The rank spectral sequence for Quillen's Q construction

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Famous conjectures still unproven:

Bass' conjecture/question (1973): X $\mathbf{Z}$-scheme separated of finite type: are the $K'_i(X)$ finitely generated?

Beilinson-Soulé conjecture (1983): X regular (separated): $\mathrm{gr}^n_\gamma K_i(X)$ torsion of finite exponent if $n \geq [i/2]$ ($\iff$ $H^j(X, \mathbf{Z}(n)) = 0$ for $n > 0$, $j \leq 0$).

Parshin conjecture (1983): X smooth projective over a finite field: $K_i(X) \otimes \mathbf{Q} = 0$ for $i > 0$.

Many dévissages, few answers.

Variant:

Weak Bass' conjecture: X $\mathbf{Z}$-scheme separated of finite type: are the $K'_i(X)$ finitely generated up to isogenies?
(f.g. up to isogeny: sum of f.g. abelian group and group of finite exponent.)

$\iff$ higher Chow groups of X finitely generated up to isogeny

$\Rightarrow$ Beilinson-Soulé conjecture.

Theorem 1 (Quillen, 1972/74). *Bass' conjecture is true in Krull dimension* ≤ 1.

Sketch of proof: dimension 0 reduces to Quillen's computation of $K_*(\mathbf{F}_q)$. Dimension 1: reduce to X = smooth affine curve over $\mathbf{F}_q$ or $\operatorname{Spec} O_F$, O_F ring of integers in a number field F; use Quillen's Q-construction

$$K'_i(X) = K_i(X) = \pi_{i+1}(BQP(X))$$

$P(X) = \{\text{locally free sheaves on } X\}$.

Abbreviate $QP(X)$ to $\mathcal{Q}$.

$B\mathcal{Q}$ = H-space: by Whitehead-Serre, $\pi_*(B\mathcal{Q})$ f.g. $\iff$ $H_*(B\mathcal{Q})$ f.g.

Rank filtration: $\mathcal{Q}_n = \{E \in \mathcal{Q} \mid \operatorname{rk} E \leq n\}$, $T_n : \mathcal{Q}_{n-1} \to \mathcal{Q}_n$ inclusion functor.

$E \in \mathcal{Q}_n$: $T_n \downarrow E := \{[F \to E] \mid F \in \mathcal{Q}_{n-1}\}$.

Proposition 2 (Quillen). *$B(T_n \downarrow E) \approx \Sigma T(E_\eta)$, $T(E_\eta)$ = Tits building of E_η (generic fibre of E).*

Theorem 3 (Solomon-Tits). *$n \geq 2$: $T(E_\eta)$ has the homotopy type of a bouquet of $(n-2)$-spheres.*

Definition 4. $\mathrm{St}(E) = H_{n-2}(T(E_\eta)) = H_{n-1}(T_n \downarrow E))$: the *Steinberg module.*

In fact, need

$$\widetilde{\mathrm{St}}(E) = \begin{cases} \mathrm{St}(E) & \text{if } n > 2 \\ \mathrm{Ker}(\mathrm{St}(E) \to \mathbf{Z}) & \text{if } n = 2 \\ \mathbf{Z} & \text{if } n = 1 \\ \mathbf{Z} & \text{if } n = 0 \end{cases}$$

Gabriel-Zisman spectral sequence

$$E^2_{p,q} = H_p(\mathcal{Q}_n, H_q(E \mapsto T_n \downarrow E)) \Rightarrow H_{p+q}(\mathcal{Q}_n)$$

$E \in \mathcal{Q}_{n-1} \Rightarrow T_n \downarrow E$ has terminal object $[E = E] \Rightarrow$ contractible, hence spectral sequence degenerates to long exact sequence

$$\cdots \to H_i(\mathcal{Q}_{n-1}) \to H_i(\mathcal{Q}_n) \to \bigoplus_{\operatorname{rk} E = n} H_{i-n}(\operatorname{Aut}(E), \widetilde{\operatorname{St}}(E)) \to \ldots \tag{1}$$

In particular, $H_i(\mathcal{Q}_{n-1}) \to H_i(\mathcal{Q}_n)$ surjective for $n > i$, bijective for $n > i+1$ (and $= H_i(\mathcal{Q})$ then).

Isomorphism classes of projective modules of rank $n \simeq \operatorname{Pic}(X)$ (finite!), hence suffices to prove $H_{i-n}(\operatorname{Aut}(E), \widetilde{\operatorname{St}}(E))$ f.g. $\forall E$.

In char. 0: follows from Borel-Serre + Raghunathan; in char. > 0: direct proof of Quillen using the Bruhat-Tits building (!)

Observation ($\approx$ 2008): the exact sequences (1) define an exact couple, hence a spectral sequence $\Rightarrow H_*(B\mathcal{Q})$. Can it give more information and be generalised?

Spectral sequence out of infinitely many degenerating spectral sequences...
???

Morally: Quillen considers $B\mathcal{Q}_{n-1} \to B\mathcal{Q}_n$ as a homotopy fibration. Homology spectral sequence suggests a homotopy cofibration.

Definition 5 (2011)**.** $T : \mathcal{C} \to \mathcal{D}$ functor. T is *cellular* if

- T is fully faithful.
- For any $d \in \mathcal{D} - \mathcal{C}$ and any $c \in \mathcal{C}$, $\mathcal{D}(d, c) = \emptyset$.

(Other terminology: *sieve.*)

Theorem 6. *T cellular: homotopy cocartesian diagram of categories*

$$\begin{array}{ccc} (\mathcal{D} - \mathcal{C}) \int \mathbf{F}_T & \xrightarrow{p} & \mathcal{C} \\ \varepsilon \Big\downarrow & & T \Big\downarrow \\ \mathcal{D} - \mathcal{C} & \xrightarrow{\iota} & \mathcal{D}. \end{array}$$

Notation: $\mathbf{F}_T : \mathcal{D} \to \mathbf{Cat}$ functor $d \mapsto T \downarrow d$, $\int$ Grothendieck construction (so $(\mathcal{D} - \mathcal{C}) \int \mathbf{F}_T \subset \mathcal{D} \int \mathbf{F}_T = T \downarrow \mathcal{D}$), ε the augmentation, p induced by first projection, $\iota =$ inclusion.

Corollary 7. $\mathcal{Q}_0 \to \mathcal{Q}_1 \to \cdots \to \mathcal{Q}_n \to \cdots \to \mathcal{Q}$ *sequence of categories. We assume:*

- *The functors* $T_n : \mathcal{Q}_{n-1} \to \mathcal{Q}_n$ *are cellular;*
- $\mathcal{Q} = \varinjlim \mathcal{Q}_n$.

Write $\mathbf{F}_n$ *for* $\mathbf{F}_{T_n}$. *Then, for any abelian group* A, *spectral sequence of homological type:*

$$E^1_{p,q} = \begin{cases} H_{p+q-1}(\mathcal{Q}_p - \mathcal{Q}_{p-1}, \tilde{\mathbf{F}}_p; A) & \textit{if } p > 0 \\ H_q(\mathcal{Q}_0, A) & \textit{if } p = 0 \end{cases} \Rightarrow H_{p+q}(\mathcal{Q}, A).$$

Here $H_*(\mathcal{Q}_p - \mathcal{Q}_{p-1}, \tilde{\mathbf{F}}_p; A)$ shorthand for the homology of the homotopy cofibre of the augmentation $(\mathcal{Q}_n - \mathcal{Q}_{n-1}) \int \mathbf{F}_p \to \mathcal{Q}_n - \mathcal{Q}_{n-1}$ as in Theorem 6.

X Noetherian integral scheme: by Quillen's resolution theorem, his Q constructions on coherent $\mathcal{O}_X$-sheaves and the full subcategory of torsion-free sheaves are homotopy equivalent. Write $\mathcal{Q}$ for the second one and define $\mathcal{Q}_n$ as the full subcategory of torsion-free sheaves of (generic) rank $\leq n$. Get *rank spectral sequence*:

$$E^1_{p,q} = \bigoplus_{\operatorname{rk} E = p} H_q(\operatorname{Aut}(E), \widetilde{\operatorname{St}}(E)) \Rightarrow H_{p+q}(B\mathcal{Q}). \tag{2}$$

(Different from Rognes' rank spectral sequence: $X = \operatorname{Spec} R$, converges to homology of $BGL(R)^+ \approx \Omega B\mathcal{Q}$.)

Vogel's argument: in Quillen's classical case, this spectral sequence is the same as the one described before.

Example 8. $X = \operatorname{Spec} \mathbf{F}_q$: one summand, $H_q(\operatorname{Aut}(E), \widetilde{\operatorname{St}}(E))$ finite for $q > 0$ and also for $q = 0$, $p > 1$ because $\widetilde{\operatorname{St}}(E)$ irreducible. $\Rightarrow H_n(BQP(\mathbf{F}_q))$ finite for $n > 1 \Rightarrow$ (Cartan-Serre) $K_i(\mathbf{F}_q)$ finite for $i > 0$.

Example 9. X projective over $\mathbf{F}_q$: $\operatorname{Aut}(E)$ still finite $\forall E \Rightarrow E^1_{p,q}$ torsion for $q > 0$, i.e. only one interesting row ($q = 0$) up to torsion. But infinitely many summands... How about E^2-terms?

How to compute the d^1 differentials?

Idea: use Ash-Rudoplh's *universal modular symbols*.

V n-dimensional vector space over field K: $(v_1, \ldots, v_n) \in (V - \{0\})^n$
$\mapsto [v_1, \ldots, v_n] \in \widetilde{\mathrm{St}}(V)$ (Ash-Rudolph, 1979).
Relations (Ash-Rudolph):

- $[v_1, \ldots, v_n] = 0$ if v_i's linearly dependent;
- If $v_0, \ldots, v_n$ all non-zero, then

$$\sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n] = 0.$$

Theorem 10 (Ash-Gunnells-McConnell 2012, K.-Sun 2014). *This is a presentation of* $\widetilde{\mathrm{St}}(V)$.

Fei Sun's thesis (2015): computation of the d^1 differentials in terms of universal modular symbols. Uses formula for "d^1 on the coefficients" (a little mysterious).

This talk: better explain Sun's results.

First tool: bootstrap idea of the rank spectral sequence.

$\mathcal{Q}(V) := \mathcal{Q} \downarrow V$ has final object $[V = V]$ hence contractible; filter it also by rank!

$$J_p(V) = \{[W \to V] \in \mathcal{Q}(V) \mid \operatorname{rk} W \leq p\}.$$

$J_p(V) = \mathcal{Q}(V)$ for $p \geq n$, $J_{n-1}(V) = T_n \downarrow V$, $J_{-1}(V) = \emptyset$, $T_p(V) : J_{p-1}(V) \to J_p(V)$ cellular.

Apply Cor. 7, get a spectral sequence $E^1_{p,q} \Rightarrow H_{p+q}(pt)$ with

$$E^1_{p,q} = \begin{cases} \bigoplus_W \widetilde{\mathrm{St}}(W) & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

i.e. $GL(V)$-equivariant resolutiuon of $\mathbf{Z}$:

$$(3)\quad 0 \to \widetilde{\mathrm{St}}(V) \xrightarrow{\eta_V} \bigoplus_{[W\to V]\in \bar{J}_{n-1}(V)} \widetilde{\mathrm{St}}(W) \xrightarrow{\partial_{n-1}} \dots$$
$$\xrightarrow{\partial_2} \bigoplus_{[W\to V]\in \bar{J}_1(V)} \widetilde{\mathrm{St}}(W) \xrightarrow{\partial_1} \bigoplus_{[W\to V]\in \bar{J}_0(V)} \widetilde{\mathrm{St}}(W) \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

$$\bar{J}_p(V) = J_p(V) - J_{p-1}(V) = \{[W \to V] \in J(V) \mid \operatorname{rk} W = p\}.$$

Proposition 11. *For $p \leq n$ and $([W \xrightarrow{u} V], [W' \xrightarrow{u'} V]) \in \bar{J}_p(V) \times \bar{J}_{p-1}(V)$, we have, with obvious notation*

$$\partial_p(u, u') = \begin{cases} \eta_W(u') & \textit{if } u' \textit{ factors through } u \\ 0 & \textit{else.} \end{cases}$$

Corollary 12. $G \subseteq \mathrm{GL}(V)$*: spectral sequence*

$$I^1_{p,q} = \bigoplus_{W \in \bar{J}_p(V)/G} H_q(\Gamma_W, \widetilde{\mathrm{St}}(W)) \Rightarrow H_{p+q}(G)$$

Γ_W *stabiliser of* W, $(-)/G = G$*-orbits.*

This spectral sequence maps to (2) (for $G = \mathrm{Aut}(E)$, $E_\eta = V$), so controlling its d^1 differentials gives control on those of (2).

Second tool: product structure.

$V, W \in \mathcal{Q}$ of dimensions n, m. $\oplus$ induces a functor

$$\mathcal{Q}(V) \times \mathcal{Q}(W) \to \mathcal{Q}(V \oplus W)$$

mapping $\mathcal{Q}_p(V) \times \mathcal{Q}_q(W)$ to $\mathcal{Q}_{p+q}(V \oplus W)$. Hence a pairing of spectral sequences, yielding $\mathrm{GL}(V) \times \mathrm{GL}(W)$-equivariant pairing of the resolutions (3). In particular, get canonical $\mathrm{GL}(V) \times \mathrm{GL}(W)$-equivariant pairing

$$\widetilde{\mathrm{St}}(V) \otimes \widetilde{\mathrm{St}}(W) \to \widetilde{\mathrm{St}}(V \oplus W) \tag{4}$$

and a pairing of the spectral sequences of Corollary 12.

Proposition 13. $(v_1, \ldots, v_n) \in V^n$, $(w_1, \ldots, w_m) \in W^m$. *Then* (4) *sends* $[v_1, \ldots, v_n] \otimes [w_1, \ldots, w_m]$ *to* $[v_1, \ldots, v_n, w_1, \ldots, w_m]$.

Corollary 14. *In* (3), *let* $\underline{v} \in V^n$ *and* $[v] = [v_1, \ldots, v_n] \in \widetilde{\mathrm{St}}(V)$ *be the corresponding symbol. Assume the* v_i*'s linearly independent. Let* $W_i = \langle v_1, \ldots, \hat{v}_i, \ldots, v_n \rangle$. *Then, for* $[W \xrightarrow{u} V] \in \bar{J}_{n-1}(V)$*:*

$$\eta_V([v])_u = \begin{cases} 0 & \text{if } W \notin \{W_1, \ldots, W_n\}; \\ [v_1, \ldots, \hat{v}_i, \ldots, v_n] & \text{if } W = W_i \text{ and } u \text{ is an admissible mono;} \\ -[v_1, \ldots, \hat{v}_i, \ldots, v_n] & \text{if } W = W_i \text{ and } u \text{ is an admissible epi.} \end{cases}$$

This is "d^1 on the coefficients. Gives d^1 (in principle) on chain level, hence controls differentials of the rank spectral sequence.

Remark 15. Ash-Doud (2018) define a $\mathrm{GL}(V)$-equivariant resolution of $\mathbf{Z}$:

$$0 \to \widetilde{\mathrm{St}}(V) \xrightarrow{\delta_n} \bigoplus_{\dim W = n-1} \widetilde{\mathrm{St}}(W) \xrightarrow{\delta_{n-1}} \dots \xrightarrow{\delta_2} \bigoplus_{\dim W = 1} \widetilde{\mathrm{St}}(W) \xrightarrow{\delta_1} \mathbf{Z} \to 0 \tag{5}$$

where the W's run through $\mathrm{Gr}(V)$ and δ_k is defined on a nonzero universal modular symbol $[v_1, \dots, v_k] \in \widetilde{\mathrm{St}}(W)$, with $\dim W = k$, by

$$\delta_k([v_1, \dots, v_k]) = \sum_{j=1}^{k} (-1)^j [w_1, \dots, \hat{w}_j, \dots, w_k]$$

where $[w_1, \dots, \hat{w}_j, \dots, w_k] \in \langle w_1, \dots, \hat{w}_j, \dots, w_k \rangle$.

Very similar to (3), but different (indexings of $\bigoplus$ are different). In fact, (5) maps to (3) but not quasi-isomorphism.

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