ON THE LICHTENBAUM-QUILLEN CONJECTURE

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ABSTRACT. The Lichtenbaum-Quillen conjecture, relating the algebraic K-theory of rings of integers in number fields to their étale cohomology, has been one of the main factors of development of algebraic K-theory in the beginning of the 1980s. Soulé and Dwyer-Friedlander mapped algebraic K-theory of a ring of integers to its ℓ-adic cohomology by means of a 'Chern character', that they proved surjective. Here, on the contrary, we map étale cohomology to algebraic K-theory, providing a right inverse to these Chern characters. This gives a different proof of surjectivity, which avoids Dwyer-Friedlander's use of 'secondary transfer'. The constructions and results of this paper concern a much wider class of rings than rings of integers in number fields.

Introduction.

It is usual to try and map algebraic K-theory to cohomology theories, for example étale cohomology. In this paper, we do the opposite: we map étale cohomology to algebraic K-theory.

This approach was initiated in [K1], where we defined under certain conditions *anti-Chern* classes:

$$\beta^{i,j} : H^j(R, \mathbb{Z}/l^\nu(i)) \to K_{2i-j}(R, \mathbb{Z}/l^\nu),$$

for a semi-local ring R in which the prime number l is invertible. Their construction was then partly conjectural, depending on results that are now available in [K2] and [K3].

Here we construct these anti-Chern classes in cases where the Kato conjecture, relating Milnor's K-theory to étale cohomology, is known. More precisely, let R be a commutative semi-local ring, in which the prime number l is invertible. If $l = 2$, assume a minor technical condition on R, for example that $\sqrt{-1} \in R$. In section 2, we construct homomorphisms:

$$\beta^{i,0} : H^0(R, \mathbb{Z}/l^\nu(i)) \to K_{2i}(R, \mathbb{Z}/l^\nu) \quad (i \geq 0)$$

$$\beta^{i,1} : H^1(R, \mathbb{Z}/l^\nu(i)) \to K_{2i-1}(R, \mathbb{Z}/l^\nu) \quad (i \geq 1)$$

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for all R,

$$\beta^{i,2} : H^2(R, \mathbb{Z}/l^\nu(i)) \to K_{2i-1}(R, \mathbb{Z}l^\nu) \quad (i \geq 2)$$

for R of geometric origin (for example a field),

$$\beta^{i,3} : H^3(F, \mathbb{Z}/2^\nu(i)) \to K_{2i-3}(F, \mathbb{Z}/2^\nu) \quad (i \geq 3)$$

for F a field,

$$\beta^{i,j} : H^j(F, \mathbb{Z}/l^\nu(i)) \to K_{2i-j}(F, \mathbb{Z}/l^\nu) \quad (i \geq j, \text{ all } j \geq 0, \text{ all } l \neq \text{ charF})$$

for F a higher local field in the sense of Kato [Ka]. The direct sum of all these homomorphisms is then shown to be *split injective*: this is the main result of the paper.

We then consider special cases, among which that of global fields and their rings of integers. If F is a number field and O_S is its ring of integers localised away from l, we get *split injections*:

$$H^2(O_S, \mathbb{Z}_l(i+1)) \to K_{2i}(O_S) \otimes \mathbb{Z}_l,$$

$$H^1(O_S, \mathbb{Z}_l(i)) \to K_{2i-1}(O_S) \otimes \mathbb{Z}_l.$$

These maps are right inverse to the Chern characters

$$ch_{i+1,2} : K_{2i}(O_S) \otimes Z_l \to H^2(O_S, Z_l(i+1)) \text{ and } ch_{i,1} : K_{2i-1}(O_S) \otimes Z_l \to H^1(O_S, Z_l(i))$$

defined by Soulé and Dwyer-Friedlander. This provides another proof that the Chern characters are surjective. Split surjectivity of $ch_{i,1}$ seems to be new. See below for a comparison of this proof with earlier ones. We prove that $H^2(O_S, \mathbb{Z}/l^\nu(i+1))$ is a direct summand of $K_{2i}(O_S)/l^\nu$, which also seems new.

Following the comments of the referee of an earlier version of this paper, I have refrained for clarity from giving results that depend on conjectures, except in §6. Yet it is worth pointing out that the (generalised) Kato conjecture, which predicts that the Galois symbols (2.1) below should be isomorphisms for a large class of semi-local rings including fields, is believed to be true by many people. Under this conjecture, the anti-Chern classes $\beta^{i,j}$ can be defined with no other restriction than $i \geq j$, and the split injectivity results can similarly be proven without other restrictions.

Also for clarity, and to keep statements as elementary as possible, I have refrained from mentioning étale K-theory when not needed, concentrating on results about algebraic K-theory. Yet étale K-theory is used in an essential way in §3 to prove the split injectivity of the anti-Chern classes.

This paper is organised as follows. In §1 we define twisted variants of Milnor's K-theory (twisted Milnor K-groups), and map them to algebraic K-theory with coefficients. This move is aimed to make the definition of the anti-Chern classes more illuminating here than in [K1]. In §2 we map the twisted Milnor K-groups to étale cohomology, and prove that these maps are isomorphisms when the Kato conjecture holds. Taking the inverse of these isomorphisms, we get under the Kato conjecture the anti-Chern classes $\beta^{i,j}$ above. In §3, we prove that the anti-Chern classes are split, using étale K-theory. Following a suggestion of Rick Jardine, I organised the proof more methodically here than in [K1], first proving that the E_2-terms of the descent spectral sequence for étale K-theory corresponding to the anti-Chern classes consist of universal cycles, then deducing the splitting.

In §4, we consider a higher local field F of dimension n and prove isomorphisms (for all $i \geq 0$):

$$H^0(F, \mathbb{Z}/l^\nu(i)) \otimes H^2(F, \mathbb{Z}/l^\nu(i+1)) \oplus \cdots \oplus H^{2s}(F, \mathbb{Z}/l^\nu(i+s)) \xrightarrow{\cong} K_{2i}(F, \mathbb{Z}/l^\nu)$$

$$H^1(F, \mathbb{Z}/l^\nu(i+1)) \oplus H^3(F, \mathbb{Z}/l^\nu(i+2) \oplus \cdots \oplus H^{2t+1}(F, \mathbb{Z}/l^\nu(i+t+1)) \xrightarrow{\cong} K_{2i+1}(F, \mathbb{Z}/l^\nu)$$

where $s = min(i, [\frac{n}{2}])$ and $t = min(i, [\frac{n-1}{2}])$, away from the characteristic p of the final residue field of F. This result seems to be new. One can conjecture that these isomorphisms still hold for $l = p$, when F is of characteristic 0; consequences of this conjecture are examined in §6 for $n = 1$.

In §5, we consider global fields and both globalise and l-adicise the earlier results, getting back the surjectivity theorems of [DF]. We also observe that Soulé's l-adic construction of cyclotomic elements in K-theory factors via the anti-Chern classes through Deligne's construction of cyclotomic elements in étale cohomology.

Finally in §6, we consider a local field F of dimension 1. Here, departing from the earlier practice, we introduce conjectures – with a vengeance. There are 3 conjectures 6.1, 6.2 and 6.3. The first one is the one mentioned two paragraphs ago: it is equivalent to the Lichtenbaum-Quillen conjecture for F. Conjecture 6.2 predicts that the torsion in $K_{2i}(F)$ is finite (for $i = 1$ this is a result of Moore, Carroll and Merkurjev), while Conjecture 6.3 relates $K_{2i-1}(F)_{ind}$ (defined in Definition 6.1) to Wagoner's $K_{2i-1}^{top}(F)$, extending a conjecture of [K4], §7.

This work builds upon earlier work of (among others) Soulé [So1]–[So4], Dwyer, Friedlander, Snaith and Thomason [DFST], Dwyer-Friedlander [DF], Thomason [T], Dwyer, Friedlander and Mitchell [DFM], Merkurjev-Suslin [MS1]–[MS3] and myself [K1]–[K4]. Concerning the surjectivity of $K_i(O_S) \otimes \mathbb{Z}_l \to \hat{K}_i^{\acute{e}t}(O_S)$, the following remarks are in order. In [So1], Soulé proved that $c_{i,2} : K_{2i-2}(O_S, \mathbb{Z}/l^\nu) \to H^2(O_S, \mathbb{Z}/l^\nu(i))$ is surjective for any ν when $l \geq i$ by an argument of cohomological dimension, hence that $c_{i,2} : K_{2i-2}(O_S) \otimes \mathbb{Z}_l \to H^2(O_S, \mathbb{Z}_l(i))$ is surjective. However he could prove surjectivity of $c_{i,1} : K_{2i-1}(O_S, \mathbb{Z}/l^\nu) \to H^1(O_S, \mathbb{Z}/l^\nu(i))$ only when O_S contains a primitive l^ν-th root of unity, thereby barring a direct proof of surjectivity of $K_{2i-1}(O_S) \otimes \mathbb{Z}_l \to H^1(O_S, \mathbb{Z}_l(i))$. He overcame this difficulty in [So2] by an argument (due originally to Lichtenbaum) using Iwasawa theory. A different argument in [Sch] makes use of Tate's duality theorems for Galois cohomology of number fields. Dwyer and Friedlander then introduced étale K-theory and proved in [DF], §8 surjectivity of $K_i(O_S) \otimes \mathbb{Z}_l \to \hat{K}_i^{\acute{e}t}(O_S) \cong H^1(O_S, \mathbb{Z}_l(i))$ by means of the "secondary transfer". Our proof of surjectivity is simpler in that it does not use deep arithmetic theorems like Iwasawa theory or Tate duality, nor a subtle object like secondary transfer. (It could however be observed that there is a certain similarity between the idea of a secondary transfer and the crucial Lemma 3.2.1 of [K2].) Also it gives a slightly stronger result than the earlier ones in that it produces a functorial splitting commuting with products and transfer. The word "slightly" is put here because the Lichtenbaum-Quillen conjecture predicts in any case that $K_i(O_S) \otimes \mathbb{Z}_l \to \hat{K}_i^{\acute{e}t}(O_S)$ is an isomorphism for all $i \geq 1$! On the other hand, it relies on a stable homotopy theory result of Dwyer-Friedlander-Mitchell [DFM] and Soulé's theorem that $K_{2i-1}(O_S) \to K_{2i-1}(F)$ is injective. In [B], Banaszak observes that the existence of a group-theoretic splitting of $K_i(O_S) \otimes \mathbb{Z}_l \to \hat{K}_i^{\acute{e}t}(O_S)$ for even i follows from purely group-theoretic considerations.

Special note for $l=2$. At several places, for example in Proposition 1.2 and Theorem 5.1, we make the restrictive assumption when $l=2$ that the ring under consideration either should contain a square root of -1 or have non-zero characteristic. This is due to the fact that, in [K3], the maps β^i of [K1], Proposition 1.4 are proven to have good properties only in the two special cases above. In fact, these results should hold under the sole assumption that the ring is not exceptional (Convention 2 below). Similarly, in §6, there should presumably be no restriction at all for $p=2$.

CONVENTIONS

1. We fix once and for all a prime number l, which is invertible on all schemes considered.
2. A connected scheme X over $\mathbb{Z}[1/2]$ is exceptional if the image of its fundamental group in $\mathbb{Z}_2^*$ by the cyclotomic character is not torsion-free. A scheme X over $\mathbb{Z}[1/2]$ is exceptional if one of its connected components is.
3. Unless necessary for the understanding, we drop the index ét from étale cohomology groups. For an affine scheme $X = SpecR$, we usually write $H^*(R)$ for $H^*(X)$, and similarly for K-theory.
4. We call an extension of rings (or a morphism of schemes) l-cyclotomic if it is covered by an extension (or morphism) corresponding to the adjunction of some l-primary roots of unity, or is a component of such a covered extension.
5. If A is an abelian group and $n \geq 1$, ${}_nA$ denotes the n-torsion of A and $A\{l\}$ its l-primary torsion.

1. Twisted Milnor K-groups.

Let F be a field of characteristic $\neq l$. Recall Milnor's K-groups $K_j^M(F) = F^{*\otimes j}/\Re$, where $\Re$ is the subgroup generated by Steinberg relations. If $l=2$, assume that F is not exceptional. We define twisted variants of $K_j^M(F)/l^\nu$:

Let $i \in \mathbb{Z}$ and E/F be the smallest extension such that $|H^0(E,\mathbb{Z}/l^\nu(i))| = l^\nu$: this is a cyclic l-cyclotomic extension with Galois group G.

Definition 1.1. $K_j^M(i)(F,\mathbb{Z}/l^\nu) = (K_j^M(E)\otimes \mathbb{Z}/l^\nu(i))_G$.

In particular, $K_0^M(1)(F,\mathbb{Z}/l^\nu)$ can be identified to the $l^\nu-$th roots of unity of F and $K_j^M(0)(F,\mathbb{Z}/l^\nu) = K_j^M(F)/l^\nu$ for all j.

(I don't know if there is a reasonable definition of $K_j^M(i)(F,\mathbb{Z}/l^\nu)$ when $l=2$ and F is exceptional.)

Proposition 1.1.

a) *If $\mu \leq \nu$, there are natural homomorphisms $K_j^M(i)(F,\mathbb{Z}/l^\mu) \to K_j^M(i)(F,\mathbb{Z}/l^\nu)$ and $K_j^M(i)(F,\mathbb{Z}/l^\nu) \to K_j^M(i)(F,\mathbb{Z}/l^\mu)$, whose composition both ways is multiplication by $l^{\nu-\mu}$. These homomorphisms are compatible in a sequence of integers $\lambda \leq \mu \leq \nu$.*

b) *$K_j^M(i)(F,\mathbb{Z}/l^\nu)$ is a functor in F.*

c) *Let F'/F be a finite extension. Then there are transfers:*

$$N_{F'/F} : K_j^M(i)(F',\mathbb{Z}/l^\nu) \to K_j^M(i)(F,\mathbb{Z}/l^\nu).$$

These transfers are functorial with respect to pull-backs in the sense that they satisfy the double coset formula.

d) *There are products*

$$K_j^M(i)(F,\mathbb{Z}/l^\nu)\otimes K_{j'}^M(i')(F,\mathbb{Z}/l^\nu)\to K_{j+j'}^M(i+i')(F,\mathbb{Z}/l^\nu),$$

extending product in Milnor's K-theory. They are associative, graded commutative (with respect to the K-theory grading), natural in F and satisfy the projection formula with respect to the product of c).

Proof of Proposition 1.1. We mostly construct the maps of the theorem, leaving functoriality claims to the reader except to point out non obvious things.

a) Let E/F correspond to μ and E'/F correspond to ν, so that $F\subseteq E\subseteq E'$. We define $K_j^M(i)(F,\mathbb{Z}/l^\mu)\to K_j^M(i)(F,\mathbb{Z}/l^\nu)$ by taking coinvariants under $Gal(E/F)$ of the composition:

$$K_j^M(E)\otimes\mathbb{Z}/l^\mu(i)\to K_j^M(E')\otimes\mathbb{Z}/l^\mu(i)\to K(E')\otimes\mathbb{Z}/l^\nu(i)\to(K_j^M(E')\otimes\mathbb{Z}/l^\nu(i))_\Delta,$$

where the first map is induced by functoriality, the second one by the inclusion

$$\mathbb{Z}/l^\mu(i)\hookrightarrow\mathbb{Z}/l^\nu(i)$$

and $\Delta=Gal(E'/E)$. Similarly, we define $K_j^M(i)(F,\mathbb{Z}/l^\nu)\to K_j^M(i)(F,\mathbb{Z}/l^\mu)$ by taking coinvariants of the composition:

$$\begin{array}{ccc}(K_j^M(E')\otimes\mathbb{Z}/l^\nu(i))_\Delta\to(K_j^M(E')\otimes\mathbb{Z}/l^\mu(i))_\Delta & & \\ \| & & \\ K_j^M(E')_\Delta\otimes\mathbb{Z}/l^\mu(i)\to K_j^M(E)\otimes\mathbb{Z}/l^\mu(i), & & \end{array}$$

where the first map is induced by the projection $\mathbb{Z}/l^\nu(i)\twoheadrightarrow\mathbb{Z}/l^\mu(i)$ and the last one is induced by transfer in Milnor's K-theory [Ka], §1.7. To check the claim about the composition both ways, we may reduce to the case $E=F$. Then the claim follows from the following facts: $[E':F]=l^{\nu-\mu}$; the composition $K_j^M(F)\to K_j^M(E')\to K_j^M(F)$ is multiplication by $[E':F]$; the composition $K_j^M(E')_\Delta\to K_j^M(F)\to K_j^M(E')_\Delta$ is multiplication by $[E':F]$.

b) Let $F\to F'$ be an extension, $E'=E\otimes_F F'$ (a Galois algebra over F') and E_1/F' be the extension analogous to E/F for the field F'. Then $G=Gal(E/F)$ acts on E', E' is a product of copies of E_1 which are permuted transitively by G and the stabiliser of one of them is $Gal(E_1/F')$. By Shapiro's lemma:

$$\begin{aligned}&H_0(G,\oplus K_j^M(E_1)\otimes\mathbb{Z}/l^\nu(i))\\&\qquad=H_0(Gal(E_1/F'),K_j^M(E_1)\otimes\mathbb{Z}/l^\nu(i))=:K_j^M(i)(F',\mathbb{Z}/l^\nu).\end{aligned}$$

We then define $K_j^M(i)(F,\mathbb{Z}/l^\nu)\to K_j^M(i)(F',\mathbb{Z}/l^\nu)$ by taking coinvariants under G of the natural homomorphism $K_j^M(E)\otimes\mathbb{Z}/l^\nu(i)\to\oplus K_j^M(E_1)\otimes\mathbb{Z}/l^\nu(i)$.

c) We proceed as in b), taking coinvariants under G of the homomorphism

$$\oplus K_j^M(E_1)\otimes \mathbb{Z}/l^\nu(i) \to K_j^M(i)(E,\mathbb{Z}/l^\nu)$$

obtained by summing and then applying the Milnor K-theory transfer relative to the extension E_1/E.

d) Let E_i and $E_{i'}$ be the extensions of F used to define

$$K_j^M(i)(F,\mathbb{Z}/l^\nu) \text{ and } K_j^M(i')(F,\mathbb{Z}/l^\nu).$$

We define the product out of the one in Milnor K-theory so that

$$(N_{E_i/F}x)\cdot(N_{E_{i'}/F}y) = N_{E_i/F}(x(N_{E_{i'}/F}y)_{E_i})$$

for $(x,y)\in K_j^M(i)(E_i,\mathbb{Z}/l^\nu)\times K_{j'}^M(i')(E_{i'},\mathbb{Z}/l^\nu)$, and, if $F=E_i$,

$$x\cdot N_{E_{i'}/F}y = N_{E_{i'}/F}(x_{E_{i'}}y)$$

(this shows that there is exactly one product extending that in Milnor's K-theory and satisfying the projection formula). $\square$

Remark 1.1. By construction of $K_j^M(i)$, $N : K_j^M(i)(E,\mathbb{Z}/l^\nu)_G \to K_j^M(i)(F,\mathbb{Z}/l^\nu)$ is an isomorphism for the extension E/F used to define $K_j^M(i)(F,\mathbb{Z}/l^\nu)$.

Proposition 1.2. *There exists a collection of homomorphisms*

$$\eta_F^{i,m} : K_{2i-m}^M(m-i)(F,\mathbb{Z}/l^\nu)\to K_m(F,\mathbb{Z}/l^\nu),$$

such that:

a) *$\eta^{1,2} : \mathbb{Z}/l^\nu(1)_G \to K_2(F,\mathbb{Z}/l^\nu)$ coincides with the Bott element construction via the homomorphism $\mathbb{Z}/l^\nu(1)_G\to\mathbb{Z}/l^\nu(i)^G$ given by the norm (here $G=Gal(F(\mu_{l^\nu})/F)$.*

b) *$\eta^{m,m}$ is the composite $K_m^M(F)/l^\nu\to K_m(F)/l^\nu\to K_m(F,\mathbb{Z}/l^\nu)$.*

c) *The homomorphisms $\eta_F^{i,m}$ commute to products, extension of scalars and transfer. If $l=2$, we must assume that fields considered either contain $\sqrt{-1}$ or have nonzero characteristic.*

Proof. We construct the $\eta_F^{i,m}$ in 4 steps:

Step 1: the case $m=2i$. We must construct a homomorphism $\eta^{i,2i} : \mathbb{Z}/l^\nu(i)_G\to K_{2i}(F,\mathbb{Z}/l^\nu)$. Since F is not exceptional if $l=2$, the G-module $\mathbb{Z}/l^\nu(i)$ is cohomologically trivial and the norm induces an isomorphism $\mathbb{Z}/l^\nu(i)_G\to\mathbb{Z}/l^\nu(i)^G = H^0(F,\mathbb{Z}/l^\nu(i))$. We compose this isomorphism with the map $\beta_F^i : H^0(F,\mathbb{Z}/l^\nu(i))\to K_{2i}(F,\mathbb{Z}/l^\nu)$ defined in [K3]. For $i=1$, we get the Bott element construction by [K3], Theorem 6.1 (ii).

Step 2: the case $m=i$. In this case, the definition is forced by b).

Step 3: the case where $|H^0(F,\mathbb{Z}/l^\nu(m-i))| = l^\nu$. In this case, we define $\eta^{i,m}$ as the composite:

$$\begin{aligned}K_{2i-m}^M(m-i)(F,\mathbb{Z}/l^\nu) &= H^0(F,\mathbb{Z}/l^\nu(m-i))\otimes K_{2i-m}^M(F)/l^\nu\\ &\to K_{2(m-i)}(F,\mathbb{Z}/l^\nu)\otimes K_{2i-m}(F,\mathbb{Z}/l^\nu)\xrightarrow{\text{product}} K_m(F,\mathbb{Z}/l^\nu),\end{aligned}$$

where the first arrow is $\eta^{m-i,2(m-i)} \otimes \eta^{2i-m,2i-m}$.

Step 4: the general case. Let E/F be the extension used to define $K^M_{2i-m}(m-i)(F,\mathbb{Z}/l^\nu)$. We define $\eta^{i,m}$ as the composite:

$$K^M_{2i-m}(m-i)(F,\mathbb{Z}/l^\nu) = K^M_{2i-m}(m-i)(E,\mathbb{Z}/l^\nu)_G \xrightarrow{\eta^{i,m}} K_m(E,\mathbb{Z}/l^\nu)_G$$

$$\to K_m(F,\mathbb{Z}/l^\nu)$$

where $G = Gal(E/F)$ and the last map induced by transfer in K-theory.

Property c) is straightforward, in view of the construction of the $\eta^{i,m}$ and [K3], Theorem 6.1. $\square$

EXTENSION TO SEMI-LOCAL RINGS

If R is a semi-local ring, its Milnor K-theory is defined exactly as for a field. However, a transfer is not yet defined generally in this larger context: this is probably related to the fact that the definition of Milnor's K-theory by means of Steinberg relations should be modified when a residue field of R has too few elements. In particular, a good transfer should exist for the Milnor K-groups as defined in a finite flat extension of semi-local rings at least when all their residue fields are infinite.

In any case, the lack of a transfer causes that some of the definitions and constructions of Proposition 1.1 and 1.2 only carry out partially for semi-local rings. So,

a) In Proposition 1.1, the definition of the morphism $K^M_m(i)(R,\mathbb{Z}/l^\nu) \to K^M_m(i)(R,\mathbb{Z}/l^\mu)$ for ν requires the existence of a transfer. Similarly (and obviously) do c) and the projection formula in d). On the contrary, and in spite of the appearances, definition of the product does not use the existence of a transfer: the two formulas used in its definition and involving transfer could be translated without it (but would become very clumsy).

b) Similarly in Proposition 1.2, the homomorphisms $\eta^{i,m}$ do not commute to an undefined transfer. But they don't need transfer in Milnor's K-theory to be defined (transfer is used in algebraic K-theory only).

Note that, for $m \leq 2$, transfer exists in Milnor's K-theory of rings with many units (in particular semi-local rings with infinite residue fields) as it coincides with its algebraic K-theory in this range [vdK].

2. Anti-Chern classes.

Let R be a commutative semi-local ring in which l is invertible. Kummer theory and cup-product in étale cohomology define homomorphisms (the Galois symbols):

$$u^j : K^M_j(R)/l^\nu \to H^j(R,\mathbb{Z}/l^\nu(m)). \tag{2.1}$$

In many cases, u^j is known to be an isomorphism. The following theorem collects some of these cases. See [K2], (0.6) for a more extensive list of such examples.

Theorem 2.1. *u^j is an isomorphism in the following cases:*

a) *(classical) $j = 1$, any l, any R.*

b) *([MS1], [Su], [L]) $j = 2$, any l, R is a field, a semi-local ring of geometric origin or a semi-localisation of a ring of integers of a number field.*

c) *([R], [MS2]) $j = 3$, $l = 2$, R is a field.*

d) *[Ka] R is a "higher local field" in the sense of Kato, any j, any l.*

It has been announced by Rost that u^4 is an isomorphism for $l = 2$ and any field.

In this section we construct homomorphisms $\beta^{i,j} : H^j(R, \mathbb{Z}/l^\nu(i)) \to K_{2i-j}(R, \mathbb{Z}/l^\nu)$ for certain pairs $j \leq i$ (depending on the nature of R) along the lines of [K1], using [K2] and the constructions of section 1.

For all $j \leq i$ there is a natural homomorphism

$$u^{i,j} : K_j^M(i)(R, \mathbb{Z}/l^\nu) \to H^j(R, \mathbb{Z}/l^\nu(i + j)) \tag{2.2}$$

defined by the composition:

$$K_j^M(i)(R, \mathbb{Z}/l^\nu) = (K_j^M(S) \otimes \mathbb{Z}/l^\nu(i))_G \to (H^j(S, \mathbb{Z}/l^\nu(j)) \otimes \mathbb{Z}/l^\nu(i))_G$$

$$= (H^j(S, \mathbb{Z}/l^\nu(i + j))_G \to H^j(R, \mathbb{Z}/l^\nu(i + j)),$$

where the first arrow is induced by the Galois symbol u^j and the second one by corestriction in cohomology. Note that we don't use a transfer in Milnor's K-theory, so the $u^{i,j}$ are defined for any semi-local ring.

Theorem 2.2. *Let R be a semi-local ring; if $l = 2$, assume that R is not exceptional. Then $u^{i,j}$ is an isomorphism in the following cases:*

a) *$j = 1$, any l, any R.*

b) *$j = 2$, any l, R is a field, a semi-local ring of geometric origin or a semi-localisation of a ring of integers of a number field.*

c) *$j = 3$, $l = 2$, R is a non-exceptional field.*

d) *R is a "higher local field" in the sense of Kato, any j, any l.*

Proof By [K2], Theorem 1 (2), in the said cases the corestriction $(H^j(S, \mathbb{Z}/l^\nu(i + j))_G \to H^j(R, \mathbb{Z}/l^\nu(i + j))$ is an isomorphism. Theorem 2.2 follows from this, Theorem 2.1 and Remark 1.1. □

Remark 2.1. Note that, in all the cases of Theorem 2.2, there is a transfer defined on Milnor's K-groups (compare end of §1). So the problems about transfer outlined at the end of §1 do not arise in these cases.

Definition 2.1. Let $j \leq i$. In the cases a)-d) of Theorem 2.2, the (i, j)-th anti-Chern class $\beta^{i,j}$ is the composition:

$$H^j(R, \mathbb{Z}/l^\nu(i)) \xrightarrow{(u^{i-j,j})^{-1}} K_j^M(i - j)(R, \mathbb{Z}/l^\nu) \xrightarrow{\eta^{i,2i-j}} K_{2i-j}(R, \mathbb{Z}/l^\nu).$$

Theorem 2.3. *The anti-Chern classes of definition 2.1 are natural in R and commute with product and transfer in étale cohomology and K-theory.*

In other words, they satisfy properties (i) - (iv) of Conjecture 3 in [K1].

3. Injectivity of the anti-Chern classes.

The aim of this section is to prove:

Theorem 3.1. *Under the conditions of Definition 2.1, for any $m \geq 0$ the direct sum of the anti-Chern classes $\beta^{i,2i-m}$ for $i \leq m$ and*

$$\begin{aligned}
&2i - m \leq 1 \text{ in case a) of Theorem 2.2,}\\
&2i - m \leq 2 \text{ in case b) of Theorem 2.2,}\\
&2i - m \leq 3 \text{ in case c) of Theorem 2.2,}\\
&2i - m \leq cd_l(F) \text{ in case d) of Theorem 2.2,}
\end{aligned}$$

is split injective.

Proof (compare [K1], proof of Theorem 4 b)). We use the étale K-theory of [DF]. Recall (*loc.cit.*) that to any scheme X over $Z[1/l]$ one associates abelian groups $K_m^{ét}(X, \mathbb{Z}/l^\nu)$ $(m \in \mathbb{Z}, \nu \geq 1)$, such that:

(i) the theory $X \mapsto K_*^{ét}(X, \mathbb{Z}/l^\nu)$ is endowed with a graded product when $l^\nu > 2$, is contravariant in X for arbitrary morphisms and covariant ("transfer") for finite morphisms;

(ii) there is a natural transformation $K_*(-, \mathbb{Z}/l^\nu) \to K_*^{ét}(-, \mathbb{Z}/l^\nu)$, which commutes with products and transfer;

(iii) for X of finite l-cohomological dimension, there is a strongly convergent spectral sequence $E_2^{pq}(X, \mathbb{Z}/l^\nu) \Rightarrow K_{-p-q}^{ét}(X, \mathbb{Z}/l^\nu)$, with

$$E_2^{pq}(X, \mathbb{Z}/l^\nu) = \begin{cases} H^p(X_{ét}, \mathbb{Z}/l^\nu(-q/2)) & \text{if } p \geq 0 \text{ and } q \text{ is even, } \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

This spectral sequence is endowed with products, contravariant in X for arbitrary morphisms and covariant for finite morphisms in a way compatible with the corresponding properties of the abutment [DF]. (Note: we use the standard "cohomological" indexing of spectral sequences as in [Q3], §8, not the Bousfield-Kan indexing as in [DF].)

Definition 3.1. We call the spectral sequence of (iii) above the descent spectral sequence for the étale K-theory of X.

Lemma 3.1. *Let R and j be as in Theorem 2.2 a), b), c), d). Assume that R has finite l-cohomological dimension. Then, all the elements of $E_2^{jq}(R, \mathbb{Z}/l^\nu)$ are universal cycles for any $q \leq -2j$.*

Proof In five steps:

1) $j=0$. Consider the composition:

$$(3.1)\qquad H^0(R,\mathbb{Z}/l^\nu(i)) \xrightarrow{\beta^i_R} K_{2i}(R,\mathbb{Z}/l^\nu) \to K^{ét}_{2i}(R,\mathbb{Z}/l^\nu)$$
$$\to E_2^{0,-2i}(R,\mathbb{Z}/l^\nu) = H^0(R,\mathbb{Z}/l^\nu(i)),$$

where β^i_R is the map constructed in [K3] and $K^{ét}_{2i}(R,\mathbb{Z}/l^\nu) \to E_2^{0i}(R,\mathbb{Z}/l^\nu)$ is the edge homomorphism of the descent spectral sequence for étale K-theory of R. In [K3], Theorem 6.1 (v), it is proven that β^i_R is a section of a natural map $ch_{i,0}$: $K_{2i}(R,\mathbb{Z}/l^\nu) \to H^0(R,\mathbb{Z}/l^\nu(i))$. It is proven in [K1], Lemma. 2.1 a) that $ch_{i,0}$ factors as the composite $K_{2i}(R,\mathbb{Z}/l^\nu) \to K^{ét}_{2i}(R,\mathbb{Z}/l^\nu) \to E_2^{0,-2i}(R,\mathbb{Z}/l^\nu)$ in (3.1). It follows that the composition (3.1) is the identity. In particular, $K^{ét}_{2i}(R,\mathbb{Z}/l^\nu) \to E_2^{0,-2i}(R,\mathbb{Z}/l^\nu)$ is surjective and $E_2^{0,-2i}(R,\mathbb{Z}/l^\nu)$ consists entirely of universal cycles.

2) $j=1$, $q=-2$. Consider the composition:

$$(3.2)\qquad H^1(R,\mathbb{Z}/l^\nu(1)) = K_1(R)/l^\nu \to K_1(R,\mathbb{Z}/l^\nu) \to K^{ét}_1(R,\mathbb{Z}/l^\nu)$$
$$\to E_2^{1,-2}(R,\mathbb{Z}/l^\nu) = H^1(R,\mathbb{Z}/l^\nu(1)),$$

where the first equality is Theorem 2.1 a) (Kummer theory) and the homomorphism

$$K^{ét}_1(R,\mathbb{Z}/l^\nu) \to E_2^{1,-2}(R,\mathbb{Z}/l^\nu)$$

is an "edge homomorphism" coming from the fact that $E_2^{0,-1}(R,\mathbb{Z}/l^\nu)=0$. It is shown in [K1], Lemma 2.1 b), that this composition is the identity. Thus $E_2^{1,-2}(R,\mathbb{Z}/l^\nu)$ consists entirely of universal cycles.

3) $q=-2j$. In the cases of Theorem 2.2, the cup-product

$$H^1(R,\mathbb{Z}/l^\nu(1))^{\otimes j} \to H^j(R,\mathbb{Z}/l^\nu(j))$$

is surjective. Since the descent spectral sequence is multiplicative, it follows that $E_2^{j,-2i}(R,\mathbb{Z}/l^\nu)$ consists entirely of universal cycles.

4) $q=-2(i+j)$, $k\geq 0$ such that $H^0(R,\mathbb{Z}/l^\nu(i))$ has order l^ν. Trivially, the product

$$E_2^{0,-2i}(R,\mathbb{Z}/l^\nu)\otimes E_2^{j,-2j}(R,\mathbb{Z}/l^\nu) = H^0(R,\mathbb{Z}/l^\nu(i))\otimes H^j(R,\mathbb{Z}/l^\nu(j))$$
$$\to H^j(R,\mathbb{Z}/l^\nu(i+j)) = E_2^{j,-2(i+j)}(R,\mathbb{Z}/l^\nu)$$

is bijective. By 2) and 3) $E_2^{j,-2(i+j)}(R,\mathbb{Z}/l^\nu)$ consists entirely of universal cycles, since the descent spectral sequence is multiplicative.

5) *The general case.* Consider i and j as in 4). Let S/R be the smallest étale extension such that the étale sheaf $\mathbb{Z}/l^\nu(i)$ becomes constant over S, and $G=Gal(S/R)$. By [K2], Theorem 1 (2), the corestriction $(H^j(S,\mathbb{Z}/l^\nu(i+j))_G \to H^j(R,\mathbb{Z}/l^\nu(i+j))$ is an isomorphism. Since the descent spectral sequence is compatible with transfer, by 4) $E_2^{j,-2(i+j)}(R,\mathbb{Z}/l^\nu)$ consists entirely of universal cycles. □

Let $F^jK^{ét}_*(X,\mathbb{Z}/l^\nu)$ be the filtration defined on $K^{ét}_*(X,\mathbb{Z}/l^\nu)$ by the descent spectral sequence.

Lemma 3.2. *Under the hypothesis of Lemma 3.1, the image of the composition*

$$H^j(R,\mathbb{Z}/l^\nu(i)) \xrightarrow{\beta^{i,j}} K_{2i-j}(R,\mathbb{Z}/l^\nu) \to K^{\acute{e}t}_{2i-j}(R,\mathbb{Z}/l^\nu)$$

is contained in $F^jK^{\acute{e}t}_{2i-j}(R,\mathbb{Z}/l^\nu)$.

Proof. By the same method as in Lemma 3.1, we reduce to the special cases $j=0$ and $i=j=1$. In both cases the lemma is trivial, because $K^{\acute{e}t}_{2i}(R,\mathbb{Z}/l^\nu)=F^0K^{\acute{e}t}_{2i}(R,\mathbb{Z}/l^\nu)$ and $K^{\acute{e}t}_1(R,\mathbb{Z}/l^\nu)=F^1K^{\acute{e}t}_1(R,\mathbb{Z}/l^\nu)$. □

Proof of Theorem 3.1. By a direct limit argument (compare [K2], (3.2), Proposition 3.2.1), we reduce to the case where R has finite cohomological dimension. Then it suffices to show that, for all (i,j), the composition

$$\begin{aligned} H^j(R,\mathbb{Z}/l^\nu(i)) &\xrightarrow{\beta^{i,j}} K_{2i-j}(R,\mathbb{Z}/l^\nu) \to F^jK^{\acute{e}t}_{2i-j}(R,\mathbb{Z}/l^\nu)\\ &\to E_\infty^{j,-2i}(R,\mathbb{Z}/l^\nu) = E_2^{j,-2i}(R,\mathbb{Z}/l^\nu) = H^j(R,\mathbb{Z}/l^\nu(i)) \end{aligned}$$

is the identity. This is checked as in Lemmas 3.1 and 3.2 by reduction to the special cases $j=0$ and $i=j=1$. These special cases have already been seen in steps 1) and 2) of the proof of Lemma 3.1. □

Remark 3.1. Let us record here that the proof of Theorem 3.1 produces "Chern characters" $ch_{i,2i-m}: K_m(R,\mathbb{Z}/l^\nu) \to H^{2i-m}(R,\mathbb{Z}/l^\nu(i))$, for the same values of (i,m) as in Theorem 3.1, which are left inverse to the $\beta^{i,2i-m}$. They are defined first when R has finite l-cohomological dimension, using the descent spectral sequence for étale K-theory, then in general by writing R as a direct limit of semi-local rings of finite l-cohomological dimension.

Remark 3.2. I don't know if, in the descent spectral sequence for the étale K-theory considered in [DF], $E_2^{jq}(R,\mathbb{Z}/l^\nu)$ consists of universal cycles for $-2j\le q\le 0$. But if one uses the unbounded below version of étale K-theory $K_*^{top}(R,\mathbb{Z}/l^\nu)$ as in [T], one can extend the construction of $\beta^{i,j}$ to values of i smaller than j, as classes with values in $K_*^{top}(R,\mathbb{Z}/l^\nu)$, and prove that E_2^{jq} consists of universal cycles for all $q\in\mathbb{Z}$ in the corresponding descent spectral sequence. See [K1] for details.

Theorem 3.2. *For any scheme X over $Z[1/l]$,*
- *a) $H^0(X,\mathbb{Z}/l^\nu(i))$ is a direct summand of $K_{2i}(X,\mathbb{Z}/l^\nu)$ for all $i\ge 0$;*
- *b) $H^1(X,\mathbb{Z}/l^\nu(i))$ is a direct summand of $H^0(X_{Zar},\mathcal{K}_{2i-1}(\mathbb{Z}/l^\nu))$ for all $i\ge 1$, where $\mathcal{K}_{2i-1}(\mathbb{Z}/l^\nu)$ denotes the Zariski sheaf associated to the presheaf $U\mapsto K_{2i-1}(U,\mathbb{Z}/l^\nu)$.*

Proof. a) has already been seen in [K3], Theorem 6.1 (v). For b), we can globalise the local anti-Chern classes $\beta^{i,1}$ into

$$\beta_X^{i,1}: H^1(X,\mathbb{Z}/l^\nu(i)) = H^0(X_{Zar},\mathcal{H}^1(\mathbb{Z}/l^\nu(i))) \to H^0(X_{Zar},\mathcal{K}_{2i-1}(\mathbb{Z}/l^\nu)),$$

where $\mathcal{H}^1(\mathbb{Z}/l^\nu(i))$ denotes the Zariski sheaf associated to étale cohomology. Composing with the globalisation of the Chern character $ch_{i,1}$ of Remark 3.1, we get the identity. □

4. Higher local fields.

Let F be a higher local field in the sense of Kato [Ka]. By definition, there is a chain of fields $F_0, \ldots, F_n = F$ such that: F_0 is a finite field; For $1 \leq r \leq n$, F_r is complete for a discrete valuation, with residue field F_{r-1}.

We call n the dimension of F and $char F_0$ its *essential residue characteristic*. Let $l \neq char F$. It follows from §3 that, for all $i \geq 0$, there are split injections (given by the anti-Chern classes of §2):

$$H^0(F,\mathbb{Z}/l^\nu(i)) \oplus H^2(F,\mathbb{Z}/l^\nu(i+1)) \oplus \cdots \oplus H^{2s}(F,\mathbb{Z}/l^\nu(i+s)) \to K_{2i}(F,\mathbb{Z}/l^\nu) \tag{4.1}$$

$$H^1(F,\mathbb{Z}/l^\nu(i+1)) \oplus H^3(F,\mathbb{Z}/l^\nu(i+2)) \oplus \cdots \oplus H^{2t+1}(F,\mathbb{Z}/l^\nu(i+t+1)) \to K_{2i+1}(F,\mathbb{Z}/l^\nu) \tag{4.2}$$

where $s = min(i, \left[\frac{n}{2}\right])$ and $t = min(i, \left[\frac{n-1}{2}\right])$. In this section, we prove:

Theorem 4.1. *Assume that l is not the essential residue characteristic of F. Then the injections (4.1) and (4.2) are isomorphisms.*

Proof. For any scheme X, let $F^jK_*(X,\mathbb{Z}/l^\nu)$ denote the filtration induced on $K_*(X,\mathbb{Z}/l^\nu)$ by the filtration on étale K-theory and the map $K_*(X,\mathbb{Z}/l^\nu) \to K_*^{\acute{e}t}(X,\mathbb{Z}/l^\nu)$.

Lemma 4.1. *For all $m \geq 0$, $F^{m+1}K_m(F,\mathbb{Z}/l^\nu) = 0$.*

Lemma 4.1 implies Theorem 4.1, since by Lemmas 3.1 and 3.2 the compositions of (4.1) and (4.2) with $K_m(F,\mathbb{Z}/l^\nu) \to K_m^{\acute{e}t}(R,\mathbb{Z}/l^\nu) \to K_m^{\acute{e}t}(R,\mathbb{Z}/l^\nu)/F^{m+1}K_m^{\acute{e}t}(R,\mathbb{Z}/l^\nu)$ are isomorphisms.

Proof of Lemma 4.1. By induction on $n = dimF$. For $n = 0$, this follows from Quillen's computation of the K-theory of finite fields [Q1]. Assume $n > 0$ and that Lemma 4.1 is true for F_{n-1}. By [Su], Corollary 3.11, there is for all $m \geq 0$ an exact sequence:

$$0 \to K_m(F_{n-1},\mathbb{Z}/l^\nu) \to K_m(F,\mathbb{Z}/l^\nu) \to K_{m-1}(F_{n-1},\mathbb{Z}/l^\nu) \to 0. \tag{4.3}$$

There are corresponding exact sequences for étale K-theory and Galois cohomology, and one can show that they are compatible with the comparison maps and descent spectral sequences. It follows that (4.3) induces for all $j \leq m$ a zero-sequence:

$$0 \to F^jK_m(F_{n-1},\mathbb{Z}/l^\nu) \to F^jK_m(F,\mathbb{Z}/l^\nu) \xrightarrow{\partial} F^{j-1}K_{m-1}(F_{n-1},\mathbb{Z}/l^\nu) \to 0.$$

By assumption on F_{n-1}, the left and right terms are 0 for $j = m+1$, so it is enough to show that this sequence is exact at the middle term for any j. Let $x \in F^jK_m(F,\mathbb{Z}/l^\nu)$ be such that $\partial x = 0$. By Suslin's theorem, x comes from $K_m(F_{n-1},\mathbb{Z}/l^\nu)$ and we have to see that it lies in $F^jK_m(F_{n-1},\mathbb{Z}/l^\nu)$. But let π be a prime element of F : then $\{\pi\}\cdot x \in F^{j+1}K_{m+1}(F,\mathbb{Z}/l^\nu)$ and $\partial(\{\pi\}\cdot x) = x \in F^jK_m(F_{n-1},\mathbb{Z}/l^\nu)$. $\square$

5. Global fields.

Theorem 5.1. *Let A be a Dedekind domain with quotient field a global field. Assume that l is invertible in A and, if $l = 2$, that either $\sqrt{-1} \in A$ or that A has nonzero characteristic. Then, for any $i \geq 1$, there is a split injection:*

$$\beta^{i,0} \oplus \beta^{i+1,2} : H^0(A, \mathbb{Z}/l^\nu(i)) \oplus H^2(A, \mathbb{Z}/l^\nu(i+1)) \to K_{2i}(A, \mathbb{Z}/l^\nu)$$

which commutes with products, change of rings and transfer. If A is finitely generated over $\mathbb{Z}$, there is a split injection $H^2(A, \mathbb{Z}_l(i+1)) \to K_{2i}(A) \otimes \mathbb{Z}_l$ which commutes with change of rings and transfer. It is right inverse to the Chern character $ch_{i+1,2} : K_{2i}(A) \otimes \mathbb{Z}_l \to H^2(A, \mathbb{Z}_l(i+1))$ constructed in [DF].

Note. Theorem 5.1 is wrong for a complete curve over a finite field.

Proof. The map $\beta^{i,0} : H^0(A, \mathbb{Z}/l^\nu(i)) \to K_{2i}(A, \mathbb{Z}/l^\nu)$ is already constructed in [K3]. By Soulé's theorem, the second Chern class $K_2(A)/l^\nu \to H^2(A, \mathbb{Z}/l^\nu(2))$ is an isomorphism for all ν ([So1], Lemma 10; [K4], app. 2). Moreover, the descent theorem of [K2], Theorem 1 (2) holds trivially for H^2 even though A is "global" because $cd_l(A) = 2$. (Here we are using that Spec$A$ is not complete.) Therefore, we get *global* anti-Chern classes

$$\beta^{i+1,2} : H^2(A, \mathbb{Z}/l^\nu(i+1)) \to K_{2i}(A, \mathbb{Z}/l^\nu)$$

in the same way as in §2. Split injectivity is proven exactly as in §3.

To go to the infinite level, we pass to the limit on $\beta^{i+1,2}$, getting a map:

$$\varprojlim H^2(A, \mathbb{Z}/l^\nu(i+1)) \to \varprojlim K_{2i}(A, \mathbb{Z}/l^\nu).$$

By the hypothesis on A, $H^1(A, \mathbb{Z}/l^\nu(i+1))$ is finite for all ν and the surjection

$$H^2(A, \mathbb{Z}_l(i+1)) \to \varprojlim H^2(A, \mathbb{Z}/l^\nu(i+1))$$

is bijective. Similarly, finite generation of $K_*(A)$ ([Q2], [Gr]) and the exact sequences

$$0 \to K_{2i}(A)/l^\nu \to K_{2i}(A, \mathbb{Z}/l^\nu) \to {}_{l^\nu}K_{2i-1}(A) \to 0$$

yield an isomorphism $K_{2i}(A) \otimes \mathbb{Z}_l \xrightarrow{\cong} \varprojlim K_{2i}(A, \mathbb{Z}/l^\nu)$. Hence the above map translates as:

$$H^2(A, \mathbb{Z}_l(i+1)) \to K_{2i}(A) \otimes \mathbb{Z}_l.$$

The last claim follows from Theorem 3.1. $\square$

Remark 5.1. Of course, the splitting of Theorem 5.1 also commutes with products in algebraic and étale K-theories at the infinite level. But this statement is empty: the product in the l-adic cohomology of A is trivial, since $cd_l(A) = 2$. The Quillen-Lichtenbaum conjecture therefore predicts that algebraic K-theory products $K_{2i}(A) \times K_{2j}(A) \to K_{2(i+j)}(A)$ are 0 (away from 2 in case A is exceptional). Similarly, it predicts that products $K_{2i}(A) \times K_{2j-1}(A) \to K_{2(i+j)-1}(A)$ are 0. Can one prove these vanishings directly?

Here is a refinement of Theorem 5.1:

Theorem 5.2. *The map $\beta^{i+1,2}: H^2(A,\mathbb{Z}/l^\nu(i+1)) \to K_{2i}(A,\mathbb{Z}/l^\nu)$ lands into $K_{2i}(A)/l^\nu$. The same holds replacing A by any integrally closed subring of $\overline{\mathbb{Q}}$ (in characteristic 0) or $\overline{\mathbb{F}_q(t)}$ (in positive characteristic).*

Proof. The first claim is a consequence of the following commutative diagram:

$$\begin{array}{ccc} H^2(A,\mathbb{Z}_l(i+1)) & \longrightarrow & K_{2i}(A)\otimes \mathbb{Z}_l \\ \downarrow & & \downarrow \\ H^2(A,\mathbb{Z}/l^\nu(i+1)) & \longrightarrow & K_{2i}(A,\mathbb{Z}/l^\nu) \end{array}$$

and the observation that i) the left vertical map is surjective (because $cd_l(A)=2$) and ii) the right vertical map factors through $K_{2i}(A)/l^\nu$ (because $\varprojlim {}_{l^\nu}K_{2i-1}(A)=0$ by Quillen's finite generation theorem). The second claim follows by taking a direct limit. □

Remark 5.2. The proof of this in [K1], Remark 4.2, in the positive characteristic case, is absurd for $i>2$ (resp. correct for $i=2$).

The following corollary seems new:

Corollary 5.1. *The composition*

$$K_{2i}(A)/l^\nu \xrightarrow{ch_{i+1,2}} K_{2i}(A,\mathbb{Z}/l^\nu) \to H^2(A,\mathbb{Z}/l^\nu(i+1))$$

is split surjective for all $i\geq 0$ and $\nu\geq 1$.

Remark 5.3. Theorem 5.2 is deep. If F is a field of cohomological dimension 2 that is not contained in $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_q(t)}$, the composite

$$H^2(F,\mathbb{Z}/l^\nu(i+1)) \to K_{2i}(F,\mathbb{Z}/l^\nu) \to {}_{l^\nu}K_{2i-1}(F)$$

is not zero in general for $i>1$. Indeed, in positive characteristic or if $\operatorname{trdeg}(F/\mathbb{Q})\geq 2$, F contains two elements t_1,t_2 which are algebraically independent over the prime field. Assume that F contains a primitive l^ν-th root of unity ζ. Taking $i=2$ and

$$\alpha=(t_1)\cdot(t_2)\cdot[\zeta]\in H^2(F,\mathbb{Z}/l^\nu(3)),$$

where $(t_i)\in H^1(F,\mathbb{Z}/l^\nu(1))$ and $[\zeta]\in H^0(F,\mathbb{Z}/l^\nu(1))$, the image of α in ${}_{l^\nu}K_3(F)$ is $\{t_1,t_2,\zeta\}$, which is in general nonzero (eg. $F=k(t_1,t_2)$, k algebraically closed). One can produce a similar counterexample in characteristic zero for a finitely generated field of transcendance degree 1 over $\mathbb{Q}$. In the arithmetic case, as the proof shows, Theorem 5.2 is true for finiteness reasons.

Theorem 5.3. *Under the hypotheses of Theorem 5.1, the local anti-Chern classes $\beta^{i,1}$ globalise as split injections $\beta^{i,1}: H^1(A,\mathbb{Z}/l^\nu(i)) \to K_{2i-1}(A,\mathbb{Z}/l^\nu)(i\geq 1)$. They yield split injections $H^1(A,\mathbb{Z}_l(i)) \to K_{2i-1}(A)\otimes\mathbb{Z}_l$. All these splittings commute to change of rings and transfer. They are right inverse to the Chern characters*

$$ch_{i,1}: K_{2i-1}(A)\otimes\mathbb{Z}_l \to H^1(A,\mathbb{Z}_l(i))$$

constructed in [DF]. The l-adic splittings for odd and even K-groups of Theorem 5.1 and this theorem commute to products.

Proof. To construct $\beta^{i,1}$ at a finite level, it suffices by Theorem 3.2 to observe that

$$K_{2i-1}(A, \mathbb{Z}/l^{\nu}) \to H^0(A_{Zar}, \mathcal{K}_{2i-1}(\mathbb{Z}/l^{\nu}))$$

is bijective. Surjectivity is obvious for dimension reasons, and injectivity follows from Soulé's theorem that

$$K_{2i-1}(A, \mathbb{Z}/l^{\nu}) \to K_{2i-1}(F, \mathbb{Z}/l^{\nu})$$

is injective, where F is the quotient field of A ([So1], Theorem 3, [So4], proof of Theorem 1). The claims about functoriality and products follow from Theorem 2.3. The l-adic case follows as in the proof of Theorem 5.1 from a passage to the limit. $\square$

Remark 5.5. Assume that $\mathrm{char} F = 0$, $A = O_F[1/l]$, let $A_n = A[\mu_{l^n}]$ and

$$\overline{E} = \varprojlim (A_n)^*/(A_n)^{*l^n}.$$

In [So2], Lemma 1 and [So3], (4.3), Soulé defines and studies a map

$$\overline{E}(i-1)_G \xrightarrow{S} K_{2i-1}(A) \otimes \mathbb{Z}_l$$

for any $i \geq 2$. Deligne [D] studies a similar map

$$\overline{E}(i-1)_G \xrightarrow{D} H^1(A, \mathbb{Z}_l(i)).$$

In [So3] it is observed that $D = ch_{i,1} \circ S$. For the same reason, one sees easily that $S = \beta^{i,1} \circ D$. This gives a slightly more precise information on the maps S and D (which are isomorphisms modulo finite groups).

6. Local fields.

Let F be a finite extension of $\mathbb{Q}_p$. Recall from Theorem 4.1 that, for $l \neq p$, the anti-Chern classes of §2 yield isomorphisms

$$H^1(F, \mathbb{Z}/l^{\nu}(i)) \xrightarrow{\cong} K_{2i-1}(F, \mathbb{Z}/l^{\nu})$$

and

$$H^0(F, \mathbb{Z}/l^{\nu}(i)) \oplus H^2(F, \mathbb{Z}/l^{\nu}(i+1)) \xrightarrow{\cong} K_{2i}(F, \mathbb{Z}/l^{\nu}).$$

Assume that F contains a square root of -1 if $p = 2$. By Remark 3.1, the following conjecture is equivalent to the "Lichtenbaum-Quillen" conjecture for the local field F.

Conjecture 6.1. *The split injections of Theorem 3.1:*

$$\beta^{i+1,1}: H^1(F,\mathbb{Z}/l^\nu(i+1)) \to K_{2i+1}(F,\mathbb{Z}/l^\nu)$$
$$\beta^{i,0}\oplus\beta^{i+1,2}: H^0(F,\mathbb{Z}/l^\nu(i))\oplus H^2(F,\mathbb{Z}/l^\nu(i+1)) \to K_{2i}(F,\mathbb{Z}/l^\nu)$$

are isomorphisms even when $l=p$.

This conjecture holds for $i=0$ and 1 by [MS1], [MS3], [Le]. It is not impossible that Panin's theorem [Pa] be sufficient to prove it, using syntomic cohomology, however I don't know how to do this.

In this section we look at consequences of Conjecture 6.1. Set $d=[F:\mathbb{Q}_p]$.

We compare algebraic and topological K-theory, as in [K4], §7. Recall from [W] the topological or p-adic K-theory of F, $K_*^{top}(F)$. There is a natural homomorphism $\phi_*: K_*(F)\to K_*^{top}(F)$. The following are basic results of Wagoner.

Proposition 6.0. *[W] For i even, $K_i^{top}(F)$ is finite. For i odd and >1, it is the direct sum of $K_i(k)$ and a finitely generated $\mathbb{Z}_p$-module of rank $d=[F:\mathbb{Q}_p]$, where k is the residue field of F. For $i=1$, it is isomorphic to the profinite completion of F^*.*

Proposition 6.1. *(Wagoner). For all $i\geq 0$, $K_i^{top}(F)\xrightarrow{\cong}\varprojlim K_i(F,\mathbb{Z}/n)$.*

Proof. By [K4], proof of Theorem 7.2, there are exact sequences:

$$0\to K_i^{top}(F)/n\to K_i(F,\mathbb{Z}/n)\to {}_nK_{i-1}^{top}(F)\to 0.$$

Since, by [W], $K_i^{top}(F)$ is the direct sum of a finite group and a finitely generated $\mathbb{Z}_p$-module, Proposition 6.1 follows. □

Corollary 6.1. *For all $i\geq 1$, the maps $\beta^{i,1}$ and $\beta^{i+1,2}$ of §2 induce split injections $\hat\beta^{i,1}: H^1(F,\hat{\mathbb{Z}}(i))\to K_{2i-1}^{top}(F)$ and $\hat\beta^{i+1,2}: H^2(F,\hat{\mathbb{Z}}(i+1))\to K_{2i}^{top}(F)$, with finite p-primary cokernel. Under Conjecture 6.1, they are isomorphisms.*

Proof. In view of Theorem 3.1, the split injections just come from Proposition 6.1 and the inverse limit over $n\geq 1$ of the maps $\beta^{i,1}$ and $\beta^{i+1,2}$ for K-theory and cohomology modulo n (note that $K_i(F,\mathbb{Z}/n)$ is finite for all n, i by [Pa]). By Theorem 4.1, $\beta^{i,1}$ is an isomorphism when n is prime to p, hence $Coker\ \hat\beta^{i,1}$ is a $\mathbb{Z}_p$-module. Its finiteness follows from Proposition 6.0 and the fact that $H^1(F,\mathbb{Z}_p(i))$ has $\mathbb{Z}_p$-rank d if $i>1$ and $d+1$ if $i=1$ (compare [Sch], Satz 4 ii)). Similarly, $\beta^{i,0}\oplus\beta^{i+1,2}$ is an isomorphism for n prime to p by Theorem 4.1. Since $H^0(F,\hat{\mathbb{Z}}(i))=0$ for $i\neq 0$, it follows similarly that $Coker\ \hat\beta^{i+1,2}$ is a $\mathbb{Z}_p$-module, finite by Proposition 6.0. The last claim is obvious. □

Variant. *For all $i\geq 0$, there is a split surjection with finite p-primary kernel*

$$K_i^{top}(F)\to\prod_l \hat K_i^{\acute et}(F)_l$$

where l runs through all primes and $\hat K_i^{\acute et}(F)_l$ denotes the l-adic étale K-theory of [DF]. Under Conjecture 6.1, it is an isomorphism.

Proof. As the $K_i^{\acute et}(F,\mathbb{Z}/n)$ are finite, $\prod_l \hat K_i^{\acute et}(F)_l\to\varprojlim K_i^{\acute et}(F,\mathbb{Z}/n)$ is an isomorphism. □

Remark 6.1. By Tate duality, $H^2(F,\hat{\mathbb{Z}}(i+1))$ is canonically isomorphic to $H^0(F,\mathbb{Q}/\mathbb{Z}(i))$. Therefore, under Conjecture 6.1, $K_{2i}^{top}(F)$ should be isomorphic to $H^0(F,\mathbb{Q}/\mathbb{Z}(i))$ (and is, away from p).

We now study the comparison maps $\phi_i: K_i(F)\to K_i^{top}(F)$.

Proposition 6.2. *For all $i > 0$, $Ker\ \phi_i$ is divisible without torsion prime to p. For i odd, it is uniquely divisible. For i even, ϕ_i is surjective.*

Proof. By [K4], Theorem 7.2, for all i $Ker\ \phi_i$ is divisible, $Coker\ \phi_i$ is torsion-free and there is a canonical isomorphism $Coker\ \phi_i \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\cong} (Ker\ \phi_{i-1})_{tors}$. By Proposition 6.0, for i even $K_i^{top}(F)$ is finite and for i odd $K_i^{top}(F) \otimes \mathbb{Q}_l/\mathbb{Z}_l = 0$ for $l \neq p$; Proposition 6.2 follows. $\square$

Corollary 6.2. *For i even > 0, $K_i(F)$ is the direct sum of the finite group $K_i^{top}(F)$ and a divisible group D_i without torsion prime to p. $K_{2i-1}(F)_{tors}$ is finite and its prime-to-p part is isomorphic to the prime-to-p part of $H^0(F, \mathbb{Q}/\mathbb{Z}(i))$. If Conjecture 6.1 holds, $K_{2i-1}(F)_{tors}$ is isomorphic to $H^0(F, \mathbb{Q}/\mathbb{Z}(i))$.*

Remark 6.2. A reformulation of Corollaries 6.1 and 6.2, under Conjecture 6.1, is an exact sequence, for all n:

$$\begin{aligned} 0 \to H^0(F,\mathbb{Z}/n(i)) \to K_{2i-1}(F) &\xrightarrow{n} K_{2i-1}(F) \to H^1(F,\mathbb{Z}/n(i)) \\ \to K_{2i-2}(F) &\xrightarrow{n} K_{2i-2}(F) \to H^2(F,\mathbb{Z}/n(i)) \to 0. \end{aligned}$$

These exact sequences exist at least when n is prime to p.

Proposition 6.3. *For all $i \geq 0$, $corank(K_{2i}(F)\{p\}) + corank(K_{2i+1}(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p) = d$.*

Proof. By [K4], proof of Theorem 7.2, there is an exact sequence:

$$0 \to K_{2i-1}(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to K_{2i-1}^{top}(F) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to K_{2i-2}(F)\{p\} \to K_{2i-2}^{top}(F)\{p\} \to 0.$$

Proposition 6.3 follows from this exact sequence and Proposition 6.0. $\square$

The following conjecture was proved by Merkurjev [Me] for $i = 1$.

Conjecture 6.2. *For all $i \geq 0$, $corank(K_{2i}(F)\{p\}) = 0$.*

We now refine the map ϕ_i, for i odd, under Conjecture 6.1, just as in [K4], Lemma 7.1. The following defines *indecomposable algebraic K-theory of odd degree.*

Definition 6.1. For a field K and an integer $i \geq 1$, we let $K_{2i-1}(K)_{ind} = K_{2i-1}(K)/\Re$, where $\Re$ is the subgroup of $K_{2i-1}(K)$ generated by the $N_{L/K}(K_1(L) \cdot K_{2i-2}(L))$, where L runs through all finite extensions of K and $N_{L/K}$ is the transfer in algebraic K-theory.

Proposition 6.4. *Under Conjecture 6.1, the homomorphism ϕ_{2i-1} factors as*

$$\overline{\phi}_{2i-1} : K_{2i-1}(F)_{ind} \to K_{2i-1}^{top}(F).$$

Proof. Considering the commutative diagram

$$\begin{array}{ccc} K_1(F) \otimes K_{2i-2}(F) & \longrightarrow & K_{2i-1}(F) \\ \downarrow & & \downarrow \\ K_1(F) \otimes K_{2i-2}^{top}(F) & \longrightarrow & K_{2i-1}^{top}(F) \\ \downarrow & & \downarrow \\ K_1(F,\hat{\mathbb{Z}}) \otimes K_{2i-2}(F,\hat{\mathbb{Z}}) & \longrightarrow & K_{2i-1}(F,\hat{\mathbb{Z}}) \end{array}$$

it is enough, in view of Proposition 6.1, to prove that the product

$$K_1(F, \hat{\mathbb{Z}}) \otimes K_{2i-2}(F, \hat{\mathbb{Z}}) \to K_{2i-1}(F, \hat{\mathbb{Z}})$$

is identically 0 under Conjecture 6.1 (then we may use the transfer). But there is another commutative diagram:

$$\begin{array}{ccc}
H^1(F, \hat{\mathbb{Z}}(1)) \otimes H^2(F, \hat{\mathbb{Z}}(i)) & \longrightarrow & H^3(F, \hat{\mathbb{Z}}(i+1)) \\
{\scriptstyle \beta^{1,1} \otimes \beta^{2,i}} \downarrow & & {\scriptstyle \beta^{3,i+1}} \downarrow \\
K_1(F, \hat{\mathbb{Z}}) \otimes K_{2i-2}(F, \hat{\mathbb{Z}}) & \longrightarrow & K_{2i-1}(F, \hat{\mathbb{Z}}).
\end{array}$$

By Conjecture 6.1, the left vertical map is an isomorphism, and by cohomological dimension $H^3(F, \hat{\mathbb{Z}}(i+1)) = 0$. This proves that the product is 0. (Is there a proof independent of Conjecture 6.1?)

Lemma 6.1. *(compare Corollary 5.1). Let $ch_{i,2} : K_{2i-2}(F, \mathbb{Z}/n) \to H^2(F, \mathbb{Z}/n(i))$ be the Chern character of Remark 3.1. Then the composition*

$$K_{2i-2}(F)/n \xrightarrow{ch_{i,2}} K_{2i-2}(F, \mathbb{Z}/n) \to H^2(F, \mathbb{Z}/n(i))$$

is surjective for all $i > 0$ and $n \geq 1$.

Proof. It suffices to show that $\beta^{i,2} : H^2(F, \mathbb{Z}/n(i)) \to K_{2i-2}(F, \mathbb{Z}/n)$ factors through $K_{2i-2}(F)/n$. Let F_0 be the algebraic closure of $\mathbb{Q}$ in F. Since F_0 and F have the same absolute Galois group, the natural map $H^2(F_0, \mathbb{Z}/n(i)) \to H^2(F, Z/n(i))$ is an isomorphism. By Theorem 5.2, $H^2(F_0, \mathbb{Z}/n(i)) \to K_{2i-2}(F_0, \mathbb{Z}/n)$ factors through $K_{2i-2}(F_0)/n$; therefore the same is true for F. $\square$

Lemma 6.2. *Ker $(K_{2i-1}(F) \to K_{2i-1}(F)_{ind})$ is divisible.*

Proof. Let E/F be a finite extension: then $N_{E/F} : K_{2i-2}(E) \to K_{2i-2}(F)$ is onto. This follows from the commutative diagram (where $n = [E : F]$)

$$\begin{array}{ccc}
K_{2i-2}(E)/n & \longrightarrow & H^2(E, \mathbb{Z}/n(i)) \\
{\scriptstyle N} \downarrow & & {\scriptstyle Cor} \downarrow \\
K_{2i-2}(F)/n & \longrightarrow & H^2(F, \mathbb{Z}/n(i))
\end{array}$$

in which the horizontal maps are surjective by Lemma 6.1 and Cor is surjective by cohomological dimension. Let $(x, y) \in K_1(F) \times K_{2i-2}(F)$, $n \geq 1$ and $E = F(\sqrt[n]{x})$. Choose $z \in K_{2i-2}(E)$ such that $N(z) = y$. Then, by the projection formula, $x \cdot y = nN_{E/F}(\xi \cdot z)$, where $\xi \in E^*$ satisfies $\xi^n = x$. This proves the claim. $\square$

Proposition 6.5. *Under Conjecture 6.1, the composite*

$$H^0(F, \mathbb{Q}/\mathbb{Z}(i)) \to K_{2i-1}(F)_{tors} \to (K_{2i-1}(F)_{ind})_{tors}$$

is an isomorphism.

This follows from Corollary 6.2 and Lemma 6.2.

Proposition 6.6. *We assume Conjecture 6.1. It follows that $Ker\ \overline{\phi}_{2i-1}$ is uniquely divisible and $Coker\ \overline{\phi}_{2i-1}$ is torsion-free, without cotorsion prime to p. If Conjecture 6.2 holds, $Coker\ \overline{\phi}_{2i-1}$ is uniquely divisible.*

Proof. Obviously, ϕ_{2i-1} and $\overline{\phi}_{2i-1}$ have the same cokernel, hence the second claim, observing that Conjecture 6.2 implies that $K_{2i-1}(F)\otimes\mathbb{Q}_p/\mathbb{Z}_p$ and $K_{2i-1}^{top}(F)\otimes\mathbb{Q}_p/\mathbb{Z}_p$ have the same corank. Since $Ker\ \phi_{2i-1}$ is divisible, $Ker\ \overline{\phi}_{2i-1}$ is divisible as well; its torsion is contained in $(K_{2i-1}(F)_{ind})_{tors}$. But the latter is finite by Proposition 6.5; therefore $Ker\ \overline{\phi}_{2i-1}$ is torsion-free. □

Just as in [K4], we propose a last conjecture on $\overline{\phi}_{2i-1}$. As in *op. cit.*, §7, one defines relative groups:

$$K_{2i-1}(A,\mathcal{M})_{ind} = Ker\ (K_{2i-1}(F)_{ind}\to K_{2i-1}(k));$$
$$K_{2i-1}^{top}(A,\mathcal{M}) = Ker\ (K_{2i-1}^{top}(A)K_{2i-1}(k));$$

where A is the valuation ring of F, $\mathcal{M}$ its maximal ideal and the first reduction map is defined in the same way as for $i=2$ in *loc. cit.* Similarly, $K_{2i-1}(A,\mathcal{M})_{ind}$ is naturally a $\mathbb{Z}_{(p)}$-module, where $\mathbb{Z}_{(p)}$ is the localisation of $\mathbb{Z}$ at p, and $\overline{\phi}_{2i-1}$ induces a map $\overline{\phi}_{2i-1}: K_{2i-1}(A,\mathcal{M})_{ind}\to K_{2i-1}^{top}(A,\mathcal{M})$ with uniquely divisible kernel and cokernel under Conjectures 6.1 and 6.2. Let $\mathbb{Z}_{(p)}^h$ be the henselisation of $\mathbb{Z}_{(p)}$.

Conjecture 6.3. *The map $\overline{\phi}_{2i-1}: K_{2i-1}(A,\mathcal{M})_{ind}\to K_{2i-1}^{top}(A,\mathcal{M})$ is injective; there exists on $K_{2i-1}(A,\mathcal{M})_{ind}$ a natural structure of a $\mathbb{Z}_{(p)}^h$-module for which it is finitely generated; moreover, $\overline{\phi}_{2i-1}$ induces an isomorphism:*

$$K_{2i-1}(A,\mathcal{M})_{ind}\otimes_{\mathbb{Z}_{(p)}^h}\mathbb{Z}_p \xrightarrow{\cong} K_{2i-1}^{top}(A,\mathcal{M})$$

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