Lefschetz pencils and crossed homomorphisms

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1. Galois action on l-adic cohomology

k finitely generated field, X/k smooth projective variety, l prime number $\neq \operatorname{char} k$: action of $G_k = Gal(k_s/k)$ on l-adic cohomology

$$H^i_l(X) = H^i_{\text{ét}}(\bar{X}, \mathbf{Q}_l), \quad \bar{X} = X \times_k k_s.$$

Conjecture 1 (Grothendieck, Serre). *This action is semi-simple.*

Special cases:

(1) Stable under products.
(2) True over E/k finite $\Rightarrow$ true over k.
(3) True for $i = 1$ (Tate, Zarhin-Mori, Faltings), hence for all i if X abelian variety, product of curves...
(4) k finite or number field: true for K3 surfaces.
(5) Deligne [Weil II]: $k \supset \mathbf{F}_p \Rightarrow Gal(k_s/k\bar{\mathbf{F}}_p)$ acts semi-simply.
(6) Conjecture over $\mathbf{F}_p \Rightarrow$ conjecture in characteristic p (follows from (5)).

Can reduce to cohomology of vanishing cycles for a Lefschetz pencil as in [Weil I, §7] (similar remark by Lei Fu, 2001):

Induction on $d = \dim X$:

(a) Can always take k large enough to have rational points.
(b) Poincaré duality + weak Lefschetz $\Rightarrow$ crucial case: $H_l^d(X)$.
(c) May replace X by X^2 because of Serre's inverse semi-simplicity theorems ($V \otimes W$ semi-simple $\Rightarrow V, W$ semi-simple). Hence may assume d *even.* (Now induction will be on even d.)

2. Review of Lefschetz pencils

Choose projective embedding $X \hookrightarrow \mathbf{P}$ and take $A \subset \mathbf{P}$, linear subspace of codimension 2. Write $\check{\mathbf{P}}$ for dual projective space, so the line $D \subset \check{\mathbf{P}}$ dual to A parametrises the hyperplanes of $\mathbf{P}$, i.e. $D \in t \mapsto H_t \supset A$. Then $\{X_t = X \cap H_t\}$ is the *pencil of hyperplane sections* associated to A (the axis). Hence incidence variety:

$$\tilde{X} = \{(x,t) \in X \times D \mid x \in H_t\}$$

sitting in a diagram

$$\begin{array}{ccc} X & \xleftarrow{\pi} & \tilde{X} \\ & & \downarrow{\scriptstyle f} \\ & & D \end{array} \tag{1}$$

with $f^{-1}(t) \xrightarrow{\sim} X_t$.

One says that $(X_t)_{t\in A}$ is a *Lefschetz pencil* if:

A) A is transverse to X, hence intersects X in a smooth sub-variety Δ. Then $\tilde{X} = Bl_\Delta(X)$ smooth and f geometrically connected.
B) There exists a finite subset S of D and for every $s \in S$ a point $x_s \in X_s$, such that f is smooth away from the x_s's.
C) x_s is an ordinary singular quadratic point of X_s.

By [SGA7, exp. XVII], the open subset of those $A \in Gr_2(\mathbf{P})$ defining a Lefschetz pencil is nonempty if $\operatorname{char} k = 0$. If $\operatorname{char} k > 0$, still true if one composes the embedding $X \hookrightarrow \mathbf{P}$ with a Veronese embedding of degree ≥ 2.

- May assume "condition (A)" holds for the Lefschetz pencil (take large enough Veronese embedding, [SGA7, Exp. XVIII, Cor. 6.4]). Implies that the vanishing cycles are $\neq 0$.
- Semi-simplicity for $\tilde{X} \Rightarrow$ semi-simplicity for X.
- By [SGA7, Exp. XVIII, Th. 5.6], the Leray spectral sequence for f degenerates (uses Hard Lefschetz and condition (A)!). So suffices to prove semi-simplicity for $E_2^{p,q}$, $p+q=d$.
- $E_2^{2,d-2}$ (resp. $E_2^{0,d}$) subspace (resp. quotient) of $H^{d-2}(\bar{Y}, \mathbf{Q}_l)$, Y smooth hyperplane section of $X_u = f^{-1}(u)$, $u \in D(k) - S$ (note: $\dim Y = d-2$ even!). So crucial case: $E_2^{1,d-1}$.

3. Vanishing cycles.

We set $d = n + 1 = 2m + 2$, $U = D - S$, take $u \in U(k)$ and let $\Pi := \pi_1(U \otimes_k k_s, u)$ (geometric fundamental group).

- $s \in S \mapsto$ *vanishing cycle* $\delta_s \in H^n(X_u, \mathbf{Q}_l(m))$, well-defined up to sign.
- If I_s inertia group of at s, for $g \in I_s$ and $x \in H^n(X_u, \mathbf{Q}_l(m))$, *Picard-Lefschetz formula* [SGA7, exp. XV, th. 3.4]:

$$gx = x + (-1)^{m+1} t_l(g)(x, \delta_s)\delta_s \tag{2}$$

where $(x, \delta_s) \in \mathbf{Q}_l(-1)$ intersection product, $t_l : I_s \to \mathbf{Q}_l(1)$ character given by action on l^ν-th roots of a uniformising parameter. In particular, *tame* action of Π.

- The $\pm\delta_s$ are conjugate under the action of Π.
- Exact sequence (ibid.)

$$\begin{aligned} 0 \to H^n(X_s, \mathbf{Q}_l(m)) \to H^n(X_u, \mathbf{Q}_l(m)) &\xrightarrow{(-,\delta_s)} \mathbf{Q}_l(-1) \\ &\to H^{n+1}(X_s, \mathbf{Q}_l(m)) \to H^{n+1}(X_u \mathbf{Q}_l(m)) \to 0. \end{aligned} \tag{3}$$

Write $E :=$ subspace of $H^n_l(X_u)$ generated by the δ_s (vanishing part of cohomology). Then

- $E^\perp = H^n_l(X_u)^\Pi$.
- $E \cap E^\perp = 0$ (uses Hard Lefschetz [Weil II, Th. 4.1.1]).
- Action of Π on E absolutely irreductible [SGA7, exp. XVIII, cor. 6.7 p. 326].
- *Kazhdan-Margulis theorem*: $\mathrm{Im}(\Pi \to \mathrm{Sp}(E))$ open [Weil I, 5.10].
- Condition (A) $\overset{\text{def}}{\iff}$ $R^n f_* \mathbf{Q}_l \xrightarrow{\sim} j_* j^* R^n f_* \mathbf{Q}_l$.

Then $R^n f_* \mathbf{Q}_l = j_* E \oplus$ constant sheaf, hence

$$E_2^{1,d-1} = E_2^{1,n} = H^1(\bar{D}, j_* E).$$

So far, nothing new…

4. Description of $H^1(\bar{D}, j_*E)$

Start from Leray exact sequence for j

$$0 \to H^1(\bar{D}, j_*E) \to H^1(\bar{U}, E) \to \bigoplus_{s \in S} H^1(I_s, E) \to 0 \tag{4}$$

Recall: action of Π factors through tame fundamental group $\Pi^t = \Pi/$ images of wild inertias at $s \in S$. Then $I_s^t = \mathrm{Im}(I_s \to \Pi^t) \simeq \prod_{l \neq p} \mathbf{Z}_l(1)$. Easy to show:

$$H^1(\Pi^t, E) \xrightarrow{\sim} H^1(\bar{U}, E); \quad H^1(\mathbf{Z}_l(1), E) \xrightarrow{\sim} H^1(I_s^t, E) \xrightarrow{\sim} H^1(I_s, E).$$

Now Π^t almost free profinite group with $|S| - 1$ generators; presentation as follows [SGA1, exp. XIII, cor. 2.12]: choose a numbering of $S = \{s_1, \ldots, s_r\}$ with $r = |S|$. Then there are generators γ_i de $I_{s_i}^t$ which generate Π^t, subject to only relation $\gamma_1 \ldots \gamma_r = 1$.

Theorem 1. *$H^1(\bar{D}, j_* E)$ canonically isomorphic to middle homology of a complex*

$$0 \to E \xrightarrow{\alpha} \bigoplus_{s \in S} \mathbf{Q}_l \xrightarrow{\tilde{\Sigma}} E \to 0$$

with $\alpha(e)_s = t_l(\gamma_s)(\delta_s, e)$ and $\tilde{\Sigma}(1_{s_j}) = \gamma_1 \dots \gamma_{j-1} \delta_{s_j}$.

Proof: Chase in commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 \to & B^1(\Pi^t, E) & \to & Z^1(\Pi^t, E) & \to & H^1(\Pi^t, E) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \bigoplus_{s \in S} B^1(I_s^t, E) & \to & \bigoplus_{s \in S} Z^1(I_s^t, E) & \to & \bigoplus_{s \in S} H^1(I_s^t, E) & \to 0.
\end{array}
\tag{5}
$$

Corollary 1. $\dim H^1(\bar{D}, j_* E) = |S| - 2 \dim E.$

5. Geometric Galois action

Want action of $\pi_1(U, u)$ on the exact sequence of Theorem 1.

G profinite group, E continuous G-module: G-action on $Z^1(G, E)$? On $Z^1(H, E)$ for $H \lhd G$?
Yes, because $Z^1(G, E) = \mathrm{Hom}_G(I_G, E)$, I_G augmentation ideal in $\hat{\mathbf{Z}}[[G]]$.

Exact sequence $0 \to I(G) \to \hat{\mathbf{Z}}[[G]] \to \hat{\mathbf{Z}} \to 0$ of left-right G-modules yields Ext exact sequence of left G-modules

$$0 \to E^G \to E \xrightarrow{\eta} Z^1(G, E) \to H^1(G, E) \to 0 \tag{6}$$

where E/E^G identified with group of coboboundaries $B^1(G, E)$ via $\eta : e \mapsto (g \mapsto (g-1)e)$.

Action of $g \in G$ on $f \in Z^1(G, E)$ (crossed homomorphism) given by

$$(gf)(h) = gf(g^{-1}hg) \tag{7}$$

or also

$$gf = f + \eta(f(g)). \tag{8}$$

$H \lhd G$: (7) defines action of G on $Z^1(H, E)$ (but (8) no longer valid.)

Still a problem: in (5), how can $G = \pi_1(U, u)$ act on $\bigoplus_{s\in S} Z^1(I_s^t, E)$???

A trick: $\tilde{\Pi}^t$ (almost) free profinite group with generators $\gamma_1, \ldots, \gamma_r$, then

$$Z^1(\tilde{\Pi}^t, E) \xrightarrow{\sim} \bigoplus_{s\in S} Z^1(I_s^t, E).$$

But can realise $\tilde{\Pi}^t$ as $\pi_1(\bar{U} - \{u_0\}, u)$, u_0 suitable rational point, then middle map of (5) corresponds to

$$Z^1(\pi_1(\bar{U}, u), E) \to Z^1(\pi_1(\bar{U} - \{u_0\}, u), E).$$

So get natural Galois action in this way.

Theorem 2. *Complex of Theorem 1 is complex of Π^t-modules for following actions:*

(i) *First term E: natural action.*

(ii) *Central term: γ_i acts by matrix*

$$\begin{pmatrix} 1 & \dots & \varepsilon t_l(\gamma_i)(\delta_1,\delta_i) & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & \varepsilon t_l(\gamma_i)(\delta_r,\delta_i) & \dots & 1 \end{pmatrix} \quad \varepsilon = (-1)^{m+1}$$

(iii) *Last term E: trivial action.*

6. Arithmetic Galois action

Situation:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{\Pi}^t & \longrightarrow & \tilde{G} & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \Pi^t & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array} \tag{9}
$$

with

$$
\begin{aligned}
\Gamma &= Gal(k_s/k) & & \\
G &= \pi_1^t(U, u) & \tilde{G} &= \pi_1^t(U - \{u_0\}, u) \\
\Pi^t &= \pi_1^t(\bar{U}, u) & \tilde{\Pi}^t &= \pi_1^t(\bar{U} - \{u_0\}, u).
\end{aligned}
$$

$\tilde{G}$ acts on complex of Theorem 1: want to describe this action.

Note: $\tilde{G}$ acts on $\tilde{\Pi}^t$ by

$$
g\gamma_i g^{-1} = \lambda_i(g)\gamma_i^{\kappa(g)}\lambda_i(g)^{-1}, \quad g\gamma g^{-1} = \lambda(g)\gamma^{\kappa(g)}\lambda(g)^{-1}
$$

($\gamma = (\gamma_1 \dots \gamma_r)^{-1} \in \tilde{\Pi}^t$), with $\lambda_i(g), \lambda(g) \in \tilde{\Pi}^t$ and $\kappa : G \to \hat{\mathbf{Z}}^*$ cyclotomic character. $\lambda_i(g)$ (resp. $\lambda(g)$) unique up to right multiplication by power of γ_i (resp. γ).

Normalise it so that $w_i(\lambda_i(g)) = 0$ (w_i: weight at γ_i).

Remark 1 (not essential for the sequel)**.** Setting $\tilde{\lambda}(g) = \pi(\lambda(g)^{-1}g)$ induces well-defined section $\Gamma \to G$ of the projection $G \to \Gamma$: corresponds geometrically to the section given by the rational point u_0.

Lemma 1. *(Maybe after finite extension of k), if $g \in \tilde{G}$ normalises γ_i, then $g\delta_i = \delta_i$.*

Proof. Follows from the construction of the vanishing cycles (and implicit in Illusie's paper [Ill]). □

Theorem 3 (Anderson, Ihara)**.** *Let* $\Lambda = \hat{\mathbf{Z}}[[\tilde{\Pi}^t]]$*. For* $i = 1, \dots, r$*, there exists a unique function* $d_i : \Lambda \to \Lambda$ *(Fox derivative at* γ_i*) verifying the identities*

$$d_i(\lambda\lambda') = d_i(\lambda) + \lambda d_i(\lambda'), \quad d_i(\gamma_j) = \delta_{ij}.$$

For all $\lambda \in \Lambda$*, one has*

$$\lambda = s(\lambda)1 + \sum_{i=1}^{r} d_i\lambda(\gamma_i - 1) \tag{10}$$

("profinite Euler-Fox formula").

Universal property:

Lemma 2. M *topological* $\tilde{\Pi}^t$*-module,* $f : G \to M$ *continuous* 1*-cocycle,* $i \in \{1, \dots, r\}$*. Then*

$$f(g) = \sum_{i=1}^{r} d_i g \cdot f(\gamma_i)$$

for any $g \in \tilde{\Pi}^t$*.*

Also, *profinite Blanchfield-Lyndon theorem* (Ihara [Ih]):

Theorem 4. *N closed normal subgroup of $\tilde{\Pi}^t$; write $\pi : \tilde{\Pi}^t \to \tilde{\Pi}^t/N$ for the projection. Exact sequence of $\tilde{\Pi}^t/N$-modules:*

$$0 \to N^{\mathrm{ab}} \xrightarrow{\varphi} \hat{\mathbf{Z}}[[\tilde{\Pi}^t/N]]^r \xrightarrow{\psi} I(\tilde{\Pi}^t/N) \to 0$$

with

$$\varphi(n) = (\pi(d_1 n), \ldots, \pi(d_r n)), \quad \psi(a_1, \ldots, a_r) = \sum a_i(\pi(\gamma_i) - 1).$$

Theorem 5. *Action of $g \in \tilde{G}$ on $\bigoplus \mathbf{Q}_l$ in the complex of Theorem 1 given by matrix*

$$a_{ij}(g) = \kappa_l(g)^{-1}\left(\delta_{ij} + (-1)^{m+1}t_l(\gamma_j)(d_i(g^{-1}\lambda_j(g)g)\delta_i, \delta_j)\right)$$

δ_{ij} Kronecker symbol.

(Cf. Theorem 3 for d_i).
(Verification: for $g = \gamma_k$, can take $\lambda_j(g) = \gamma_k$ for all j, then $d_i(g^{-1}\lambda_j(g)g) = \delta_{ik}$ and get back formula of Theorem 1.)

Note: also have $g^{-1}\lambda_j(g)g = \lambda_j(g^{-1})^{-1}$.

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