta) = 0$ by the Albert–Hasse–Brauer–Noether theorem. Hence, $\mathrm{Pic}(R^\Delta)/l^v \to H^2(R^\Delta, \mathbf{Z}/l^v(1))$ is an isomorphism, and so is $K_0(R^\Delta, \mathbf{Z}/l^v) \to \beta^{-1}K_0(R^\Delta, \mathbf{Z}/l^v)$. □

From Proposition 4.6, we can recover the formulation of [7] or [22].

APPENDIX E. A REFINEMENT

Theorem 9.3(v) says that the maps β_X^i have retractions. We shall refine this result to the space level for certain types of schemes. We take notation as at the end of Section 9.

E.1. THEOREM. *Assume that X is a scheme of finite type over $\mathbf{Z}[1/l]$ ($\mathbf{Z}[\frac{1}{2}, \sqrt{-1}]$ if $l = 2$). Then there exist morphisms of spectra $\alpha : \mathbf{L}KX \to \mathrm{j}_l(X)$ such that $(\Omega^\infty\alpha)\beta_X \simeq Id$.*

Proof. Up to normalising X, we may assume that it is normal and irreducible. There are two cases to consider:

(a) *The generic fibre of* $f: X \to \operatorname{Spec} \mathbf{Z}[1/l]$ *is empty*. Then X is defined over $\mathbf{F}_q$ for some finite field $\mathbf{F}_q$. We may assume that $\mathbf{F}_q$ is the field of constants of X. Then $\mathrm{j}_l(X) = \mathrm{j}_l(\mathbf{F}_q) \simeq \mathbf{L}K\mathbf{F}_q$ (see Proposition 9.7). Any closed point $x \in X$ defines a composite $\sigma_x: KX \to K\mathbf{F}_q(x) \to K\mathbf{F}_q$, where the last map is the transfer. The composite $K\mathbf{F}_q \to KX \xrightarrow{\sigma_x} K\mathbf{F}_q$ is multiplication by $\deg(x)$. By [17, Corollary 3], there exists on X a rational zero-cycle of degree 1. Taking the corresponding linear combination of the σ_x, we get the desired map α.

(b) *The generic fibre of f is nonempty*. We may assume that X is affine and integral. Let R be the maximal ring of S-integers through which f factors. Choose a prime $\mathfrak{p}$ of R such that $\Delta_{R/\mathfrak{p}} = \Delta_R$ and the closed fibre of X at $\mathfrak{p}$ is nonempty and geometrically connected over $R/\mathfrak{p}$ (since X is of finite type, there are infinitely many of them, see [25, Satz XVII]). Taking this closed fibre, we are reduced to case (a).

E.2. Corollary. *Under the hypothesis of Theorem* 9.5, $H^0(X, \mathbf{Q}_l/\mathbf{Z}_l(i))$ *is a direct summand of* $K_{2i-1}(X)$.

E.3. Note that in general the maps α of Theorem 9.5 are not ring spectra homomorphisms. However, by Proposition 8.1 and Theorem 9.3(v), the composition $\mathbf{L}KX \xrightarrow{\alpha} \mathrm{j}_l(X) \to \mathrm{j}(l, 1) \to \mathrm{bu}^\wedge$ is independent of α and is multiplicative. If $X = \operatorname{Spec} \mathcal{O}_S$ where $\mathcal{O}_S$ is a ring of S-integers in a number field, any prime $\mathfrak{p}$ as in (b) above defines a retraction $\alpha_\mathfrak{p}$ which is a homomorphism of ring spectra. The dependence of $\alpha_\mathfrak{p}$ on $\mathfrak{p}$ is a subtle question to which I hope to come back to in a future paper.

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