# Motives and adjoints

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RMS/SMF/IMSc Indo-French Conference in Mathematics 2008
Chennai, December 16, 2008

$\mathcal{M} : \{fields\} \to Cat$ pseudo-functor. Recall:

- $k$ field $\mapsto \mathcal{M}(k)$ category.
- $f : k \to K$ field extension $\mapsto \mathcal{M}(f) = f^*_{\mathcal{M}} = f^* : \mathcal{M}(k) \to \mathcal{M}(K)$.
- $k \xrightarrow{f} K \xrightarrow{g} L$ successive extensions: natural isomorphism

$$c_{f,g} : g^* \circ f^* \Rightarrow (g \circ f)^*$$

with 2-cocycle relation between the $c_{f,g}$ (for 3 successive extensions).

We call this a *motivic theory*.

$\mathcal{M}, \mathcal{N}$ motivic theories: morphism of motivic theories $\varphi : \mathcal{M} \to \mathcal{N}$:

- $\forall k \ \varphi_k : \mathcal{M}(k) \to \mathcal{N}(k)$.
- $f : k \to K$ extension: natural isomorphism

$$v_f^{\varphi} = v_f : \varphi_K f^*_{\mathcal{M}} \xrightarrow{\sim} f^*_{\mathcal{N}} \varphi_k$$

with 1-cocycle relation w.r.t. $c^{\mathcal{M}}_{f,g}$ and $c^{\mathcal{N}}_{f,g}$ (for $f, g$ composable).

**Two questions:**

Q1. $\mathcal{M}$ motivic theory, $f : k \to K$: when does $f^*$ have a (left or right) adjoint? (Notation: usually $f_\#$ for left adjoint, $f_*$ for right adjoint.)

Q2. $\varphi : \mathcal{M} \to \mathcal{N}$ morphism of motivic theories, $f : k \to K$. Assume that $f^*_{\mathcal{M}}, f^*_{\mathcal{N}}$ have (say) left adjoints $f^{\mathcal{M}}_\#, f^{\mathcal{N}}_\#$. Get *base change morphism*

$$w^{\varphi}_{f} : f^{\mathcal{N}}_\# \varphi_K \to \varphi_k f^{\mathcal{M}}_\#. \tag{1}$$

When is (1) an isomorphism?

**In this talk:**

Q1. Very often yes (but disparate proofs). Also some counterexamples.

Q2. Sometimes yes, but no in interesting cases.

Applications:

- To Bloch-Rovinsky-Beilinson's "correspondences at the generic point".
- (in progress:) to $L$-functions.

# 1. Examples of adjoints

**1.1. $K/k$ finite separable.** basically all theories one can think of (left and right adjoints):

**Examples 1.** a) $\mathcal{M}(k) = \mathrm{Sm}(k)$: $f_\# =$ naïve restriction of scalars, $f_*=$ Weil restriction of scalars.

b) $\mathcal{M}(k) =$ étale sheaves of abelian groups: $f_* \xrightarrow{\sim} f_\#$ (via trace). Yields examples like categories of commutative group schemes...

c) Categories of pure motives ($f_* \xrightarrow{\sim} f_\#$).

d) Voevodsky's triangulated categories of motives (ditto).

Etc. (Not the most exciting.)

**1.2. Categories of pure motives.** $(A, \sim)$ adequate pair: $A$ commutative ring, $\sim$ adequate equivalence relation on algebraic cycles with coefficients in $A$.

Morphisms of motivic theories:

$$\mathrm{Sm}^{\mathrm{proj}}(k) \to \mathrm{Cor}_\sim(k, A) \to \mathrm{Mot}_\sim^{\mathrm{eff}}(k, A) \to \mathrm{Mot}_\sim(k, A)$$

(last 2 fully faithful.)

**Theorem 2.** *a) If* $A = \mathbf{Q}$ *and* $\sim = \text{num}$, $\exists f_* \xrightarrow{\sim} f_\#$ *for any* $f : k \to K$ *primary, for* $\mathcal{M} = \text{Cor}_{\text{num}}, \text{Mot}^{\text{eff}}_{\text{num}}, \text{Mot}_{\text{num}}$, *and base change morphisms are isomorphisms.*

*b) If* $A = \mathbf{Q}$, $\sim = \text{rat}$, $\mathcal{M} = \text{Mot}^{\text{eff}}_{\text{rat}}$, $k = \bar{k}$, $K = k(C)$ *with* $g(C) > 0$, $\exists f_\# \mathbf{1} \Rightarrow k$ *is the algebraic closure of a finite field. (*$\mathbf{1}$ *unit motive.)*

*c) In b), "*$\Leftarrow$*" if the (Tate-)Beilinson conjecture* $\text{rat} = \text{num}$ *over* $k$ *holds.*

*d) In b), if* $k = \bar{\mathbf{F}}_q$, $\exists C$ *such that* $\nexists f_\# \mathbb{L}$ *(*$\mathbb{L}$ *Lefschetz motive).*

**Idea of proofs**: For a): By Jannsen, $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q})$ is abelian semi-simple.

**Lemma 3.** *$T : \mathcal{M} \to \mathcal{N}$ $\mathbf{Q}$-linear functor between $\mathbf{Q}$-linear abelian semi-simple categories. If $T$ is fully faithful, it has isomorphic left and right adjoints.*

If $K/k$ is primary, $\mathrm{Mot}_{\mathrm{num}}(k, \mathbf{Q}) \to \mathrm{Mot}_{\mathrm{num}}(K, \mathbf{Q})$ is fully faithful basically because cycles modulo numerical eq. are invariant under algebraically closed extensions.

For b), main point: $k = \bar{k}$, $A$ abelian variety over $k$. Then $A(k) \otimes \mathbf{Q} = 0$ $\iff$ $k$ is the algebraic closure of a finite field.

For c), uses birational motives (see below).

For d), uses example of Srinivas: if $X/\bar{\mathbf{F}}_q$, smooth cubic hypersurface of dimension 3 and $C$ smooth hyperplane section of the Fano surface parametrising the lines of $X$, then $CH_1(X_K) \otimes \mathbf{Q} \neq A_1^{\mathrm{num}}(X_K, \mathbf{Q})$ for $K = \bar{\mathbf{F}}_q(C)$.

Properties of adjoints in case of numerical motives:

- Commute with twist (follows from projection formula).
- Respect weights.

In particular, $\mathrm{Ab}(k)$ category of abelian $k$-varieties: fully faithful morphism of motivic theories

$$h_1 : \mathrm{Ab} \otimes \mathbf{Q} \to \mathrm{Mot}_{\mathrm{num}}\,.$$

This implies:

$$f_* h_1(A) = h_1(\mathrm{Tr}_{K/k} A), \quad f_{\#} h_1(A) = h_1(\mathrm{Im}_{K/k} A)$$

($K/k$-trace and image).

**1.3. 1-motives.** (under construction).

**Future theorem 4.** *$\mathcal{M}(k)$ = Deligne's 1-motives, $f : k \to K$ primary: $\exists f_{\#}$ and $f_*$, with*

$$f_{\#}[0 \to A] = [0 \to \mathrm{Im}_{K/k} A]$$
$$f_*[0 \to A] = [0 \to \mathrm{Tr}_{K/k} A].$$

*Bonus*: $K/k$ finitely generated $\Rightarrow$

- $f_{\#}$ has a first left derived functor.
- $f_*$ has a first right derived functor.

$$R^1 f_*[0 \to A] = [\mathrm{LN}(A, K/k) \to 0]$$

$\mathrm{LN}(A, K/k) = A(K\bar{k})/(\mathrm{Tr}_{K/k} A)(\bar{k})$ the *Lang-Néron group of A* viewed as $G_k$-module (finitely generated by Lang-Néron).

Idea of proofs: glueing (start from lattices, dualise to tori, treat semi-abelian varieties, etc.)

**1.4. Birational categories.** $\mathcal{M}$ motivic theory, $\varphi : \mathrm{Sm}^{\mathrm{proj}} \to \mathcal{M}$ or $\varphi : \mathrm{Sm} \to \mathcal{M}$ morphism.

$\forall k$ $S_b(k) \subset \mathrm{Sm}^{\mathrm{proj}}(k)$ or $\mathrm{Sm}(k)$, set of birational morphisms

$\mapsto \varphi(S_b(k) \subset \mathcal{M}(k)$.

Yields naturally commutative diagram of motivic theories

$$\begin{array}{ccc} \mathrm{Sm}^* & \xrightarrow{\varphi} & \mathcal{M} \\ \downarrow & & \downarrow \\ S_b^{-1}\mathrm{Sm}^* & \xrightarrow{\bar{\varphi}} & S_b^{-1}\mathcal{M} \end{array} \qquad * = \emptyset \text{ or proj}$$

$S_b^{-1}\mathrm{Sm}$: the *birational theory* associated to $(\mathcal{M}, \varphi)$.

**Theorem 5.** $f : k \to K$ *such that* $K = k(U)$ *for* $U \in \mathrm{Sm}(k)$*. Then* $f_{\#}$ *exists for* $\mathcal{M} = S_b^{-1}\,\mathrm{Sm}$*.*

*Sketch.* Two proofs:

1) $\mathrm{Sm}(U)$ category of smooth $U$-schemes: pair of adjoint functors

$$\mathrm{Sm}(k) \leftrightarrows \mathrm{Sm}(U)$$

(extension and restriction of scalars). They preserve birational morphisms, hence induce other pair of adjoint functors:

$$S_b^{-1}\,\mathrm{Sm}(k) \leftrightarrows S_b^{-1}\,\mathrm{Sm}(U).$$

Finally, the "generic fibre" functor $\mathrm{Sm}(U) \to \mathrm{Sm}(K)$ induces an equivalence of categories

$$S_b^{-1}\,\mathrm{Sm}(U) \xrightarrow{\sim} S_b^{-1}\,\mathrm{Sm}(K)$$

(techniques developed with Sujatha).

2) (In characteristic 0): Two results obtained with Sujatha:

$$S_b^{-1}\,\mathrm{Sm}^{\mathrm{proj}} \xrightarrow{\sim} S_b^{-1}\,\mathrm{Sm}$$

$$X, Y \in S_b^{-1}\,\mathrm{Sm}^{\mathrm{proj}} : \mathrm{Hom}(X,Y) = Y(k(X))/R$$

$R = R$-equivalence.

$X \in S_b^{-1}\,\mathrm{Sm}^{\mathrm{proj}}(K)$: is the functor

$$S_b^{-1}\,\mathrm{Sm}^{\mathrm{proj}}(k) \ni Y \mapsto \mathrm{Hom}(X, Y_K) = Y_K(K(X))/R$$

corepresentable?

Yes, because $Y_K(K(X))/R = Y(k(\mathcal{X}))/R$ for $\mathcal{X} \in \mathrm{Sm}^{\mathrm{proj}}(k)$ such that $k(\mathcal{X}) = K(X)$ (so, it is corepresented by $\mathcal{X}$).

Resolution of singularities used 3 times in this second proof!

**Theorem 6.** *$f : k \to K$ finitely generated: $\exists f_\#$ for $\mathcal{M} = \mathrm{Mot}^o_{\mathrm{rat}} := (S_b^{-1}\,\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{rat}})^\natural$ (birational motives). ($\natural$ = pseudo-abelian envelope.) Coefficients* $\mathbf{Z}$ *if* $\operatorname{char} k = 0$, $\mathbf{Q}$ *if* $\operatorname{char} k > 0$.

*Proof.* Use that, for $X, Y \in \mathrm{Sm}^{\mathrm{proj}}(k)$, in $\mathrm{Mot}^o_{\mathrm{rat}}(k, A)$

$$\mathrm{Hom}(h^o(X), h^o(Y)) = CH_0(Y_{k(X)}) \otimes A$$

(proven with Sujatha).

In char. 0, same proof as second proof of previous theorem. In char. $p$, use de Jong (more intricate). $\square$

**Examples 7.** $K = k(C)$, $C$ curve:

1) $f_{\#}\mathbf{1} = h^{o}(C) = h_0(C) \oplus h_1(C) \oplus h_2(C) = \mathbf{1} \oplus h_1(C)$.

(Note: $h_2(C) = \mathbb{L} = 0$ in $S_b^{-1}\,\mathrm{Mot}^{\mathrm{eff}}_{\mathrm{rat}}(k, A)$.)

2) $\Gamma/K$ curve; $S/k$ surface such that $k(S) = K(\Gamma)$. Then

$$f_{\#}h_1(\Gamma) = h_1(\mathrm{Im}_{K/k}\, J) \oplus t_2(S)$$

$J$ Jacobian of $\Gamma$, $t_2(S)$ transcendental part of $h_2(S)$ (orthogonal complement of the Néron-Severi part).

Empirically: $f_{\#}(w) \in \{w, w+1\}$.

$\Rightarrow$ base change for $\mathrm{Mot}^{o}_{\mathrm{rat}} \to \mathrm{Mot}^{o}_{\mathrm{num}}$ not isomorphism!

# 2. Applications

## 2.1. Motives at the generic point.

**Definition 8.** $n \geq 0$:

a) $d_{\leq n} \operatorname{Mot}^{o}_{\mathrm{rat}}(k, A)$ thick subcategory generated by $h^{o}(X)$, $\dim X \leq n$.

b) $d_{n} \operatorname{Mot}^{o}_{\mathrm{rat}}(k, A) = (d_{\leq n} \operatorname{Mot}^{o}_{\mathrm{rat}}(k, A)/I_n)^{\natural}$, $I_n$ ideal of morphisms factoring through an object of $d_{\leq n-1} \operatorname{Mot}^{o}_{\mathrm{rat}}(k, A)$.

Fact: $X, Y \in \mathrm{Sm}^{\mathrm{proj}}(k)$ of dimension $n$. In $d_n \operatorname{Mot}^o_{\mathrm{rat}}(k, A)$, have

$$\operatorname{Hom}(\bar{h}^o(X), \bar{h}^o(Y)) = CH^n(X \times Y) \otimes A / \equiv$$

$\equiv$ subgroup generated by classes of irreducible cycles not dominant on either $X$ or $Y$: these are the *correspondences at the generic point.*

**Conjecture 9** (Bloch, Rovinsky, Beilinson)**.** *$\forall k, n$, $d_n \operatorname{Mot}^o_{\mathrm{rat}}(k, \mathbf{Q})$ is abelian semi-simple (in particular, $\dim_{\mathbf{Q}} \operatorname{Hom} < \infty$).*

**Examples 10.** a) $n = 0$: Artin motives.

b) $n = 1$: $\mathrm{Ab}(k) \otimes \mathbf{Q}$.

In both cases the conjecture is true.

$f : k \to K$ finitely generated extension of transcendence degree $d$:

$f_{\#} d_{\leq n} \operatorname{Mot}^{o}_{\mathrm{rat}}(K, \mathbf{Q}) \subseteq d_{\leq n+d} \operatorname{Mot}^{o}_{\mathrm{rat}}(K, \mathbf{Q})$.

$\Rightarrow f_{\#}$ induces functors

$$f^{n}_{\#} : d_{n} \operatorname{Mot}^{o}_{\mathrm{rat}}(K, \mathbf{Q}) \to d_{n+d} \operatorname{Mot}^{o}_{\mathrm{rat}}(k, \mathbf{Q}).$$

**Theorem 11.** *Let* $A = \mathbf{Z}$ *if* $\operatorname{char} k = 0$ *and* $A = \mathbf{Q}$ *if* $\operatorname{char} k = p > 0$. *Let* $X, Y \in \mathrm{Sm}^{\mathrm{proj}}(K)$ *of dimension* $n$. *Suppose* $K/k$ *has a smooth projective model* $S$ *and that* $X, Y$ *spread to projective* $S$*-schemes* $\mathcal{X}, \mathcal{Y}$, *smooth over* $k$.

*Then exact sequence*

$$d_n \operatorname{Mot}^o_{\mathrm{rat}}(K, A)(\bar{h}^o(X), \bar{h}^o(Y)) \xrightarrow{f^n_\#} d_{n+d} \operatorname{Mot}^o_{\mathrm{rat}}(k, A)(f^n_\# \bar{h}^o(X), f^n_\# \bar{h}^o(Y)) \to CH^{n+d}(\mathcal{X} \times_k \mathcal{Y} - \mathcal{X} \times_S \mathcal{Y}) \otimes A / \overline{\equiv} \to 0$$

$\overline{\equiv}$ *image of* $\equiv \subset CH^{n+d}(\mathcal{X} \times_k \mathcal{Y}) \otimes A$.

**2.2. $L$-functions.** Situation: $k = \bar{k}$,

$$\begin{array}{ccc} \Gamma & \longrightarrow & S \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ \operatorname{Spec} K & \xrightarrow{j} & C \\ & & {\scriptstyle p}\downarrow \\ & & \operatorname{Spec} k \end{array}$$

$C$ curve, $S$ surface, $f$ projective flat generically smooth with geometrically connected fibres, $\Gamma$ generic fibre of $f$, $K = k(C)$.

**Theorem 12.** *$J$ Jacobian of $C$, $l$ prime $\neq \operatorname{char} k$. Isomorphisms*

$$H^0(C, j_*R^1 f'_* \mathbf{Q}_l(1)) \simeq V_l(\mathrm{Tr}_{K/k} J)$$
$$H^2(C, j_*R^1 f'_* \mathbf{Q}_l(1)) \simeq V_l(\mathrm{Tr}_{K/k} J)(-1)$$

*and exact sequence*

$$0 \to \mathrm{LN}(J, K/k) \otimes \mathbf{Q}_l \to H^1(C, j_*R^1 f'_* \mathbf{Q}_l(1)) \to H^2_{\mathrm{tr}}(S, \mathbf{Q}_l(1)) \to 0$$

$H^2_{\mathrm{tr}}(S, \mathbf{Q}_l(1)) := H^2(S, \mathbf{Q}_l(1)) / \mathrm{NS}(S) \otimes \mathbf{Q}_l.$

Now assume $k = \bar{k}_0$; $G := Gal(k/k_0$.

$\mathbf{K}_l$ Grothendieck group of (continuous, finite-dimensional) $\mathbf{Q}_l$-representations of $G$. Consider in $\mathbf{K}_l$:

(1) $A_l = [H^*(S)]$, alternating sum of cohomology groups of $S$;

(2) $B_l = [R^* p_* j_* R^* f'_* \mathbf{Q}_l]$ (9 terms).

These classes lift to classes $A, B \in \mathbf{K}$, $\mathbf{K}$ Grothendieck group of Chow motives (by Murre for $A_l$ and by Theorem 12 for $B_l$).

Consider the complex

$$0 \to \mathbf{Z} = \mathrm{NS}(C) \xrightarrow{f^*} \mathrm{NS}(S) \to \mathrm{NS}(\Gamma) = \mathbf{Z}.$$

Its homology $D$ at $\mathrm{NS}(S)$ controls multiple fibres of $f$: may view $D$ as $G_k$-lattice, hence $D \otimes \mathbf{Q}$ as Artin motive.

**Theorem 13.** $A - B = [D \otimes \mathbf{Q}(1)]$.

## 2.3. Case of a finite field. Suppose $k_0$ finite.

Corollary to Theorem 12 (motivic expression of $L(K, H^1(\Gamma), s)$):

**Corollary 14.**

$$L(K, H^1(\Gamma), s) = \frac{\zeta(h_1(\mathrm{Tr}_{K/k} J), s)\zeta(h_1(\mathrm{Tr}_{K/k} J), s-1)}{\zeta(t_2(S), s)\zeta(\mathrm{LN}(J, K/k), s-1)}. \qquad \square$$

This formula involves terms appearing in the various adjoints above...

Corollary to Theorem 13:

**Corollary 15.**

$$\frac{\zeta(S, s)}{L(K, H^*(\Gamma), s)} = \zeta(D(1), s) = \zeta(D, s-1).$$