A SHEAF-THEORETIC REFORMULATION OF THE TATE CONJECTURE

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Contents

Abstract. We present a conjecture which unifies several conjectures on motives in characteristic p.

Introduction

In a talk at the June 1996 Oberwolfach algebraic K-theory conference, I explained that, in view of Voevodsky's proof of the Milnor conjecture, the Bass conjecture on finite generation of K-groups of regular schemes of finite type over the integers implies the rational Beilinson-Soulé conjecture on vanishing of algebraic K-theory of low weights in characteristic $\neq 2$. This argument is reproduced in the appendix. The next day, Thomas Geisser explained that, in characteristic > 0, the Tate conjecture on surjectivity of the $\mathbb{Q}_l$-adic cycle map for smooth, projective varieties over a finite field, together with the conjecture that rational and numerical equivalences agree for such varieties, also implies the Beilinson-Soulé conjecture. His work is now available in [10]. The present paper stems from an attempt to understand the relationship between these two facts.

Let l be a prime number. We present a conjecture (conjecture 8.12) which is equivalent to the conjunction of three well-known conjectures on smooth, projective varieties X over $\mathbb{F}_p$ ($p \neq l$):

1. *Tate's conjecture*: the geometric cycle map
$$CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}(\overline{X}, \mathbb{Q}_l(n))^G \tag{*}$$
is surjective ($\overline{X} = X \times_{\mathbb{F}_p} \bar{\mathbb{F}}_p$, $G = Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$).
2. *Partial semi-simplicity*: the characteristic subspace of $H^n(\overline{X}, \mathbb{Q}_l(n))$ corresponding to the eigenvalue 1 of Frobenius is semi-simple.

3. *rational equivalence = homological equivalence*: the map (*) is injective.

We note that the conjunction of (1), (2) and (3) (in codimensions n and $\dim X - n$) is equivalent to the so-called strong form of the Tate conjecture: the order of the pole of the Hasse-Weil zeta function $\zeta(X,s)$ at $s = n$ equals the rank of the group $A^n(X)$ of codimension n cycles modulo numerical equivalence, *cf.* [49, th. 2.9]. In this light, conjecture 8.12 below can really be thought of as a sheaf-theoretic reformulation of the (strong) Tate conjecture.

The main result of this paper is that conjecture 8.12 (and therefore the three above together) also implies a host of other familiar conjectures:

- The Tate conjecture and the partial semi-simplicity conjecture (see definition 6.7) for smooth, projective varieties over *any* finitely generated field of characteristic p (theorem 8.32).
- The injectivity of the (arithmetic) $\mathbb{Q}_l$-adic cycle map for *any* smooth variety X over $\mathbb{F}_p$, and a description of its image. In particular, that rational equivalence equals homological equivalence on X (proposition 8.20 and corollary 8.22).
- The Bass conjecture (finite generation of the algebraic K-groups) after tensoring by $\mathbb{Q}$ for such X (corollary 8.28).
- The rational Beilinson-Soulé conjecture for such X (*ibid.*).
- The vanishing of $K_n^M(F) \otimes \mathbb{Q}$ for $n > \operatorname{trdeg}(F/\mathbb{F}_p)$, when F is a field of characteristic p (Bass-Tate conjecture in characteristic p, *cf.* corollary 8.30).
- Soulé's conjecture on the order of the zeroes and poles of the zeta function of a variety over $\mathbb{F}_p$ at integers [42] (in fact a refined version of it, th. 8.41).
- The existence of a canonical integral structure on continuous étale cohomology (proposition 8.19).

It is known that these conjectures imply other ones: for example, the Tate conjecture plus semi-simplicity (in a range) imply that homological and numerical equivalences agree ([32, prop. 8.4], [49, (2.6)]), that the Künneth components of the diagonal are algebraic [49, §3] and that Hard Lefschetz holds for cycles modulo numerical equivalence (see theorem 8.32 b)). On the other hand, the injectivity of the cycle map implies the existence of Beilinson's conjectural filtration on Chow groups (a variant of [21, lemma 2.7], see also [21, lemma 2.2], *cf.* theorems 8.32 b) and 8.36 here), which is in turn equivalent to Murre's idempotents conjecture ([35, 1.4], [21, th. 5.2], *cf.* corollary 8.33 here).

One conjecture we don't know to follow from conjecture 8.12 is the Hodge-type standard conjecture in characteristic p [27, §5]. This is perhaps moral, since this conjecture implies the semi-simplicity of the full Frobenius action on cohomology groups, not just at the eigenvalue 1 (*ibid.*, th. 5-6 (2)); we feel that conjecture 8.12 is not strong enough to imply such a fact *per se*.

There is a variant of conjecture 8.12, involving Voevodsky's motivic cohomology (conjecture 9.6). We show that, under resolution of singularities, it is equivalent to conjecture 8.12. It implies Lichtenbaum's conjectures of [28, §7] on the nature of the (étale) motivic cohomology of a smooth projective variety over $\mathbb{F}_p$ and on values of its zeta function at nonnegative integers; in fact, we extend this consequence to arbitrary smooth varieties (proposition 9.13 and theorem 9.16). Under resolution of singularities, these properties therefore follow from the Tate conjecture, the partial semi-simplicity conjecture and the rational equivalence=homological equivalence conjecture.

Let us now describe conjectures 8.12 and 9.6. In [17], Jannsen defines continuous étale cohomology as the derived functors of the composition of left exact functors

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{N}} & \xrightarrow{\Gamma} & Ab^{\mathbb{N}} \\ & & \Big\downarrow{\scriptstyle \varprojlim} \\ & & Ab \end{array}$$

where $\mathcal{A}$ is the category of abelian sheaves over the small étale site of some scheme X. Our starting point, only briefly considered in [17, p. 219], is to observe that one can equally well define this composition by going the other way:

$$\begin{array}{ccc} \mathcal{A}^{\mathbb{N}} & & \\ {\scriptstyle \varprojlim}\Big\downarrow & & \\ \mathcal{A} & \xrightarrow{\Gamma} & Ab. \end{array}$$

In general the category $\mathcal{A}$ does not satisfy Grothendieck's axiom $AB4^*$ (products are not exact), but nevertheless the inverse limit functor is left exact and preserves injectives, so it makes sense to study its higher derived functors with values in $\mathcal{A}$ or, better, its total derived functor $R\varprojlim$ with values in the derived category $\mathcal{D}^+(\mathcal{A})$. In particular, we can define

$$\mathbb{Z}_l(n)^c_X := R\varprojlim \mu_{l^\nu}^{\otimes n} \in \mathcal{D}^+(\mathcal{A})$$

provided the prime l is invertible on X, a blanket assumption here. Note that $\mathbb{Z}_l(n)^c_X$ is not an l-adic sheaf in the sense of SGA5, but a complex of honest sheaves (up to quasi-isomorphism).

It is more convenient to work at once over the big étale site of $\operatorname{Spec}\mathbb{F}_p$ (restricted to smooth schemes): denote by $\mathbb{Z}_l(n)^c$ the corresponding object. We prove in section 6 that $\mathbb{Z}_l(n)^c = \mathbb{Q}_l/\mathbb{Z}_l(n)[-1]$ for $n < 0$, that

$$\mathbb{Z}_l(0)^c \simeq \mathbb{Z}^c \otimes \mathbb{Z}_l$$

where $\mathbb{Z}^c$ is the class of a certain explicit complex of length 1 coming from the small étale site of $\operatorname{Spec}\mathbb{F}_p$, such that

$$\mathbb{Q}^c := \mathbb{Z}^c \otimes \mathbb{Q} \simeq \mathbb{Q}[0] \oplus \mathbb{Q}[-1],$$

and that $\mathbb{H}^i(X, \mathbb{Z}_l(n)^c)$ is finite for $i \notin [n, 2n+1]$, $n > 0$ and X smooth over $\mathbb{F}_p$.

In section 8, we define a map

$$\mathbb{Q}_l(0)^c \overset{L}{\otimes} \alpha^*\mathcal{K}_n^M[-n] \to \mathbb{Q}_l(n)^c$$

where $\mathcal{K}_n^M$ is the sheaf of Milnor K groups over the big smooth Zariski site of $\operatorname{Spec}\mathbb{F}_p$ and α is the projection of the big étale site onto the big Zariski site. Conjecture 8.12 says that this map is an *isomorphism*.

In section 9 we analogously define a map

$$\mathbb{Z}_l(0)^c \overset{L}{\otimes} \alpha^*\mathbb{Z}(n) \to \mathbb{Z}_l(n)^c.$$

Here, $\mathbb{Z}(n)$ is the n-th motivic complex of Suslin-Voevodsky [47] and α is as above. Conjecture 9.6 also states that this map is an isomorphism.

This paper is organised as follows. In section 1 we go through basic facts concerning inverse limits and their higher derived functors. In sections 2 and 3 we "recall" some properties of continuous étale cohomology, cohomology with proper supports and Borel-Moore homology. In section 4, we do some computations over finite fields. In section 5 we develop some elementary useful formalism. In section 6 we give our main unconditional results. In section 7, we generalise the main result of Milne [32, th. 0.1] on the principal part of the zeta function $\zeta(X, s)$ at $s = n$ from smooth, projective varieties to arbitrary varieties over $\mathbb{F}_p$ (theorem 7.2, theorem 7.8 and corollary 7.10); I am particularly fond of the statement in theorem 7.8 b). In section 8, we present our main conjecture and derive the consequences listed above, and others. Perhaps the most important ones are contained in theorems 8.32 and 8.36. In section 9 we present the motivic variant of our main conjecture and explore some further consequences of it. In particular, in proposition 9.25 we give as a consequence of conjecture 9.6 (and resolution of singularities) a description of the kernel and cokernel of the integral cycle map $CH^n(X) \otimes \mathbb{Z}_l \to H^{2n}_{\mathrm{cont}}(X, \mathbb{Z}_l(n))$. In section 10, we prove the restriction of conjecture 8.12 to curves over $\mathbb{F}_p$. In section 11 we propose a "Zariski" variant of conjecture 9.6 related to the Kato conjecture. In section 12 we briefly discuss the much more open case of schemes of finite type over $\operatorname{Spec}\mathbb{Z}[1/l]$. Here it is clear that l-adic cohomology does not give the full picture; in order to explain the contribution of archimedean places, we expect that Arakelov geometry will play its rôle. Finally, in the appendix, we give a proof of our announcement that the Bass conjecture implies the Beilinson-Soulé conjecture, in all characteristics.

A pendant of the present investigation for $l = p$ could doubtlessly be developed; we don't tackle this here. Another thing which should definitely be done is to investigate analogues of conjectures 8.12 and 9.6 over p-adic fields, $p \neq l$ and $p = l$.

It will be obvious that this paper owes much to Uwe Jannsen's previous work on motives. In particular, it is by reading the last chapter of [18], where Jannsen extends the Tate and Hodge conjectures from smooth projective varieties to arbitrary varieties, that I got the courage to look for a sheaf-theoretic version of the Tate conjecture. But it would be unfair not to acknowledge the influence of Vladimir Voevodsky's work as well. I also wish to thank Bernhard Keller and Amnon Neeman for a discussion which helped clarify some confusion related to section 4. Finally, I would like to thank Thomas Geisser for a large number of helpful comments.

1. Continuous cohomology

Notation. If $\mathcal{A}$ is an abelian category and $\mathcal{D}(\mathcal{A})$ is its derived category, we usually denote by $A \mapsto A[0]$ the canonical functor $\mathcal{A} \to \mathcal{D}(\mathcal{A})$.

1.1. **Inverse limits.** Let $\mathcal{A}$ be an abelian category, and let $\mathcal{A}^{\mathbb{N}}$ be the category of functors from $\mathbb{N}$ to $\mathcal{A}$, where the category structure on $\mathbb{N}$ is given by its ordering. An object in $\mathcal{A}^{\mathbb{N}}$ is the same as a projective system of objects of $\mathcal{A}$.

The constant functor $\mathcal{A} \to \mathcal{A}^{\mathbb{N}}$ (induced by the projection $\mathbb{N} \to \{1\}$) is exact. If $\mathcal{A}$ is complete ($\iff AB3^*$ holds), this functor has as a left adjoint the left exact functor:

$$\begin{aligned} \varprojlim : \mathcal{A}^{\mathbb{N}} &\to \mathcal{A} \\ (A_n) &\mapsto \varprojlim A_n. \end{aligned}$$

which carries injectives to injectives.

By [17, (1.1)], $\mathcal{A}^{\mathbb{N}}$ has enough injectives if and only if $\mathcal{A}$ has. In this case, we can derive $\varprojlim$ in

$$\varprojlim{}^i : \mathcal{A}^{\mathbb{N}} \to \mathcal{A} \qquad (i \geq 0).$$

More globally, we can define a total derived functor

$$R\varprojlim : \mathcal{D}^+(\mathcal{A}^{\mathbb{N}}) \to \mathcal{D}^+(\mathcal{A}).$$

1.2. **Extending functors in one variable.** Let $\mathcal{B}$ be another complete abelian category with enough injectives and $T : \mathcal{A} \to \mathcal{B}$ a left exact functor. Then T induces a left exact functor from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{B}^{\mathbb{N}}$, that we still denote by T. We have a natural transformation of left exact functors

$$T \circ \varprojlim \to \varprojlim \circ T \tag{1.1}$$

hence a chain of natural transformations in $\mathcal{D}(\mathcal{B})$

$$RT \circ R\varprojlim \leftarrow R(T \circ \varprojlim) \to R(\varprojlim \circ T) \to R\varprojlim \circ RT. \tag{1.2}$$

1.1. Lemma. *a) The left transformation in* (1.2) *is a natural isomorphism, hence a spectral sequence*

$$R^pT(\varprojlim{}^q A) \Rightarrow R^{p+q}(T \circ \varprojlim)(A)$$

for any $A \in \mathcal{A}^{\mathbb{N}}$*.*

b) If T carries injectives to $\varprojlim$-acyclics, the right transformation in (1.2) *is a natural isomorphism, hence a spectral sequence*

$$\varprojlim{}^p R^qT(A) \Rightarrow R^{p+q}(\varprojlim \circ T)(A).$$

c) If T commutes with products, then (1.1) *and the middle transformation in* (1.2) *are natural isomorphisms.*

d) If T has a left adjoint which respects monomorphisms, then (1.1) *and all transformations in* (1.2) *are natural isomorphisms.*

Proof. a) holds because $\varprojlim$ carries injectives to injectives. b) and c) are clear; the assumption in d) implies those of b) and c). $\square$

1.3. **Extending functors in two variables.** Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be three abelian categories and let $T : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be an biadditive, bicovariant functor. Then T extends naturally to a functor

$$T : \mathcal{A}^{\mathbb{N}} \times \mathcal{B}^{\mathbb{N}} \to \mathcal{C}^{\mathbb{N}}$$

by the formula

$$T((a_\nu), (b_\nu)) = (T(a_\nu, b_\nu)).$$

This formula can be derived in the usual way. For example, suppose that $\mathcal{A} = \mathcal{B} = \mathcal{C}$ is endowed with a right exact tensor product

$$\otimes = \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

for which $\mathcal{A}$ has enough acyclic objects. Suppose moreover that $\otimes$ has finite homological dimension. One can then derive the natural transformation

$$\otimes \circ (\varprojlim, \varprojlim) \to \varprojlim \circ \otimes$$

between functors from $\mathcal{A}^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}}$ to $\mathcal{A}$ into a natural transformation

$$\overset{L}{\otimes} \circ (R \varprojlim, R \varprojlim) \to R \varprojlim \circ \overset{L}{\otimes} \tag{1.3}$$

between functors from $\mathcal{D}^+(\mathcal{A}^{\mathbb{N}}) \times \mathcal{D}^+(\mathcal{A}^{\mathbb{N}})$ to $\mathcal{D}^+(\mathcal{A})$.

If T is contravariant in $\mathcal{A}$ and covariant in $\mathcal{B}$, like Hom, its extension should be defined by an end, as in [30, ch. IX §5].

1.4. **Sheaves.** Let X be a site and R a ring. We consider the category $\mathcal{A}$ of sheaves of R-modules on X. Since the functor "global sections" Γ has as a left adjoint the exact functor "constant sheaf", we can apply lemma 1.1. As in [17], we define

1.2. Definition. Let $(\mathcal{F}_\nu)$ be a projective system of abelian sheaves over X. The *continuous cohomology* of X with values in $(\mathcal{F}_\nu)$ is

$$H^n_{\text{cont}}(X, (\mathcal{F}_\nu)) = R^n(\Gamma \circ \varprojlim)((\mathcal{F}_\nu)) = R^n(\varprojlim \circ \Gamma)((\mathcal{F}_\nu))$$

(see lemma 1.1 c)).

Since products are exact in $R - mod$, $\varprojlim^i = 0$ for $i > 1$ and the spectral sequence of lemma 1.1 b) yields "Milnor" exact sequences [17, (1.6)]

$$0 \to \varprojlim{}^1 H^{n-1}(X, \mathcal{F}_\nu) \to H^n_{\text{cont}}(X, (\mathcal{F}_\nu)) \to \varprojlim H^n(X, \mathcal{F}_\nu) \to 0. \tag{1.4}$$

But the spectral sequence of lemma 1.1 a) also says that continuous étale cohomology can be computed by a hypercohomology spectral sequence:

$$H^p(X, \varprojlim{}^q(\mathcal{F}_\nu)) \Rightarrow H^{p+q}_{\text{cont}}(X, (\mathcal{F}_\nu)).$$

This gives us a formula for the *stalks* of $\varprojlim^q(\mathcal{F}_\nu)$ (*cf.* [17, (3.12)]):

1.3. Lemma. *For x a point of X, we have*

$$(\varprojlim{}^q(\mathcal{F}_\nu))_x = \varinjlim_{U \ni x} H^q_{\text{cont}}(U, (\mathcal{F}_\nu)). \qquad \square$$

We also have:

1.4. Proposition. *Suppose that* $\mathrm{cd}_R(X) \le n$. *Then,*

a) For any inverse system of R-sheaves $(\mathcal{F}_\nu)$ and any $U \in X$, we have $H^q_{\text{cont}}(U, (\mathcal{F}_\nu)) = \varprojlim^q(\mathcal{F}_\nu) = 0$ for $q > n + 1$.

b) If moreover $(\mathcal{F}_\nu)$ satisfies the Mittag-Leffler condition, then $H^q_{\text{cont}}(U, (\mathcal{F}_\nu)) = \varprojlim^q(\mathcal{F}_\nu) = 0$ for $q > n$.

Proof. a) This follows from the Milnor exact sequence (1.4).

b) By [17, (1.14)], we may assume that the transition morphisms of $(\mathcal{F}_\nu)$ are surjective. Then the same holds for the inverse system $(H^n_{\mathrm{cont}}(U,\mathcal{F}_\nu))$. The conclusion now follows again from (1.4). □

Let $f : Y \to X$ be a morphism of sites. By lemma 1.1 c), the natural transformation

$$f_* \circ \varprojlim \to \varprojlim \circ f_* \tag{1.5}$$

is an isomorphism; denote by f_*^{cont} either side of (1.5). Suppose that f^* respects monomorphisms (*e.g.* is left exact). Applying lemma 1.1 d), we get similarly two spectral sequences

$$\varprojlim{}^p(R^q f_*\mathcal{F}_n) \Rightarrow R^{p+q} f_*^{\mathrm{cont}}(\mathcal{F}_n) \Leftarrow R^p f_* \varprojlim{}^q(\mathcal{F}_n).$$

In general, the natural transformation

$$f^* \circ \varprojlim \to \varprojlim \circ f^*$$

is not an isomorphism. When f^* has a left adjoint which respects monomorphisms, we can however apply lemma 1.1 d). This is the case for example if the associated topos $Shv(Y)$ is induced from $Shv(X)$ via f (*cf.* [54, exposé IV, §5]).

Suppose R commutative of finite Tor-dimension. Let $(\mathcal{F}_\nu),(\mathcal{G}_\nu)$ be two inverse systems of sheaves. Applying (1.3), we get a morphism in $\mathcal{D}^+(\mathcal{A})$

$$R\varprojlim(\mathcal{F}_\nu) \overset{L}{\otimes} R\varprojlim(\mathcal{G}_\nu) \to R\varprojlim(\mathcal{F}_\nu[0] \overset{L}{\otimes} \mathcal{G}_\nu[0]).$$

Let $(\mathcal{H}_\nu)$ be a third inverse system of sheaves and $\alpha : (\mathcal{F}_\nu)\times(\mathcal{G}_\nu) \to (\mathcal{H}_\nu)$ a bilinear map in $\mathcal{A}^{\mathbb{N}}$. Using the composite

$$\mathcal{F}_\nu[0] \overset{L}{\otimes} \mathcal{G}_\nu[0] \to (\mathcal{F}_\nu \otimes \mathcal{G}_\nu)[0] \xrightarrow{\alpha[0]} \mathcal{H}_\nu[0]$$

in $\mathcal{D}^+(\mathcal{A})$, we get an induced morphism

$$\alpha : R\varprojlim(\mathcal{F}_\nu) \overset{L}{\otimes} R\varprojlim(\mathcal{G}_\nu) \to R\varprojlim(\mathcal{H}_\nu). \tag{1.6}$$

2. Continuous étale cohomology

2.1. Definition. Let l be a prime number and S be a scheme over $\operatorname{Spec}\mathbb{Z}[1/l]$. Let $S_{\text{Ét}}$ denote the big étale site of S, with underlying category the category of all S-schemes locally of finite type. For $n \in \mathbb{Z}$, we denote by $\mathbb{Z}_l(n)^c_S$ (*resp.* $\mathbb{Q}_l(n)^c_S$) the object $R\varprojlim(\mathbb{Z}/l^\nu(n))$ (*resp.* $\mathbb{Z}_l(n)^c \otimes \mathbb{Q}$) of $\mathcal{D}^+(Ab(S_{\text{Ét}}))$. Here, $\mathbb{Z}/l^\nu(n)$ denotes the sheaf of l^ν-th roots of unity twisted n times.

For X of locally finite type over S, we shall usually denote by $H^i_{\mathrm{cont}}(X,\mathbb{Z}_l(n))$ and $H^i_{\mathrm{cont}}(X,\mathbb{Q}_l(n))$ the hypercohomology groups $\mathbb{H}^i_{\text{ét}}(X,\mathbb{Z}_l(n)^c_S)$ and $\mathbb{H}^i_{\text{ét}}(X,\mathbb{Q}_l(n)^c_S)$.

Note that, by definition, $\mathbb{Z}_l(n)^c_S$ and $\mathbb{Q}_l(n)^c_S$ are complexes of honest sheaves over $S_{\text{Ét}}$ (up to quasi-isomorphism). By subsection 1.4, their hypercohomology coincides with Jannsen's continuous étale cohomology [17]. They are related by the obvious

2.2. Lemma. *There are exact triangles in $\mathcal{D}^+(Ab(S_{\text{Ét}}))$:*

$$\mathbb{Z}_l(n)^c \xrightarrow{l^\nu} \mathbb{Z}_l(n)^c \longrightarrow \mathbb{Z}/l^\nu(n)[0] \longrightarrow \mathbb{Z}_l(n)^c[1]$$

$$\mathbb{Z}_l(n)^c \longrightarrow \mathbb{Q}_l(n)^c \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l(n)[0] \longrightarrow \mathbb{Z}_l(n)^c[1]$$

where $\mathbb{Q}_l/\mathbb{Z}_l(n) := \varinjlim \mathbb{Z}/l^\nu(n)$. □

Let $X \xrightarrow{f} S$ be a morphism. Since $f^*\mathbb{Z}/l^\nu(n) = \mathbb{Z}/l^\nu(n)$, we have morphisms in $\mathcal{D}^+(Ab(X_{\text{Ét}}))$

$$f^*\mathbb{Z}_l(n)^c_S \to \mathbb{Z}_l(n)^c_X$$
$$f^*\mathbb{Q}_l(n)^c_S \to \mathbb{Q}_l(n)^c_X.$$

If f is locally of finite type, then the topos $Shv(X_{\text{Ét}})$ is induced [54, exposé IV, §5] from $Shv(S_{\text{Ét}})$ via f and these are isomorphisms by the remarks after proposition 1.4. This justifies to write $H^*_{\text{cont}}(X)$ rather than $H^*_{\text{cont}}(X/S)$. In other cases, as a rule they are (violently) *not* isomorphisms. For example, if $S = \operatorname{Spec} k$, k a field, $X = \operatorname{Spec} \bar{k}$, $\bar{k}$ its algebraic closure, then $\mathbb{Q}_l(n)^c_X = 0$ while $f^*\mathbb{Q}_l(n)^c_S$ is usually nonzero. However, lemma 2.2 implies

2.3. Lemma. *The cone of $f^*\mathbb{Z}_l(n)^c_S \to \mathbb{Z}_l(n)^c_X$ has uniquely divisible cohomology sheaves.* □

In general, we shall denote by

$$\tilde{H}^*_{\text{cont}}(X/S, \mathbb{Z}_l(n))$$

the cohomology groups $H^*_{\text{ét}}(X, f^*\mathbb{Z}_l(n)^c_S)$. By the above, they coincide with $H^*_{\text{cont}}(X, \mathbb{Z}_l(n))$ when f is locally of finite type, and map naturally to them in any case. The same holds with $\mathbb{Q}_l$ coefficients. If S is affine Noetherian and f is quasi-compact and quasi-separated, then, by [51, C.9], X is the filtering inverse limit of a system of S-schemes X_i of finite presentation, with affine transition maps. Since étale cohomology commutes with this type of inverse limits [54, Exposé VII, cor. 5.8], we have

$$\tilde{H}^*_{\text{cont}}(X/S, \mathbb{Z}_l(n)) = \varinjlim H^*(X_i, \mathbb{Z}_l(n)).$$

Therefore, the groups $\tilde{H}^*_{\text{cont}}(X/S, \mathbb{Z}_l(n))$ generalise those introduced by Jannsen in [18, §11]. In particular, using lemma 1.3, we can describe the stalks of $\mathbb{Z}_l(n)^c_S$ at a geometric S-point s:

$$\mathcal{H}^q(\mathbb{Z}_l(n)^c_S)_s = \tilde{H}^q_{\text{cont}}(\operatorname{Spec} \mathcal{O}^{sh}_{S,s}/S, \mathbb{Z}_l(n)). \tag{2.1}$$

Let $m, n \in \mathbb{Z}$. From (1.6) we deduce canonical pairings:

$$\begin{aligned} \mathbb{Z}_l(m)^c_S \overset{L}{\otimes} \mathbb{Z}_l(n)^c_S &\longrightarrow \mathbb{Z}_l(m+n)^c_S \\ \mathbb{Q}_l(m)^c_S \overset{L}{\otimes} \mathbb{Q}_l(n)^c_S &\longrightarrow \mathbb{Q}_l(m+n)^c_S. \end{aligned} \tag{2.2}$$

l-adic sheaves. For any $\mathbb{Z}[1/l]$-scheme X, let $\mathbb{Z}_l(X_{\text{ét}})$ be the full subcategory of the category $Ab(X_{\text{ét}})^{\mathbb{N}}$ (*small* étale site) consisting of AR-$\mathbb{Z}_l$-constructible sheaves in the sense of [56, exposé VI, déf. 1.5.1] (this category is denoted by $U(X)$ in *loc. cit.*, not. 1.5.4). This is an abelian subcategory. Suppose X connected, let x be a geometric point of X and let $\Pi = \pi_1(X, x)$ be its étale fundamental group at x. Let $\mathbb{Z}_l(\Pi)_{fg}$ be the category of continuous representations of Π onto finitely generated $\mathbb{Z}_l$-modules. Then

$$L \mapsto (L/l^\nu)$$

defines a functor $\mathbb{Z}_l(\Pi)_{fg} \to \mathbb{Z}_l(X_{\text{ét}})$. We have

2.4. Proposition. [56, exposé VI, prop. 1.2.5 and 1.5.5] *This functor is a full embedding; moreover, any object of $\mathbb{Z}_l(X_{\text{ét}})$ is AR-equivalent to the image of an object of $\mathbb{Z}_l(\Pi)_{fg}$, in the sense of* [56, exposé V, 2.2]. □

Ekedahl [8] has constructed a triangulated category $\mathcal{D}_{\mathbb{Z}_l}(X_{\text{ét}})$ with a t-structure such that

- $\mathbb{Z}_l(X_{\text{ét}})$ is the heart of $\mathcal{D}_{\mathbb{Z}_l}(X_{\text{ét}})$ for this t-structure.
- There is a triangulated functor

  $$\mathcal{D}_{\mathbb{Z}_l}(X_{\text{ét}}) \to \mathcal{D}^+(Ab(X_{\text{ét}})^{\mathbb{N}})$$

  which respects the t-structures and whose restriction to hearts coincides with the full embedding $\mathbb{Z}_l(X_{\text{ét}}) \hookrightarrow Ab(X_{\text{ét}})^{\mathbb{N}}$.

This is not completely true, as Ekedahl works modulo essential (ML) isomorphisms. However one can replace his construction by that where one keeps essential isomorphisms uninverted, so that $\mathcal{D}_{\mathbb{Z}_l}(X_{\text{ét}})$ actually maps to $\mathcal{D}^+(Ab(X_{\text{ét}})^{\mathbb{N}})$.

For all $n \in \mathbb{Z}$, the projective system of sheaves $\mathbb{Z}_l(n) = (\mathbb{Z}/l^\nu(n))$ belongs to $\mathbb{Z}_l(X_{\text{ét}})$. By definition, we have

$$\mathbb{Z}_l(n)^c_{|X} = R \varprojlim \mathbb{Z}_l(n)$$

where ${}_{|X}$ denotes restriction to the small étale site of X.

Although we need $\mathbb{Z}_l(n)^c$ to express our conjectures, the reader should be aware that all serious computations are actually done with l-adic sheaves, notably with the help of proposition 2.4. Also beware of similar-looking spectral sequences which are in fact different. For example, if $f : Y \to X$ is a morphism, there are two spectral sequences

$$H^p_{\mathrm{cont}}(X, R^q f_*^{l\text{-adic}}\mathbb{Z}_l(n)) \Rightarrow H^{p+q}_{\mathrm{cont}}(Y, \mathbb{Z}_l(n)) \Leftarrow H^p_{\text{ét}}(X, R^q f_*\mathbb{Z}_l(n)^c).$$

Here we have written $R^q f_*^{l\text{-adic}}$ for the higher derived images of f_* from $\mathbb{Z}_l(Y_{\text{ét}}) \to \mathbb{Z}_l(X_{\text{ét}})$. They are best explained by a naturally commutative diagram of functors

$$\begin{array}{ccccc}
\mathbb{Z}_l(n)_Y \in \mathcal{D}_{\mathbb{Z}_l}(Y_{\text{ét}}) & \longrightarrow & \mathcal{D}^+(Ab(Y_{\text{ét}})^{\mathbb{N}}) & \xrightarrow{R\varprojlim} & \mathcal{D}^+(Ab(Y_{\text{ét}})) \ni \mathbb{Z}_l(n)^c_Y \\
{\scriptstyle Rf_*^{l\text{-adic}}}\downarrow & & {\scriptstyle Rf_*}\downarrow & & {\scriptstyle Rf_*}\downarrow \\
\mathcal{D}_{\mathbb{Z}_l}(X_{\text{ét}}) & \longrightarrow & \mathcal{D}^+(Ab(X_{\text{ét}})^{\mathbb{N}}) & \xrightarrow{R\varprojlim} & \mathcal{D}^+(Ab(Y_{\text{ét}}))
\end{array}$$

The left spectral sequence is obtained from the composition of $Rf_*^{l\text{-adic}}$ with the bottom horizontal composition, applied to $\mathbb{Z}_l(n)_Y$, while the right one is simply the Leray spectral sequence for the right Rf_*, applied to $\mathbb{Z}_l(n)^c_Y$. These two spectral sequences will generally yield different filtrations on $H^*_{\mathrm{cont}}(Y, \mathbb{Z}_l(n))$. In practice it is the l-adic spectral sequence that is the most useful.

3. Classical theorems

This section is included for future reference and can be skipped at first reading. We give a few basic results on continuous étale cohomology, which directly involve $\mathbb{Z}_l(n)^c$ or variants of it. One notable exception is Poincaré duality, for which we are not aware of a formulation involving this object.

Notation. Let $\pi_X : X_{\text{Ét}} \to X_{\text{ét}}$ be the projection of the big étale site of X onto its small étale site. If $C \in \mathcal{D}^+(X_{\text{Ét}})$, we write

$$C_{|X} := R(\pi_X)_* C$$

for the restriction of C to $X_{\text{ét}}$. We simply write

$$\mathbb{Z}_l(n)^c_{|X} = (\mathbb{Z}_l(n)^c_X)_{|X}$$
$$\mathbb{Q}_l(n)^c_{|X} = (\mathbb{Q}_l(n)^c_X)_{|X}.$$

3.1. Theorem. (Proper base change) *Let*

$$\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
{\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\
X' & \xrightarrow{g} & X
\end{array}$$

be a cartesian diagram of schemes, where f is proper and g is locally of finite type. Then, for all $n \in \mathbb{Z}$, the base change morphism

$$g^* Rf_*\mathbb{Z}_l(n)^c_Y \to Rf'_* g'^*\mathbb{Z}_l(n)^c_Y$$

in $\mathcal{D}^+(X'_{\text{Ét}})$ is an isomorphism.

Proof. This follows from applying $R\varprojlim$ to the classical proper base change theorem [54, Exposé XII, th. 5.1]

$$g^* Rf_*\mathbb{Z}/l^\nu(n) \xrightarrow{\sim} Rf'_* g'^*\mathbb{Z}/l^\nu(n)$$

extended to the big étale sites. □

3.2. Corollary. *Let $f:\tilde X\to X$ be a proper morphism and $U\subset X$ be an open subset such that $f_{|f^{-1}(U)}: f^{-1}(U)\to U$ is an isomorphism. Let $Z=X\setminus U$ and $\tilde Z=f^{-1}(Z)$, both with their reduced structure. Then there is an exact triangle*

$$\mathbb{Z}_l(n)^c_X\to i_*\mathbb{Z}_l(n)^c_Z\oplus Rf_*\mathbb{Z}_l(n)^c_{\tilde X}\to \tilde\imath_*\mathbb{Z}_l(n)^c_{\tilde Z}\to \mathbb{Z}_l(n)^c_X[1]$$

where i (resp. $\tilde\imath$ is the closed immersion $Z\hookrightarrow X$ (resp. $\tilde Z\hookrightarrow\tilde X$). Hence a long exact sequence

$$\begin{aligned}\cdots\to H^{i-1}_{\mathrm{cont}}(\tilde Z,\mathbb{Z}_l(n))\to H^i_{\mathrm{cont}}(X,\mathbb{Z}_l(n))&\\ \to H^i_{\mathrm{cont}}(Z,\mathbb{Z}_l(n))\oplus H^i_{\mathrm{cont}}(\tilde X,\mathbb{Z}_l(n))&\to H^i_{\mathrm{cont}}(\tilde Z,\mathbb{Z}_l(n))\to\dots\end{aligned}$$

Proof. (compare *e.g.* [36, proof of prop. 2.1]) Let $j:U\to X$ be the open immersion complementary to i. Apply the exact triangle of functors $j_!j^*\to Id\to i_*i^*\to j_!j^*[1]$ to the morphism $\mathbb{Z}_l(n)^c_X\to Rf_*\mathbb{Z}_l(n)^c_{\tilde X}$ to get a commutative diagram of exact triangles

$$\begin{array}{ccccccc} j_!j^*\mathbb{Z}_l(n)^c_X & \longrightarrow & \mathbb{Z}_l(n)^c_X & \longrightarrow & i_*i^*\mathbb{Z}_l(n)^c_X & \longrightarrow & j_!j^*\mathbb{Z}_l(n)^c_X[1]\\ \downarrow && \downarrow && \downarrow && \downarrow\\ j_!j^*Rf_*\mathbb{Z}_l(n)^c_{\tilde X} & \longrightarrow & Rf_*\mathbb{Z}_l(n)^c_{\tilde X} & \longrightarrow & i_*i^*Rf_*\mathbb{Z}_l(n)^c_{\tilde X} & \longrightarrow & j_!j^*Rf_*\mathbb{Z}_l(n)^c_{\tilde X}[1]\end{array}$$

By theorem 3.1 and the assumption, the left vertical morphism is an isomorphism and

$$i_*i^*Rf_*\mathbb{Z}_l(n)^c_{\tilde X}\xrightarrow{\sim} i_*Rf'_*\mathbb{Z}_l(n)_{\tilde Z},$$

where f' is the restriction of f to $\tilde Z$. Moreover, $i_*i^*\mathbb{Z}_l(n)^c_X\simeq i_*\mathbb{Z}_l(n)^c_Z$. We deduce from this the exact triangle of corollary 3.2. □

3.3. Theorem. (Homotopy invariance) *Let $f:X\to S$ be a morphism locally of finite type whose geometric fibres are isomorphic to affine spaces (e.g. a torsor under a vector bundle). Then, the natural map*

$$\mathbb{Z}_l(n)^c_S\to Rf_*f^*\mathbb{Z}_l(n)^c_S\xrightarrow{\sim} Rf_*\mathbb{Z}_l(n)^c_X$$

is an isomorphism.

Proof. This is clear from the homotopy invariance of étale cohomology with coefficients $\mathbb{Z}/l^\nu(n)$ [54, Exposé XV, cor. 2.2]. □

3.4. Theorem. (Purity, *cf.* [17, (3.17)]) *Let (X,Z) be a regular S-pair of codimension c and $i:Z\hookrightarrow X$ the corresponding closed immersion.*
a) Suppose (X,Z) is a smooth S-pair. Then, for all $n\in\mathbb{Z}$ is an isomorphism

$$\mathbb{Z}_l(n-c)^c_{|Z}[-2c]\xrightarrow{\sim} Ri^!\mathbb{Z}_l(n)^c_{|X}$$

with the usual naturality properties.
b) Assume S and X are quasi-compact, quasi-separated over $\operatorname{Spec}\mathbb{Z}[1/l]$ *and that X has étale l-cohomological dimension N. Then there are maps*

$$Ri^!\mathbb{Z}_l(n)^c_{|X}\leftarrow M(N)\mathbb{Z}_l(n-c)^c_{|Z}[-2c]\to\mathbb{Z}_l(n-c)^c_{|Z}[-2c]$$

whose cones are killed by $M(N)$. Here $M(N)$ is the integer described in [50, 3.3]. *In particular,*

$$Ri^!\mathbb{Q}_l(n)^c_{|X}\simeq\mathbb{Q}_l(n-c)^c_{|Z}[-2c]$$

Proof. a) follows from classical purity ([54, Exposé XVI, cor. 3.8] and [31, ch. VI, §6]), since $i^!$ has as a left adjoint the exact functor i_*, *cf.* lemma 1.1 d). b) follows from Thomason's theorem on absolute cohomological purity [50, 3.5] and the limit result already quoted [51, C.9]. □

3.5. Remark. O. Gabber has announced a proof of absolute cohomological purity in general. Using this, one can get rid of the integer $M(N)$ in theorem 3.4 b).

3.6. Theorem. (Cohomological dimension) *If* $\mathrm{cd}_l(X) \leq N$*, then* $H^q_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) = 0$ *for* $q > N$*.*

Proof. This follows from proposition 1.4. □

3.7. Theorem. (Trace maps) *Let* $f : X' \to X$ *be a finite flat morphism of constant separable degree* d*. Then there exists a morphism*

$$Tr_f : f_*\mathbb{Z}_l(n)^c_{X'} \to \mathbb{Z}_l(n)^c_X$$

whose composition with the adjunction morphism

$$\mathbb{Z}_l(n)^c_X \to f_*\mathbb{Z}_l(n)^c_{X'}$$

is multiplication by d*. This morphism commutes with base change. If* $g : X'' \to X'$ *is another such morphism, then*

$$Tr_{f\circ g} = Tr_f \circ f_* Tr_g.$$

Moreover, if $Z \xrightarrow{i} X$ *is a smooth* S*-pair of codimension* c*,* $f : X' \to X$ *is finite and flat and* $Z' = f^{-1}(Z)$*, then the diagram*

$$\begin{array}{ccc}
\mathbb{Z}_l(n-c)^c_{|Z'}[-2c] & \xrightarrow{\sim} & Ri'^{!}\mathbb{Z}_l(n)^c_{|X'} \\
{\scriptstyle Tr_{f'}}\downarrow & & {\scriptstyle Tr_f}\downarrow \\
\mathbb{Z}_l(n-c)^c_{|Z}[-2c] & \xrightarrow{\sim} & Ri^!\mathbb{Z}_l(n)^c_{|X}
\end{array}$$

commutes, where i' *is the closed immersion* $Z' \hookrightarrow X'$ *and* f' *is the restriction of* f *to* Z'*.*

Proof. Except for the last claim, this has nothing to do with continuous cohomology: the way the trace is described in [54, exp. IX, §5.1] shows that it defines a natural transformation

$$Tr_f : f_*f^* \to Id$$

on $\mathcal{D}^b(X_{\text{Ét}})$, which commutes with base change. In fact, [54, exp. IX, §5.1] only deals with the case when f is étale, but the extension to a finite morphism reduces by the same method to the case of a radicial morphism, when f^* is an equivalence of categories.

The last claim is local for the étale topology and to prove it we reduce as usual to the case when $c = 1$ and Z is defined say, by a rational function $a \in \Gamma(X - Z, \mathbb{G}_m)$. Then the isomorphism

$$i_*\mathbb{Z}/l^\nu(n-1)_Z \xrightarrow{\sim} \mathcal{H}^2_Z(\mathbb{Z}/l^\nu(n))$$

is given for all ν by the "extraordinary cup-product" [16, 9.14] by the Kummer class of a in $H^2_Z(X, \mathbb{Z}/l^\nu(1))$, and similarly for (X', Z'). The diagram of sheaves

$$\begin{array}{ccc}
f_*i'_*\mathbb{Z}/l^\nu(n-1)_{Z'} & \xrightarrow{(a)} & f_*\mathcal{H}^2_{Z'}(\mathbb{Z}/l^\nu(n)) \\
{\scriptstyle f_*i'_*Tr_{f'}}\downarrow & & {\scriptstyle Tr_f}\downarrow \\
i_*\mathbb{Z}/l^\nu(n-1)_Z & \xrightarrow{(a)} & \mathcal{H}^2_Z(\mathbb{Z}/l^\nu(n))
\end{array}$$

clearly commutes for all $\nu \geq 1$; applying $Ri^! R\varprojlim$ to this commutative diagram of l-adic sheaves yields the result. □

3.8. Theorem. (Gersten's conjecture) *Let* k *be a field of characteristic* $\neq l$ *and* X *a smooth affine variety over* k*. Let* $\{x_1, \dots, x_r\}$ *be a finite set of points of* X *and* $Y = \operatorname{Spec} \mathcal{O}_{X,x_1,\dots,x_r}$*. Then the Gersten complex*

$$0 \to \tilde H^i_{\mathrm{cont}}(Y/k, \mathbb{Z}_l(n)) \to \coprod_{y\in Y^{(0)}} \tilde H^i_{\mathrm{cont}}(k(y)/k, \mathbb{Z}_l(n)) \to \coprod_{y\in Y^{(1)}} \tilde H^{i-1}_{\mathrm{cont}}(k(y)/k, \mathbb{Z}_l(n-1)) \to \dots$$

is universally exact in the sense of Grayson [13] *for all* i*. If* $X' \xrightarrow{f} X$ *is finite and flat, the diagram*

$$\begin{array}{ccccccc} 0\to\tilde H^i_{\mathrm{cont}}(Y'/k,\mathbb{Z}_l(n))\to & \coprod_{y\in Y'^{(0)}} \tilde H^i_{\mathrm{cont}}(k(y)/k,\mathbb{Z}_l(n))\to & \coprod_{y\in Y'^{(1)}} \tilde H^{i-1}_{\mathrm{cont}}(k(y)/k,\mathbb{Z}_l(n-1))\to\dots \\ f_*\downarrow & f_*\downarrow & f_*\downarrow \\ 0\to\tilde H^i_{\mathrm{cont}}(Y/k,\mathbb{Z}_l(n))\to & \coprod_{y\in Y^{(0)}} \tilde H^i_{\mathrm{cont}}(k(y)/k,\mathbb{Z}_l(n))\to & \coprod_{y\in Y^{(1)}} \tilde H^{i-1}_{\mathrm{cont}}(k(y)/k,\mathbb{Z}_l(n-1))\to\dots \end{array}$$

commutes, where $Y' = Y\times_X X'$*.*

Proof. The first claim follows from [4, cor. 5.1.11, prop. 5.3.2 and ex. 7.1.8], theorem 3.3 and theorem 3.4, plus [4, th. 6.2.5] and theorem 3.7 in case k is finite. More precisely, by homotopy invariance we get universally exact Cousin complexes

$$0\to\tilde H^i_{\mathrm{cont}}(Y/k,\mathbb{Z}_l(n))\to\coprod_{y\in Y^{(0)}}\tilde H^i_{y,\mathrm{cont}}(Y/k,\mathbb{Z}_l(n))\to\coprod_{y\in Y^{(1)}}\tilde H^{i+1}_{y,\mathrm{cont}}(Y/k,\mathbb{Z}_l(n-1))\to\dots$$

and then purity allows us to rewrite them as Gersten complexes. The second claim is clear for the Cousin complexes, and follows for Gersten complexes from the last part of theorem 3.7. □

3.9. Remark. With more effort one could check that

$$K\mapsto \tilde H^*_{\mathrm{cont}}(K/k,\mathbb{Z}_l(n+*)),$$

where k is a base field and K runs through finitely generated extensions of K, defines a cycle module in the sense of Rost [38]. We skip this since it will not be needed here.

3.1. **Cohomology with proper supports.** For $X\xrightarrow{f} S$ separated of finite type, we define *continuous cohomology with proper supports* as

$$H^i_{\mathrm{cont},c}(X/S,\mathbb{Z}_l(n)) := H^i_{\text{ét}}(S,R\varprojlim Rf_!\mathbb{Z}/l^\nu(n)) \tag{3.1}$$

where $Rf_!$ is the functor "higher direct image with proper support" from [54, appendix to exposé XVII]. By abuse of notation, we shall denote the object $R\varprojlim Rf_!\mathbb{Z}/l^\nu(n)$ by $Rf_!\mathbb{Z}_l(n)^c_X$. We have

3.10. Lemma. *There are natural homomorphisms*

$$H^i_{\mathrm{cont},c}(X/S,\mathbb{Z}_l(n))\to H^i_{\mathrm{cont}}(X,\mathbb{Z}_l(n))$$

which are isomorphisms when X/S *is proper.*

Proof. This follows from the same fact with finite coefficients. □

Using [54, exp. XVII, (6.2.7.3)], one can define trace maps for finite flat morphisms in continuous cohomology with proper supports, verifying the same properties as in cohomology without supports (see theorem 3.7). We shall not need them here, so skip the details.

3.2. **(Borel-Moore) étale homology.** For any $X\xrightarrow{f} S$ locally of finite type, define

$$L_l(n)^c_{X/S} := R\varprojlim Rf^!\mathbb{Z}/l^\nu(n)$$

where $Rf^!$ is the extraordinary inverse image of [54, Exposé XVIII]. We define *continuous étale homology* as

$$H_i^{\mathrm{cont}}(X/S,\mathbb{Z}_l(n)) := H^{-i}_{\text{ét}}(X,L_l(-n)^c_{X/S}). \tag{3.2}$$

This coincides with Laumon's definition of l-adic étale homology [29] in the case $S=\operatorname{Spec} k$, k an algebraically closed field.

3.11. Proposition. *a) Let $i : Z \hookrightarrow X$ be a closed immersion and $j : U \hookrightarrow X$ be the complementary open immersion. Then there are long exact sequences*

$$\cdots \to H_i^{\mathrm{cont}}(Z/S, \mathbb{Z}_l(n)) \xrightarrow{i_*} H_i^{\mathrm{cont}}(X/S, \mathbb{Z}_l(n)) \xrightarrow{j^!} H_i^{\mathrm{cont}}(U/S, \mathbb{Z}_l(n)) \to H_{i-1}^{\mathrm{cont}}(Z/S, \mathbb{Z}_l(n)) \to \dots$$

b) Continuous étale homology is contravariant for étale S-morphisms or finite flat S-morphisms and covariant for finite flat S-morphisms; if $u : X' \to X$ is finite and flat, of constant separable degree d, then the composition

$$H_*^{\mathrm{cont}}(X/S, \mathbb{Z}_l(n)) \xrightarrow{u^*} H_*^{\mathrm{cont}}(X'/S, \mathbb{Z}_l(n)) \xrightarrow{u_*} H_*^{\mathrm{cont}}(X/S, \mathbb{Z}_l(n))$$

is multiplication by d.
c) If X is smooth compactifiable of pure dimension d, there are canonical isomorphisms

$$H_i^{\mathrm{cont}}(X/S, \mathbb{Z}_l(n)) \simeq H^{2d-i}_{\mathrm{cont}}(X, \mathbb{Z}_l(d-n)).$$

Proof. a) This follows from applying $R\varprojlim$ to the classical exact triangles

$$i_*Ri^!Rf^!\mathbb{Z}/l^\nu(n) \to Rf^!\mathbb{Z}/l^\nu(n) \to Rj_*j^!Rf^!\mathbb{Z}/l^\nu(n) \to i_*Ri^!Rf^!\mathbb{Z}/l^\nu(n)[1].$$

b) If $u : X' \to X$ is étale or finite and flat, the map

$$u^* : H_*^{\mathrm{cont}}(X/S, \mathbb{Z}_l(n)) \to H_*^{\mathrm{cont}}(X'/S, \mathbb{Z}_l(n))$$

is defined through the unit morphism

$$L_l(-n)^c_{X/S} \to Ru_*u^*L_l(-n)^c_{X/S} = Ru_*u^!L_l(-n)^c_{X/S} = Ru_*L_l(-n)^c_{X'/S}.$$

If u is finite and flat, the map

$$u_* : H_*^{\mathrm{cont}}(X'/S, \mathbb{Z}_l(n)) \to H_*^{\mathrm{cont}}(X/S, \mathbb{Z}_l(n))$$

is defined through the trace map (see proof of theorem 3.7)

$$u_*L_l(-n)^c_{X'/S} = u_*u^*L_l(-n)^c_{X/S} \xrightarrow{Tr_u} L_l(-n)^c_{X/S}.$$

The formula $u_*u^* = d$ is then clear.

c) This follows from applying $R\varprojlim$ to the isomorphisms of [54, exposé XVIII, th. 3.2.5]

$$Rf^!\mathbb{Z}/l^\nu(n) \simeq f^*\mathbb{Z}/l^\nu(n+d)[2d].$$

□

3.3. **Duality.** Recall [54, exposé XVIII, th. 3.1.4] that, for $f : X \to S$ separated of finite type, the functors $Rf_!$ and $Rf^!$ are (partially) adjoint on the derived categories of sheaves of $\mathbb{Z}/l^\nu$-Modules. Let $T_f : Rf_!Rf^! \to Id$ be the counit of this adjunction. It induces an isomorphism

$$Rf_*RHom_{\mathcal{D}^b(X_{\text{ét}},\mathbb{Z}/l^\nu)}(A, Rf^!B) \xrightarrow{(T_f)_*} RHom_{\mathcal{D}^b(S_{\text{ét}},\mathbb{Z}/l^\nu)}(Rf_!A, B)$$

for any A, B. Taking $A = \mathbb{Z}/l^\nu, B = \mathbb{Z}/l^\nu(-n)$, this translates as

$$Rf_*Rf^!\mathbb{Z}/l^\nu(-n) \xrightarrow{\sim} RHom_S(Rf_!\mathbb{Z}/l^\nu, \mathbb{Z}/l^\nu(-n)) \simeq RHom_S(Rf_!\mathbb{Z}/l^\nu(n), \mathbb{Z}/l^\nu).$$

These isomorphisms are compatible when ν varies in the sense that, for $\mu \geq \nu$, the diagram of pairings

$$\begin{array}{ccc} Rf_*Rf^!\mathbb{Z}/l^\mu(-n) \overset{L}{\otimes} Rf_!\mathbb{Z}/l^\mu(n) & \longrightarrow & \mathbb{Z}/l^\mu \\ \downarrow & & \downarrow \\ Rf_*Rf^!\mathbb{Z}/l^\nu(-n) \overset{L}{\otimes} Rf_!\mathbb{Z}/l^\nu(n) & \longrightarrow & \mathbb{Z}/l^\nu \end{array}$$

commutes, where the left map is the tensor product of the two projections. This translates as a pairing of $\mathbb{Z}_l$-sheaves, in $\mathcal{D}_{\mathbb{Z}_l}(S_{\text{ét}})$

$$Rf_*^{l\text{-adic}}L_l(-n)_{X/S} \overset{L}{\otimes} Rf_!^{l\text{-adic}}\mathbb{Z}_l(n)_X \to \mathbb{Z}_l$$

whose adjunction

$$Rf_*^{l\text{-adic}}L_l(-n)_{X/S} \to R\underline{Hom}_{\mathcal{D}_{\mathbb{Z}_l}(S_{\text{ét}})}(Rf_!^{l\text{-adic}}\mathbb{Z}_l(n)_X, \mathbb{Z}_l) \tag{3.3}$$

is an isomorphism in $\mathcal{D}_{\mathbb{Z}_l}(S_{\text{ét}})$.

Note that, more generally, there are pairings of l-adic sheaves

$$Rf_*^{l\text{-adic}}L_l(-p)_{X/S} \overset{L}{\otimes} Rf_!^{l\text{-adic}}\mathbb{Z}_l(q)_X \to \mathbb{Z}_l(q-p)$$

which induce products in cohomology

$$H_i^{\text{cont}}(X/S, \mathbb{Z}_l(p)) \times H^j_{\text{cont},c}(X/S, \mathbb{Z}_l(q)) \to H^{j-i}_{\text{cont}}(S, \mathbb{Z}_l(q-p)). \tag{3.4}$$

3.4. **Schemes over $\mathbb{F}_p$.** In this paper, we shall mainly concentrate on the case $S = \operatorname{Spec}\mathbb{F}_p$, where p is a prime number $\neq l$. The next results are special to this case.

3.12. Lemma. *Let X be a scheme of finite type over $\operatorname{Spec}\mathbb{F}_p$. Then for all $n \in \mathbb{Z}$, $i \geq 0$,*
a) $H^i_{\text{cont}}(X, \mathbb{Z}_l(n)) \xrightarrow{\sim} \varprojlim H^i_{\text{ét}}(X, \mathbb{Z}/l^\nu(n))$.
b) $H^i_{\text{cont}}(X, \mathbb{Z}_l(n))$ is a finitely generated $\mathbb{Z}_l$-module.
The same assertions hold for étale cohomology with proper supports and étale homology.

Proof. a) follows from Deligne's theorem that the $H^i_{\text{ét}}(X, \mathbb{Z}/l^\nu(n))$ are finite [55, Th. finitude]. b) By a), this $\mathbb{Z}_l$-module is compact. On the other hand, the exact sequence

$$H^i_{\text{cont}}(X, \mathbb{Z}_l(n)) \xrightarrow{l} H^i_{\text{cont}}(X, \mathbb{Z}_l(n)) \to H^i_{\text{ét}}(X, \mathbb{Z}/l(n))$$

shows that it is finite modulo l. Therefore it is finitely generated. Similarly for the other cases. □

3.13. Proposition. *Let X be a (not necessarily smooth) variety over a finite field k, with* $\operatorname{char} k \neq l$. *Let $\bar{k}$ be the algebraic closure of k, $G_k = Gal(\bar{k}/k)$ and $\overline{X} = \bar{k} \otimes_k X$. Then, for all $n \in \mathbb{Z}$ and $q \geq 0$, there is an exact sequence*

$$0 \to H^{q-1}_{\text{cont}}(\overline{X}, \mathbb{Z}_l(n))_{G_k} \to H^q_{\text{cont}}(X, \mathbb{Z}_l(n)) \to H^q_{\text{cont}}(\overline{X}, \mathbb{Z}_l(n))^{G_k} \to 0.$$

There are similar exact sequences for continuous cohomology with proper supports and continuous homology.

Proof. (Note how we use essentially the l-adic sheaf definition of continuous étale cohomology, and not the complexes $\mathbb{Z}_l(n)^c$.) We do it only for continuous cohomology, the other cases being similar. By [17, (3.4)], there is a "Hochschild-Serre" spectral sequence

$$E_2^{p,q} = H^p_{\text{cont}}(k, H^q_{\text{cont}}(\overline{X}, \mathbb{Z}_l(n))) \Rightarrow H^{p+q}_{\text{cont}}(X, \mathbb{Z}_l(n)).$$

Moreover, by [17, §2], we can identify the E_2-terms to continuous cohomology of the profinite group $G_k = Gal(\bar{k}/k)$ in the sense of Tate [48].

Since k is finite, we have

$$E_2^{p,q} = \begin{cases} 0 & \text{if } p \neq 0,1 \\ H^q_{\text{cont}}(\overline{X}, \mathbb{Z}_l(n))^{G_k} & \text{if } p = 0 \\ H^q_{\text{cont}}(\overline{X}, \mathbb{Z}_l(n))_{G_k} & \text{if } p = 1. \end{cases}$$

This concludes the proof. □

3.14. Corollary. *With X as in proposition 3.13, let $d_i(n) = \dim_{\mathbb{Q}_l} H^i_{\text{cont}}(X, \mathbb{Q}_l(n))$. Then we have*

$$\dim H^i_{\text{cont}}(\overline{X}, \mathbb{Q}_l(n))^{G_k} = \dim H^i_{\text{cont}}(\overline{X}, \mathbb{Q}_l(n))_{G_k} = d_i(n) - d_{i-1}(n) + \dots.$$

There are similar identities for continuous cohomology with proper supports and continuous homology.

Proof. Since $H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))$ is a finite-dimensional $\mathbb{Q}_l$-vector space, we have

$$\dim_{\mathbb{Q}_l} H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))^{G_k} = \dim_{\mathbb{Q}_l} H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))_{G_k}$$

for all i, as follows from the exact sequence

$$0\to H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))^{G_k}\to H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))\xrightarrow{1-F} H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))\to H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))_{G_k}\to 0.$$

The result then follows from proposition 3.13 which shows that

$$d_i(n)=\dim H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))^{G_k}+\dim H^{i-1}_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))_{G_k}$$

□

3.15. Corollary. *Let X be a (not necessarily smooth) variety over a finite field k, with $\operatorname{char} k\neq l$. Then, for all $n\in\mathbb{Z}$,*

$$\sum_{i\geq 0}(-1)^i\dim_{\mathbb{Q}_l} H^i_{\mathrm{cont}}(X,\mathbb{Q}_l(n))=0.$$

The same holds for continuous cohomology with proper supports and for continuous étale homology. □

3.5. **Duality again.** Let $B\mapsto B^\#[1]$ be the functor from $\mathcal{D}^b(\mathbb{Z}/l^\nu-mod)$ to $\mathcal{D}^b((\operatorname{Spec}\mathbb{F}_p)_{\text{ét}},\mathbb{Z}/l^\nu)$ which associates to a complex of $\mathbb{Z}/l^\nu$-modules the corresponding complex of constant $(\operatorname{Spec}\mathbb{F}_p)_{\text{ét}}$-sheaves, shifted once to the left. This is right adjoint to $R\Gamma$, with counit

$$R\Gamma(\mathbb{F}_p,B^\#[1])\to B$$

induced by the canonical isomorphism

$$H^1(\mathbb{F}_p,\mathbb{Z}/l^\nu)\xrightarrow{\sim}\mathbb{Z}/l^\nu$$

given by evaluation on the Frobenius. Applying $R\Gamma(\mathbb{F}_p,-)$ to (3.3) and then applying this adjunction, we get a chain of adjunction isomorphisms in $\mathcal{D}^b_{\mathbb{Z}_l}(*)$:

$$\begin{aligned}R\Gamma(X,L_l(-n))&\simeq R\Gamma(\mathbb{F}_p,Rf_*^{l\text{-adic}}L_l(-n)_{X/\operatorname{Spec}\mathbb{F}_p})\\&\simeq R\Gamma(\mathbb{F}_p,R\underline{Hom}_{\mathcal{D}_{\mathbb{Z}_l}((\operatorname{Spec}\mathbb{F}_p)_{\text{ét}})}(Rf_!^{l\text{-adic}}\mathbb{Z}_l(n)_X,\mathbb{Z}_l))\\&\simeq RHom_{\mathcal{D}_{\mathbb{Z}_l}((\operatorname{Spec}\mathbb{F}_p)_{\text{ét}})}(Rf_!^{l\text{-adic}}\mathbb{Z}_l(n)_X,\mathbb{Z}_l))\\&\simeq RHom_{\mathcal{D}^b_{\mathbb{Z}_l}(*)}(R\Gamma(\mathbb{F}_p,Rf_!^{l\text{-adic}}\mathbb{Z}_l(n)_X),\mathbb{Z}_l)[1].\end{aligned}$$

By [8, th. 7.2], the two sides can be identified with objects of the derived category of finitely generated $\mathbb{Z}_l$-modules. We then get isomorphisms

$$H_i^{\mathrm{cont}}(X,\mathbb{Z}_l(n))\xrightarrow{\sim}H^{i+1}(RHom_{\mathcal{D}^b(\mathbb{Z}_l-mod)}(R\Gamma(\mathbb{F}_p,Rf_!^{l\text{-adic}}\mathbb{Z}_l(n)_X),\mathbb{Z}_l)).$$

By the standard Ext spectral sequence, this yields in a short exact sequence

$$0\to Ext(H^{i+2}_{c,\mathrm{cont}}(X,\mathbb{Z}_l(n)),\mathbb{Z}_l)\to H_i^{\mathrm{cont}}(X,\mathbb{Z}_l(n))\to Hom(H^{i+1}_{c,\mathrm{cont}}(X,\mathbb{Z}_l(n)),\mathbb{Z}_l)\to 0. \tag{3.5}$$

For any $\mathbb{Z}_l$-module A, let $\bar{A}$ denote A/tors.

3.16. Lemma. *Let A be a finitely generated $\mathbb{Z}_l$-module. Then there is a canonical isomorphism*

$$Ext(A,\mathbb{Z}_l)\simeq Hom(A_{\mathrm{tors}},\mathbb{Q}_l/\mathbb{Z}_l).$$

Proof. This follows from the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & Hom(A_{\mathrm{tors}}, \mathbb{Q}_l/\mathbb{Z}_l) & \longrightarrow & Ext(A_{\mathrm{tors}}, \mathbb{Z}_l) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
Hom(A, \mathbb{Q}_l) & \longrightarrow & Hom(A, \mathbb{Q}_l/\mathbb{Z}_l) & \longrightarrow & Ext(A, \mathbb{Z}_l) & \longrightarrow & Ext(A, \mathbb{Q}_l) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
Hom(\bar{A}, \mathbb{Q}_l) & \longrightarrow & Hom(\bar{A}, \mathbb{Q}_l/\mathbb{Z}_l) & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

(We used that A_{tors} has finite exponent and that $\bar{A}$ is free). □

In view of lemma 3.16, the first group in the exact sequence (3.5) is isomorphic to

$$Hom(H^{i+2}_{c,\mathrm{cont}}(X, \mathbb{Z}_l(n))_{\mathrm{tors}}, \mathbb{Q}_l/\mathbb{Z}_l)$$

while the last one is obviously isomorphic to

$$Hom(\bar{H}^{i+1}_{c,\mathrm{cont}}(X, \mathbb{Z}_l(n)), \mathbb{Z}_l).$$

We finally get the following theorem:

3.17. Theorem. (*cf.* [33, lemma 5.3] in the smooth, proper case) *For any X separated of finite type over $\mathbb{F}_p$, the geometric adjunction between $Rf_!$ and $Rf^!$ and the arithmetic adjunction over* $\operatorname{Spec}\mathbb{F}_p$ *induce perfect pairings*

$$\bar{H}_i^{\mathrm{cont}}(X, \mathbb{Z}_l(n)) \times \bar{H}^{i+1}_{c,\mathrm{cont}}(X, \mathbb{Z}_l(n)) \to \mathbb{Z}_l$$

given by the products (3.4) *followed by the trace, as well as isomorphisms*

$$Hom(H^{i+2}_{c,\mathrm{cont}}(X, \mathbb{Z}_l(n))_{\mathrm{tors}}, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\sim} H_i^{\mathrm{cont}}(X, \mathbb{Z}_l(n))_{\mathrm{tors}}.$$

In particular, if X is smooth and proper of pure dimension d, $H^{2d+1}_{\mathrm{cont}}(X, \mathbb{Z}_l(d))$ is canonically isomorphic to $\mathbb{Z}_l$ and the resulting pairings induced by cup-product

$$H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) \times H^{2d+1-i}_{\mathrm{cont}}(X, \mathbb{Z}_l(d-n)) \to \mathbb{Z}_l$$

are perfect modulo torsion. □

4. Restriction to $\mathbb{F}_p$

We now start our investigation. Let $p \neq l$. We shall begin by computing the restriction of $\mathbb{Z}_l(n)^c_{\mathbb{F}_p}$ and $\mathbb{Q}_l(n)^c_{\mathbb{F}_p}$ to the *small* étale site of $\operatorname{Spec}\mathbb{F}_p$.

Let $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n$. Note that the discrete abelian group $\mathbb{Q}/\mathbb{Z}$ is naturally a module over this profinite ring, with continuous action. Consider $\hat{\mathbb{Z}}$ as an additive profinite group. We introduce the discrete topological $\hat{\mathbb{Z}}$-module

$$\tilde{M} = \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$$

provided with the following action: for $(a, r, s) \in \hat{\mathbb{Z}} \times \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$, $a(r, s) = (r, ar + s)$. So we have a short exact sequence

$$0 \to \mathbb{Q}/\mathbb{Z} \to \tilde{M} \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{4.1}$$

whose kernel and cokernel are trivial $\hat{\mathbb{Z}}$-modules.

Let M be the pull-back of this extension under the projection map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, so that we have another short exact sequence of $\hat{\mathbb{Z}}$-modules

$$0 \to \mathbb{Q}/\mathbb{Z} \to M \to \mathbb{Q} \to 0. \tag{4.2}$$

4.1. Definition. We denote by $\mathbb{Z}^c$ the object of $\mathcal{D}^b(Ab(\hat{\mathbb{Z}}))$ represented by the complex of length 1

$$\mathbb{Q} \xrightarrow{\gamma} M \tag{4.3}$$

where M is as above and γ is the composition $\mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \hookrightarrow M$. ($\mathbb{Q}$ sits in degree 0 and M in degree 1.)
We set $\mathbb{Q}^c = \mathbb{Z}^c \otimes \mathbb{Q}$.

4.2. Proposition. *We have*

$$\mathcal{H}^i(\mathbb{Z}^c) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Q} & \text{if } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbb{H}^i(\hat{\mathbb{Z}}, \mathbb{Z}^c) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first formulas are obvious in view of the definition of $\mathbb{Z}^c$. For the next ones, we need:

4.3. Lemma. $H^i(\hat{\mathbb{Z}}, M) = 0$ *for* $i > 0$*; there is an exact sequence*

$$0 \to \mathbb{Q}/\mathbb{Z} \to H^0(\hat{\mathbb{Z}}, M) \to \mathbb{Z} \to 0.$$

Indeed, $H^2(\hat{\mathbb{Z}}, M) = 0$ since M is divisible, and the exact sequence (4.2) yields a long cohomology exact sequence

$$0 \to H^0(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \to H^0(\hat{\mathbb{Z}}, M) \to H^0(\hat{\mathbb{Z}}, \mathbb{Q}) \xrightarrow{\delta} H^1(\hat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \to H^1(\hat{\mathbb{Z}}, M) \to H^1(\hat{\mathbb{Z}}, \mathbb{Q}).$$

The definition of the action of $\hat{\mathbb{Z}}$ on $\tilde{M}$ shows that δ is surjective, which gives the claim.

To compute the hypercohomology of $\mathbb{Z}^c$, we use lemma 4.3 and the long exact sequence

$$\begin{aligned} 0 \to \mathbb{H}^0(\hat{\mathbb{Z}}, \mathbb{Z}^c) \to H^0(\hat{\mathbb{Z}}, \mathbb{Q}) \xrightarrow{\gamma_*} H^0(\hat{\mathbb{Z}}, M) \to \mathbb{H}^1(\hat{\mathbb{Z}}, \mathbb{Z}^c) \\ \to H^1(\hat{\mathbb{Z}}, \mathbb{Q}) \xrightarrow{\gamma_*} H^1(\hat{\mathbb{Z}}, M) \to \mathbb{H}^2(\hat{\mathbb{Z}}, \mathbb{Z}^c) \to 0. \end{aligned}$$

We have $H^i(\hat{\mathbb{Z}}, \mathbb{Q}) = H^i(\hat{\mathbb{Z}}, M) = 0$ for $i > 0$ and the map

$$H^0(\hat{\mathbb{Z}}, \mathbb{Q}) \xrightarrow{\gamma_*} H^0(\hat{\mathbb{Z}}, M)$$

sends $\mathbb{Q}$ onto the kernel $\mathbb{Q}/\mathbb{Z}$ of the extension in lemma 4.3. Hence the claims on $\mathbb{H}^*(\hat{\mathbb{Z}}, \mathbb{Z}^c)$. □

The canonical generator e of $H^1_{\text{cont}}(\hat{\mathbb{Z}}, \hat{\mathbb{Z}})$ represented by the continuous homomorphism

$$e : \hat{\mathbb{Z}} \xrightarrow{Id} \hat{\mathbb{Z}}$$

will play an important rôle in the sequel. We now use it to describe the boundary homomorphism associated to $\mathbb{Z}^c$:

4.4. Proposition. *Let ∂ be the morphism defined by the exact triangle*

$$\mathbb{Z} \to \mathbb{Z}^c \to \mathbb{Q}[-1] \xrightarrow{\partial} \mathbb{Z}[1]$$

stemming from proposition 4.2. Then the diagram

$$\begin{tikzcd} \mathbb{Q}[-1] \arrow[r, "\partial"] \arrow[d] & \mathbb{Z}[1] \\ \mathbb{Q}/\mathbb{Z}[-1] \arrow[r, "\cdot e"] & \mathbb{Q}/\mathbb{Z}[0] \arrow[u, "\beta"'] \end{tikzcd}$$

commutes, where $\cdot e$ is cup-product by e, β is the Bockstein and the left vertical map is induced by the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$.

Proof. This follows from the two commutative diagrams of exact triangles

$$
\begin{tikzcd}
 & \mathbb{Q}[-1] \arrow[r, equal] \arrow[d] & \mathbb{Q}[-1] \arrow[d, "\partial"'] & \\
\mathbb{Q}[0] \arrow[r] \arrow[d, equal] & \mathbb{Q}/\mathbb{Z}[0] \arrow[r, "\beta"] \arrow[d] & \mathbb{Z}[1] \arrow[r] \arrow[d] & \mathbb{Q}[1] \arrow[d, equal] \\
\mathbb{Q}[0] \arrow[r, "\gamma"] & M[0] \arrow[r] \arrow[d] & \mathbb{Z}^c[1] \arrow[r] \arrow[d] & \mathbb{Q}[1] \\
 & \mathbb{Q}[0] \arrow[r, equal] & \mathbb{Q}[0] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Q}[-1] \arrow[r] \arrow[d] & \mathbb{Q}/\mathbb{Z}[-1] \arrow[d] \\
\mathbb{Q}/\mathbb{Z}[0] \arrow[r, equal] \arrow[d] & \mathbb{Q}/\mathbb{Z}[0] \arrow[d] \\
M[0] \arrow[r] \arrow[d] & \tilde{M}[0] \arrow[d] \\
\mathbb{Q}[0] \arrow[r] & \mathbb{Q}/\mathbb{Z}[0]
\end{tikzcd}
$$

and the fact that, by definition of $\tilde{M}$, the boundary map $\mathbb{Q}/\mathbb{Z}[-1] \to \mathbb{Q}/\mathbb{Z}[0]$ corresponding to $\tilde{M}$ is given by cup-product by e. □

4.5. Corollary. $\mathbb{Q}^c \simeq \mathbb{Q}[0] \oplus \mathbb{Q}[-1]$.

Proof. Indeed, proposition 4.4 shows that $\partial \otimes \mathbb{Q} = 0$. □

We can consider $\mathbb{Z}^c$ as a complex of sheaves over $\operatorname{Spec} \mathbb{F}_p$, by identifying G with $\hat{\mathbb{Z}}$ by means of the absolute Frobenius.

4.6. Theorem. *Denote by $\mathbb{Z}_l(n)^c_{|\mathbb{F}_p}$ (resp. $\mathbb{Q}_l(n)^c_{|\mathbb{F}_p}$) the object $R\pi_*\mathbb{Z}_l(n)^c_{\mathbb{F}_p}$ (resp. $R\pi_*\mathbb{Q}_l(n)^c_{\mathbb{F}_p}$) of the derived category $\mathcal{D}^+(Ab(\operatorname{Spec} \mathbb{F}_p)_{\text{ét}})$, where $\pi : (\operatorname{Spec} \mathbb{F}_p)_{\text{Ét}} \to (\operatorname{Spec} \mathbb{F}_p)_{\text{ét}}$ is the natural projection. Then*
a) $\mathbb{Z}_l(n)^c_{|\mathbb{F}_p} = \mathbb{Q}_l/\mathbb{Z}_l(n)[-1]$ if $n \neq 0$.
b) $\mathbb{Z}_l(0)^c_{|\mathbb{F}_p} = \mathbb{Z}^c \otimes \mathbb{Z}_l$.

Proof. We first compute the cohomology sheaves of $\mathbb{Z}_l(n)^c_{|\mathbb{F}_p}$ for $n \neq 0$. Let k be a finite field of characteristic p. We note that the groups $H^0(k, \mathbb{Z}/l^\nu(n))$ and $H^1(k, \mathbb{Z}/l^\nu(n))$ have order bounded independently of ν. Using lemma 3.12, it follows that

$$
\begin{aligned}
&H^0_{\text{cont}}(k, \mathbb{Z}_l(n)) = 0 \\
&H^1_{\text{cont}}(k, \mathbb{Z}_l(n)) \quad \text{is finite} \\
&H^q_{\text{cont}}(k, \mathbb{Z}_l(n)) = 0 \quad \text{for } q > 1.
\end{aligned}
$$

It follows from this that $H^q_{\text{cont}}(k, \mathbb{Q}_l(n)) = 0$ for all q, hence that $\mathbb{Q}_l(n)^c_{|\mathbb{F}_p} = 0$. We conclude by using lemma 2.2.

To prove b), we note that, after proposition 4.2, $\mathbb{Z}^c \overset{L}{\otimes} \mathbb{Z}/l^\nu \simeq \mathbb{Z}/l^\nu[0]$ for all $\nu \geq 1$. From this we deduce a morphism $\mathbb{Z}^c \otimes \mathbb{Z}_l \to \mathbb{Z}_l(0)^c_{|\mathbb{F}_p}$ out of the composite morphisms

$$\mathbb{Z}^c \otimes \mathbb{Z}_l \to \mathbb{Z}^c \overset{L}{\otimes} \mathbb{Z}/l^\nu \to \mathbb{Z}/l^\nu[0].$$

We claim that this morphism is an isomorphism. To see this, we note that it inserts into an exact triangle

$$C \to \mathbb{Z}^c \otimes \mathbb{Z}_l \to \mathbb{Z}_l(0)^c_{|\mathbb{F}_p} \to C[1]$$

where $C = R\varprojlim(\mathbb{Z}^c \otimes \mathbb{Z}_l, l)$. The claim now follows from

4.7. Lemma. $C = 0$.

Indeed, we shall show that the map $R\varprojlim(\mathbb{Q}_l, l) \xrightarrow{R\varprojlim\gamma\otimes 1} R\varprojlim(M\otimes\mathbb{Z}_l, l)$ is an isomorphism. We have evidently $R\varprojlim(\mathbb{Q}_l, l) = \mathbb{Q}_l[0]$. On the other hand, lemma 4.3 and a little bit of computation show that

$$\begin{aligned}
&\mathbb{H}^0(k, R\varprojlim(M\otimes\mathbb{Z}_l, l)) = \mathbb{Q}_l\\
&\mathbb{H}^q(k, R\varprojlim(M\otimes\mathbb{Z}_l, l)) = 0 \quad \text{for } q>0
\end{aligned}$$

for any finite extension $k/\mathbb{F}_p$, that is

$$R\varprojlim(M\otimes\mathbb{Z}_l, l) \simeq \mathbb{Q}_l[0].$$

To conclude, it suffices to notice that the map $\mathbb{Q}_l[0] \to \mathbb{Q}_l[0]$ defined by $\gamma\otimes 1$ and the above computation is the inverse limit of the projections $\mathbb{Q}_l \to \mathbb{Q}_l/\mathbb{Z}_l$ (for multiplication by l), hence the identity. □

4.8. Corollary. *The canonical generator e of $H^1_{\text{cont}}(\mathbb{F}_p, \mathbb{Z}_l)$ is the image of the canonical generator of $\mathbb{H}^1(\mathbb{F}_p, \mathbb{Z}^c)$ (also denoted by e). The composition*

$$\mathbb{Q}^c \to \mathbb{Q}[-1] \to \mathbb{Q}^c[-1]$$

is given by cup-product by e, where the left map is "projection onto H^1" and the right one is "inclusion of H^0". □

5. Cohomology and homology theories

Let k be a field. We shall only need a basic concept of a "pure cohomology theory":

5.1. Definition. Let Sm/k denote the category of smooth k-schemes of finite type, and let $\mathcal{A}$ be an abelian category. A *pure cohomology theory* with values in $\mathcal{A}$ is an assignment

$$h = (h^n) : Ob(Sm/k) \to Ob(\mathcal{A}^{\mathbb{Z}})$$

such that, for any closed immersion $Z \hookrightarrow X$ of pure codimension c of smooth schemes, with complementary open set U, there is a long exact sequence

$$\dots \to h^{i-2c}(Z) \to h^i(X) \to h^i(U) \to h^{i-2c+1}(Z) \to \dots$$

b) *h has transfers* if, for any finite morphism $f : Y \to X$ in Sm/k, of pure degree d, there are maps

$$h^i(X) \xrightarrow{f^*} h^i(X), \quad h^i(Y) \xrightarrow{f_*} h^i(X) \quad i\in\mathbb{Z}$$

such that $f_*\circ f^*$ is multiplication by d.

5.2. Theorem. *Let $\mathcal{B}$ be a thick subcategory of $\mathcal{A}$. Let $N>0$, and let h be a pure cohomology theory such that $h^*(X)\in\mathcal{B}$ for any smooth, projective X with $\dim X\le N$.*
a) Suppose $\operatorname{char} k = 0$. Then $h^*(X)\in\mathcal{B}$ for any $X\in Sm/k$ with $\dim X\le N$.
b) Suppose $\operatorname{char} k > 0$. Then the conclusion still holds if h has transfers and $\mathcal{A}/\mathcal{B}$ is a $\mathbb{Q}$-linear category.

Proof. Passing to the quotient category $\mathcal{A}/\mathcal{B}$, we reduce to $\mathcal{B}=0$. We show that $h^*(X)=0$ for all equidimensional $X\in Sm/k$ by induction on $n=\dim X$. The case $n=0$ follows from the assumption. Suppose $n>0$ and the claim known in dimensions $<n$. Let $X\in Sm/k$ of dimension n. We first prove that, for any proper closed subset of X, $h^*(X) \xrightarrow{\sim} h^*(X-Z)$. Suppose first Z smooth. Then we get the result by purity and the induction hypothesis. In general, let $Z'\subseteq Z$ be the singular locus of Z. If $\operatorname{char} k=0$, then $Z'\neq Z$; if $\operatorname{char} k>0$, then $Z'\neq Z$ after a suitable finite radicial extension of k, which is allowable in view of the additional hypotheses. Then $Z\setminus Z'$ is smooth in $X\setminus Z'$. Writing $X\setminus Z = (X\setminus Z')\setminus(Z\setminus Z')$, we get $h^*(X\setminus Z') \xrightarrow{\sim} h^*(X\setminus Z)$. We conclude by Noetherian induction on Z.

If $\operatorname{char} k=0$, then, by resolution of singularities [15], a suitable open subset of X embeds into a smooth, projective k-variety and the proof is complete. If $\operatorname{char} k>0$, then by de Jong's theorem [22], a suitable open subset of X is finite-covered by an open subset of a smooth, projective k-variety. We now conclude by a transfer argument. □

We have the following variants of definition 5.1 and theorem 5.2.

5.3. Definition. a) Let $\mathcal{A}$ be an abelian category. A *pure graded cohomology theory* with values in $\mathcal{A}$ is a pure cohomology theory

$$h = (h_n)_{n\in\mathbb{Z}}$$
$$X \mapsto (h_n^*(X)) := (h^*(X,n))$$

with values in $\mathcal{A}^{\mathbb{N}}$, satisfying the following condition: for (X,Z,U) as in definition 5.1 a), the exact sequences of *loc. cit.* split into exact sequences

$$\dots \to h^{i-2c}(Z,n-c) \to h^i(X,n) \to h^i(U,n) \to h^{i-2c+1}(Z,n-c) \to \dots$$

c) h *has transfers* if all h_n have transfers.

5.4. Theorem. *Let $\mathcal{B}$ be a thick subcategory of $\mathcal{A}$. Let $N > 0$, $n \in \mathbb{Z}$ and let h be a pure graded cohomology theory such that $h^i(X,m) \in \mathcal{B}$ for all $i \le m \le n$ and any smooth, projective X with* $\dim X \le d$.
a) Suppose $\operatorname{char} k = 0$. *Then $h^i(X,m) \in \mathcal{B}$ for any $i \le m \le n$ and any $X \in Sm/k$ with* $\dim X \le d$.
b) Suppose $\operatorname{char} k > 0$. *Then the conclusion still holds if h has transfers and $\mathcal{A}/\mathcal{B}$ is a $\mathbb{Q}$-linear category.*

Proof. The same as that of theorem 5.2, noting that if $i \le m$ and $c > 0$, then $i - 2c + 1 \le m - c$. □

5.5. Definition. Let Sch/k be the category of schemes of finite type over k and $\mathcal{A}$ an abelian category.
a) A *homology theory* with values in $\mathcal{A}$ is an assignment

$$h = (h_n)_{n\in\mathbb{Z}} : Ob(Sch/k) \to Ob(\mathcal{A}^{\mathbb{Z}})$$

such that, for any closed immersion $Z \hookrightarrow X$ with complementary open set U, there is a long exact sequence

$$\dots \to h_i(Z) \to h_i(X) \to h_i(U) \to h_{i-1}(Z) \to \dots$$

b) A homology theory h *has transfers* if, for any finite morphism $f : Y \to X$ in Sm/k, of pure degree d, there are maps

$$h_i(X) \xrightarrow{f^*} h_i(Y), \quad h_i(Y) \xrightarrow{f_*} h_i(X), \quad i \in \mathbb{Z}$$

such that $f_* \circ f^*$ is multiplication by d.

5.6. Theorem. *Let h be a homology theory with values in $\mathcal{A}$, and let $\mathcal{B}$ be a thick subcategory of $\mathcal{A}$ such that $h_i(X) \in \mathcal{B}$ for all $i < m$ for all smooth, projective X of dimension $\le N$.*
a) Suppose $\operatorname{char} k = 0$. *Then $h_i(X) \in \mathcal{B}$ for all $i < m$ for all $X \in Sch/k$ of dimension $\le N$.*
b) Suppose $\operatorname{char} k > 0$. *Then the conclusion still holds if h has transfers and $\mathcal{A}/\mathcal{B}$ is a $\mathbb{Q}$-linear category.*

Proof. By induction on N, we have $h_i(X) \xrightarrow{\sim} h_i(U)$ for any open subscheme U of X and $i < m$. If $\operatorname{char} k = 0$, we can choose such an U smooth and embedded in a smooth, projective scheme of dimension N (by resolution of singularities [15]). If $\operatorname{char} k > 0$, the extra assumptions allow us to pass to a finite, radicial extension of k if necessary. Then X gets a nonempty smooth open subscheme, and we can conclude as before, using de Jong's theorem [22]. □

6. Back to $\mathbb{Z}_l(n)^c$ and $\mathbb{Q}_l(n)^c$

In this section, we fix a prime number $p \neq l$ and denote by $\mathcal{S}$ the category of smooth schemes of finite type over $\mathbb{F}_p$. We consider the restriction of $\mathbb{Z}_l(n)^c_{\mathbb{F}_p}$ and $\mathbb{Q}_l(n)^c_{\mathbb{F}_p}$ to $\mathcal{S}$, considered as a subsite of the big étale site of $\operatorname{Spec}\mathbb{F}_p$ (the *smooth big étale site* of $\operatorname{Spec}\mathbb{F}_p$); for simplicity, we denote these objects by $\mathbb{Z}_l(n)^c$ and $\mathbb{Q}_l(n)^c$. Let G still denote the absolute Galois group of $\mathbb{F}_p$.

6.1. Proposition. *Let X be a smooth, projective variety over $\mathbb{F}_p$. Then, for $n \in \mathbb{Z}$,*

$$H^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)) = \begin{cases} 0 & \text{if } i \neq 2n, 2n+1 \\ H^{2n}(\overline{X}, \mathbb{Q}_l(n))^G & \text{if } i = 2n \\ H^{2n}(\overline{X}, \mathbb{Q}_l(n))_G & \text{if } i = 2n+1. \end{cases}$$

Proof. By Deligne's proof of the Weil conjectures [5], the eigenvalues of the Frobenius action on $H^q(\overline{X}, \mathbb{Q}_l)$ are algebraic integers whose infinite absolute values are all equal to $q/2$. The proposition then follows from proposition 3.13. □

6.2. Corollary. *$H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(n))$ and $H^i_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$ are finite for $i \neq 2n, 2n+1$.*

Proof. The first claim follows from proposition 6.1 and lemma 3.12; the second one follows from proposition 6.1, lemma 3.12 and the long exact sequence

$$\dots \to H^i_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) \to H^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)) \to H^i_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to H^{i+1}_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) \to \dots$$

stemming from lemma 2.2. □

6.3. Theorem. *Let π be the projection of $\mathcal{S}$ onto the small étale site of* $\operatorname{Spec}\mathbb{F}_p$. *Then the natural map*

$$\pi^*\mathbb{Z}_l(n)^c_{|\mathbb{F}_p} = \pi^*\pi_*\mathbb{Z}_l(n)^c \xrightarrow{\alpha} \mathbb{Z}_l(n)^c$$

is an isomorphism for $n \leq 0$. ($\mathbb{Z}_l(n)^c$ is "locally constructible".)

Proof. Let $K(n)$ be the cone of α. We have to prove that $H^*(X, K(n)) = 0$ for any $X \in \mathcal{S}$. By lemma 2.2, $K(n)$ has uniquely divisible cohomology sheaves. By theorem 3.7 (or rather its proof), $X \mapsto H^*(X, K(n))$ therefore defines a pure graded cohomology theory with transfers with values in $\mathbb{Q}$-vector spaces, in the sense of definition 5.3. (To see that it is pure, it is enough to see that the complexes of sheaves $\pi^*\mathbb{Q}_l(n)^c_{|\mathbb{F}_p}$ verify purity for $n \leq 0$, which is trivial from theorem 4.6). By theorem 5.2 b), it is enough to prove the claim for X smooth and projective. In this case, it follows immediately from proposition 6.1 and theorem 4.6. □

6.4. Corollary. *Over $\mathcal{S}$, we have*

$$\mathbb{Q}_l(0)^c \simeq \mathbb{Q}_l[0] \oplus \mathbb{Q}_l[-1].$$

Proof. This follows from corollary 4.5, theorem 4.6 and theorem 6.3. □

Recall from section 4 the canonical generator $e \in H^1_{\mathrm{cont}}(\mathbb{F}_p, \mathbb{Z}_l)$. Cup-product by e induces endomorphisms of degree 1 and square 0

$$\mathbb{Z}_l(n)^c \xrightarrow{e} \mathbb{Z}_l(n)^c[1]. \tag{6.1}$$

6.5. Proposition. (*cf.* [32, prop. 6.5]) *For any scheme X of finite type over $\mathbb{F}_p$, the diagram*

$$\begin{array}{ccc} H^q_{\mathrm{cont}}(\overline{X}, \mathbb{Z}_l(n))^G & \longrightarrow & H^q_{\mathrm{cont}}(\overline{X}, \mathbb{Z}_l(n))_G \\ \uparrow & & \downarrow \\ H^q_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) & \xrightarrow{\cdot e} & H^{q+1}_{\mathrm{cont}}(X, \mathbb{Z}_l(n)) \end{array}$$

commutes, where the top horizontal map is the natural one and the vertical maps come from the Hochschild-Serre spectral sequence. The same holds for continuous cohomology with proper supports and continuous homology.

Proof. This is an immediate consequence of the multiplicativity of the Hochschild-Serre spectral sequence (view e as an element of $H^1(\mathbb{F}_p, H^0_{\text{cont}}(\overline{X}, \mathbb{Z}_l)) \subseteq H^1_{\text{cont}}(X, \mathbb{Z}_l)$) and the following lemma:

6.6. Lemma. *Let A be a continuous G-module. Then the diagram*

$$\begin{array}{ccc} H^0(G,A) & \xrightarrow{\cdot e} & H^1(G,A) \\ \| & & \wr\downarrow \\ A^G & \longrightarrow & A_G \end{array}$$

commutes. Here, the bottom horizontal map is the composition $A^G \hookrightarrow A \twoheadrightarrow A_G$.

Indeed, the right vertical isomorphism is induced by the map sending a continuous cocycle c to $c(F)$. On the other hand, cup-product by e is defined at the level of cocycles by the formula $(a \cdot e)(g) = e(g)a$. □

Recall condition S^n from [49]:

6.7. Definition. a) Let G be a profinite group and A a topological G-module. We say that A is *semi-simple at* 1 if the composition

$$A^G \hookrightarrow A \twoheadrightarrow A_G$$

is bijective.
b) Let k be a field and X/k be smooth and projective. Let k_s be a separable closure of k, $G_k = Gal(k_s/k)$, $\overline{X} = X \otimes_k \bar{k}_s$ and $n \in \mathbb{Z}$. We say that X *satisfies condition* $S^n(X)$ if the topological G_k-module $H^{2n}_{\text{cont}}(\overline{X}, \mathbb{Q}_l(n))$ is semi-simple at 1.

6.8. Corollary. *Let X be smooth, projective over $\mathbb{F}_p$. Then, condition $S^n(X)$ is equivalent to the bijectivity of $H^{2n}_{\text{cont}}(X, \mathbb{Q}_l(n)) \xrightarrow{\cdot e} H^{2n+1}_{\text{cont}}(X, \mathbb{Q}_l(n))$.* □

In the remainder of this section, we shall freely use the notion of a $\mathbb{Q}_l$-sheaf appearing in Deligne [6]; we refer to this article for a precise definition.

6.9. Theorem. *Let $\mathcal{F}$ be a $\mathbb{Q}_l$-sheaf over $\mathbb{F}_p$, pure of weight m. Consider $\mathcal{F}$ as a $\mathbb{Q}_l$-sheaf over the big étale site of $\mathbb{F}_p$ by pull-back. Then,*
a) For any smooth variety $X/\mathbb{F}_p$ and $q \geq 0$, $H^q_{\text{cont}}(\overline{X}, \mathcal{F})$ is mixed of weights between $m + q$ and $m + 2q$.
b) For any variety $X/\mathbb{F}_p$ of dimension d and $q \geq 0$, $H^q_{\text{cont}}(\overline{X}, \mathcal{F})$ is mixed of weights between $m + 2q - 2d$ and $m + 2q$.
c) If $\mathcal{F}^\vee(-m)$ is entire, then for any $X/\mathbb{F}_p$ of dimension d and $q \geq 0$, $H^q_{\text{cont}}(\overline{X}, \mathcal{F})$ is mixed of weights $\leq 2m + 2d$.

In c), $\mathcal{F}^\vee$ denotes the dual of $\mathcal{F}$ and "entire" means that the eigenvalues of the action of Frobenius are algebraic integers.

Proof. a) The bound $m+q$ follows from [6, cor. 3.3.5]. To prove the other, we consider the graded pure cohomology theory with transfers (definition 5.3)

$$h^i(X, n) = H^i_{\text{cont}}(\overline{X}, \mathcal{F}(n))$$

with values in the category $\mathcal{A}$ of $\mathbb{Q}_l$-sheaves over $\mathbb{F}_p$. Let $\mathcal{B}$ be the thick subcategory consisting of sheaves of weights $\leq m$. We want to prove that $h^i(X, n) \in \mathcal{B}$ for $i \leq n$. By theorem 5.4, we reduce to the case where X is projective; then it follows from the main result of [5].

b) The proof should be along the same lines, but we could not find a suitable formalisation. Suppose first X smooth. By the known l-cohomological dimension of $\overline{X}$, we may assume $q \leq 2d$. Then the lower bound follows from a) since $m + 2q - 2d \leq m + q$.

In general, we argue by induction on $\dim X$. Recall [22] that an *alteration* is a proper morphism $f : X' \to X$ such that $f^{-1}(U) \to U$ is finite and flat for some nonempty open subset U of X. Let

$f : X' \to X$ be an alteration, with X' smooth over $\mathbb{F}_p$: it exists by [22]. Let U be the maximal open subset of X such that $U' = f^{-1}(U)$ is flat over U, and $Z = X \setminus U$ (the "center" of the alteration). Set $Z' = f^{-1}(Z)$. So we have a diagram

$$\begin{array}{ccccc} Z' & \xrightarrow{i'} & X' & \xleftarrow{j'} & U' \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle f''}\big\downarrow \\ Z & \xrightarrow{i} & X & \xleftarrow{j} & U. \end{array}$$

We argue as in the proof of corollary 3.2. By proper base change, we have a commutative diagram of long exact sequences

$$\begin{array}{ccccccc} \dots \to H^{q-1}_{\text{cont}}(\overline{Z}, \mathcal{F}) \to & & H^q(\overline{X}, j_!\mathcal{F}_U) & & \to H^q_{\text{cont}}(\overline{X}, \mathcal{F}) \to & & H^q_{\text{cont}}(\overline{Z}, \mathcal{F}) \to \dots \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \dots \to H^{q-1}_{\text{cont}}(\overline{Z'}, \mathcal{F}) \to & & H^q(\overline{X}, j_! Rf''_* f''^* \mathcal{F}_{U'}) & & \to H^q_{\text{cont}}(\overline{X'}, \mathcal{F}) \to & & H^q_{\text{cont}}(\overline{Z'}, \mathcal{F}) \to \dots \end{array}$$

By induction and the smooth case, the sheaves $H^q_{\text{cont}}(\overline{X'}, \mathcal{F})$, $H^q_{\text{cont}}(\overline{Z}, \mathcal{F})$ and $H^{q-1}_{\text{cont}}(\overline{Z'}, \mathcal{F})$ are all of weights $\leq m + 2q$. Therefore $H^q(\overline{X}, j_! Rf''_* f''^* \mathcal{F}_{U'})$ is of weights $\leq m + 2q$ as well. On the other hand, since f'' is finite and flat, there is a trace morphism $Rf''_* f''^* \mathcal{F}_{U'} \to \mathcal{F}_U$ whose composition on the left with the pull-back map is multiplication by $\deg f''$. Therefore $H^q(X, j_!\mathcal{F}_U)$ is of weights $\leq m + 2q$ as well and finally $H^q_{\text{cont}}(X, \mathcal{F})$ is of weights $\leq m + 2q$, as desired. For the lower bound we argue similarly, noting that $m + 2(q-1) - 2\dim Z' \geq m + 2q - 2d$.

c) If X is smooth, this follows from [6, cor. 3.3.3] and Poincaré duality (compare [21, proof of th. 2]); in general, we proceed as in the proof of b). □

6.10. Corollary. *a) The sheaf $\mathcal{H}^i(\mathbb{Q}_l(n)^c)$ is 0 for $i < n$.*
b) For all $X \in Sch/\mathbb{F}_p$, the groups $H^i_{\text{cont}}(X, \mathbb{Z}_l(n))$ are finite for $i \notin [n, n+d+1]$, and for $i \notin [n, 2n+1]$ if X is smooth.
c) For all $X \in Sch/\mathbb{F}_p$, the groups $H^i_{\text{cont}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$ are finite for $i \notin [n, n+d+1]$, and for $i \notin [n, 2n+1]$ if X is smooth.

Proof. This follows from theorem 6.9, applied to $\mathcal{F} = \mathbb{Q}_l(n)$, proposition 3.13 and lemma 3.12 b). □

6.11. Definition. For $d \geq 0$, let Sch_d be the category of schemes of finite type of dimension $\leq d$ over $\mathbb{F}_p$ and $\pi_d : (\operatorname{Spec} \mathbb{F}_p)_{\text{Ét}} = (Sch/\mathbb{F}_p)_{\text{ét}} \to (Sch_d)_{\text{ét}}$ the projection of the big étale site of $\mathbb{F}_p$ over Sch_d provided with the étale topology. If $C \in \mathcal{D}((\operatorname{Spec} \mathbb{F}_p)_{\text{Ét}})$, we denote $R(\pi_d)_* C$ simply by $C_{|Sch_d}$.

6.12. Corollary. *For $n > d$, $\mathbb{Q}_l(n)^c_{|Sch_d} = 0$ and $\mathbb{Z}_l(n)^c_{|Sch_d} = \mathbb{Q}_l/\mathbb{Z}_l(n)[-1]$.*

Proof. By theorem 6.9 c), applied to $\mathcal{F} = \mathbb{Q}_l$, for any $X \in Sch_d$ and any $q \geq 0$, $H^q_{\text{cont}}(\overline{X}, \mathbb{Q}_l(n))$ is mixed of weights $\leq 2d - 2n$. By proposition 3.13, this implies that $H^*_{\text{cont}}(X, \mathbb{Q}_l(n)) = 0$ for $n > d$. The statement for $\mathbb{Z}_l(n)^c$ follows from this by lemma 2.2. □

6.13. Corollary. *For any $X \in Sch/\mathbb{F}_p$, one has $H_*^{\text{cont}}(X, \mathbb{Q}_l(n)) = H^*_{c,\text{cont}}(X, \mathbb{Q}_l(n)) = 0$ for any $n < 0$.*

Proof. If X is smooth, this follows from corollary 6.12 and proposition 3.11 c). In general, it follows from this, theorem 5.6 (for homology) and theorem 3.17 (for cohomology with proper supports). □

6.14. Corollary. *Let $f : X \to U$ be a smooth, projective morphism of varieties over $\mathbb{F}_p$. Let $\delta = \dim U$ and d be the relative dimension of f. Then we have*

$$H^p_{\text{cont}}(U, R^q f_*^{l\text{-adic}} \mathbb{Q}_l(n)) = 0 \quad \textit{for} \quad p + q < n, p + d < n, d + \delta < n \quad \textit{or} \quad p + q > 2n + 1.$$

If U is smooth, this group vanishes for $\delta+q<n$ as well. Finally, if f is "defined over $\mathbb{F}_p$", i.e. $X=X_0\times_{\mathbb{F}_p} U$ with X_0 smooth and projective over $\mathbb{F}_p$, it also vanishes for $2p+q<2n$.

Proof. By Hard Lefschetz [6, th. 4.1.1], smooth and proper base change and Deligne's degeneracy criterion for spectral sequences, the Leray spectral sequence degenerates at E_2, hence the group $H^p_{\mathrm{cont}}(U, R^q f_*^{l\text{-adic}}\mathbb{Q}_l(n))$ is a direct summand of $H^{p+q}(X,\mathbb{Q}_l(n))$ (compare [7, §4]). The vanishings for $p+q<n$ and $p+q>2n+1$ then follow from corollary 6.10. The vanishing for $d+\delta<n$ follows from corollary 6.12. The vanishing for $p+d<n$ follows from Hard Lefschetz, which implies that $R^q f_*^{l\text{-adic}}\mathbb{Q}_l(n)\simeq R^{2d-q}f_*^{l\text{-adic}}\mathbb{Q}_l(n+d-q)$ (apply the inequality $p+q<n$ with these new values of p,q,n). For the bound $\delta+q<n$, we note that the result of theorem 6.9 c) in the smooth case does not necessitate $\mathcal{F}$ to be defined over $\mathbb{F}_p$. Finally, the bound $2p+q<2n$ follows from theorem 6.9 b), this time applied with $\mathcal{F}=H^q_{\mathrm{cont}}(\overline{X}_0,\mathbb{Q}_l(n))$, which is pure of weight $q-2n$ by the main result of [5], and proposition 3.13. □

6.15. Remark. This extends Jannsen's inequalities in [19, th. 1]: *loc. cit.* says that, for U a smooth curve over $\mathbb{F}_p$ and X as above, $H^2_{\mathrm{cont}}(\pi_1(U), H^q(\overline{X},\mathbb{Q}_l(n)))=0$ whenever $q+2\le n$ or $q+2>2n+1$.

Since continuous étale cohomology acts by products on continuous étale cohomology with supports and étale homology, for any $X\in Sch/\mathbb{F}_p$ we have complexes

$$\begin{array}{ccccc}
\dots\to H^{i-1}_{\mathrm{cont}}(X,\mathbb{Q}_l(n)) & \xrightarrow{\cdot e} & H^{i}_{\mathrm{cont}}(X,\mathbb{Q}_l(n)) & \xrightarrow{\cdot e} & H^{i+1}_{\mathrm{cont}}(X,\mathbb{Q}_l(n))\to\dots\\
\dots\to H^{i-1}_{c,\mathrm{cont}}(X,\mathbb{Q}_l(n)) & \xrightarrow{\cdot e} & H^{i}_{c,\mathrm{cont}}(X,\mathbb{Q}_l(n)) & \xrightarrow{\cdot e} & H^{i+1}_{c,\mathrm{cont}}(X,\mathbb{Q}_l(n))\to\dots\\
\dots\to H_{i+1}^{\mathrm{cont}}(X,\mathbb{Q}_l(n)) & \xrightarrow{\cdot e} & H_{i}^{\mathrm{cont}}(X,\mathbb{Q}_l(n)) & \xrightarrow{\cdot e} & H_{i-1}^{\mathrm{cont}}(X,\mathbb{Q}_l(n))\to\dots
\end{array}\tag{6.2}$$

6.16. Theorem. *a) If S^n holds for any smooth, projective variety over $\mathbb{F}_p$, then the last two complexes of* (6.2) *are acyclic for any $X\in Sch/\mathbb{F}_p$, and the first one is acyclic for X smooth of pure dimension d of one replaces n by $d-n$.*
b) If S^n holds for $n\le d$ and any smooth, projective variety of dimension $\le d$, then the first complex of (6.2) *is acyclic for $n\le d$ and any $X\in Sch_d$.*

Proof. a) First assume that X is smooth projective. Then the first sequence of (6.2) is exact by proposition 6.1 and corollary 6.8; so is the second one by lemma 3.10 and so is also the third one by proposition 3.11 c) and theorem 3.17.

In general, we first deal with continuous homology. Let $X\in Sch/\mathbb{F}_p$, Z a closed subset and U the open complement. Proposition 3.11 a) and a little inductive argument involving a big diagram chase shows that, if theorem 6.16 holds for two among X,Z,U, then it holds for the third. Moreover, if $f:U'\to U$ is a finite and flat morphism and theorem 6.16 holds for U', then it holds for U by a transfer argument. An argument as in the proof of theorem 5.6 now yields the conclusion.

The case of cohomology with proper supports follows from this and theorem 3.17; the case of cohomology for X smooth follows from this and proposition 3.11 c).

For the sake of the proof of b), we note that the above argument works if we restrict to varieties of dimension $\le d$ for some d.

b) By a) and the assumption, the first sequence of (6.2) is exact for any X smooth of dimension $\le d$. For an arbitrary X of dimension $\le d$, we can argue as in the proof of theorem 6.9 b), using the ideas in the second paragraph of the proof of a). □

6.17. Corollary. *If S^n holds for any smooth, projective variety over $\mathbb{F}_p$, then, for any $X\in Sch/\mathbb{F}_p$ and any $i\in\mathbb{Z}$, $H^i_{c,\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))$ and $H_i^{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))$ are semi-simple at* 1. *If S^n holds for all $n\le d$ and all smooth, projective varieties of dimension $\le d$, then $H^i_{\mathrm{cont}}(\overline{X},\mathbb{Q}_l(n))$ is also semi-simple at* 1 *for all $n\le d$ and $X\in Sch_d$.*

Proof. This follows from theorem 6.16 and proposition 6.5. □

6.18. Remark. One should compare this with [18, th. 12.7], where a similar result is proven with heavier assumptions.

7. Values of the zeta function

In this section, we generalise the l-primary part of [32, th. 0.1] to arbitrary varieties.

7.1. Definition. a) Let $u : A \to B$ be a homomorphism of $\mathbb{Z}_l$-modules with finite kernel and cokernel. The *index* of u is

$$\operatorname{ind}(u) = \frac{|\operatorname{Ker} u|}{|\operatorname{Coker} u|}.$$

b) Let C be a bounded cochain complex of $\mathbb{Z}_l$-modules with finite cohomology groups. We set

$$\chi(C) = \prod_{i\in\mathbb{Z}} |H^i(C)|^{(-1)^i}.$$

Note that a) is a special case of b).

Let $X \in Sch/\mathbb{F}_p$ of dimension d and $\zeta(X,s)$ be its zeta function. By [56, exposé XV], we have $\zeta(X,s) = Z(X,p^{-s})$, where

$$Z(X,t) = \prod_{i=0}^{2d} \det(1 - tF|H^i_c(\overline{X},\mathbb{Q}_l))^{(-1)^{i+1}} \tag{7.1}$$

where F denotes the geometric Frobenius acting on the $\mathbb{Q}_l$-adic cohomology with proper supports of $\overline{X} := X \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$[1]. In this section, we shall prove:

7.2. Theorem. *Let $r_i(n) = \dim_{\mathbb{Q}_l} H^i_c(X,\mathbb{Q}_l(n))$ and $a_i(n)$ be the order of the zero of $\det(1-tF|H^i_c(\overline{X},\mathbb{Q}_l))$ at $t = p^{-n}$. Assume that $S^n(X)$ holds for any smooth, projective $X/\mathbb{F}_p$ (cf. definition 6.7). Then, for any $X \in Sch/\mathbb{F}_p$ and $i \geq 0$:*
a) $a_i(n) = r_i(n) - r_{i-1}(n) + \dots$
b) Let $\det(1 - tF|H^i_c(\overline{X},\mathbb{Q}_l)) = (1-p^nt)^{a_i(n)} f_i(t)$. Then

$$|f_i(p^{-n})|_l = \left|\operatorname{ind}(H^i_c(\overline{X},\mathbb{Z}_l(n))^G \to H^i_c(\overline{X},\mathbb{Z}_l(n))_G)\right|_l^{-1}$$

where $||_l$ is the l-adic absolute value.

7.3. Remark. Condition S^n is necessary in theorem 7.2, as is easily seen from the special case where X is smooth and projective.

Proof. a) The statement is clear from corollaries 6.17 and 3.14. The proof of b) will be divided into a series of lemmas.

7.4. Lemma. *a) Let $0 \to C' \to C \to C'' \to 0$ be a short exact sequence of complexes satisfying the assumption of definition 7.1 b). Then*

$$\chi(C) = \chi(C')\chi(C'').$$

b) Let $A \xrightarrow{u} B \xrightarrow{v} C$ be a chain of $\mathbb{Z}_l$-homomorphisms such that u and v have finite kernel and cokernel. Then

$$\operatorname{ind}(v\circ u) = \operatorname{ind}(v)\operatorname{ind}(u).$$

Proof. a) is classical; b) follows from the exact sequence

$$0 \to \operatorname{Ker} u \to \operatorname{Ker} vu \to \operatorname{Ker} v \to \operatorname{Coker} u \to \operatorname{Coker} vu \to \operatorname{Coker} v \to 0.$$

∎

7.5. Lemma. *If $A = B$ in definition 7.1, then $|\operatorname{ind}(u)|_l = |\det(u\otimes\mathbb{Q}_l)^{-1}|_l$.*

[1] Throughout this section, we drop the index cont from continuous cohomology groups, since we shall not consider any other cohomology.

Proof. Lemma 7.4 a) shows that, if $A' \subset A$ is stable under u, we have

$$\mathrm{ind}(u) = \mathrm{ind}(u_{|A'})\mathrm{ind}(\bar{u})$$

where $\bar{u}$ is the induced endomorphism of A/A'. Therefore we reduce to the cases where A is finite or torsion-free. In the first case, the claim is clear in view of the exact sequence

$$0 \to \operatorname{Ker} u \to A \xrightarrow{u} A \to \operatorname{Coker} u \to 0.$$

In the second case, it follows *e.g.* from [40, ch. III, prop. 2]. □

7.6. Lemma. *Under S^n, we have* $\det(1 - tF|H^i_c(\overline{X}, \mathbb{Q}_l)) = (1 - p^n t)^{a_i(n)} f_i(t)$*, with*

$$|f_i(p^{-n})|_l = \left| \mathrm{ind}\left(\frac{H^i_c(\overline{X}, \mathbb{Z}_l(n))}{H^i_c(\overline{X}, \mathbb{Z}_l(n))^G} \xrightarrow{1-F} \frac{H^i_c(\overline{X}, \mathbb{Z}_l(n))}{H^i_c(\overline{X}, \mathbb{Z}_l(n))^G} \right)^{-1} \right|_l .$$

Proof. For simplicity, set $H = \dfrac{H^i_c(\overline{X}, \mathbb{Z}_l(n))}{H^i_c(\overline{X}, \mathbb{Z}_l(n))^G}$. The claim follows from lemma 7.5 and the exact sequence of G-modules

$$0 \to H^i_c(\overline{X}, \mathbb{Z}_l(n))^G \to H^i_c(\overline{X}, \mathbb{Z}_l(n)) \to H \to 0. \tag{7.2}$$

□

Theorem 7.2 b) now follows from lemma 7.6 by taking the cohomology of (7.2), which gives an exact sequence

$$0 \to H^G \to H^i_c(\overline{X}, \mathbb{Z}_l(n))^G \to H^i_c(\overline{X}, \mathbb{Z}_l(n))_G \to H_G \to 0.$$

□

We shall need the following corollary in section 9:

7.7. Corollary. *With notation as in theorem 7.2, we also have*

$$|f_i(p^{-n})|_l = \left|\mathrm{ind}(H_i(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))^G \to H_i(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))_G)\right|_l .$$

If X is smooth of pure dimension d, we have

$$|f_i(p^{-n})|_l = \left|\mathrm{ind}(H^{2d-i}(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(d-n))^G \to H^{2d-i}(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(d-n))_G)\right|_l .$$

Proof. The first formula follows from the duality between $H^i_c(\overline{X}, \mathbb{Z}_l(n))$ and $H_i(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))$, while the second one follows from the isomorphism between étale homology and étale cohomology for smooth varieties. □

7.8. Theorem. *With assumptions and notation as in theorem 7.2, we have*
a) $a(n) := \mathrm{ord}_{s=n}\zeta(X, s) = \sum_{i \geq 0}(-1)^{i+1}(2d + 1 - i)r_i(n)$*, where* $d = \dim X$*.*
b) Let $\zeta(X, s) = (1 - p^{n-s})^{a(n)}\varphi(s)$*. Then*

$$|\varphi(n)|_l = \left|\chi\left(H^*_c(X, \mathbb{Z}_l(n)), e\right)\right|_l .$$

Proof. a) follows immediately from theorem 7.2 a). b) follows from theorem 7.2 b) and a little diagram chase in the commutative diagram of exact sequences (see proposition 6.5)

$$\begin{tikzcd}
0 \arrow[d] & 0 & 0 \arrow[d] \\
\to H_c^{i-2}(\overline{X},\mathbb{Z}_l(n))_G \arrow[d] & H_c^i(\overline{X},\mathbb{Z}_l(n))^G \arrow[u] \arrow[r] & H_c^i(\overline{X},\mathbb{Z}_l(n))_G \arrow[d] \\
\ldots \to H_c^{i-1}(X,\mathbb{Z}_l(n)) \arrow[d] \arrow[r, "\cdot e"] & H_c^i(X,\mathbb{Z}_l(n)) \arrow[u] \arrow[r, "\cdot e"] & H_c^{i+1}(X,\mathbb{Z}_l(n)) \to \ldots \arrow[d] \\
H_c^{i-1}(\overline{X},\mathbb{Z}_l(n))^G \arrow[d] \arrow[r] & H_c^{i-1}(\overline{X},\mathbb{Z}_l(n))_G \arrow[u] & H_c^{i+1}(\overline{X},\mathbb{Z}_l(n))^G \to \arrow[d] \\
0 & 0 \arrow[u] & 0
\end{tikzcd}$$

□

7.9. Corollary. *With assumptions and notation as in theorem 7.8, we have*

$$|\varphi(n)|_l = \left|\prod_{i\geq 0} |H_c^i(X,\mathbb{Z}_l)_{\mathrm{tors}}|^{(-1)^i} \chi\left(\overline{H}_c^*(X,\mathbb{Z}_l(n)),\bar{e}\right)\right|_l$$

where $\overline{H}_c^i(X,\mathbb{Z}_l(n)) = H_c^i(X,\mathbb{Z}_l(n))/\mathrm{tors}$ *and* $\bar{e}$ *is the induced homomorphism.*

Proof. This follows from theorem 7.8 and lemma 7.4 a). □

7.10. Corollary. *Keep assumptions and notation as in corollary 7.9. Assume X smooth and projective. Let $R_n(X)$ be the determinant of the pairing*

$$\overline{H}^{2n}(X,\mathbb{Z}_l(n)) \times \overline{H}^{2d-2n}(X,\mathbb{Z}_l(d-n)) \to \overline{H}^{2d}(X,\mathbb{Z}_l(d)) \xrightarrow{\sim} H^{2d}(\overline{X},\mathbb{Z}_l(d)) \simeq \mathbb{Z}_l$$

with respect to any bases of $\overline{H}^{2n}(X,\mathbb{Z}_l(n))$ *and* $\overline{H}^{2d-2n}(X,\mathbb{Z}_l(d-n))$ *(this is well-defined up to a unit of $\mathbb{Z}_l$). Then*

$$|\varphi(n)|_l = \left|\prod_{i\neq 2n,2n+1} |H^i(X,\mathbb{Z}_l(n))|^{(-1)^i} \cdot \frac{|H^{2n}(X,\mathbb{Z}_l(n))_{\mathrm{tors}}|}{|H^{2n+1}(X,\mathbb{Z}_l(n))_{\mathrm{tors}}|R_n(X)}\right|_l.$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
\overline{H}_c^i(X,\mathbb{Z}_l(n)) \arrow[r, "\bar{e}"] \arrow[d, "\alpha"'] & \overline{H}_c^{i+1}(X,\mathbb{Z}_l(n)) \arrow[d, "\beta"'] \\
Hom(\overline{H}^{2d-2n}(X,\mathbb{Z}_l(d-n)),\overline{H}^{2d}) \arrow[r, "Hom(Id,\bar{e})"] & Hom(\overline{H}^{2d-2n}(X,\mathbb{Z}_l(d-n)),\overline{H}^{2d+1})
\end{tikzcd}$$

where $\overline{H}^{2d}$ and $\overline{H}^{2d+1}$ respectively stand for $H^{2d}(X,\mathbb{Z}_l(d))$ and $H^{2d+1}(X,\mathbb{Z}_l(d))$ (in order for the diagram to fit in the page). By lemma 7.5, we have $|\mathrm{ind}(\alpha)|_l = |R_n(X)^{-1}|_l$ and β is an isomorphism by theorem 3.17, hence $\mathrm{ind}(\beta) = 1$. Finally, $\overline{H}^{2d} \xrightarrow{\bar{e}} \overline{H}^{2d+1}$ is also an isomorphism by proposition 6.5. Applying lemma 7.4 b) once again, we get

$$|\mathrm{ind}(\bar{e})|_l = |R_n(X)^{-1}|_l$$

and the result now follows from corollary 7.9. □

Corollary 7.10 is a variant of [32, th. 0.1].

8. The conjecture

In this section and the next one, it will be occasionally convenient to use the following definition:

8.1. Definition. For any integer $d \geq 0$, we denote by $\mathcal{S}_d$ the full subcategory of $Sm/\mathbb{F}_p$ formed of smooth varieties of dimension $\leq d$; we consider $\mathcal{S}_d$ as a subsite of $Sm/\mathbb{F}_p$ (for the étale or Zariski topology).

8.1. **Milnor's K-theory.** For any commutative ring R, denote by $K_n^M(R)$ the group defined by generators and relations as in the case of a field [34] (to be on the safe side, include the relation $\{x, -x\} = 0$). The Zariski sheaf $\mathcal{K}_n^M$ associated to the presheaf $\operatorname{Spec} R \mapsto K_n^M(R)$ is called the *sheaf of Milnor K-groups*: it is a sheaf on the big Zariski site of $\operatorname{Spec} \mathbb{Z}$. We have

8.2. Theorem. *Suppose X is a smooth variety over a field k. Then*
a) There is a complex of Zariski sheaves over X:

$$0 \to \mathcal{K}_n^M \to \coprod_{x \in X^{(0)}} (i_x)_* K_n^M(k(x)) \to \coprod_{x \in X^{(1)}} (i_x)_* K_{n-1}^M(k(x)) \to \dots$$

which is exact, except perhaps at $\mathcal{K}_n^M$, where its kernel is killed by $(n-1)!$. Here $X^{(p)}$ denotes the set of points of X of codimension p, $i_x : \{x\} \hookrightarrow X$ is the natural immersion and $K_n^M(k(x))$ is considered as a constant sheaf on x_{Zar}. In particular the sequence

$$0 \to \mathcal{K}_n^M \otimes \mathbb{Q} \to \coprod_{x \in X^{(0)}} (i_x)_* K_n^M(k(x)) \otimes \mathbb{Q} \to \coprod_{x \in X^{(1)}} (i_x)_* K_{n-1}^M(k(x)) \otimes \mathbb{Q} \to \dots$$

is exact.
b) We have

$$H^p_{\mathrm{Zar}}(X, \mathcal{K}_n^M) \otimes \mathbb{Q} = \begin{cases} 0 & \text{for } p > n \\ CH^n(X) \otimes \mathbb{Q} & \text{for } p = n. \end{cases}$$

Proof. a) The complex was defined by Kato [26]. By Rost [38, th. 6.1], it is acyclic, except perhaps at $\mathcal{K}_n^M$ and $\coprod_{x \in X^{(0)}} (i_x)_* K_n^M(k(x))$. By a result of Gabber (unpublished), it is exact at $\coprod_{x \in X^{(0)}} (i_x)_* K_n^M(k(x))$ as well. Finally, $\mathcal{K}_n^M$ maps to the sheaf $\mathcal{K}_n$ of algebraic K-groups with kernel killed by $(n-1)!$, by results of Suslin [46] and Guin [14]. The claim on $\operatorname{Ker}(\mathcal{K}_n^M \to \coprod_{x \in X^{(0)}} (i_x)_* K_n^M(k(x)))$ follows from this and Gersten's conjecture for the algebraic K-theory of X (Quillen [37, th. 7.5.11]). The statement after tensoring by $\mathbb{Q}$, hence b), readily follow. □

8.3. Remark. If one is interested only in $\mathcal{K}_*^M \otimes \mathbb{Q}$, one can get another proof of a) by using the fact that this complex is the weight n-part for the Adams operations of the corresponding Gersten complex for algebraic K-theory, *cf.* [44]. Moreover, b) holds without tensoring by $\mathbb{Q}$ for $n = \dim X$, see [26].

8.4. Corollary. *Let k be a field and $f : Y \to X$ a finite flat morphism of smooth k-schemes, of constant separable degree d. Then there exists a morphism*

$$Tr_f : f_* \mathcal{K}_n^M \otimes \mathbb{Q} \to \mathcal{K}_n^M \otimes \mathbb{Q}$$

whose left composition with the natural morphism $\mathcal{K}_n^M \otimes \mathbb{Q} \to f_* \mathcal{K}_n^M \otimes \mathbb{Q}$ is multiplication by d. If $g : Z \to Y$ is another such morphism, we have

$$Tr_{f \circ g} = Tr_f \circ f_* Tr_g.$$

Proof. This follows from the result in the case of a finite field extension (Bass-Tate [1, §5], Kato [25, prop. 5]) and theorem 8.2 b). One should use [25, lemma 16], which implies that the Milnor K-theory transfer is compatible with the Gersten resolution of theorem 8.2 a). □

8.5. Corollary. *Let (X, Z) be a smooth pair of codimension c of k-schemes, where k is a field, and $i : Z \hookrightarrow X$ the corresponding closed immersion. Then*

$$Ri^!(\mathcal{K}_n^M)_X \otimes \mathbb{Q} \simeq (\mathcal{K}_{n-c}^M)_Z \otimes \mathbb{Q}[-c].$$

Proof. Applying the functor $i^!$ to the flasque resolution of $(\mathcal{K}_n^M)_X \otimes \mathbb{Q}$ given by theorem 8.2 a), we get the corresponding resolution of $(\mathcal{K}_{n-c}^M)_Z \otimes \mathbb{Q}$ shifted by c to the right. □

8.6. Corollary. *If k is a field, the map over the big smooth Zariski site of* $\operatorname{Spec} k$

$$\mathcal{K}_n^M \otimes \mathbb{Q} \to R\alpha_*\alpha^*\mathcal{K}_n^M \otimes \mathbb{Q}$$

is an isomorphism, where α is the projection of the big étale site onto the big Zariski site.

Proof. Over the small Zariski site of $\operatorname{Spec} k$, we have $(R^q\alpha_*\alpha^*\mathcal{K}_n^M\otimes\mathbb{Q})(k) = H^q(k, K_n^M(k_s)\otimes\mathbb{Q}) = 0$ for $q > 0$ and $\mathcal{K}_n^M(k)\otimes\mathbb{Q} \xrightarrow{\sim} (\alpha_*\alpha^*\mathcal{K}_n^M\otimes\mathbb{Q})(k) = (K_n^M(k_s)\otimes\mathbb{Q})^{G_k}$ by a transfer argument, where k_s is a separable closure of k and $G_k = Gal(k_s/k)$. Over the small étale site of a smooth k-scheme X, the isomorphism of corollary 8.6 follows from this (applied to all residue fields of X) and theorem 8.2 a). □

Corollary 8.6 implies that the map

$$H^i_{\mathrm{Zar}}(X, \mathcal{K}_n) \otimes \mathbb{Q} \to H^i_{\text{ét}}(X, \mathcal{K}_n) \otimes \mathbb{Q}$$

is an isomorphism for all i, n; we shall take this opportunity to write both groups $H^i(X, \mathcal{K}_n) \otimes \mathbb{Q}$.

8.7. Corollary. *If k is a field, over the big smooth Zariski site of* $\operatorname{Spec} k$*, we have*

$$\mathcal{K}_n^M \otimes \mathbb{Q} \xrightarrow{\sim} R\pi_*\pi^*\mathcal{K}_n^M \otimes \mathbb{Q}$$

where π is the projection $\mathbb{A}^1_k \to \operatorname{Spec} k$.

Proof. This follows from [38, prop. 8.6]. □

8.2. **$\mathcal{K}^M$-homology.** Let Z be an excellent scheme. For all $n \in \mathbb{Z}$, we denote by $\mathcal{M}_{n,Z}$ the (chain) complex of Zariski sheaves

$$\dots \to \coprod_{x\in Z_{(i)}} (i_x)_*K^M_{n+i}(\kappa(x)) \to \coprod_{x\in Z_{(i-1)}} (i_x)_*K^M_{n+i-1}(\kappa(x)) \to \dots$$

defined by Kato [26]. Since its terms are flasque, its hypercohomology is computed by the cohomology of its global sections[2]. In other terms,

$$\mathbb{H}^{-i}_{\mathrm{Zar}}(Z, \mathcal{M}_{n,Z}) = A_i(Z, K_*^M, n)$$

where the right hand group is the one defined in [38, §5]. For the purpose of future compatibility with étale homology (see subsection 8.4), we set:

8.8. Definition. $H_i(Z, \mathcal{K}_n^M) = \mathbb{H}^{-i}_{\mathrm{Zar}}(Z, \mathcal{M}_{-n,Z})$.

We collect in the following proposition some properties of $\mathcal{K}^M$-homology.

8.9. Proposition. *a) If X is smooth over a field of pure dimension d, one has a quasi-isomorphism*

$$\mathcal{M}_{n,X} \otimes \mathbb{Q} \simeq \mathcal{K}^M_{n+d,X} \otimes \mathbb{Q}[d].$$

In particular,

$$H_i(X, \mathcal{K}_n^M) \otimes \mathbb{Q} \simeq H^{d-i}_{\mathrm{Zar}}(X, \mathcal{K}^M_{d-n}) \otimes \mathbb{Q}.$$

b) Let $i : Z \hookrightarrow Y$ be a closed immersion. One has

$$Ri^!\mathcal{M}_{n,Y} = \mathcal{M}_{n,Z}. \tag{8.1}$$

c) The assignment $Z \mapsto \mathcal{M}_{n,Z}$ is covariant for proper morphisms and contravariant for flat, equidimensional morphisms of schemes of finite type over a field.

[2]Recall that the hypercohomology of a chain complex C_i is the hypercohomology of the cochain complex C^i defined by $C^i = C_{-i}$.

Proof. a) follows from theorem 8.2; b) is obvious and c) follows from [38, (4.5)]. □

8.3. **The conjecture.** For $n = 1$, $\alpha^*\mathcal{K}_n^M = \mathbb{G}_m$; the Kummer exact sequences

$$1 \to \mu_{l^\nu} \to \mathbb{G}_m \xrightarrow{l^\nu} \mathbb{G}_m \to 1 \quad (\nu \geq 1)$$

induce a "Kummer" morphism

$$\mathbb{G}_m[0] \to \mathbb{Z}_l(1)^c[1]$$

in the derived category, and in particular a morphism of sheaves

$$u_1 : \alpha^*\mathcal{K}_1^M \to \mathcal{H}^1(\mathbb{Z}_l(1)^c).$$

We denote by $r \mapsto (r)$ the induced morphism $R^* \to H^1_{\mathrm{cont}}(\operatorname{Spec} R, \mathbb{Z}_l(1))$ for any ring R containing $1/l$.

8.10. Proposition. *a) The morphism u_1 and cup-product define morphisms*

$$u_n : \alpha^*\mathcal{K}_n^M \to \mathcal{H}^n(\mathbb{Z}_l(n)^c).$$

b) u_n is compatible with the transfer of corollary 8.4 and the direct image morphism of theorem 3.7 in continuous étale cohomology.

Proof. a) It is enough to show that, for any affine scheme $U = \operatorname{Spec} R$ and any $r \in R^*$, the cup-product

$$(r) \cdot (1 - r)$$

is 0 in $H^2_{\mathrm{cont}}(U, \mathbb{Z}_l(2))$. To do this we just mimic Tate's classical argument [48, th. 3.1]. We use the étale covering $U_1 = \operatorname{Spec} R_1$, with $R_1 = R[t]/(t^l - r)$, noting that, if r_1 is the image of t in R_1, $r_1^l = r$ and $N_{R_1/R}(1 - r_1) = 1 - r$, hence

$$(r) \cdot (1 - r) = (r) \cdot f_*(1 - r_1) = f_*(f^*(r) \cdot (1 - r_1)) = l f_*((r_1 \cdot (1 - r_1))$$

where f is the projection $U_1 \to U$. This shows that $(r) \cdot (1 - r)$ is contained in a divisible subgroup of $H^2_{\mathrm{cont}}(U, \mathbb{Z}_l(2))$; by [17, (4.9)], we conclude that $(r) \cdot (1 - r) = 0$.

b) This follows from theorem 3.8 and the construction of the transfer on the Milnor K-sheaves (corollary 8.4). □

By corollary 6.10 a), the morphism of proposition 8.10 tensored with $\mathbb{Q}$ defines a morphism

$$\alpha^*\mathcal{K}_n^M[-n] \otimes \mathbb{Q} \to \mathbb{Q}_l(n)^c$$

Tensoring this with $\mathbb{Q}_l(0)^c$ and using the pairing

$$\mathbb{Q}_l(0)^c \overset{L}{\otimes} \mathbb{Q}_l(n)^c \to \mathbb{Q}_l(n)^c$$

(*cf.* (2.2)), we get a morphism in $\mathcal{D}^+(Ab(\operatorname{Spec} \mathbb{F}_p)_{\text{Ét}})$

$$\mathbb{Q}_l(0)^c \overset{L}{\otimes} \alpha^*\mathcal{K}_n^M[-n] \to \mathbb{Q}_l(n)^c. \tag{8.2}$$

Let X be a smooth variety. Applying $H^{2n}_{\text{ét}}(X, -)$ to (8.2) we get a map

$$H^n_{\text{ét}}(X, \alpha^*\mathcal{K}_n^M) \otimes \mathbb{Q}_l \to H^{2n}_{\mathrm{cont}}(X, \mathbb{Q}_l(n)).$$

By theorem 8.2 b) and corollary 8.6, the left hand side is isomorphic to $CH^n(X) \otimes \mathbb{Q}_l$, hence we get a map

$$CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}_{\mathrm{cont}}(X, \mathbb{Q}_l(n)).$$

8.11. Lemma. *This map is the l-adic cycle map of* [17, (6.14)] *tensored with* $\mathbb{Q}$.

Proof. In view of the way the cycle class is defined ([55, cycle 2.2.8], [17, (3.23)]), we reduce to $n = 1$, in which case the result is trivial. □

8.12. Conjecture. *The restriction of* (8.2) *to the smooth big étale site is an isomorphism.*

Conjecture 8.12 is trivially true for $n = 0$. In more concrete terms, it implies:

8.13. Proposition. *If conjecture 8.12 holds, then there is an exact triangle*

$$\mathbb{Z}_l(n)^c \to \alpha^*\mathcal{K}_n^M \otimes \mathbb{Q}_l[-n] \oplus \alpha^*\mathcal{K}_n^M \otimes \mathbb{Q}_l[-n-1] \to \mathbb{Q}_l/\mathbb{Z}_l[0] \to \mathbb{Z}_l(n)^c[1].$$

In particular, for any $X \in \mathcal{S}$, there is a long exact sequence

$$\cdots \to H^i_{\text{cont}}(X, \mathbb{Z}_l(n)) \to H^{i-n}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}_l \oplus H^{i-n-1}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}_l \\ \to H^i_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to H^{i+1}_{\text{cont}}(X, \mathbb{Z}_l(n)) \to \dots$$

Proof. This follows from lemma 2.2, corollary 6.4 and corollary 8.6. □

8.14. Remark. Conjecture 8.12 extended to all of $Sch/\mathbb{F}_p$ is false. For example, continuous étale cohomology is homotopy invariant on $Sch/\mathbb{F}_p$, while $\mathcal{K}^M$-cohomology is not. See subsection 8.4 for a *homological* formulation.

8.15. Proposition. *Suppose conjecture 8.12 holds. Then, for any smooth, projective variety $X/\mathbb{F}_p$,*

- *$H^i(X, \mathcal{K}_n^M)$ is torsion for $i < n$.*
- *The cycle map $CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}_{\text{cont}}(X, \mathbb{Q}_l(n))$ is an isomorphism.*
- *Cup-product by e: $H^{2n}_{\text{cont}}(X, \mathbb{Q}_l(n)) \to H^{2n+1}_{\text{cont}}(X, \mathbb{Q}_l(n))$ is an isomorphism.*

Proof. The first two claims directly follow from proposition 6.1 and lemma 8.11. The last one follows from the same (obvious) fact with $\mathbb{Q}_l(n)^c$ replaced by $\mathbb{Q}_l(0)^c \overset{L}{\otimes} \alpha^*\mathcal{K}_n^M[-n]$ (*cf.* corollary 4.8). □

8.16. Corollary. *Conjecture 8.12 implies the Tate conjecture in codimension n, the equality of rational and homological equivalences and condition $S^n(X)$ for any smooth, projective variety X over $\mathbb{F}_p$.*

Proof. Follows from proposition 8.15, proposition 6.1 and corollary 6.8. □

Conversely:

8.17. Theorem. *Suppose that the Tate conjecture in codimension n, the equality of rational and homological equivalences and condition $S^n(X)$ hold for $n \le d$ and any smooth, projective variety X over $\mathbb{F}_p$ of dimension $\le d$. Then conjecture 8.12 holds for $n \le d$ on $\mathcal{S}_d$ (see definition 8.1).*

Proof. Fix $n \le d$. Let $K(n)$ be the cone of (8.2): we need to show that the restriction of $K(n)$ to $\mathcal{S}_d$ is 0. We note that $X \mapsto H^*(X, K(n))$ defines (for varying n) a graded pure cohomology theory with transfers with values in $\mathbb{Q}$-vector spaces in the sense of definition 5.3: this follows from theorem 3.4, corollary 8.5 and proposition 8.10 b). Applying theorem 5.2 b), we see that it is enough to prove that $\mathbb{H}^*(X, (8.2))$ is an isomorphism for any smooth, projective variety X of dimension $\le d$. By the above arguments, the only thing which is left to prove is the vanishing of $H^i(X, \mathcal{K}_n^M) \otimes \mathbb{Q}$ for $i < n$.

For this, we observe that the assumptions imply that numerical and homological equivalences agree over X ([32, prop. 8.4], [49, (2.6)]). Therefore, the category of Chow motives of dimension $\le d$ over $\mathbb{F}_p$ is abelian and semi-simple [20] and we can apply the Soulé-Geisser argument [43], [10, proof of th. 3.3 b)]. We have to prove that, for any simple motive M of dimension $\le d$, we have

$$H^i(M, \mathcal{K}_n^M) \otimes \mathbb{Q} = 0 \quad \text{for} \quad i < n \quad \text{and} \quad n \le d.$$

Suppose first that M is the Lefschetz motive L^n. Then M is a direct summand of $\mathbb{P}^n_{\mathbb{F}_p}$ and one sees easily that the conclusion is valid. Suppose now that $M \neq L^n$. Let P be the minimum polynomial of the geometric Frobenius F_M of M, viewed as an element of the (finite dimensional $\mathbb{Q}$-algebra) $End(M)$. By [10, th. 2.8], we have $P(p^m) \neq 0$. On the other hand, F_M acts on $H^i(M, \mathcal{K}_n^M) \otimes \mathbb{Q}$ by multiplication by p^n [43, prop. 2 (iv)]. So $H^i(M, \mathcal{K}_n^M) \otimes \mathbb{Q}$ is killed by multiplication by a nonzero rational number, and therefore is 0. □

In the opposite direction:

8.18. Theorem. *Conjecture 8.12 holds if and only if, for any finitely generated field F of characteristic p,*

$$\tilde{H}^i_{\mathrm{cont}}(F/\mathbb{F}_p, \mathbb{Q}_l(n)) = \begin{cases} 0 & \text{if } i \neq n, n+1 \\ K_n^M(F) \otimes \mathbb{Q}_l & \text{if } i = n, n+1. \end{cases}$$

Proof. Necessity is clear. For sufficiency, let $K(n)$ be as in the proof of theorem 8.17. The functor $(X, Z) \mapsto H^*_Z(X, K(n))$ defines a cohomology theory with supports in the sense of [4, def. 5.1.1], which verifies axioms COH1 (étale excision), COH3 (homotopy invariance) and COH6 (transfers) of *loc. cit.*, 5.1, 5.3 and 6.2. By *loc. cit.*, theorem 6.2.5, for any local ring R of a smooth variety over $\mathbb{F}_p$, with field of fractions F, the map

$$H^*(R, K(n)) \to H^*(F, K(n))$$

is injective. This concludes the proof. □

We now give some further consequences of conjecture 8.12.

8.19. Proposition. *If conjecture 8.12 holds, then, for all $X \in \mathcal{S}$, $H^*_{\mathrm{cont}}(X, \mathbb{Z}_l(n))$ carries a canonical (i.e. independent of l) integral structure.*

Proof. Consider the morphism

$$\mathbb{Q}_l(0)^c \otimes \alpha^* \mathcal{K}_n^M[-n] \to \mathbb{Q}_l/\mathbb{Z}_l(n)[0]$$

obtained by composing (8.2) with the morphism $\mathbb{Q}_l(n)^c \to \mathbb{Q}_l/\mathbb{Z}_l(n)[0]$. Writing $\mathbb{Q}_l/\mathbb{Z}_l(n)$ as a direct summand of $(\mathbb{Q}/\mathbb{Z})'(n) := \coprod_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l(n)$, it factors as

$$\mathbb{Q}_l(0)^c \otimes \alpha^* \mathcal{K}_n^M[-n] \to (\mathbb{Q}/\mathbb{Z})'(n)[0] \to \mathbb{Q}_l/\mathbb{Z}_l(n)[0].$$

Let $\mathbb{Q}^c$ be as in definition 4.1. The way (8.2) is defined show that the composite morphism

$$\mathbb{Q}^c \otimes \alpha^* \mathcal{K}_n^M[-n] \xrightarrow{\otimes \mathbb{Q}_l} \mathbb{Q}_l(0)^c \otimes \alpha^* \mathcal{K}_n^M[-n] \to (\mathbb{Q}/\mathbb{Z})'(n)[0]$$

does not depend on $l \neq p$. If we denote by C the fibre of this morphism, then conjecture 8.12 says that there is an isomorphism

$$\mathbb{Z}_l(n)^c \xrightarrow{\sim} \mathbb{Z}_l \otimes C.$$

□

8.20. Proposition. *Suppose conjecture 8.12 holds. Let X be a smooth variety. Then*
a) We have canonical isomorphisms for all $i \in \mathbb{Z}$:

$$H^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)) \simeq H^{i-n}_{\mathrm{Zar}}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}_l \oplus H^{i-n-1}_{\mathrm{Zar}}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}_l.$$

b) Under this decomposition of $H^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$, cup-product by e has matrix

$$\begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix}.$$

Proof. a) is clear from the exact sequence in proposition 8.13; b) follows immediately from a). □

8.21. Remark. We recover the conclusion of theorem 6.16 in a more concrete way.

8.22. Corollary. *If conjecture 8.12 holds, then the $\mathbb{Q}_l$-adic cycle map of* [17, (6.14)]

$$CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$$

is injective for any smooth variety X over $\mathbb{F}_p$, and the composition

$$CH^n(X) \otimes \mathbb{Q}_l \to H^{2n}_{\mathrm{cont}}(X, \mathbb{Q}_l(n)) \xrightarrow{\cdot e} H^{2n+1}_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$$

is bijective. □

8.23. Corollary. *Conjecture 8.12 implies the existence of Beilinson's conjectural filtration in its weak form (cf.* [21, conj. 2.1]*) on $CH^n(X)$ for any smooth, projective variety X over a field F of characteristic p.*

Proof. We may assume F finitely generated. The proof is then a variant of [21, lemma 2.7], by applying corollary 8.22 to a extension of X to a smooth, projective scheme over a smooth model of F over $\mathbb{F}_p$ (in other words, we use $\tilde{H}^{2n}_{\text{cont}}(X/\mathbb{F}_p, \mathbb{Q}_l(n))$ instead of $H^{2n}_{\text{cont}}(X, \mathbb{Q}_l(n))$ as in *loc. cit.*).□

8.24. Corollary. *Assume conjecture 8.12. Then, for any smooth variety X and any i, the composition*

$$H^{i-n}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}_l \to H^i_{\text{cont}}(X, \mathbb{Q}_l(n)) \to H^i_{\text{cont}}(\overline{X}, \mathbb{Q}_l(n))^G$$

is an isomorphism.

Proof. This follows from propositions 8.20 and 6.5. □

8.25. Remark. Corollary 8.24 extends the Tate conjecture.

8.26. Corollary. *For any smooth variety X over $\mathbb{F}_p$, let $d_i(n) = \dim_{\mathbb{Q}_l} H^i_{\text{cont}}(X, \mathbb{Q}_l(n))$. Assume conjecture 8.12. Then, for all $i \leq n$, $H^i(X, \mathcal{K}_n^M) \otimes \mathbb{Q}$ is a finite-dimensional vector space of dimension $d_{i+n}(n) - d_{i+n-1}(n) + \dots$.*

Proof. This is clear from corollary 8.24 and proposition 3.14. □

8.27. Corollary. *Assume conjecture 8.12. Let $X/\mathbb{F}_p$ be smooth of pure dimension d. Then, with notation as in* (7.1)*, the order of* $\det(1 - tF|H^i_c(\overline{X}, \mathbb{Q}_l))$ *at $t = p^{n-d}$ is* $\dim_{\mathbb{Q}} H^{2d-n-i}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}$.

Proof. Since conjecture 8.12 implies condition $S^n(X)$, this order equals $\dim_{\mathbb{Q}_l} H^i_c(\overline{X}, \mathbb{Q}_l(d-n))^G = \dim_{\mathbb{Q}_l} H^i_c(\overline{X}, \mathbb{Q}_l(d-n))_G$. Since X is smooth, by (geometric) Poincaré duality, there is an isomorphism

$$H^i_c(\overline{X}, \mathbb{Q}_l(d-n))_G \simeq (H^{2d-i}(\overline{X}, \mathbb{Q}_l(n))^G)^*.$$

The claim now follows from corollary 8.24. □

8.28. Corollary. *If conjecture 8.12 holds for all $n \geq 0$, then*

(i) *For X smooth projective over $\mathbb{F}_p$, $K_i(X)$ is torsion for $i > 0$ (Parshin's conjecture).*

(ii) *For X smooth over $\mathbb{F}_p$, there are isomorphisms for $i, n \geq 0$*

$$K_i(X)^{(n)}_{\mathbb{Q}} \simeq H^{n-i}(X, \mathcal{K}_n) \otimes \mathbb{Q} \simeq H^{n-i}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}.$$

(iii) *The Beilinson-Soulé conjecture holds rationally: for any smooth X, $K_i(X)^{(n)}_{\mathbb{Q}} = 0$ for $i \geq 2n$.*

(iv) *The Bass conjecture holds rationally: for any smooth variety X over $\mathbb{F}_p$, the groups $K_i(X) \otimes \mathbb{Q}$ are finite dimensional vector spaces.*

(v) *For any field F of characteristic p and any $n \geq 0$, the map $K_n^M(F) \otimes \mathbb{Q} \to K_n(F) \otimes \mathbb{Q}$ is an isomorphism.*

Proof. (i) follows from corollary 8.16 and Geisser's theorem [10, th. 3.3 b)]. (ii) also follows from results of Geisser [10, th. 3.4 (ii) and cor. 3.6]. This implies (iii) trivially, and (iv) follows from (ii) and corollary 8.26 Finally, (v) also follows from [10, th. 3.4 (ii)]. □

8.29. Corollary. *Assume conjecture 8.12. If $X/\mathbb{F}_p$ is smooth affine of dimension d, then $H^i(X, \mathcal{K}_n^M)$ is torsion for $i > d - n$. In particular, $CH^n(X)$ is torsion if $d < 2n$.*

Proof. This follows from proposition 8.20 a), theorem 3.6 and the cohomological dimension of affine schemes [54, exposé XIX]. □

8.30. Corollary. (Bass-Tate conjecture, *cf.* [1, question p. 390]) *If conjecture 8.12 holds in weight n, then $K_n^M(F) \otimes \mathbb{Q} = 0$ for any extension F of $\mathbb{F}_p$, of transcendence degree $< n$.*

Proof. We may assume F finitely generated; then this follows from corollary 8.29 (or from corollary 6.12). □

8.31. Remarks.

1. The bound is sharp by [1, prop. 5.10].
2. This also follows from corollary 8.16 and Geisser's theorem [10, th. 3.4], but the above reason seems more enlightening.

8.32. Theorem. *Let F be a field of characteristic p and let X_F be a smooth, projective variety of dimension d over F.*
a) Conjecture 8.12 in weight n implies the Tate conjecture in codimension n and condition S^n for X_F.
b) If Conjecture 8.12 holds for $n \le d$, it implies the algebraicity of the Künneth components of the diagonal, Hard Lefschetz for cycles modulo numerical equivalence and the strong version of Beilinson's conjectural filtration (cf. [21, strong conj. 2.1]*) on all Chow groups of X_F.*

Proof. a) As in the proof of corollary 8.23, we assume F finitely generated over $\mathbb{F}_p$ and extend X_F to a smooth, projective morphism

$$f : X \to U$$

over a smooth model U of F. We have the Leray spectral sequence for l-adic cohomology (l-adic sheaves)

$$E_2^{p,q} = H^p_{\text{cont}}(U, R^q f_*^{l\text{-adic}} \mathbb{Q}_l(n)) \Rightarrow H^{p+q}_{\text{cont}}(X, \mathbb{Q}_l(n)). \tag{8.3}$$

As seen in the proof of corollary 6.14, this spectral sequence degenerates at E_2 [7]. Using proposition 8.20, we therefore get a commutative diagram

$$\begin{CD}
H^{n-2}_{\text{Zar}}(X, \mathcal{K}^M_n) \otimes \mathbb{Q}_l \oplus H^{n-1}_{\text{Zar}}(X, \mathcal{K}^M_n) \otimes \mathbb{Q}_l @>\sim>> H^{2n-1}_{\text{cont}}(X, \mathbb{Q}_l(n)) @>\sim>> \bigoplus_{p+q=2n} E_2^{p-1,q} \\
@V{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}VV @V{\cdot e}VV @V{\cdot e}VV \\
H^{n-1}_{\text{Zar}}(X, \mathcal{K}^M_n) \otimes \mathbb{Q}_l \oplus CH^n(X) \otimes \mathbb{Q}_l @>\sim>> H^{2n}_{\text{cont}}(X, \mathbb{Q}_l(n)) @>\sim>> \bigoplus_{p+q=2n} E_2^{p,q}.
\end{CD} \tag{8.4}$$

Here, cup-product by e maps $E_2^{p-1,q}$ to $E_2^{p,q}$. We therefore get an isomorphism

$$CH^n(X) \otimes \mathbb{Q}_l \xrightarrow{\sim} \bigoplus_{p+q=2n} E_2^{p,q} / e \cdot E_2^{p-1,q}.$$

In particular, the right hand side surjects onto

$$E_2^{0,2n} = H^0_{\text{cont}}(U, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n)) = H^{2n}_{\text{cont}}(\overline{X}, \mathbb{Q}_l(n))^{\pi_1(U)}.$$

Passing to the limit over U, we get the Tate conjecture and some information in passing on the conjectural filtration.

As for $S^n(X_F)$, the degeneration of the spectral sequence, lemma 6.6 and theorem 6.16 yield a commutative diagram

$$\begin{CD}
H^0_{\text{cont}}(\overline{U}, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n))^G @>>> H^0_{\text{cont}}(\overline{U}, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n))_G \\
@A{\wr}AA @VVV \\
H^0_{\text{cont}}(U, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n)) @>{\cdot e}>{\sim}> H^1_{\text{cont}}(U, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n))
\end{CD}$$

in which the right vertical map is injective. Here $\overline{U} = U \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$. This yields

$$H^0_{\text{cont}}(\overline{U}, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n))^G \xrightarrow{\sim} H^0_{\text{cont}}(\overline{U}, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n))_G. \tag{8.5}$$

On the other hand, by [6, cor. 3.4.13], the $\pi_1(\overline{U})$-module $R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n)$ is semi-simple, hence

$$H^0_{\text{cont}}(\overline{U}, R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n)) = R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n)^{\pi_1(\overline{U})} \xrightarrow{\sim} R^{2n} f_*^{l\text{-adic}} \mathbb{Q}_l(n)_{\pi_1(\overline{U})} \tag{8.6}$$

and then (8.5) and (8.6) yield

$$R^{2n}f_*^{l\text{-adic}}\mathbb{Q}_l(n)^{\pi_1(U)} \xrightarrow{\sim} R^{2n}f_*^{l\text{-adic}}\mathbb{Q}_l(n)_{\pi_1(U)}.$$

Passing back to the generic fibre, we get $S^n(X_F)$.

b) The algebraicity of the Künneth components follows from [49, §3]. We prove the assertion on Hard Lefschetz, since we didn't trace a proof in the literature. We first note that a) and [49, prop. 2.6] imply that homological equivalence equals numerical equivalence on X_F. Let h be the cycle class of a hyperplane section of X_F (relative to some projective embedding), and let us denote by $A^n(X_F)$ the group of cycles of codimension n on X_F modulo numerical equivalence. Let $d = \dim X_F$. We have to show that, for all $n < d/2$, the product by h^{d-2n}

$$A^n(X_F)\otimes\mathbb{Q} \xrightarrow{\cdot h^{d-2n}} A^{d-n}(X_F)\otimes\mathbb{Q}$$

is bijective. By the Tate conjecture and the equality of homological and numerical equivalences, this translates as the bijectivity of

$$H^{2n}(\overline{X}_F,\mathbb{Q}_l(n))^{G_F} \xrightarrow{\cdot h^{d-2n}} H^{2d-2n}(\overline{X}_F,\mathbb{Q}_l(d-n))^{G_F}.$$

By the geometric Hard Lefschetz, this map is injective. On the other hand, Poincaré duality gives an isomorphism

$$H^{2n}(\overline{X}_F,\mathbb{Q}_l(n))^{G_F} \xrightarrow{\sim} Hom(H^{2d-2n}(\overline{X}_F,\mathbb{Q}_l(d-n))_{G_F},\mathbb{Q}_l).$$

Hence the two $\mathbb{Q}_l$-vector spaces $H^{2n}(\overline{X}_F,\mathbb{Q}_l(n))^{G_F}$ and $H^{2d-2n}(\overline{X}_F,\mathbb{Q}_l(d-n))_{G_F}$ have the same dimension. But $\dim H^{2d-2n}(\overline{X}_F,\mathbb{Q}_l(d-n))_{G_F} = \dim H^{2d-2n}(\overline{X}_F,\mathbb{Q}_l(d-n))^{G_F}$ by $S^n(X_F)$ and cup-product by h^{d-2n} must be bijective.

The strong form of Beilinson's conjecture follows from this, corollary 8.23 and [17, lemma 2.2]. □

8.33. Corollary. *With notation as in theorem 8.32 and under the assumptions of theorem 8.32 b), Murre's conjecture* [35, 1.4] *holds for X_F.*

Proof. This follows from theorem 8.32 b) and [21, th. 5.2]. □

8.34. Remarks.

1. Using $H^{2n+1}_{\text{cont}}(X,\mathbb{Q}_l(n))$ yields a different presentation of the filtration on $CH^n(X_F)\otimes\mathbb{Q}_l$.
2. One can mimic the proof of [21, lemma 3.1], thereby obtaining extra information on the direct limit of the spectral sequences (8.3) and a direct proof of the strong form of Beilinson's filtration conjecture. More precisely we note that, for (U,f,X) be as in the proof of theorem 8.32, when U shrinks to smaller and smaller neighbourhoods of $\operatorname{Spec} F$, the spectral sequences (8.3) have a direct limit

$$\tilde E_2^{p,q} = \tilde H^p_{\text{cont}}(F/\mathbb{F}_p, H^q(\overline{X}_F,\mathbb{Q}_l(n))) \Rightarrow \tilde H^{p+q}_{\text{cont}}(X_F/\mathbb{F}_p,\mathbb{Q}_l(n)) \tag{8.7}$$

which degenerates and only depends on X_F.

8.35. Corollary. *Let F, X_F be as in theorem 8.32, and let $\delta = \operatorname{trdeg}(F/\mathbb{F}_p)$. If conjecture 8.12 holds for all $n \le d$, then, in the spectral sequence* (8.7)*,*

(i) $\tilde E_2^{p,q} = 0$ *for* $p+q<n$, $p+d<n$, $\delta+q<n$, $\delta+d<n$ *or* $p>n+1$*;*

(ii) *For all q,* $\tilde E_2^{n,q} \xrightarrow{\cdot e} \tilde E_2^{n+1,q}$ *is surjective.*

Moreover, if X is defined over a finite field, then $\tilde E_2^{p,q}=0$ for $2p+q<2n$.

Proof. All inequalities of (i), except the last one, and the last statement follow from corollary 6.14. For the last inequality of (i) and for (ii), let us write more precisely $\tilde E_2^{p,q}(n)$ in order to keep track of the Tate twist n. By Hard Lefschetz for l-adic cohomology, we have an isomorphism

$$\tilde E_2^{p,q}(n) \xrightarrow{\sim} \tilde E_2^{p,2d-q}(n+d-q).$$

The right hand side group is a direct summand of $\tilde H^{p+2d-q}_{\rm cont}(X_F/\mathbb{F}_p,\mathbb{Q}_l(n+d-q))$; under conjecture 8.12 (for $d-n$), this group is 0 for $p>n+1$ by proposition 8.20 a). On the other hand, by proposition 8.20 b), cup-product by e

$$\tilde H^{n+2d-q}_{\rm cont}(X_F/\mathbb{F}_p,\mathbb{Q}_l(n+d-q)) \xrightarrow{\cdot e} \tilde H^{n+2d-q+1}_{\rm cont}(X_F/\mathbb{F}_p,\mathbb{Q}_l(n+d-q))$$

is surjective, which proves the second assertion of corollary 8.35. □

We use this to get much more precise information on the conjectural Beilinson filtration, not only on Chow groups but also on $\mathcal{K}^M$-cohomology groups. Note how the structure obtained is reminiscent of a pure Hodge structure.

8.36. Theorem. *With notation and assumptions as in corollary 8.35, there is for all i a filtration on $H^i(X_F,\mathcal{K}^M_n)\otimes\mathbb{Q}$*

$$F^pH^i(X_F,\mathcal{K}^M_n)_\mathbb{Q},\quad p\geq 0$$

such that

(i) *The action of correspondences on the associated graded $\operatorname{gr}^p H^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}$ factors through numerical equivalence.*
(ii) $\operatorname{gr}^p H^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}=0$ *for $p>n$, $p+d<n$, $p>\delta+i$ or $n>\delta+d$.*
(iii) *Usual naturality properties (contravariance, compatibility with products).*
(iv) *There are "higher Abel-Jacobi maps"*

$$\operatorname{gr}^p H^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}\to \tilde H^p_{\rm cont}(F/\mathbb{F}_p,H^{i+n-p}_{\rm cont}(\overline{X}_F,\mathbb{Q}_l(n)))/e\cdot H^{p-1}_{\rm cont}(F/\mathbb{F}_p,H^{i+n-p}_{\rm cont}(\overline{X}_F,\mathbb{Q}_l(n)))$$

such that the induced maps on $\operatorname{gr}^p H^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{Q}_l$ are isomorphisms.

In particular, there is a canonical isomorphism

$$H^i(X_F,\mathcal{K}^M_n)\otimes\mathbb{Q}_l\simeq\coprod_{p\leq n}\operatorname{gr}^p H^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{Q}_l.$$

Moreover, if X is defined over a finite field, then $\operatorname{gr}^p H^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}=0$ for $p<n-i$.

Proof. Using a diagram analogous to (8.4), we get an isomorphism

$$H^i(X_F,\mathcal{K}^M_n)\otimes\mathbb{Q}_l\simeq\coprod_{p+q=i+n}\tilde E_2^{p,q}(n)/e\cdot\tilde E_2^{p-1,q}(n).$$

This isomorphism depends on the Hard Lefschetz theorem; it can be made canonical by [7]. By corollary 8.35, we have

$$\tilde E_2^{p,q}(n)/e\cdot\tilde E_2^{p-1,q}(n)=0\quad\text{for}\quad p\notin[n-i,n].$$

Define $F^pH^i(X_F,\mathcal{K}^M_n)\otimes\mathbb{Q}_l$ as the sub-vector space $\coprod_{r\geq p}\tilde E_2^{r,i+n-r}(n)/e\cdot\tilde E_2^{r-1,i+n-r}(n)$.

8.37. Lemma. *This filtration is $\mathbb{Q}$-rational; the corresponding filtration $F^pH^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}$ on $H^i(X_F,\mathcal{K}^M_n)\otimes\mathbb{Q}$ does not depend on the choice of l.*

Proof. Choose $f:X\to U$ as in the proof of theorem 8.32. Applying Rf_* to the isomorphism (8.2) restricted to X, we get an isomorphism

$$Rf_*\mathbb{Q}_l(0)^c\overset{L}{\otimes}\alpha^*\mathcal{K}^M_n[-n]\xrightarrow{\sim}Rf_*\mathbb{Q}_l(n)^c$$

and the rational filtration comes from the Leray spectral sequence

$$R^pf_*(\mathcal{H}^q(\mathbb{Q}_l(0)^c\overset{L}{\otimes}\alpha^*\mathcal{K}^M_n[-n]))\Rightarrow R^{p+q}f_*(\mathbb{Q}_l(0)^c\overset{L}{\otimes}\alpha^*\mathcal{K}^M_n[-n]).$$

□

Lemma 8.37 shows that $F^pH^i(X_F,\mathcal{K}^M_n)_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{Q}_l\xrightarrow{\sim}F^pH^i(X_F,\mathcal{K}^M_n)\otimes\mathbb{Q}_l$. □

8.4. **Homological transformation.** Let $i : Z \hookrightarrow X$ be a closed injection in $Sch/\mathbb{F}_p$, with X smooth of pure dimension d. Applying $Ri^!$ to (8.2) restricted to X, we get a morphism in $\mathcal{D}^+(Ab(Z_{\text{ét}}))$:

$$Ri^!(\mathbb{Q}_l(0)^c \overset{L}{\otimes} \alpha^*\mathcal{K}^M_{n,X}[-n]) \to Ri^!\mathbb{Q}_l(n)^c_X.$$

By proposition 3.11 c), the right hand side can be rewritten as $L_l(n-d)_Z[-2d] \otimes \mathbb{Q}_l$. On the other hand, since $\mathbb{Q}_l(0)^c = \mathbb{Q}_l[0] \oplus \mathbb{Q}_l[-1]$, the left hand side can be rewritten

$$\mathbb{Q}_l(0)^c \overset{L}{\otimes} Ri^!\alpha^*\mathcal{K}^M_{n,X}[-n].$$

Using the natural transformation $\alpha^*Ri^! \to Ri^!\alpha^*$, we get a composition

$$\mathbb{Q}_l(0)^c \overset{L}{\otimes} \alpha^*Ri^!\mathcal{K}^M_{n,X}[-n] \to L_l(n-d)_Z[-2d] \otimes \mathbb{Q}_l.$$

By proposition 8.9, the left hand side is isomorphic to $\mathbb{Q}_l(0)^c \overset{L}{\otimes} \alpha^*\mathcal{M}_{n-d,Z}[-n-d]$. After shifting and changing n into $n-d$, we therefore get a natural transformation

$$\mathbb{Q}_l(0)^c \overset{L}{\otimes} \alpha^*\mathcal{M}_{n,Z}[-n] \to L_l(n)_Z \otimes \mathbb{Q}_l. \tag{8.8}$$

It can be shown by the method of [12, proof of th. 4.1] that for Z quasi-projective (8.8) does not depend on the choice of the closed embedding i and has the usual naturality properties; we shall not need this here, so we skip the proof.

8.38. Lemma. *The natural map*

$$\alpha^*Ri^!\mathcal{K}^M_{n,X} \otimes \mathbb{Q} \to Ri^!\alpha^*\mathcal{K}^M_{n,X} \otimes \mathbb{Q}$$

is an isomorphism.

Proof. This follows from corollary 8.6 and the classical isomorphism of functors

$$Ri^!R\alpha_* \xrightarrow{\sim} R\alpha_*Ri^!.$$

□

8.39. Theorem. *Conjecture 8.12 for all n is equivalent to the following:* (8.8) *is an isomorphism for all n and all $Z \in Sch/\mathbb{F}_p$. In particular, under conjecture 8.12, one has isomorphisms*

$$H_i^{\text{cont}}(Z, \mathbb{Q}_l(n)) \simeq H_{i-n}(Z, \mathcal{K}^M_n) \otimes \mathbb{Q}_l \oplus H_{i-n+1}(Z, \mathcal{K}^M_n) \otimes \mathbb{Q}_l.$$

Proof. This follows from the construction of (8.8) and lemma 8.38. □

8.40. Proposition. *Under conjecture 8.12, there are isomorphisms for all quasi-projective $X \in Sch/\mathbb{F}_p$:*

$$H_i(X, \mathcal{K}^M_n) \otimes \mathbb{Q} \xrightarrow{\sim} H_i(X, \mathcal{K}_n) \otimes \mathbb{Q} \simeq \operatorname{gr}_n K'_{i-n}(X) \otimes \mathbb{Q}$$

where the second group is defined analogously to the first (with K-groups) and the third one is defined in [44, th. 7 and 7.4].

Proof. The first isomorphism follows from corollary 8.28 (v); the second follows from the spectral sequence of [44, th. 8 (iv)] and corollary 8.28 (v) again, which implies that this spectral sequence degenerates at E^2. □

With the help of theorem 8.39, we can now extend corollary 8.27 to singular varieties:

8.41. Theorem. *Assume conjecture 8.12. Let X be a quasi-projective variety over $\mathbb{F}_p$. Then, with notation as in* (7.1)*, the order of* $\det(1 - tF|H^i_c(\overline{X}, \mathbb{Q}_l))$ *at $t = p^{-n}$ is* $\dim_{\mathbb{Q}} H_{i-n}(X, \mathcal{K}^M_n) \otimes \mathbb{Q} = \dim_{\mathbb{Q}} \operatorname{gr}_n K'_{i-2n}(X) \otimes \mathbb{Q}$.

Proof. This follows from theorem 7.2, theorem 8.39 and proposition 8.40. □

Theorem 8.41 implies (and precises) Soulé's conjecture that

$$\mathrm{ord}_{s=n}\zeta(X,s) = \sum_i (-1)^{i+1} \dim_{\mathbb{Q}} \mathrm{gr}_n\, K'_i(X) \otimes \mathbb{Q}$$

[42, conj. 2.2].

9. Integral refinement

In all this section, we assume resolution of singularities for varieties over $\mathbb{F}_p$. We stress this assumption occasionally. The reader who does not want to make such an assumption is advised to skip this section.

9.1. **Review of motivic cohomology.** Recall the motivic complexes $\mathbb{Z}(n)$ of Suslin and Voevodsky [47]. We shall denote by $H^*(X, \mathbb{Z}(n))$ (*resp.* $H^*_{\text{ét}}(X, \mathbb{Z}(n))$) the groups respectively denoted by $H^*_B(X, \mathbb{Z}(n))$ and $H^*_L(X, \mathbb{Z}(n))$ in [53], and similarly when replacing $\mathbb{Z}(n)$ by other $A(n) := A \overset{L}{\otimes} \mathbb{Z}(n)$, where A is an abelian group. Occasionally we shall write H^*_M instead of H^*.

9.1. Proposition. *We have quasi-isomorphisms*

$$\mathbb{Q}(n) \xrightarrow{\sim} R\alpha_*\alpha^*\mathbb{Q}(n) \tag{9.1}$$

and

$$\alpha^*\mathbb{Z}(n) \overset{L}{\otimes} \mathbb{Z}/l^\nu \xrightarrow{\sim} \mu_{l^\nu}^{\otimes n}[0] \tag{9.2}$$

where α is the projection of the big étale site of $\operatorname{Spec}\mathbb{F}_p$ *onto its big Zariski site; in* (9.1), $\mathbb{Q}(n)$ *denotes* $\mathbb{Z}(n) \otimes \mathbb{Q}$.

Proof. This follows from [53, th. 2.5 and 2.6]. □

From this we get some information on the étale motivic cohomology groups of a smooth scheme X:

9.2. Proposition. *Let $\mathbb{Z}_{(l)}(n)$ denote $\mathbb{Z}(n) \otimes \mathbb{Z}_{(l)}$, where $\mathbb{Z}_{(l)}$ is the localisation of $\mathbb{Z}$ at l. Then, for any smooth scheme X over $\mathbb{F}_p$, $H^i_{\text{ét}}(X, \mathbb{Z}_{(l)}(n))$ is uniquely divisible-by-finite for $i < n$, torsion for $i = 2n + 1$, torsion of cofinite type for $i = 2n + 2$ and finite for $i > 2n + 2$. If moreover X is projective, then $H^i_{\text{ét}}(X, \mathbb{Z}_{(l)}(n))$ is*

(i) *uniquely divisible-by-finite for $i < 2n$;*
(ii) *with finite torsion for $i = 2n$.*

Proof. By [53, cor. 2.3], we have $H^i_{\mathrm{Zar}}(X, \mathbb{Z}(n)) = 0$ for $i > 2n$. It follows from this and (9.1) that $H^i_{\text{ét}}(X, \mathbb{Z}(n))$ is torsion for $i > 2n$. The claims now follow from proposition 6.1, corollaries 6.2 and 6.10 and the exact sequence

$$\dots \to H^{i-1}_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \xrightarrow{\beta} H^i_{\text{ét}}(X, \mathbb{Z}_{(l)}(n)) \to H^i_{\text{ét}}(X, \mathbb{Q}(n)) \to H^i_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to \dots \tag{9.3}$$

□

9.3. Proposition. *Let X be a smooth variety over $\mathbb{F}_p$. Then,*

(i) $\mathcal{H}^i(\mathbb{Z}(n)) = 0$ *for $i > n$.*
(ii) $\mathcal{H}^n(\mathbb{Q}(n)) \simeq \mathcal{K}^M_n \otimes \mathbb{Q}$
(iii) $H^p(X, \mathbb{Z}(n)) = 0$ *for $p > 2n$*
(iv) $H^{2n}(X, \mathbb{Z}(n))$ *is canonically isomorphic to $CH^n(X)$.*

Proof. See [53, cor. 2.3 and cor. 2.4]. □

For any $\mathbb{F}_p$-scheme X, let F_X denote the geometric Frobenius of X [56, exposé XV]. We note that F_X defines a finite correspondence from X to itself in the sense of [52]. It is easy to check that $X \mapsto F_X$ "extends" to an endofunctor F of the category of effective geometrical motives $DM_{gm}^{eff}(\mathbb{F}_p)$. It is clear that, for $M, M' \in DM_{gm}^{eff}(\mathbb{F}_p)$,

$$F_M \otimes F_{M'} \mapsto F_{M\otimes M'}$$

under the natural map

$$End_{DM}(M) \otimes_{\mathbb{Z}} End_{DM}(M') \to End_{DM}(M \otimes M').$$

9.4. Proposition. *For any smooth scheme X over $\mathbb{F}_p$, the geometric Frobenius of X acts on $H^*(X, \mathbb{Z}(n))$ by multiplication by p^n.*

Proof. We interpret motivic cohomology groups as Homs in $DM_{gm}^{eff}(\mathbb{F}_p)$. So we have

$$H^i(X, \mathbb{Z}(n)) = Hom_{DM}(M(X), \mathbb{Z}(n)[i])$$

By naturality of the geometric Frobenius, its action on the latter groups is the same as the action induced by the geometric Frobenius of $\mathbb{Z}(n)$. Therefore it suffices to show that its class in $End_{DM}(\mathbb{Z}(n))$ is p^n times the identity. Since $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$, we are reduced to the case $n = 1$ by the remarks just before the statement of proposition 9.4. Then $\mathbb{Z}(1)$ is a shift of the Lefschetz motive L. By [52, prop. 2.1.4], we can compute F_L in the category of effective Chow motives; then the result is classical [43]. □

9.2. **The conjecture.** From (9.2) we get compatible maps ($\nu \geq 0$)

$$\alpha^*\mathbb{Z}(n) \to \mu_{l^\nu}^{\otimes n}[0].$$

Tensoring it with $\mathbb{Z}_l(0)^c$, we get compatible maps

$$\mathbb{Z}_l(0)^c \overset{L}{\otimes} \alpha^*\mathbb{Z}(n) \to \mu_{l^\nu}^{\otimes n}[0].$$

which yield a morphism

$$\mathbb{Z}_l(0)^c \overset{L}{\otimes} \alpha^*\mathbb{Z}(n) \to \mathbb{Z}_l(n)^c. \tag{9.4}$$

The following lemma is obvious from (9.2) and lemma 2.2.

9.5. Lemma. *The morphism* $(9.4)\overset{L}{\otimes}\mathbb{Z}/l$ *is an isomorphism.* □

9.6. Conjecture. *The morphism* (9.4) *is an isomorphism.*

In view of lemma 9.5, conjecture 9.6 is equivalent to

9.7. Conjecture. *The morphism* $(9.4)\otimes\mathbb{Q}$ *is an isomorphism.*

Assume resolution of singularities. By proposition 9.3 and corollary 6.4, the morphism $(9.4)\otimes\mathbb{Q}$ induces a homomorphism of étale sheaves

$$\alpha^*\mathcal{K}_n^M \otimes \mathbb{Q}_l \xrightarrow{u_n} \mathcal{H}^n(\mathbb{Q}_l(n)^c). \tag{9.5}$$

It is easy to check that this morphism coincides with that of proposition 8.10 (tensored with $\mathbb{Q}$).

9.8. Proposition. *Under resolution of singularities, conjectures 9.6 and 9.7 are equivalent to the following:*

(i) $\mathcal{H}^i(\mathbb{Z}(n)) \otimes \mathbb{Q} = 0$ *for* $i < n$.
(ii) *The morphism* u_n *from* (9.5) *is an isomorphism.*

(iii) *The morphism of étale sheaves*

$$\mathcal{H}^n(\mathbb{Q}_l(n)^c) \xrightarrow{e} \mathcal{H}^{n+1}(\mathbb{Q}_l(n)^c)$$

given by cup-product by e is an isomorphism.

(iv) $\mathcal{H}^i(\mathbb{Q}_l(n)^c) = 0$ *for* $i > n+1$.

Proof. Assume conjecture 9.7. Then (i) follows from theorem 6.9 and [53, th. 2.5], (ii) is clear, (iii) follows from the fact that it is true for $n = 0$ and (iv) follows from the fact that $\mathbb{Z}(n)$ is acyclic in degrees $> n$ [47]. Conversely, (i)–(iv) imply that (9.4)$\otimes\mathbb{Q}$ induces an isomorphism on cohomology sheaves, hence is an isomorphism. □

9.9. Corollary. *Under conjecture 9.7,* $H^i(X, \mathbb{Q}(n)) \simeq H^{i-n}(X, \mathcal{K}_n^M) \otimes \mathbb{Q}$. □

9.10. Theorem. *Under resolution of singularities, conjectures 8.12 and 9.6 are equivalent.*

Proof. From proposition 9.8 it easily follows that conjecture 9.6 implies conjecture 8.12. Conversely, conjecture 8.12 implies (ii)–(iv) in proposition 9.8 and we are left to see that it also implies (i). By theorem 5.4 and the purity theorem for motivic cohomology, we have to show that

$$H^i(X, \mathbb{Q}(n)) = 0 \quad \text{for} \quad i < n$$

for any smooth, projective X. The proof is similar to that of theorem 8.17, using corollary 8.16 and proposition 9.4. □

The following proposition is trivial from theorem 4.6 b).

9.11. Proposition. *If conjecture 9.6 holds, then* $\mathbb{Z}_l(n)^c \simeq \mathbb{Z}^c \overset{L}{\otimes} \alpha^*\mathbb{Z}(n) \overset{L}{\otimes} \mathbb{Z}_l$. □

In particular, we recover proposition 8.19 in a more explicit way.

9.12. Proposition. *Suppose conjecture 9.6 holds. Let X be a smooth variety. Then there is a long exact sequence*

$$\cdots \to H^{i-2}_{\text{ét}}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_l \xrightarrow{\partial} H^i_{\text{ét}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l$$
$$\to H^i_{\text{cont}}(X, \mathbb{Z}_l(n)) \to H^{i-1}_{\text{ét}}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_l \to \dots$$

Moreover, the connecting homomorphism ∂ is given by tensoring by $\mathbb{Z}_l$ the composition

$$\begin{tikzcd} H^{i-2}_{\text{ét}}(X, \mathbb{Q}(n)) \arrow[r, "\partial"] \arrow[d] & H^i_{\text{ét}}(X, \mathbb{Z}_{(l)}(n)) \\ H^{i-2}_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[r, "\cdot e"] & H^{i-1}_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[u, "\beta"] \end{tikzcd}$$

where β is the Bockstein map of (9.3). *In particular, the image of ∂ is torsion.*

Proof. The exact sequence is clear; the computation of ∂ follows from proposition 4.4. □

9.13. Proposition. *Keep the assumptions and notation as in proposition 9.12. Then*

a) For any X,

(i) *The map* $H^i_{\text{ét}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to H^i_{\text{cont}}(X, \mathbb{Z}_l(n))$ *is bijective for* $i \leq n$ *and injective for* $i = n+1$.

(ii) *As a module over* $\mathbb{Z}_{(l)}$, $H^i_{\text{ét}}(X, \mathbb{Z}_{(l)}(n))$ *is finite for* $i < n$, *finitely generated for* $i = n$ *and an extension of a finitely generated module by a torsion divisible module of finite corank for* $i > n$. *We have*

$$\operatorname{corank}_{\mathbb{Q}_l/\mathbb{Z}_l} H^i_{\text{ét}}(X, \mathbb{Z}(n)) = \operatorname{rank}_{\mathbb{Z}} H^{i-2}_{\text{ét}}(X, \mathbb{Z}(n)).$$

b) If X is moreover projective, then

(i) *The map* $H^i_{\text{ét}}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \to H^i_{\text{cont}}(X, \mathbb{Z}_l(n))$ *is bijective for* $i \neq 2n+1, 2n+2$ *and injective for* $i = 2n+1$.

(ii) *As a module over* $\mathbb{Z}_{(l)}$, $H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))$ *is finite for* $i\neq 2n,2n+2$, *finitely generated for* $i=2n$ *and cofinitely generated for* $i=2n+2$.

(iii) *Suppose conjecture 9.6 holds for all* $n\le d$. *If* $d=\dim X$, *the group* $H^{2d+2}_{\text{ét}}(X,\mathbb{Z}_{(l)}(d))$ *is canonically isomorphic to* $\mathbb{Q}_l/\mathbb{Z}_l$ *and the pairings*

$$H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))\times H^{2d+2-i}_{\text{ét}}(X,\mathbb{Z}_{(l)}(d-n))\to H^{2d+2}_{\text{ét}}(X,\mathbb{Z}_{(l)}(d))$$

are perfect.

(iv) *We have isomorphisms*

$$\begin{aligned} H^{2n}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{tors}} &\xrightarrow{\sim} H^{2n}_{\text{cont}}(X,\mathbb{Z}_l(n))_{\text{tors}}\\ H^{2n+1}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n)) &\xrightarrow{\sim} H^{2n+1}_{\text{cont}}(X,\mathbb{Z}_l(n))_{\text{tors}}\\ H^{2n+2}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{cotors}} &\xrightarrow{\sim} H^{2n+2}_{\text{cont}}(X,\mathbb{Z}_l(n)).\end{aligned}$$

Proof. Everything follows easily from propositions 9.2 and 9.12 plus the fact that the ring extension $\mathbb{Z}_l/\mathbb{Z}_{(l)}$ is faithfully flat, except for the corank computation. We have exact sequences

$$H^{i-1}_{\text{cont}}(X,\mathbb{Z}_l(n))\to H^{i-2}_{\text{ét}}(X,\mathbb{Z}(n))\otimes\mathbb{Q}\to \mathit{Div}(H^i_{\text{ét}}(X,\mathbb{Z}(n))\otimes\mathbb{Z}_l)\to 0.$$

Since the last group is torsion and the first one is finitely generated, the image of the left map must be a lattice in $H^{i-2}_{\text{ét}}(X,\mathbb{Z}(n))\otimes\mathbb{Q}$. This gives the claim. □

9.14. Remark. In particular, propsition 9.13 b) shows that conjecture 9.6 implies S. Lichtenbaum's conjectures 1) to 7) of [28, §7] on the values of étale motivic cohomology, after localisation at l. By [33], this implies conjecture 8) of *loc. cit.* on the values of the zeta function of X at nonnegative integers (note that there is an obvious misprint in the formula of *loc. cit.*). We now extend this to get a formula for the principal part of $\zeta(X,s)$ at $s=n$ in terms of motivic cohomology for *any* smooth variety, thereby completing the result of corollary 8.27.

9.15. Lemma. *The homomorphism* ∂ *of proposition 9.12 induces a homomorphism*

$$\bar\partial^{i-2}: H^{i-2}_{\text{ét}}(X,\mathbb{Z}(n))\otimes\mathbb{Q}_l/\mathbb{Z}_l\to H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{tors}}$$

with finite kernel and cokernel.

Proof. The existence of $\bar\partial^{i-2}$ is clear from the factorisation of ∂ given in proposition 9.12. By the exact sequence in this proposition, $\operatorname{Ker}\partial$ is a lattice in $H^{i-2}_{\text{ét}}(X,\mathbb{Q}(n))$ and $\operatorname{Coker}\bar\partial^{i-2}\xrightarrow{\sim} H^i_{\text{cont}}(X,\mathbb{Z}_l(n))_{\text{tors}}$, which is finite. □

9.16. Theorem. *Let* X *be a smooth variety of pure dimension* d *over* $\mathbb{F}_p$. *If conjecture 9.6 holds, then:*

(i) *the order of* $\det(1-tF|H^i_c(\overline{X},\mathbb{Q}_l))$ *at* $t=p^{n-d}$ *is* $\operatorname{rank} H^{2d-i}(X,\mathbb{Z}(n))$.

(ii) $\zeta(X,s)=(1-p^{n-d-s})^{a_{d-n}}\varphi(s)$, *with* $a_{d-n}=\sum(-1)^{i+1}\operatorname{rank} H^i(X,\mathbb{Z}(n))$ *and*

(iii) $|\varphi(d-n)|_l=\left|\prod_{i=0}^{2d}\operatorname{ind}(\bar\partial^i)^{(-1)^{i+1}}\right|_l$.

Note that we don't get a separate formula for the principal part of $\det(1-tF|H^i_c(\overline{X},\mathbb{Q}_l))$ at $t=p^{-n}$ in terms of motivic cohomology. See remark 9.18 for more on this point.

Proof. (i) and (ii) follow from corollaries 8.27 and 9.9. (iii) With the help of lemma 9.15, we extend the commutative square of proposition 9.12 into a bigger commutative diagram:

$$\begin{array}{ccc} H^{i-2}_{\text{ét}}(X,\mathbb{Z}(n))\otimes\mathbb{Q}_l/\mathbb{Z}_l & \xrightarrow{\bar\partial^{i-2}} & H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{tors}}\\ \downarrow & & \uparrow{\scriptstyle\beta}\\ H^{i-2}_{\text{ét}}(X,\mathbb{Q}_l/\mathbb{Z}_l(n)) & \xrightarrow{\cdot e} & H^{i-1}_{\text{ét}}(X,\mathbb{Q}_l/\mathbb{Z}_l(n))\\ \downarrow & & \uparrow\\ H^{i-2}_{\text{ét}}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n))^G & \xrightarrow{\rho^{i-2}} & H^{i-2}_{\text{ét}}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n))_G \end{array}$$

which follows from lemma 6.6 as in the proof of proposition 6.5. By corollary 7.7, the bottom horizontal map has finite kernel and cokernel, and so has the top one by lemma 9.15. By the cross of exact sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^{i-1}(X,\mathbb{Z}(n))\otimes\mathbb{Q}_l/\mathbb{Z}_l \arrow[d] \arrow[dr, "\alpha^{i-1}"] & & \\
0 \arrow[r] & H^{i-2}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n))_G \arrow[r] \arrow[dr, "\beta^{i-2}"'] & H^{i-1}(X,\mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[r] \arrow[d] & H^{i-1}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n))^G \arrow[r] & 0 \\
 & & H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{tors}} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

the compositions

$$H^{i-1}(X,\mathbb{Z}(n))\otimes\mathbb{Q}_l/\mathbb{Z}_l \xrightarrow{\alpha^{i-1}} H^{i-1}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n))^G$$
$$H^{i-2}(\overline{X},\mathbb{Q}_l/\mathbb{Z}_l(n))_G \xrightarrow{\beta^{i-2}} H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{tors}}$$

have same kernel and cokernel. Since, by the above, $\operatorname{Ker}\alpha^{i-2}$ and $\operatorname{Coker}\beta^{i-2}$ are finite for any i, this proves

9.17. Lemma. *For all i, α^i and β^i have finite kernel and cokernel; moreover* $\operatorname{ind}(\alpha^{i-1}) = \operatorname{ind}(\beta^{i-2})$. $\square$

By lemmas 9.17, 7.4 b) and the above commutative diagram, we have

$$\operatorname{ind}(\bar{\partial}^{i-2}) = \operatorname{ind}(\alpha^{i-2})\operatorname{ind}(\rho^{i-2})\operatorname{ind}(\beta^{i-2}) = \operatorname{ind}(\alpha^{i-2})\operatorname{ind}(\rho^{i-2})\operatorname{ind}(\alpha^{i-1}).$$

By corollary 7.7, $|f_i(p^{n-d})|_l = |\operatorname{ind}(\rho^{2d-i})|_l$. It follows that

$$|f(p^{n-d})|_l = \left|\prod_{i=0}^{2d} f_i(p^{n-d})^{(-1)^{i+1}}\right|_l = \left|\prod_{i=0}^{2d}\operatorname{ind}(\bar{\partial}^i)^{(-1)^{i+1}}\right|_l$$

as desired. $\square$

9.18. Remark. Theorem 9.16 (i) becomes much more suggestive if one compares the group $H^{2d-i}(X,\mathbb{Z}(n))$ to

$$H^i_c(X,\mathbb{Z}(d-n)),$$

motivic cohomology with compact supports [52]. It is likely that these two groups are dual to each other after tensoring by $\mathbb{Q}$. More precisely, by [9, §9] we have

$$H^i_c(X,\mathbb{Z}(d-n)) \simeq H_{2d-i}(X,\mathbb{Z}(n))$$

since X is smooth, where the right hand group is motivic homology, and there is a pairing

$$\begin{aligned}
&H_{2d-i}(X,\mathbb{Z}(n)) \times H^{2d-i}(X,\mathbb{Z}(n)) \\
&\qquad = Hom_{DM}(\mathbb{Z}(n)[2d-i], M(X)) \times Hom_{DM}(M(X),\mathbb{Z}(n)[2d-i]) \\
&\qquad\qquad \to End_{DM}(\mathbb{Z}(n)[2d-i]) = \mathbb{Z}.
\end{aligned}$$

Here the motivic groups are Zariski, but they do or should coincide with the étale groups after tensoring by $\mathbb{Q}$. This strongly hints that one should formulate a version of conjecture 9.6 for continuous étale homology analogous to theorem 8.39, involving this time Borel-Moore motivic homology. This would presumably allow one to remove the smoothness assumption in theorem 9.16. In this generality, there is a pairing between étale motivic cohomology with compact supports and étale motivic Borel-Moore homology, analogous to the above, that one can conjecture to be perfect after tensoring by $\mathbb{Q}$. What would then be the relationship between the principal part of $\det(1-tF|H^i_c(\overline{X},\mathbb{Q}_l))$ at $t=p^{n-d}$ and the determinant of this pairing (together with the torsion of the two motivic groups)?

We even suspect that this conjectural picture formally follows from conjecture 9.6.

Specialising to the case where X is projective and taking account of proposition 9.13 and the functional equation of $\zeta(X,s)$, we get:

9.19. Corollary. *With the assumptions and notation of theorem 9.16, if X is moreover projective, then $\zeta(X,s)=(1-p^{n-s})^{a_n}\varphi(s)$, with $a_n=\operatorname{rank}H^{2n}(X,\mathbb{Z}(n))$ and*

$$|\varphi(n)|_l=\left|\prod_{i\neq 2n}|H^i_{\text{ét}}(X,\mathbb{Z}(n))|^{(-1)^i}\operatorname{ind}\left(H^{2n}(X,\mathbb{Z}_{(l)}(n))\otimes\mathbb{Q}_l/\mathbb{Z}_l\xrightarrow{\bar\partial^{2n}}H^{2n+2}(X,\mathbb{Z}(n))\right)^{-1}\right|_l.$$

Alternatively, using corollary 7.10 and proposition 9.13 b) (iv), we get the following formula, which is the l-primary part of [28, §7, conj. 8] (after correcting the weight $n+2$ to n in *loc. cit.*):

9.20. Theorem. *With the assumptions and notation of corollary 9.19, we have*

$$|\varphi(n)|_l=\left|\prod_{i\neq 2n,2n+2}|H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))|^{(-1)^i}\cdot\frac{|H^{2n}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{tors}}||H^{2n+2}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{cotors}}|}{|R_n(X)|_l}\right|_l$$

where $H^{2n+2}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))_{\text{cotors}}$ is the quotient of $H^{2n+2}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))$ by its maximal divisible subgroup (a finite group by corollary 6.2 and proposition 9.13 b) (iv)) and $R_n(X)$ is the discriminant of the pairing

$$H^{2n}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))/\text{tors}\times H^{2d-2n}_{\text{ét}}(X,\mathbb{Z}_{(l)}(d-n))/\text{tors}\to H^{2d}_{\text{ét}}(X,\mathbb{Z}_{(l)}(d))/\text{tors}\simeq\mathbb{Z}_{(l)}$$

with respect to any bases of $H^{2n}_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))/\text{tors}$ and $H^{2d-2n}_{\text{ét}}(X,\mathbb{Z}_{(l)}(d-n))/\text{tors}$ (this is well-defined up to a unit of $\mathbb{Z}_{(l)}$). □

Proposition 9.13 shows that, under resolution of singularities and conjecture 9.6, the étale motivic cohomology groups $H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))$ are finitely generated over $\mathbb{Z}_{(l)}$ for $i\le 2n+1$ and any smooth, projective variety $X/\mathbb{F}_p$. What about the converse? The answer is surprisingly negative and is given by theorem 9.22 and corollary 9.24 below.

9.21. Proposition. *Conjecture 9.6 holds for $n\le m$ when restricted to $\mathcal{S}_d$ if and only if, for any smooth, projective X of dimension $\le d$, the homomorphisms*

$$H^*_{\text{ét}}(X,\mathbb{Z}_l(0)^c\overset{L}{\otimes}\alpha^*\mathbb{Z}(n))\to H^*_{\text{cont}}(X,\mathbb{Z}_l(n))$$

are isomorphisms for all $n\le m$.

Proof. Let $K(n)$ be the cone of (9.4): conjecture 9.6 holds on $\mathcal{S}_d$ if and only if, for any X smooth of dimension $\le d$, the groups $H^*(X,K(n))$ are 0. The proof is similar to that of theorem 8.17. Note that $X\mapsto H^*(X,K(n))$ defines a pure graded cohomology theory with values in $\mathbb{Q}_l$-vector spaces. The first claim follows from purity for continuous étale cohomology (theorem 3.4) and purity for étale motivic cohomology [47], while the second one follows from lemma 9.5. Proposition 9.21 now follows once again from theorem 5.4 b), noting that $K(n)$ has transfers. □

9.22. Theorem. *Let X be a smooth, projective variety over $\mathbb{F}_p$. Assume that the groups $H^i_{\text{ét}}(X,\mathbb{Z}_{(l)}(n))$ are finitely generated $\mathbb{Z}_{(l)}$-modules for all $i\le 2n+1$. Then:*
a) The Tate conjecture holds for X in codimension n;
b) Let $K(n)$ be the cone of (9.4). Then $\mathbb{H}^i_{\text{ét}}(X,K(n))=0$ for $i\neq 2n,2n+1$ and there is an exact sequence

$$0\to\mathbb{H}^{2n}_{\text{ét}}(X,K(n))\to H^{2n}_{\text{cont}}(X,\mathbb{Q}_l(n))\xrightarrow{\cdot e}H^{2n+1}_{\text{cont}}(X,\mathbb{Q}_l(n))\to\mathbb{H}^{2n+1}(X,K(n))\to 0.$$

Proof. By the assumptions and proposition 9.2, $H^i_{\text{ét}}(X, \mathbb{Z}_{(l)}(n))$ is finite for $i \neq 2n, 2n+2$ and torsion for $i = 2n+2$. It follows that

$$H^i(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \xrightarrow{\sim} H^i(X, \mathbb{Z}(n) \otimes \mathbb{Z}_l(0)^c)$$

for $i \neq 2n+1, 2n+2$, hence that $\mathbb{H}^i_{\text{ét}}(X, \mathbb{Z}(n) \overset{L}{\otimes} \mathbb{Z}_l(0)^c)$ is a finitely generated $\mathbb{Z}_l$-module for $i \neq 2n, 2n+1$. So are the $H^i_{\text{cont}}(X, \mathbb{Z}_l(n))$ for all i. Since the $\mathbb{H}^i_{\text{ét}}(X, K(n))$ are uniquely divisible, it follows that they are 0, except perhaps for $i = 2n, 2n+1$. We have a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & H^{2n}_{\text{cont}}(X, \mathbb{Z}_l(n)) \arrow[d, "A"'] & \\
 & H^{2n}(X, K(n)) \arrow[d, "B"'] & \\
0 \to H^{2n+1}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l \arrow[r] & H^{2n+1}(X, \mathbb{Z}(n) \otimes \mathbb{Z}_l(0)^c) \arrow[r] \arrow[d] & H^{2n}(X, \mathbb{Q}(n)) \otimes \mathbb{Z}_l \\
 & H^{2n+1}_{\text{cont}}(X, \mathbb{Z}_l(n)) \arrow[d] & \\
 & H^{2n+1}(X, K(n)) \arrow[d] & \\
 & H^{2n+2}(X, \mathbb{Z}(n) \otimes \mathbb{Z}_l(0)^c). &
\end{tikzcd}$$

Let $M = \operatorname{Coker} A$ and N its torsion subgroup. Then $B(N)$ maps to 0 in $H^{2n}(X, \mathbb{Q}(n)) \otimes \mathbb{Z}_l$, hence is contained in $H^{2n+1}(X, \mathbb{Z}(n)) \otimes \mathbb{Z}_l$. But this group is finite and $B(N)$ is divisible, hence $B(N) = 0$. This shows that M is torsion-free and, since $H^{2n}_{\text{cont}}(X, \mathbb{Z}_l(n))$ is finitely generated, this shows that $A = 0$. We first deduce that

$$H^{2n}(X, \mathbb{Z}(n)) \otimes \mathbb{Q}_l \xrightarrow{\sim} H^{2n}(X, \mathbb{Z}(n) \otimes \mathbb{Q}_l(0)^c) \xrightarrow{\sim} H^{2n}_{\text{cont}}(X, \mathbb{Q}_l(n))$$

which gives a). Using this and tensoring the above vertical exact sequence by $\mathbb{Q}$, we get an exact sequence

$$0 \to \mathbb{H}^{2n}_{\text{ét}}(X, K(n)) \to H^{2n}_{\text{cont}}(X, \mathbb{Q}_l(n)) \xrightarrow{C} H^{2n+1}_{\text{cont}}(X, \mathbb{Q}_l(n)) \to \mathbb{H}^{2n+1}(X, K(n)) \to 0$$

(note that $H^{2n+2}(X, \mathbb{Z}(n) \otimes \mathbb{Z}_l(0)^c)$ is torsion). Theorem 9.22 results from this and the following lemma, which is an easy consequence of corollary 4.8:

9.23. Lemma. *The map C is cup-product by e.* □

9.24. Corollary. *Conjecture 9.6 holds for weights $\leq d$ and smooth varieties of dimension $\leq d$ if and only if the condition of proposition 9.22 and condition S^n are verified for projective ones for all $n \leq d$.* □

It is difficult in general to get more precise information on the cokernel of the l-adic cycle map

$$CH^n(X) \otimes \mathbb{Z}_l \to H^{2n}_{\text{cont}}(X, \mathbb{Z}_l(n)).$$

We shall do so when the ground field is finite. (Compare [33, remark 5.6 a)].)

9.25. Proposition. *If conjecture 9.6 holds, then, for any smooth, projective variety X over $\mathbb{F}_p$, there is an exact sequence*

$$\begin{aligned} 0 \to H^{2n-1}_M(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to CH^n(X) \otimes \mathbb{Z}_l \oplus H^{2n}_{\text{cont}}(X, \mathbb{Z}_l(n))_{\text{tors}} \\ \to H^{2n}_{\text{cont}}(X, \mathbb{Z}_l(n)) \to CH^n(X) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^{2n}_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)). \end{aligned}$$

Proof. This follows readily from the commutative diagram of exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & H^{2n-1}_M(X,\mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[r] \arrow[d] & H^{2n-1}_{\text{ét}}(X,\mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[d] \\
 & H^{2n}(X,\mathbb{Z}(n))\otimes\mathbb{Z}_l \arrow[r] \arrow[d] & H^{2n}_{\text{cont}}(X,\mathbb{Z}_l(n)) \arrow[d] \\
 & H^{2n}(X,\mathbb{Z}(n))\otimes\mathbb{Q}_l \arrow[r, "\sim"] \arrow[d] & H^{2n}_{\text{cont}}(X,\mathbb{Q}_l(n)) \arrow[d] \\
 & H^{2n}_M(X,\mathbb{Q}_l/\mathbb{Z}_l(n)) \arrow[r] \arrow[d] & H^{2n}_{\text{ét}}(X,\mathbb{Q}_l/\mathbb{Z}_l(n)) \\
 & 0 &
\end{tikzcd}$$

Here the top 0s and the middle isomorphism come from proposition 9.13 b), and the bottom 0 comes from proposition 9.3 (iii). □

10. The one-dimensional case

In this section, we prove:

10.1. Theorem. *Conjecture 8.12 is true when restricted to smooth curves over $\mathbb{F}_p$.*

Proof. We reduce as usual to the case of a smooth, projective curve $X/\mathbb{F}_p$, that we may suppose connected. For $n \geq 2$, $\mathbb{Q}_l(n)^c_{|X} = 0$ by corollary 6.12. On the other hand, $(\mathcal{K}^M_n)_X \otimes \mathbb{Q} = 0$: to see this, we reduce by theorem 8.2 to showing that $K^M_n(\mathbb{F}_p(X)) \otimes \mathbb{Q} = 0$. This follows from the case $n = 2$, which is a consequence of Harder's theorem. Theorem 10.1 is proven for $n \geq 2$.

Assume now $n = 1$. Let k be the field of constants of X. We have $\mathcal{K}^M_1 = \mathbb{G}_m$ and:

$$H^0_{\text{Zar}}(X,\mathbb{G}_m)\otimes\mathbb{Q} = k^*\otimes\mathbb{Q} = 0;$$
$$H^1_{\text{Zar}}(X,\mathbb{G}_m)\otimes\mathbb{Q} = \operatorname{Pic}(X)\otimes\mathbb{Q} \xrightarrow[\sim]{\deg} \mathbb{Q};$$
$$H^i_{\text{Zar}}(X,\mathbb{G}_m)\otimes\mathbb{Q} = 0 \quad \text{for} \quad i > 1.$$

On the other hand, we can compute $H^i_{\text{cont}}(X,\mathbb{Q}_l(1))$ with the help of lemma 3.12 a) and the Kummer exact sequence. By proposition 6.1, it suffices to consider $i = 2, 3$. For $i = 2$, we have short exact sequences

$$0 \longrightarrow \operatorname{Pic}(X)/l^\nu \longrightarrow H^2_{\text{ét}}(X,\mu_{l^\nu}) \longrightarrow {}_{l^\nu}Br(X) \longrightarrow 0$$
$$0 \longrightarrow Br(X)/l^\nu \longrightarrow H^3_{\text{ét}}(X,\mu_{l^\nu}) \longrightarrow {}_{l^\nu}H^3_{\text{ét}}(X,\mathbb{G}_m) \longrightarrow 0$$

and $Br(X) = 0$, $H^3_{\text{ét}}(X,\mathbb{G}_m) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ by class field theory. This gives:

$$H^2_{\text{cont}}(X,\mathbb{Q}_l(1)) = \operatorname{Pic}(X)\hat{}_l\otimes\mathbb{Q} \xrightarrow[\sim]{\deg} \mathbb{Q}_l;$$
$$H^3_{\text{cont}}(X,\mathbb{Q}_l(1)) = T_l(\mathbb{Q}/\mathbb{Z})\otimes\mathbb{Q} \simeq \mathbb{Q}_l.$$

One checks easily from these computations that (8.2) induces isomorphisms of cohomology groups. □

11. A conjecture in the Zariski topology

In this section as in section 9, we assume resolution of singularities.

Assume conjecture 9.6 holds. We can rewrite it as an exact triangle

$$\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l\to\mathbb{Z}_l(n)^c\to\alpha^*\mathbb{Z}(n)\otimes\mathbb{Q}_l[-1]\to\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l[1].$$

Taking propositions 9.3 (i) and 9.8 (i), (ii) into account, this can be rewritten as another exact triangle

$$\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l\to\mathbb{Z}_l(n)^c\to\alpha^*\mathcal{K}_n^M\otimes\mathbb{Q}_l[-n-1]\to\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l[1]$$

or an isomorphism

$$\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l\xrightarrow{\sim}\operatorname{fibre}(\mathbb{Z}_l(n)^c\to\alpha^*\mathcal{K}_n^M\otimes\mathbb{Q}_l[-n-1]).$$

Still by proposition 9.8, the right hand side can equally be written $\tau_{\le n}\mathbb{Z}_l(n)^c$. Applying $R\alpha_*$, we therefore get as a consequence of conjecture 9.6 a conjectural isomorphism

$$R\alpha_*\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l\xrightarrow{\sim}R\alpha_*(\tau_{\le n}\mathbb{Z}_l(n)^c).$$

Recall now the Kato conjecture: for any field F of characteristic $\neq l$, the Galois symbol

$$K_n^M(F)/l\to H^n(F,\mu_l^{\otimes n})$$

is bijective, and the Beilinson-Lichtenbaum conjecture [47], [53]

$$\mathbb{Z}(n)\xrightarrow{\sim}\tau_{\le n+1}R\alpha_*\alpha^*\mathbb{Z}(n).$$

By the main result of [47], the Kato conjecture is equivalent to the Beilinson-Lichtenbaum conjecture under resolution of singularities. Moreover, by the main result of [53], the Kato conjecture holds for $l=2$ (without assuming resolution of singularities).

Assuming the Kato conjecture and conjecture 9.6, we therefore get a new conjectural isomorphism:

$$\mathbb{Z}(n)\otimes\mathbb{Z}_l\xrightarrow{\sim}\tau_{\le n+1}R\alpha_*(\tau_{\le n}\mathbb{Z}_l(n)^c).$$

This is equivalent to an isomorphism $\mathbb{Z}(n)\otimes\mathbb{Z}_l\xrightarrow{\sim}\tau_{\le n}R\alpha_*\mathbb{Z}_l(n)^c$ plus an injection of Zariski sheaves

$$R^{n+1}\alpha_*\mathbb{Z}_l(n)^c\hookrightarrow\mathcal{K}_n^M\otimes\mathbb{Q}_l$$

the latter being the same as asking for $R^{n+1}\alpha_*\mathbb{Z}_l(n)^c$ to be torsion-free. We can now state our conjecture in the Zariski topology:

11.1. Conjecture. *a) The natural morphism in the derived category of abelian sheaves over the big Zariski site of* $\operatorname{Spec}\mathbb{F}_p$

$$\mathbb{Z}(n)\otimes\mathbb{Z}_l\to\tau_{\le n}R\alpha_*\mathbb{Z}_l(n)^c$$

is an isomorphism.
b) The Zariski sheaf $R^{n+1}\alpha_*\mathbb{Z}_l(n)^c$ *is torsion-free.*

11.2. Remark. b) is closely related to 'higher Hilbert 90'.

We have proven part a) of the following (under resolution of singularities):

11.3. Proposition. *Under resolution of singularities,*
a) Conjecture 9.6 and the Kato conjecture imply conjecture 11.1.
b) Conjecture 11.1 is equivalent to the conjunction of the Kato conjecture and either

(i) *finite generation of the* $\mathbb{Z}_{(l)}$*-module* $H^i(X,\mathbb{Z}_{(l)}(n))$ *for all* $i\le n+1$ *and all smooth* X
(ii) *same as (ii), but only for smooth, projective* X.

Proof. Assume conjecture 11.1. Tensoring the isomorphism by $\mathbb{Z}/l$, we get the Beilinson-Lichtenbaum conjecture, hence the Kato conjecture. Moreover, applying lemma 3.12 b), we get the finite generation statement. Without assumptions, the cone of $R\alpha_*\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l \to R\alpha_*\mathbb{Z}_l(n)^c$ has uniquely divisible cohomology sheaves. Therefore, the same holds in degree $\leq n$ for the cone of the truncations $\tau_{\leq n+1}R\alpha_*\alpha^*\mathbb{Z}(n)\otimes\mathbb{Z}_l \to \tau_{\leq n+1}R\alpha_*\mathbb{Z}_l(n)^c$. Assuming the Kato conjecture, hence the Beilinson-Lichtenbaum conjecture, this means that the cohomology sheaves of

$$K(n) = \mathrm{cone}(\mathbb{Z}(n)\otimes\mathbb{Z}_l \to \tau_{\leq n+1}R\alpha_*\mathbb{Z}_l(n)^c)$$

are uniquely divisible in degrees $\leq n$ and that $R^{n+1}\alpha_*\mathbb{Z}_l(n)^c$ is torsion-free. Let now X be a smooth, projective variety over $\mathbb{F}_p$. Using lemma 3.12 b), the finite generation assumption implies that $\mathbb{H}^i(X, K(n)) = 0$ for $i \leq n$. Applying theorem 5.4 b), we get the same result for all smooth X. It follows that $\mathcal{H}^i(K(n)) = 0$ for $i \leq n$, which is equivalent to the statement of conjecture 11.1. $\square$

11.4. Corollary. *Under resolution of singularities and the Kato conjecture, finite generation of the groups $H^i(X, \mathbb{Z}_{(l)}(n))$ for $n > 0$, $i \leq 0$ and smooth, projective varieties X implies their vanishing for any smooth X (motivic form of the Beilinson-Soulé vanishing conjecture).* $\square$

By proposition 11.3 a), conjecture 11.1 implies that the groups $H^i(X, \mathbb{Z}_{(l)}(n))$ are finitely generated over $\mathbb{Z}_{(l)}$ for $i \leq n+1$ and any smooth variety X over $\mathbb{F}_p$. What about $i > n+1$? Conjecture 9.6 implies that these groups have finite rank, but not that they are finitely generated. In particular, it does not imply the finite generation of the Chow groups of X. We shall now formulate a last conjecture and show that, together with conjecture 9.6, it implies this finite generation.

11.5. Conjecture. *For any smooth scheme X over a reasonable base ($\mathbb{F}_p$, $\mathbb{Z}[1/l]$, a separably closed field are reasonable), the groups*

$$H^i_{\mathrm{Zar}}(X, \mathcal{H}^j_{\text{ét}}(\mathbb{Z}/l^\nu(n)))$$

are finite for all i, j, ν, n.

11.6. Theorem. (Under resolution of singularities) *Assume that the Kato conjecture holds, conjecture 9.6 holds and conjecture 11.5 holds for X projective and $\nu = 1$, $n = 0$. Then the groups $H^i(X, \mathbb{Z}_{(l)}(n))$ are finitely generated over $\mathbb{Z}_{(l)}$ for all i and any smooth $X/\mathbb{F}_p$.*

Proof. Once again, by theorem 5.4 it is sufficient to deal with the projective case. Assume first that $\Gamma(X, \mathbb{G}_m)$ contains a primitive l-th root of unity. Then, by the Beilinson-Lichtenbaum conjecture, we have for any n an exact triangle on the small Zariski site of X:

$$\mathbb{Z}/l(n-1)_M \to \mathbb{Z}/l(n)_M \to \mathcal{H}^n(\mathbb{Z}/l) \to \mathbb{Z}/l(n-1)_M[1]$$

where the index M indicates motivic complexes. From the assumption, we get inductively that $H^i_M(X, \mathbb{Z}/l(n))$ is finite for all i, n. If $\Gamma(X, \mathbb{G}_m)$ does not contain a primitive l-th root of unity, we reduce to this case by a classical transfer argument. By induction on ν we then get the same result for $H^i_M(X, \mathbb{Z}/l^\nu(n))$ and all $\nu \geq 1$.

The Soulé-Geisser motivic argument implies that $H^i_M(X, \mathbb{Z}_{(l)}(n))$ has finite exponent for $i < 2n$. The injection

$$H^i_M(X, \mathbb{Z}_{(l)}(n))/l^\nu \hookrightarrow H^i_M(X, \mathbb{Z}/l^\nu(n))$$

for ν sufficiently large then shows that this group is finite.

For $i = 2n$, we use proposition 9.25: it suffices to show that $H^{2n-1}_M(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$ is finite. We do this by showing that it has finite exponent, just as above. $\square$

12. The case of rings of integers

In this section, let $S = \operatorname{Spec} \mathbb{Z}[1/l]$ and $\mathcal{S}$ be the category of regular S-schemes. We don't have even a conjectural description of $\mathbb{Z}_l(n)^c$ over $\mathcal{S}_{\text{ét}}$. We shall indicate the little knowledge we have, mostly for the restriction of $\mathbb{Z}_l(n)^c$ to the *small* étale site of S.

We begin with an elementary result which partially reduces the problem to understanding what happens at the generic point of S. Unfortunately, we have to state it in terms of étale homology.

12.1. Proposition. *Let $\mathcal{X}$ be a scheme of finite type over S, X its generic fibre and, for $p \neq l$, $\mathcal{X}_p$ its closed fibre at p. Then, for all $n \in \mathbb{Z}$, there is a long exact sequence*

$$\cdots \to H_i^{\text{cont}}(\mathcal{X}/S, \mathbb{Z}_l(n)) \to \tilde{H}_i^{\text{cont}}(X/S, \mathbb{Z}_l(n)) \to \coprod_{p\neq l} H_{i-1}^{\text{cont}}(\mathcal{X}_p/\mathbb{F}_p, \mathbb{Z}_l(n)) \to H_{i-1}^{\text{cont}}(\mathcal{X}/S, \mathbb{Z}_l(n)) \to \dots$$

Proof. This follows from proposition 3.11 a). □

12.2. Corollary. *Let $\mathcal{X}$ be a scheme of finite type over S and X its generic fibre. Then for all $n < 0$ and $i \in \mathbb{Z}$, the natural map*

$$H_i^{\text{cont}}(\mathcal{X}/S, \mathbb{Q}_l(n)) \to \tilde{H}_i^{\text{cont}}(X/S, \mathbb{Q}_l(n))$$

is an isomorphism.

Proof. This follows from proposition 12.1 and corollary 6.13. □

Recall, for each prime p, the canonical generator of $H^1_{\text{cont}}(\mathbb{F}_p, \mathbb{Z}_l)$ that we now denote by e_p. Given its importance in the theory in characteristic p, the first question that comes to mind is: do the e_p lift compatibly to characteristic 0? The answer is, predictably, yes but only up to a factor.

More precisely, the cyclotomic character gives a continuous homomorphism

$$e : \pi_1(S) \to \mathbb{Z}_l^*$$

which is surjective by Gauss' theorem on the irreducibility of cyclotomic polynomials. This defines a class in $H^1_{\text{cont}}(S, \mathbb{Z}_l^*)$. Consider the continuous homomorphism

$$\begin{aligned} \ell : \mathbb{Z}_l^* &\to \mathbb{Z}_l \\ u &\mapsto \begin{cases} \frac{1}{l^2}\log(u^l) & \text{if } l > 2 \\ \frac{1}{8}\log(u^2) & \text{if } l = 2. \end{cases} \end{aligned}$$

This has kernel the roots of unity of $\mathbb{Z}_l^*$ and coincides with $1/l$ of the usual logarithm when restricted to $1 + l\mathbb{Z}_l$ ($l > 2$) and with $1/4$ of the usual logarithm when restricted to $1 + 4\mathbb{Z}_2$ ($l = 2$). In all cases, it is surjective. Via ℓ, we associate to e a class, still denoted by e:

$$e \in H^1_{\text{cont}}(S, \mathbb{Z}_l).$$

12.3. Lemma. *For all $p \neq l$, the image of e in $H^1_{\text{cont}}(\mathbb{F}_p, \mathbb{Z}_l)$ is $\ell(p)e_p$.*

Proof. The absolute Frobenius automorphism at p raises roots of unity to the p-th power. □

12.4. Definition. Let $\mathbb{R}_0$ be "the" field of real algebraic numbers ($\mathbb{R}_0 = \mathbb{R} \cap \overline{\mathbb{Q}}$ for a suitable embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$). For $n \in \mathbb{Z}/2$, we denote by $\mathbb{Z}(n)$ the $G_{\mathbb{R}_0}$-module with support $\mathbb{Z}$ on which the action of $G_{\mathbb{R}_0}$ is given by ε^n, where ε is its nontrivial character of order 2. We set

$$A(n) = Ind_{G_{\mathbb{R}_0}}^{G_{\mathbb{Q}}} \mathbb{Z}(n).$$

12.5. Theorem. *Let* $j : \operatorname{Spec} \mathbb{Q} \hookrightarrow \operatorname{Spec} \mathbb{Z}[1/2]$ *be the inclusion of the generic point. For* $n \geq 2$*,*
a) There is a non-canonical isomorphism

$$\mathbb{Q}_l(n)^c_{|S} \simeq j_* A(n+1) \otimes \mathbb{Q}_l[-1].$$

b) The natural morphism

$$Rj_* \alpha^* \mathbb{Z}(n)_{|\mathbb{Q}} \otimes \mathbb{Q}_l \to \mathbb{Q}_l(n)^c_{|S}$$

is an isomorphism.

Proof. a) Let R be a ring of S-integers in a number field F (we assume $1/l \in R$). It is clear that $H^0_{\text{cont}}(R, \mathbb{Q}_l(n)) = 0$ and $H^i_{\text{cont}}(R, \mathbb{Q}_l(n)) = 0$ for $i \geq 3$ as well for reasons of cohomological dimension. By Soulé's main theorem [41], we have

- $H^2(R, \mathbb{Q}_l(n)) = 0$;
- the Chern character induces an isomorphism

$$K_{2n-1}(R) \otimes \mathbb{Q}_l \xrightarrow{\sim} H^1_{\text{cont}}(R, \mathbb{Q}_l(n)).$$

On the other hand, the localization exact sequence in algebraic K-theory shows that

$$K_{2n-1}(R) \otimes \mathbb{Q} \xrightarrow{\sim} K_{2n-1}(F) \otimes \mathbb{Q}.$$

Therefore, the Chern character induces an isomorphism

$$j_* \alpha^* \mathcal{K}_{2n-1} \otimes \mathbb{Q}_l[-1] \xrightarrow{\sim} \mathbb{Q}_l(n)^c_{|S}. \tag{12.1}$$

On the other hand, the Borel regulator can be interpreted as a $Gal(\mathbb{C}/\mathbb{R})$-equivariant homomorphism

$$K_{2n-1}(\mathbb{C}) \xrightarrow{\rho_n} \mathbb{R}(n+1)$$

where $\mathbb{R}(n) := \mathbb{Z}(n) \otimes \mathbb{R}$. By [3], the composition

$$K_{2n-1}(F) \to \coprod_{v|\infty} K_{2n-1}(F_v) \xrightarrow{(\rho_{2n-1})} \coprod_{v|\infty} \mathbb{R}(n+1)^{G_v}$$

where $G_v = Gal(\mathbb{C}/\mathbb{F}_v)$ indices an isomorphism

$$K_{2n-1}(F) \otimes \mathbb{R} \xrightarrow{\sim} \coprod_{v|\infty} \mathbb{R}(n+1)^{G_v}$$

hence, in the limit, an isomorphism of $G_\mathbb{Q}$-modules

$$K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{R} \xrightarrow{\sim} A(n+1) \otimes \mathbb{R}.$$

By the theory of characters, this isomorphism implies an abstract isomorphism

$$K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q} \simeq A(n+1) \otimes \mathbb{Q}$$

hence an isomorphism

$$K_{2n-1}(\overline{\mathbb{Q}}) \otimes \mathbb{Q}_l \simeq A(n+1) \otimes \mathbb{Q}_l$$

and a).

b) This follows from (12.1) and [24, th. 4.1 and 4.2]. □

12.6. Remark. Behind the noncanonical isomorphism of theorem 12.5 a) are of course the determinants of the Borel regulators. This, and the prominent rôle of archimedean places, shows how Arakelov geometry lurks in the background.

Theorem 12.5 shows that the situation is markedly different in characteristic 0 and in characteristic p:

12.7. Corollary. *For* $n \geq 2$*, cup-product by* e

$$\mathcal{H}^i(\mathbb{Q}_l(n)^c_{|S}) \xrightarrow{\cdot e} \mathcal{H}^{i+1}(\mathbb{Q}_l(n)^c_{|S})$$

is 0 *for all* i. □

For $n < 0$, contrary to characteristic p, it is well-known that $\mathbb{Q}_l(n)^c_{|S} \neq 0$ [39]. It is widely conjectured (*loc. cit.*):

12.8. Conjecture. *Theorem 12.5 a) extends to $n < 0$.*

To describe $\mathbb{Z}_l(1)^c_{|S}$, it is convenient to introduce more definitions. Let D be the discrete topological $G_{\mathbb{Q}}$-module defined by

$$D = \varinjlim_{[F:\mathbb{Q}]<+\infty} Div(O_F)$$

and define the étale *sheaf of codivisors* $\underline{\text{Codiv}}$ as the kernel of the surjective map

$$j_*D \to \underline{\text{Div}}.$$

The degree map on D induces a map

$$\deg : \underline{\text{Codiv}} \to j_*\mathbb{Z}.$$

We also define $\tilde{A}(0)$ as the kernel of the augmentation $A(0) \to \mathbb{Z}$. Finally, let Br be the *Zariski* sheaf associated to the presheaf $U \mapsto Br(U)$ on the small étale site of S.

12.9. Proposition. *There are exact sequences*

$$0 \to j_*\tilde{A}(0) \otimes \mathbb{Q} \to \mathbb{G}_m \otimes \mathbb{Q} \to \underline{\text{Codiv}} \to 0$$

$$0 \to Br \to \underline{\text{Codiv}} \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\deg} j_*\mathbb{Q}/\mathbb{Z} \to 0$$

except for the contribution of real places to the Brauer group.

Proof. This is a reformulation of well-known results. In the first sequence, the left hand sheaf corresponds to $O_F^* \otimes \mathbb{Q}$ for F running through number fields; its description follows from Dirichlet's units theorem (hence is once again transcendental). The right hand side can be obtained similarly, using the S-units theorem. The exact sequence concerning the Brauer group is a reformulation of the Albert-Brauer-Hasse-Noether theorem. □

12.10. Theorem. *a) $\mathcal{H}^i(\mathbb{Q}_l(1)^c_{|S}) = 0$ for $i \neq 1, 2$.*
b) The natural map

$$\mathbb{G}_m[-1] \otimes \mathbb{Z}_l \to \mathcal{H}^1(\mathbb{Z}_l(1)^c_{|S})$$

given by Kummer theory is an isomorphism.
c) There is an exact sequence

$$0 \to \mathcal{H}^2(\mathbb{Q}_l(1)^c_{|S}) \to \underline{\text{Codiv}} \otimes \mathbb{Q}_l \xrightarrow{\deg} j_*\mathbb{Q}_l \to 0.$$

Proof. a) follows as in the proof of theorem 12.5 a). To see b), consider $R \subset F$ as in the said proof. From the Kummer exact sequence, we get short exact sequences

$$0 \to R^*/(R^*)^{l^\nu} \to H^1(R, \mu_{l^\nu}) \to {}_{l^\nu}\operatorname{Pic}(R) \to 0.$$

Since all groups are finite, there are no $\varprojlim^1$ and we get a resulting short exact sequence

$$0 \to \varprojlim R^*/(R^*)^{l^\nu} \to H^1(R, \mathbb{Z}_l(1)) \to T_l(\operatorname{Pic}(R)) \to 0.$$

But R^* is finitely generated and $\operatorname{Pic}(R)$ is finite (Dirichlet's theorem); hence the left hand side is $R^* \otimes \mathbb{Z}_l$ and the right hand side is 0.

Finally, to see c), we use the similar exact sequences one degree higher:

$$0 \to \operatorname{Pic}(R)/l^\nu \to H^2(R, \mu_{l^\nu}) \to {}_{l^\nu}Br(R) \to 0$$

and proposition 12.9. □

A nontrivial question is whether cup-product by e

$$\mathcal{H}^1(\mathbb{Q}_l(1)^c_{|S}) \xrightarrow{\cdot e} \mathcal{H}^2(\mathbb{Q}_l(1)^c_{|S})$$

is surjective. This seems related to the Leopoldt conjecture. Out of laziness we don't examine this in more detail and just give a conjectural value for $\mathbb{Q}_l(0)^c_{|S}$, which is easily seen to be equivalent to the Leopoldt conjecture.

12.11. Conjecture. *There is an exact triangle*

$$j_*\mathbb{Q}_l[-1] \xrightarrow{\cdot e} \mathbb{Q}_l(0)^c_{|S} \longrightarrow j_*A(1) \otimes \mathbb{Q}_l[-1] \longrightarrow j_*\mathbb{Q}_l[0].$$

In particular, $\mathcal{H}^i(\mathbb{Q}_l(0)^c_{|S}) = 0$ *for* $i \neq 1$.

Further insight on the conjectural behaviour of $\mathbb{Q}_l(n)^c$ in characteristic 0 can be found in [19] and [18, §13]. These two references, together with theorems 12.5 and 12.10, at least suggest:

12.12. Conjecture. *For* $n \geq 0$, *the map*

$$\alpha^*\mathbb{Z}(n) \otimes \mathbb{Z}_l \to \tau_{\leq n} j^*\mathbb{Z}_l(n)_S$$

restricted to smooth varieties over $\mathbb{Q}$ *is an isomorphism.*

The case $n = 0$ is trivial and the case $n = 1$ is easily checked as in the proof of theorem 12.10 b) (use the finite generation of $\Gamma(X, \mathcal{O}_X^*)$ and of $Pic(X)$ for X regular of finite type over $\operatorname{Spec}\mathbb{Z}$, which follow from Dirichlet, Mordell-Weil and Néron-Severi). On the other hand, I have no idea of the nature of $\tau_{>n} j^*\mathbb{Z}_l(n)_S$.

Appendix A. The Bass conjecture implies the rational Beilinson-Soulé conjecture

Let $n, i \geq 0$ be two integers; for any regular scheme X, denote by $K_n(X)^{(i)}$ the i-th eigenspace of the Adams operations acting on $K_n(X)$ [44]. For any prime number l and any abelian group A, denote by $A_{(l)}$ the localisation $A \otimes \mathbb{Z}_{(l)}$ of A at l. In this appendix, we prove:

A.1. Theorem. *Assume that* $K_n(X)^{(i)}$ *is finitely generated when* X *is of finite type over* $\mathbb{Z}$.
a) If $1 \leq i \leq n/2$, *then for any* $d \geq 0$ *there exists an effectively computable integer* $M(n,d)$ *such that* $M(n,d)K_n(X)^{(i)}_{(2)} = 0$ *for any regular scheme of dimension* $\leq d$.
b) One can choose $M(n,d)$ *such that, if* $n = 2i-1 > 1$, *for any regular scheme* X *of dimension* $\leq d$, *in the commutative diagram*

$$\begin{array}{ccc} K_{2i-1}(X)^{(i)}_{(2)} & \longrightarrow & K_{2i-1}(\kappa(X))^{(i)}_{(2)} \\ \uparrow & & \uparrow \\ K_{2i-1}(X_0)^{(i)}_{(2)} & \longrightarrow & K_{2i-1}(\kappa(X_0))^{(i)}_{(2)} \end{array}$$

all maps have kernel and cokernel killed by $M(n,d)$. *Here* X_0 *is the scheme of constants of* X, *i.e. the normalisation of* $\operatorname{Spec}\mathbb{Z}$ *in* X, *and* $\kappa(X)$ *denotes the total ring of functions of* X.

The proof goes along the same lines as the one of corollary 11.4, but is more involved. The game is to dodge resolution of singularities. The method goes back to [23].

The proof will be divided into a series of lemmas.

A.2. Lemma. *For all* $n \geq 0$ *there is an effectively computable constant* $M(n)$ *such that, if* F *is a field of characteristic* 0,

$$M(n)K_n(F)^{(i)}$$

is uniquely 2*-divisible for* $i \leq n/2$ *and, for* $n = 2i-1 > 1$, *the natural map*

$$K_n(F_0)^{(i)}_{(2)} \to K_n(F)^{(i)}_{(2)}$$

multiplied by $M(n)$ *is injective with uniquely divisible cokernel.*

Proof. We have the Bloch-Lichtenbaum-Friedlander-Suslin-Voevodsky spectral sequence

$$E_2^{p,q} = H^{p-q}(F, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(F) \quad (p, q \le 0)$$

([2], [9], [52], see the discussion in [24, 2.1]). By Soulé [45], for all $k > 0$ the Adams operation ψ^k acts on this spectral sequence, by multiplication by k^{-q} on $E_2^{p,q}$ and converging to the standard action on $K_{-p-q}(F)$. It follows that, for all $r \ge 2$, the differentials d_r are killed by

$$w_{r-1} := gcd_{k\ge 2} k^N (k^{r-1} - 1)$$

where N is large with respect to r (*cf.* [44, p. 498]). It follows that, for any $p, q \le 0$, the group

$$w_1 w_2 \dots w_{-p} E_2^{p,q}$$

consists of universal cycles, while universal boundaries of degree (p, q) are killed by

$$w_1 w_2 \dots w_{-q}.$$

To summarise, there is a chain of inclusions

$$w_1 \dots w_{-p} w_1 \dots w_{-q} E_2^{p,q} \subseteq w_1 \dots w_{-q} E_\infty^{p,q} \subseteq w_1 \dots w_{-q} E_2^{p,q}.$$

On the other hand, let $(F^i K_n(F))_{i\ge 0}$ be the filtration on $K_n(F)$ induced by the spectral sequence. The Adams operation ψ^k respects this filtration and acts on $\operatorname{gr}^i K_n(F) = E_\infty^{i-n,-i}$ by k^i. Let φ be the composition

$$K_n(F)^{(i)} \hookrightarrow K_n(F) \twoheadrightarrow K_n(F)/F^{i+1}K_n(F).$$

Then $w_1 \dots w_{n-i} \operatorname{Ker} \varphi = 0$ and $w_1 \dots w_{i-1} \operatorname{Im} \varphi \subseteq \operatorname{gr}^i K_n(F)$. Conversely, by the argument of [44, p. 499], this subgroup has index $w_1 \dots w_{n-i}$. In other words, there is a chain of inclusions

$$w_1 \dots w_{n-i} w_1 \dots w_{i-1} K_n(F)^{(i)} \subseteq w_1 \dots w_{n-i} E_\infty^{i-n,-i} \subseteq w_1 \dots w_{i-1} \overline{K_n(F)}^{(i)}$$

where $\overline{K_n(F)}^{(i)}$ is a quotient of $K_n(F)^{(i)}$ by a subgroup of exponent $w_1 \dots w_{n-i}$.

Let $M(n) = (w_1 \dots w_n)^4$. We see from the above that there exists for all i a chain of homomorphisms

$$H^{2i-n}(F, \mathbb{Z}(n)) \to ? \leftarrow K_n(F)^{(i)}$$

with kernel and cokernel killed by $M(n)$.

By [24, th. 3.1 a)], $H^{2i-n}(F, \mathbb{Z}(n))$ is uniquely 2-divisible for $2i - n \le 0$, and by *loc. cit.*, th. 7.1, $H^1(F_0, \mathbb{Z}_{(2)}(n)) \to H^1(F, \mathbb{Z}_{(2)}(n))$ is injective with uniquely divisible cokernel (this relies on [47] and [53]). The result follows. □

A.3. Lemma. *Lemma A.2 also holds for F of characteristic $p > 0$.*

Proof. We distinguish two cases:

1) $p = 2$. By results of Geisser and Levine [11], $\operatorname{Coker}(K_n^M(F) \to K_n(F))$ is uniquely 2-divisible. The result then follows from [44, p. 498, cor. 1] (and its proof).

2) $p > 2$. By the long homotopy exact sequence, it is enough to show that, for all $i \le n/2$ and $\nu \ge 1$, $M(n)K_n(F, \mathbb{Z}/2^\nu)^{(i)}$ is 0 if $i < n/2$ and that $M(n)K_n(F_0, \mathbb{Z}/2^\nu)^{(i)} \to M(n)K_n(F, \mathbb{Z}/2^\nu)^{(i)}$ is bijective (*resp.* injective) for $n = 2i - 1$ (*resp.* $2i - 2$). We may assume F perfect (transfer argument). Let E be the field of fractions of the Witt vectors on F. We have split short exact sequences

$$0 \to K_n(F, \mathbb{Z}/2^\nu) \to K_n(E, \mathbb{Z}/2^\nu) \to K_{n-1}(F, \mathbb{Z}/2^\nu) \to 0$$

which are respected by the Adams operations, and similarly for F_0. The result follows from this, up to increasing $M(n)$ a bit. □

A.4. Lemma. *For any n, d, there is a constant $M(n,d)$ such that, for any regular scheme X of dimension d,*

$$M(n,d)K_n(X)^{(i)}$$

is uniquely 2-divisible for $i \le n/2$ and, for $n = 2i-1 > 1$, in the commutative diagram

$$\begin{array}{ccc} K_{2i-1}(X)^{(i)}_{(2)} & \longrightarrow & K_{2i-1}(\kappa(X))^{(i)}_{(2)} \\ \uparrow & & \uparrow \\ K_{2i-1}(X_0)^{(i)}_{(2)} & \longrightarrow & K_{2i-1}(\kappa(X_0))^{(i)}_{(2)} \end{array}$$

with natural maps multiplied by $M(n,d)$, all maps are injective with uniquely divisible cokernel.

Proof. To do this, we reduce to the field case by using the Quillen spectral sequence [37] (for X and X_0)

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} K_{-p-q}(\kappa(x)) \Rightarrow K_{-p-q}(X),$$

Soulé's theorem that the Adams operations act on this spectral sequence [44, p. 521, th. 4 (i), (ii)] and an argument similar to that in step 1 (*cf.* also [23]). □

A.5. Lemma. *Theorem A.1 holds for X of finite type over $\mathbb{Z}$.*

Proof. This follows immediately from lemma A.4. □

A.6. Lemma. *Theorem A.1 holds for $X = \operatorname{Spec} F$, F a field.*

Proof. We may assume F finitely generated. Write $F = \kappa(X)$ for X regular of finite type over $\mathbb{Z}$. The result follows from lemma A.5, a direct limit argument and lemma A.3. □

Proof of theorem A.1. Follows from step 5 and a new application of the Quillen spectral sequence. □

A.7. Corollary. *Under the assumption of theorem A.1, for any smooth scheme X over a field F of characteristic 0,*
a) $H^{2i-n}(X, \mathbb{Z}_{(2)}(i)) = 0$ for $i > 0$ and $2i - n \le 0$.
b) In the diagram

$$\begin{array}{ccc} H^1(X, \mathbb{Z}_{(2)}(i)) & \longrightarrow & H^1(F(X), \mathbb{Z}_{(2)}(i)) \\ & & \uparrow \\ & & H^1(F_0, \mathbb{Z}_{(2)}(i)) \end{array}$$

the two maps are isomorphisms.The same holds for F of characteristic > 0, under resolution of singularities.

Proof. The case $X = \operatorname{Spec} F$ follows from theorem A.1 by applying the argument in the proof of lemma A.1 backwards. The general case follows from the coniveau spectral sequence plus purity for motivic cohomology. □

By the same method as in the proof of theorem A.1, one can further show that the Bass conjecture implies the Parshin conjecture for smooth, projective varieties X over $\mathbb{F}_p$ ($K_i(X) \otimes \mathbb{Q} = 0$ for $i > 0$), hence recover the consequences Geisser deduces from this conjecture in [10, §§3.3 and 3.4] (see corollaries 8.28 and 8.30 here).

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