VOEVODSKY'S MOTIVES AND WEIL RECIPROCITY

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Abstract

We describe Somekawa's K-group associated to a finite collection of semiabelian varieties (or more general sheaves) in terms of the tensor product in Voevodsky's category of motives. While Somekawa's definition is based on Weil reciprocity, Voevodsky's category is based on homotopy invariance. We apply this to explicit descriptions of certain algebraic cycles.

1. Introduction

1.1

In this article, we construct an isomorphism

$$K(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}\big(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]\big). \tag{1.1}$$

Here k is a perfect field, and $\mathcal{F}_1, \dots, \mathcal{F}_n$ are homotopy invariant Nisnevich sheaves with transfers in the sense of [31]. On the right-hand side, the tensor product $\mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]$ is computed in Voevodsky's triangulated category $\mathbf{DM}_-^{\mathrm{eff}}$ of effective motivic complexes. The group $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ will be defined in Definition 5.1 by an explicit set of generators and relations: it is a generalization of the group which was defined by K. Kato and studied by M. Somekawa [24] when $\mathcal{F}_1, \dots, \mathcal{F}_n$ are semiabelian varieties.

1.2

In [24, Introduction], Somekawa wrote that he expected an isomorphism of the form

$$K(k; G_1, \dots, G_n) \simeq \mathrm{Ext}^n_{MM}\big(\mathbf{Z}, G_1[-1] \otimes \cdots \otimes G_n[-1]\big),$$

where MM is a conjectural abelian category of mixed motives over k, $G_1, \dots, G_n$ are semiabelian varieties over k, and $G_1[-1], \dots, G_n[-1]$ are the corresponding 1-motives.

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Since we do not have such a category MM at hand, (1.1) provides the closest approximation to Somekawa's expectation.

1.3
The most basic case of (1.1) is $\mathcal{F}_1 = \cdots = \mathcal{F}_n = \mathbf{G}_m$. By [24, Theorem 1.4], the left-hand side is isomorphic to the usual Milnor K-group $K_n^M(k)$. The right-hand side is almost by definition the motivic cohomology group $H^n(k, \mathbf{Z}(n))$. Thus, when k is perfect, we get a new and less combinatorial proof of the Suslin–Voevodsky isomorphism (see [27, Theorem 3.4], [17, Theorem 5.1])

$$K_n^M(k) \simeq H^n\big(k, \mathbf{Z}(n)\big). \tag{1.2}$$

1.4
The isomorphism (1.1) also has the following application to algebraic cycles. Let X be a k-scheme of finite type. Write $\underline{CH}_0(X)$ for the homotopy invariant Nisnevich sheaf with transfers (see [8, Theorem 2.2])

$$U \mapsto CH_0\big(X \times_k k(U)\big) \quad (U \text{ smooth connected}).$$

Let $i, j \in \mathbf{Z}$. We write $CH_i(X, j)$ for Bloch's homological higher Chow group (see [15, Section 1.1]): if X is equidimensional of dimension d, it agrees with the group $CH^{d-i}(X, j)$ of [5].

THEOREM 1.5
Suppose that $\operatorname{char} k = 0$. *Let* $X_1, \ldots, X_n$ *be quasi-projective* k*-schemes. Put* $X = X_1 \times \cdots \times X_n$. *For any* $r \geq 0$, *we have an isomorphism*

$$K\big(k; \underline{CH}_0(X_1), \ldots, \underline{CH}_0(X_n), \mathbf{G}_m, \ldots, \mathbf{G}_m\big) \xrightarrow{\sim} CH_{-r}(X, r), \tag{1.3}$$

where we put r *copies of* $\mathbf{G}_m$ *on the left-hand side.*†

1.6
When $X_1, \ldots, X_n$ are smooth projective,†† special cases of (1.3) were previously known. The case $r = 0$ was proved by Raskind and Spiess [21, Corollary 4.2.6], and the case $n = 1$ was proved by Akhtar [1, Theorem 6.1] (without assuming k to be perfect). The extension to quasi-projective case is new and nontrivial.

†Using recent results of Shane Kelly [14], one may remove the characteristic zero hypothesis if we invert the exponential characteristic of k.
††In this case, Theorem 1.5 is valid in any characteristic (see Remark 12.4).

1.7

Theorem 1.5 is proven using the Borel–Moore motivic homology introduced in [7, Section 9]. We also have a variant which involves motivic homology (see Theorem 12.3). Here is an application. Let C_1, C_2 be two smooth connected curves over our perfect field k, and put $S = C_1 \times C_2$. Assume that C_1 and C_2 both have a 0-cycle of degree 1. Then the special case $n = 2$ and $r = 0$ of Theorem 12.3 gives an isomorphism

$$\mathbf{Z} \oplus \mathrm{Alb}_S(k) \oplus K(k; \mathrm{Alb}_{C_1}, \mathrm{Alb}_{C_2}) \xrightarrow{\sim} H_0(S, \mathbf{Z}).$$

Here, for a smooth variety X, we denote by Alb_X the Albanese variety of X in the sense of Serre [22]; it is a semiabelian variety universal for morphisms from X to semiabelian varieties (see [31, Theorem 3.4.2]). The right-hand side in this case is Suslin homology (see [28] and Section 12.2).

Since Somekawa's groups are defined in an explicit manner, one can sometimes determine the structure of $K(k; \mathrm{Alb}_{C_1}, \mathrm{Alb}_{C_2})$ completely. For instance, when k is finite, we have $K(k; \mathrm{Alb}_{C_1}, \mathrm{Alb}_{C_2}) = 0$ by [10]. This immediately implies the bijectivity of the generalized Albanese map

$$a_S : H_0(S, \mathbf{Z})^{\deg=0} \to \mathrm{Alb}_S(k)$$

of Ramachandran and Spiess–Szamuely [25]. Note that a_S is not bijective for a smooth projective surface S in general (see [13, Proposition 9]).

1.8

We conclude this introduction by pointing out the main difficulty and main ideas in the proof of (1.1).

The definitions of the two sides of (1.1) are quite different: the left-hand side is based on *Weil reciprocity*, while the right-hand side is based on *homotopy invariance*. Thus it is not even obvious how to define map (1.1) to start with. Our solution is to write both sides as quotients of a common larger group and to prove that one quotient factors through the other. This provides map (1.1), which is automatically surjective (Theorem 5.3).

The proof of its injectivity turns out to be much more difficult. We need to find many relations coming from Weil reciprocity. Our main idea, inspired by [24, Theorem 1.4] (recalled in Section 1.3), is to use the *Steinberg relation* to create Weil reciprocity relations. To show that this provides us with enough such relations, we need to carry out a heavy computation of symbols in Section 11.

2. Mackey functors and presheaves with transfers

2.1

A *Mackey functor* over k is a contravariant additive (i.e., commuting with coproducts) functor A from the category of étale k-schemes to the category of abelian groups, provided with a covariant structure verifying the following exchange condition: if

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"'] \\ X' \arrow[r, "f"] & X \end{tikzcd}$$

is a Cartesian square of étale k-schemes, then the diagram

$$\begin{tikzcd} A(Y') \arrow[r, "f'^*"] \arrow[d, "g'_*"'] & A(Y) \arrow[d, "g_*"'] \\ A(X') \arrow[r, "f^*"] & A(X) \end{tikzcd}$$

commutes. Here, * denotes the contravariant structure while $_*$ denotes the covariant structure. The Mackey functor A is *cohomological* if we further have

$$f_* f^* = \deg(f)$$

for any $f : X' \to X$, with X connected. We denote by **Mack** the abelian category of Mackey functors, and by $\mathbf{Mack}_c$ its full subcategory of cohomological Mackey functors.

2.2

A Mackey functor may be viewed as a contravariant additive functor on the category **Span** of "spans" on étale k-schemes, defined as follows [29, (1.4)]: objects are étale k-schemes. A morphism from X to Y is an equivalence class of diagram (span)

$$X \xleftarrow{g} Z \xrightarrow{f} Y \tag{2.1}$$

where, as usual, two spans (Z, f, g) and (Z', f', g') are equivalent if there exists an isomorphism $Z \xrightarrow{\sim} Z'$ making the obvious diagram commute. The composition of spans is defined via fiber product in an obvious manner (compare Quillen's Q-construction in [19, Section 2]).

If A is a Mackey functor, the corresponding functor on **Span** has the same value on objects, while its value on span (2.1) is given by $g_* f^*$.

Note that **Span** is a preadditive category: one may add (but not subtract) two morphisms with the same source and target. We may as well view a Mackey functor as a contravariant additive functor on the associated additive category $\mathbf{Z}\,\mathbf{Span}$. In the notation of the appendix, we thus have $\mathbf{Mack} = \mathrm{Mod}\text{–}\mathbf{Z}\,\mathbf{Span}$.

2.3

Let **Cor** be Voevodsky's category of finite correspondences on smooth k-schemes, denoted by $\mathrm{SmCor}(k)$ in [31, Section 2.1]. Recall that the objects of **Cor** are the same as those of the category Sm/k of smooth varieties over k. Following [31, Section 2.1], we denote by $c(X, Y)$ the group of morphisms in **Cor** from X to Y, which is, by definition, the free abelian group on the set of closed integral subschemes of $X \times Y$ which are finite and surjective over some irreducible component of X.

Let **PST** be the category of presheaves with transfers (i.e., contravariant additive functors from **Cor** to abelian groups), denoted by $\mathrm{PreShv}(\mathrm{SmCor}(k))$ in [31, Section 3.1]. In the same style as Section 2.2, we have, by definition, $\mathbf{PST} = \mathrm{Mod}-\mathbf{Cor}$.

2.4

The category $\mathbf{Z}\,\mathbf{Span}$ is isomorphic to the full subcategory of **Cor** consisting of smooth k-schemes of dimension 0 (= étale k-schemes). In particular, any presheaf with transfers in the sense of Voevodsky [31, Definition 3.1.1] restricts to a Mackey functor over k. By [30, Corollary 3.15], the restriction of a *homotopy invariant* presheaf with transfers yields a cohomological Mackey functor. In other words, we have exact functors

$$\rho : \mathbf{PST} \to \mathbf{Mack}, \tag{2.2}$$

$$\rho : \mathbf{HI} \to \mathbf{Mack}_c, \tag{2.3}$$

where **HI** is the full subcategory of **PST** consisting of homotopy invariant presheaves with transfers.

2.5

By definition, the functor (2.2) equals i^*, where i is the inclusion $\mathbf{Z}\,\mathbf{Span} \to \mathbf{Cor}$. This inclusion has a left adjoint π_0 (scheme of constants). Both functors i and π_0 are symmetric monoidal: for π_0, reduce the problem to the case where k is algebraically closed.

2.6

Let $\mathcal{A}$ be an additive symmetric monoidal category. In the appendix, we show that the symmetric monoidal structure of $\mathcal{A}$ extends canonically to the category $\mathrm{Mod}\text{–}\mathcal{A}$ of (right) $\mathcal{A}$-modules (Section A.8). Given a symmetric monoidal functor $f : \mathcal{A} \to \mathcal{B}$, its extension $f_!$ to right modules is symmetric monoidal (Section A.12).

2.7

If $\mathcal{A} = \mathbf{Cor}$ in Section 2.6, we get a tensor structure in **PST**: we show in Example A.11 that it agrees with the one defined by Voevodsky [31, p. 206].

For the reader's convenience, we recall how to compute this tensor product (see Example A.11 and [31, Section 3.2]). For a smooth variety X over k, denote as usual by $L(X)$ the Nisnevich sheaf with transfers represented by X. Let $\mathcal{F}$ and $\mathcal{F}'$ be presheaves with transfers. There are exact sequences of the form

$$\bigoplus_j L(Y_j) \to \bigoplus_i L(X_i) \to \mathcal{F} \to 0,$$

$$\bigoplus_{j'} L(Y'_{j'}) \to \bigoplus_{i'} L(X'_{i'}) \to \mathcal{F}' \to 0.$$

Then the tensor product $\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{F}'$ of $\mathcal{F}$ and $\mathcal{F}'$ is given by

$$\operatorname{Coker}\Big(\bigoplus_{j,i'} L(Y_j \times X'_{i'}) \oplus \bigoplus_{i,j'} L(X_i \times Y'_{j'}) \to \bigoplus_{i,j} L(X_i \times X'_{i'})\Big).$$

2.8

If $\mathcal{A} = \mathbf{Z}\mathbf{Span}$ in Section 2.6, we get a tensor structure on **Mack**. Another tensor product of Mackey functors $\overset{M}{\otimes}$ was originally defined by L. G. Lewis (unpublished); it was used in [9, Section 5] and [10]. If either A or B is cohomological, $A \overset{M}{\otimes} B$ is cohomological. In Example A.18, we show that $\overset{M}{\otimes}$ agrees with the tensor structure from Section 2.6.

For the reader's convenience, we recall the definition of $\overset{M}{\otimes}$. Let $A_1, \dots, A_n$ be Mackey functors. For any étale k-scheme X, we define

$$(A_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} A_n)(X) := \Big[\bigoplus_{Y \to X} A_1(Y) \otimes \cdots \otimes A_n(Y)\Big] \Big/ R,$$

where $Y \to X$ runs through all finite étale morphisms, and R is the subgroup generated by all elements of the form

$$a_1 \otimes \cdots \otimes f_*(a_i) \otimes \cdots \otimes a_n - f^*(a_1) \otimes \cdots \otimes a_i \otimes \cdots \otimes f^*(a_n),$$

where $Y_1 \xrightarrow{f} Y_2 \to Y$ is a tower of étale morphisms, $1 \le i \le n$, $a_i \in A_i(Y_1)$, and $a_j \in A_j(Y_2)$ $(j = 1, \dots, i-1, i+1, \dots, n)$.

2.9
The functor $\rho = i^* = (\pi_0)_*$ of (2.2) is symmetric monoidal; namely, if $\mathcal{F}$ and $\mathcal{G}$ are presheaves with transfers, then

$$\rho\mathcal{F} \overset{M}{\otimes} \rho\mathcal{G} \xrightarrow{\sim} \rho(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}). \tag{2.4}$$

Indeed, by (A.1) and the right exactness of $\overset{M}{\otimes}$ and $\otimes_{\mathbf{PST}}$ we reduce to $\mathcal{F}$ and $\mathcal{G}$ representable. But if $L(X) \in \mathbf{PST}$ is the presheaf represented by a smooth k-scheme X, then i^* converts the "atomization" homomorphism

$$\bigoplus_{x \in X_{(0)}} L(x) \to L(X)$$

into an isomorphism, and the monoidality of ρ follows. (This also shows the exactness of i_*, which we shall not use here.)

2.10
Let $\mathcal{F} \in \mathbf{PST}$. We define $C_1(\mathcal{F}) \in \mathbf{PST}$ by $C_1(\mathcal{F})(X) = \mathcal{F}(X \times \mathbf{A}^1)$ for all smooth X. For $a \in k = \mathbf{A}^1(k)$, the morphism $X \to X \times \mathbf{A}^1, x \mapsto (x, a)$ defines a morphism $i_a^* : C_1(\mathcal{F}) \to \mathcal{F}$ in $\mathbf{PST}$.

The inclusion functor $\mathbf{HI} \to \mathbf{PST}$ has a left adjoint h_0 given by $h_0(\mathcal{F}) = \operatorname{Coker}(i_0^* - i_1^* : C_1(\mathcal{F}) \to \mathcal{F})$, and the symmetric monoidal structure of $\mathbf{PST}$ induces one on $\mathbf{HI}$ via h_0. In other words, if $\mathcal{F}, \mathcal{G} \in \mathbf{HI}$, we define

$$\mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G} = h_0(\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}). \tag{2.5}$$

Note that (2.3) is *not* symmetric monoidal (since it is the restriction of (2.2)).

2.11
For any $\mathcal{F} \in \mathbf{PST}$, the unit morphism $\mathcal{F} \to h_0(\mathcal{F})$ induces a surjection

$$\mathcal{F}(k) \twoheadrightarrow h_0(\mathcal{F})(k). \tag{2.6}$$

This is obvious from the formula $h_0(\mathcal{F}) = \operatorname{Coker}(C_1(\mathcal{F}) \to \mathcal{F})$.

2.12
We shall also need to work with Nisnevich sheaves with transfers. We denote by $\mathbf{NST}$ the category of Nisnevich sheaves with transfers (objects of $\mathbf{PST}$ which are sheaves in the Nisnevich topology). By [31, Theorem 3.1.4], the inclusion functor $\mathbf{NST} \to \mathbf{PST}$ has an exact left adjoint $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{Nis}}$ (sheafification). The category $\mathbf{NST}$ then inherits a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{NST}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{\mathrm{Nis}}.$$

Similarly, we define $\mathbf{HI}_{\mathrm{Nis}} = \mathbf{HI} \cap \mathbf{NST}$. The sheafification functor restricts to an exact functor $\mathbf{HI} \to \mathbf{HI}_{\mathrm{Nis}}$ (see [31, Theorem 3.1.11]), and $\mathbf{HI}_{\mathrm{Nis}}$ gets a tensor product by the formula

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G} = (\mathcal{F} \otimes_{\mathbf{HI}} \mathcal{G})_{\mathrm{Nis}}.$$

To summarize, all functors in the following naturally commutative diagram are symmetric monoidal:

$$\begin{tikzcd} \mathbf{PST} \arrow[r, "\mathrm{Nis}"] \arrow[d, "h_0"'] & \mathbf{NST} \arrow[d, "h_0^{\mathrm{Nis}}"'] \\ \mathbf{HI} \arrow[r, "\mathrm{Nis}"] & \mathbf{HI}_{\mathrm{Nis}}, \end{tikzcd} \tag{2.7}$$

where each functor is left adjoint to the corresponding inclusion.

2.13
Let $\mathcal{F}$ be a presheaf on Sm/k, and let $\mathcal{F}_{\mathrm{Nis}}$ be the associated Nisnevich sheaf. Then we have an isomorphism

$$\mathcal{F}(k) \xrightarrow{\sim} \mathcal{F}_{\mathrm{Nis}}(k). \tag{2.8}$$

Indeed, any covering of $\operatorname{Spec} k$ for the Nisnevich topology refines to a trivial covering. In particular, the functor $\mathcal{F} \mapsto \mathcal{F}_{\mathrm{Nis}}(k)$ is exact.

This applies in particular to a presheaf with transfers and the associated Nisnevich sheaf with transfers.

2.14
Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. Then (2.4) yields a canonical isomorphism

$$(\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k) \simeq (\mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k). \tag{2.9}$$

Composing (2.9) with the unit morphism $\mathrm{Id} \Rightarrow h_0^{\mathrm{Nis}}$ from (2.7) and taking (2.5) into account, we get a canonical morphism

$$(\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k) \to (\mathcal{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{F}_n)(k), \tag{2.10}$$

which is surjective by Sections 2.11 and 2.13.

2.15

If G is a commutative k-group scheme whose identity component is a quasi-projective variety, then G has a canonical structure of Nisnevich sheaf with transfers (see [25, proof of Lemma 3.2] completed by [3, Lemma 1.3.2]). This applies in particular to semiabelian varieties and also to the "full" Albanese scheme (see [20]) of a smooth variety (which is an extension of a lattice by a semiabelian variety). In particular, if $G_1, \dots, G_n$ are such k-group schemes, (2.10) yields a canonical surjection

$$(G_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} G_n)(k) \to (G_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} G_n)(k), \tag{2.11}$$

where the G_i's are considered on the left as Mackey functors, and on the right as homotopy invariant Nisnevich sheaves with transfers.

3. Presheaves with transfers and motives

3.1

The left adjoint h_0^{Nis} in (2.7) "extends" to a left adjoint C_* of the inclusion

$$\mathbf{DM}_-^{\mathrm{eff}} \to D^-(\mathbf{NST})$$

where the left-hand side is Voevodsky's triangulated category of effective motivic complexes [31, Section 3, especially Proposition 3.2.3].

More precisely, $\mathbf{DM}_-^{\mathrm{eff}}$ is defined as the full subcategory of objects of $D^-(\mathbf{NST})$ whose cohomology sheaves are homotopy invariant. The canonical t-structure of $D^-(\mathbf{NST})$ induces a t-structure on $\mathbf{DM}_-^{\mathrm{eff}}$, with heart $\mathbf{HI}_{\mathrm{Nis}}$. The functor C_* is right exact with respect to these t-structures, and if $\mathcal{F} \in \mathbf{NST}$, then $H_0(C_*(\mathcal{F})) = h_0^{\mathrm{Nis}}(\mathcal{F})$.

3.2

The tensor structure of Section 2.12 on $\mathbf{NST}$ extends to one on $D^-(\mathbf{NST})$ (see [31, p. 206]). Via C_*, this tensor structure descends to a tensor structure on $\mathbf{DM}_-^{\mathrm{eff}}$ (see [31, p. 210]), which will simply be denoted by $\otimes$. The relationship between this tensor structure and the one of Section 2.12 is as follows: If $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$, then

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G} = H^0\big(\mathcal{F}[0] \otimes \mathcal{G}[0]\big), \tag{3.1}$$

where $\mathcal{F}[0], \mathcal{G}[0]$ are viewed as complexes of Nisnevich sheaves with transfers concentrated in degree 0.

We shall need the following lemma, which is not explicit in [31].

LEMMA 3.3

The tensor product $\otimes$ of $\mathbf{DM}_-^{\mathrm{eff}}$ is right exact with respect to the homotopy t-structure.

Proof
By definition,

$$C \otimes D = C_*(C \overset{L}{\otimes} D)$$

for $C, D \in \mathbf{DM}_-^{\mathrm{eff}}$, where $\overset{L}{\otimes}$ is the tensor product of $D^-(\mathbf{NST})$ defined in [31, p. 206]. We want to show that if C and D are concentrated in degrees at most 0, then so is $C \otimes D$. Using the canonical left resolutions of [31, p. 206], it is enough to do it for C and D of the form $C_*(L(X))$ and $C_*(L(Y))$ for two smooth schemes X, Y. Since C_* is symmetric monoidal, we have that

$$C_*\bigl(L(X)\bigr) \otimes C_*\bigl(L(Y)\bigr) \overset{\sim}{\longleftarrow} C_*\bigl(L(X) \overset{L}{\otimes} L(Y)\bigr) = C_*\bigl(L(X \times Y)\bigr),$$

and the claim is obvious in view of the formula for C_* (see [31, p. 207]). □

3.4
Let $C \in \mathbf{DM}_-^{\mathrm{eff}}$. For any $X \in \mathrm{Sm}/k$ and any $i \in \mathbf{Z}$, we have that

$$\mathbb{H}^i_{\mathrm{Nis}}(X, C) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}\bigl(M(X), C[i]\bigr),$$

where $M(X) = C_*(L(X))$ is the motive of X computed in $\mathbf{DM}_-^{\mathrm{eff}}$ (cf. [31, Proposition 3.2.7]).

Specializing to the case $X = \operatorname{Spec} k$ ($M(X) = \mathbf{Z}$) and taking Section 2.13 into account, we get that

$$\mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}\bigl(\mathbf{Z}, C[i]\bigr) \simeq H^i(C)(k). \tag{3.2}$$

Combining (3.1), (2.8), and (3.2), we get the following.

LEMMA 3.5
Let $\mathcal{F}_1, \dots, \mathcal{F}_n$ be homotopy invariant Nisnevich sheaves with transfers. Then we have a canonical isomorphism

$$(\mathcal{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{F}_n)(k) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}\bigl(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]\bigr). \tag{3.3}$$

3.6
Summarizing, for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$, we get a surjective homomorphism

$$(\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k) \to \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}\bigl(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]\bigr) \tag{3.4}$$

by composing (2.9), (2.6), (2.5), (2.8), and (3.3).

4. Presheaves with transfers and local symbols

4.1

Given a presheaf with transfers $\mathcal{G}$, recall from [30, p. 96] the presheaf with transfers $\mathcal{G}_{-1}$ defined by the formula

$$\mathcal{G}_{-1}(U) = \operatorname{Coker}\big(\mathcal{G}(U \times \mathbf{A}^1) \to \mathcal{G}\big(U \times \big(\mathbf{A}^1 - \{0\}\big)\big)\big). \tag{4.1}$$

If $\mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$, then $\mathcal{G}(U \times (\mathbf{A}^1 - \{0\})) \simeq \mathcal{G}(U) \oplus \mathcal{G}_{-1}(U)$ for all smooth U. Thus $\mathcal{G}_{-1} \in \mathbf{HI}_{\mathrm{Nis}}$ and $\mathcal{G} \mapsto \mathcal{G}_{-1}$ is an exact endofunctor of $\mathbf{HI}_{\mathrm{Nis}}$.

Suppose that $\mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$. Let $X \in \mathrm{Sm}/k$ (connected), let $K = k(X)$, and let $x \in X$ be a point of codimension 1. By [30, Lemma 4.36], there is a canonical isomorphism

$$\mathcal{G}_{-1}\big(k(x)\big) \simeq H^1_{\mathrm{Zar},x}(X, \mathcal{G}) \tag{4.2}$$

yielding a canonical map

$$\partial_x : \mathcal{G}(K) \to \mathcal{G}_{-1}\big(k(x)\big). \tag{4.3}$$

The following lemma follows from the construction of the isomorphisms (4.2). It is part of the general fact that $\mathcal{G}$ defines a cycle module in the sense of Rost (cf. [6, Proposition 5.4.64]).

LEMMA 4.2

(a) *Let $f : Y \to X$ be a dominant morphism, with Y smooth and connected. Let $L = k(Y)$, and let $y \in Y^{(1)}$ be such that $f(y) = x$. Then the diagram*

$$\begin{tikzcd} \mathcal{G}(L) \arrow[r, "(\partial_y)"] & \mathcal{G}_{-1}\big(k(y)\big) \\ \mathcal{G}(K) \arrow[u, "f^*"] \arrow[r, "\partial_x"] & \mathcal{G}_{-1}\big(k(x)\big) \arrow[u, "ef^*"'] \end{tikzcd}$$

commutes, where e is the ramification index of v_y relative to v_x.

(b) *If f is finite surjective, the diagram*

$$\begin{tikzcd} \mathcal{G}(L) \arrow[r, "(\partial_y)"] \arrow[d, "f_*"'] & \displaystyle\bigoplus_{y \in f^{-1}(x)} \mathcal{G}_{-1}\big(k(y)\big) \arrow[d, "f_*"] \\ \mathcal{G}(K) \arrow[r, "\partial_x"] & \mathcal{G}_{-1}\big(k(x)\big) \end{tikzcd}$$

commutes.

PROPOSITION 4.3
Let $\mathscr{G} \in \mathbf{HI}_{\mathrm{Nis}}$. There is a canonical isomorphism

$$\mathscr{G}_{-1} = \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathscr{G}).$$

Proof
The statement means that $\mathscr{G}_{-1}$ represents the functor

$$\mathscr{H} \mapsto \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathscr{H} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m, \mathscr{G}).$$

Sheafifying (4.1) for the Nisnevich topology and using homotopy invariance, we obtain an isomorphism

$$\mathrm{Coker}(\mathscr{G} \to p_* p^* \mathscr{G}) \xrightarrow{\sim} \mathscr{G}_{-1},$$

where $p : \mathbf{A}^1 - \{0\} \to \operatorname{Spec} k$ is the structural morphism. Moreover, [30, Proposition 5.4] shows that $R^i p_* p^* \mathscr{G}(K) = H^i_{\mathrm{Nis}}(\mathbf{A}^1_K - \{0\}, p^*\mathscr{G}) = 0$ for any field K/k and $i > 0$; hence by [30, Proposition 4.20] one has that $R^i p_* p^* \mathscr{G} = 0$ for any $i > 0$. It follows that $p_* p^* \mathscr{G}[0] \xrightarrow{\sim} Rp_* p^* \mathscr{G}[0]$.

By [31, Proposition 3.2.8], we have that

$$Rp_* p^* \mathscr{G}[0] = \underline{\mathrm{Hom}}\big(M\big(\mathbf{A}^1 - \{0\}\big), \mathscr{G}[0]\big),$$

where $\underline{\mathrm{Hom}}$ is the (partially defined) internal Hom of $\mathbf{DM}^{\mathrm{eff}}_{-}$. By [31, Proposition 3.5.4] (Gysin triangle) and homotopy invariance, we have an exact triangle, split by any rational point of $\mathbf{A}^1 - \{0\}$:

$$\mathbf{Z}(1)[1] \to M\big(\mathbf{A}^1 - \{0\}\big) \to \mathbf{Z} \xrightarrow{+1}.$$

To get a canonical splitting, we may choose the rational point $1 \in \mathbf{A}^1 - \{0\}$.

By [31, Corollary 3.4.3], we have an isomorphism $\mathbf{Z}(1)[1] \simeq \mathbf{G}_m[0]$. Hence, in $\mathbf{DM}^{\mathrm{eff}}_{-}$, we have an isomorphism

$$\mathscr{G}_{-1}[0] \simeq \underline{\mathrm{Hom}}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{G}_m[0], \mathscr{G}[0]\big). \tag{4.4}$$

Let $\mathscr{H} \in \mathbf{HI}_{\mathrm{Nis}}$. We get that

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathscr{H}[0], \mathscr{G}_{-1}[0]\big) \\
&\quad \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathscr{H}[0] \otimes \mathbf{G}_m[0], \mathscr{G}[0]\big) && \text{by (4.4)} \\
&\quad \simeq \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}\big(H^0\big(\mathscr{H}[0] \otimes \mathbf{G}_m[0]\big), \mathscr{G}\big) && \text{by Lemma 3.3} \\
&\quad = \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathscr{H} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m, \mathscr{G}) && \text{by (3.1)},
\end{aligned}$$

as desired. □

Remark 4.4
The proof of Proposition 4.3 also shows that, in $\mathbf{DM}_-^{\mathrm{eff}}$, we have an isomorphism

$$\underline{\mathrm{Hom}}\big(\mathbf{G}_m[0], \mathcal{G}[0]\big) \simeq \underline{\mathrm{Hom}}(\mathbf{G}_m, \mathcal{G})[0],$$

where the left $\underline{\mathrm{Hom}}$ is computed in $\mathbf{DM}_-^{\mathrm{eff}}$ and the right $\underline{\mathrm{Hom}}$ is computed in $\mathbf{HI}_{\mathrm{Nis}}$. In particular, $\underline{\mathrm{Hom}}(\mathbf{G}_m[0], -) : \mathbf{DM}_-^{\mathrm{eff}} \to \mathbf{DM}_-^{\mathrm{eff}}$ is t-exact.

PROPOSITION 4.5
Let C be a smooth, proper, connected curve over k, with function field K. For any $\mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$, there exists a canonical homomorphism

$$\mathrm{Tr}_{C/k} : H^1_{\mathrm{Zar}}(C, \mathcal{G}) \to \mathcal{G}_{-1}(k)$$

such that, for any $x \in C$, the composition

$$\mathcal{G}_{-1}\big(k(x)\big) \simeq H^1_x(C, \mathcal{G}) \to H^1_{\mathrm{Zar}}(C, \mathcal{G}) \xrightarrow{\mathrm{Tr}_C} \mathcal{G}_{-1}(k)$$

equals the transfer map $\mathrm{Tr}_{k(x)/k}$ associated to the finite surjective morphism $\operatorname{Spec} k(x) \to \operatorname{Spec} k$.

Proof
By [31, Proposition 3.2.7], we have that

$$H^1_{\mathrm{Zar}}(C, \mathcal{G}) \xrightarrow{\sim} H^1_{\mathrm{Nis}}(C, \mathcal{G}) \simeq \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}\big(M(C), \mathcal{G}[1]\big).$$

The structural morphism $C \to \operatorname{Spec} k$ yields a morphism of motives $M(C) \to \mathbf{Z}$, which, by Poincaré duality [31, Theorem 4.3.2], yields a canonical morphism

$$\mathbf{G}_m[1] \simeq \mathbf{Z}(1)[2] \to M(C).$$

(One may view this morphism as the image of the canonical morphism $\mathbb{L} \to h(C)$ in the category of Chow motives.)

Therefore, by Proposition 4.3 and Remark 4.4, we get a map

$$\mathrm{Tr}_{C/k} : H^1_{\mathrm{Zar}}(X, \mathcal{G}) \to \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}\big(\mathbf{G}_m[1], \mathcal{G}[1]\big) = \mathcal{G}_{-1}(k).$$

It remains to prove the claimed compatibility. Let $M^x(C)$ be the motive of C with supports in x, defined as $C_*(\mathrm{Coker}(L(C - \{x\}) \to L(C))$. Let $\mathbf{Z}_{k(x)} = M(\operatorname{Spec} k(x))$. By [31, Proof of Proposition 3.5.4], we have an isomorphism $M^x(C) \simeq \mathbf{Z}_{k(x)}(1)[2]$, and we have to show that the composition

$$\mathbf{Z}(1)[2] \to M(C) \xrightarrow{g_x} \mathbf{Z}_{k(x)}(1)[2]$$

induces $\mathrm{Tr}_{k(x)/k}$, up to twisting and shifting. To see this, we observe that g_x is the image of the morphism of Chow motives

$$h(C) \to h\bigl(\operatorname{Spec} k(x)\bigr)(1)$$

dual to the morphism $h(\operatorname{Spec} k(x)) \to h(C)$ induced by the inclusion $\operatorname{Spec} k(x) \to C$. This is easy to check from the definition of g_x in [31]. (Observe that, in this special case, $Bl_x(C) = C$, and note that we may use a variant of said construction by replacing $C \times \mathbf{A}^1$ with $C \times \mathbf{P}^1$ to stay within smooth projective varieties.) The conclusion now follows from the fact that the composition

$$\operatorname{Spec} k(x) \to C \to \operatorname{Spec} k$$

is the structural morphism of $\operatorname{Spec} k(x)$. □

PROPOSITION 4.6 (Reciprocity)
Let C be a smooth, proper, connected curve over k, with function field K. Then the sequence

$$\mathscr{G}(K) \xrightarrow{(\partial_x)} \bigoplus_{x \in C} \mathscr{G}_{-1}\bigl(k(x)\bigr) \xrightarrow{\sum_x \mathrm{Tr}_{k(x)/k}} \mathscr{G}_{-1}(k)$$

is a complex.

Proof
This follows from Proposition 4.5, since the composition

$$\mathscr{G}(K) \to \bigoplus_{x \in C} H^1_x(C, \mathscr{G}) \xrightarrow{(g_x)} H^1(C, \mathscr{G})$$

is 0. □

THEOREM 4.7
Suppose that $\mathscr{F} \in \mathbf{HI}_{\mathrm{Nis}}$. Then there exists a canonical isomorphism

$$\mathscr{F} \simeq (\mathscr{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m)_{-1}.$$

Proof
We compute again in $\mathbf{DM}^{\mathrm{eff}}_{-}$. As recalled in the proof of Proposition 4.3, we have that $\mathbf{G}_m[0] = \mathbf{Z}(1)[1]$. Hence the cancellation theorem from [33, Corollary 4.10] yields a canonical isomorphism

$$\mathscr{F}[0] \simeq \underline{\mathrm{Hom}}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\bigl(\mathbf{G}_m[0], \mathscr{F}[0] \otimes \mathbf{G}_m[0]\bigr).$$

By taking H^0, we obtain that

$$\mathcal{F} \simeq \underline{\mathrm{Hom}}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathbf{G}_m, \mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m).$$

The right-hand side is isomorphic to $(\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m)_{-1}$ by Proposition 4.3. □

4.8
If $\mathcal{F}, \mathcal{G}$ are presheaves with transfers, there is a bilinear morphism of presheaves with transfers (i.e., a natural transformation over $\mathbf{PST} \times \mathbf{PST}$):

$$\begin{aligned}
&\mathcal{F}(U) \otimes \mathcal{G}_{-1}(V) \\
&\quad = \mathrm{Coker}\big(\mathcal{F}(U) \otimes \mathcal{G}(V \times \mathbf{A}^1) \to \mathcal{F}(U) \otimes \mathcal{G}\big(V \times \big(\mathbf{A}^1 - \{0\}\big)\big)\big) \\
&\to \mathrm{Coker}\big((\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})(U \times V \times \mathbf{A}^1) \to (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})\big(U \times V \times \big(\mathbf{A}^1 - \{0\}\big)\big)\big) \\
&\qquad = (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{-1}(U \times V),
\end{aligned}$$

which induces a morphism

$$\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G}_{-1} \to (\mathcal{F} \otimes_{\mathbf{PST}} \mathcal{G})_{-1}. \tag{4.5}$$

Notation 4.9
Let $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$, and let $\mathcal{H} = \mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}$. Let X, K, x be as in Section 4.1. For $(a, b) \in \mathcal{F}(K) \times \mathcal{G}(K)$, we denote by $a \cdot b$ the image of $a \otimes b$ in $\mathcal{H}(K)$ by the map

$$\mathcal{F}(K) \otimes \mathcal{G}(K) \to \mathcal{H}(K). \tag{4.6}$$

We define the *local symbol* on $\mathcal{F}$

$$\mathcal{F}(K) \times K^* \to \mathcal{F}\big(k(x)\big)$$

to be the composition

$$\mathcal{F}(K) \times K^* \xrightarrow{\cdot} (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m)(K) \xrightarrow{\partial_x} (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathbf{G}_m)_{-1}\big(k(x)\big) \simeq \mathcal{F}\big(k(x)\big),$$

where the first map is given by (4.6) with $\mathcal{G} = \mathbf{G}_m$ and the last isomorphism is given by Theorem 4.7. The image of $(a, b) \in \mathcal{F}(K) \times K^*$ by the local symbol is denoted by $\partial_x(a, b) \in \mathcal{F}(k(x))$.

PROPOSITION 4.10 (cf. [6, Proposition 5.5.27])
Let $\mathcal{F}, \mathcal{G} \in \mathbf{HI}_{\mathrm{Nis}}$, and consider the morphism induced by (4.5),

$$\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}_{-1} \xrightarrow{\Phi} (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})_{-1}.$$

Let X, K, x be as in Section 4.1. Then the diagram

$$\begin{tikzcd}
\mathcal{F}(\mathcal{O}_{X,x}) \otimes \mathcal{G}(K) \arrow[rr] \arrow[d, "i_x^* \otimes \partial_x"] & & (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})(K) \arrow[dd, "\partial_x"] \\
\mathcal{F}\big(k(x)\big) \otimes \mathcal{G}_{-1}\big(k(x)\big) \arrow[d] & & \\
(\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G}_{-1})\big(k(x)\big) \arrow[rr, "\Phi"] & & (\mathcal{F} \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{G})_{-1}\big(k(x)\big)
\end{tikzcd}$$

commutes, where i_x^* is induced by the reduction map $\mathcal{O}_{X,x} \to k(x)$. In other words, with Notation 4.9 we have the identity

$$\partial_x(a \cdot b) = \Phi(i_x^* a \cdot \partial_x b) \tag{4.7}$$

for $(a, b) \in \mathcal{F}(\mathcal{O}_{X,x}) \times \mathcal{G}(K)$.

COROLLARY 4.11
Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$, let X, K, x be as in Section 4.1, and let $(a, f) \in \mathcal{F}(K) \times K^*$.

(a) *Suppose that there is $\tilde{a} \in \mathcal{F}(\mathcal{O}_{X,x})$ whose image in $\mathcal{F}(K)$ is a.[†] Then we have that*

$$\partial_x(a, f) = v_x(f) a(x),$$

where $a(x)$ is the image of $\tilde{a}$ in $\mathcal{F}(k(x))$.

(b) *Suppose that $v_x(f - 1) > 0$. Then $\partial_x(a, f) = 0$.*

Proof
Since (4.3) is given by v_x when $\mathcal{G} = \mathbf{G}_m$, (a) follows from Proposition 4.10 (applied with $\mathcal{G} = \mathbf{G}_m$) and Theorem 4.7. Part (b) follows again from Proposition 4.10, after switching the roles of $\mathcal{F}$ and $\mathcal{G}$. $\square$

PROPOSITION 4.12
Let G be a semiabelian variety. The local symbol on G defined in Notation 4.9 agrees with Somekawa's local symbol [24, (1.1)] (generalizing the Rosenlicht–Serre local symbol) on G.

Proof
Up to base changing from k to $\bar{k}$ (see Lemma 4.2(a)), we may assume k algebraically

[†] By [30, Corollary 4.19], $\tilde{a}$ is unique.

closed. By [23, Chapter III, Proposition 1], it suffices to show that the local symbol in Notation 4.9 satisfies the properties in [23, Chapter III, Definition 2], which characterize the Rosenlicht–Serre local symbol. In this definition, Condition (i) is obvious, Condition (ii) is Corollary 4.11(b), Condition (iii) is Corollary 4.11(a), and Condition (iv) is Proposition 4.6. □

5. K-groups of Somekawa type

Definition 5.1

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$.

(a) A *relation datum of Somekawa type* for $\mathcal{F}_1, \dots, \mathcal{F}_n$ is a collection $(C, h, (g_i)_{i=1,\dots,n})$ of the following objects: (1) a smooth proper connected curve C over k, (2) $h \in k(C)^*$, and (3) $g_i \in \mathcal{F}_i(k(C))$ for each $i \in \{1, \dots, n\}$, which satisfies the condition

$$\text{for any } c \in C \text{, there is } i(c) \text{ such that } c \in R_i \text{ for all } i \neq i(c), \tag{5.1}$$

where $R_i := \{c \in C \mid g_i \in \mathrm{Im}(\mathcal{F}_i(\mathcal{O}_{C,c}) \to \mathcal{F}_i(k(C)))\}$.

(b) We define the *K-group of Somekawa type* $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ to be the quotient of $(\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k)$ by its subgroup generated by elements of the form

$$\sum_{c \in C} \mathrm{Tr}_{k(c)/k}\big(g_1(c) \otimes \cdots \otimes \partial_c(g_{i(c)}, h) \otimes \cdots \otimes g_n(c)\big), \tag{5.2}$$

where $(C, h, (g_i)_{i=1,\dots,n})$ runs through all relation data of Somekawa type.

Remark 5.2

In view of Proposition 4.12, our group $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ coincides with the Milnor K-group defined in [24] when $\mathcal{F}_1, \dots, \mathcal{F}_n$ are semiabelian varieties over k.[†] (Note that Somekawa works with all finite extensions but over an arbitrary field; we work with finite separable extensions but are assuming k perfect.)

THEOREM 5.3

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. The homomorphism (2.10) factors through $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Consequently, we get a surjective homomorphism (1.1).

Proof

Put $\mathcal{F} := \mathcal{F}_1 \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \cdots \otimes_{\mathbf{HI}_{\mathrm{Nis}}} \mathcal{F}_n$. Let $(C, h, (g_i)_{i=1,\dots,n})$ be a relation datum of Somekawa type. We must show that the element (5.2) goes to 0 in $\mathcal{F}(k)$ via (2.10). Consider

[†] As was observed by W. Raskind, the signs appearing in [24, (1.2.2)] should not be there (cf. [21, p. 10, footnote]).

the element $g = g_1 \otimes \cdots \otimes g_n \in \mathcal{F}(K)$. It follows from (4.7) that, for any $c \in C$, we have

$$g_1(c) \otimes \cdots \otimes \partial_c(g_{i(c)}, h) \otimes \cdots \otimes g_n(c)$$
$$= g_1(c) \otimes \cdots \otimes \partial_c\big(g_{i(c)} \otimes \{h\}\big) \otimes \cdots \otimes g_n(c) = \partial_c\big(g \otimes \{h\}\big).$$

The claim now follows from Proposition 4.6. □

6. K-groups of geometric type

Definition 6.1

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathbf{PST}$.

(a) A *relation datum of geometric type* for $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is a collection $(C, f, (g_i)_{i=1,\ldots,n})$ of the following objects: (1) a smooth projective connected curve C over k, (2) a surjective morphism $f : C \to \mathbf{P}^1$, and (3) $g_i \in \mathcal{F}_i(C')$ for each $i \in \{1, \ldots, n\}$, where $C' = f^{-1}(\mathbf{P}^1 \setminus \{1\})$.

(b) We define the *K-group of geometric type* $K'(k; \mathcal{F}_1, \ldots, \mathcal{F}_n)$ to be the quotient of $(\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k)$ by its subgroup generated by elements of the form

$$\sum_{c \in C'} v_c(f)\, \mathrm{Tr}_{k(c)/k}\big(g_1(c) \otimes \cdots \otimes g_n(c)\big), \tag{6.1}$$

where $(C, f, (g_i)_{i=1,\ldots,n})$ runs through all relation data of geometric type. (Here we used the notation $g_i(c) = \iota_c^*(g_i) \in \mathcal{F}(k(c))$, where $\iota_c : c = \operatorname{Spec} k(c) \to C'$ is the closed immersion.)

The rest of this section is devoted to a proof of the following theorem.

THEOREM 6.2

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. The homomorphism (2.10) induces an isomorphism

$$K'(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0]\big). \tag{6.2}$$

Remark 6.3

By combining Theorems 5.3 and 6.2, we obtain a surjective homomorphism

$$K(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) \to K'(k; \mathcal{F}_1, \ldots, \mathcal{F}_n) \tag{6.3}$$

for any $\mathcal{F}_1, \ldots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. The existence of this surjection is *not* clear from the definition.

6.4

Let $\mathcal{F} \in \mathbf{PST}$. Suppose that we are given the following data: (a) a smooth projective connected curve C over k, (b) a surjective morphism $f : C \to \mathbf{P}^1$, and (c) a map $\alpha : L(C') \to \mathcal{F}$ in $\mathbf{PST}$, where $C' = f^{-1}(\Delta)$ and $\Delta = \mathbf{P}^1 \setminus \{1\} (\simeq \mathbf{A}^1)$. To such a triple (C, f, α), we associate an element

$$\alpha\big(\operatorname{div}(f)\big) \in \mathcal{F}(k), \tag{6.4}$$

where we regard $\operatorname{div}(f)$ as an element of $Z_0(C') = c(\operatorname{Spec} k, C') = L(C')(k)$.

One can rewrite the element (6.4) as follows. The map $\alpha : L(C') \to \mathcal{F}$ can be regarded as a section $\alpha \in \mathcal{F}(C')$. To each closed point $c \in C'$, we write $\alpha(c)$ for the image of α in $\mathcal{F}(k(c))$ by the map induced by $c = \operatorname{Spec} k(c) \to C'$. Now (6.4) is rewritten as

$$\sum_{c \in C'} v_c(f) \operatorname{Tr}_{k(c)/k} \alpha(c). \tag{6.5}$$

PROPOSITION 6.5

Let $\mathcal{F} \in \mathbf{PST}$. We define $\mathcal{F}(k)_{\mathrm{rat}}$ to be the subgroup of $\mathcal{F}(k)$ generated by elements (6.4) for all triples (C, f, α) as in Section 6.4. Then we have that

$$h_0(\mathcal{F})(k) = \mathcal{F}(k)/\mathcal{F}(k)_{\mathrm{rat}}.$$

Proof

By definition we have that

$$h_0(\mathcal{F})(k) = \operatorname{Coker}\big(i_0^* - i_\infty^* : \mathcal{F}(\Delta) \to \mathcal{F}(k)\big), \tag{6.6}$$

where $\Delta = \mathbf{P}^1 \setminus \{1\} (\simeq \mathbf{A}^1)$ and i_a^* is the pullback by the inclusion $i_a : \{a\} \to \Delta$ for $a \in \{0, \infty\}$.

Suppose that we are given a triple (C, f, α) as in Section 6.4, and set $C' = f^{-1}(\Delta)$. The graph $\gamma_{f|_{C'}}$ of $f|_{C'}$ defines an element of $c(\Delta, C') = L(C')(\Delta)$. In the commutative diagram

$$\begin{array}{ccc} L(C')(\Delta) & \xrightarrow{\alpha} & \mathcal{F}(\Delta) \\ {\scriptstyle i_0^*-i_\infty^*} \downarrow & & \downarrow {\scriptstyle i_0^*-i_\infty^*} \\ L(C')(k) & \xrightarrow{\alpha} & \mathcal{F}(k), \end{array}$$

the image of $\gamma_{f|_{C'}}$ in $L(C')(k) = Z_0(C')$ is $\operatorname{div}(f)$, which shows the vanishing of $\alpha(\operatorname{div}(f))$ in $h_0(\mathcal{F})(k)$.

Conversely, given $\alpha \in \mathcal{F}(\Delta)$, (6.4) for the triple $(\mathbf{P}^1, \mathrm{id}_{\mathbf{P}^1}, \alpha)$ coincides with $(i_0^* - i_\infty^*)(\alpha)$. This completes the proof. $\square$

LEMMA 6.6
Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$. Put $\mathcal{F} := \mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n$. Let (C, f, α) be a triple considered in Section 6.4. Then $\alpha \in \mathcal{F}(C')$ is the sum of a finite number of elements of the form

$$\mathrm{Tr}_h(g_1 \otimes \cdots \otimes g_n), \tag{6.7}$$

where D is a smooth projective curve, $h : D \to C$ is a surjective morphism, $g_i \in \mathcal{F}_i(h^{-1}(C'))$ for $i = 1, \dots, n$, and $\mathrm{Tr}_h : \mathcal{F}(h^{-1}(C')) \to \mathcal{F}(C')$ is the transfer with respect to $h|_{h^{-1}(C')}$.

Proof
By the facts recalled in Section 2.7, we reduce the problem to the case $\mathcal{F}_i = L(X_i)$, where X_i is a smooth variety over k for each $i = 1, \dots, n$. Then we have $\mathcal{F} = L(X)$ with $X = X_1 \times \cdots \times X_n$. Let Z be an integral closed subscheme of $C' \times X$ which is finite and surjective over C'. It suffices to show that $Z \in c(C', X) = L(X)(C')$ can be written as (6.7).

Let $q : D' \to Z$ be the normalization, and let $h : D' \to C'$ be the composition $D' \to Z \to C'$, so that h is a finite surjective morphism. For $i = 1, \dots, n$, we define $g_i \in c(D', X_i) = L(X_i)(D')$ to be the graph of $D' \to X \to X_i$. If we set $g = g_1 \otimes \cdots \otimes g_n \in L(X)(D')$, then by definition we have that $\mathrm{Tr}_h(g) = Z$ in $L(X)(C')$. The assertion is proved. □

6.7
Now it follows from Definition 6.1(b), Proposition 6.5, Lemma 6.6, and (6.5) that (2.9) and (2.6) induce an isomorphism

$$K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \simeq h_0(\mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k)$$

for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{PST}$. If $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$, the right-hand side is canonically isomorphic to $\mathrm{Hom}_{\underline{\mathbf{DM}}^{\mathrm{eff}}}(\mathbf{Z}, \mathcal{F}_1[0] \otimes \cdots \otimes \mathcal{F}_n[0])$ by (3.3) and (2.8). This completes the proof of Theorem 6.2. □

7. Milnor K-theory

7.1
Our aim is to show that the map (6.3) is bijective. The first step is the special case of the multiplicative groups.

Recall that if C is a smooth projective connected curve over k, then the composition

$$K^M_{r+1}\big(k(C)\big) \xrightarrow{\partial_c} \bigoplus_{c\in C} K^M_r\big(k(c)\big) \xrightarrow{\oplus N_{k(c)/k}} K^M_r(k)$$

is the zero map by Weil reciprocity (see [4, Chapter I, (5.4)]). Here, for each closed point $c \in C$, we write $\partial_c : K^M_{r+1}(k(C)) \to K^M_r(k(c))$ and $N_{k(c)/k} : K^M_r(k(c)) \to K^M_r(k)$ for the tame symbol and the norm map. The tame symbol satisfies (and is characterized by) the property

$$\partial_c\big(\{a_1,\dots,a_n,f\}\big) = v_c(f)\big\{a_1(c),\dots,a_n(c)\big\}$$

for any $a_1,\dots,a_n \in \mathcal{O}^*_{C,c}$ and $f \in k(C)^*$.

PROPOSITION 7.2
When $\mathcal{F}_1 = \cdots = \mathcal{F}_n = \mathbf{G}_m$, the map (6.3) is bijective.

Proof
It suffices to show that relations (6.1) vanish in $K(k;\mathbf{G}_m,\dots,\mathbf{G}_m)$. Because of Somekawa's isomorphism [24, Theorem 1.4]

$$K(k;\mathbf{G}_m,\dots,\mathbf{G}_m) \simeq K^M_n(k) \tag{7.1}$$

given by $\{x_1,\dots,x_n\}_{E/k} \mapsto N_{E/k}(\{x_1,\dots,x_n\})$, it suffices to show this vanishing in the usual Milnor K-group $K^M_n(k)$, which follows from the Weil reciprocity recalled above. □

The following lemmas appear to be crucial in the proof of the main theorem.

LEMMA 7.3
Let C be a smooth projective connected curve over k, and let $Z = \{p_1,\dots,p_s\}$ be a finite set of closed points of C. If k is infinite, then we have that $K^M_2(k(C)) = \{k(C)^*, \mathcal{O}^*_{C,Z}\}$.

Proof
Let $\mathfrak{p}_i$ be the maximal ideal of $A = \mathcal{O}_{C,Z}$ corresponding to p_i. Since A is a semilocal principal ideal domain, we can choose generators $\pi_1,\dots,\pi_s$ of $\mathfrak{p}_1,\dots,\mathfrak{p}_s$. Since k is infinite, we can change π_i into $\mu_i\pi_i$ for suitable $\mu_1,\dots,\mu_s \in k^*$ to achieve $\pi_i + \pi_j \not\equiv 0 \pmod{\mathfrak{p}_k}$ for i,j,k all distinct. (Indeed, the set of bad $(\mu_1,\dots,\mu_s)$ is contained in a finite union of hyperplanes in $\bar{k}^s$.) It follows that $\pi_i + \pi_j \in A^*$ for all $i \neq j$.

By the relation $\{f,-f\} = 0$ $(f \in k(C)^*)$, we have that $K^M_2(k(C)) = \{A^*,A^*\} + \sum_{i<j}\{\pi_i,\pi_j\}$. Now the identity

$$\{\pi_i, \pi_j\} = \{\pi_i, \pi_j\} - \{-\pi_j, \pi_j\} = \{-\pi_i/\pi_j, \pi_j\}$$
$$= \{-\pi_i/\pi_j, \pi_j\} + \big\{-\pi_i/\pi_j, 1 + (\pi_i/\pi_j)\big\} = \{-\pi_i/\pi_j, \pi_i + \pi_j\}$$

proves the lemma. □

LEMMA 7.4
Let C be a smooth projective connected curve over k, let $Z \subset C$ be a proper closed subset, and let $r > 0$. If k is an infinite field, then $K^M_{r+1}k(C)$ is generated by elements of the form $\{a_1, \dots, a_{r+1}\}$ where the $a_i \in k(C)^*$ satisfy $\mathrm{Supp}(\mathrm{div}(a_i)) \cap Z = \emptyset$ *for all* $1 \le i \le r$ *and* $\mathrm{Supp}(\mathrm{div}(a_i)) \cap \mathrm{Supp}(\mathrm{div}(a_j)) = \emptyset$ *for all* $1 \le i < j \le r$.

Proof
We proceed by induction on r. The assertion follows from Lemma 7.3 when $r = 1$. Suppose that $r > 1$. Take $a_1, \dots, a_{r+1} \in k(C)^*$. By induction, there exist $b_{m,i} \in k(C)^*$ such that $\mathrm{Supp}(\mathrm{div}(b_{m,i})) \cap Z = \emptyset$ for all $i < r$ and m, $\mathrm{Supp}(\mathrm{div}(b_{m,i})) \cap \mathrm{Supp}(\mathrm{div}(b_{m,j})) = \emptyset$ for all $i < j < r$ and m, and

$$\{a_1, \dots, a_r\} = \sum_m \{b_{m,1}, \dots, b_{m,r}\}$$

holds in $K^M_r k(C)$. For each m, Lemma 7.3 shows that there exist $c_{m,i}, d_{m,i} \in k(C)^*$ such that

$$\mathrm{Supp}\big(\mathrm{div}(c_{m,i})\big) \cap \Big(Z \cup \bigcup_{j=1}^{r-1} \mathrm{Supp}\big(\mathrm{div}(b_{m,j})\big)\Big) = \emptyset$$

and that

$$\{b_{m,r}, a_{r+1}\} = \sum_i \{c_{m,i}, d_{m,i}\}$$

holds in $K^M_2 k(C)$. Then we have that

$$\{a_1, \dots, a_{r+1}\} = \sum_{m,i} \{b_{m,1}, \dots, b_{m,r-1}, c_{m,i}, d_{m,i}\}$$

in $K^M_{r+1}k(C)$, and we are done. □

8. K-groups of Milnor type

We now generalize the notion of Milnor K-groups to arbitrary homotopy invariant Nisnevich sheaves with transfers, although we shall seriously use this generalization only for special, representable sheaves.

8.1

Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$. We shall call a homomorphism $\mathbf{G}_m \to \mathcal{F}$ a *cocharacter* of $\mathcal{F}$. (By Proposition 4.3, the group $\mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathbf{G}_m, \mathcal{F})$ is canonically isomorphic to $\mathcal{F}_{-1}(k)$.)

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. Denote by $St(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ the subgroup of $(\mathcal{F}_1 \otimes_{\mathbf{PST}} \cdots \otimes_{\mathbf{PST}} \mathcal{F}_n)(k)$ generated by the elements

$$a_1 \otimes \cdots \otimes \chi_i(a) \otimes \cdots \otimes \chi_j(1-a) \otimes \cdots \otimes a_n \tag{8.1}$$

where $\chi_i : \mathbf{G}_m \to \mathcal{F}_i$, $\chi_j : \mathbf{G}_m \to \mathcal{F}_j$ are two cocharacters with $i < j$, $a \in k^* \setminus \{1\}$, and $a_m \in \mathcal{F}_m(k)$ $(m \neq i, j)$.

Definition 8.2

For $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$, we write $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ for the quotient of $(\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k)$ by the subgroup generated by $\mathrm{Tr}_{E/k}\, St(E; \mathcal{F}_1, \dots, \mathcal{F}_n)$, where E runs through all finite extensions of k. This is the *K-group of Milnor type* associated to $\mathcal{F}_1, \dots, \mathcal{F}_n$.

8.3

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. We have a canonical isomorphism

$$(\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k) \simeq (\mathbf{Z} \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathbf{Z} \overset{M}{\otimes} \mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k)$$

because $\mathbf{Z}$ is the unit object for the tensor structure of Mackey functors. Since there is no nontrivial cocharacter of $\mathbf{Z}$, it induces an isomorphism

$$\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \simeq \tilde{K}(k; \mathbf{Z}, \cdots, \mathbf{Z}, \mathcal{F}_{r+1}, \dots, \mathcal{F}_n).$$

8.4

The assignment $k \mapsto \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ inherits the structure of a cohomological Mackey functor, which is natural in $(\mathcal{F}_1, \dots, \mathcal{F}_n)$. In particular, the choice of elements $f_i \in \mathcal{F}_i(k) = \mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathbf{Z}, \mathcal{F}_i)$ for $i = 1, \dots, r$ induces a homomorphism

$$\tilde{K}(k; \mathcal{F}_{r+1}, \dots, \mathcal{F}_n) \simeq \tilde{K}(k; \mathbf{Z}, \dots, \mathbf{Z}, \mathcal{F}_{r+1}, \dots, \mathcal{F}_n) \to \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n). \tag{8.2}$$

LEMMA 8.5

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. The image of $St(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Consequently, we have a surjective homomorphism $\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \to K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ and a composite surjective homomorphism

$$\tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n) \longrightarrow\!\!\!\!\rightarrow K'(k; \mathcal{F}_1, \dots, \mathcal{F}_n). \tag{8.3}$$

Proof
This is a straightforward generalization of Somekawa's proof of [24, Theorem 1.4]. We need to show that the image of (8.1) vanishes in $K(k;\mathcal{F}_1,\dots,\mathcal{F}_n)$. By functoriality, we may assume that $\mathcal{F}_i=\mathcal{F}_j=\mathbf{G}_m$ for some $i<j$ and that χ_i,χ_j are the identity cocharacters. Given $a_m\in\mathcal{F}_m(k)$ $(m\neq i,j)$ and $a\in k^*\setminus\{1\}$, we put $a_i=1-at^{-1}, a_j=1-t\in\mathbf{G}_m(k(\mathbf{P}^1))=k(t)^*$. Then $(\mathbf{P}^1,t,(a_1,\dots,a_n))$ is a relation datum of Somekawa type. Note that $a_i\in\mathbf{G}_m(\mathbf{P}^1\setminus\{0,a\})$, and note that $a_j\in\mathbf{G}_m(\mathbf{P}^1\setminus\{1,\infty\})$. Direct computation shows that

$$a_j(0)=\partial_\infty(a_j,t)=\partial_1(a_j,t)=1,\qquad \partial_a(a_i,t)=a^{-1},\qquad a_j(a)=1-a.$$

Thus this relation datum yields the vanishing of

$$\{a_1,\dots,a_{i-1},a^{-1},a_{i+1},\dots,a_{j-1},1-a,a_{j+1},\dots,a_n\}_{k/k},$$

which is the negative of the image of (8.1) in $K(k;\mathcal{F}_1,\dots,\mathcal{F}_n)$. □

LEMMA 8.6
Let $\mathcal{F}_1,\dots,\mathcal{F}_n\in\mathbf{HI}_{\mathrm{Nis}}$, and let $\mathcal{G}'\twoheadrightarrow\mathcal{G}''$ be an epimorphism in $\mathbf{HI}_{\mathrm{Nis}}$. If (8.3) is bijective for $(\mathcal{G}',\mathcal{F}_1,\dots,\mathcal{F}_n)$, it is bijective for $(\mathcal{G}'',\mathcal{F}_1,\dots,\mathcal{F}_n)$.

Proof
Let $\mathcal{G}=\mathrm{Ker}(\mathcal{G}'\to\mathcal{G}'')$. For $\mathcal{H}\in\{\mathcal{G},\mathcal{G}',\mathcal{G}''\}$, we put $\tilde{K}_{\mathcal{H}}=\tilde{K}(k;\mathcal{H},\mathcal{F}_1,\dots,\mathcal{F}_n)$, $K'_{\mathcal{H}}=K'(k;\mathcal{H},\mathcal{F}_1,\dots,\mathcal{F}_n)$. In the commutative diagram

$$\begin{tikzcd}
\tilde{K}_{\mathcal{G}} \arrow[r] \arrow[d] & \tilde{K}_{\mathcal{G}'} \arrow[r, "f"] \arrow[d] & \tilde{K}_{\mathcal{G}''} \arrow[r] \arrow[d] & 0 \\
K'_{\mathcal{G}} \arrow[r] & K'_{\mathcal{G}'} \arrow[r] & K'_{\mathcal{G}''} \arrow[r] & 0
\end{tikzcd}$$

the upper row is a complex and f is surjective. The lower row is exact because of Theorem 6.2 and Lemma 3.3, and all vertical arrows are surjective. The assertion now follows from a diagram chase. □

8.7
Let E/k be an étale k-algebra, and let $\mathcal{F}\in\mathbf{HI}_{\mathrm{Nis}}$. We define the *Weil restriction* $R_{E/k}\mathcal{F}\in\mathbf{HI}_{\mathrm{Nis}}$ of $\mathcal{F}$ by the formula $R_{E/k}\mathcal{F}(U)=\mathcal{F}(U\times_k E)$ for all smooth varieties U. If $\mathcal{F}$ is a semiabelian variety, then $R_{E/k}\mathcal{F}$ is the (usual) Weil restriction of $\mathcal{F}$.

LEMMA 8.8
Let E/k be a finite extension. Let $\mathcal{F}_1,\dots,\mathcal{F}_{n-1}\in\mathbf{HI}_{\mathrm{Nis}}$, and let $\mathcal{F}_n$ be a Nisnevich

sheaf with transfers over E. We have canonical isomorphisms

$$K(k;\mathcal{F}_1,\dots,\mathcal{F}_{n-1},R_{E/k}\mathcal{F}_n)\simeq K(E;\mathcal{F}_1,\dots,\mathcal{F}_n),$$
$$K'(k;\mathcal{F}_1,\dots,\mathcal{F}_{n-1},R_{E/k}\mathcal{F}_n)\simeq K'(E;\mathcal{F}_1,\dots,\mathcal{F}_n),$$
$$\tilde{K}(k;\mathcal{F}_1,\dots,\mathcal{F}_{n-1},R_{E/k}\mathcal{F}_n)\simeq \tilde{K}(E;\mathcal{F}_1,\dots,\mathcal{F}_n).$$

Proof

The first isomorphism was constructed in [26, Lemma 4] when $\mathcal{F}_1,\dots,\mathcal{F}_n$ are semi-abelian varieties. The same construction works for arbitrary $\mathcal{F}_1,\dots,\mathcal{F}_n$ and also for K' and $\tilde{K}$. □

8.9

If $\mathcal{F}_1=\cdots=\mathcal{F}_n=\mathbf{G}_m$, (8.3) is bijective by Proposition 7.2. This is false in general, for example, if all the $\mathcal{F}_i$'s are proper (Definition 10.1) and $n>1$. However, we have the following.

PROPOSITION 8.10

(a) *Let $\mathcal{F}_1=\mathcal{F}_1'\oplus\mathcal{F}_1''$. Then the natural map*

$$\tilde{K}(k;\mathcal{F}_1,\dots,\mathcal{F}_n)\to\tilde{K}(k;\mathcal{F}_1',\dots,\mathcal{F}_n)\oplus\tilde{K}(k;\mathcal{F}_1'',\dots,\mathcal{F}_n)$$

is bijective.

(b) *Let $T_1,\dots,T_n$ be tori. Assume that, for each i, there exists an exact sequence of tori*

$$0\to P_i^1\to P_i^0\to T_i\to 0$$

where P_i^0 and P_i^1 are invertible tori (i.e., direct summands of permutation tori). Then (8.3) is bijective for $\mathcal{F}_i=T_i$.

Proof

(a) This is formal, as $\tilde{K}(k;\mathcal{F}_1,\dots,\mathcal{F}_n)$ is a quotient of the multiadditive multifunctor $(\mathcal{F}_1\overset{M}{\otimes}\cdots\overset{M}{\otimes}\mathcal{F}_n)(k)$ (see Section 8.4).

(b) Note that, by Hilbert's Theorem 90, the sequences $0\to P_i^1\to P_i^0\to T_i\to 0$ are exact in $\mathbf{HI}_{\mathrm{Nis}}$. Lemma 8.6 reduces the problem to the case where all the T_i's are permutation tori. Then Lemma 8.8 reduces the problem to the case where all the T_i's are split tori. Finally, we reduce the problem to $\mathcal{F}_1=\cdots=\mathcal{F}_n=\mathbf{G}_m$ by (a). □

Question 8.11

Is Proposition 8.10(b) true for general tori?

8.12
Let $T_1,\dots,T_n$ be as in Proposition 8.10(b). Let C/k be a smooth projective connected curve with function field K, and let $v\in C$ be a closed point. Put $\mathcal{F}=T_1\otimes_{\mathbf{HI}_{\mathrm{Nis}}}\cdots\otimes_{\mathbf{HI}_{\mathrm{Nis}}}T_n$. By Theorem 4.7 and (4.3), we get a residue map $\partial_v:\mathcal{F}\otimes_{\mathbf{HI}_{\mathrm{Nis}}}\mathbf{G}_m(K)\to\mathcal{F}(k(v))$. From Lemma 3.5, Theorem 6.2, and Proposition 8.10(b), this can be reinterpreted as

$$\partial_v:\tilde{K}(K;T_1,\dots,T_n,\mathbf{G}_m)\to\tilde{K}\big(k(v);T_1,\dots,T_n\big). \tag{8.4}$$

As v varies, these maps satisfy the reciprocity law of Proposition 4.6 and the compatibility of Lemma 4.2.

9. Reduction to the representable case

Following [31, p. 207], we write that $h_0^{\mathrm{Nis}}(X):=h_0^{\mathrm{Nis}}(L(X))$ for a smooth variety X over k.

PROPOSITION 9.1
The following statements are equivalent:

(a) *The homomorphism (6.3) is bijective for any $\mathcal{F}_1,\dots,\mathcal{F}_n\in\mathbf{HI}_{\mathrm{Nis}}$.*

(b) *Let $\mathcal{F}_1=\cdots=\mathcal{F}_n=h_0^{\mathrm{Nis}}(C')$ for a smooth connected curve C'/k. Then (6.3) is bijective.*

(c) *Let C be a smooth projective connected curve over k, and let $f:C\to\mathbf{P}^1$ be a surjective morphism. Let $C'=f^{-1}(\mathbf{P}^1\setminus\{1\})$. Let $\iota:L(C')\to h_0^{\mathrm{Nis}}(C')=:\mathcal{A}$ be the canonical surjection, which we regard as an element of $\mathcal{A}(C')$. These data define a relation datum of geometric type $(C,f,(\iota,\dots,\iota))$ for $\mathcal{F}_1=\cdots=\mathcal{F}_n=\mathcal{A}$, and its associated element (6.1) is*

$$\sum_{c\in C'}v_c(f)\,\mathrm{Tr}_{k(c)/k}\big(\iota(c)\otimes\cdots\otimes\iota(c)\big)\in\mathcal{A}\overset{M}{\otimes}\cdots\overset{M}{\otimes}\mathcal{A}(k). \tag{9.1}$$

Then the image of (9.1) in $K(k;\mathcal{A},\dots,\mathcal{A})$ vanishes.

Proof
Only the implication (c) $\Rightarrow$ (a) requires a proof. Let $(C,f,(g_i))$ be a relation datum of geometric type for $\mathcal{F}_1,\dots,\mathcal{F}_n$. We need to show the vanishing of

$$\sum_{c\in C'}v_c(f)\big\{g_1(c),\dots,g_n(c)\big\}_{k(c)/k}\quad\text{in } K(k;\mathcal{F}_1,\dots,\mathcal{F}_n). \tag{9.2}$$

For each $i=1,\dots,n$, the section $g_i:L(C')\to\mathcal{F}_i$ factors through a morphism $\varphi_i:\mathcal{A}\to\mathcal{F}_i$ since $\mathcal{F}_i$ is homotopy invariant. The homomorphism $K(k;\mathcal{A},\dots,\mathcal{A})\to K(k;\mathcal{F}_1,\dots,\mathcal{F}_n)$ defined by $(\varphi_1,\dots,\varphi_n)$ sends the image of (9.1) in $K(k;\mathcal{A},\dots,\mathcal{A})$ to (9.2). Hence (9.2) vanishes by the assumption (c). $\square$

10. Proper sheaves

Definition 10.1
Let $\mathcal{F}$ be a Nisnevich sheaf with transfers. We call $\mathcal{F}$ *proper* if, for any smooth curve C over k and any closed point $c \in C$, the induced map $\mathcal{F}(\mathcal{O}_{C,c}) \to \mathcal{F}(k(C))$ is surjective. We say that $\mathcal{F}$ is *universally proper* if the above condition holds when replacing k by any finitely generated extension K/k and C by any regular K-curve.

Example 10.2

(a) A semiabelian variety G over k is proper in the sense of Definition 10.1 if and only if G is an abelian variety.

(b) Recall from [12] that $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ is called *birational* if $\mathcal{F}(X) \to \mathcal{F}(U)$ is bijective for any smooth k-variety X and any open dense subset $U \subset X$. A birational sheaf $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ is by definition proper. Examples of birational sheaves will be given in Lemma 11.2(b) below. In particular, if C is a smooth proper curve, then $h_0^{\mathrm{Nis}}(C)$ is proper.

In fact, the following holds.

LEMMA 10.3
Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$.

(a) *$\mathcal{F}$ is proper if and only if $\mathcal{F}(C) \xrightarrow{\sim} \mathcal{F}(k(C))$ for any smooth k-curve C.*

(b) *$\mathcal{F}$ is universally proper if and only if it is birational (see Example 10.2(b)).*

Proof
Let us prove (b), as the proof of (a) is a subset of it. It is obvious from the definitions that birational sheaves are universally proper. Conversely, assume $\mathcal{F}$ to be universally proper. Let X be a smooth k-variety. By [30, Corollary 4.19], the map $\mathcal{F}(X) \to \mathcal{F}(U)$ is injective for any dense open subset of X. Let $x \in X^{(1)}$, and let $p : X \to \mathbf{A}^{d-1}$ be a dominant rational map defined at x, where $d = \dim X$. (We may find such a p thanks to Noether's normalization theorem.) Applying the hypothesis to the generic fiber of p, we find that $\mathcal{F}(\mathcal{O}_{X,x}) \to \mathcal{F}(k(X))$ is surjective. Since this is true for all points $x \in X^{(1)}$, we get the surjectivity of $\mathcal{F}(X) \to \mathcal{F}(k(X))$ from Voevodsky's Gersten resolution [30, Theorem 4.37]. □

The following proposition is not necessary for the proof of the main theorem, but its proof is much simpler than the general case.

PROPOSITION 10.4

Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$. Assume that $\mathcal{F}_1, \dots, \mathcal{F}_{n-1}$ are proper. Then the homomorphism (6.3) is bijective.

Proof

Suppose that $(C, f, (g_i)_{i=1,\dots,n})$ is a relation datum of geometric type for $(\mathcal{F}_1, \dots, \mathcal{F}_n)$. It suffices to show the vanishing of the image of

$$\sum_{c \in C'} v_c(f)\,\mathrm{Tr}_{k(c)/k}\big(g_1(c) \otimes \cdots \otimes g_n(c)\big) \in (\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n)(k) \tag{10.1}$$

in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$. Let $\bar{g}_i$ be the image of g_i in $\mathcal{F}(k(C))$. By assumption we have that $\bar{g}_i \in \mathrm{Im}(\mathcal{F}_i(\mathcal{O}_{C,c}) \to \mathcal{F}_i(k(C)))$ for all $c \in C$ and $i = 1, \dots, n-1$. Hence $(C, f, (\bar{g}_i)_{i=1,\dots,n})$ is a relation datum of Somekawa type (with $i(c) = n$ for all $c \in C$). By Corollary 4.11, the element (10.1) coincides with

$$\sum_{c \in C'} \mathrm{Tr}_{k(c)/k}\big(g_1(c) \otimes \cdots \otimes g_{n-1}(c) \otimes \partial_c(g_n, f)\big);$$

hence, its image in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes by Definition 5.1. □

11. Main theorem

Definition 11.1

Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$. We say that $\mathcal{F}$ is *curvelike* if there exists an exact sequence in $\mathbf{HI}_{\mathrm{Nis}}$

$$0 \to T \to \mathcal{F} \to \bar{\mathcal{F}} \to 0, \tag{11.1}$$

where $\bar{\mathcal{F}}$ is proper (Definition 10.1) and T sits in an exact sequence in $\mathbf{HI}_{\mathrm{Nis}}$

$$0 \to R_{E_1/k}\mathbf{G}_m \to R_{E_2/k}\mathbf{G}_m \to T \to 0, \tag{11.2}$$

where E_1 and E_2 are étale k-algebras.[†]

This terminology is justified by the following lemma.

LEMMA 11.2

(a) *If C is a smooth curve over k, then $h_0^{\mathrm{Nis}}(C)$ is the Nisnevich sheaf associated to the presheaf of relative Picard groups*

$$U \mapsto \mathrm{Pic}(\bar{C} \times U, D \times U),$$

[†]It then follows from Hilbert's Theorem 90 applied to $R_{E_1/k}\mathbf{G}_m$ that $T = T_{\text{ét}}$; hence, T agrees with the cokernel of $R_{E_1/k}\mathbf{G}_m \to R_{E_2/k}\mathbf{G}_m$ as tori. We shall not need this remark in the remainder of our paper.

where $\bar{C}$ is the smooth projective completion of C, $D = \bar{C} \setminus C$, and U runs through smooth k-schemes.

(b) *If X is a smooth projective variety over k, then, for any smooth variety U over k, we have that*

$$h_0^{\mathrm{Nis}}(X)(U) = CH_0(X_{k(U)}), \tag{11.3}$$

where $k(U)$ denotes the total ring of fractions of U. In particular, $h_0^{\mathrm{Nis}}(X)$ is birational.

(c) *For any smooth curve C, $h_0^{\mathrm{Nis}}(C)$ is curvelike.*

Proof

Parts (a) and (b) are proven in [28, Theorem 3.1] and [8, Theorem 2.2], respectively. We prove (c). This follows from (b) if C is projective over k. We assume that C is affine. With the notation of (a), we have the Gysin exact triangle [31, Proposition 3.5.4]

$$M(D)(1)[1] \to M(C) \to M(\bar{C}) \xrightarrow{+1}.$$

By [31, Theorem 3.4.2], we have that $h_i^{\mathrm{Nis}}(C) = 0$ for all $i \neq 0$ and $h_1^{\mathrm{Nis}}(\bar{C}) = R_{E/k}\mathbf{G}_m$, where $E = H^0(\bar{C}, \mathcal{O}_{\bar{C}})$. Hence we get an exact sequence

$$0 \to R_{E/k}\mathbf{G}_m \to R_{D/k}\mathbf{G}_m \to h_0^{\mathrm{Nis}}(C) \to h_0^{\mathrm{Nis}}(\bar{C}) \to 0,$$

which proves (c). □

Remark 11.3

Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ be curvelike. The sheaves T and $\bar{\mathcal{F}}$ in (11.1) are uniquely determined by $\mathcal{F}$ up to unique isomorphism. Indeed, this amounts to showing that any morphism $T \to \bar{\mathcal{F}}$ is trivial. This is reduced to the case $T = R_{E/k}\mathbf{G}_m$ as in (11.2), and further to $T = \mathbf{G}_m$ by adjunction as in Lemma 8.8. Then $\mathrm{Hom}_{\mathbf{HI}_{\mathrm{Nis}}}(\mathbf{G}_m, \bar{\mathcal{F}}) \simeq \bar{\mathcal{F}}_{-1}(k) = 0$ by definition (see (4.1) and Definition 10.1).

We call T and $\bar{\mathcal{F}}$ the *toric* and *proper* parts of $\mathcal{F}$, respectively (see the footnote on p. 2778).

LEMMA 11.4

(a) *Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$ be curvelike with toric part T, and let C be a smooth proper connected k-curve. Let Z be a proper closed subset of C, let $A = \mathcal{O}_{C,Z}$, and let $K = k(C)$. Then the sequence*

$$0 \to T(A) \xrightarrow{f} T(K) \oplus \mathcal{F}(A) \xrightarrow{g} \mathcal{F}(K) \to 0$$

is exact, where f and g are given by $f(a) = (a,a)$ and $g(b,c) = b - c$, under the identification $T(A) \subset \mathcal{F}(A) \subset \mathcal{F}(K)$ and $T(A) \subset T(K) \subset \mathcal{F}(K)$.

(b) *Let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ be curvelike with toric parts $T_1, \dots, T_n$, and let C, Z, A, K be as in (a). Then the group $\mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K)$ has the following presentation:*

Generators: *For each subset $I \subseteq \{1, \dots, n\}$, elements $[I; f_1, \dots, f_n]$ with $f_i \in \mathcal{F}_i(A)$ if $i \in I$ and $f_i \in T_i(K)$ if $i \notin I$.*

Relations:

- *Multilinearity:*

$$[I; f_1, \dots, f_i + f_i', \dots, f_n] = [I; f_1, \dots, f_i, \dots, f_n] + [I; f_1, \dots, f_i', \dots, f_n].$$

- *Let $I \subsetneq \{1, \dots, n\}$, and let $i_0 \notin I$. Let $[I; f_1, \dots, f_n]$ be a generator. Suppose that $f_{i_0} \in T_{i_0}(A)$. Then $[I; f_1, \dots, f_n] = [I \cup \{i_0\}; f_1, \dots, f_n]$.*

Proof

Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T(A) \arrow[r] \arrow[d, "a"] & \mathcal{F}(A) \arrow[r] \arrow[d, "b"] & \bar{\mathcal{F}}(A) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & T(K) \arrow[r] \arrow[d] & \mathcal{F}(K) \arrow[r] \arrow[d] & \bar{\mathcal{F}}(K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus T_{-1}(k_i) \arrow[r] \arrow[d] & \bigoplus \mathcal{F}_{-1}(k_i) \arrow[r] \arrow[d] & \bigoplus \bar{\mathcal{F}}_{-1}(k_i) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here the k_i's run through the residue fields of points of Z and the (exact) vertical sequences are those from [30, Theorem 4.37]. Since $0 \to T \to \mathcal{F} \to \bar{\mathcal{F}} \to 0$ is an exact sequence of Nisnevich sheaves and any field has Nisnevich cohomological dimension 0, all horizontal sequences are left exact and the middle one is also exact at $\bar{\mathcal{F}}(K)$. (See Section 4.1 for the exactness of the bottom row.) Since $\bar{\mathcal{F}}$ is proper, we have that $\bar{\mathcal{F}}_{-1}(k_i) = 0$ (see the end of Remark 11.3); it follows that c is an isomorphism and that the upper horizontal sequence is also exact at $\bar{\mathcal{F}}(A)$. Now (a) follows from a diagram chase and (b) follows from (a). □

Remark 11.5

A shorter but more delicate proof is that the maps a, b, c have compatible retractions. Since C is a curve, this may be deduced from the proof of [31, Lemma 4.5] (see also [31, Corollary 4.18]).

PROPOSITION 11.6

Let C/k be a smooth proper connected curve, let $v \in C$, and let $K = k(C)$. Then there exists a unique law associating to a system $(\mathcal{F}_1, \dots, \mathcal{F}_n)$ of n curvelike sheaves a homomorphism

$$\partial_v : \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* \to \tilde{K}\big(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n\big)$$

such that

(a) *If σ is a permutation of $\{1, \dots, n\}$, the diagram*

$$\begin{array}{ccc} \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}\big(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n\big) \\ \sigma\big\downarrow & & \sigma\big\downarrow \\ \mathcal{F}_{\sigma(1)}(K) \otimes \cdots \otimes \mathcal{F}_{\sigma(n)}(K) \otimes K^* & \xrightarrow{\partial_v} & \tilde{K}\big(k(v); \mathcal{F}_{\sigma(1)}, \dots, \mathcal{F}_{\sigma(n)}\big) \end{array}$$

commutes.

(b) *If $[I, f_1, \dots, f_n]$ is a generator of $\mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K)$ as in Lemma 11.4(b) for some Z containing v, with $I = \{1, \dots, i\}$, then*

$$\partial_v(f_1 \otimes \cdots \otimes f_n \otimes f) = \big\{f_1(v), \dots, f_i(v), \partial_v\big(\{f_{i+1}, \dots, f_n, f\}_{K/K}\big)\big\}_{k(v)/k}$$

where $\partial_v(\{f_{i+1}, \dots, f_n, f\}_{K/K})$ is the residue (8.4).

Proof

By Lemma 11.4(b), it suffices to check that ∂_v agrees on relations. Up to permutation, we may assume that $I = \{1, \dots, i\}$ and $i_0 = i + 1$. The claim then follows from Proposition 4.10. □

LEMMA 11.7

(a) *Keep the notation of Proposition 11.6. Let L/K be a finite extension, write D for the smooth projective model of L, and write $h : D \to C$ for the corresponding morphism. Let $Z = h^{-1}(v)$. Write $\mathcal{F}_{n+1} = \mathbf{G}_m$. Then, for any $i \in \{1, \dots, n+1\}$, the diagram*

$$\begin{tikzcd}
& \mathcal{F}_1(L)\otimes\cdots\otimes\mathcal{F}_{n+1}(L) \arrow[r, "(\partial_w)"] & \bigoplus_{w\in Z}\tilde{K}\big(k(w);\mathcal{F}_1,\dots,\mathcal{F}_n\big) \arrow[dd, "(\mathrm{Tr}_{k(w)/k(v)})"] \\
\mathcal{F}_1(K)\otimes\cdots\otimes\mathcal{F}_i(L)\otimes\cdots\otimes\mathcal{F}_{n+1}(K) \arrow[ur, "u"] \arrow[dr, "d"'] & & \\
& \mathcal{F}_1(K)\otimes\cdots\otimes\mathcal{F}_{n+1}(K) \arrow[r, "\partial_v"] & \tilde{K}\big(k(v);\mathcal{F}_1,\dots,\mathcal{F}_n\big)
\end{tikzcd}$$

commutes, where u is given componentwise by functoriality for $j \neq i$ and by the identity for $j = i$, and d is given componentwise by the identity for $j \neq i$ and by $\mathrm{Tr}_{L/K}$ for $j = i$.

(b) *The homomorphisms ∂_v's induce residue maps*

$$\partial_v : (\mathcal{F}_1 \overset{M}{\otimes} \cdots \overset{M}{\otimes} \mathcal{F}_n \overset{M}{\otimes} \mathbf{G}_m)(K) \to \tilde{K}\big(k(v);\mathcal{F}_1,\dots,\mathcal{F}_n\big),$$

which verify the compatibility of Lemma 4.2(b).

Proof

(a) For clarity, we distinguish two cases: $i < n+1$ and $i = n+1$. In the former case, up to permutation we may assume that $i = n$. It is enough to check commutativity on generators in the style of Lemma 11.4(b). Let T_l denote the toric part of $\mathcal{F}_l$. In view of Lemma 11.4(a) and Proposition 11.6(a), it suffices to check the commutativity for $x = f_1 \otimes \cdots \otimes f_n \otimes f$ when one of the following two conditions is satisfied:

(1) For some $j \in \{0,\dots,n-1\}$, $f_l \in \mathcal{F}_l(\mathcal{O}_{C,v})$ $(1 \le l \le j)$, $f_l \in T_l(K)$ $(j+1 \le l \le n-1)$, $f_n \in T_n(L)$, and $f \in K^*$.

(2) For some $j \in \{0,\dots,n-1\}$, $f_l \in \mathcal{F}_l(\mathcal{O}_{C,v})$ $(1 \le l \le j)$, $f_l \in T_l(K)$ $(j+1 \le l \le n-1)$, $f_n \in \mathcal{F}_n(\mathcal{O}_{D,Z})$, and $f \in K^*$.

Let $w \in Z$. If (1) holds, we have that

$$\partial_w\big(u(x)\big) = \big\{f_1(w),\dots,f_j(w),\partial_w\big(\{f_{j+1},\dots,f_n,f\}_{L/L}\big)\big\}_{k(w)/k(w)}$$

and

$$\partial_v\big(d(x)\big) = \big\{f_1(v),\dots,f_j(v),\partial_v\big(\big\{f_{j+1},\dots,\mathrm{Tr}_{L/K}(f_n),f\big\}_{K/K}\big)\big\}_{k(v)/k(v)}.$$

Observe that the restriction of $f_l(v)$ to $k(w)$ is $f_l(w)$ for every $w \in Z$ and $l = 1,\dots,j$. Since the residue maps (∂_w) (8.4) verify the compatibility of Lemma 4.2, the commutativity for x follows. (Recall that $\mathrm{Tr}_{k(w)/k(v)}(\{a_1,\dots,a_n\}_{k(w)/k(w)}) = \{a_1,\dots,a_n\}_{k(w)/k(v)}$.)

If (2) holds, we have that

$$\partial_w\big(u(x)\big) = \big\{f_1(w), \dots, f_j(w), \partial_w\big(\{f_{j+1}, \dots, f_{n-1}, f\}_{L/L}\big), f_n(w)\big\}_{k(w)/k(w)}$$

and

$$\partial_v\big(d(x)\big) = \big\{f_1(v), \dots, f_j(v), \\ \partial_v\big(\{f_{j+1}, \dots, f_{n-1}, f\}_{K/K}\big), \mathrm{Tr}_{L/K}(f_n)(v)\big\}_{k(v)/k(v)}.$$

In addition to the observation mentioned in (1), we remark that the restriction of $\partial_v(\{f_{j+1}, \dots, f_{n-1}, f\}_{K/K})$ to $k(w)$ is $\partial_w(\{f_{j+1}, \dots, f_{n-1}, f\}_{L/L})$ for every $w \in Z$. The commutativity for x follows from Lemma 4.2(b) applied to $\mathcal{F}_n$.

If $i = n+1$, the check is similar, with the projection formula working on the last variable.

Now (b) follows from (a) and the definition of $\overset{M}{\otimes}$ from Section 2.8. □

LEMMA 11.8

The homomorphisms ∂_v's of Lemma 11.7 induce residue maps

$$\partial_v : \tilde{K}(K; \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m) \to \tilde{K}\big(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n\big),$$

which verify the compatibility of Lemma 4.2(b).

Proof

Set $\mathcal{F}_{n+1} = \mathbf{G}_m$. Let $i < j$ be two elements of $\{1, \dots, n+1\}$, and let $\chi_i : \mathbf{G}_m \to \mathcal{F}_i$, $\chi_j : \mathbf{G}_m \to \mathcal{F}_j$ be two cocharacters. Let $f \in K^* - \{1\}$. We must show that ∂_v vanishes on

$$x = f_1 \otimes \cdots \otimes \chi_i(f) \otimes \cdots \otimes \chi_j(1-f) \otimes \cdots \otimes f_{n+1}$$

for any $(f_1, \dots, f_{n+1}) \in \mathcal{F}_1(K) \times \cdots \times \mathcal{F}_{n+1}(K)$ (product excluding (i, j)). By functoriality, we may assume that χ_i, χ_j are the identity cocharacters. We distinguish two cases for clarity: $j < n+1$ and $j = n+1$. But exactly the same argument works for both cases. Presently we suppose $j < n+1$.

Up to permutation, we may assume that $i = n-1$, $j = n$. Let us say that an element $(x_1, \dots, x_{n-2}) \in \mathcal{F}_1(K) \times \cdots \times \mathcal{F}_{n-2}(K)$ is *in normal form* if, for each $s = 1, \dots, n-2$, either $x_s \in \mathcal{F}_s(\mathcal{O}_v)$ or $x_s \in T_s(K)$. (Here T_s is the toric part of $\mathcal{F}_s$.) Then Lemma 11.4 reduces the problem to the case where $(f_1, \dots, f_{n-2})$ is in normal form. Up to permutation, we may assume that $f_s \in \mathcal{F}_s(\mathcal{O}_v)$ for $s \le r$ and $f_s \in T_s(K)$ for $r < s \le n-2$. Then

$$\partial_v x = \big\{f_1(v), \dots, f_r(v), \partial_v\big(\big\{f_{r+1}, \dots, f_{n-2}, f, (1-f), f_{n+1}\big\}_{K/K}\big)\big\}_{k(v)/k(v)}.$$

Let $\varphi_v : \tilde{K}(k(v), T_{r+1}, \dots, T_n) \to \tilde{K}(k(v), \mathcal{F}_1, \dots, \mathcal{F}_n)$ be the homomorphism induced by $(f_1(v), \dots, f_r(v))$ via (8.2), and let $\varphi_K : T_{r+1}(K) \otimes \cdots \otimes T_n(K) \otimes K^* \to \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^*$ be the analogous homomorphism defined by $(f_1, \dots, f_r)$. The diagram

$$\begin{tikzcd} T_{r+1}(K) \otimes \cdots \otimes T_n(K) \otimes K^* \arrow[r, "\partial_v"] \arrow[d, "\varphi_K"'] & \tilde{K}\big(k(v); T_{r+1}, \dots, T_n\big) \arrow[d, "\varphi_v"'] \\ \mathcal{F}_1(K) \otimes \cdots \otimes \mathcal{F}_n(K) \otimes K^* \arrow[r, "\partial_v"] & \tilde{K}\big(k(v); \mathcal{F}_1, \dots, \mathcal{F}_n\big) \end{tikzcd}$$

commutes. But the top map factors through

$$\partial_v : \tilde{K}(K; T_{r+1}, \dots, T_n, \mathbf{G}_m) \to \tilde{K}\big(k(v); T_{r+1}, \dots, T_n\big)$$

obtained in (8.4), hence the desired vanishing.

Thus we have shown that the map ∂_v of Proposition 11.6 vanishes on $St(K; \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$. The conclusion now follows from Lemma 11.7(b). □

11.9

Let $\mathcal{F} \in \mathbf{HI}_{\mathrm{Nis}}$, and let C be a smooth proper k-curve. The *support* of a section $f \in \mathcal{F}(k(C))$ is the finite set

$$\operatorname{Supp}(f) = \big\{c \in C \;\big|\; f \notin \mathcal{F}(\mathcal{O}_{C,c})\big\}.$$

The following lemma and proposition generalize Lemma 7.4.

LEMMA 11.10

Let $T_1, \dots, T_r$ be r curvelike tori. Put $T_{r+1} = \mathbf{G}_m$. Let D be a smooth proper k-curve, and let $Z \subset D$ be a proper closed subset. If the field k is infinite, the group $\tilde{K}(k(D); T_1, \dots, T_r, \mathbf{G}_m)$ is generated by elements $\{f_1, \dots, f_{r+1}\}_{k(E)/k(D)}$ where E is another curve, $p : E \to D$ is a finite surjective morphism, and $f_i \in T_i(k(E))$ satisfy

$$\begin{aligned} \operatorname{Supp}(f_i) \cap p^{-1}(Z) &= \emptyset \quad \textit{for all } 1 \le i \le r, \\ \operatorname{Supp}(f_i) \cap \operatorname{Supp}(f_j) &= \emptyset \quad \textit{for all } 1 \le i < j \le r. \end{aligned} \tag{11.4}$$

Proof

As in the proof of Proposition 8.10(b), we reduced the problem to the case where all the T_i's are $R_{E_i/k}\mathbf{G}_m$ for some étale k-algebras E_i/k. Using the formula

$$(R_{E_1/k}\mathbf{G}_{m,E_1})_{E_2} \simeq R_{E_1 \otimes_k E_2/E_2}\mathbf{G}_{m,E_1 \otimes E_2}$$

and Lemma 8.8 repeatedly, we further reduce the problem to the case where all the T_i's are $\mathbf{G}_m$. Then it follows from Lemma 7.4. □

PROPOSITION 11.11
Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be n curvelike sheaves, and let C be a smooth proper k-curve. Put $\mathcal{F}_{n+1} = \mathbf{G}_m$. If the field k is infinite, the group $\tilde{K}(k(C); \mathcal{F}_1, \ldots, \mathcal{F}_n, \mathbf{G}_m)$ is generated by elements $\{f_1, \ldots, f_{n+1}\}_{k(D)/k(C)}$, where D is another curve, $D \to C$ is a finite surjective morphism, and $f_i \in \mathcal{F}_i(k(D))$ satisfy

$$\operatorname{Supp}(f_i) \cap \operatorname{Supp}(f_j) = \emptyset \quad \textit{for all } 1 \le i < j \le n. \tag{11.5}$$

Proof
Let T_i be the toric part of $\mathcal{F}_i$. Given a finite surjective morphism $D \to C$ and $f_i \in \mathcal{F}_i(k(D))$ $(i = 1, \ldots, n+1)$, we construct a sequence $(Z_i, g_i^{(1)}, g_i^{(2)})_{i=1,\ldots,n+1}$ of closed subsets $Z_i \subset D$ and sections $g_i^{(1)} \in \mathcal{F}_i(\mathcal{O}_{D,Z_i})$, $g_i^{(2)} \in T_i(k(D))$ such that $f_i = g_i^{(1)} + g_i^{(2)}$ by induction. First we put $Z_1 = \emptyset$, $g_1^{(1)} = f_1$, and $g_1^{(2)} = 0$. Suppose that we have constructed $(Z_{i-1}, g_{i-1}^{(1)}, g_{i-1}^{(2)})$. We define $Z_i = Z_{i-1} \cup \operatorname{Supp}(g_{i-1}^{(1)})$. Then we apply Lemma 11.4(a) to find $g_i^{(1)} \in \mathcal{F}_i(\mathcal{O}_{D,Z_i})$ and $g_i^{(2)} \in T_i(k(D))$ such that $f_i = g_i^{(1)} + g_i^{(2)}$. By construction, we have that

$$\operatorname{Supp}(g_i^{(1)}) \cap \operatorname{Supp}(g_j^{(1)}) = \emptyset$$

for all $1 \le i < j \le n+1$, and

$$\{f_1, \ldots, f_{n+1}\}_{k(D)/k(C)} = \sum_{\mathbf{e} \in \{1,2\}^n} \{g_1^{(e_1)}, \ldots, g_{n+1}^{(e_{n+1})}\}_{k(D)/k(C)},$$

where $\mathbf{e} = (e_1, \ldots, e_{n+1})$. Given $\mathbf{e} \in \{1,2\}^n$, let $I = \{i \in \{1, \ldots, n\} \mid e_i = 1\}$. The collection of $g_i^{(1)}$ for $i \in I$ defines a homomorphism

$$\tilde{K}\big(k(D); T_{i_1}, \ldots, T_{i_m}, \mathbf{G}_m\big) \to \tilde{K}\big(k(D); \mathcal{F}_1, \ldots, \mathcal{F}_n, \mathbf{G}_m\big),$$

where $i_1 < \cdots < i_m$ are the elements of $\{1, \ldots, n\} \setminus I$. The proposition then follows by applying Lemma 11.10 with $Z = \bigcup_{e_i=1} \operatorname{Supp}(g_i^{(1)})$ for each $\mathbf{e}$. □

LEMMA 11.12
Let $C, D, \mathcal{F}_1, \ldots, \mathcal{F}_n$ be as in Proposition 11.11. Let $f_i \in \mathcal{F}_i(k(D))$, and let $v \in D$. Put $\xi := \{f_1, \ldots, f_{n+1}\}_{k(D)/k(C)}$, regarded as an element of $\tilde{K}(k(C); \mathcal{F}_1, \ldots, \mathcal{F}_n, \mathbf{G}_m)$.

(a) *If $v(f_{n+1} - 1) > 0$, then $\partial_v(\xi) = 0$.*

(b) *Suppose that (11.5) holds. If $v \in \operatorname{Supp}(f_i)$ for some $1 \le i \le n$, then*

$$\partial_v(\xi) = \big\{f_1(v), \ldots, \partial_v(f_i, f_{n+1}), \ldots, f_n(v)\big\}_{k(v)/k}.$$

Proof
This follows from Corollary 4.11 and Proposition 4.10. □

PROPOSITION 11.13
Let C be a smooth projective connected curve, and let $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$ be curvelike. The composition

$$\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v : \tilde{K}\big(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m\big) \to \tilde{K}(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$$

$$\to K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$$

is the zero map.

Proof
(a) Assume first that k is infinite. If $\xi = \{f_1, \dots, f_{n+1}\}_{k(D)/k(C)}$ satisfies (11.5), then we have that $\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v(\xi) = 0$ by Definition 5.1 and Lemma 11.12(b). Hence the claim follows from Proposition 11.11.

(b) If k is finite, we use a classical trick: let p_1, p_2 be two distinct prime numbers, and let k_i be the $\mathbf{Z}_{p_i}$-extension of k. Let $x \in \tilde{K}(k(C); \mathcal{F}_1, \dots, \mathcal{F}_n, \mathbf{G}_m)$. By (a), the image of x in $K(k; \mathcal{F}_1, \dots, \mathcal{F}_n)$ vanishes in $K(k_1; \mathcal{F}_1, \dots, \mathcal{F}_n)$ and $K(k_2; \mathcal{F}_1, \dots, \mathcal{F}_n)$, hence is 0 by a transfer argument. □

Finally, we arrive at the following.

THEOREM 11.14
The homomorphism (1.1) is an isomorphism for any $\mathcal{F}_1, \dots, \mathcal{F}_n \in \mathbf{HI}_{\mathrm{Nis}}$.

Proof
It suffices to show the statement in Proposition 9.1(c). With the notation therein, $\mathcal{A}$ is curvelike by Lemma 11.2(c). The image of (9.1) in $K(k; \mathcal{A}, \dots, \mathcal{A})$ is seen to be $\sum_{v \in C} \mathrm{Tr}_{k(v)/k} \circ \partial_v(\{\iota, \dots, \iota, f\}_{k(C)/k(C)})$ by Lemma 11.12, hence trivial by Proposition 11.13. □

12. Application to algebraic cycles

12.1
We assume that k is of characteristic zero. Let X be a k-scheme of finite type, and let $M^c(X) := C_*^c(X) \in \mathbf{DM}_-^{\mathrm{eff}}$ be the motive of X with compact supports (see [31, Section 4.1]). Then the sheaf $\underline{CH}_0(X)$ of Section 1.4 agrees with $H_0(M^c(X))$ by [8, Theorem 2.2]. If X is quasi-projective, we have an isomorphism

$$CH_{-i}(X, j+2i) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, M^c(X)(i)[-j]\big)$$

for all $i \in \mathbf{Z}_{\geq 0}$, $j \in \mathbf{Z}$ by [31, Proposition 4.2.9].[†]

Proof of Theorem 1.5
Using Lemma 3.3, we see that

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, \underline{CH}_{\mathbf{0}}(X_1)[0] \otimes \cdots \otimes \underline{CH}_{\mathbf{0}}(X_n)[0] \otimes \mathbf{G}_m[0]^{\otimes r}\big)\\
&\quad\simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, M^c(X_1) \otimes \cdots \otimes M^c(X_n) \otimes \mathbf{G}_m[0]^{\otimes r}\big)\\
&\quad\simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, M^c(X)(r)[r]\big) \simeq CH_{-r}(X, r).
\end{aligned}$$

(Here we used $\mathbf{G}_m[0] \simeq \mathbf{Z}(1)[1]$.) Now the theorem follows from Theorem 11.14. ☐

12.2
Let X be a k-scheme of finite type. Recall that for $i \in \mathbf{Z}_{\geq 0}$, $j \in \mathbf{Z}$ the motivic homology of X is defined by [7, Definition 9.4]

$$H_j\big(X, \mathbf{Z}(-i)\big) := \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, M(X)(i)[-j]\big). \tag{12.1}$$

When $i = 0$, $H_j(X, \mathbf{Z}(0))$ agrees with Suslin homology (see [28]).

THEOREM 12.3
Let $X_1, \ldots, X_n$ be k-schemes of finite type. Suppose that either the X_i are smooth or char $k = 0$. *Put $X = X_1 \times \cdots \times X_n$. For any $r \geq 0$, we have an isomorphism*

$$K\big(k; h_{\mathbf{0}}^{\mathrm{Nis}}(X_1), \ldots, h_{\mathbf{0}}^{\mathrm{Nis}}(X_n), \mathbf{G}_m, \ldots, \mathbf{G}_m\big) \xrightarrow{\sim} H_{-r}\big(X, \mathbf{Z}(-r)\big).$$

Proof
Using Lemma 3.3, we see that

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, h_{\mathbf{0}}^{\mathrm{Nis}}(X_1)[0] \otimes \cdots \otimes h_{\mathbf{0}}^{\mathrm{Nis}}(X_n)[0] \otimes \mathbf{G}_m[0]^{\otimes r}\big)\\
&\quad\simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, M(X_1) \otimes \cdots \otimes M(X_n) \otimes \mathbf{G}_m[0]^{\otimes r}\big)\\
&\quad\simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{-}}\big(\mathbf{Z}, M(X)(r)[r]\big) \simeq H_{-r}\big(X, \mathbf{Z}(-r)\big).
\end{aligned}$$

Now the theorem follows from Theorem 11.14. ☐

[†] The proof of [31, Proposition 4.2.9] is written for equidimensional schemes but is the same in general. Moreover, the quasi-projective assumption can be removed if one replaces higher Chow groups by the Zariski hypercohomology of the cycle complex as in [15, after Theorem 1.7].

Remark 12.4
If $X_1, \dots, X_n$ are smooth projective varieties, then (1.3) is valid in any characteristic. Indeed, we have that $M(X_i) = M^c(X_i)$ and hence $\underline{CH}_0(X_i) = h_0^{\mathrm{Nis}}(X_i)$. Moreover, [32] and [8, Appendix B] show that $H_{-r}(X, \mathbf{Z}(-r)) \simeq CH_{-r}(X, r)$. Thus (1.3) follows from Theorem 12.3.

Appendix. Extending monoidal structures

A.1
Let $\mathcal{A}$ be an additive category. We write $\mathrm{Mod}\text{–}\mathcal{A}$ for the category of contravariant additive functors from $\mathcal{A}$ to abelian groups. This is a Grothendieck abelian category. We have the additive Yoneda embedding

$$y_{\mathcal{A}} : \mathcal{A} \to \mathrm{Mod}\text{–}\mathcal{A}$$

sending an object to the corresponding representable functor.

A.2
An object of $\mathrm{Mod}\text{–}\mathcal{A}$ is *free* if it is a direct sum of representable objects. Let $M \in \mathrm{Mod}\text{–}\mathcal{A}$. For any $A \in \mathcal{A}$, the Yoneda isomorphism

$$M(A) \simeq \mathrm{Mod}\text{–}\mathcal{A}\big(y_{\mathcal{A}}(A), M\big)$$

realizes M canonically as a quotient of a free module:

$$L_0(M) = \bigoplus_{(A,f)} y_{\mathcal{A}}(A) \twoheadrightarrow M,$$

where (A, f) runs through pairs of an object $A \in \mathcal{A}$ and an element $f \in M(A)$. Iterating, we get a canonical and functorial free resolution

$$\cdots \to L_n(M) \to \cdots \to L_0(M) \to M \to 0 \tag{A.1}$$

as in [17, Lemma 8.1].

A.3
Let $f : \mathcal{A} \to \mathcal{B}$ be an additive functor. We have an induced functor $f^* : \mathrm{Mod}\text{–}\mathcal{B} \to \mathrm{Mod}\text{–}\mathcal{A}$ (composition with f). As in [2, Example 1, Propositions 5.1 and 5.4], the functor f^* has a left adjoint $f_!$ and a right adjoint f_*, and the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{y_{\mathcal{A}}} & \mathrm{Mod}-\mathcal{A} \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f_!}\big\downarrow \\ \mathcal{B} & \xrightarrow{y_{\mathcal{B}}} & \mathrm{Mod}-\mathcal{B} \end{array}$$

is naturally commutative.

A.4

If f is fully faithful, then $f_!$ and f_* are fully faithful and f^* is a localization, as in [2, Example 1, Proposition 5.6].

A.5

Suppose that f has a left adjoint g. Then we have natural isomorphisms

$$g^* \simeq f_!, \qquad g_* \simeq f^*$$

as in [2, Example 1, Proposition 5.5].

A.6

Suppose further that f is fully faithful. Then $g^* \simeq f_!$ is fully faithful. From the composition

$$g^* g_* \Rightarrow \mathrm{Id}_{\mathrm{Mod}-\mathcal{A}} \Rightarrow g^* g_!$$

of the unit with the counit, one then deduces a canonical morphism of functors

$$g_* \Rightarrow g_!.$$

A.7

Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. Their *tensor product* is the category $\mathcal{A} \boxtimes \mathcal{B}$ whose objects are finite collections (A_i, B_i) with $(A_i, B_i) \in \mathcal{A} \times \mathcal{B}$, and

$$(\mathcal{A} \boxtimes \mathcal{B})\big((A_i, B_i), (C_j, D_j)\big) = \bigoplus_{i,j} \mathcal{A}(A_i, C_j) \otimes \mathcal{B}(B_i, D_j).$$

We have a "cross-product" functor

$$\boxtimes : \mathrm{Mod}-\mathcal{A} \times \mathrm{Mod}-\mathcal{B} \to \mathrm{Mod}-(\mathcal{A} \boxtimes \mathcal{B})$$

given by

$$(M \boxtimes N)\big((A_i, B_i)\big) = \bigoplus_i M(A_i) \otimes N(B_i).$$

A.8

Let $\mathcal{A}$ be provided with a biadditive bifunctor $\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. We may view $\bullet$ as an additive functor $\mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$. We may then extend $\bullet$ to $\mathrm{Mod}\text{–}\mathcal{A}$ by the composition

$$\mathrm{Mod}\text{–}\mathcal{A} \times \mathrm{Mod}\text{–}\mathcal{A} \xrightarrow{\boxtimes} \mathrm{Mod}\text{–}(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\bullet_!} \mathrm{Mod}\text{–}\mathcal{A}.$$

This is an extension in the sense that the diagram

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{A} \arrow[r, "y_{\mathcal{A}} \times y_{\mathcal{A}}"] \arrow[d, "\bullet \times \bullet"'] & \mathrm{Mod}\text{–}\mathcal{A} \times \mathrm{Mod}\text{–}\mathcal{A} \arrow[d, "\bullet"'] \\
\mathcal{A} \arrow[r, "y_{\mathcal{A}}"] & \mathrm{Mod}\text{–}\mathcal{A}
\end{tikzcd}$$

is naturally commutative.

If $\bullet$ is monoidal (resp., monoidal symmetric), then its associativity and commutativity constraints canonically extend to $\mathrm{Mod}\text{–}\mathcal{A}$.

A.9

As a composition of right exact functors, $\bullet$ is right exact in the abelian category $\mathrm{Mod}\text{–}\mathcal{A}$. The tensor product of two free modules (as in Section A.2) is free. On the other hand, free objects have no reason to be flat in general (see caveat in [17, Remark 8.6]). We shall see in Corollary A.15 that they are flat when $\mathcal{A}$ is rigid.

A.10

For $M \in \mathrm{Mod}\text{–}\mathcal{A}$, let $L_\bullet(M) \to M$ be its canonical free resolution of M from (A.1). If $N \in \mathrm{Mod}\text{–}\mathcal{A}$ is another object, then the sequence

$$L_1(M) \bullet L_0(N) \oplus L_0(M) \bullet L_1(N) \to L_0(M) \bullet L_0(N) \to M \bullet N \to 0$$

is exact, yielding a presentation of $M \bullet N$ by free objects.

Example A.11

If $\mathcal{A} = \mathbf{Cor}$, then $\mathrm{Mod}\text{–}\mathcal{A} = \mathbf{PST}$. The free resolution $L_\bullet(M)$ of an object $M \in \mathbf{PST}$ is Voevodsky's resolution $\mathcal{L}(M)$ in [31, p. 206]; from Section A.10, we recover his definition of $M \otimes N$ in [31, p. 206] or in [17, Definition 8.2].

A.12

Let $\mathcal{A}, \mathcal{B}$ be two additive symmetric monoidal categories, and let $f : \mathcal{A} \to \mathcal{B}$ be an additive symmetric monoidal functor. The above definition shows that the functor $f_! : \mathrm{Mod}\text{–}\mathcal{A} \to \mathrm{Mod}\text{–}\mathcal{B}$ is also symmetric monoidal.

A.13
In Section A.8, let us write $\bullet_! = \int$ for clarity. Let $P \in \mathrm{Mod}-(\mathcal{A} \boxtimes \mathcal{A})$. Then $\int P$ is the *left Kan extension of* P *along* $\bullet$ in the sense of [16, Chapter X, Section 3]. This gives a formula for $\int P$ as a *coend* [16, Theorem X.4.1]; for $A \in \mathcal{A}$:

$$\int P(A) = \int^{(B,B')} \mathcal{A}(A, B \bullet B') \otimes P(B, B'). \tag{A.2}$$

In particular, we have the following.

PROPOSITION A.14
Suppose is $\mathcal{A}$ rigid. Then (A.2) simplifies as

$$\int P(A) = \int^{B} P(B, A \bullet B^*),$$

where B^* is the dual of $B \in \mathcal{A}$. In particular, if $P = M \boxtimes N$ for $M, N \in \mathrm{Mod}-\mathcal{A}$, we have, for $A \in \mathcal{A}$, that

$$(M \bullet N)(A) = \int^{B} M(B) \otimes N(A \bullet B^*), \tag{A.3}$$

which describes $M \bullet N$ as a "convolution." In particular, for $N = y_{\mathcal{A}}(C)$, $M \bullet y_{\mathcal{A}}(C)$ is given by the formula

$$\big(M \bullet y_{\mathcal{A}}(C)\big)(A) = M(A \otimes C^*). \tag{A.4}$$

Proof
Applying (A.2) and rigidity, we have that

$$\begin{aligned} \int P(A) &= \int^{(B,B')} \mathcal{A}(A, B \bullet B') \otimes P(B, B') \\ &= \int^{(B,B')} \mathcal{A}(A \bullet B^*, B') \otimes P(B, B') \\ &= \int^{B} P(B, A \bullet B^*), \end{aligned}$$

because in the third formula, the variable B' is dummy. (This simplification is not in Mac Lane [16]!)

We get (A.4) from (A.3), since

$$\begin{aligned} \int^{B} M(B) \otimes y_{\mathcal{A}}(C)(A \bullet B^*) &= \int^{B} M(B) \otimes \mathcal{A}(A \bullet B^*, C) \\ &\simeq \int^{B} M(B) \otimes \mathcal{A}(A \bullet C^*, B) = M(A \bullet C^*), \end{aligned}$$

because the variable B is dummy in the penultimate term. $\square$

COROLLARY A.15
If $\mathcal{A}$ is rigid, any free object of $\mathrm{Mod}\text{–}\mathcal{A}$ *is flat.*

Proof
This is an immediate consequence of (A.4). □

A.16
We shall need a refinement of (A.3). For this, we first have the probably well-known lemma, of which we include a proof for lack of reference. (Unfortunately it is not in Mac Lane's book either.)

LEMMA A.17 (Change of variables)
Let $\mathcal{C} \overset{R}{\underset{L}{\leftrightarrows}} \mathcal{D}$ be a pair of adjoint functors between small categories (L is left adjoint and R is right adjoint); moreover, let $T : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{X}$ be a functor, where small colimits are representable in $\mathcal{X}$. Then there is a canonical isomorphism

$$\int^c T(Lc, c) \simeq \int^d T(d, Rd).$$

Proof
Let $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow RL$ and $\varepsilon : LR \Rightarrow \mathrm{Id}_{\mathcal{D}}$ be the unit and the counit of the adjunction, respectively. Using ε we get a natural transformation

$$\varepsilon^*_{d,d'} = T(\varepsilon_d, 1) : T(d, Rd') \to T(LRd, Rd').$$

By the universal property of coends, this yields a morphism $\varphi : \int^d T(d, Rd) \to \int^c T(Lc, c)$. Using η, we similarly get a natural transformation $\eta_*^{c,c'} : T(Lc, c') \to T(Lc, RLc')$ and a morphism $\psi : \int^c T(Lc, c) \to \int^d T(d, Rd)$.

Write $X = \int^d T(d, Rd)$. Checking that $\psi \circ \varphi = 1$ amounts to checking that, for any $d_0 \in \mathcal{D}$, the composition

$$T(d_0, Rd_0) \xrightarrow{\varepsilon^*_{d_0,d_0}} T(LRd_0, Rd_0) \to \int^c T(Lc, c) \xrightarrow{\psi} X$$

equals the canonical map $\rho_{d_0} : T(d_0, Rd_0) \to X$. By definition, this composition is equal to

$$T(d_0, Rd_0) \xrightarrow{\varepsilon^*_{d_0,d_0}} T(LRd_0, Rd_0) \xrightarrow{\eta_*^{Rd_0,Rd_0}} T(LRd_0, RLRd_0) \xrightarrow{\rho_{LRd_0}} X,$$

that is,

$$\rho_{LRd_0} \circ T(1, \eta_{Rd_0}) \circ T(\varepsilon_{d_0}, 1) = \rho_{LRd_0} \circ T(\varepsilon_{d_0}, 1) \circ T(1, \eta_{Rd_0}).$$

By the universal property of X, we have the identity

$$\rho_{LRd_0} \circ T(\varepsilon_{d_0}, 1) = \rho_{d_0} \circ T\big(1, R(\varepsilon_{d_0})\big);$$

hence,

$$\rho_{LRd_0} \circ T(\varepsilon_{d_0}, 1) \circ T(1, \eta_{Rd_0}) = \rho_{d_0} \circ T\big(1, R(\varepsilon_{d_0})\big) \circ T(1, \eta_{Rd_0}) = \rho_{d_0}$$

because of the adjunction identity $R(\varepsilon_{d_0}) \circ \eta_{d_0} = 1$. The proof that $\varphi \circ \psi = 1$ is similar. $\square$

Example A.18
In Proposition A.14, take $\mathcal{A} = \mathbf{Z}\mathbf{Span}(k)$. This category is rigid, all objects being self-dual. (Duality acts on morphisms by converting a span (f, g) into (g, f).) For any étale k-scheme X, the obvious forgetful functor $\omega : \mathbf{Z}\mathbf{Span}(X) \to \mathbf{Z}\mathbf{Span}(k)$ has a left adjoint $Y \mapsto X \times_k Y$. For a Mackey functor $M \in \mathbf{Mack}(k)$, write $M^X = M \circ \omega$. Applying Lemma A.17 with $\mathcal{C} = \mathbf{Z}\mathbf{Span}(k)$, $\mathcal{D} = \mathbf{Z}\mathbf{Span}(X)$, and $T(Y, Z) = N^X(Y) \otimes M(Z)$ and using that all objects are self-dual, we convert (A.3) into the formula

$$(M \overset{M}{\otimes} N)(X) = \int^{Y \in \mathbf{Z}\mathbf{Span}(X)} M^X(Y) \otimes N^X(Y).$$

Unfolding the definition of the coend, we immediately get the formula of Section 2.8. The case of more than two factors follows by associativity.

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