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Some finiteness results for étale cohomology

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Abstract

We prove some finiteness theorems for the étale cohomology, Borel–Moore homology and cohomology with proper supports with divisible coefficients of schemes of finite type over a finite or p-adic field. This yields vanishing results for their l-adic cohomology, proving part of a conjecture of Jannsen.

Introduction

Let p be a prime number and $n \in \mathbb{Z}$. We denote by $(\mathbb{Q}/\mathbb{Z})'(n)$ the étale sheaf $\varinjlim \mu_m^{\otimes n}$ over the big étale site of Spec $\mathbb{Z}_{(p)}$, where m runs through the integers prime to p and $\mu_m^{\otimes n}$ denotes the sheaf of mth roots of unity, twisted n times. We denote by

$$e \in H^1(\mathbb{F}_p, \hat{\mathbb{Z}})$$

the generator sending Frobenius to 1.

We shall consider étale cohomology, Borel–Moore étale homology and cohomology with proper supports. Recall that the first notion is absolute while the two others are relative to a base. More precisely, if $f : X \to S$ is a compactifiable morphism, with m invertible on S, then one defines

- $H_c^i(X/S, \mathbb{Z}/m(n)) = H^i(S, Rf_! \mu_m^{\otimes n})$;
- $H_i^c(X/S, \mathbb{Z}/m(n)) = H^{-i}(X, Rf^! \mu_m^{\otimes -n})$.

Here $Rf_!$ is higher direct image with proper supports and $Rf^!$ is its right adjoint [2,3]. When S is unambiguous, we shall usually drop it from the notation.

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For simplicity, we abbreviate the étale cohomology groups $H^i_{\text{ét}}(X, (\mathbb{Q}/\mathbb{Z})'(n))$ into $H^i(X,n)$ for any X over $\operatorname{Spec}\mathbb{Z}_{(p)}$; similarly for Borel–Moore homology and cohomology with proper supports. We shall use without further mention that all the groups with these coefficients we encounter here are of cofinite type, i.e. direct sums of a finite group and finitely many copies of $\mathbb{Q}_l/\mathbb{Z}_l$, for all $l \neq p$. This follows from the classical finiteness results for étale cohomology [6, Th. finitude].

In this paper, we prove finiteness results for the groups $H^i(X,n)$, $H^i_c(X,n)$ and $H_i^c(X,n)$ in certain ranges, where X is a variety over $\mathbb{F}_p$ or $\mathbb{Q}_p$. A typical such result is of the form: "$H^i(X,n)$ is finite unless $n \in [a,b]$ and $i \in [a',b']$", where a,b,a',b' are specific integers depending on the situation. See Theorems 1–3 and 6 for precise statements. Theorem 6 relies on Theorems 2 and 3.

The method of proof is not especially original: it can be described as a dévissage from the case of smooth projective varieties over $\mathbb{F}_p$ [5], using purity results and de Jong's alteration theorems. With the above notation, while our results over $\mathbb{F}_p$ for individual l-primary components of $H^i_c(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$ readily follow from [7] for $i < a'$, it is not clear to us that they do for $i > b'$; analogously for the finiteness results on $H^i(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$. Getting finiteness not only for individual l-primary components but for the full group relies of course on Gabber's theorem [9].

From Theorem 6 we deduce finiteness results for the groups $H^j(G, H^{i-j}(\bar{X}, n))$, where X is a smooth projective variety over $\mathbb{Q}_p$ and $G = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$: see Theorem 10. This proves part of a conjecture of Jannsen [13, Conjecture 3 and Remark 5] on the range of vanishing of the groups $H^j(G, H^{i-j}(\bar{X}, \mathbb{Q}_l(n)))$. In fact, the range in which we find that these groups vanish overlaps with that predicted by Jannsen: see picture in Section 4. On the other hand, Jannsen's conjecture does not imply any more vanishing for the groups $H^i(X, \mathbb{Q}_l(n))$ than what we get as a consequence of Theorem 6, which is therefore optimal in this sense.

By an easy extension of [13, Corollary 5] using Hard Lefschetz, Poincaré duality and local Tate duality, we get corresponding finiteness statements for $H^j(G, H^{i-j}(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ when $j = 0, 2$ (this relies on Faltings' Hodge–Tate decomposition). Surprisingly, the range obtained is exactly the same as the one away from p.

The main motivation for writing this paper is the application of Theorem 1 to an equivalence between three arithmetic conjectures on varieties over $\mathbb{F}_p$: [16, Theorem 3.4]. The extension of the finiteness results from varieties over $\mathbb{F}_p$ to varieties over $\mathbb{Q}_p$ was motivated by the search for a similar set of conjectures for the latter varieties; the one issue that prevents us to formulate them at the moment is the lack of a clear understanding of the question raised in Section 6. Theorems 2 and 6 are also used in the proof of [15, Corollary 2].

1. Varieties over a finite field

Theorem 1. *Let X be a smooth variety of dimension d over $\mathbb{F}_p$ and let $n \in \mathbb{Z}$. Then the group $H^i(X,n)$ is finite in the following cases*:

(1) $i \notin [n, 2n+1]$;
(2) $n<0$;
(3) $n>d$.

This group is 0 *for* $i>2d+1$, *and for* $i>d+1$ *if* X *is affine.*
Moreover, cup-product by e

$$H^0(X,0) \to H^1(X,0)$$

has finite kernel and cokernel.

Proof. We argue by induction on d, starting with case (1).

(a) By [1, Theorem 2], the statement is true for X projective: in this case, $H^i(X,n)$ is actually finite for all $i \neq 2n, 2n+1$. In particular, Theorem 1(1) is true when $\dim X = 0$.

(b) Suppose that X is an open subset of a smooth variety X': we claim that Theorem 1(1) holds for X if and only if it holds for X'. It is convenient to reason modulo finite abelian groups, that is, within the category A quotient of the category of abelian groups by the thick subcategory of finite abelian groups. Let $Z = X' - X$, with reduced structure. Let Z_1 be the singular locus of Z and $X_1 = X' - Z_1$, so that $X_1 - X = Z - Z_1$ is smooth. We have a long exact sequence of cohomology groups with supports:

$$\begin{aligned} \cdots \to H^i_{Z-Z_1}(X_1, n) \to H^i(X_1, n) \\ \to H^i(X, n) \to H^{i+1}_{Z-Z_1}(X_1, n) \to \cdots \end{aligned} \tag{1.1}$$

By purity, $H^i_{Z-Z_1}(X_1, n)$ decomposes according to the connected components Z_α of the smooth variety $Z - Z_1$ into a direct sum $\oplus_\alpha H^{i-2c_\alpha}(Z_\alpha, n - c_\alpha)$, where c_α is the codimension of Z_α in X_1. By induction on $\dim X$, these groups are all finite as long as $i - 2c_\alpha \notin [n - c_\alpha, 2n - 2c_\alpha + 1]$ or $i \notin [n + c_\alpha, 2n+1]$. In particular, the two extreme groups in (1.1) are finite provided we assume that $i \notin [n, 2n+1]$. So the map

$$H^i(X_1, n) \to H^i(X, n)$$

is an isomorphism in A. By induction on $\dim Z$, this shows that $H^i(X', n) \to H^i(X, n)$ is an isomorphism in A, and so the two groups are finite together.

In particular, Theorem 1(1) is true if X is an open subset of a smooth, projective variety.

(c) In general, de Jong's theorem [14] allows us to find an alteration $\pi : Y \to X$ which is generically étale, with Y open in a smooth projective variety. Let U be an open subset of X such that $\pi_{|\pi^{-1}(U)}$ is finite étale. We shall show that Theorem 1 holds for U; hence, by (b), it will hold for X and the proof will be complete. Since $H^i(U, \mathbb{Q}_l/\mathbb{Z}_l(n))$ is known to be a co-finitely generated $\mathbb{Z}_l$-module for all $l \neq p$ (a sum of a finite l-primary group and a finite number of copies of $\mathbb{Q}_l/\mathbb{Z}_l$), it will suffice to show that $H^i(U, n)$ is annihilated by some nonzero integer.

By (b), Theorem 1(1) is true for $\pi^{-1}(U)$. The composition

$$H^i(U,n) \xrightarrow{\pi^*} H^i(\pi^{-1}(U),n) \xrightarrow{\pi_*} H^i(U,n)$$

is multiplication by the degree N of $\pi_{|\pi^{-1}(U)}$. Hence $H^i(U,n)$ is annihilated by NM, where M is the order of the finite group $H^i(\pi^{-1}(U),n)$. This completes the proof of Theorem 1 in case (1).

Case (2) is a special case of case (1).

The two vanishing statements are clear from the known cohomological dimension of varieties and affine varieties over a field.

To prove the statement on cup-product by e, we may assume that X is connected. It suffices to prove that the map

$$H^i(k,0) \to H^i(X,0)$$

is an isomorphism in A for all i, where k is the field of constants of X. The proof goes along the same lines: the isomorphism is clear in case X is projective, and then in general by using the finiteness of $H^i(Y,n)$ for Y smooth and $n<0$. (In fact, one easily checks that cup-product by e is injective on H^0.)

It remains to prove Theorem 1 in case (3). Assume $n>d$. As noted above, for X smooth projective, $H^i(X,n)$ is finite for all $i \neq 2n, 2n+1$. But for $i=2n$ or $2n+1$ it is 0 as seen above (cohomological dimension), hence also finite. The fact that this remains true for all smooth X can then be proven along the same steps as for case (1). □

Note that the proof of [1, Theorem 2] rests on Gabber's theorem that, for X smooth projective over $\bar{\mathbb{F}}_p$ and $i \geqslant 0$, the group $H^i_{\text{ét}}(\bar{X}, \mathbb{Z}_l)$ is torsion-free for almost all l [9]. The above proof avoids the issue whether Gabber's theorem remains true for X smooth, not necessarily projective.

Theorem 2. *Let X be a variety of dimension d over $\mathbb{F}_p$ and let $n \in \mathbb{Z}$. Then the group $H^i(X,n)$ is finite in the following cases*:

(1) $n \notin [0,d]$
(2) $i \notin [n, n+d+1]$.

This group is 0 *for* $i>2d+1$, *and for* $i>d+1$ *if X is affine.*

Proof. We again argue by induction on d.

We first reduce to X reduced, then to X integral by closed Mayer–Vietoris, which is valid for étale cohomology with torsion coefficients (a trivial application of the proper base change theorem!) Then, de Jong [14, Theorem 7.3], there exists a finite extension $k/\mathbb{F}_p$ and a connected smooth projective k-variety Y provided with the action of a finite group G, an open G-invariant subscheme $\tilde{X} \subseteq Y$, a proper map

$f:\tilde{X}\to X_k$ and a subset $Z\subsetneq X_k$ such that $f^{-1}(U)\to U$ is a quasi-Galois covering of group G, with $U=X_k-Z$.

Let $\tilde{Z}=f^{-1}(Z)$, so that we have the commutative diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d, "hg"'] & \widetilde{X} \arrow[d, "f"'] & \widetilde{Z} \arrow[l] \arrow[d, "f'"'] \\ U \arrow[r, "j"] & X_k & Z. \arrow[l, "i"'] \end{tikzcd}$$

By using proper base change, we obtain a long exact sequence

$$\begin{aligned}\cdots &\to H^{i-1}(\tilde{Z},G,n)\to H^i(X_k,n)\\ &\to H^i(Z,n)\oplus H^i(\tilde{X},G,n)\to H^i(\tilde{Z},G,n)\to\dots\end{aligned}$$

where $H^i(\tilde{X},G,n)$ is equivariant étale cohomology [11]. More precisely, let $D_G(X_k)$ be the derived category of G-equivariant étale sheaves on X_k, $D(X_k)$ the derived category of ordinary étale sheaves, $\iota: D(X_k)\to D_G(X_k)$ the "trivial G-action" functor and $R\Gamma_G: D_G(X_k)\to D(X_k)$ its right adjoint (total derived functor of "fixed points under G"). Applying the exact triangle of functors $j_!j^*\to Id\to i_*i^*\to j_!j^*[1]$ to the morphism $\iota(\mathbb{Q}/\mathbb{Z})'(n)_{X_k}\to Rf_*(\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}}$, we get a commutative diagram of exact triangles in $D_G(X_k)$

$$\begin{tikzcd} \iota j_!j^*(\mathbb{Q}/\mathbb{Z})'(n)_{X_k} \arrow[r] \arrow[d] & \iota(\mathbb{Q}/\mathbb{Z})'(n)_{X_k} \arrow[r] \arrow[d] & \iota i_*i^*(\mathbb{Q}/\mathbb{Z})'(n)_{X_k} \arrow[r, "+1"] \arrow[d] & {} \\ j_!j^*Rf_*(\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}} \arrow[r] & Rf_*(\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}} \arrow[r] & i_*i^*Rf_*(\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}} \arrow[r, "+1"] & {} \end{tikzcd}$$

hence by adjunction a similar diagram in $D(X_k)$

$$\begin{tikzcd} j_!j^*(\mathbb{Q}/\mathbb{Z})'(n)_{X_k} \arrow[r] \arrow[d] & (\mathbb{Q}/\mathbb{Z})'(n)_{X_k} \arrow[r] \arrow[d] & i_*i^*(\mathbb{Q}/\mathbb{Z})'(n)_{X_k} \arrow[r, "+1"] \arrow[d] & {} \\ R\Gamma_G(j_!j^*Rf_*(\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}}) \arrow[r] & R\Gamma_G(Rf_*(\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}}) \arrow[r] & R\Gamma_G(i_*i^*Rf_*(\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}}) \arrow[r, "+1"] & {} \end{tikzcd}$$

By definition of a quasi-Galois covering of group G, $f^{-1}(U)\to U$ can be decomposed as $f^{-1}(U)\xrightarrow{g} V\xrightarrow{h} U$, where g is Galois of group G and h is radicial. Since radicial maps induce isomorphisms in étale cohomology, a small argument using proper base change shows that the left vertical map in the diagram is an isomorphism. Similarly,

$$i_* i^* Rf_* (\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{X}} \xrightarrow{\sim} i_* Rf'_* (\mathbb{Q}/\mathbb{Z})'(n)_{\tilde{Z}},$$

where f' is the restriction of f to $\tilde{Z}$, and $i_* i^*(\mathbb{Q}/\mathbb{Z})'(n)_{X_k} = i_*(\mathbb{Q}/\mathbb{Z})'(n)_Z$.

A hypercohomology spectral sequence now shows that the natural maps $H^i(\tilde{X}, G, n) \to H^i(\tilde{X}, n)^G$ and $H^i(\tilde{Z}, G, n) \to H^i(\tilde{Z}, n)^G$ are isomorphisms modulo finite groups.

By induction, the groups $H^{i-1}(\tilde{Z}, n)$ and $H^i(Z, n)$ are finite in the desired range. This is also the case for $H^i(\tilde{X}, n)$, thanks to Theorem 1. It follows that $H^i(X_k, n)$ is finite. A transfer argument now implies that $H^i(X, n)$ has finite exponent, hence is finite. The two vanishing statements are clear. □

Theorem 3. *Let X be a variety of dimension d over $\mathbb{F}_p$ and let $n \in \mathbb{Z}$.*

(a) *The Borel–Moore étale homology group $H_i^c(X, n)$ is finite in the following cases:*

(1) $n \notin [0, d]$;
(2) $i \notin [2n - 1, n + d]$.

This group is 0 *for* $i \notin [-1, 2d]$, *and for* $i \notin [d - 1, 2d]$ *if X is affine.*

(b) *The cohomology group with proper supports $H_c^i(X, n)$ is finite in the following cases:*

(1) $n \notin [0, d]$;
(2) $i \notin [2n, n + d + 1]$.

This group is 0 *for* $i \notin [0, 2d + 1]$, *and for* $i \notin [d, 2d + 1]$ *if X is affine.*

Proof. (a) In case X is smooth, this follows from the isomorphism (geometric Poincaré duality)

$$H_i^c(X, n) \simeq H^{2d-i}(X, d - n)$$

and Theorem 1. In general, X contains an smooth open subset U (choose U affine if X is affine). Let Z be the reduced complement. Then the claims follow from the long exact sequences

$$\cdots \to H_i^c(Z, n) \to H_i^c(X, n) \to H_i^c(U, n) \to H_{i-1}^c(Z, n) \to \cdots$$

and induction on d.

(b) For any X, there are perfect parings (arithmetic Poincaré duality)

$$H_i^c(X, \mu_m^{\otimes n}) \times H_c^{i+1}(X, \mu_m^{\otimes n}) \to \mathbb{Z}/m.$$

The claims follow from (a) and this duality. □

2. Duality for regular schemes

Let $X \xrightarrow{f} S$ be a flat compactifiable morphism of pure relative dimension d and n an integer invertible on S. According to [3, (3.2.1.2)], there is a canonical natural transformation

$$t_f : f^*(d)[2d] \to Rf^! \tag{2.1}$$

between functors from $D^+(S, \mathbb{Z}/n)$ to $D^+(X, \mathbb{Z}/n)$, the derived categories of étale sheaves of $\mathbb{Z}/n$-modules over S and X. If f is smooth, t_f is an isomorphism of functors [25, Theorem 3.2.5]).

In the next theorem, we shall apply Thomason's absolute cohomological purity [25]. Recall the conditions of [25, 2.2]:

(a) Either X is of finite type over $\mathbb{Z}$, or over a local or global field, or over a separably closed field, or over a ring of integers in a local field

(b) or X is the inverse limit scheme of an inverse system of schemes X_α with affine étale transition maps $X_\alpha \to X_\beta$, and each X_α satisfies (a).

Theorem 4. *Suppose that X satisfies condition* (a) *or* (b) *and is regular. Let N be the étale n-cohomological dimension of X, i.e. the supremum of the étale l-cohomological dimensions of X for l prime dividing n. Then there exists an integer $m(N)$, depending only on N, such that* Ker t_f *and* Coker t_f *are annihilated by $m(N)$.*

Proof. Embed f into a smooth compactifiable morphism $\tilde{f}$ of pure dimension D:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & \tilde{X} \arrow[dl, "\tilde{f}"] \\ S & \end{tikzcd}$$

(i is a closed immersion).

Let $c = D - d$. By Thomason [25, Theorem 3.5], there exists an integer $M(N)$ and a map with kernels and cokernels killed by $M(N)^2$:

$$M(N) i_* i^* \mathbb{Z}/n \to \underline{H}^{2c}_X(\tilde{X}, \mathbb{Z}/n(c));$$

moreover, $\underline{H}^j_X(\tilde{X}, \mathbb{Z}/n(c))$ is killed by $M(N)$ for $j \neq 2c$. From this and the full faithfulness of i_*, one derives a natural transformation with kernel and cokernel killed by $M(N)^2$:

$$M(N) i^* \to Ri^!(c)[2c].$$

Composing with $\tilde{f}^*(d)[2d]$, we get a new map

$$M(N)f^*(d)[2d] \to Ri^!\tilde{f}^*(D)[2D]$$

with the same properties. The proof of [25, Theorem 3.5] makes it clear that the diagram

$$\begin{tikzcd} M(N)Rf^! \arrow[r] \arrow[d] & Rf^! = Ri^!R\widetilde{f}^! \arrow[d, "\wr"'] \\ M(N)f^*(d)[2d] \arrow[r] & Ri^!\widetilde{f}^*(D)[2D] \end{tikzcd}$$

commutes. This completes the proof. □

Remark 5. Gabber (unpublished) has proven absolute cohomological purity for étale cohomology with finite coefficients. We could have appealed to his theorem to get rid of the factor $m(N)$ in Theorem 4, but this is pointless in this context since we reason up to groups of finite exponent anyway.

3. Varieties over $\mathbb{Q}_p$

Theorem 6. *Let X be a variety of dimension d over $\mathbb{Q}_p$.*

(a) *The group $H^i(X,n)$ is finite in the following cases*:

(1) $n \notin [0, d+1]$;
(2) $i \notin [n, n+d+1]$.

This group is 0 for $i > 2d+2$, and for $i > d+2$ if X is affine.

(b) *The Borel–Moore étale homology group $H_i^c(X,n)$ is finite in the following cases*:

(1) $n \notin [-1, d]$;
(2) $i \notin [n-1, n+d]$.

This group is 0 for $i \notin [-2, 2d]$, and for $i \notin [d-2, 2d]$ if X is affine.

(c) *The cohomology group with proper supports $H_c^i(X,n)$ is finite in the following cases*:

(1) $n \notin [0, d+1]$;
(2) $i \notin [n, n+d+1]$.

This group is 0 for $i \notin [0, 2d+2]$, and for $i \notin [d, 2d+2]$ if X is affine.

(d) *If X is smooth and U is an open subset of X, then, for all i, the kernel and cokernel of the map*

$$H^i(X,0) \to H^i(U,0)$$

are finite.

Proof. We argue as usual by induction on d.

(a) We first assume X smooth projective and strictly semi-stable over some finite extension K of $\mathbb{Q}_p$. Let $\mathscr{X}$ be a strict semi-stable model over O_K and Y the closed fibre. Consider the long exact sequence of cohomology with supports:

$$\cdots \to H^i_Y(\mathscr{X}, n) \to H^i(\mathscr{X}, n) \to H^i(X_K, n) \to H^{i+1}_Y(\mathscr{X}, n) \to \cdots$$

By proper base change, the map $H^i(\mathscr{X}, n) \to H^i(Y, n)$ is an isomorphism, hence $H^i(\mathscr{X}, n)$ is finite for $n \notin [0, d]$ or $i \notin [n, n + d + 1]$ by Theorem 2. On the other hand, Theorem 4 implies that the natural map

$$H^{i+1}_Y(\mathscr{X}, n) \to H^c_{2d-i-1}(Y/\mathbb{Z}_p, d - n)$$

is an isomorphism up to groups of finite exponent. Moreover, $H^c_{2d-i-1}(Y/\mathbb{Z}_p, d - n) \simeq H^c_{2d-i+1}(Y/\mathbb{F}_p, d - n + 1)$ since $Ri^!(\mathbb{Q}/\mathbb{Z})'(d - n) = (\mathbb{Q}/\mathbb{Z})'(d - n - 1)[-2]$, where $i = \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}_p$ is the closed immersion. From Theorem 3(a), we deduce that $H^{i+1}_Y(\mathscr{X}, n)$ is finite for $n \notin [1, d + 1]$ or $i \notin [n, 2n - 1]$. It follows that $H^i(X_K, n)$ is finite, hence that $H^i(X, n)$ is finite by a transfer argument. (We have not used that Y is a divisor with normal crossings.)

Next, the usual purity argument shows by induction on d that, given a smooth variety V of dimension d over $\mathbb{Q}_p$ and an open subset U, Theorem 6(a) holds for V if and only if it holds for U. In particular, Theorem 6(a) is true for open subsets of semi-stable smooth projective varieties.

Now assume X smooth, and let $\mathscr{X}$ be a flat model of X over $\mathbb{Z}_p$. By de Jong [14], there exists an alteration $\mathscr{X}_1 \to \mathscr{X}$ with $\mathscr{X}_1$ an open subset of a strict semi-stable variety over a suitable finite extension $\mathcal{O}$ of $\mathbb{Z}_p$. In particular, $X_1 \to X$ is an alteration, where X_1 is the generic fibre of $\mathscr{X}_1$, and X_1 is open in a strictly semi-stable smooth projective variety over the field of fractions of $\mathcal{O}$. Then the first two steps show that Theorem 6(a) holds for X.

The case X arbitrary is dealt with exactly as in the proof of Theorem 2, except that things are now simpler as one may use resolution of singularities instead of de Jong's theorem.

(b) is proven exactly as Theorem 3(a), and (c) follows from (b) by the corresponding duality:

$$H^c_i(X, \mu_m^{\otimes n}) \times H^{i+2}_c(X, \mu_m^{\otimes n+1}) \to \mathbb{Z}/m.$$

Finally, (d) follows from (a) and purity. $\square$

Remark 7. As Illusie pointed out, in the first part of the proof of (a) the recourse to Theorem 4 is in fact not necessary; instead, we could have appealed to the purity result of Rapoport–Zink [19, Theorem 2.21], using the fact that Y is a divisor with normal crossings.

Remark 8. For X of finite type over $\mathbb{F}_p$ or $\mathbb{Q}_p$, the group $H^i(X,\mathbb{Q}_l(n))$ for $l\neq p$ vanishes exactly when $H^i(X,\mathbb{Q}_l/\mathbb{Z}_l(n))$ is finite. This is obvious from the exact sequence

$$H^i(X,\mathbb{Z}_l(n))\to H^i(X,\mathbb{Q}_l(n))\to H^i(X,\mathbb{Q}_l/\mathbb{Z}_l(n))\to H^{i+1}(X,\mathbb{Z}_l(n))$$

and the fact that the groups $H^*(X,\mathbb{Z}_l(n))$ are finitely generated $\mathbb{Z}_l$-modules. Similarly for Borel–Moore homology and cohomology with proper supports.

(For continous étale cohomology in general, see [12]; here, finiteness theorems for étale cohomology with finite coefficients cause that the naïve definition with inverse limits works fine—similarly for Borel–Moore homology and cohomology with proper supports.)

Lemma 9. *Let X be a smooth projective variety of dimension d over an algebraically closed field k of characteristic* 0. *Let L be the class in* $H^2(\bar{X},\hat{\mathbb{Z}}(1))$ *of an ample line bundle on X. Then for any* $i\leqslant d$, *the kernel and cokernel of the map*

$$H^i(\bar{X},\mathbb{Q}/\mathbb{Z}(n))\xrightarrow{L^{d-i}} H^{2d-i}(\bar{X},\mathbb{Q}/\mathbb{Z}(n+d-i))$$

are finite.

Proof. (I am indebted to the referee for pointing out this simple argument.) We may assume that k is the algebraic closure of a field finitely generated over $\mathbb{Q}$, hence admits an embedding $\sigma : k\to\mathbb{C}$. Let $X_\sigma = X\times_{k,\sigma}\mathbb{C}$. Then there are isomorphism

$$H^*_{\text{ét}}(X,\mathbb{Q}/\mathbb{Z}(n))\xrightarrow{\sim} H^*_{\text{ét}}(X_\sigma,\mathbb{Q}/\mathbb{Z}(n))\simeq H^*_B(X_\sigma,\mathbb{Q}/\mathbb{Z}(n)),$$

where H^*_B is Betti (singular) cohomology. These isomorphisms commute with cup-product by L, if L is interpreted in Betti cohomology as the class of the given hyperplane section in $H^2_B(X_\sigma,\mathbb{Z}(1))$ (where $\mathbb{Z}(1)=(2\pi\sqrt{-1})\mathbb{Z}$). The result now easily follows from the classical Hard Lefschetz theorem and the finite generation of the cohomology of X with integer coefficients. □

Theorem 10. *Let X be smooth projective of dimension d over* $\mathbb{Q}_p$. *Let* $G = Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ *and* $\bar{X} = X\times_{\mathbb{Q}_p}\bar{\mathbb{Q}}_p$. *For* $j=0,1,2,$ *the group* $H^j(G,H^{i-j}(\bar{X},n))$ *is finite in the following cases*:

$j=0$: (1) $n\notin[0,d]$.
(2) $i\notin[n,n+d]$.
$j=1$: (1) $n\notin[0,d+1]$.
(2) $i\notin[n,n+d+1]$.
$j=2$: (1) $n\notin[1,d+1]$.
(2) $i\notin[n+1,n+d+1]$.

This extends to the group $H^j(G, H^{i-j}(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ *for* $j=0$ *and* $j=2$. *On the other hand, for* $j=1$ *and* (n,i) *in the above range, the corank of this group is equal to* $\dim_{\mathbb{Q}_p} H^{i-1}(\bar{X}, \mathbb{Q}_p)$.

Proof. Everywhere, we shall use without mention that the cohomology groups of G with finite coefficients are finite.

We first deal with coefficients $(\mathbb{Q}/\mathbb{Z})'(n)$. By Lemma 9, the Hochschild–Serre spectral sequence

$$H^a(G, H^b(\bar{X}, n)) \Rightarrow H^{a+b}(X, n))$$

degenerates in the category A already used in the proof of Theorem 1 [8]. Applying Theorem 6(a), we get the estimate $H^j(G, H^{i-j}(\bar{X}, n))$ finite at least in the cases

(1) $n \notin [0, d+1]$;
(2) $i \notin [n, n+d+1]$.

Let L be the class in $H^2(\bar{X}, \hat{\mathbb{Z}}'(1))$ of an ample line bundle on X, where $\hat{\mathbb{Z}}'(1) = \prod_{l \neq p} \mathbb{Z}_l(1)$. Then the corresponding maps in Lemma 9 are G-equivariant and may be reformulated as isomorphisms in A

$$H^{i-j}(\bar{X}, n)) \simeq H^{2d-i+j}(\bar{X}, n+d-i+j).$$

(For $i-j \leqslant d$, the isomorphism is given by cup-product by L^{d-i+j}; for $i-j \geqslant d$, it is given by cup-product by L^{i-j-d}.) We therefore get that $H^j(G, H^{i-j}(\bar{X}, n))$ is finite also in the cases

(1) $n+d-i+j \notin [0, d+1]$;
(2) $2d-i+2j \notin [n+d-i+j, n+2d-i+j+1]$

or

(1) $i \notin [n+j-1, n+d+j]$;
(2) $n \notin [j-1, d+j]$.

We now consider the case of coefficients $\mathbb{Q}_p/\mathbb{Z}_p(n)$. By an analogue of Remark 8 (compare [13, Lemma 1]), the finiteness of $H^j(G, H^{i-j}(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ is equivalent to the vanishing of $H^j(G, H^{i-j}(\bar{X}, \mathbb{Q}_p(n)))$. By Jannsen [13, Corollary 5], the case $j=0$ holds for $n<0$ or $i<n$ as an application of Faltings' Hodge–Tate decomposition (see also [24, Proof of Theorem 2 in 2.1.3]). We get the remaining cases $n>d$ or $i>n+d$ by the same application of Hard Lefschetz as above. From the case $j=0$, we then get the case $j=2$ by local Tate duality and Poincaré duality for X.

Finally, we get the remaining case $j=1$ by an Euler–Poincaré characteristic argument: for any finite-dimensional $\mathbb{Q}_p$-vector space V with continuous action of G, we have

$$\dim H^0(G, V) - \dim H^1(G, V) + \dim H^2(G, V) = -\dim V.$$

This may be reduced to Tate's corresponding theorem for finite G-modules [21, p. 109, Theorem 5] in the same way as [24, proof of Proposition 2] or [13, proof of Lemma 2] (compare [24, p. 127]). We leave it to the reader to check the rather tedious bookkeeping in order to see that the range for (i, n) is correct.

This completes the proof of Theorem 10. $\square$

Corollary 11 (compare Soulé [24, 2.1.3, Theorem 2]). *Let X be smooth projective over $\mathbb{Q}_p$. Then, for (i, n) as in Theorem* 6(a), *we have*

$$\operatorname{corank} H^i(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \beta^{i-1}(X),$$

where $\beta^{i-1}(X)$ is the $(i-1)$-st Betti number of $\bar{X}$.

Remark 12. As the referee pointed out, one can easily extend Theorems 2 and 6(a) to cohomology groups with supports: this is left to the interested reader.

Remark 13. A plausible extension of Corollary 11 to arbitrary $\mathbb{Q}_p$-varieties X could be the formula

$$\operatorname{corank} H^i_c(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \beta^{i-1}(X)$$

for (i, n) as in Theorem 6(c), where $\beta^{i-1}(X)$ is the $(i-1)$-st virtual Betti number of $\bar{X}$ (cf. [10, 3.3.1]). If this is true, I do not see how to prove it by the methods of this paper. At least the formula

$$\sum_i (-1)^i \operatorname{corank} H^i_c(X, \mathbb{Q}_p/\mathbb{Z}_p(n)) = \sum_i (-1)^i \beta^{i-1}(X)$$

holds, by reduction to Corollary 11.

4. Jannsen's conjecture

In [13], Jannsen proposes the following conjecture:

Conjecture 14 (Jannsen [13, Conjecture 3, Remark 5]). *With assumptions and notation as in Theorem* 10, *the group $H^0(G, H^i(\bar{X}, \mathbb{Q}_l(n)))$ is* 0 *for $n<0$ or $i<2n$ and any prime number l.*

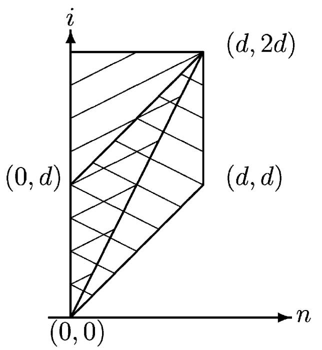
As pointed out by Jannsen, this conjecture would follow for $l \neq p$ from the monodromy conjecture [13, pp. 342–343]. In particular, it holds in this case for abelian varieties and also for $\dim X \leqslant 2$ [19, Satz 2.13, 14]), hence for $i \leqslant 2$ by a Bertini–Lefschetz argument (I am indebted to the referee for pointing this out). In view of [13, Lemma 1], Theorem 10 proves part of it unconditionally, with in fact a stronger result.

What does it imply for the vanishing of $H^*(X, \mathbb{Q}_l(n))$ when $l \neq p$? By Theorem 6(a), we may assume $n \in [0, d+1]$. First, since $\dim H^0(G, H^i(\bar{X}, \mathbb{Q}_l(n))) = \dim H^2(G, H^i(\bar{X}, \mathbb{Q}_l(i+1-n)))$ [13, Lemma 11(a)], we get the estimate $H^2(G, H^{i-2}(\bar{X}, \mathbb{Q}_l(n))) = 0$ for $i > 2n$. Next, since the Euler–Poincaré characteristic of G is 0 for $\mathbb{Q}_l[[G]]$-modules when $l \neq p$, we have

$$\begin{aligned}&\dim H^1(G, H^{i-1}(\bar{X}, \mathbb{Q}_l(n)))\\&\quad = \dim H^0(G, H^{i-1}(\bar{X}, \mathbb{Q}_l(n))) + \dim H^2(G, H^{i-1}(\bar{X}, \mathbb{Q}_l(n)))\end{aligned}$$

for any (n, i). However, even taking the estimates of Theorem 10 into account, one sees that Conjecture 14 does not add any refinement to the bounds in Theorem 6(a).

For the benefit of the reader, we include a picture of the region for Jannsen's conjecture and the region where we get vanishing. For clarity we have in fact filled with slanted lines the regions complementary to Jannsen's vanishing (positive slope) and our vanishing (negative slope). The region where $H^0(G, H^i(\bar{X}, \mathbb{Q}_l(n)))$ may be nonzero according to our results plus Jannsen's conjecture is the triangle at the intersection of the two hatched regions (with summits $(0,0), (0,d)$ and $(d, 2d)$), and the region in which Jannsen's conjecture remains open is the hatched triangle under the line $i = 2n$ (with summits $(0,0), (d,d)$ and $(d, 2d)$). It is mysterious that outside this small region, Jannsen's conjecture may be obtained for $l \neq p$ essentially by dévissage from Deligne's Weil I and for $l = p$ from Faltings' Hodge–Tate decomposition!



5. Varieties over a number field

In [13, Conjecture 1], Jannsen also proposes a global conjecture:

Conjecture 15. *Let X be a smooth projective variety over $\mathbb{Q}$ and p a prime number.*[1] *Let S be a finite set of places of $\mathbb{Q}$, including p and the prime at infinity, and such that X has*

[1] We do not assume X geometrically connected: the case of a variety over an arbitrary number field can be easily recovered from this statement by restriction of scalars.

good reduction outside S. Let G_S *be the Galois group of the maximal S-ramified extension of* $\mathbb{Q}$. *Then the group* $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ *is finite if* $i \notin [n-1, 2n-1]$.

Theorem 10 gives the following result in the direction of this conjecture:

Theorem 16. *Let* X, p, S, G_S *be as in Conjecture* 15, *and suppose* X *irreducible of dimension d. Then the group* $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ *is, up to a finite group, dual to the group*

$$\operatorname{Ker}(H^1(\mathbb{Q}, N) \to H^1(\mathbb{Q}_p, N))$$

with $N = H^{2d-i}(\bar{X}, \mathbb{Z}_p(d-n+1))/tors$, *except perhaps if* $0 \leqslant n \leqslant d+1$ *and* $n-1 \leqslant i \leqslant n+d$.

Remark 17. By Hard Lefschetz, the group N is also isomorphic to $H^i(\bar{X}, \mathbb{Z}_l(i+1-n))/tors$.

Proof. Of course this proof uses well-known methods and we do not claim any originality for it. Suppose that (n, i) is not as in the exceptional cases. For all $v \in S$, let G_v be the decomposition group at v. By Theorem 10, $H^2(G_S, H^i(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ contains

$$\operatorname{Ker}(H^2(G_S, H^i(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) \to \prod_{v \in S} H^2(\mathbb{Q}_v, H^i(\bar{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))))$$

as a subgroup of finite index. By Poitou–Tate duality [18, Theorem 4.10, p. 70] and a limit argument using the fact that S is finite, this kernel is dual to

$$\operatorname{Ker}(H^1(G_S, M^*(1)) \to \prod_{v \in S} H^1(\mathbb{Q}_v, M^*(1)))$$

with

$$M^* = \varprojlim H^i(\bar{X}, \mathbb{Z}/p^\nu(n))^* \simeq H^{2d-i}(\bar{X}, \mathbb{Z}_p(d-n)),$$

where we have now used Poincaré duality.

Let $N = M^*(1)/tors$: since N_{tors} is finite, the map $H^1(G_S, M^*(1)) \to H^1(G_S, N)$ has finite kernel and cokernel. By Jannsen [13, Lemma 4], the inflation map $H^1(G_S, N) \to H^1(\mathbb{Q}, N)$ is an isomorphism provided $2d - i \neq 2(d-n+1)$, i.e. $i \neq 2(n-1)$. But we cannot have $i = 2(n-1)$: if $n = 0$ this is clear and if $1 \leqslant n \leqslant d+1$ this would imply $n-1 \leqslant i \leqslant n+d$, contradicting the hypothesis. So the inflation map is always an isomorphism. On the other hand, applying Theorem 10 a second time, we see that $H^1(\mathbb{Q}_v, N))$ is finite for all $v \neq p$. Since it is obviously finite also for $v = \infty$, Theorem 16 follows. $\square$

For $d = 0$, Theorem 6 boils down the well-known fact that for a number field K, $H^2(G_S, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is dual to $\mathrm{Ker}(H^1(K, \mathbb{Z}_p(1-n)) \to \prod_{v|p} H^1(K_v, \mathbb{Z}_p(1-n)))$ up to a finite group for $n \neq 0, 1$, compare [20, Section 5, Corollary 4]. For $n > 1$ the first group is 0 by Soulé [22, Theorem 5] and for $n < 0$ the second one is isomorphic to the kernel of the "p-adic regulator map"

$$K_{2m-1}(K) \otimes \mathbb{Z}_p \to \prod_{v|p} K_{2m-1}(K_v, \mathbb{Z}_p)$$

with $m = 1 - n$. For K totally real, its finiteness is equivalent to the nonvanishing of a certain p-adic L-function at $s = m$, at least when K is abelian, cf. Soulé [23, Section 3].

6. An open question in weight 0

For X smooth connected over $\mathbb{F}_p$ and $l \neq p$, the cohomology groups $H^i(X, \mathbb{Q}_l)$ are very simple: they are 0 for $i > 1$, isomorphic to $\mathbb{Q}_l$ for $i = 0, 1$, and moreover, cup-product by $e \in H^1(\mathbb{F}_p, \mathbb{Q}_l)$ gives an isomorphism $H^0(X, \mathbb{Q}_l) \xrightarrow{\sim} H^1(X, \mathbb{Q}_l)$ (Theorem 1 and Remark 8).

This is not true anymore when X is not smooth. It would be highly desirable to understand the groups $H^*(X, \mathbb{Q}_l)$ better, in terms of the singularities of X, and for applications to smooth varieties over $\mathbb{Q}_p$. We shall simply make an elementary step in this direction.

Let Sm be the category of smooth $\mathbb{F}_p$-schemes of finite type, Sm^0 the full subcategory of smooth schemes of dimension 0 and $Sm_\bullet$, $Sm^0_\bullet$ the corresponding categories of simplicial schemes. The inclusion functor $Sm^0 \to Sm$ has a left adjoint which is also a retraction

$$\pi_0 : Sm \to Sm^0,$$

$$X \mapsto \coprod_{i \in I} \mathrm{Spec}\, k_i,$$

where I is the set of connected components X_i of X and, for all i, k_i is the field of constants of X_i. There corresponds to it a functor

$$\pi_0 : Sm_\bullet \to Sm^0_\bullet,$$

which is a left adjoint and a retraction of the natural inclusion. We may view $Sm^0_\bullet$ as the category of simplicial G-sets finite in each degree, where $G = Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$.

Let $Y_\bullet \to X$ be a smooth simplicial resolution of X [4]. There is a spectral sequence

$$E_1^{p,q} = H^q(\bar{Y}_p, \mathbb{Q}_l) \Rightarrow H^{p+q}(\bar{X}, \mathbb{Q}_l),$$

where ¯ denotes the geometric fibre. Tracing the Frobenius action, one sees that the weight 0 part of $E_1^{p,q}$ is precisely $H^q(\pi_0(\bar{Y}_p), \mathbb{Q}_l)$, which is of course 0 for $q > 0$. Since taking weight 0 is an exact functor, the spectral sequence yields an isomorphism

$$H^*(\pi_0(\bar{Y}_\bullet), \mathbb{Q}_l) \simeq H^*(\bar{X}, \mathbb{Q}_l)_{(0)},$$

where the index (0) denotes the weight 0 part (note that the spectral sequence computing the cohomology of the simplicial set $\pi_0(\bar{Y}_\bullet)$ degenerates at E_2, with the same E_2-terms as the weight 0 part of the above spectral sequence). In particular, *the rational homology type of the simplicial G-set $\pi_0(Y_\bullet)$ only depends on X.*

One might hope that the G-homotopy type of $\pi_0(Y_\bullet)$ only depends on X, at least when we restrict to special types of simplicial resolutions $Y_\bullet$. Hélène Esnault and Ofer Gabber have independently pointed out that, at any rate, alterations are not sufficient for this. For instance, we might take $X = \operatorname{Spec} \mathbb{F}_p$ and for $Y_\bullet$ the hypercovering associated to some finite Galois extension with group Δ: then the G-homotopy type of $\pi_0(\bar{Y}_\bullet)$ is obviously $B\Delta$ (with the trivial action of G). It seems that something like envelopes [10] is needed, forcing some recourse to resolution of singularities.

Besides understanding $H^*(X, \mathbb{Z}_l)$ for smooth varieties X over $\mathbb{Q}_p$, this question is also closely related to a conjecture of Kato [17, Conjecture 0.3]. We hope to come back to it in a future paper.

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