THE QUILLEN-LICHTENBAUM CONJECTURE AT THE PRIME 2

BRUNO KAHN

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Abstract. Using recent major advances in algebraic K-theory, we prove the Quillen-Lichtenbaum conjecture at the prime 2 for rings of S-integers of totally imaginary number fields. For formally real number fields, we get corresponding results under a compatibility assumption of certain spectral sequences with products.

Introduction

Let F be a number field, l a prime number, S a finite set of places of F containing the places at infinity and the places dividing l, and O_S the rings of S-integers in F. If l is odd, the Quillen-Lichtenbaum conjecture predicts the existence of isomorphisms [15, conj. 2.5]:

$$K_{2i-j}(O_S)\otimes \mathbf{Z}_l \xrightarrow{ch_{i,j}} H^j_{\text{ét}}(O_S,\mathbf{Z}_l(i)) \qquad (j=1,2, i\geq j).$$

Here $K_*(O_S)$ denotes Quillen's K-theory of O_S [20] and $H^j_{\text{ét}}(O_S,\mathbf{Z}_l(i))$ its l-adic étale cohomology.

Such homomorphisms have been constructed by Soulé and Dwyer-Friedlander, and proven surjective with finite kernel [25], [5]. Soulé used higher Chern class maps, which approximate the $ch_{i,j}$, and Dwyer-Friedlander used étale K-theory.

For $l=2$, such a conjecture is reasonable only if F is totally imaginary: the Dwyer-Friedlander proof works in fact when $\sqrt{-1}\in F$.

In this paper, we establish such isomorphisms for $l = 2$ when F is totally imaginary; we also draw a number of consequences. Our proof does not rely on the above-mentioned work, but on more recent fundamental work by several authors, the most notable being Vladimir Voevodsky's proof of the Milnor conjecture [39]. We get similar results when F is formally real; for this, however, we have to assume a compatibility of certain spectral sequences with products, see below. We now state our results (under this assumption in the formally real case).

Theorem 1. *Let r_1 be the number of real places of F. Then there exist homomorphisms*

$$K_{2i-j}(O_S)\otimes \mathbf{Z}_2 \xrightarrow{ch_{i,j}} H^j_{\text{ét}}(O_S,\mathbf{Z}_2(i)) \qquad (j=1,2, i\geq j)$$

which are

(i) *bijective for $2i-j\equiv 0,1,2,7 \pmod 8$*
(ii) *surjective with kernel isomorphic to $(\mathbf{Z}/2)^{r_1}$ for $2i-j\equiv 3 \pmod 8$*
(iii) *injective with cokernel isomorphic to $(\mathbf{Z}/2)^{r_1}$ for $2i-j\equiv 6 \pmod 8$.*

Moreover, for $i\equiv 3 \pmod 4$ there is an exact sequence

$$0\to K_{2i-1}(O_S)\otimes\mathbf{Z}_2\to H^1_{\text{ét}}(O_S,\mathbf{Z}_2(i))\to(\mathbf{Z}/2)^{r_1}$$
$$\to K_{2i-2}(O_S)\otimes\mathbf{Z}_2\to H^2_{\text{ét}}(O_S,\mathbf{Z}_2(i))\to 0$$

in which $\operatorname{Im}(H^1_{\text{ét}}(O_S,\mathbf{Z}_2(i))\to(\mathbf{Z}/2)^{r_1})$ has 2-rank $\rho_i\geq 1$ if $r_1\geq 1$.
The homomorphisms $ch_{i,j}$ are natural in O_S.

One may ask:

Question. Let F be a number field. Is it true that $\rho_i = 1$ if $r_1 \geq 1$?

This amounts to asking whether, for $i \equiv 3 \pmod 4$, the image of $H^1(O_S,\mathbf{Z}_2(i))$ in $H^1(O_S,\mathbf{Z}/2)\subset F^*/F^{*2}$ is contained in $\{\pm 1\}\times\{\text{totally positive elements}\}$.

Theorem 1 has a number of corollaries, the first of them being arithmetic:

Corollary 1. ("Lichtenbaum's conjecture" at $l = 2$, compare [15, question 4.2]) *Let F be totally real. Assume Federer's "main conjecture" for $l = 2$ [7, 3.4] holds for F (by [18], this is true if F is "2-regular"[1]). Then, for i even > 0, one has*

$$\zeta_F(1-i) =_2 2^{r_1}\frac{|K_{2i-2}(O_F)|}{|K_{2i-1}(O_F)|}$$

where ζ_F is the zeta function of F and $=_2$ means that the two sides have the same 2-primary part.

Corollary 2. *Let N be a power of 2. Then there are natural homomorphisms*

$$K_{2i-2}(O_S)/N\to H^2_{\text{ét}}(O_S,\mu_N^{\otimes i})$$

which are

(i) *injective with cokernel isomorphic to $(\mathbf{Z}/2)^{r_1}$ for $2i-2\equiv 0 \pmod 8$*

[1] See also [42, footnote p. 499] for abelian fields.

(ii) *bijective for* $2i-2 \equiv 2 \pmod 8$
(iii) *with kernel isomorphic to* $(\mathbf{Z}/2)^{r_1-\rho_i}$ *and cokernel isomorphic to* $(\mathbf{Z}/2)^{r_1}$ *for* $2i-2 \equiv 4 \pmod 8$
(iv) *surjective with kernel a quotient of* $(\mathbf{Z}/2)^{r_1}$ *for* $2i-2 \equiv 6 \pmod 8$.

Corollary 3. *a) (compare* [37, th. 6.2] *for* $2i-2=2$*) Let* $r(F,S) = rg_2 \operatorname{Pic} O_S + |S_f| - 1$*, where* rg_2 *denotes the* 2*-rank of a finite abelian group and* S_f *is the set of finite places in* S*. Then the* 2*-rank of* $K_{2i-2}(O_S)$ *is*

(i) $r(F,S)$ *for* $2i-2 \equiv 0 \pmod 8$, $2i-2>0$
(ii) $r(F,S)+r_1$ *for* $2i-2 \equiv 2 \pmod 8$
(iii) *between* $r(F,S)$ *and* $r(F,S)+r_1-\rho_i$ *for* $2i-2 \equiv 4 \pmod 8$
(iv) *between* $r(F,S)$ *and* $r(F,S)+r_1$ *for* $2i-2 \equiv 6 \pmod 8$.

b) Suppose F *formally real and let* N *be the number of* 2*-primary roots of unity in* $F(\sqrt{-1})$*. Then the* 2*-torsion of* $K_{2i-1}(O_S)$ *is*

(i) *cyclic of order* $\frac{N}{|i|_2}$ *for* $i \equiv 0 \pmod 4$*, where* $|i|_2$ *is the dyadic absolute value of* i
(ii) *cyclic of order* 2 *for* $i \equiv 1 \pmod 4$
(iii) *isomorphic to* $(\mathbf{Z}/2)^{r_1-1} \oplus C$ *with* C *cyclic of order* $4N$ *for* $i \equiv 2 \pmod 4$
(iv) 0 *for* $i \equiv 3 \pmod 4$.

Corollary 4. *If* F *is totally imaginary, the product* $K_i(O_S) \times K_j(O_S) \to K_{i+j}(O_S)$ *is identically* 0 *after localization at* 2 *if* i *and* j *are both* >0 *and one of them is even.*

Corollary 5. *Suppose* $\sqrt{-1} \in F$*. Let* $\mathbb{S}$ *be the sphere spectrum and* $\mathrm{j}_2(O_S)$ *be the cyclotomic spectrum defined in* [14]*. Then the unit map* $\mathbb{S} \to K(O_S)$ *factors canonically through a map of spectra* $\mathrm{j}_2(O_S) \to K(O_S)_{(2)}$*. Here* $_{(2)}$ *denotes Bousfield localization at* 2.

In [6], Dwyer and Mitchell give a description of the Bousfield localization of $K(O_S)$ with respect to mod l complex K-theory as a module spectrum over what is essentially the localization of $\mathrm{j}_l(O_S)$ with respect to mod l complex K-theory. It seems that theorem 1 and corollary 5, together with their work, yield a similar description of $K(O_S)_{(2)}$ as module spectrum over $\mathrm{j}_2(O_S)$, at least after 2-completion.

The case $F = \mathbf{Q}$ of corollaries 1 and 3 was obtained earlier by Weibel [41].

As by-products of the proof, we also get some results of independent interest:

Theorem 2. *The Beilinson-Soulé conjecture holds after localization at* 2 *for any subfield* F *of* $\overline{\mathbf{Q}}$: $H^i(F, \mathbf{Z}_{(2)}(n)) = 0$ *for* $i \le 0$ *and* $n \ge 1$.

Theorem 3. *For any field* F *of characteristic* 0,
a) The map $K_3^M(F) \to K_3(F)$ *is injective.*
b) $3\operatorname{Ker}(K_4^M(F) \to K_4(F))$ *is a quotient of* $3\operatorname{Ker}(K_4^M(F_0) \to K_4(F_0))$*, where* F_0 *is the algebraic closure of* $\mathbf{Q}$ *in* F*. In particular, this group is* 0 *if* $\sqrt{-1} \in F$ *and finite if* F *is finitely generated.*

A positive answer to the question above would imply that, for any field F of characteristic 0, the sequence

$$K_4^M(\mathbf{Q}) \to K_4^M(F) \to K_4(F)$$

is exact up to 3-torsion.

We also get a new (although expensive) proof of the Bass-Tate theorem $K_i^M(F) \simeq (\mathbf{Z}/2)^{r_1}$ for F a number field and $i \geq 3$ [1], after localization at 2.

As indicated above, our proofs rely on major recent work done, among others, by S. Bloch, E. Friedlander, S. Lichtenbaum, A. Suslin and V. Voevodsky, alone and in collaboration. We refer to section 2 for a detailed description of what we use: in this introduction we confine to a quick outline of our proof.

Our strategy is to use the Bloch-Lichtenbaum-Friedlander-Suslin-Voevodsky spectral sequences [3], [33], [8], [38]

$$H^{p-q}(F, \mathbf{Z}(-q)) \Rightarrow K_{-p-q}(F) \qquad (p, q \leq 0) \tag{0.1}$$

$$H^{p-q}(F, \mathbf{Z}/m(-q)) \Rightarrow K_{-p-q}(F, \mathbf{Z}/m) \qquad (p, q \leq 0) \tag{0.2}$$

together with the Suslin-Voevodsky theorem [35], [39]

$$H^i(F, \mathbf{Z}/m(n)) \xrightarrow{\sim} H^i_{\text{ét}}(F, \mu_m^{\otimes n}) \qquad (i \leq n) \tag{0.3}$$

for m a power of 2 (see section 2). Here $H^i(F, \mathbf{Z}(n))$ and $H^i(F, \mathbf{Z}/m(n))$ are Voevodsky's *motivic cohomology groups*. Using a little technique from topology and (in an essential way) Quillen's finiteness result for $K_*(O_S)$ [21], we prove in theorem 4.2 a motivic version of theorem 1 for F.

The difficulty is now to descend from F to O_S. In an perfect world, we wouldn't have this difficulty, working directly with spectral sequences analogous to (0.1) and (0.2) for O_S, together with the analogue of (0.3) for O_S. However, none of these results is available at the moment and we have to use a rather convoluted argument, whose crux is proposition 5.2.

As indicated above, our approach is conditional in the formally real case, because it appeals to a multiplicative property of the spectral sequences (0.1) and (0.2) (in fact, a rather weak property: see section 2.1). This multiplicative property appears non-trivial to prove given the present construction of these spectral sequences. It has been quietly used also by other authors earlier (e.g. [34, p. 350], [41, proof of prop. 4]). However, there is the hope that the BLFSV spectral sequences can be constructed more naturally as Atiyah-Hirzebruch spectral sequences, using the homotopy theory of schemes currently developed by Morel and Voevodsky [17]; this would make obvious the property we need.

While I was completing this paper, I learnt from J. Rognes and C. Weibel that they were preparing a paper which contained closely related material, including corollary 1. We considered joining efforts; however the idea was rejected for practical reasons, so that their work will appear separately [23].

Theorem 1 is proved in section 5. We don't bother to give a proof of theorem 2, which follows from theorem 4.1 a) (i) by a direct limit argument. Theorem 3 a) is proven in section 3 and theorem 3 b) is proven in section 7. Corollaries 1, 2, 3, 4 and 5 are proven in section 6.

Finally, it will be clear to the reader that our results extend to fields of positive characteristic $\neq 2$, provided one admits resolution of singularities.

1. Notation

We denote by $\mathbf{Z}_{(2)}$ (resp. $\mathbf{Z}_2$) the localization (resp. the completion) of $\mathbf{Z}$ at 2. If A is an abelian group, we shall sometimes denote the group $A \otimes \mathbf{Z}_{(2)}$ by $A_{(2)}$.

Let α be the projection of the big étale site of $\operatorname{Spec} \mathbf{Z}[1/2]$ onto its big Zariski site. Following Suslin and Voevodsky [35], if m is a power of 2 and $n \geq 0$, we denote by $B/m(n)$ the truncation $\tau_{\leq n} R\alpha_* \mu_m^{\otimes n}$ of the total direct image by α of the étale sheaf of twisted m-th roots of unity. This is a complex of Zariski sheaves (up to quasi-isomorphism), acyclic in degrees $> n$. For any field F of characteristic $\neq 2$, one has

$$H^i(F, B/m(n)) = \begin{cases} H^i_{\text{ét}}(F, \mu_m^{\otimes n}) & \text{if } i \leq n \\ 0 & \text{if } i > n. \end{cases}$$

For $n \geq 0$, we denote by $\mathbf{Z}(n)$ the n-th motivic complex of Voevodsky: this is a complex of sheaves over the big Zariski site of $\operatorname{Spec} \mathbf{Z}$. One has $\mathbf{Z}(0) = \mathbf{Z}$, $\mathbf{Z}(1) \simeq \mathbb{G}_m[-1]$ and $\mathbf{Z}(n)$ is acyclic in degrees $> n$ [35]. We have [35, §3]

$$H^n(F, \mathbf{Z}(n)) \simeq K_n^M(F). \tag{1.1}$$

For $m > 0$, we denote the complex $\mathbf{Z}(n) \overset{L}{\otimes} \mathbf{Z}/m$ by $\mathbf{Z}/m(n)$. For m a power of 2, there is a natural morphism of complexes over the big Zariski site of $\operatorname{Spec} \mathbf{Z}[1/2]$ [35], [39]

$$\mathbf{Z}/m(n) \to B/m(n) \tag{1.2}$$

which for $m = 1$ corresponds to the Kummer exact sequence for multiplication by m on $\mathbb{G}_m$.

We also denote the complex $\mathbf{Z}(n) \otimes \mathbf{Z}_{(2)}$ by $\mathbf{Z}_{(2)}(n)$.

We define

$$\begin{aligned} \mathbf{Z}_2(n) &= R \varprojlim \mathbf{Z}/2^\nu(n) \\ B_2(n) &= R \varprojlim B/2^\nu(n) \simeq \tau_{\leq n} R\alpha_* \mathbf{Z}_2(n)_{\text{ét}} \end{aligned}$$

where

$$\mathbf{Z}_2(n)_{\text{ét}} = \varprojlim \mu_{2^\nu}^{\otimes n}.$$

There are natural morphisms

$$\mathbf{Z}_{(2)}(n) \to \mathbf{Z}_{(2)}(n) \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2 \to \mathbf{Z}_2(n) \to B_2(n). \tag{1.3}$$

We shall need two kinds of continuous dyadic étale cohomology theories for a subfield F of $\overline{\mathbf{Q}}$. The first is

$$H^i_{\text{Zar}}(F, B_2(n)) = \begin{cases} H^i_{\text{ét}}(F, \mathbf{Z}_2(n)_{\text{ét}}) & \text{if } i \le n \\ 0 & \text{if } i > n \end{cases}$$

where $H^i_{\text{ét}}(F, \mathbf{Z}_2(n)_{\text{ét}})$ is Tate's continuous cohomology [37]. For the second, we define a group $\tilde{H}^i(F, B_2(n))$ as follows:

- If F is a number field,
$$\tilde{H}^i(F, B_2(n)) = \varinjlim H^i(O_S, B_2(n))$$
where S runs through the finite sets of places of F.
- In general,
$$\tilde{H}^i(F, B_2(n)) = \varinjlim \tilde{H}^i(F_\alpha, B_2(n))$$
where F_α runs through the finitely generated subfields of F.

This is closely related to Jannsen's "tame" p-adic étale cohomology [9, 11.6]. Since motivic cohomology commutes with direct limits, the natural maps $H^i(F, \mathbf{Z}(n)) \otimes \mathbf{Z}_2 \to H^i(F, B_2(n))$ factor as

$$H^i(F, \mathbf{Z}(n)) \otimes \mathbf{Z}_2 \to \tilde{H}^i(F, B_2(n)) \to H^i(F, B_2(n)). \tag{1.4}$$

Finally, we shall use for convenience a nonstandard definition of "local field".

1.1. Definition. (for this paper) A *local field* of characteristic 0 is a subfield of $\overline{\mathbf{Q}}$, henselian for a discrete valuation and with finite residue field.

2. Foundational results

2.1. **The Bloch-Lichtenbaum-Friedlander-Suslin-Voevodsky spectral sequences.** Recall Bloch's higher Chow groups $CH^i(X, m)$ [2]. In [3], Bloch and Lichtenbaum construct a strongly convergent spectral sequence

$$E_2^{p,q} = CH^{-q}(F, -p-q) \Rightarrow K_{-p-q}(F)$$

for any field F. There is a similar spectral sequence with finite coefficients:

$$CH^{-q}(F, -p-q; \mathbf{Z}/n) \Rightarrow K_{-p-q}(F, \mathbf{Z}/n).$$

On the other hand, it follows from the work of Suslin [33], Friedlander-Voevodsky [8] and Voevodsky [38] that there are isomorphisms

$$CH^i(X, n) \simeq H^{2i-n}(X, \mathbf{Z}(i))$$
$$CH^i(X, n; \mathbf{Z}/n) \simeq H^{2i-n}(X, \mathbf{Z}/n(i)).$$

for any smooth variety X over a field F with resolution of singularities (e.g. char $F = 0$). For the convenience of the reader, we give some details: by [33] and [38], for any quasiprojective F-scheme X, equidimensional of dimension d, there is an isomorphism

$$CH^i(X,j) \simeq H^{BM}_{j+2(d-i)}(X,\mathbf{Z}(d-i))$$

where $H^{BM}_{j+2(d-i)}(X,\mathbf{Z}(d-i))$ is *Borel-Moore motivic homology* ([33] deals with the affine case and [38] with the general case). If X is moreover smooth, there is an isomorphism [8], [38]

$$H^{BM}_j(X,\mathbf{Z}(i)) \simeq H^{2d-j}(X,\mathbf{Z}(d-i)).$$

These isomorphisms stem from quasi-isomorphisms between certain complexes of abelian groups. There are similar isomorphisms for finite coefficients, obtained by tensoring the quasi-isomorphisms between the relevant complexes by $\mathbf{Z}/m$ (in the derived sense).

Putting all this together, we get for any field F with resolution of singularities two spectral sequences

$$E_2^{p,q} = H^{p-q}(F,\mathbf{Z}(-q)) \Rightarrow K_{-p-q}(F) \qquad (2.1)$$

$$E_2^{p,q} = H^{p-q}(F,\mathbf{Z}/m(-q)) \Rightarrow K_{-p-q}(F,\mathbf{Z}/m) \qquad (2.2)$$

that we call the *Bloch-Lichtenbaum-Friedlander-Suslin-Voevodsky (or BLFSV) spectral sequences*. By the 2*-local BLFSV spectral sequence*, we shall mean the spectral sequence (2.1) tensored by $\mathbf{Z}_{(2)}$.

These spectral sequences are compatible with transfer, which follows easily from their construction. For F formally real, we shall moreover have to assume the following property in the proof of lemma 4.3: cup-product by a cycle of the E_2-terms commutes with d_2-differentials.

2.2. **Soulé's contribution.** Soulé [28] uses the Adams operations to split the BLFSV spectral sequence up to torsion. His result is:

2.1. Theorem. (Soulé) *For every $k > 0$ there is an action of the Adams operation ψ^k on the spectral sequence* (2.1)*. It converges to the action of ψ^k on $K_{-p-q}(F)$ and acts upon $E_r^{p,q}$, $r \geq 2$, by multiplication by k^{-q}.*

From this, one deduces in a standard way:

2.2. Corollary. *For any field F with resolution of singularities, the differentials of the BLFSV spectral sequence are torsion. For any q, there is an isomorphism*

$$K_q(F) \simeq H^q(F,\mathbf{Z}(q)) \oplus H^{q-2}(F,\mathbf{Z}(q-1)) \oplus \cdots \oplus H^{q-2r}(F,\mathbf{Z}(q-r)) \oplus \dots$$

(a finite sum), up to groups of finite exponent. □

2.3. **The Suslin-Voevodsky theorems.** Recall the Kato conjecture: the natural map (norm residue homomorphism)

$$K_n^M(F)/m \to H^n_{\text{ét}}(F, \mu_m^{\otimes n})$$

is an isomorphism for any field F. In [35], Suslin and Voevodsky prove

2.3. Theorem. (Suslin-Voevodsky) *Let the Kato conjecture hold mod m in degree n. Then the map* (1.2) *is a quasi-isomorphism when restricted to any smooth scheme over a field with resolution of singularities.*

On the other hand, Voevodsky proves in [39]:

2.4. Theorem. (Voevodsky) *The Kato conjecture holds when m is a power of* 2.

2.5. Corollary. *Over any field with resolution of singularities, the map* (1.2) *is a quasi-isomorphism when m is a power of* 2*, and the map*

$$\mathbf{Z}_2(n) \to B_2(n)$$

from (1.3) *is a quasi-isomorphism as well.* □

2.4. **Other results.** We collect here various results that we shall need in the rest of the paper.

2.6. Theorem. (Suslin [29, cor. 5.3]) *For any field F, the kernel of $K_n^M(F) \to K_n(F)$ is killed by $(n-1)!$.*

2.7. Theorem. (Soulé [27, th. 1]) *For any $i > 1$, the sequence from Quillen's localization exact sequence*

$$0 \to K_i(O_S) \to K_i(F) \to \coprod_{v \notin S} K_{i-1}(\kappa(v)) \to 0$$

is exact, where $\kappa(v)$ is the residue field at v.

Similarly, passing to the inverse limit on the localization exact sequences for étale cohomology, one gets:

2.8. Theorem. *Let T be a finite set of places of F containing S. Then the sequence*

$$0 \to H^j(O_S, B_2(i)) \to H^j(O_T, B_2(i)) \to \coprod_{v \in T-S} H^{j-1}(\kappa(v), B_2(i-1)) \to 0$$

is exact for $j = 1, 2$ and $i \geq 2$.

3. Divisibility of motivic cohomology; proof of theorem 3 a)

3.1. Theorem. *Let F be a field of characteristic* 0.
a) For $i \leq 0$, $H^i(F, \mathbf{Z}(n))$ is uniquely 2*-divisible.*
b) If F has étale 2*-cohomological dimension $d < +\infty$, then $H^i(F, \mathbf{Z}(n))$ is* 2*-divisible for $i \geq d+1$ and uniquely* 2*-divisible for $i \geq d+2$.*
c) If F is finitely generated and has étale 2*-cohomological dimension $d < +\infty$, then $H^i(F, \mathbf{Z}(n))$ is uniquely* 2*-divisible for $i \geq d+1$ and $H^d(F, \mathbf{Z}(n))$ is an extension of a uniquely* 2*-divisible group by a torsion group.*

Proof. In a), it is obvious that $H^i(F,\mathbf{Z}(n))$ is uniquely 2-divisible for $i<0$ and that $H^0(F,\mathbf{Z}(n))$ is 2-torsion-free, since $H^i(F,\mathbf{Z}/2(n))=0$ for $i<0$ by corollary 2.5. To prove that $H^0(F,\mathbf{Z}(n))$ is 2-divisible, we may assume F finitely generated. There is a commutative diagram with injective horizontal arrows (coming from multiplication by 2 on $\mathbf{Z}(n)$ and $\mathbf{Z}_2(n)$)

$$\begin{array}{ccc} H^0(F,\mathbf{Z}_2(n))/2 & \hookrightarrow & H^0(F,\mathbf{Z}/2(n)) \\ \uparrow & & \Vert\uparrow \\ H^0(F,\mathbf{Z}(n))/2 & \hookrightarrow & H^0(F,\mathbf{Z}/2(n)). \end{array}$$

We have $H^0(F,B_2(n))=0$ since F contains only finitely many roots of unity, hence $H^0(F,\mathbf{Z}_2(n))=0$ as well.

b) This is clear since $H^i(F,\mathbf{Z}/2(n)) \xrightarrow{\sim} H^i_{\text{ét}}(F,\mu_2^{\otimes n})=0$ for $i>d$.

c) By [12, prop. 4], $H^d_{\text{ét}}(F,\varinjlim \mu_{2^\nu}^{\otimes n})=0$ for $n\neq d-1$. This gives the claim when $n\neq d-1$. But if $n=d-1$, the group $H^i(F,\mathbf{Z}(d-1))$ is 0 for $i>d$, so the claim also holds (trivially) in this case. □

3.2. Corollary. *Let notation and hypotheses be as in theorem 3.1. Then in the 2-local BLFSV spectral sequence, all differentials starting from $E_r^{p,q}$ for $p\leq q$ are 0. If $cd_2(F)=d$, this is also true for the differentials arriving at $E_r^{p,q}$ for $p-q\geq d+2$; if moreover F is finitely generated we can replace this inequality by $p-q\geq d+1$.*

Indeed, all differentials in the spectral sequence are torsion by corollary 2.2; on the other hand, $E_2^{p,q}$ is divisible for $p\leq q$ by theorem 3.1 a) and torsion-free in the cases considered by theorem 3.1 b) and c). □

Proof of theorem 3 a). By theorem 2.6, it is sufficient to prove this after localization at 2. The 2-local BLFSV spectral sequence yields an exact sequence (compare (1.1))

$$H^0(F,\mathbf{Z}_{(2)}(2)) \xrightarrow{d_2^{-2,-2}} K_3^M(F)_{(2)} \to K_3(F)_{(2)} \to H^1(F,\mathbf{Z}_{(2)}(2)) \to 0 \tag{3.1}$$

in which the differential $d_2^{-2,-2}$ is 0 by corollary 3.2. We note that the middle map, an edge homomorphism, is indeed the natural map from Milnor to Quillen's K-theory: this follows from the multiplicativity of the spectral sequence and the similar fact for the edge homomorphism in degree 1. Theorem 3 a) follows. □

4. Generic results

4.1. Theorem. *Let F be a number field with r_1 real embeddings and r_2 complex embeddings. Then*
a) $H^i(F,\mathbf{Z}_{(2)}(n))$ is
(i) 0 *if $i\leq 0$ or $i>n$.*
(ii) 0 *if $i\geq 3$, $n\not\equiv i \pmod 2$.*

(iii) *isomorphic to* $(\mathbf{Z}/2)^{r_1}$ *if* $i \geq 3$, $n \equiv i \pmod 2$, $i \leq n$.
(iv) *torsion if* $i = 2$.
(v) *finitely generated over* $\mathbf{Z}_{(2)}$ *if* $i = 1$. *In this case, its rank is* r_2 *if* n *is even and* $r_1 + r_2$ *if* n *is odd; its 2-torsion subgroup is isomorphic to* $H^0(F, \varinjlim \mu_{2^\nu}^{\otimes n})$.

Moreover, let ε *be the class of* -1 *in* $H^1(F, \mathbf{Z}(1)) = F^*$. *Then cup-product by* ε *induces isomorphisms* $H^i(F, \mathbf{Z}_2(n)) \xrightarrow{\sim} H^{i+1}(F, \mathbf{Z}_2(n+1))$ *for* $3 \leq i \leq n$ *and cup-product by* ε^2 *induces a surjection* $H^1(F, \mathbf{Z}(1)) \twoheadrightarrow H^3(F, \mathbf{Z}(3))$.
b) The homomorphisms

$$H^i(F, \mathbf{Z}(n)) \otimes_{\mathbf{Z}} \mathbf{Z}_2 \to \tilde{H}^i(F, B_2(n))$$

from (1.4) *are isomorphisms for all* $n \geq 1$.

Proof. a) By Quillen and Borel's theorems [21], [4], $K_{2q}(F)$ is torsion for $q > 0$ and $K_{2q-1}(F)$ is finitely generated of rank r_2 or $r_1 + r_2$ according as q is even or odd. Applying theorem 3.1 and corollary 2.2, this gives (i) and the fact that $H^i(F, \mathbf{Z}(n))$ is torsion for $i \geq 2$.[2] To compute it in cases (ii) and (iii), consider the exact sequence

$$\cdots \to H^{i-1}(F, \mathbf{Q}_2/\mathbf{Z}_2(n)) \to H^i(F, \mathbf{Z}_{(2)}(n)) \to H^i(F, \mathbf{Q}(n)) \to H^i(F, \mathbf{Q}_2/\mathbf{Z}_2(n)) \to \dots$$

Since $H^i(F, \mathbf{Z}(n))$ is torsion for $i \geq 2$, $H^i(F, \mathbf{Q}(n)) = 0$ and this exact sequence degenerates into isomorphisms

$$H^{i-1}(F, \mathbf{Q}_2/\mathbf{Z}_2(n)) \xrightarrow{\sim} H^i(F, \mathbf{Z}_{(2)}(n)) \qquad (i \geq 3).$$

Claim (iii) now follows from corollary 2.5 and the well-known value of Galois cohomology of number fields [36, th. 3.1].

To prove (v) we use the BLFSV spectral sequence and the above results.

Finally, let us prove the claims concerning cup-product by ε. In the first claim, there is nothing to prove if $i \not\equiv n \pmod 2$ in view of (ii). If $i \equiv n \pmod 2$, we have a commutative diagram

$$\begin{array}{ccc} H^i(F, \mathbf{Z}_{(2)}(n)) & \xrightarrow{\varepsilon} & H^{i+1}(F, \mathbf{Z}_{(2)}(n+1)) \\ \wr\downarrow & & \wr\downarrow \\ H^i(F, \mathbf{Z}/2(n)) & \xrightarrow{\varepsilon} & H^{i+1}(F, \mathbf{Z}/2(n+1)) \end{array}$$

in which the vertical maps are isomorphisms in view of (iii). So we reduce to proving the bijectivity of the bottom horizontal map, which follows from corollary 2.5 and the well-known corresponding result in Galois cohomology [36, th. 3.1]. In the second claim, we use a similar diagram plus the surjectivity of $H^1(F, \mathbf{Z}(1)) \to H^1(F, \mathbf{Z}/2(1))$.

[2] Actually we cannot apply theorem 3.1 c) directly if F is formally real; however, $cd_2(F(\sqrt{-1})) = 2$ in any case and we can get back to F by a transfer argument.

b) Observe that there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{2} & H^i(F,\mathbf{Z}(n))\otimes\mathbf{Z}_2 & \longrightarrow & H^i(F,\mathbf{Z}/2(n)) & \longrightarrow & H^{i+1}(F,\mathbf{Z}(n))\otimes\mathbf{Z}_2 & \xrightarrow{2} & \dots\\
 & & \downarrow{\scriptstyle f_i} & & \wr\downarrow & & \downarrow{\scriptstyle f_{i+1}} & & \\
\dots & \xrightarrow{2} & \tilde H^i(F,B_2(n)) & \longrightarrow & H^i(F,B/2(n)) & \longrightarrow & \tilde H^{i+1}(F,B_2(n)) & \xrightarrow{2} & \dots
\end{array}$$

The top row is the long cohomology exact sequence for multiplication by 2 on $\mathbf{Z}(n)$, tensored by the flat $\mathbf{Z}$-module $\mathbf{Z}_2$. The bottom row is the direct limit of the corresponding long exact sequences for O_S. By a cone argument (compare [13, proof of th. 7.2]), we get that, for all i:

- Ker f_i is divisible;
- Coker f_i is torsion-free;
- $(\text{Ker } f_{i+1})_{\text{tors}} \xrightarrow{\sim} \text{Coker } f_i\otimes\mathbf{Q}/\mathbf{Z}$.

We now proceed to prove that f_i is an isomorphism inductively on i. If $i\le 0$, both sides are 0 and there is nothing to prove. This implies that Ker f_1 is uniquely divisible. If $n=1$, the bijectivity of f_1 is checked directly. If $n>1$, since the source and target of f_1 are finitely generated $\mathbf{Z}_2$-modules of the same rank [26], Ker $f_1=0$ and Coker f_1 is torsion, hence 0. Since $H^2(F,\mathbf{Z}(n))$ is torsion, this implies Ker $f_2=0$. Since $\tilde H^2(F,B_2(n))$ is also torsion [26], we also have Coker $f_2=0$. Finally, for $i\ge 3$, both the source and target of f_i are killed by 2, hence f_i is bijective as well. □

Remark. As a special case of theorem 4.1 b) (iii), we recover the Bass-Tate theorem localized at 2: $K_i^M(F)\otimes\mathbf{Z}_{(2)}\simeq(\mathbf{Z}/2)^{r_1}$ for $i\ge 3$ (compare (1.1)).

4.2. Theorem. *For any number field, the 2-local BLFSV spectral sequence degenerates from E_3 on. There are homomorphisms*

$$K_{2i-j}(F)_{(2)}\to H^j(F,\mathbf{Z}_{(2)}(i))\qquad (j=1,2,\ i\ge j)$$

which are

(i) *bijective for $2i-j\equiv 0,1,2,7 \pmod 8$*
(ii) *surjective with kernel isomorphic to $(\mathbf{Z}/2)^{r_1}$ for $2i-j\equiv 3 \pmod 8$*
(iii) *injective with cokernel isomorphic to $(\mathbf{Z}/2)^{r_1}$ for $2i-j\equiv 6 \pmod 8$.*

Moreover, for $i\equiv 3 \pmod 4$ there is an exact sequence

$$0\to K_{2i-1}(F)_{(2)}\to H^1(F,\mathbf{Z}_{(2)}(i))\to(\mathbf{Z}/2)^{r_1}\to K_{2i-2}(F)_{(2)}\to H^2(F,\mathbf{Z}(i)_{(2)})\to 0$$

in which $\operatorname{Im}(H^1(F,\mathbf{Z}_{(2)}(i))\to(\mathbf{Z}/2)^{r_1})$ has 2-rank $\rho_i\ge 1$ if $r_1\ge 1$.

Proof. (compare [11, appendix]) The homomorphisms $K_{2i-j}(F)\to H^j(F,\mathbf{Z}(i))$ are edge homomorphisms in the BLFSV spectral sequence, whose existence follows from theorem 4.1 a) (i). To prove the rest of theorem 4.2, we need:

4.3. Lemma. *Let $p\equiv 2 \pmod 4$. In the BLFSV spectral sequence localized at 2, the differential $d_2^{p,q}$ is surjective for $q=p-2$ and bijective for $q<p-2$.*

It is in the proof of this lemma that we use our assumption that cup-product by a cycle of the E_2-terms commutes with d_2-differentials.

Proof. We first deal with the case $p = -2$. By topology [22], we know that $K_4^M(\mathbf{Q}) \to K_4(\mathbf{Q})$ is 0. Since $K_i^M(F) = \{-1\}^{i-1}K_1(F)$ for $i \geq 3$ (Bass-Tate), we get that $K_i^M(F) \to K_i(F)$ is 0 for $i \geq 5$. This implies that the differential $d_2^{-2,-i+1}$ is surjective for $i = 5$. We now observe that $\varepsilon \in H^1(F, \mathbf{Z}(1))$ is a cycle in $E_2^{0,-1}$, because it comes from $\{-1\} \in K_1(F)$. By our compatibility assumption, cup-product by ε commutes with the d_2-differentials. It follows that $d_2^{-2,-i+1}$ is also surjective for $i \geq 6$, hence bijective by theorem 4.1 a) (iii).

In general, we shall use a generator $\bar\beta$ of $H^0(F, \mathbf{Z}/16(4)) = E_2^{-4,-4}$ in the BLFSV spectral sequence with $\mathbf{Z}/16$ coefficients to reduce to the special case above. To show that $\bar\beta$ is a d_2-cycle, we show that it comes from an element of order 16

$$\beta \in K_8(F, \mathbf{Z}/16)$$

via the edge homomorphism. To construct β, we lift an element of order 16, $\tilde\beta \in \pi_7^S(pt) \simeq \mathbf{Z}/240$, to $\pi_8^S(pt, \mathbf{Z}/16)$; since $\pi_8^S(pt) \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$, there are 4 choices for such a lift. By Quillen's computation [22], the image of $\tilde\beta$ in $\pi_8(BU, \mathbf{Z}/16)$ generates the latter group. We take for β the image of $\tilde\beta$ in $K_8(F, \mathbf{Z}/16)$. (The above ambiguity then disappears by [40] and [16], although we won't need this.) The claim now follows from the commutative diagram

$$\begin{array}{ccc}
\pi_8(BU, \mathbf{Z}/16) & & \\
\wr\uparrow & & \\
K_8(\mathbf{C}, \mathbf{Z}/16) & & \\
\wr\uparrow & & \\
K_8(\overline{\mathbf{Q}}, \mathbf{Z}/16) & \xrightarrow{\sim} & H^0(\overline{\mathbf{Q}}, \mathbf{Z}/16(4)) \\
\uparrow & & \wr\uparrow \\
K_8(F, \mathbf{Z}/16) & \longrightarrow & H^0(F, \mathbf{Z}/16(4)).
\end{array}$$

Here the left vertical isomorphisms follow from Suslin's results [30], [31], the horizontal one follows, say, from the BLFSV spectral sequence while the right vertical one follows from corollary 2.5 plus the obvious fact that $\mu_{16}^{\otimes 4}$ is a trivial Galois module.

Consider the natural morphism from the 2-local BLFSV spectral sequence to the mod 16 BLFSV spectral sequence. By theorem 4.1 b) (iii), this morphism is bijective on the $E_2^{p,q}$-terms for $p - q \geq 3$ and $p - q$ even. It follows that the differential $d_2^{-2,-i+1}$ is still surjective for $i = 5$ and bijective for $i > 5$ in the mod 16 spectral sequence. Using the bijective operator $\bar\beta$, we find that $d_2^{-2+4q,-i+1+4q}$ is surjective for $i = 5$ and bijective for $i > 5$ and for any $q \geq 0$ in the mod 16 spectral sequence.

The morphism of spectral sequences is also surjective on the $E_2^{-2,-4-4q}$-terms as $H^3(F,\mathbf{Z}(4+4q))=0$ (theorem 4.1 a) (ii)). Therefore, going back we get the same result in the 2-local spectral sequence. □

Lemma 4.3 implies that, in the 2-local BLFSV spectral sequence, the differentials $E_2^{p,q}$ are 0 for $p\equiv 0 \pmod 4, q\le p$, and that $E_3^{p,q}=0$ for $p-q>2$, except perhaps for $E_3^{p,p-3}$ and $E_3^{p,p-4}$ if $p\equiv 0 \pmod 4$. The rest of theorem 4.2 easily follows, except for the fact that $\rho_i(F)>0$ for $i\equiv 3 \pmod 4$ when $r_1>0$. This will follow from

4.4. Lemma. *Let $r_1>0$. For $i\equiv 3 \pmod 4$, the differential*

$$H^1(F,\mathbf{Z}(i)) \xrightarrow{d_2^{1-i,-i}} H^4(F,\mathbf{Z}(i+1))$$

of the BLFSV spectral sequence is nonzero on the 2-torsion element of $H^1(F,\mathbf{Z}(i))$.

Proof. Let v be a real place of F and $F_v\subset\overline{\mathbf{Q}}$ be the corresponding real closure. In view of the commutative diagram

$$\begin{array}{ccc} H^0(F_v,\mathbf{Z}/2(i)) & \xrightarrow{\sim} & {}_2H^1(F_v,\mathbf{Z}_{(2)}(i)) \\ \wr\uparrow & & \uparrow \\ H^0(F,\mathbf{Z}/2(i)) & \xrightarrow{\sim} & {}_2H^1(F,\mathbf{Z}_{(2)}(i)) \end{array}$$

it is sufficient to show the same fact when replacing F by F_v. Since F_v is a direct limit of number fields, we get a similar exact sequence

$$0\to K_{2i-1}(F_v)\to H^1(F_v,\mathbf{Z}_{(2)}(i))\to H^4(F_v,\mathbf{Z}_{(2)}(i+1))$$
$$\to K_{2i-2}(F_v)\otimes\mathbf{Z}_{(2)}\to H^2(F_v,\mathbf{Z}_{(2)}(i))\to 0.$$

The map $K_q(F_v)\to K_q(\mathbf{R})$ is an isomorphism on torsion (Jannsen [10]), and by Suslin's theorem [31], $K_{2i-2}(\mathbf{R})$ is torsion-free for $i\equiv 3 \pmod 4$. (We could avoid the recourse to Jannsen's theorem at the price of a slightly uglier argument.) Since $K_{2i-2}(F_v)$ is torsion, it is 0. This implies that $H^2(F_v,\mathbf{Z}_{(2)}(i))=0$. One checks easily, by comparing with the exact sequence of Galois modules

$$0\to\mu_2\to\mu_4^{\otimes i}\to\mu_2\to 0$$

that the composition $H^0(F_v,\mathbf{Z}/2(i))\to H^1(F_v,\mathbf{Z}_{(2)}(i))\to H^1(F_v,\mathbf{Z}/2(i))$ is an isomorphism. This implies that $H^1(F_v,\mathbf{Z}_{(2)}(i))$ is the direct sum of a cyclic group of order 2 and a uniquely 2-divisible group. Since the map $H^1(F_v,\mathbf{Z}_{(2)}(i))\to H^4(F_v,\mathbf{Z}_{(2)}(i+1))$ is surjective, it must be nonzero on 2-torsion.

This completes the proof of lemma 4.4, hence of theorem 4.2. □

4.5. Corollary. *Let K be a local field (see definition 1.1). Then the 2-local BLFSV spectral sequence degenerates from E_3 on. There are natural isomorphisms*

$$K_{2i-j}(K)\otimes\mathbf{Z}_{(2)}\xrightarrow{\sim} H^j(K,\mathbf{Z}_{(2)}(i)) \qquad (i\ge 1, j=1,2).$$

Moreover, the natural map $H^j(K,\mathbf{Z}(i))\otimes\mathbf{Z}_2\to\tilde{H}^j(K,B_2(i))$ is an isomorphism for $j\le i$.

Proof. This follows from theorem 4.1 b) and theorem 4.2 by a passage to the limit, as K is a union of non-formally real global fields. □

5. Proof of theorem 1

First we define the maps $ch_{i,j} : K_{2i-j}(O_S)\otimes\mathbf{Z}_2 \to H^j(O_S, B_2(i)) \simeq H^j_{\text{ét}}(O_S, \mathbf{Z}_2(i))$. For $j = 1$ this is easy, since $K_{2i-1}(O_S) \xrightarrow{\sim} K_{2i-1}(F)$ (theorem 2.7) and similarly $H^1(O_S, B_2(i)) \xrightarrow{\sim} \tilde{H}^1(F, B_2(i))$ (theorem 2.8).

Suppose $j = 2$. We shall need some preparation:

5.1. Lemma. *The sequence*

$$0 \to H^2(O_S, B_2(i)) \to \tilde{H}^2(F, B_2(i)) \to \coprod_{v\notin S} H^1(\kappa(v), B_2(i-1)) \to 0$$

is exact.

This follows from theorem 2.8. □

Let κ be a finite field of characteristic $\neq 2$ with q elements. By Quillen's theorem [19], then $K_{2i-1}(\kappa) \xrightarrow{\sim} K_{2i-1}(\bar\kappa)^{G_\kappa}$, where $\bar\kappa$ is an algebraic closure of κ and $G_\kappa = Gal(\bar\kappa/\kappa)$. Moreover, Frobenius acts on $K_{2i-1}(\bar\kappa)$ by multiplication by q^i. It follows that the Galois module $K_{2i-1}(\bar\kappa)_{(2)}$ can be identified with $\varinjlim \mu_{2^\nu}^{\otimes i}$. Using the exact sequence $0 \to B_2(i) \to B_2(i) \otimes \mathbf{Q} \to B_2(i) \otimes \mathbf{Q}/\mathbf{Z} \to 0$, this yields an isomorphism

$$\rho : K_{2i-1}(\kappa) \xrightarrow{\sim} H^1(\kappa, B_2(i)).$$

5.2. Proposition. *Let $i > 1$ and K be a local field with residue field κ. There is an odd integer m such that the diagram*

$$\begin{array}{ccc} K_{2i-2}(K) \otimes \mathbf{Z}_2 & \longrightarrow & K_{2i-3}(\kappa) \otimes \mathbf{Z}_2 \\ \downarrow & & \downarrow{\scriptstyle m\rho} \\ H^2(K, B_2(i)) & \longrightarrow & H^1(\kappa, B_2(i-1)) \end{array}$$

commutes. Here the horizontal maps are the residue maps in K-theory and étale cohomology, while the left vertical map is the map of corollary 4.5 composed with $H^2(K, \mathbf{Z}_{(2)}(i)) \otimes_{\mathbf{Z}_{(2)}} \mathbf{Z}_2 \to H^2(K, B_2(i))$.

Proof. Note that the right vertical and bottom horizontal maps are isomorphisms. The three groups $K_{2i-3}(\kappa) \otimes \mathbf{Z}_2$, $H^2(K, B_2(i))$ and $H^1(\kappa, B_2(i-1))$ are therefore cyclic of the same order, say, N.

5.3. Lemma. *The induced map $K_{2i-2}(K)/N \to H^2(K, B_2(i))$ is bijective.*

This follows from the commutative diagram

$$\begin{array}{ccccccc} K_{2i-2}(K) \otimes \mathbf{Z}_{(2)} & \xrightarrow{\sim} & H^2(K, \mathbf{Z}_{(2)}(i)) & & \longrightarrow & & H^2(K, B_2(i)) \\ \downarrow & & \downarrow & & & & \wr\downarrow \\ K_{2i-2}(K)/N & \xrightarrow{\sim} & H^2(K, \mathbf{Z}_{(2)}(i))/N & \xrightarrow{\sim} & H^2(K, \mathbf{Z}/N(i)) & \xrightarrow{\sim} & H^2(K, B/N(i)) \end{array}$$

in which the top left horizontal map is an isomorphism by corollary 4.5, the right vertical map is an isomorphism because $cd_2(K) = 2$, the bottom left horizontal map is the top left isomorphism tensored by $\mathbf{Z}/N$, the bottom middle horizontal map is an isomorphism by the vanishing of $H^3(K, \mathbf{Z}_{(2)}(i))$ (which follows from corollary 4.5) and the bottom right horizontal map is an isomorphism by corollary 2.5. □

Going back to proposition 5.2, we get a diagram

$$\begin{array}{ccc} K_{2i-2}(K)/N & \twoheadrightarrow & K_{2i-3}(\kappa) \otimes \mathbf{Z}_2 \\ \wr\downarrow & & \wr\downarrow \\ H^2(K, B_2(i)) & \xrightarrow{\sim} & H^1(\kappa, B_2(i-1)) \end{array}$$

in which the top horizontal map is surjective by the localization exact sequence for local fields. Therefore all edges of the square are isomorphisms. Since all vertices are finite cyclic groups, the two composite isomorphisms $K_{2i-2}(K)/N \to H^1(\kappa, B_2(i-1))$ must differ by an invertible constant. □

Remark. With more work one can show that $m = 1$.

From proposition 5.2 one deduces a commutative diagram

$$\begin{array}{ccccc} 0 \to K_{2i-2}(O_S) \otimes \mathbf{Z}_2 & \longrightarrow & K_{2i-2}(F) \otimes \mathbf{Z}_2 & \longrightarrow & \coprod_{v \notin S} K_{2i-3}(\kappa(v)) \otimes \mathbf{Z}_2 \to 0 \\ & & \downarrow & & \downarrow{\scriptstyle (m_v\rho_v)} \\ 0 \to H^2(O_S, B_2(i)) & \longrightarrow & \tilde{H}^2(F, B_2(i)) & \longrightarrow & \coprod_{v \notin S} H^1(\kappa(v), B_2(i-1)) \to 0 \end{array}$$

where the m_vs are odd integers and the ρ_vs are the isomorphisms defined just before proposition 5.2. The right vertical map is an isomorphism, hence there is a unique map $ch_{i,2} : K_{2i-2}(O_S) \otimes \mathbf{Z}_2 \to H^2(O_S, B_2(i))$ making the diagram commute. By construction, this map has the same kernel and cokernel as the corresponding map for F, and the same holds for $ch_{i,1}$, hence theorem 4.2. □

6. Proofs of corollaries 1, 2, 3, 4 and 5

Proof of corollary 1. Let S be the set of real and dyadic places of F. Note that $K_{2i-1}(O_F)$ is finite and that we have a chain of equalities

$$\frac{|K_{2i-2}(O_F)\{2\}|}{|K_{2i-1}(O_F)\{2\}|} = \frac{|K_{2i-2}(O_S)\{2\}|}{|K_{2i-1}(O_S)\{2\}|} = 2^{-r_1} \frac{|H^2_{\text{ét}}(O_S, \mathbf{Z}_2(i))|}{|H^1(O_S, \mathbf{Z}_2(i))|} = 2^{-r_1} \frac{|H^1_{\text{ét}}(O_S, \mathbf{Q}_2/\mathbf{Z}_2(i))|}{|H^0(O_S, \mathbf{Q}_2/\mathbf{Z}_2(i))|}.$$

Here the first equality comes from the localization exact sequence, the second one follows from theorem 1 and the third follows from the finiteness of the dyadic cohomology groups. The proof now goes exactly as in [18, §2]. □

Proof of corollary 2. This follows easily from theorem 1 and the fact that $H^3_{\text{ét}}(O_S,\mathbf{Z}_2(i)) \simeq \begin{cases}(\mathbf{Z}/2)^{r_1} & \text{if } i \text{ is odd}\\ 0 & \text{if } i \text{ is even.}\end{cases}$ □

Proof of corollary 3. a) This follows easily from theorem 1 and the exact sequences

$$0 \to H^2(O_S,\mathbf{Z}_2(i))/2 \to H^2(O_S,\mu_2^{\otimes i}) \to {}_2H^3(O_S,\mathbf{Z}_2(i)) \to 0$$

observing that the last group is 0 or elementary abelian of rank r_1 according as i is even or odd, and the 2-rank of the middle group can be computed by identifying it to $H^2(O_S,\mu_2)$ and using the Kummer exact sequence, which gives

$$rk_2 H^2(O_S,\mu_2) = r(F,S) + r_1.$$

(Note that the Brauer group $H^2(O_S,\mathbb{G}_m)$ is isomorphic to $\operatorname{Ker}((\mathbf{Z}/2)^{r_1}\oplus(\mathbf{Q}/\mathbf{Z})^{S_f} \xrightarrow{\Sigma} \mathbf{Q}/\mathbf{Z})$.)

b) Everything follows easily from theorem 1, except for (iii) and (iv). The latter follows from lemma 4.4. For the former, theorem 1 gives an exact sequence

$$0 \to (\mathbf{Z}/2)^{r_1} \to K_{2i-1}(O_S)\{2\} \to H^0(F,\mathbf{Q}_2/\mathbf{Z}_2(i)) \to 0$$

where $H^0(F,\mathbf{Q}_2/\mathbf{Z}_2(i))$ is cyclic of order $2N$, and we have to show that this sequence is not split. Choose a real place v of F. We get correspondingly a commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & K_{2i-1}(\mathbf{R})\{2\} & \longrightarrow & H^0(\mathbf{R},\mathbf{Q}_2/\mathbf{Z}_2(i)) & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (\mathbf{Z}/2)^{r_1} & \longrightarrow & K_{2i-1}(O_S)\{2\} & \longrightarrow & H^0(F,\mathbf{Q}_2/\mathbf{Z}_2(i)) & \longrightarrow & 0.
\end{array}$$

Here the top row is a non-split exact sequence, as follows from Suslin's computation of $K_*(\mathbf{R})$ [31]. Since the left vertical map is surjective and the right vertical map is injective, this forces the bottom sequence to be non-split as well. □

Proof of corollary 4. Let $(x,y) \in K_i(O_S)_{(2)} \times K_j(O_S)_{(2)}$, where i,j are as in the statement of corollary 4. By theorems 1 and 2.1, x (resp. y) is pure of weight $[i/2]+1$ (resp. $[j/2]+1$), where $[\alpha]$ denotes the integral part of α. Hence their product $x\cdot y \in K_{i+j}(O_S)_{(2)}$ is pure of weight $[i/2]+[j/2]+2$, and also of weight $[\frac{i+j}{2}]+1$. The assumption on (i,j) implies that these two weights differ by 1. For any $k \in \mathbf{Z}-\{0\}$, we have

$$\psi^k(x\cdot y) = k^{[i/2]+[j/2]+2}x\cdot y = k^{[\frac{i+j}{2}]+1}x\cdot y.$$

Taking $k=-1$ gives $2x\cdot y = 0$. To get rid of the factor 2, we need a lemma:

6.1. Lemma. *Let $m>0$.*
a) For any $x \in K_m(F)$, there exists a finite extension E/F such that x_E is divisible by 2 in $K_m(E)$.
b) For any finite extension E/F, the transfer map $N_{E/F} : K_{2m}(E)_{(2)} \to K_{2m}(F)_{(2)}$ is surjective.

Proof. a) follows from Suslin's theorem that $K_m(\overline{\mathbf{Q}})$ is divisible [31]. For b), it suffices to prove the corresponding result for $K_{2m}(F)/N$, where N is the 2-primary part of $[E:F]$: this follows from the fact that $N_{E/F}(x_E) = [E:F]x$ for $x \in K_{2m}(F)$. But corollary 2 gives us an isomorphism

$$K_{2m}(F)/N \xrightarrow{\sim} H^2(F, \mu_N^{\otimes(m+1)}).$$

This isomorphism is compatible with transfer because the BLFST spectral sequences are. The conclusion now follows from the fact that $cd_2(F) = 2$ [24, p. 17, lemma 4]. □

Getting back to the proof of corollary 4, let $(x, y) \in K_i(O_S)_{(2)} \times K_j(O_S)_{(2)}$ as before. We may assume that j is even. To prove that $x \cdot y = 0$, it is sufficient by theorem 2.7 to show that its image in $K_{i+j}(F)_{(2)}$ is 0. Denote the images of x and y in $K_i(F)_{(2)}$ and $K_j(F)_{(2)}$ still be x, y, and let by lemma 6.1 a) E be a finite extension of F such that $x_E = 2\xi$ for some $\xi \in K_i(E)_{(2)}$. By lemma 6.1 b), there exists $\eta \in K_j(E)_{(2)}$ such that $y = N_{E/F}\eta$. By the projection formula,

$$x \cdot y = x \cdot N_{E/F}\eta = N_{E/F}(x_E \cdot \eta) = N_{E/F}(2\xi \cdot \eta) = 0$$

from the first part of the proof. □

Proof of corollary 5. This follows directly from [14, th. D.2]. □

7. An application; proof of theorem 3 b)

7.1. Theorem. *Let F be a field of characteristic 0 and F_0 its field of constants, i.e. the algebraic closure of $\mathbf{Q}$ in F. Then, for $n \geq 2$, the natural map $H^1(F_0, \mathbf{Z}_{(2)}(n)) \to H^1(F, \mathbf{Z}_{(2)}(n))$ is injective with uniquely divisible cokernel, and the map $H^2(F_0, \mathbf{Z}_{(2)}(n)) \to H^2(F, \mathbf{Z}_{(2)}(n))$ is injective.*

Proof. We may assume F to be finitely generated. We first prove that there is an exact sequence

$$0 \to H^1(F_0, \mathbf{Z}_{(2)}(n))/2 \to H^1(F, \mathbf{Z}_{(2)}(n))/2 \to {}_2H^2(F_0, \mathbf{Z}_{(2)}(n)) \to {}_2H^2(F, \mathbf{Z}_{(2)}(n)). \tag{7.1}$$

The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(F, \mathbf{Z}_2(n))/2 & \longrightarrow & H^1(F, \mathbf{Z}/2(n)) & \longrightarrow & {}_2H^2(F, \mathbf{Z}_2(n)) & \longrightarrow & 0 \\
 & & \uparrow & & \| \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^1(F, \mathbf{Z}_{(2)}(n))/2 & \longrightarrow & H^1(F, \mathbf{Z}/2(n)) & \longrightarrow & {}_2H^2(F, \mathbf{Z}_{(2)}(n)) & \longrightarrow & 0
\end{array}$$

and the snake lemma yield an exact sequence

$$0 \to H^1(F, \mathbf{Z}_{(2)}(n))/2 \to H^1(F, \mathbf{Z}_2(n))/2 \to {}_2H^2(F, \mathbf{Z}_{(2)}(n)) \to {}_2H^2(F, \mathbf{Z}_2(n)) \to 0.$$

By a result of Suslin [32, cor. 2.7], $H^1(F_0, B_2(n)) \to H^1(F, B_2(n))$ is an isomorphism and $H^2(F_0, B_2(n)) \to H^2(F, B_2(n))$ is injective, hence the same holds by replacing $B_2(n)$ by $\mathbf{Z}_2(n)$ by corollary 2.5. The commutative diagram with exact rows

$$\begin{array}{ccccccc}
0\to H^1(F,\mathbf{Z}_{(2)}(n))/2 & \longrightarrow & H^1(F,\mathbf{Z}_2(n))/2 & \longrightarrow & {}_2H^2(F,\mathbf{Z}_{(2)}(n)) & \longrightarrow & {}_2H^2(F,\mathbf{Z}_2(n))\to 0\\
\uparrow & & \wr\uparrow & & \uparrow & & \text{inj}\uparrow\\
0\to H^1(F_0,\mathbf{Z}_{(2)}(n))/2 & \longrightarrow & H^1(F_0,\mathbf{Z}_2(n))/2 & \longrightarrow & {}_2H^2(F_0,\mathbf{Z}_{(2)}(n)) & \longrightarrow & {}_2H^2(F_0,\mathbf{Z}_2(n))\to 0
\end{array}$$

then yields (7.1).

We now show that the map $H^1(F_0,\mathbf{Z}_{(2)}(n))/2\to H^1(F_0,\mathbf{Z}_2(n))/2$ is surjective; in view of the commutative diagram

$$\begin{array}{ccc}
H^1(F,\mathbf{Z}_{(2)}(n))/2 & \hookrightarrow & H^1(F,\mathbf{Z}_2(n))/2\\
\uparrow & & \wr\uparrow\\
H^1(F_0,\mathbf{Z}_{(2)}(n))/2 & \longrightarrow & H^1(F_0,\mathbf{Z}_2(n))/2
\end{array}$$

this will show that $H^1(F_0,\mathbf{Z}_{(2)}(n))/2\to H^1(F,\mathbf{Z}_{(2)}(n))/2$ is surjective, hence theorem 7.1, using (7.1) and the classical fact that $\operatorname{Ker}(H^i(F_0,\mathbf{Z}_{(2)}(n))\to H^i(F,\mathbf{Z}_{(2)}(n)))$ is a priori a torsion group.

From the morphism $\mathbf{Z}_{(2)}(n)\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2\to\mathbf{Z}_2(n)$, we get a commutative diagram

$$\begin{array}{ccccc}
H^1(F_0,\mathbf{Z}_2(n)) & \longrightarrow & \varprojlim H^1(F_0,\mathbf{Z}_2(n))/2^\nu & \longrightarrow & \varprojlim H^1(F_0,\mathbf{Z}/2^\nu(n))\\
\uparrow & & \uparrow & & \|\uparrow\\
H^1(F_0,\mathbf{Z}_{(2)}(n))\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2 & \longrightarrow & \varprojlim H^1(F_0,\mathbf{Z}_{(2)}(n))/2^\nu & \longrightarrow & \varprojlim H^1(F_0,\mathbf{Z}/2^\nu(n)).
\end{array}$$

By theorem 4.1 a) (v), the bottom left horizontal map is an isomorphism. On the other hand, it is classical that the Tate module $T_2(K_{2n-2}(F))$ is 0, and in view of theorem 4.2, the same holds for $H^2(F,\mathbf{Z}_{(2)}(n))$. Using the short exact sequences

$$0\to H^1(F_0,\mathbf{Z}_{(2)}(n))/2^\nu\to H^1(F_0,\mathbf{Z}/2^\nu(n))\to {}_{2^\nu}H^2(F_0,\mathbf{Z}_{(2)}(n))\to 0$$

this implies that the bottom right map is an isomorphism as well. In the top row, the composite of both maps is bijective because $\varprojlim{}^1 H^0(F_0,\mathbf{Z}/2^\nu(n))=0$. This implies that $H^1(F_0,\mathbf{Z}_{(2)}(n))\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2\to H^1(F_0,\mathbf{Z}_2(n))$ is bijective, hence so is $H^1(F_0,\mathbf{Z}_{(2)}(n))/2=(H^1(F_0,\mathbf{Z}_{(2)}(n))\otimes_{\mathbf{Z}_{(2)}}\mathbf{Z}_2)/2\to H^1(F_0,\mathbf{Z}_2(n))/2$. □

Proof of theorem 3 b). Using the 2-local BLFSV spectral sequence and corollary 3.2, we get a commutative diagram with exact rows

$$\begin{array}{ccccc}
H^1(F,\mathbf{Z}_{(2)}(3)) & \longrightarrow & K_4^M(F)_{(2)} & \longrightarrow & K_4(F)_{(2)}\\
\uparrow & & \uparrow & & \uparrow\\
H^1(F_0,\mathbf{Z}_{(2)}(3)) & \longrightarrow & K_4^M(F_0)_{(2)} & \longrightarrow & K_4(F_0)_{(2)}
\end{array}$$

where F_0 is the field of constants of F. By theorem 2.6, the map $K_4^M(F)\to K_4(F)$ is killed by 6. By theorem 7.1, the left vertical map has uniquely divisible cokernel. This concludes the proof. □

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Institut de Mathématiques de Jussieu, Equipe Théories Géométriques, Université Paris 7, Case 7012, 75251 Paris Cedex 05, France

E-mail address: `kahn@math.jussieu.fr`