THE BLOCH-OGUS–GABBER THEOREM

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To the memory of Robert W. Thomason

ABSTRACT. The purpose of this paper is twofold. Firstly we give an exposition of Gabber's proof of the Bloch-Ogus theorem for étale cohomology of a smooth variety X over a field k, with torsion coefficients defined over k. Secondly we discuss the axioms used in that proof (étale excision and cohomology of the projective line), and apply them to other cohomology theories. We give several extensions of the theorem: the torsion hypothesis is irrelevant, the basic exact sequences for semi-local rings of X are universally exact in the sense of Grayson, and they remain exact after multiplying X by an arbitrary k-variety.

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INTRODUCTION

Stemming directly from Quillen's proof of the Gersten conjecture in the geometric case [43], the Bloch-Ogus theorem is a fundamental result of modern algebraic geometry. Its simplest consequence is that, for a local ring A of a smooth algebraic variety with function field K, $h^*(A)$ injects into $h^*(K)$ for all "cohomology theories" h^* satisfying a list of natural axioms (étale cohomology with coefficients in roots of unity is such a theory).

The Bloch-Ogus theorem, briefly described, is as follows. Given a smooth algebraic variety X and a cohomology theory h^* as above, filtration by codimension of support yields *Cousin complexes* which form the E_1-terms of the *coniveau spectral sequence*, converging to $h^*(X)$ (compare [24]). Restriction of the Cousin complexes to the open subsets of X defines complexes of flasque Zariski sheaves. The Bloch-Ogus theorem says that these complexes of sheaves are *acyclic*, except in degree 0 where their cohomology is the Zariski sheaf $\mathcal{H}^*$ associated to $U \mapsto h^*(U)$. This identifies the E_2-term of the coniveau spectral sequence to $H^*(X_{Zar}, \mathcal{H}^*)$. By a standard argument, the proof reduces to proving an "effacement theorem" for the version of h^* with supports.

Bloch and Ogus' proof of the effacement theorem [2] uses a geometric presentation lemma devised by Quillen for his proof of the analogous Gersten conjecture for algebraic K-theory (a strengthening of Noether's normalization theorem), see [43, Lemma 7.5.12]. This step reduces the proof to a "split" case. However, Quillen's simple K-theoretic argument in this split case has no counterpart in the cohomology setting, and is replaced in [2] by a much longer argument, involving a delicate diagram chase (*op. cit.*, section 5).

At the beginning of the eighties, Gabber provided a different proof of the effacement theorem for étale cohomology, which has now appeared in print as [14] (he considers there a more general situation of vanishing cycles). Gabber uses another, more powerful variant of the Noether normalization theorem, which reduces one to a situation where the ambiant space is the affine line over some base and the relevant closed subset is finite over the base. In this special case, Gabber's argument to prove effacement is different from that of Bloch-Ogus. It essentially uses the section at infinity coming from an embedding of the affine line into the projective line, as well as a computation of the cohomology of the latter. A geometric presentation lemma very close to Gabber's was independently found at about the same time by Ojanguren, who used it to prove Gersten-like properties of quadratic forms [40]. Unlike Gabber, Ojanguren does not use the projective line, but rather the affine line and homotopy invariance proved for the Witt ring by Karoubi; see remark after Lemma 4.1.3. This kind of idea has been reemployed to go further in that direction, notably for non-abelian cohomology [8], [7]. See also the articles of Nashier [37] and Dutta [11].

Gabber's proof makes it clear that the Bloch-Ogus theorem holds for considerably more general cohomology theories than those considered in [2], notably in cases where purity or homotopy invariance does not hold. As an example, his method was then used by Gros and Suwa to prove Gersten's conjecture for logarithmic Hodge-Witt sheaves [22].

Somewhat later, Grayson, looking at Quillen's proof again, observed that the argument gives more: the K-theoretic Zariski sheaves of complexes analogous to the sheafified Cousin complexes have stalks which are not only exact, but even "universally exact" [19]. This means that they remain exact after applying any finitely presented additive (not necessarily exact) functor to them. Gabber also states that the sheaves of Cousin complexes for étale cohomology have universally exact stalks [14, 1.6], but he does not elaborate. Finally, by unrelated methods, Gillet, using Suslin's rigidity theorem for the algebraic K-theory with coefficients of henselian discrete valuation rings, obtained a proof of Gersten's conjecture for the K-theory with coefficients of an arbitrary discrete valuation ring

[16].

The scope of this paper is twofold. Firstly, we give a simple and detailed exposition of the proof of the Bloch-Ogus theorem for étale cohomology, following Gabber's method: this is done in Part I. In contrast with [2], étale homology is not used. Secondly, we give in Part II an axiomatic treatment of Gabber's argument. We show that it applies to any "cohomology theory with supports" which satisfies two simple axioms: étale excision and an axiom dubbed "key lemma". The latter follows either from homotopy invariance or from a good behaviour of the cohomology of $\mathbf{P}^1$.

In section 1, we construct the Cousin complexes and the coniveau spectral sequence. In section 2, we formulate the effacement theorem for torsion coefficients defined over the base field, and derive the Bloch-Ogus theorem from it. In section 3, we prove a geometric presentation theorem which is a little stronger than Gabber's. In section 4, we prove Gabber's effacement theorem in a special case, and deduce it in general thanks to the geometric presentation theorem.

In section 5, we set up the axioms described above. In section 6, we formulate a "universally exact" version of the Bloch-Ogus theorem for cohomology theories with supports which are defined by a "substratum" (see subsections 5.1 and 5.2). In section 7, we give several examples to which our framework applies. Classical examples include étale cohomology with finite coefficients, Betti cohomology, de Rham cohomology, algebraic K- and G-theory. Other examples include Hodge, de Rham-Witt, Hodge-Witt and logarithmic Hodge-Witt cohomology, Rost's cycle cohomology and Voevodsky's motivic cohomology. In section 8, we give a few corollaries which partially motivated this paper (see table of contents): the one in subsection 8.1 was announced in [7], and those in subsections 8.2 and 8.4 were used in [30].

There are 3 appendices. In Appendix A, we extend the effacement theorem to arbitrary (not necessarily torsion) complexes of sheaves coming from the small étale site of k. This includes a self-contained proof of Gabber's computation of the étale cohomology of the projective line [13], which was not given in section 4. In Appendix B, we prove a refined version of the Bloch-Ogus theorem over a semi-local Dedekind domain, Gillet style. Appendix C is technical: it exposes a homological theory of unbounded complexes of objects of an abelian category, which allows for a smoother exposition in sections 5 and 6.

As the expert reader will already be aware after having read this introduction, we do not claim much originality in many results and proofs given here. The main purpose of writing this text was for us to make those easily accessible to the general public. We strongly encourage the reader to also have a look at V. Voevodsky's preprint [52], whose methods and results are different yet closely related.

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Part 1. **Etale cohomology**

1. The coniveau spectral sequence

Let X be a scheme and A a sheaf of abelian groups on the small étale site of X. In this section, we recall the construction of the coniveau spectral sequence over X with coefficients in A in a leisurely way.

1.1. **An exact couple.**

First consider a chain of closed subsets of X

$$\overrightarrow{Z} : \emptyset \subset Z_d \subset Z_{d-1} \subset \cdots \subset Z_0 = X.$$

Take the convention that $Z_i = \emptyset$ for $i > d$ and $Z_i = X$ for $i < 0$. For a pair $(Z_{p+1} \subset Z_p)$, we have a long exact sequence of cohomology with supports:

$$\cdots \to H^{p+q}_{Z_{p+1}}(X, A) \xrightarrow{i^{p+1,q-1}} H^{p+q}_{Z_p}(X, A) \xrightarrow{j^{p,q}} H^{p+q}_{Z_p - Z_{p+1}}(X - Z_{p+1}, A) \xrightarrow{k^{p,q}} H^{p+q+1}_{Z_{p+1}}(X, A) \to \dots$$

We construct an exact couple $C_{\overrightarrow{Z}}(D, E, i, j, k)$ [27, ch. VIII, §4], [35, ch. 2, §2.3] by setting $D^{p,q} = H^{p+q}_{Z_p}(X, A)$ and $E^{p,q} = H^{p+q}_{Z_p - Z_{p+1}}(X - Z_{p+1}, A)$:

$$\begin{array}{ccc} D^{p+1,q-1} & \xrightarrow{i^{p+1,q-1}} & D^{p,q} \\ & k^{p,q} \nwarrow_{(0,+1)} \qquad j^{p,q} \swarrow & \\ & E^{p,q} & \end{array}$$

This exact couple yields a spectral sequence of cohomological type, converging to $D^{0,n} = H^n(X, A)$ with respect to the filtration

$$F^p = \mathrm{Im}[H^n_{Z_p}(X, A) \to H^n(X, A)] = \mathrm{Ker}[H^n(X, A) \to H^n(X - Z_p, A)]$$

The E_1-terms of this spectral sequence are $E_1^{p,q} = E^{p,q}$, and the differential $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is the composite

$$H^{p+q}_{Z_p - Z_{p+1}}(X - Z_{p+1}, A) \xrightarrow{k} H^{p+q+1}_{Z_{p+1}}(X, A) \xrightarrow{j} H^{p+q+1}_{Z_{p+1} - Z_{p+2}}(X - Z_{p+2}, A).$$

1.2. **Passing to the limit.**

We now assume that X is equidimensional and noetherian of dimension d and that for all p, $\mathrm{codim}_X\, Z_p \geq p$. Order the set of $(d+1)$-tuples $\overrightarrow{Z}$ by $\overrightarrow{Z} \leq \overrightarrow{Z}'$ if $Z_p \subseteq Z'_p$ for all p.

The construction of the exact couple $C_{\vec{Z}}$ is covariant with respect to this ordering. Passing to the limit now defines a new exact couple $\underset{\to}{C}$ with

$$\begin{aligned} \underset{\to}{D}^{p,q} &= \varinjlim_{\vec{Z}} H^{p+q}_{Z_p}(X, A) := H^{p+q}_{X^{(p)}}(X, A) \text{ and} \\ \underset{\to}{E}^{p,q} &= \varinjlim_{\vec{Z}} H^{p+q}_{Z_p - Z_{p+1}}(X - Z_{p+1}, A) \end{aligned}$$

where $X^{(p)}$ denotes the set of points of codimension p in X. The following lemma describes the second direct limit more concretely:

Lemma 1.2.1. *a) If $T_1, \dots, T_r$ are pairwise disjoint closed subsets of X, then*

$$\bigoplus H^*_{T_i}(X, A) \xrightarrow{\sim} H^*_{\bigcup T_i}(X, A).$$

b) We have

$$\underset{\to}{E}^{p,q} \simeq \coprod_{x \in X^{(p)}} H^{p+q}_x(X, A) \tag{1.1}$$

where, for $x \in X^{(p)}$, $H^{p+q}_x(X, A) := \varinjlim_{U \ni x} H^{p+q}_{\overline{x} \cap U}(U, A)$.

Proof. a) By induction on r we may assume $r = 2$. We have a commutative diagram

$$\begin{array}{ccccc} & & H^*_{T_2}(X, A) & & \\ & & \downarrow & \searrow & \\ H^*_{T_1}(X, A) & \longrightarrow & H^*_{T_1 \cup T_2}(X, A) & \longrightarrow & H^*_{T_2}(X - T_1, A) \\ & \searrow & \downarrow & & \\ & & H^*_{T_1}(X - T_2, A) & & \end{array}$$

in which the row and column are exact and the two diagonal maps are isomorphisms by excision. The claim follows.

b) Note that, if the irreducible components of codimension p of Z_p are $Y_1, \dots, Y_r$, then $Z_p \setminus Z_{p+1} = \coprod (Y_i \setminus Z_{p+1})$ (disjoint union) as soon as Z_{p+1} contains the $Y_i \cap Y_j$ and the higher codimensional components of Z_p. The isomorphism now follows from a). $\square$

The spectral sequence associated to the exact couple $\underset{\to}{C}$ still converges to $H^*(X, A)$. It is called the *coniveau spectral sequence* (compare [2, remark 3.10], [15, p. 239]):

$$E_1^{p,q} = \coprod_{x \in X^{(p)}} H^{p+q}_x(X, A) \Rightarrow H^{p+q}(X, A). \tag{1.2}$$

The associated filtration

$$N^p H^n(X, A) = \operatorname{Im}(H^n_{X^{(p)}}(X, A) \to H^n(X, A))$$

is called the *coniveau filtration* or *filtration by codimension of support*. Its E_1-terms yield *Cousin complexes*:

$$\begin{aligned} 0 \to \coprod_{x\in X^{(0)}} H^q_x(X,A) \xrightarrow{d_1^{0,q}} \coprod_{x\in X^{(1)}} H^{1+q}_x(X,A) \xrightarrow{d_1^{1,q}} \dots \\ \dots \xrightarrow{d_1^{p-1,q}} \coprod_{x\in X^{(p)}} H^{p+q}_x(X,A) \xrightarrow{d_1^{p,q}} \dots \end{aligned} \tag{1.3}$$

In the next section, we shall need:

Lemma 1.2.2. *For all n,p, the presheaf*

$$U \mapsto \coprod_{x\in U^{(p)}} H^n_x(U,A)$$

is a sheaf for the Zariski topology of X. This sheaf is flasque and can be identified with

$$\coprod_{x\in X^{(p)}} i_{x*}H^n_x(X,A)$$

where i_x is the immersion $x \hookrightarrow X$ and the abelian group $H^n_x(X,A)$ is considered as a (constant) sheaf on x for the Zariski topology.

Proof. For $x \in X^{(p)}$, define a presheaf F_x on the category of Zariski open subsets of X by

$$F_x(U) = \begin{cases} H^n_x(U,A) & \text{if } U \ni x \\ 0 & \text{if } U \not\ni x. \end{cases}$$

By definition of $H^n_x(X,A)$ (see Lemma 1.2.1 b)), $F_x(U) = F_x(X)$ if $U \ni x$, hence F_x is the sheaf $i_{x*}H^n_x(X,A)$, which is obviously flasque. □

Suppose now that X is a smooth, irreducible variety over a field k, A is a locally constant, constructible sheaf and the stalks of A are m-torsion, with m prime to the characteristic of k. We shall use cohomological purity to transform these complexes into ones which involve only étale cohomology without supports. For $i \in \mathbf{Z}$, we write

$$A(i) = A \otimes \mu_m^{\otimes i}$$

where μ_m is the sheaf of m-th roots of unity. Let Z be a smooth irreducible closed subvariety of X of codimension p. Cohomological purity ([36, ch. VI, §§ 5 and 6], [SGA4 1/2, p. 63, th. V.3.4]) then gives canonical isomorphisms:

$$H^n_Z(X,A) \xleftarrow{\sim} H^{n-2p}(Z, A(-p)).$$

Noting that, for an arbitrary irreducible closed subvariety Z of X, the intersection $Z\cap U$ is smooth for small enough open subsets U, hence defines a smooth pair $Z\cap U \subset U$, this yields isomorphisms

$$H^{p+q}_x(X,A) \simeq H^{q-p}(k(x), A(-p))^1$$

[1]Strictly speaking, this argument is only valid when the ground field k is perfect. Otherwise a closed point of X whose residue field is inseparable over k will produce a counterexample to the statement just before this equation. However, if k is imperfect, the isomorphism will hold after passing to its perfect

for $x \in X^{(p)}$ (where $k(x)$ is the residue field of x), so that the complex (1.3) takes the following, perhaps more familiar form (compare [2, prop. 3.9]):

$$\begin{aligned} 0 \to H^q(k(X), A) \to \coprod_{x \in X^{(1)}} H^{q-1}(k(x), A(-1)) \to \dots \\ \dots \to \coprod_{x \in X^{(p)}} H^{q-p}(k(x), A(-p)) \to \dots \end{aligned} \tag{1.4}$$

Here, $k(X)$ is the function field of X. Note that $H^{q-p}(k(x), A(-p))$ is simply Galois cohomology of the residue field of x. So the E_1-terms of the coniveau spectral sequence have taken an especially simple form. Note also that

$$E_1^{p,q} = 0 \text{ for } p > q.$$

2. THE EFFACEMENT THEOREM AND THE BLOCH-OGUS THEOREM

2.1. **Effaceable sheaves.**

In this paper, we are interested in a special property of the sheaf A:

Definition 2.1.1. Let X be a variety over k. Let $t_1, \dots, t_r \in X$ be a finite number of points contained in some affine open subset of X. An étale sheaf A over X is *effaceable at* $t_1, \dots, t_r$ if the following condition is satisfied:
Given $p \geq 0$, for any small enough affine open neighbourhood W of $t_1, \dots, t_r$ and any closed subset $Z \subseteq W$ of codimension $\geq p+1$, there exists a smaller open neighbourhood of $t_1, \dots, t_r$, $U \subseteq W$, and a closed subset $Z' \subseteq U$ containing $Z \cap U$ such that

(1) $\operatorname{codim}_U(Z') \geq p$;
(2) the composite $H^n_Z(W, A) \to H^n_{Z\cap U}(U, A) \to H^n_{Z'\cap U}(U, A)$ is 0 for all $n \geq 0$.

The sheaf A is *effaceable* if it is effaceable at $t_1, \dots, t_r$ for all $t_1, \dots, t_r$ as above.

This condition looks very technical, but it has far-reaching consequences:

Proposition 2.1.2. *Let $R = \mathcal{O}_{X,(t_1,\dots,t_r)}$ be the semi-local ring of X at $(t_1, \dots, t_r)$ and $Y = \operatorname{Spec} R$. Suppose the sheaf A is effaceable at $t_1, \dots, t_r$. Then, in the exact couple defining the coniveau spectral sequence for (Y, A), the map $i^{p,q}$ is identically 0 for all $p > 0$. In particular,*

$$E_2^{p,q} = \begin{cases} H^q(Y, A) & \text{if } p = 0 \\ 0 & \text{if } p > 0. \end{cases}$$

The Cousin complex (1.3) *yields an exact sequence:*

$$0 \to H^q(Y, A) \xrightarrow{e} \coprod_{x \in Y^{(0)}} H^q_x(Y, A) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} H^{q+1}_x(Y, A) \xrightarrow{d_1^{1,q}} \dots$$

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closure. Since étale cohomology is invariant under purely inseparable extensions [36, Ch. II, p. 77, remark 3.17], the isomorphism holds in general. Compare [2, remark 4.7].

Proof. Consider the diagram

$$\begin{array}{ccccc}
H^n_Z(W,A) & \longrightarrow & H^n_{Z\cap U}(U,A) & \longrightarrow & H^n_{Z'\cap U}(U,A)\\
\downarrow & & \downarrow & & \downarrow\\
H^n_{W^{(p+1)}}(W,A) & \longrightarrow & H^n_{U^{(p+1)}}(U,A) & \longrightarrow & H^n_{U^{(p)}}(U,A)\\
 & & \downarrow & & \downarrow\\
 & & H^n_{Y^{(p+1)}}(Y,A) & \longrightarrow & H^n_{Y^{(p)}}(Y,A)
\end{array}$$

The composition of arrows in the first row is identically 0 for any n. Therefore, the compositions $H^n_Z(W,A) \to H^n_{Y^{(p+1)}}(Y,A) \to H^n_{Y^{(p)}}(Y,A)$ are 0. Passing to the limit over Z, this gives that the compositions $H^n_{W^{(p+1)}}(W,A) \to H^n_{Y^{(p+1)}}(Y,A) \to H^n_{Y^{(p)}}(Y,A)$ are 0. Passing to the limit over W, we get that the map

$$H^n_{Y^{(p+1)}}(Y,A) \xrightarrow{i^{p+1,n-p-1}} H^n_{Y^{(p)}}(Y,A)$$

is itself 0. □

Corollary 2.1.3 (The Bloch-Ogus theorem)**.** *Let A be effaceable on X. Then, the E_2-term of the coniveau spectral sequence for (X,A) is*

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{H}^q(A))$$

where $\mathcal{H}^q(A)$ is the Zariski sheaf associated to the presheaf $U \mapsto H^q(U,A)$.

Proof. Consider the complex of flasque Zariski sheaves associated to the Cousin complexes (1.3) (compare Lemma 1.2.2):

$$0 \to \coprod_{x\in X^{(0)}} i_{x*}H^q_x(X,A) \to \coprod_{x\in X^{(1)}} i_{x*}H^{1+q}_x(X,A) \to \cdots \to \coprod_{x\in X^{(p)}} i_{x*}H^{p+q}_x(X,A) \to \dots \tag{2.1}$$

Proposition 2.1.2 implies that (2.1) is a resolution of $\mathcal{H}^q(A)$, with global sections (1.3). The conclusion follows. □

2.2. **The effacement theorem.**

The main result of [14] is:

Theorem 2.2.1. (Gabber) *For X smooth over k, any torsion sheaf (on the small étale site of X) of the form p^*A_0 is effaceable, where $p: X \to \operatorname{Spec} k$ is the structural morphism.*

Specializing to twisted roots of unity and using sequence (1.4), we get a more familiar case:

Corollary 2.2.2. (Bloch-Ogus, [2]) *Let X be smooth, irreducible over k, R and Y as in Proposition 2.1.2, and let m be an integer prime to the characteristic of k. Then, for all $i \in \mathbf{Z}$ and $q \geq 0$, we have an exact sequence:*

$$0 \to H^q(Y, \mu_m^{\otimes i}) \to H^q(k(Y), \mu_m^{\otimes i}) \to \coprod_{x\in Y^{(1)}} H^{q-1}(k(x), \mu_m^{\otimes(i-1)}) \to \dots$$

Remarks 2.2.3.

(1) In Appendix A we shall remove the hypothesis that A_0 is torsion in Theorem 2.2.1.

(2) Effaceable sheaves have the following trivial stability properties:
- A sheaf A is effaceable at $t_1, \dots, t_r$ if and only if $A_{|W}$ is effaceable for a small enough open neighbourhood W of $t_1, \dots, t_r$ (effaceability is a local condition).
- Let A, B be two étale sheaves over X. Then $A \oplus B$ is effaceable if and only if A and B are.

(3) Over a smooth k-variety X, it is not true that all étale sheaves are effaceable. As an example, take $k = \mathbf{R}$ and for X the affine line $\mathbf{A}^1_{\mathbf{R}}$. Let $f : X' \to X$ be the two-fold covering given by the equation $x^2 + y^2 = 0$, where x is the parameter of $\mathbf{A}^1_{\mathbf{R}}$, and let $A = f_*\mathbf{Z}/2$. Let Y be the localization of X at 0 and $Y' = X' \times_X Y$. Since f is finite, there is an isomorphism

$$H^*(Y, A) \xrightarrow{\sim} H^*(Y', \mathbf{Z}/2).$$

On the other hand, the structural morphism $Y' \to \operatorname{Spec} \mathbf{R}$ is split by the inclusion of the closed point of Y', hence $H^*(Y', \mathbf{Z}/2)$ contains $H^*(\mathbf{R}, \mathbf{Z}/2) \neq 0$ as a direct summand. However, the two generic points η'_1, η'_2 of Y' are isomorphic to $\operatorname{Spec} \mathbf{C}(x)$, hence the Kummer theory class of -1 in $H^1(Y', \mathbf{Z}/2)$ goes to 0 in both $H^1(\eta'_1, \mathbf{Z}/2)$ and $H^1(\eta'_2, \mathbf{Z}/2)$. Correspondingly, if η denotes the generic point of Y, the map

$$H^1(Y, A) \to H^1(\eta, A)$$

is not injective. One can give a similar example with $X = \mathbf{A}^2_{\mathbf{R}}$ and X' defined by the equation $x^2 + y^2 + z^2 = 0$ (from H^2 onwards), etc. See also [9, p.173].

(4) One may however produce effaceable sheaves which are more general than those of Theorem 2.2.1:

Proposition 2.2.4. *Let $\tilde{X} \xrightarrow{f} X$ be a finite map between schemes of pure dimension d with $\tilde{X}$ smooth, and let B be an étale sheaf over $\tilde{X}$. If B is effaceable at $f^{-1}(\{t_1, \dots, t_r\})$, then f_*B is effaceable at $t_1, \dots, t_r$.*

Proof. Let $T = \{t_1, \dots, t_r\}$, Z be as in Definition 2.1.1, $\tilde{T} = f^{-1}(T)$ and $\tilde{Z} = f^{-1}(Z)$. Apply the effacement theorem to $(\tilde{X}, \tilde{T}, \tilde{Z}, B)$ and get a pair $(\tilde{U}, \tilde{Z}')$ such that $\tilde{T} \subset \tilde{U}$ and the composition

$$H^n_{\tilde{Z}}(\tilde{X}, B) \to H^n_{\tilde{Z} \cap \tilde{U}}(\tilde{U}, B) \to H^n_{\tilde{Z}' \cap \tilde{U}}(\tilde{U}, B)$$

is 0 for all $n \geq 0$. Let

$$U = X - f(\tilde{X} - \tilde{U})$$
$$Z' = f(\tilde{Z}')$$

so that $T \subset U$, $Z \subset Z'$, $\operatorname{codim}_X Z' \geq p$ and $f^{-1}(U) \subseteq \tilde{U}$, $\tilde{Z}' \subseteq f^{-1}(Z')$. We then get a commutative diagram

$$\begin{array}{ccccccc}
H^n_Z(X,f_*B) & \longrightarrow & H^n_{Z\cap U}(U,f_*B) & & \longrightarrow & & H^n_{Z'\cap U}(U,f_*B) \\
 & & & & & & \wr\downarrow \\
\wr\downarrow & & & & & & H^n_{f^{-1}(Z')\cap f^{-1}(U)}(f^{-1}(U),B) \\
 & & & & & & \uparrow \\
H^n_{\tilde{Z}}(\tilde{X},B) & \longrightarrow & H^n_{\tilde{Z}\cap\tilde{U}}(\tilde{U},B) & \xrightarrow{0} & H^n_{\tilde{Z}'\cap\tilde{U}}(\tilde{U},B) & \longrightarrow & H^n_{f^{-1}(Z')\cap\tilde{U}}(\tilde{U},B)
\end{array}$$

where the left vertical map and top right vertical map are isomorphisms by Shapiro's lemma for étale cohomology (exactness of f_* for a finite morphism). Proposition 2.2.4 follows. □

Note that in the proof of Proposition 2.2.4, it is not necessary to assume X smooth. So it provides non-smooth cases in which the effacement theorem holds.

Corollary 2.2.5 (Shapiro's lemma for Zariski cohomology)**.** *Let $X' \xrightarrow{f} X$ be a finite flat map between two smooth varieties over k, and let $\mathcal{F}$ be a sheaf of abelian groups over X' for the Zariski topology. Suppose that $\mathcal{F}$ is of the form $\mathcal{H}^q(A)$ for some effaceable étale sheaf A over X'. Then the natural homomorphism*

$$H^p_{\text{Zar}}(X, f_*\mathcal{F}) \to H^p_{\text{Zar}}(X', \mathcal{F})$$

is an isomorphism for any $p \geq 0$.

Proof. Shapiro's lemma for étale cohomology yields isomorphisms of cohomology groups with supports, for $Z \subset X$ a closed subset and $Z' = f^{-1}(Z)$:

$$H^p_Z(X, f_*A) \xrightarrow{\sim} H^p_{Z'}(X', A).$$

Localizing, we get isomorphisms of the E_1-terms of the coniveau spectral sequences for A (over X') and f_*A (over X):

$$\coprod_{x\in X^{(p)}} H^{p+q}_x(X, f_*A) \xrightarrow{\sim} \coprod_{x\in X'^{(p)}} H^{p+q}_x(X', A).$$

This isomorphism of Cousin complexes induces an isomorphism of their homology groups:

$$H^p_{\text{Zar}}(X, \mathcal{H}^q(f_*A)) \xrightarrow{\sim} H^p_{\text{Zar}}(X', \mathcal{H}^q(A))$$

by the Bloch-Ogus theorem over semi-local rings (note that f_*A is effaceable by Proposition 2.2.4). Finally, there is an isomorphism of Zariski sheaves

$$\mathcal{H}^q(f_*A) \xrightarrow{\sim} f_*\mathcal{H}^q(A)$$

which is merely Shapiro's lemma for étale cohomology applied at the local rings of X. □

Remarks 2.2.6.

(1) An alternative argument would be to show that $R^q f_*\mathcal{F} = 0$ for $q > 0$. This is what we shall do in subsection 8.4, removing the flatness hypothesis.
(2) "Shapiro's lemma for Zariski cohomology" is false for arbitrary Zariski sheaves! For example, take $X = \mathbf{A}^1_k$ and X' some 2-fold covering of X split at 0 (e.g. $X' = \mathbf{A}^1_k$, $f(x) = x^2 - 1$ if $\operatorname{char} k \neq 2$). Let Y be the localization of X at 0 and $Y' = f^{-1}(Y)$. Let $\mathcal{F} = j_! A$ (extension by 0), where $j : \eta \to X'$ is the inclusion of the generic point and A is a constant sheaf. Then we have:

$$H^i(Y, f_*\mathcal{F}) = 0 \text{ for } i > 0 \text{ (since } Y \text{ is local);}$$
$$H^1(Y', \mathcal{F}) = A.$$

The latter is easily seen by a Čech cohomology computation.

The proof of Theorem 2.2.1 is given in sections 3 and 4. We shall in fact prove something slightly stronger (and simpler to state):

Theorem 2.2.7 (Effacement theorem). *Let X be a smooth, affine variety over k, $t_1, \dots, t_r \in X$ a finite number of points, $p \geq 0$ an integer and Z a closed subvariety of codimension $\geq p+1$. Let A be a sheaf of torsion abelian groups over the (small) étale site of X. Assume that $A = p^*A_0$, where $p : X \to \operatorname{Spec} k$ is the structural morphism and A_0 is a $Gal(k_s/k)$-module. If k is infinite, then there exist an open subset U of X containing all t_i and a closed subvariety $Z' \subseteq X$ containing Z such that*

(1) $\operatorname{codim}_X(Z') \geq p$;
(2) *the map* $H^n_{Z\cap U}(U, A) \to H^n_{Z'\cap U}(U, A)$ *is* 0 *for all* $n \geq 0$.

If k is finite, then there exists (U, Z') as above such that (at least) the composite

$$H^n_Z(X, A) \to H^n_{Z\cap U}(U, A) \to H^n_{Z'\cap U}(U, A) \tag{2.2}$$

is 0 *for all* $n \geq 0$.

Remark 2.2.8. We would like to point out that (contrary to the definition of effaceability) the statement in Theorem 2.2.7 is not local: the proof by no means implies that the map of Theorem 2.2.7 (2) remains 0 when U is replaced by a smaller open set. Therefore, if in Theorem 2.2.7 one replaces X by Y as in Proposition 2.1.2, it is not at all clear that the conclusion still holds. In other words, given a closed subset $Z \subset Y$ of codimension $\geq p+1$, although the maps $H^n_Z(Y, A) \to H^n_{Y^{(p)}}(Y, A)$ are all 0 by the proof of prop. 2.1.2, it is not clear whether one can find a single $Z' \subset Y$ as in Theorem 2.2.7 such that the maps $H^n_Z(Y, A) \to H^n_{Z'}(Y, A)$ are 0. This shows the subtlety of the situation and probably why Gersten's conjecture is so difficult for general regular local rings of dimension ≥ 2.

3. Some geometry

3.1. **The geometric presentation theorem.**

The key to the proof of Theorem 2.2.7 is a geometric presentation theorem which follows from lemmas of Gabber [14] supplemented with some remarks of Gros and Suwa [22]. This section is devoted to a detailed proof of this theorem. For simplicity, we write $\mathbf{A^n}$ for $\mathbf{A^n_k}$ and $S \times S'$ rather than $S \times_k S'$ for the product of two k-schemes S and S'.

Theorem 3.1.1 (Geometric Presentation Theorem)**.** *Let X be a smooth, affine, irreducible variety of dimension d over an infinite field k; let $t_1, \dots, t_r \in X$ be a finite set of points and Z a closed subvariety of codimension > 0. Then there exists a map $\varphi = (\psi, v) : X \to \mathbf{A^{d-1}} \times \mathbf{A^1}$, an open set $V \subset \mathbf{A^{d-1}}$, and an open set $U \subset \psi^{-1}(V)$ containing $t_1, \dots, t_r$ such that*

(1) $Z \cap U = Z \cap \psi^{-1}(V)$*;*
(2) $\psi|_Z$ *is finite;*
(3) $\varphi|_U$ *is étale and defines a closed immersion* $Z \cap U \hookrightarrow \mathbf{A}^1_V$*;*
(4) $\varphi(t_i) \notin \varphi(Z)$ *if* $t_i \notin Z$ $(1 \le i \le r)$*;*
(5) $\varphi^{-1}(\varphi(Z \cap U)) \cap U = Z \cap U$*.*

(If no t_i lies on Z, it is quite possible that $Z \cap U$ is empty in Theorem 3.1.1. See remark after the proof of Lemma 3.5.1.)

Corollary 3.1.2. *With notation as in Theorem 3.1.1, $\psi|_{Z\cap U} : Z \cap U \to V$ is a finite morphism and one has a cartesian square:*

$$\begin{array}{ccc} Z \cap U & \hookrightarrow & U \\ \wr\downarrow & & \downarrow{\scriptstyle \varphi|_U} \\ \varphi(Z \cap U) & \hookrightarrow & \mathbf{A}^1_V \end{array}$$

where the horizontal arrows are closed immersions, the left vertical one is an isomorphism and the right vertical one is étale. (One could say that φ defines an étale neighbourhood of $Z \cap U \subset \mathbf{A}^1_V$, that is, it induces an analytic isomorphism along $Z \cap U$.)

This theorem and its corollary can be summarized by the diagram below:

$$\begin{array}{ccccc} X & & \xrightarrow{\varphi} & & \mathbf{A^{d-1}} \times \mathbf{A^1} \\ \cup & & & & \cup \\ U & & \longrightarrow & & \mathbf{A}^1_V \\ & \nwarrow & & \nearrow & \\ {\scriptstyle \psi|_U}\downarrow & & Z \cap U & & \downarrow{\scriptstyle p_1} \\ & \swarrow & & \searrow & \\ V & & = & & V \subset \mathbf{A^{d-1}} \end{array}$$

We shall see that the pair (Z', U) of Theorem 2.2.7 may be taken as $(\psi^{-1}(\psi(Z)), U)$ where ψ and U are as in Theorem 3.1.1.

The proof of Theorem 3.1.1 is essentially a (quite involved) series of exercises of commutative algebra and elementary algebraic geometry; it elaborates on those of [14, §3] and [22, §2]. We briefly outline it:

(1) Reduction to the case where the t_i are closed points and Z a principal divisor.
(2) Securing ψ.
(3) Securing φ.

(4) Constructing V.
(5) Constructing U.

3.2. **Reduction to closed points.**

Lemma 3.2.1. *a) With notation as in Theorem 3.1.1, there exist closed points $s_1, \dots, s_r \in X$ such that*

(1) $s_i \in \overline{\{t_i\}}$ *for all* i;
(2) $s_i \notin Z$ *if* $t_i \notin Z$.

b) Let the t_i be closed. Then there is a non-zero $f \in \Gamma(X, \mathcal{O}_X)$ such that $Z \subset V(f)$ and $t_i \notin V(f)$ if $t_i \notin Z$.

Proof. a) follows from [34, p. 34, th. 5.5] and b) follows from [6, ch. II, §1, prop. 2].□

Reduction. Thanks to Lemma 3.2.1, we may assume that the t_i's are closed points in Theorem 3.1.1 (note that since $\psi|_Z$ is finite, $\varphi|_Z$ is also finite, hence $\varphi(Z)$ is closed; moreover, any U containing the s_i's also contains the t_i's) and also that Z is a principal divisor. Therefore:

Hypothesis. *From now on, we assume that*

- *the t_i's are closed in X;*
- *Z is a principal divisor in X.*

Let $A = \Gamma(X, \mathcal{O}_X)$, so that $X = \operatorname{Spec} A$. Write A as a quotient of $k[X_1, \dots, X_N]$ (for N large enough), hence X as a closed subvariety of $\mathbf{A^N}$. Any $u \in k[X_1, \dots, X_N]$ defines a morphism $\mathbf{A^N} \to \mathbf{A^1}$. This is the case in particular for $u \in E$, where E is the sub-vector space of $k[X_1, \dots, X_N]$ spanned by $X_1, \dots, X_N$. Hence any r-tuple $\varphi = (u_1, ..., u_r) \in E^r$ defines a morphism $\mathbf{A^N} \to \mathbf{A^r}$. Composing it with the closed immersion $X \hookrightarrow \mathbf{A^N}$, we get a morphism $X \to \mathbf{A^r}$ that we still denote by φ.

Let $\mathcal{E}$ be the affine variety associated to E (so that $E = \mathcal{E}(k)$). We may view $\mathcal{E}$ as the dual space of $\mathbf{A}^N$. We shall in fact prove the following more precise statement:

Theorem 3.2.2. *Let $X, Z, t_1, \dots, t_r$ be as in Theorem 3.1.1. We assume that the t_i are closed and that Z is a principal divisor. Then there exists a non-empty Zariski open subset Ω of $\mathcal{E}^{d-1} \times \mathcal{E}$, such that for any $\varphi = (\psi, \upsilon) \in \Omega(k)$:*

(1) $\psi|_Z : Z \to \mathbf{A^{d-1}}$ *is finite;*
(2) φ *is étale at all* t_i $(1 \le i \le r)$ *and at all points of the finite set* $S = (\bigcup_{1 \le i \le r} \psi^{-1}(\psi(t_i))) \cap Z$;
(3) $\varphi|_S : S \to \varphi(S)$ *is radicial;*
(4) *For* $1 \le i \le r$, $t_i \notin Z \Rightarrow \varphi(t_i) \notin \varphi(Z)$.

Moreover, for any $\varphi \in E^d$ satisfying 1)–4), there exists a pair (U, V) such that (φ, U, V) satisfies the remaining conditions of the Geometric Presentation Theorem.

(Note that $\Omega(k) \neq \emptyset$ since k is infinite.)

Condition 3) means that $\varphi|_S$ separates the points in S and that for $P \in S$, the residue field extension $k(P)/k(\varphi(P))$ is purely inseparable. This theorem slightly improves on the lemmas of Gabber and Gros-Suwa: it says that the set of φ satisfying 1)–4) contains a Zariski open set, while the former only say that this set is non-empty.

Remark 3.2.3. It can be shown that Theorem 3.2.2 holds even without assuming the t_i's to be closed and Z to be a principal divisor. We don't need this refinement here.

3.3. **Securing ψ.**

Lemma 3.3.1. (compare [18, prop. 1.1]) *There exists a non-empty open set $\Omega_1 \subseteq \mathcal{E}^{d-1}$ such that, for $\psi \in \Omega_1(k)$, $\psi|_Z : Z \to \mathbf{A^{d-1}}$ is finite.*

Proof. Let $B = k[Z]$ be the affine algebra of Z, and x_i the image of X_i in B. For $\psi = (u_1, \dots, u_{d-1}) \in E^{d-1}$, $\psi|_Z$ is finite if and only if, for all i, x_i is integral over $k[\overline{u}_1, \dots, \overline{u}_{d-1}]$, where $\overline{u}_j$ is the image of u_j in B.

Let $K = k(\mathcal{E}^{d-1})$ be the function field of $\mathcal{E}^{d-1}$ and η the generic point of $\mathcal{E}^{d-1}$. We view η as a rational point of $\mathcal{E}^{d-1}$ over K, so that $\eta = (\eta_1, \dots, \eta_{d-1})$ with $\eta_i \in \mathcal{E}(K)$. For simplicity, we still write η_j for the image of η_j in $K \otimes_k B$. Since $\dim Z \leq d-1$, there is for all i a non-zero algebraic relation in $K \otimes_k B$:

$$f_i(\eta_1, \dots, \eta_{d-1}, x_i) = 0$$

with $f_i \in K[T_1, \dots, T_d]$. We claim that f_i can be chosen so that it gives an integral dependence relation on x_i. To see this, we argue as in [34, p. 262, proof of Lemma 2]: let $n = \deg f_i$ and $f_i^{(n)}$ be the homogeneous part of degree n of f_i. Since k is infinite, we can find $(t_1, \dots, t_{d-1}) \in k^{d-1}$ such that $f_i^{(n)}(t_1, \dots, t_{d-1}, 1) \neq 0$ in K. Letting $g_i = f_i(T_1 + t_1 T_d, \dots, T_{d-1} + t_{d-1} T_d, T_d)$, the coefficient of T_d in g_i is $f_i^{(n)}(t_1, \dots, t_{d-1}, 1)$ and we get

$$g_i(\eta'_1, \dots, \eta'_{d-1}, x_i) = 0$$

with $\eta'_j = \eta_j - t_j x_i$. Substituting back η_j instead of η'_j gives the desired integral dependence relation. Therefore there exists a polynomial $g'_i \in k[\mathcal{E}^{d-1}][T_1, \dots, T_d]$ such that

- $g'_i(\eta_1, \dots, \eta_{d-1}, x_i) = 0 \in k[\mathcal{E}^{d-1}] \otimes_k B$;
- the coefficient a_i of $T_d^{\deg g'_i}$ in g'_i is $\neq 0$.

Then, we may take $\Omega_1 = \{(u_1, \dots, u_{d-1}) \in \mathcal{E}^{d-1} \mid a_i(u_1, \dots, u_{d-1}) \neq 0 \text{ for } 1 \leq i \leq N\}$.
□

3.4. **Securing φ.**

Lemma 3.4.1. *Assume k is algebraically closed. With notation as in Theorem 3.1.1, there exists a non-empty open set $\Omega_2 \subseteq \mathcal{E}^d$ such that, for $\varphi = (\psi, \upsilon) \in \Omega_2(k)$, φ is étale at $t_i, 1 \leq i \leq p$ and all points of S.*

Proof. Recall that the Jacobian criterion says that $\varphi = (u_1, ..., u_d)$ is étale at $x \in X$ if and only if $(du_1 \wedge ... \wedge du_d) \mid_x \neq 0$ in $\Omega^d_{X/k} \otimes k(x)$. Throughout this proof, for a $\varphi \in \mathcal{E}^d$, we write $\varphi = (\psi, \upsilon)$ with $\psi = (u_1, \dots, u_{d-1}) \in \mathcal{E}^{d-1}$ and $\upsilon = u_d \in \mathcal{E}$.

For $i = 1, \dots, p$, define

$$T^i = \left\{(\varphi, y) \in \mathcal{E}^d \times Z \mid \psi(y) = \psi(t_i) \text{ and } (du_1 \wedge ... \wedge du_d) \mid_y = 0\right\},$$

i.e., for any k-algebra R,

$$T^i(R) = \left\{(\varphi, y) \in (E \otimes_k R)^d \times Z(R) \mid \psi(y) = \psi(t_i) \in R^{d-1} \text{ and } (du_1 \wedge ... \wedge du_d) \mid_y = 0\right\},$$

and also

$$T'^i = \left\{\varphi \in \mathcal{E}^d \mid (du_1 \wedge ... \wedge du_d)|_{t_i} = 0\right\}.$$

It is clear that T^i is a closed subset of $\mathcal{E}^d \times Z$. By Chevalley's theorem, its projection on $\mathcal{E}^d$:

$$F^i = \left\{\varphi \in \mathcal{E}^d \mid \text{ for some } y \in Z, \psi(y) = \psi(t_i) \text{ and } (du_1 \wedge ... \wedge du_d) \mid_y = 0\right\}$$

is constructible. On the other hand, T'^i is closed in $\mathcal{E}^d$. Let $\Omega_2^i = \mathcal{E}^d - (\overline{F}^i \cup T'^i)$. By definition, $\varphi \in \Omega_2^i$ implies that φ is étale at t_i and all points of $\psi^{-1}(\psi(t_i)) \cap Z$.

We shall show that $\dim T^i < \dim \mathcal{E}^d$, hence that $\dim \overline{F}^i < \dim \mathcal{E}^d$, and also that $\dim T'^i < \dim \mathcal{E}^d$, hence that Ω_2^i is non-empty. For $y \in X$, let

$$F_y^i = \left\{\varphi = (u_1, ..., u_d) \in \mathcal{E}^d \mid \psi(y) = \psi(t_i) \text{ and } (du_1 \wedge ... \wedge du_d) \mid_y = 0\right\} \subseteq \mathcal{E}^d \times_k k(y).$$

For $y = t_i$, this is just T'^i. For $y \in Z$ this is the fibre at y of the projection $T^i \to Z$. To prove that $\dim T^i < \dim \mathcal{E}^d$, it is enough to show that for all $y \in Z$, $\operatorname{codim}_{\mathcal{E}^d_{k(y)}} F_y^i > \dim \overline{\{y\}}$. (It would be enough to let y run through the projections of the generic points of T^i.)

Let y be such a point. First assume $y \neq t_i$. Then the linear space $\mathcal{H} = \{u \in \mathcal{E}_{k(y)} \mid u(y) = u(t_i)\}$ is of codimension one in $\mathcal{E}_{k(y)}$ and so $\mathcal{H}^{d-1}$ is of codimension $d-1$ in $\mathcal{E}_{k(y)}^{d-1}$. Since the differentials du, $u \in \mathcal{E}$, generate $\Omega^1_{X/k}$ at each point of X, the subspace spanned by $du, u \in \mathcal{H}$, is of codimension ≤ 1 in $\Omega^1_{X/k} \otimes k(y)$. In particular we can find $u_i \in \mathcal{H}(k(y)) \subseteq k(y)[X_1, \dots, X_N]$ $(i = 1, \dots, d-1)$ such that $du_1, ..., du_{d-1} \in \Omega^1_{X/k} \otimes k(y)$ are independent at y. Complete this system by a $u_d \in \mathcal{E}(k(y))$ such that $du_1, ..., du_d$ are linearly independent at y. Then $\varphi = (u_1, ..., u_d) \in \mathcal{E}(k(y))^d$ is such that:

- $\psi(y) = \psi(t_i)$;
- $(du_1 \wedge ... \wedge du_d) \mid_y \neq 0$.

Thus F_y^i is of codimension > 0 in $\mathcal{H}^{d-1} \times \mathcal{E}_{k(y)}$ and so is of codimension $\geq d > \dim Z$ in $\mathcal{E}^d_{k(y)}$. As $\dim Z \geq \dim \overline{\{y\}}$, we are done.

The case $y = t_i$ is easier, since then $\dim \overline{\{y\}} = 0$: we need only use the fact that $\Omega^1_{X/k}$ is generated by $d\mathcal{E}$ at t_i. This also shows that $\dim T'^i < \dim \mathcal{E}^{d-1}$ for all i, which we wanted. Thus $\Omega_2^i \neq \emptyset$.

Let $\Omega_2 = \bigcap_{i=1}^p \Omega_2^i$. For $\varphi \in \Omega_2(k)$, φ is étale at all t_i and all points of $\psi^{-1}(\psi(t_i)) \cap Z$ ($1 \le i \le p$). This completes the proof of Lemma 3.4.1. □

Lemma 3.4.2. *Assume k is algebraically closed. There exist non-empty open sets $\Omega_3, \Omega_4 \subseteq \mathcal{E}^d$ such that:*

(1) *For $\varphi = (\psi, \upsilon) \in \Omega_3(k)$, $\varphi|_S$ is injective.*
(2) *For $\varphi \in \Omega_4(k)$ and $1 \le i \le p$, $t_i \notin Z \Rightarrow \varphi(t_i) \notin \varphi(Z)$.*

Proof. For $s, s' \in X(k)$, consider

$$M(s,s') = \{(\psi, \upsilon, t, t') \in \mathcal{E}^{d-1} \times \mathcal{E} \times (Z \times Z \setminus \Delta_Z) \mid \begin{cases} \psi(t) = \psi(s) \\ \psi(t') = \psi(s') \\ \upsilon(t) = \upsilon(t') \end{cases}\},$$

where Δ_Z is the diagonal of $Z \times Z$. Let p be the projection of $M(s,s')$ on $\mathcal{E}^d = \mathcal{E}^{d-1} \times \mathcal{E}$. Let us fix $s, s' \in X(k)$ and for simplicity write M instead of $M(s,s')$. Let p be the projection of M on $\mathcal{E}^d = \mathcal{E}^{d-1} \times \mathcal{E}$; we are going to show that p is not dominant.

Let q be the composite of p with the projection $\mathcal{E}^{d-1} \times \mathcal{E} \to \mathcal{E}^{d-1}$, $\eta = \operatorname{Spec} K$ the generic point of $\mathcal{E}^{d-1}$ and M_η the generic fibre of q. Then p induces a map $M_\eta \to \mathcal{E}_K$ which is dominant if p is dominant.

Let $\overline{K}$ be an algebraic closure of K. This gives rise to the geometric point $\overline{\eta} : \operatorname{Spec} \overline{K} \to \operatorname{Spec} K \to \mathcal{E}^{d-1}$, and we have an induced morphism $M_{\overline{\eta}} \to \mathcal{E}_{\overline{K}}$ (where $M_{\overline{\eta}} = M_\eta \otimes_K \overline{K}$), which is still dominant if p is dominant.

Now the point $\overline{\eta}$ defines a $\overline{K}$-morphism $\psi_0 : Z_{\overline{K}} \to \mathbf{A}^{\mathbf{d-1}}_{\overline{\mathbf{K}}}$ which is finite as a consequence of 3.3.1. View $s \in X(k)$ as a point of $X(\overline{K})$. Let y_α ($\alpha = 1, \dots, n$) be the finitely many ($\overline{K}$-rational) points of $Z_{\overline{K}}$ such that $\psi_0(y_\alpha) = \psi_0(s)$. Let z_β ($\beta = 1, \dots, m$) be the finitely many ($\overline{K}$-rational) points of $Z_{\overline{K}}$ such that $\psi_0(z_\beta) = \psi_0(s')$. We have:

$$\begin{aligned} M_{\overline{\eta}} &= \{(\upsilon, t, t') \in (\mathcal{E} \times (Z \times Z \setminus \Delta_Z)) \times_k \overline{K} \mid \begin{cases} \psi_0(t) = \psi_0(s) \\ \psi_0(t') = \psi_0(s') \\ \upsilon(t) = \upsilon(t') \end{cases}\} \\ &= \bigcup_{y_\alpha \neq z_\beta} V_{\alpha,\beta} \end{aligned}$$

where

$$V_{\alpha,\beta} = \{\upsilon \in \mathcal{E}_{\overline{K}} \mid \upsilon(y_\alpha) = \upsilon(z_\beta)\} \times \{(y_\alpha, z_\beta)\}.$$

Thus the projection $p(M_{\overline{\eta}}) \subset \mathcal{E}_{\overline{K}}$ decomposes as

$$p(M_{\overline{\eta}}) = \bigcup_{y_\alpha \neq z_\beta} \{\upsilon \in \mathcal{E}_{\overline{K}} \mid \upsilon(y_\alpha) = \upsilon(z_\beta)\}.$$

This is a finite union of proper linear subspaces of $\mathcal{E}_{\overline{K}}$. Hence the projection map $M_{\overline{\eta}} \to \mathcal{E}_{\overline{K}}$ is not dominant, so $p : M \to \mathcal{E}^{d-1} \times \mathcal{E}$ is not dominant and the Zariski closure of the constructible set $p(M)$ is a proper closed subset of $\mathcal{E}^d$.

Let now Ω_3 be the complement of $\bigcup_{i,j} \overline{p(M(t_i, t_j))}$ in $\mathcal{E}^d$: this is a proper open set that satisfies condition 1) of Lemma 3.4.2.

The proof of condition 2) is entirely similar, using the sets

$$N(s) = \{(\varphi, t) \in \mathcal{E}^d \times Z \mid \varphi(t) = \varphi(s)\} \quad (s \notin Z),$$

for $s = t_i$, $t_i \notin Z$. □

Lemma 3.4.3. *There exists a non-empty open subset Ω of $\mathcal{E}^d$ such that all $\varphi \in \Omega(k)$ satisfy conditions 1)–4) of Theorem 3.2.2.*

Proof. Let $\overline{k}$ be an algebraic closure of k, $\Omega_1 \subseteq \mathcal{E}^{d-1}_{\overline{k}}$ as in Lemma 3.3.1 and $\Omega_2, \Omega_3, \Omega_4 \subseteq \mathcal{E}^d_{\overline{k}}$ as in Lemmas 3.4.1 and 3.4.2. Let $\overline{\Omega} = (\Omega_1 \times \mathcal{E}) \cap \Omega_2 \cap \Omega_3 \cap \Omega_4$. There exists a finite normal extension K/k such that $\overline{\Omega}$ is defined over K. Let Ω be the intersection of its conjugates under Aut(K/k). This is a non-empty open subset of $\mathcal{E}^d_K$, which is defined over the radicial closure L of k in K. But Ω is even defined over k, since we can raise the equations of its complement to an appropriate p^n-th power, where $p =$ char k. Then $\Omega_{\overline{k}} \subseteq \overline{\Omega}$.

Let $\varphi \in \Omega(k)$. By construction, φ satisfies conditions 1)–4) of Theorem 3.2.2 after extending scalars to $\overline{k}$. We conclude the proof of Lemma 3.4.3 by observing that each of these conditions descends to k. □

3.5. **Constructing V.**

Lemma 3.5.1. *Let $\varphi = (\psi, \upsilon) \in E^d$ satisfy conditions 1)–4) of Theorem 3.2.2. Then there exists $V \subseteq \mathbf{A^{d-1}}$ such that*

(1) *φ is étale at all points of $Z \cap \psi^{-1}(V)$;*
(2) *$\varphi|_{Z\cap\psi^{-1}(V)} \to \mathbf{A}^1_V$ is a closed immersion;*
(3) *$\psi(t_i) \in V$ for all i.*

Proof. Let Z_φ be the intersection of Z with the (closed) locus where φ is not étale. By condition 2) of Theorem 3.2.2, $Z_\varphi \cap \psi^{-1}(\psi(t_i)) = \emptyset$ for all i. By condition 1), $\psi|_Z$ is finite and $\psi(Z_\varphi)$ is closed in $\mathbf{A^{d-1}}$. Its complement $V_1 \subseteq \mathbf{A^{d-1}}$ is such that φ is étale at all points of $Z \cap \psi^{-1}(V_1)$, and that $\psi(t_i) \in V_1$ for all i. Consider the commutative diagram

$$\begin{array}{ccc} k[U_1, \dots, U_d] & \xrightarrow{{}^t\varphi} & B \\ & \nwarrow & \uparrow {\scriptstyle {}^t\psi} \\ & & k[U_1, \dots, U_{d-1}] \end{array}$$

where $B = k[Z]$ as above, and ${}^t\varphi, {}^t\psi$ are the homomorphisms corresponding to φ and ψ on coordinate rings. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the maximal ideals of $k[U_1, \ldots, U_{d-1}]$ corresponding to $\psi(t_1), \ldots, \psi(t_r)$. Let $\mathfrak{a} = \bigcap \mathfrak{p}_i$.

Since φ is étale at all points of S and $\varphi|_S : S \to \varphi(S)$ is radicial,

$$k[U_1, \ldots, U_d]/\mathfrak{a}\mathfrak{k}[\mathfrak{U}_1, \ldots, \mathfrak{U}_\mathfrak{d}] \to \mathfrak{B}/\mathfrak{a}\mathfrak{B}$$

is an isomorphism. Since B is a finitely generated $k[U_1, \ldots, U_{d-1}]$-module, by Nakayama's lemma there is an $f \in k[U_1, \ldots, U_{d-1}] - (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_r)$ such that $k[U_1, \ldots, U_d][1/f] \to B[1/f]$ is surjective. Let $V_2 = \{f \neq 0\} \subseteq \mathbf{A}^{\mathbf{d-1}}$: then V_2 contains $\psi(t_i)$ for all i and has the property that φ induces a closed immersion $Z \cap \psi^{-1}(V_2) \hookrightarrow \mathbf{A}^1_{V_2}$.

Let $V = V_1 \cap V_2$. Then φ is étale at all points of $Z \cap \psi^{-1}(V)$, induces a closed immersion $Z \cap \psi^{-1}(V) \hookrightarrow \mathbf{A}^1_V$ and contains $\psi(t_i)$ for all i. Hence V satisfies conditions 1), 2), 3) of Lemma 3.5.1. □

Remark 3.5.2. If $S = \emptyset$, $Z \cap \psi^{-1}(V)$ may well be empty.

3.6. **Constructing U.**

Lemma 3.6.1. *With notation as in Lemma 3.5.1, let*

$$\Phi = \varphi^{-1}\Big(\varphi\Big(Z \cap \psi^{-1}(V)\Big)\Big) - Z \cap \psi^{-1}(V).$$

Then:

(1) Φ *is closed in* $\psi^{-1}(V)$*;*
(2) $U_1 = \psi^{-1}(V) - \Phi$ *contains all the* t_i*, satisfies* $Z \cap \psi^{-1}(V) = Z \cap U_1$ *and* $\varphi^{-1}\Big(\varphi(Z \cap U_1)\Big) \cap U_1 = Z \cap U_1$.

Proof. For simplicity, let $T = Z \cap \psi^{-1}(V)$. By Lemma 3.5.1, $\varphi|_{\varphi^{-1}(\varphi(T))} : \varphi^{-1}(\varphi((T)) \to \varphi(T)$ is étale and $T \to \varphi(T)$ is an isomorphism. Therefore T is open in $\varphi^{-1}(\varphi((T))$ and $\Phi = \varphi^{-1}(\varphi(T)) - T$ is closed in $\varphi^{-1}(\varphi(T))$. Since $\psi|_Z$ is finite, $\psi|_T$ is finite over V and $\varphi|_T$ is finite over $\mathbf{A}^1_V$. So $\varphi(T)$ is closed in $\mathbf{A}^1_V$, hence in $\mathbf{A}^1_{\psi(T)}$. It follows that $\varphi^{-1}(\varphi(T))$ is closed in $\varphi^{-1}(\mathbf{A}^1_{\psi(T)}) = \psi^{-1}(\psi(T))$. Thus Φ is closed in $\psi^{-1}(\psi(T))$. Still by finiteness of $\psi\,|_Z$, $\psi^{-1}(\psi(T))$ is closed in $\psi^{-1}(V)$. This proves 1).

If $t_i \in Z$, then $t_i \notin \Phi$, hence $t_i \in U_1$; if $t_i \notin Z$, $t_i \notin \Phi$ by condition 4) of Theorem 3.2.2. In both cases, U_1 is a Zariski neighborhood of t_i. It is obvious that $Z \cap U_1 = Z \cap \psi^{-1}(V)$; the last condition of Lemma 3.6.1 holds by construction. □

End of proof of Theorem 3.2.2. Let $U_2 \subseteq X$ be the (open) locus where φ is étale. By Lemma 3.5.1, $t_1, \ldots, t_r \in U_2$ and $Z \cap \psi^{-1}(V) \subseteq U_2$. It follows that $U = U_1 \cap U_2$, with U_1 as in Lemma 3.6.1, also satisfies condition 2) of this lemma; moreover, $\varphi|_U$ is étale. So the triple (φ, V, U) satisfies all conditions of Theorem 3.1.1. □

4. Proof of the effacement theorem

For the convenience of the reader, we state the effacement theorem once again:

Effacement theorem. *Let X be a smooth, affine variety over a field k, $t_1, \dots, t_r \in X$ a finite number of points, $p \geq 0$ an integer and Z a closed subvariety of codimension $\geq p+1$. Let A be a sheaf of torsion abelian groups over the étale site of X. Assume that $A = p^* A_0$, where $p : X \to \operatorname{Spec} k$ is the structural morphism and A_0 is a $Gal(k_s/k)$-module. If k is infinite, then there exists an open subset U of X containing all t_i and a closed subvariety $Z' \subseteq X$ containing Z such that*

(1) $\operatorname{codim}_X(Z') \geq p$;
(2) *the map* $H^n_{Z\cap U}(U, A) \to H^n_{Z'\cap U}(U, A)$ *is* 0 *for all* $n \geq 0$.

If k is finite, then there exists (U, Z') as above such that (at least) the composite

$$H^n_Z(X, A) \to H^n_{Z\cap U}(U, A) \to H^n_{Z'\cap U}(U, A)$$

is 0 for all $n \geq 0$.

4.1. A key lemma.

Key lemma (compare [13, Lemma 2], [22, p. 621]) *Let V be k-scheme and A be as above. Let $\pi : \mathbf{A}^1_V \to V$, $\tilde{\pi} : \mathbf{P}^1_V \to V$ be the natural projections, $j : \mathbf{A}^1_V \to \mathbf{P}^1_V$ the inclusion and $s_\infty : V \to \mathbf{P}^1_V$ the section at infinity of $\tilde{\pi}$. Finally, let $F \subset V$ be a closed subset of V. Assume that V and $V - F$ are quasi-compact and quasi-separated. Then the diagram*

$$\begin{array}{ccc} H^n_{\mathbf{A}^1_F}(\mathbf{A}^1_V, A) & & \\ & \nwarrow^{\pi^*} & \\ j^* \uparrow & & H^n_F(V, A) \\ & \nearrow^{s_\infty^*} & \\ H^n_{\mathbf{P}^1_F}(\mathbf{P}^1_V, A) & & \end{array}$$

is commutative.

Proof. We may clearly limit ourselves to the cases where

(1) A is torsion prime to the characteristic of k, or
(2) A is p-primary torsion, where $p > 0$ is the characteristic of k.

In case (1), we use the following computation of $H^n(\mathbf{P}^1_V, A)$. For $i \in \mathbf{Z}$, let $A(i) = \varinjlim_m \operatorname{Hom}(\mu_m^{\otimes -i}, A)$. Recall the étale first Chern class of $\mathcal{O}(1)$ mod m:

$$c^{(m)} = c_1(\mathcal{O}(1))_m \in H^2(\mathbf{P}^1_V, \mu_m^{\otimes i})$$

defined as the boundary of the canonical class $[\mathcal{O}(1)] \in \operatorname{Pic}(\mathbf{P}^1_V) = H^1(\mathbf{P}^1_V, \mathbb{G}_{\gtrdot})$ in the long exact sequence associated to the Kummer exact sequence

$$1 \to \mu_m \to \mathbb{G}_{\gtrdot} \xrightarrow{\gtrdot} \mathbb{G}_{\gtrdot} \to \nVdash.$$

As V is quasi-compact and quasi-separated, the cup-products $H^i(V, {}_mA) \xrightarrow{\tilde{\pi}^*} H^i(\mathbf{P}^1_V, {}_mA) \xrightarrow{\cdot c^{(m)}} H^{i+2}(\mathbf{P}^1_V, {}_mA(1))$ for various m fit together and give a map

$$H^i(V, A) \xrightarrow{c_1(\mathcal{O}(1))} H^{i+2}(\mathbf{P}^1_V, A(1)).$$

We then have:

Proposition 4.1.1. (compare [SGA5, exposé VII, cor. 2.2.4]) *In case* (1), *the natural map*

$$H^n(V, A) \oplus H^{n-2}(V, A(-1)) \xrightarrow{(\tilde{\pi}^*, c_1(\mathcal{O}(1)))} H^n(\mathbf{P}^1_V, A)$$

is an isomorphism for all $n \geq 0$. □

In case (2), things are even simpler:

Proposition 4.1.2. (compare [13, Lemma 3]) *In case* (2), *the natural map*

$$H^n(V, A) \to H^n(\mathbf{P}^1_V, A)$$

is an isomorphism for all $n \geq 0$. □

Using the exact sequence for cohomology with supports and applying Propositions 4.1.1 and 4.1.2 to V and $V - F$, we get canonical decompositions:

$$H^n_{\mathbf{P}^1_F}(\mathbf{P}^1_V, A) \simeq \begin{cases} H^n_F(V, A) \oplus H^{n-2}_F(V, A(-1)) & \text{in case (1)} \\ H^n_F(V, A) & \text{in case (2).} \end{cases} \tag{4.1}$$

The key lemma now follows from

Lemma 4.1.3. *Suppose* A *is torsion prime to the characteristic of* k. *In the diagram of the key lemma, the restrictions of* s^*_∞ *and* j^* *to the factor* $H^{n-2}_F(V, A(-1))$ *of* (4.1) *are* 0.

Indeed, the map $H^{n-2}_F(V, A(-1)) \to H^n_{\mathbf{P}^1_F}(\mathbf{P}^1_V, A)$ is given by cup-product by the first Chern class of $\mathcal{O}(1)$. But $s^*_\infty \mathcal{O}(1)$ is trivial, and so is $j^*\mathcal{O}(1)$. □

Remark 4.1.4. If A is torsion invertible on V, the key lemma has a much simpler proof: in this case π^* is an isomorphism by homotopy invariance [36, ch. VI, p. 240, cor. 4.20] (for the definition of acyclicity, see [36, ch. VI, p. 232, section 4]). Applying this homotopy invariance to the projection $(\mathbf{P}^1 - \{1\})_V \to V$, we see that s_0 and s_∞, as right inverses of this projection, induce the same map on cohomology. Now replace s^*_∞ by s^*_0 in the diagram of the key lemma and add a map on the top

$$\begin{array}{ccc}
H^n_F(V, A) & & \\
\wr\big\uparrow s_0^* & & \\
H^n_{\mathbf{A}^1_F}(\mathbf{A}^1_V, A) & & \\
& \simeq\nwarrow \pi^* & \\
\big\uparrow j^* & & H^n_F(V, A) \\
& \nearrow s_0^* & \\
H^n_{\mathbf{P}^1_F}(\mathbf{P}^1_V, A) & &
\end{array}$$

to make its commutativity obvious.

So Gabber's argument going via the cohomology of $\mathbf{P}^1$ can be thought of as a substitute for homotopy invariance when the latter does not hold.

See [7, § 5] for more on the homotopy invariant point of view, notably in non-abelian situations.

4.2. **The proof.**

Theorem 4.2.1. *Let V be a k-scheme, F a closed subset of V and F' a closed subset of $\mathbf{A}^1_F$ such that the projection $f : F' \to F$ is finite.*

$$\begin{array}{ccccc} F' & \hookrightarrow & \mathbf{A}^1_F & \overset{i'}{\hookrightarrow} & \mathbf{A}^1_V \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi_F} & & \downarrow{\scriptstyle \pi} \\ & & F & \overset{i}{\hookrightarrow} & V \end{array}$$

Then, for any torsion étale sheaf A of abelian groups on the small étale site of V, the map

$$H^n_{F'}(\mathbf{A}^1_V, A) \to H^n_{\mathbf{A}^1_F}(\mathbf{A}^1_V, A)$$

is identically 0.

Caution. When we write $H^n_{F'}(\mathbf{A}^1_V, A)$ and so on, we really mean $H^n_{F'}(\mathbf{A}^1_V, \pi^*A)$ and so on, where π^* denotes the pull-back morphism from sheaves over the small étale site of V to that of $\mathbf{A}^1_V$ via π. Being this fastidious would quickly become cumbersome notationally; hence we allow ourselves to abbreviate π^*A to A, and similarly for the other morphisms to V; this should cause no confusion. Another way to present things is to consider the morphism α from the big étale site to the small étale site of V. What we do is consider α^*A and restrict it to the small étale site of T for any V-scheme T (and then we sneakily change the notation α^*A back to A).

Proof. Note that $s_\infty(V) \cap F' = \emptyset$. Therefore we can factor s_∞ into

$$s_\infty = k \circ s'$$

where k is the open immersion $\mathbf{P}^1_V - F' \hookrightarrow \mathbf{P}^1_V$, and insert the diagram above into the bigger commutative diagram:

$$\begin{array}{ccccccc} H^n_{F'}(\mathbf{A}^1_V, A) & \xrightarrow{\alpha} & H^n_{\mathbf{A}^1_F}(\mathbf{A}^1_V, A) & & & & \\ & & & \nwarrow{\scriptstyle \pi^*} & & & \\ \text{excision}\uparrow\simeq & & {\scriptstyle j^*}\uparrow & & H^n_F(V, A) & & \\ & & & \nearrow{\scriptstyle s_\infty^*} & & \nwarrow{\scriptstyle s'^*} & \\ H^n_{F'}(\mathbf{P}^1_V, A) & \xrightarrow{\beta} & H^n_{\mathbf{P}^1_F}(\mathbf{P}^1_V, A) & & \xrightarrow{k^*} & & H^n_{\mathbf{P}^1_F - F'}(\mathbf{P}^1_V - F', A) \end{array}$$

where the bottom row is part of an exact sequence for cohomology with supports. Since $k^* \circ \beta = 0$, it obviously follows that $\alpha = 0$. □

Proof of Theorem 2.2.7. We may assume X irreducible. Suppose first k infinite. Let ψ,φ,V,U be as in Theorem 3.1.1, and set $Z'=\psi^{-1}(\psi(Z))$. We apply Theorem 4.2.1 with $V=V$, $F=\psi(Z)$ and $F'=\varphi(Z\cap U)$. In the commutative diagram

$$\begin{array}{ccc} H^n_{Z\cap U}(U,A) & \longrightarrow & H^n_{Z'\cap U}(U,A)\\ {\scriptstyle\varphi^*}\uparrow\wr & & {\scriptstyle\varphi^*}\uparrow\\ H^n_{F'}(\mathbf{A}^1_V,A) & \longrightarrow & H^n_{\mathbf{A}^1_F}(\mathbf{A}^1_V,A)\end{array}$$

the left vertical map is an isomorphism by Corollary 3.1.2 and étale excision ([36, ch. III, p. 92, prop. 1.27] and [8, prop. 4.4]), and the bottom horizontal map is 0 by Theorem 4.2.1. So the top horizontal map is 0 as well.

Suppose now k finite. We reduce to the infinite case by the following standard argument. Let p,q be two distinct prime numbers and let K_1,K_2 denote respectively the $\mathbf{Z}_p$ and $\mathbf{Z}_q$-extensions of k. Let $(\psi_1,\varphi_1,V_1,U_1)$, $(\psi_2,\varphi_2,V_2,U_2)$ be as in Theorem 3.1.1 and Z'_1, Z'_2 be as above for (X_{K_1},Z_{K_1}) and (X_{K_2},Z_{K_2}) respectively. There are finite subextensions $k\subset k_1\subset K_1$, $k\subset k_2\subset K_2$ such that $(\psi_1,\varphi_1,V_1,U_1,Z'_1)$ and $(\psi_2,\varphi_2,V_2,U_2,Z'_2)$ are respectively defined over k_1 and k_2. Note that the effacement theorem holds respectively over k_1 and k_2 with these choices, by the above. Define

$$U=X-(p_1(X_{k_1}-U_1)\cup p_2(X_{k_2}-U_2))\,,\ Z'=p_1(Z'_1)\cup p_2(Z'_2)$$

where $p_1:X_{k_1}\to X$, $p_2:X_{k_2}\to X$ are the two projections. In other words, U and $X-Z'$ are those parts of $(U_1)_{\overline{k}}\cap(U_2)_{\overline{k}}$ and $X_{\overline{k}}\setminus((Z_1)_{\overline{k}}\cup(Z_2)_{\overline{k}})$ which are rational over k. Note that U contains all t_is. We have

$$U_{k_i}\subseteq U_i,\ Z_i\subseteq Z_{k_i}\ (i=1,2).$$

Considering the commutative diagrams ($i=1,2$)

$$\begin{array}{ccc} H^n_{(Z\cap U)_{k_i}}(U_{k_i},A) & \longrightarrow & H^n_{(Z'\cap U)_{k_i}}(U_{k_i},A)\\ \uparrow & & \uparrow\\ H^n_{Z\cap U_i}(U_i,A) & \xrightarrow{0} & H^n_{Z'\cap U_i}(U_i,A)\\ \uparrow & & \\ H^n_Z(X,A) & & \end{array}$$

shows that the composite $H^n_Z(X,A)\to H^n_{Z_{k_i}}(X_{k_i},A)\to H^n_{(Z\cap U)_{k_i}}(U_{k_i},A)\to H^n_{(Z'\cap U)_{k_i}}(U_{k_i},A)$ is 0 for $i=1,2$. Equivalently, the composite $H^n_Z(X,A)\to H^n_{Z\cap U}(U,A)\to H^n_{Z'\cap U}(U,A)\to H^n_{(Z'\cap U)_{k_i}}(U_{k_i},A)$ is 0 for $i=1,2$. But the composite

$$H^n_{Z'\cap U}(U,A)\to H^n_{(Z'\cap U)_{k_i}}(U_{k_i},A)\to H^n_{Z'\cap U}(U,A)$$

equals multiplication by $[k_i:k]$, where the second map is transfer. Since these two degrees are coprime, this shows that the composite $H^n_Z(X,A)\to H^n_{Z\cap U}(U,A)\to H^n_{Z'\cap U}(U,A)$ is 0 indeed. □

Part 2. Other cohomology theories

5. Axiomatizing Gabber's proof

5.1. Basic axioms.

In [2], Bloch and Ogus prove their main theorem not only for étale cohomology with coefficients twisted roots of unity, but also for other "cohomology theories with supports". Counterexample 2.2.3 (3) and the method of proof in the present paper show that this point of view should be taken seriously. On the other hand, the Bloch-Ogus axioms are very complicated, and the present proof shows that many of them are unnecessary. In this section, we want to indulge in the exercise of finding a convenient and simpler set of axioms which is enough to make the proof of the effacement theorem work.

Let $\mathcal{S}_k$ be a full subcategory of the category of algebraic k-schemes, stable under étale extensions. In practice, $\mathcal{S}_k$ will be either the category Var/k of all separated algebraic k-schemes or the category Sm/k of smooth k-schemes. Let $\mathcal{P}_k$ the category of pairs (X, Z), where $X \in \mathcal{S}_k$ and Z is a closed subset of X. By definition, a morphism $(X', Z') \xrightarrow{f} (X, Z)$ of $\mathcal{P}_k$ is any k-morphism $f : X' \to X$ such that $f^{-1}(Z) \subseteq Z'$ (example: $X' = X$, f the identity, $Z \subseteq Z'$).

The most naïve is to ask for a "cohomology theory with supports"

$$(X, Z) \mapsto h^*_Z(X)$$

a contravariant functor from $\mathcal{P}_k$ to **Z**-graded abelian groups, satisfying conditions abstracted from the proof of Theorem 2.2.7. It is more natural, however, and more powerful, as we shall see in section 6, to give such a theory a cohomological support

$$(X, Z) \mapsto C_Z(X)$$

where $C_Z(X)$ is, for example, a complex of abelian groups. It may be useful to allow $C_Z(X)$ to be a complex of objects in more general abelian categories $\mathcal{A}$ than abelian groups, for example if we want to have some ring action on the situation. So we give the following general definitions:

Definition 5.1.1. Let $\mathcal{A}$ be an abelian category.
a) A *cohomology theory with supports* is a contravariant functor $(X, Z) \mapsto h^*_Z(X)$ from $\mathcal{P}_k$ to $\mathcal{A}$ satisfying
For any triple $Z \subseteq Y \subseteq X$, where Y, Z are closed in X, there is a long exact sequence

$$\cdots \to h^q_Z(X) \to h^q_Y(X) \to h^q_{Y-Z}(X - Z) \to h^{q+1}_Z(X) \to \dots$$

which is natural in (X, Y, Z) in an obvious sense.
b) A *substratum* is a contravariant functor $X \mapsto C(X)$ from $\mathcal{S}_k$ to complexes of objects of $\mathcal{A}$.

In b), define $C_Z(X)$ as the homotopy fibre of $C(X) \to C(X-Z)$, i.e. $C[-1]$, where C is the mapping cone of this morphism. This definition is natural in (X, Z). For all triples (X, Y, Z) as in a), the sequence of complexes

$$0 \to C_Z(X) \to C_Y(X) \to C_{Y-Z}(X - Z) \to 0$$

is exact up to homotopy ([20, p. 47, prop. 5.12], [29, p. 32, I.4.22]). Hence, defining

$$h^q_Z(X) = H^q(C_Z(X)) \tag{5.1}$$

yields a cohomology theory with supports in the sense of a).

Recall that a contravariant functor T to an additive category is *additive* if it commutes with finite coproducts, i.e. $T(X \coprod Y) \to T(X) \times T(Y)$ is an isomorphism for all X, Y.

We now introduce a first axiom for a cohomology theory with supports h^* (resp. a substratum C) in the sense of definition 5.1.1.

COH1 (Etale excision). h^* is additive and for any diagram

$$\begin{array}{ccc} & & X' \\ & \nearrow & \Big\downarrow f \\ Z & \hookrightarrow & X \end{array}$$

where f is étale and $f^{-1}(Z) \xrightarrow{f} Z$ is an isomorphism, the induced map

$$h^q_Z(X) \xrightarrow{f^*} h^q_Z(X')$$

is an isomorphism for all q.

SUB1 (Etale Mayer-Vietoris). C is additive and for Z, X', X, f as in a), the commutative square

$$\begin{array}{ccc} C(X') & \xrightarrow{v} & C(X'-Z) \\ f\Big\uparrow & & f\Big\uparrow \\ C(X) & \xrightarrow{u} & C(X-Z) \end{array}$$

is homotopy cartesian.

Recall that a commutative square of complexes

$$\begin{array}{ccc} A & \longrightarrow & B \\ \Big\uparrow & & \Big\uparrow \\ C & \longrightarrow & D \end{array}$$

is *homotopy cartesian* if the natural map from the mapping cone of $[C \to A \oplus D]$ to B is a homotopy equivalence.

Lemma 5.1.2. *The square of axiom SUB1 is homotopy cartesian if and only if the induced map*

$$C_Z(X) \xrightarrow{f} C_Z(X')$$

is a homotopy equivalence.

Proof. This follows from the triangulated category version of the nine diagram. More precisely, consider the map of exact triangles

$$\begin{array}{ccccc}
C(X')\oplus C(X-Z) & \xrightarrow{diag(v,Id)} & C(X'-Z)\oplus C(X-Z) & \longrightarrow & C_Z(X')[1]\\
{\scriptstyle (f,-u)}\big\uparrow & & {\scriptstyle (f,-Id)}\big\uparrow & & {\scriptstyle f}\big\uparrow\\
C(X) & \xrightarrow{u} & C(X-Z) & \longrightarrow & C_Z(X)[1]
\end{array}$$

in the homotopy category $K(\mathcal{A})$. By [29, Proposition 5.6 of chapter XI], we can complete this diagram, up to isomorphism, into

$$\begin{array}{ccccc}
D & \longrightarrow & D' & \longrightarrow & D''\\
\big\uparrow & & \big\uparrow & & \big\uparrow\\
C(X')\oplus C(X-Z) & \xrightarrow{diag(v,Id)} & C(X'-Z)\oplus C(X-Z) & \longrightarrow & C_Z(X')[1]\\
{\scriptstyle (f,-u)}\big\uparrow & & {\scriptstyle (f,-Id)}\big\uparrow & & {\scriptstyle f}\big\uparrow\\
C(X) & \xrightarrow{u} & C(X-Z) & \longrightarrow & C_Z(X)[1]
\end{array}$$

in which all rows and columns are exact triangles. It is clear that:

- the middle top vertical map induces an isomorphism $C(X'-Z) \xrightarrow{\sim} D'$.
- $D'' \simeq 0 \iff C_Z(X) \xrightarrow{\sim} C_Z(X')$.
- **SUB1** holds $\iff D \xrightarrow{\sim} D' \iff D'' \simeq 0$.

The claim follows. □

Remarks 5.1.3.

(1) We say that a cohomology theory (resp. a substratum) satisfies *Zariski excision* (resp. *Zariski Mayer-Vietoris*) if axiom **COH1** (resp. **SUB1**) holds when we let f run through open immersions. The obvious analogue of Lemma 5.1.2 (ordinary excision) holds.

(2) Note that in **SUB1** one can replace the condition "C is additive" by "$C(\emptyset) = 0$" (take the case $X' = Z$ in the commutative square).

(3) Definition 5.1.1 allows us to set up a coniveau exact couple and spectral sequence as in section 1. Zariski excision allows us to recognize the E_1-terms of the coniveau spectral sequence in the form of equation (1.1), producing *Cousin complexes* in the sense of [24] by Zariski sheafification. In particular, if h^* satisfies Zariski excision, we get a convergent *coniveau spectral sequence*, for all $X \in \mathcal{S}_k$

$$E_1^{p,q} = \coprod_{x\in X^{(p)}} h_x^{p+q}(X) \Rightarrow h^{p+q}(X)$$

analogous to (1.2), where $h_x^n(X) = \varinjlim_{U\ni x} h^n_{\overline{x}\cap U}(U)$ (note that $E_1^{p,q} = 0$ for $p \notin [0, \dim X]$).

As we shall see in section 7, if h^* is defined by a substratum, Zariski excision gives rise to an (a priori unrelated) *Brown-Gersten* spectral sequence as well.

Lemma 5.1.4. *For a substratum C and the associated cohomology theory with supports h^*,*
a) Axiom SUB1 implies axiom COH1.
b) The converse is true if, for all X, $C(X)$ is fibrant in the sense of definition C.1.1 b) of Appendix C.

Proof. Part a) follows trivially from Lemma 5.1.2. Part b) also follows from this lemma, Lemma C.1.4 b) and Corollary C.2.7. □

Remarks 5.1.5.

(1) Obviously, Lemma 5.1.4 holds when étale Mayer-Vietoris is replaced by Zariski Mayer-Vietoris.
(2) By Theorem C.3.1, if $\mathcal{A}$ verifies axiom AB5 and has a generator in the sense of [23, 1.5 and 1.6] and if moreover countable products are exact in $\mathcal{A}$ (for example, $\mathcal{A}$ satisfies AB4^*), then there exists another substratum F and a natural transformation $C \xrightarrow{\psi} F$ such that, for all X,
    (a) $F(X)$ is fibrant;
    (b) ψ_X is a monomorphism and a quasi-isomorphism.
  This applies to the case where $\mathcal{A}$ is the category of left R-modules over some ring [23, §1].

Let us now introduce our second axiom. To do this, we need an assumption on $\mathcal{S}_k$.

Assumption 5.1.6.

(i) $\operatorname{Spec} k \in \mathcal{S}_k$.
(ii) If $X \in \mathcal{S}_k$, then $\mathbf{P}^1_X \in \mathcal{S}_k$.

Lemma 5.1.7. *If $\mathcal{S}_k$ satisfies assumption 5.1.6, then*
a) $X \in \mathcal{S}_k \Rightarrow \mathbf{A}^1_X \in \mathcal{S}_k$.
b) For any $n \geq 1$, the open subsets of $\mathbf{A}^n_k$ are in $\mathcal{S}_k$. □

In axioms **COH2** and **SUB2**, we assume that $\mathcal{S}_k$ satisfies assumption 5.1.6.

COH2 ("Key lemma" for cohomology) Let V be an open subset of $\mathbf{A}^n_k$ (for some n) and

$$\begin{array}{ccccc} \mathbf{A}^1_V & \xrightarrow{j} & \mathbf{P}^1_V & \xleftarrow{s_\infty} & V \\ & {\scriptstyle\pi}\searrow & {\scriptstyle\tilde\pi}\downarrow & \swarrow{\scriptstyle =} & \\ & & V & & \end{array}$$

be the diagram representing the inclusion of $\mathbf{A}^1_V$ and the section at infinity into $\mathbf{P}^1_V$. Let F be a closed subset of V. Then the diagram of the key lemma of subsection 4.1

$$\begin{array}{ccc} h^q_{\mathbf{A}^1_F}(\mathbf{A}^1_V) & & \\ & \nwarrow^{\pi^*} & \\ j^*\uparrow & & h^q_F(V) \\ & \nearrow^{s^*_\infty} & \\ h^q_{\mathbf{P}^1_F}(\mathbf{P}^1_V) & & \end{array}$$

is commutative.

SUB2 ("Key lemma" for substrata) Let V, F be as in axiom **COH2**. Then the diagram

$$\begin{array}{ccc} C_{\mathbf{A}^1_F}(\mathbf{A}^1_V) & & \\ & \nwarrow^{\pi^*} & \\ j^*\uparrow & & C_F(V) \\ & \nearrow^{s^*_\infty} & \\ C_{\mathbf{P}^1_F}(\mathbf{P}^1_V) & & \end{array}$$

is homotopy commutative.

We give a last definition.

Definition 5.1.8. Let $X \in \mathcal{S}_k$ be affine and let $t_1, \dots, t_r \in X$ be a finite set of points. A cohomology theory with supports h^* (resp. a substratum C) over k is *strictly effaceable at* $(t_1, \dots, t_r)$ if, given $p \geq 0$, for any open neighbourhood $W \subseteq X$ of $t_1, \dots, t_r$ and for any closed subset $Z \subseteq W$ of codimension $\geq p+1$, there exist an open neighbourhood $U \subseteq W$ of $t_1, \dots, t_r$ and a closed subset $Z' \subseteq W$ containing Z such that $\operatorname{codim}_W(Z') \geq p$ and the map $h^q_{Z\cap U}(U) \to h^q_{Z'\cap U}(U)$ is 0 for all $q \in \mathbf{Z}$ (resp. the map $C_{Z\cap U}(U) \to C_{Z'\cap U}(U)$ is nullhomotopic). It is *strictly effaceable* if this condition is satisfied for any $(X, t_1, \dots, t_r)$ as above, with X smooth.

Example 5.1.9. Suppose k infinite and let A be a sheaf of torsion abelian groups over the small étale site of $\operatorname{Spec} k$. By Theorem 2.2.7, the cohomology theory with supports

$$(X, Z) \mapsto H^*_Z(X_{\acute{e}t}, \alpha^* A)$$

is strictly effaceable, where α is the projection of the big étale site on the small étale site.

The following theorem is immediate from the arguments of section 4.

Theorem 5.1.10. *Let k be infinite and S_k satisfy assumption 5.1.6. A cohomology theory with supports h^* (resp. a substratum C) satisfying étale excision COH1 (resp. étale Mayer-Vietoris SUB1) and the key lemma for cohomology COH2 (resp. the key lemma for substrata SUB2) is strictly effaceable.*

The point here is that, in diagrams where there is an excision map as in subsection 4.2, one can use a homotopy inverse of this map to show that the map one wishes to be nullhomotopic is indeed nullhomotopic.

Corollary 5.1.11. *Let k be infinite, and $\mathcal{S}_k$ verify assumption 5.1.6 and let the cohomology theory with supports h^* satisfy axioms COH1 and COH2. Then, for any smooth $X \in \mathcal{S}_k$, the Cousin complexes are flasque resolutions of the Zariski sheaves $\mathcal{H}^q$ associated to the presheaves $U \mapsto h^q(U)$, and the E_2-terms of the spectral sequence of remark 5.1.3* (3) *are*

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, \mathcal{H}^q).$$

There is a need for something extra, like transfer maps, to deal with finite fields: see Theorem 6.2.5.

Remarks 5.1.12.

(1) The axioms above are much more economical than those of Bloch and Ogus in [2, § 1]. Definition 5.1.1 corresponds to axioms (1.1.1) and (1.1.2) of Bloch-Ogus. Axiom **COH1** corresponds to axiom (1.1.3). Axiom **COH2** has no counterpart in [2], but might be compared with [2, (1.5)]. On the other hand, we do not need to introduce any twists, nor a corresponding homology theory. This means that purity, let alone Poincaré duality, is irrelevant for strict effaceability.

(2) However, the "key lemma" we axiomatized in **COH2** and **SUB2** is unsatisfactory, because it is not obvious how to check it in practice. Moreover, **COH2** need not imply **SUB2** even for fibrant substrata. In subsections 5.3 and 5.4, we introduce stronger axioms that do not have this defect.

5.2. **Spectra.**

In order to include algebraic K-theory in the formalism of this section, it is necessary to consider substrata with values not only in complexes, but also in the category $\mathcal{E}$ of *spectra* in the sense of algebraic topology. We refer to [4, § 2] for the definition of a suitable such category $\mathcal{E}$, provided with an appropriate closed model structure (fibrations, cofibrations, weak equivalences). Recall (e.g. [50, 5.32]) that the Dold-Kan correspondence gives rise to an embedding DK of the category of complexes of abelian groups into the category of spectra such that $\pi_n(DK(C^\cdot)) = H^{-n}(C^\cdot)$ for any complex $C^\cdot$ of abelian groups.

The preceding subsection "extends" to substrata with values in spectra by

- replacing "complexes of objects of $\mathcal{A}$" by "objects of $\mathcal{E}$" in definition 5.1.1 b);
- defining $h^q_Z(X) = \pi_{-q}(C_Z(X))$ in (5.1);
- replacing the cases in Lemma 5.1.4 b) by "For all $X \in \mathcal{S}_k$, the spectrum $C(X)$ is fibrant and cofibrant."
- replacing "Lemma C.1.4 b)" in the proof of Lemma 5.1.4 b) by "the following fact: a weak equivalence between two fibrant and cofibrant spectra is a homotopy equivalence".
- replacing remark 5.1.5 (2) by "For any substratum $C : \mathcal{S}_k \to \mathcal{E}$, there exist two substrata C', C'' and natural transformations $C \xrightarrow{\varphi} C' \xleftarrow{\varphi'} C''$ such that for all X,
  (a) $C'(X)$ is fibrant;

(b) $C''(X)$ is fibrant and cofibrant;
(c) φ_X, φ'_X are weak equivalences.
This follows from the folklore result, for which we have no reference, that any map between objects of $\mathcal{E}$ can be factored *in a functorial way* into a cofibration followed by a trivial fibration, and also into a trivial cofibration followed by a fibration (small object argument)."

In the next sections, we shall allow substrata C to take their values in spectra, and comment on this only when necessary.

5.3. **Homotopy invariance.**

In this subsection we discuss two new axioms **COH3/SUB3** and **COH4/SUB4** for cohomology theories/substrata. As we shall see in section 7, the axiom **COH3/SUB3** below is satisfied by many theories. Axiom **COH4/SUB4** is auxiliary and merely serves to give a smooth proof that **COH3** $\Rightarrow$ **COH2** (resp. **SUB3** $\Rightarrow$ **SUB2**).

We assume that $\mathcal{S}_k$ satisfies assumption 5.1.6.

COH3 (Homotopy invariance for cohomology). Let V, π be as in axiom **COH2**. Then $h^q_F(V) \xrightarrow{\pi^*} h^q_{\mathbf{A}^1_F}(\mathbf{A}^1_V)$ is an isomorphism for all q.
COH4 (Rigidity for cohomology). Let $V, \tilde{\pi}, F$ be as in axiom **COH2**, and let s_0, s_∞ be the sections at 0 and ∞ of $\tilde{\pi}$. Then $s_0^*, s_\infty^* : h^q_{\mathbf{P}^1_F}(\mathbf{P}^1_V) \to h^q_F(V)$ coincide for all q.

SUB3 (Homotopy invariance for substrata). Let V, π be as in axiom **COH2**. Then $C(V) \xrightarrow{\pi^*} C(\mathbf{A}^1_V)$ is a homotopy equivalence.
SUB4 (Rigidity for substrata). Let $V, \tilde{\pi}, F$ be as in axiom **COH2**, and let s_0, s_∞ be the sections at 0 and ∞ of $\tilde{\pi}$. Then $s_0^*, s_\infty^* : C_{\mathbf{P}^1_F}(\mathbf{P}^1_V) \to C_F(V)$ are homotopic.

Lemma 5.3.1. *Let h^* be the cohomology theory with supports associated to the substratum C.*
a) Axiom **SUB3** *implies axiom* **COH3***.*
b) If $C(X)$ is fibrant and cofibrant for all X, then axiom **COH3** *implies axiom* **SUB3***.*

Proof. Part a) : Axiom **SUB3** implies the same property for substrata with support, by the same argument as in the proof of Lemma 5.1.2. Part b): the proof is the same as for Lemma 5.1.4. □

Proposition 5.3.2. *a) Axiom* **COH3** *implies axiom* **COH2** *and axiom* **COH2** *implies axiom* **COH4***.*
b) Axiom **SUB3** *implies axiom* **SUB2** *and axiom* **SUB2** *implies axiom* **SUB4***.*

Proof. Part a): compare remark 4.1.4. Part b) is analogous but we give a detailed proof for the convenience of the reader. First we show that **SUB2** implies **SUB4**. Generally, for $x \in \mathbf{P}^1(k)$ (resp. $x \in \mathbf{A}^1(k)$), let us denote by s_x (resp. s'_x) the section of $\tilde{\pi}$ (resp. of π)

determined by x. We can complete the diagram of **SUB2** as

$$\begin{array}{ccc}
C_F(V) & & \\
{\scriptstyle s'^*_0}\uparrow & & \\
C_{\mathbf{A}^1_F}(\mathbf{A}^1_V) & & \\
 & \nwarrow{\scriptstyle \pi^*} & \\
{\scriptstyle j^*}\uparrow & & C_F(V). \\
 & \nearrow{\scriptstyle s^*_\infty} & \\
C_{\mathbf{P}^1_F}(\mathbf{P}^1_V) & &
\end{array} \tag{5.2}$$

It is then clear that the vertical composition is s_0^* while the right composition is s_∞^*.

We now show that **SUB3** implies **SUB4**. Axiom **SUB3** implies that s'^*_x is a homotopy inverse of π^* for all $x \in \mathbf{A}^1(k)$. Consider now the inclusion $j_1 : \mathbf{A}^1_X \hookrightarrow \mathbf{P}^1_X$ given by $t \mapsto t/(t-1)$. We have $j_1(0) = 0$ and $j_1(1) = \infty$, or in other words:

$$j_1 \circ s'_0 = s_0, \qquad j_1 \circ s'_1 = s_\infty.$$

Since s'^*_0 and s'^*_1 are homotopic, it follows that s_0^* and s_∞^* are homotopic.

Finally, we show that **SUB3** implies **SUB2**. Using **SUB3** $\Rightarrow$ **SUB4**, we may replace, up to homotopy, s_∞^* by s_0^* in diagram 5.2, which then becomes obviously commutative. But **SUB3** implies that s'^*_0 is a homotopy equivalence, hence the triangle of **SUB2** is homotopy commutative as desired. □

5.4. **Cohomology of $\mathbf{P}^1$.**

In order to express our axiom on the cohomology of $\mathbf{P}^1$, we need to introduce more material. We still assume $\mathcal{S}_k$ to satisfy assumption 5.1.6.

a) *Cohomology theories*. We suppose given a cohomology theory h^*, a cohomology theory e^* and, for any $(X, Z) \in \mathcal{P}_k$, a map

$$\operatorname{Pic} X \to \operatorname{Hom}(e^*_Z(X), h^*_Z(X))$$

which is natural in (X, Z) (we do not require this map to be additive). Taking $X = \mathbf{P}^1_V$, $Z = \mathbf{P}^1_F$, we get a homomorphism

$$e^*_{\mathbf{P}^1_F}(\mathbf{P}^1_V) \xrightarrow{[\mathcal{O}(1)]-[\mathcal{O}]} h^*_{\mathbf{P}^1_F}(\mathbf{P}^1_V)$$

hence, composing with $\tilde{\pi}^*$, a homomorphism

$$e^*_F(V) \xrightarrow{\alpha_{(V,F)}} h^*_{\mathbf{P}^1_F}(\mathbf{P}^1_V)$$

natural in (V, F).

COH5 (cohomology of $\mathbf{P}^1$, cohomological version) Let $V, F, \tilde{\pi}$ be as in axiom **COH2**. Then the natural map

$$h^q_F(V) \oplus e^q_F(V) \xrightarrow{(\tilde{\pi}^*, \alpha_{(V,F)})} h^q_{\mathbf{P}^1_F}(\mathbf{P}^1_V)$$

is an isomorphism for all q.

b) *Substrata*. We suppose given a substratum C, a substratum D and, for any $X \in \mathcal{S}_k$, a map

$$\operatorname{Pic} X \to \operatorname{Hom}_{\mathcal{E}}(D(X), C(X)) \tag{5.3}$$

natural in X, where $\mathcal{E}$ is either the category of complexes of objects of our abelian category $\mathcal{A}$ or the category of spectra of subsection 5.2. Taking $X = \mathbf{P}^1_V$ we get a map (for spectra, in the stable homotopy category)

$$D(\mathbf{P}^1_V) \xrightarrow{[\mathcal{O}(1)]-[\mathcal{O}]} C(\mathbf{P}^1_V)$$

hence, composing with $\tilde{\pi}^*$, a map (for spectra, in the stable homotopy category)

$$D(V) \xrightarrow{\alpha_V} C(\mathbf{P}^1_V)$$

natural in V.

SUB5 (cohomology of $\mathbf{P}^1$, substratum version) Let $V, \tilde{\pi}$ be as in axiom **COH2**. Then the natural map (for spectra, in the stable homotopy category)

$$C(V) \oplus D(V) \xrightarrow{(\tilde{\pi}^*, \alpha_V)} C(\mathbf{P}^1_V)$$

is a homotopy equivalence.

(To be correct, we should use wedge $\vee$ rather than direct sum $\oplus$ in **SUB5** when C and D are given by spectra.)

Remarks 5.4.1.

(1) The map (5.3) induces, by functoriality, a map on cones

$$\operatorname{Pic} X \to \operatorname{Hom}_{\mathcal{E}}(D_Z(X), C_Z(X))$$

for any $(X, Z) \in \mathcal{P}_k$. Hence we get a map (for spectra, in the stable homotopy category)

$$C_F(V) \oplus D_F(V) \xrightarrow{(\tilde{\pi}^*, \alpha_V)} C_{\mathbf{P}^1_F}(\mathbf{P}^1_V)$$

generalizing that of axiom **SUB5**, and the latter implies by the usual argument (cf proof of Lemma 5.1.2) that this generalized map is a homotopy equivalence as well.

(2) Axiom **COH5** implies that the cohomology theory e^* is uniquely determined by h^* up to isomorphism. For example, e^* verifies Zariski (resp. étale) excision if h^* does. Similarly, axiom **SUB5** implies that D is uniquely determined by C up to homotopy. But the action of Pic is not determined by these axioms in an obvious way.

Lemma 5.4.2. *Let h^* be the cohomology theory with supports associated to the substratum C and e^* the cohomology theory with supports associated to the substratum D.*
a) Axiom **SUB5** *implies axiom* **COH5***.*
b) If $C(X)$ is fibrant and cofibrant for all X, then axiom **COH5** *implies axiom* **SUB5***.*

Proof. a) follows from remark 5.4.1 (1); b) is proven as in Lemma 5.1.4 b). □

Proposition 5.4.3. *a) Axiom* **COH5** *implies axiom* **COH2***.*
b) Axiom **SUB5** *implies axiom* **SUB2***.*

Proof. For a), compare proof of Lemma 4.1.3. Part b) is similar but we give a detailed proof, as in subsection 5.3. By remark 5.4.1, we are reduced to checking that in the diagram

$$\begin{array}{ccccc}
 & & C_{\mathbf{A}^1_F}(\mathbf{A}^1_V) & & \\
 & & & \nwarrow{\scriptstyle \pi^*} & \\
 & & j^*\uparrow & & C_F(V) \\
 & & & \nearrow{\scriptstyle s_\infty^*} & \\
C_F(V)\oplus D_F(V) & \xrightarrow{(\tilde\pi^*,\alpha_V)} & C_{\mathbf{P}^1_F}(\mathbf{P}^1_V) & &
\end{array}$$

the two paths from $C_F(V)\oplus D_F(V)$ to $C_{\mathbf{A}^1_F}(\mathbf{A}^1_V)$ are homotopic. It is enough to check this on both components $C_F(V)$ and $D_F(V)$. On $C_F(V)$ this is trivial (the two paths are actually equal). On $D_F(V)$, the two paths are nullhomotopic, because the pull-backs of $\mathcal{O}(1)$ by s_∞ and j are both trivial. □

5.5. Generating new theories out of old.

The following remarks show how to construct some strictly effaceable cohomology theories and substrata. Here substrata take their values either in $C(\mathcal{A})$, where $\mathcal{A}$ is a suitable abelian category, or in the category of spectra $\mathcal{E}$ of subsection 5.2.

(1) Let $\mathcal{S}_k = Var/k$, let h^* (resp. C) be a cohomology theory with supports (resp. a substratum) over $\mathcal{S}_k$, and let $T \in Var/k$. Define a new cohomology theory with supports h^T (resp. substratum C^T) by

$$(h^T)^*_Z(X) = h^*_{Z\times_k T}(X\times_k T)$$

(resp.

$$C^T(X) = C(X\times_k T)).$$

Assume that h^* (resp. C) satisfies axiom **COH1** (resp. **SUB1**) (for all k-schemes). Then h^T (resp. C^T) also does. This is obvious.

Suppose now that h^* (resp. C) satisfies axiom **COHi** (resp. **SUBi**) for some $i \in \{2,3,5\}$, not only for open subsets of $\mathbf{A}^n_k$ but for any $V \in Var/k$. Then the same holds for h^T (resp. C^T). This is equally obvious.

(2) Let $C \xrightarrow{f} C'$ be a morphism of substrata, and let C'' be the homotopy fibre of f. Then, for $i = 1, 3$, if two among C, C', C'' verify axiom **SUBi**, so does the third. This is *not* clear (and probably wrong) for axioms **SUB2** and **SUB4**, or for "strictly effaceable".

As for axiom **SUB5**, the following holds:
Let D, D' be the substrata attached respectively to C and C' in **COH5**. Assume given a morphism $f : D \to D'$ such that, for any $X \in \mathcal{S}_k$ and $\alpha \in \operatorname{Pic} X$, the diagram

$$\begin{array}{ccc} D(X) & \xrightarrow{\alpha_*} & C(X) \\ {\scriptstyle f_X}\downarrow & & {\scriptstyle f_X}\downarrow \\ D'(X) & \xrightarrow{\alpha_*} & C'(X) \end{array}$$

commutes. Let D'' be the homotopy fibre of $D \xrightarrow{f} D'$. From the assumption above, we get a natural transformation

$$\operatorname{Pic} X \to \operatorname{Hom}(D''(X), C''(X)).$$

Then, if two among the pairs $(C, D), (C', D'), (C'', D'')$ (together with the actions of Pic) verify axiom **COH5**, so does the third.

(3) Let $(h_\alpha)_{\alpha \in A}$ (resp. $(C_\alpha)_{\alpha \in A}$) be a filtered direct system of cohomology theories with supports (resp. substrata) and $h = \varinjlim h_\alpha$ (resp. $C = \varinjlim C_\alpha$). If all h_α (resp. all C_α) verify axiom **COHi** for some i (resp. **SUBi** for $i = 1, 3$), then so does h^* (resp. C). The same claim for **SUB2** and **SUB4** in the case of substrata is not clear. As for **COH5**, we must request that the D_α attached to the C_α form a direct system compatible with that of the C_α via the actions of Pic.

(4) Let C be a substratum, and suppose given a direct system of substrata

$$\cdots \to C^{(n)} \to C^{(n+1)} \to \dots \qquad (n \geq 0)$$

with a homotopy equivalence $\varinjlim C^{(n)} \xrightarrow{\sim} C$. Suppose that $C^{(0)}$ and, for all n, the homotopy fibre of $C^{(n)} \to C^{(n+1)}$ satisfies axiom **SUBi** for $i = 1, 3$ or 5 (for **SUB5**, we request analogous conditions on the Ds, as above). Then so does C. This follows by induction from remarks 2 and 3.

6. Universal exactness

In this section, we want to show how strict effaceability of a substratum (rather than a cohomology theory) implies not only exactness, but even *universal exactness* of the associated Cousin complexes. Recall that a complex $A^\cdot$ is *contractible* if there is a homotopy from the identity to 0 on $A^\cdot$.

6.1. Generalities.

We take from [19] the definition of universal exactness, actually in slightly greater generality:

Definition 6.1.1. Let $\mathcal{A}$ be an abelian category. A complex $C^\cdot$ of objects of $\mathcal{A}$ is *universally exact* if the following condition is satisfied:
For any abelian category $\mathcal{B}$ and any additive functor $T : \mathcal{A} \to \mathcal{B}$ commuting with filtering direct limits, the complex $T(C^\cdot)$ is exact.

(In the case $\mathcal{A}$ is the category of left modules over a ring, one should compare this notion with Lazard's pure sequences [32, Ch. I, §2, esp. Th. 2.3].)

Note that the exactness of $T(C)$ is automatic if T is exact, but we only require it to be *additive*. Here are some examples:

Examples 6.1.2.

(1) A contractible complex is universally exact. Indeed, any additive functor will transform a homotopy into a homotopy.
(2) If $\mathcal{A}$ satisfies AB5, a filtering direct limit of universally exact complexes is universally exact.
(3) Let $C^\cdot : \quad 0 \to C^0 \to \cdots \to C^{n-1} \to C^n \to 0$ be a bounded exact complex, so that $B^i(C^\cdot) \xrightarrow{\sim} Z^i(C^\cdot)$ for all i. Suppose all the exact sequences $0 \to Z^i(C^\cdot) \to C^i \to B^{i+1}(C) \to 0$ are filtering direct limits of split exact sequences. Then $C^\cdot$ is universally exact. This follows from the previous two examples.

Conversely:

Proposition 6.1.3. *Suppose $\mathcal{A}$ satisfies AB5 and any object of $\mathcal{A}$ is a filtering direct limit of finitely presented objects (e.g. $\mathcal{A}$ is the category of left modules over a ring). Then any bounded universally exact complex $C^\cdot$ of objects of $\mathcal{A}$ can be described as in example 6.1.2* (3).

Recall that an object X of $\mathcal{A}$ is of finite presentation if the functor $Y \mapsto \operatorname{Hom}(X,Y)$ commutes with direct limits.

Proof. Applying definition 6.1.1 with $T =$ identity, we see that $C^\cdot$ is exact. Let X be a finitely presented object of $\mathcal{A}$. Applying the functor

$$T(M) = \operatorname{Hom}(X,M)$$

to $C^\cdot$, we see in particular that $\operatorname{Hom}(X,C^{n-1}) \to \operatorname{Hom}(X,C^n)$ is surjective. It follows that, for any $f : X \to C^n$, the pull-back of the exact sequence

$$0 \to Z^{n-1}(C^\cdot) \to C^{n-1} \to C^n \to 0 \tag{D}$$

by f is split. By the assumption in Proposition 6.1.3, (D) is a filtering direct limit of split exact sequences, and in particular is universally exact. This now implies that the sequence

$$0 \to C^0 \to \cdots \to C^{n-2} \to B^{n-1}(C^\cdot) \to 0$$

is universally exact. We get the conclusion by induction on n. □

6.2. **Universal exactness of Cousin complexes.**

Theorem 6.2.1. *Let $\mathcal{S}_k$ satisfy assumption 5.1.6. Let $X \in \mathcal{S}_k$ be an affine variety, $t_1, \dots, t_r \in X$ a finite set of points and h^* a cohomology theory with supports on $\mathcal{P}_k$. Suppose that h^* is given by a substratum C which is strictly effaceable at $t_1, \dots, t_r$. Then the Cousin complexes*

$$0 \to h^q(Y) \xrightarrow{e} \coprod_{x\in Y^{(0)}} h^q_x(Y) \xrightarrow{d_1^{0,q}} \coprod_{x\in Y^{(1)}} h^{q+1}_x(Y) \xrightarrow{d_1^{1,q}} \dots$$

are universally exact, where $Y = \operatorname{Spec} \mathcal{O}_{X,t_1,\dots,t_r}$.

The proof uses the following well-known lemma:

Lemma 6.2.2. *Let $\mathcal{T}$ be a triangulated category and $A \to B \to C \to A[1]$ an exact triangle in $\mathcal{T}$. Suppose that the map $C \to A[1]$ is 0. Then the map $B \to C$ has a section. ("Every epimorphism is split".)*

Proof. Apply the functor $\operatorname{Hom}(C, ?)$ to the triangle and get an exact sequence

$$\operatorname{Hom}(C,B) \to \operatorname{Hom}(C,C) \xrightarrow{0} \operatorname{Hom}(C,A[1]).$$

which shows that $\operatorname{Hom}(C,B) \to \operatorname{Hom}(C,C)$ is surjective. Let $s : C \to B$ be an element that maps to Id_C. Then by definition, s is a section, as wanted. □

To prove Theorem 6.2.1, we go a little more carefully than in the proof of Proposition 2.1.2. We note that the Cousin complex of Theorem 6.2.1 is obtained by pasting together complexes

$$0 \to h^q_{Y^{(p)}}(Y) \to \coprod_{y \in Y^{(p)}} h^q_y(Y) \to h^{q+1}_{Y^{(p+1)}}(Y) \to 0 \tag{6.1}$$

which in turn are obtained as direct limits of the complexes

$$0 \to h^q_{Z'\cap W}(W) \to h^q_{(Z'-Z)\cap W}(W \setminus Z) \to h^{q+1}_{Z\cap W}(W) \to 0 \tag{6.2}$$

coming from the long exact cohomology sequence of definition 5.1.1. Here W varies among the open neighbourhoods of $(t_1, \dots, t_r)$ and $Z \subseteq Z' \subseteq W$ vary among closed subsets of codimensions respectively $\geq p$ and $\geq p+1$.

Lemma 6.2.3. *Let $t_1, \dots, t_r$ and Y be as in Theorem 6.2.1. Suppose the substratum C is strictly effaceable at $t_1, \dots, t_r$. Then for any $p \geq 0$ and $q \in \mathbf{Z}$, the complex* (6.1) *is a direct limit of split exact sequences, where h^* is the cohomology theory associated to C. In particular, it is universally exact.*

Proof. Let $W \subseteq X$ be an open neighbourhood of $(t_1, \dots, t_r)$, $Z \subseteq W$ a closed subset of codimension $\geq p$ and take U, Z' as given by definition 5.1.8. In the triangulated category $K(\mathcal{A})$ of complexes of objects of $\mathcal{A}$ up to homotopy ([26, chap. I] and [20, § 5]), or in the homotopy category of $\mathcal{E}$ if C is given by spectra [4], consider the triangle or fibre sequence

$$C_{Z'\cap U}(U) \to C_{(Z'-Z)\cap U}(U \setminus Z) \to C_{Z\cap U}(U)[1] \xrightarrow{0} C_{Z'\cap U}(U)[1].$$

Here $C_{Z'\cap U}(U)[1]$ means $\Sigma C_{Z'\cap U}(U)$ if C is given by spectra. Lemma 6.2.2 shows that the map $C_{(Z'-Z)\cap U}(U \setminus Z) \to C_{Z\cap U}(U)[1]$ has a homotopy section. Correspondingly, the sequence

$$0 \to h^q_{Z'\cap U}(U) \to h^q_{(Z'-Z)\cap U}(U \setminus Z) \to h^{q+1}_{Z\cap U}(U) \to 0$$

is split exact for all q. And such sequences are cofinal in the direct system of complexes (6.2). □

Corollary 6.2.4. *Suppose k is infinite and $\mathcal{S}_k$ verifies assumption 5.1.6. Let h^* be a cohomology theory with supports on $\mathcal{P}_k$, with values in an abelian category $\mathcal{A}$ satisfying axiom AB5 and having a generator. If h^* satisfies axioms* **COH1** *(étale excision) and either* **COH3** *(homotopy invariance) or* **COH5** *(cohomology of $\mathbf{P}^1$), and can be defined by a substratum of complexes or spectra (the latter assuming $\mathcal{A} = \{$abelian groups$\}$), then the Cousin complexes of Theorem 6.2.1 are universally exact for X smooth.*

Proof. If h^* can be defined by a substratum of complexes, it can be defined by a fibrant substratum C by remark 5.1.5 (2). Similarly, for a substratum of spectra, it can be defined by a fibrant and cofibrant substratum by subsection 5.2. By Lemmas 5.1.4 b), 5.3.1 b) and 5.4.2 b), C satisfies axioms **SUB1** and either **SUB3** or **SUB5**, hence axiom **SUB2** by Propositions 5.3.2 and 5.4.3. By Theorem 5.1.10, it is strictly effacable. The corollary now follows from Theorem 6.2.1.

Note that Theorem 6.2.1 does not cover the case of finite fields. For this, we introduce another axiom, which was already used in section 4:

COH6 For any finite field extension ℓ/k and any $(X, Z) \in \mathcal{P}_k$, there is given a map

$$\mathrm{Cor}_{\ell/k} : h^*_{Z_\ell}(X_\ell) \to h^*_Z(X)$$

such that $\mathrm{Cor}_{\ell/k} \circ \mathrm{Res}_{\ell/k} = [\ell : k]$, where $\mathrm{Res}_{\ell/k}$ corresponds to extension of scalars. This map is natural in $(X, Z) \in \mathcal{P}_k$.

Theorem 6.2.5. *Let k be a finite field and h^* a cohomology theory with supports on $\mathcal{P}_k$, with values in an abelian category satisfying axiom AB5 and having a generator. Suppose $\mathcal{S}_k$ verifies assumption 5.1.6 and h^* satisfies axioms* **COH1**, **COH6** *and either* **COH3** *or* **COH5**, *and can be defined by a substratum of complexes or spectra. Then, for any connected smooth affine $X \in \mathcal{S}_k$ and any finite set of points $t_1, \dots, t_r \in X$, the Cousin complexes*

$$0 \to h^q(Y) \xrightarrow{e} h^q_\eta(Y) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} h^{q+1}_x(Y) \xrightarrow{d_1^{1,q}} \dots$$

are universally exact, where $Y = \operatorname{Spec} \mathcal{O}_{X,t_1,\dots,t_p}$.

Proof. Extend C to $\mathcal{S}_K$ for infinite algebraic extensions K/k by setting $C(X) = \varinjlim C(X_0 \otimes_{k_0} \ell)$, where k_0 is a suitable finite subextension of K such that $X = X_0 \otimes_{k_0} K$ for some X_0, and ℓ runs through the finite subextensions of K/k_0. This extends h^* to a cohomology theory with supports on $\mathcal{P}_K$, admitting a substratum and satisfying axioms **COH1**, **COH6** and either **COH3** or **COH5**. By Corollary 6.2.4, the Cousin complexes of Theorem 6.2.1 are universally exact for K-varieties.

Let T be an additive functor (with values in some abelian category satisfying AB5) which commutes with filtering direct limits. We have to prove that the complex

$$0 \to T(h^q(Y)) \xrightarrow{e} T(h^q_\eta(Y)) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} T(h^{q+1}_x(Y)) \xrightarrow{d_1^{1,q}} \dots \tag{6.3}$$

is acyclic, for X a smooth k-variety and Y as in Theorem 6.2.1. We use the same trick as in section 4. Let p_1, p_2 be two different primes and K_1, K_2 the $\mathbf{Z}_{p_1}$ and $\mathbf{Z}_{p_2}$-extensions of k respectively. Let A be some homology group of the sequence (6.3). For an algebraic extension K/k, let (6.3)$_K$ denote (6.3) "pushed over K". By assumption on T, we have

$$A_K = \varinjlim A_\ell$$

where ℓ runs through finite subextensions of K/k. On the other hand, the transfer condition shows that, if $[K : k] = N < +\infty$, then

$$N \operatorname{Ker}(A \to A_K) = 0.$$

It follows that $\operatorname{Ker}(A \to A_{K_i})$ is p_i-primary torsion and therefore $\operatorname{Ker}(A \to A_{K_1} \oplus A_{K_2}) = 0$. Finally, since K_1 and K_2 are infinite, we have $A_{K_1} = A_{K_2} = 0$ as observed above. So $A = 0$, as was to be proven. □

7. Examples

7.1. **Hypercohomology of sheaves.**

In this subsection as in subsection 7.5, the category $\mathcal{S}_k$ need not satisfy assumption 5.1.6.

7.1.1. Let ν be a Grothendieck topology on $\mathcal{S}_k$. To a complex of sheaves of abelian groups $\mathcal{C}$ over ν one can associate a cohomology theory with supports h^*, given by the ν-hypercohomology of $\mathcal{C}$ with supports:

$$h^*_Z(X) = \mathbb{H}^*_Z(X_\nu, \mathcal{C}).$$

7.1.2. Let $f : \mathcal{C} \to \mathcal{C}'$ be a morphism. Then f induces a morphism f_* of associated cohomology theories. If f is a quasi-isomorphism, f_* is an isomorphism in the following two cases:

- $\mathcal{C}$ and $\mathcal{C}'$ are bounded below;
- for all $X \in \mathcal{S}_k$, the ν-cohomological dimension of X is finite.

Indeed, we have a morphism of hypercohomology spectral sequences:

$$\begin{array}{ccc} {}'E_2^{p,q} = H^p_Z(X_\nu, \mathcal{H}^q(\mathcal{C}')) & \Rightarrow & \mathbb{H}^{p+q}_Z(X_\nu, \mathcal{C}') \\ {\scriptstyle f_*}\uparrow & & {\scriptstyle f_*}\uparrow \\ E_2^{p,q} = H^p_Z(X_\nu, \mathcal{H}^q(\mathcal{C})) & \Rightarrow & \mathbb{H}^{p+q}_Z(X_\nu, \mathcal{C}) \end{array}$$

which is an isomorphism on E_2-terms by assumption. Here, $\mathcal{H}^q(\mathcal{C})$ and $\mathcal{H}^q(\mathcal{C}')$ are the cohomology sheaves of $\mathcal{C}$ and $\mathcal{C}'$ respectively. In both cases, the two spectral sequences converge, hence the map on abutments is an isomorphism.

7.1.3. A cohomology theory given by a complex of ν-sheaves $\mathcal{C}$ can always be defined by a substratum C of complexes of abelian groups (this is in fact the way ν-hypercohomology is defined!) There are several well-known constructions for C:

- Suppose $\mathcal{C}$ is bounded below. Choose a Cartan-Eilenberg injective resolution $\mathcal{I}$ of $\mathcal{C}$ and define $C(X) = Tot(\mathcal{I})(X)$, where $Tot(\mathcal{I})$ is the total complex associated to the bicomplex $\mathcal{I}$. Note that the terms of $C(X)$ are injective abelian groups and $C(X)$ is bounded below. Hence $C(X)$ is fibrant in the sense of definition C.1.1 (compare Proposition C.1.2).
- Replace $\mathcal{C}$ by a fibrant complex of sheaves $\mathcal{F}$. By Theorem C.3.1, we can do this functorially in $\mathcal{C}$. Define now $C(X)$ as $\mathcal{F}(X)$; note that $C(X)$ is fibrant for all X as a complex of abelian groups. This construction does not require $\mathcal{C}$ to be bounded below.
- The Godement resolution. Suppose the topos associated to ν has enough points (this is the case for Zariski, Nisnevich, étale, complex topologies). To $\mathcal{C}$ one associates a new complex of sheaves

$$T\mathcal{C} : U \mapsto \prod_{f\in\Pi} \prod_{f^*(U)} f^*\mathcal{C}$$

where Π is the set of points of ν (compare [50, 1.31], especially for set-theoretic problems). The terms of this complex are flabby in the sense of [36, ex. III.1.9 (c)]. Iterating T yields a cosimplicial complex of flabby sheaves $T^{\cdot}\mathcal{C}$, which in turn yields a bicomplex of flabby sheaves $T^*\mathcal{C}$ in the usual way. One defines $C(X) = Tot(T^*\mathcal{C})(X)$. This is essentially the object denoted by $\mathbb{H}^{\cdot}(X, \mathcal{C})$ in [50].

By the usual arguments, there is a commutative diagram (for $\mathcal{C}$ bounded below)

$$\begin{array}{ccc} & & Tot(\mathcal{I}) \\ & \nearrow & \\ \mathcal{C} & & \uparrow \\ & \searrow & \\ & & Tot(T^*\mathcal{C}) \end{array}$$

in which the vertical map induces a quasi-isomorphism on global sections.

The last two constructions are natural in $\mathcal{C}$. All constructions have the following virtue: if $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$ defines an exact triangle in the derived category of ν-sheaves, then so does $C'(X) \to C(X) \to C''(X)$, for all X, in the derived category of abelian groups, where C', C, C'' are the associated substrata.

7.1.4. Let C be associated to $\mathcal{C}$ as in 7.1.3, and suppose we sheafify it for the ν-topology. In the first construction C is already the complex of sheaves $Tot(\mathcal{I})$. In the second one, the stalk of $T^{\cdot}\mathcal{C}$ at a point x is homotopy equivalent to the constant cosimplicial complex of abelian groups defined by $\mathcal{C}_x$. In both cases, the resulting complex of sheaves is quasi-isomorphic to $\mathcal{C}$.

7.1.5. Instead of taking complexes of sheaves of abelian groups, one can take complexes of sheaves with values in an abelian category with enough injectives, or *sheaves of spectra* [50] in the line of 5.2. All the above holds in these contexts, mutatis mutandis. In the case of spectra, for 7.1.2 use the spectral sequence of [50, prop. 1.36]. For the second construction in 7.1.3, use [50, def. 1.33]. Nisnevich repeats these constructions in [38], because he uses a different notion of point of a topos for the Nisnevich topology.

7.1.6. Let ν' be another Grothendieck topology on $\mathcal{S}_k$ which is finer than ν. Then the identity functor of $\mathcal{S}_k$ defines a morphism of sites $\nu' \xrightarrow{\alpha} \nu$. If $\mathcal{C}$ is a complex of sheaves for the ν' topology, there is an isomorphism

$$\mathbb{H}^*_Z(X_{\nu'}, \mathcal{C}) \xrightarrow{\sim} \mathbb{H}^*_Z(X_\nu, R\alpha_*\mathcal{C})$$

where $R\alpha_*\mathcal{C}$ is the total direct image of $\mathcal{C}$ (in the derived category). In the case $\mathcal{C}$ is a sheaf of spectra, one should use the object $\mathbb{R}^{\cdot}\alpha\mathcal{C}$ of [50, def. 1.55] instead of $R\alpha_*\mathcal{C}$, cf [50, th. 1.56]. So we can view ν'-hypercohomology as ν-hypercohomology.

7.1.7. Suppose that ν is the big Zariski site of $\operatorname{Spec} k$. Then ν-hypercohomology of $\mathcal{C}$ verifies Zariski excision. Similarly, suppose that ν is the big Nisnevich site Nis on $\operatorname{Spec} k$. Then ν-hypercohomology verifies étale excision, i.e. axiom **COH1**. This is known when $\mathcal{C}$ is reduced to a single sheaf (for the Nisnevich case, cf [8, prop. 4.4], which applies to Nisnevich cohomology; recall that the proofs of [36, prop. III.1.27] and [38, th. 1.27] have a gap). In general, the proof follows from a comparison of convergent hypercohomology spectral sequences, as in 7.1.2. The two spectral sequences converge without boundedness conditions on $\mathcal{C}$, because the Zariski or Nisnevich cohomological dimensions of k-schemes of finite type are finite [38]. See [50, ex. 1.49] for Zariski excision in the case of a sheaf of spectra (in [38] Nisnevich does not give the corresponding statement for étale excision explicitly).

7.1.8. By 7.1.6 and 7.1.7, ν-hypercohomology satisfies Zariski (resp. étale) excision as soon as ν is finer than the Zariski (resp. the Nisnevich) topology.

7.2. **Generating new theories out of old, continued.**

Let $\mathcal{C}$ be as in 7.1.1, let A be a bounded below complex of abelian groups, viewed as a complex of constant Nisnevich sheaves, and let $\mathcal{C}' = \mathcal{C} \overset{L}{\otimes} A$ (in the derived category). Then, if the cohomology theory associated to $\mathcal{C}$ verifies axiom **COH3** or **COH5**, the same is true for $\mathcal{C}'$. In the case of axiom **COH5**, if $\mathcal{D}$ is a complex of sheaves associated to $\mathcal{C}$, we associate to $\mathcal{C}'$ the complex $\mathcal{D}' = \mathcal{D} \overset{L}{\otimes} A$ and take for the action of Pic the original action tensored by A. The claim can be justified in a few steps:

(a) $A = \mathbf{Z}[0]$. This is trivial.
(b) $A = \mathbf{Z}/n$. Follows from the previous case, item (2) of subsection 5.5 , 7.1.3 and the exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/n \to 0$.
(c) A consists of a single finitely generated abelian group placed in degree 0. Follows from the previous cases.
(d) A consists of a single abelian group placed in degree 0. Follows from the previous case and item (3) of subsection 5.5 by a passage to the limit.
(e) The general case. Follows from the previous case and item (4) of subsection 5.5.

In case $\mathcal{C}$ is a sheaf of spectra as in 7.1.5, one has the same by taking for $\mathcal{D}$ the sheaf of spectra $\mathcal{C} \wedge A$, where A is an arbitrary spectrum, viewed as a constant sheaf of spectra.

The method is the same, reducing to the case where A is an Eilenberg-Mac Lane spectrum by dévissage from its Postnikov tower, cf [50, proof of th. 1.46].

7.3. **Homotopy invariant examples.**

In all examples of this subsection, $\mathcal{S}_k = Var/k$.

(1) *Etale cohomology with coefficients in a sheaf defined over k and torsion prime to the characteristic of k.* Etale excision follows from 7.1.8 and homotopy invariance from [36, cor. VI.4.20]. More generally, by 7.1.8 and item (4) of subsection 5.5, one may take étale hypercohomology of a bounded below complex of sheaves whose cohomology is torsion prime to the characteristic of k.

(2) (over **C**:) *Classical hypercohomology with coefficients in a bounded below complex of abelian groups*. Here, étale excision again comes from 7.1.8 and the fact that, for X a **C**-variety, the topological space $X(\mathbf{C})$ essentially maps to the small étale site of X (cf [SGA4-III, exposé XI, 4.0]). Homotopy invariance is known for **Z** as coefficients, and the general case follows from remark 7.2.

(3) ($\operatorname{char} k = 0$:) *De Rham cohomology*. Recall that, for a k-variety X, $H^*_{dR}(X/k) = \mathbb{H}^*_{\mathrm{Zar}}(X, \Omega^\cdot_{X/k})$, where $\Omega^\cdot_{X/k}$ is the de Rham complex. To check étale excision, we note that, since the $\Omega^i_{X/k}$ are coherent sheaves, the maps $\mathbb{H}^*_{\mathrm{Zar}}(X, \Omega^\cdot_{X/k}) \to \mathbb{H}^*_{\mathrm{Nis}}(X, \Omega^\cdot_{X/k})$ are isomorphisms [36, remark III.3.8], so we can apply 7.1.8. Actually, since $\operatorname{char} k = 0$, we even have purity [25]. Homotopy invariance is proven in [25, remark p. 54].

(4) *Motivic cohomology*. Let k admit resolution of singularities in the sense of [12, def. 3.4] (for example, $\operatorname{char} k = 0$), and let $i \geq 0$. In [49, section 2], motivic cohomology of weight i is defined by $H^*_Z(X, \mathbf{Z}(i)) = \mathbb{H}^*_Z(X_{\mathrm{cdh}}, \mathbf{Z}(i)_{\mathrm{cdh}})$ for $(X, Z) \in \mathcal{P}_k$, where cdh is the Grothendieck topology introduced in [12, def. 3.2], $\mathbf{Z}(i)$ is a certain complex of presheaves with transfers with homotopy invariant cohomology presheaves in the sence of [52] and $\mathbf{Z}(i)_{\mathrm{cdh}}$ is its sheafification for the cdh topology. Therefore motivic cohomology is given by a substratum, is homotopy invariant and satisfies étale excision since the cdh topology is stronger than the Nisnevich topology. It also satisfies purity by [49, prop. 2.4].

(5) *Cycle modules*. Let M_* be a cycle module in the sense of [44]. For $Z \in Var/k$ and $j \in \mathbf{Z}$, denote by $C.(Z, M_j)$ the (homological) Gersten complex associated to Z and ending with $\coprod_{x \in X_{(0)}} M_j(k(x))$. For X of pure dimension d, define

$$C(X) = C^\cdot(X, M_j) = C.(X, M_{j-d})$$

viewed as a cohomological complex. For $Z \subset X$ a closed subset, there is an obvious short exact sequence of complexes

$$0 \to C.(Z, M_{j+d}) \to C^\cdot(X, M_j) \to C^\cdot(X - Z, M_j) \to 0.$$

So $C.(Z, M_{j+d})$ is homotopy equivalent to the homotopy fibre $C_Z(X)$. In particular, the substratum C verifies étale excision (axiom **SUB 1**), and even purity. By a result of Rost [44, prop. 8.6], it is also homotopy invariant. Therefore, the Gersten complexes on a smooth semi-local scheme are universally exact ([44, th. 6.1] for the exactness). Universal exactness actually follows directly from replacing the cycle module M_* by $T \circ M_*$, where T is a given additive functor which commutes with filtering direct limits.

This example includes as a special case Milnor's K-groups, Milnor's K-groups mod m, m-torsion in Milnor's K-groups ...

(6) *Algebraic G- (= K'-)theory*. This is the only case in this list of examples where the substratum is given by spectra, not complexes. Etale excision **COH1** is implied by the much stronger *localization theorem* of Quillen, akin to purity [43, prop. 7.3.2]. Homotopy invariance **COH3** also follows from Quillen [43, prop. 7.4.1]. By smashing the algebraic G-theory spectrum by the Moore spectrum $M(\mathbf{Z}/n)$, we get the case of algebraic G-theory with coefficients $\mathbf{Z}/n$ (compare subsection 7.2).

7.4. Non homotopy invariant examples.

In all examples of this subsection, $\mathcal{S}_k = Var/k$, except in examples (4) and (5) where $\mathcal{S}_k = Sm/k$.

(1) *Etale (hyper-)cohomology with bounded below coefficients coming from k*. As above, étale excision follows from 7.1.8. Axiom **COH5** is proven in Appendix A. More precisely, in subsection A.2, we define an étale sheaf $\mathbf{Q}/\mathbf{Z}(-1)$ (over the big étale site of $\operatorname{Spec} \mathbf{Z}$) and a map

$$\operatorname{Pic} X \to \Gamma(X, R\operatorname{Hom}_{\acute{e}t}(\mathbf{Q}/\mathbf{Z}(-1)[-3], \mathbf{Z})) \tag{7.1}$$

for any scheme X. Let now $\mathcal{C}_0$ be a bounded below complex of sheaves over the *small* étale site of $\operatorname{Spec} k$, and $\mathcal{C}$ its inverse image to the *big* étale site. Let h^* be the cohomology theory with supports defined by $\mathcal{C}$ and e^* the cohomology theory with supports defined by $\mathcal{C} \overset{L}{\otimes} \mathbf{Q}/\mathbf{Z}(-1)[-3]$. The map (7.1) induces a map $\operatorname{Pic} X \to \operatorname{Hom}(e, h)$, and we show in subsection A.3 that this map satisfies axiom **COH5**.

(2) *Hodge and de Rham cohomology in any characteristic*. Etale excision is seen as above. Axiom **COH5** is due to Illusie: if h^* is the cohomology theory associated to the Zariski sheaf $\Omega^i_{X/k}$, then axiom **COH5** holds for h^* with e^* associated to $\Omega^{i-1}_{X/k}[-1]$ (i.e. $e^j_Z(X) = H^{j-1}_Z(X_{\mathrm{Zar}}, \Omega^{i-1}_{X/k})$). Here the map $\operatorname{Pic} X \to \operatorname{Hom}(e, h)$ is given by cup product with the first Chern class, defined through the map $H^1_{\mathrm{Zar}}(X, \mathcal{O}^*_X) \xrightarrow{d\log} H^1_{\mathrm{Zar}}(X, \Omega^1_{X/k})$. Using item (4) of subsection 5.5, one can then extend axiom **COH5** to the de Rham-Witt complex itself, or truncations of it. Compare [21, p. 22, proof of (4.2.7)]. Note that X need not be smooth since, in [SGA7, exposé XI, th. 1.1], X is arbitrary. Note also that, even in characteristic 0, Hodge cohomology is not homotopy invariant. In characteristic p one can then "escalate the ladder" to get the same result for Deligne-Illusie's Hodge-Witt and de Rham-Witt cohomology ([28], compare [21]).

(3) ($\operatorname{char} k = p$) *Logarithmic Hodge-Witt and de Rham-Witt cohomology* [22]. Etale excision is proven as above. Axiom **COH5** follows from [21, th. I.2.1.11] and is proven there in the spirit of item (2) of subsection 5.5, using the description of the logarithmic de Rham-Witt pro-complex as Frobenius fixed points of the de Rham-Witt pro-complex [21, I. (1.3.2)] .

(4) *Cohomology of a torus*. Let $\mathbb{T}$ be a k-torus. Consider the cohomology theory with supports h^* given by the sheaf associated to $\mathbb{T}$ on the big étale site of $\operatorname{Spec} k$. As above, h^* satisfies étale excision. Let $M = \operatorname{Hom}(\mathbb{G}_m, \mathbb{T})$ be the group of cocharacters of

$\mathbb{T}$, also viewed as a big étale sheaf. Let e^* be the cohomology theory with supports given by

$$e^q_Z(X) = H^{q-1}_Z(X_{ét}, M).$$

Cup-product defines a map

$$\operatorname{Pic} X \to \operatorname{Hom}_X(e, h)$$

for all X. It can be shown that axiom **COH5** is verified for h, e and this natural transformation.

(5) *Etale weight-two motivic cohomology.* For an affine scheme X, let $\Gamma(X, 2)$ be the weight-two motivic complex introduced by Lichtenbaum in [33]. For X smooth, its Zariski sheafification $\Gamma(2)_{\mathrm{Zar}}$ is quasi-isomorphic to $\tau^{>0}\mathbf{Z}(2)_{\mathrm{Zar}}$, where $\mathbf{Z}(2)_{\mathrm{Zar}}$ is the Zariski sheafification of the complex of 7.3 (4). The main steps in the proof of this are [1, th. 7.2], [48] and [53, proof of prop. 4.9 and subsection 4.3]. (Conjecturally $\Gamma(2)_{\mathrm{Zar}}$ and $\mathbf{Z}(2)_{\mathrm{Zar}}$ coincide). It is proven in [31] that the cohomology theory $(X, Z) \mapsto \mathbb{H}^*_Z(X_{ét}, \Gamma(2)_{ét})$ satisfies axiom **COH5**, where $\Gamma(2)_{ét}$ is the étale sheafification of $X \mapsto \Gamma(X, 2)$.

(6) *Algebraic K^B-theory*. Here K^B denotes the Bass extension of Quillen's algebraic K-theory, which coincides with the latter for regular Noetherian schemes, see [51, §6]. Axiom **COH1** is one of the main results of Thomason-Trobaugh: it applies generally to X, X' quasi-compact and quasi-separated such that $X - Z$ is quasi-compact as well [51, Th. 7.1 and 7.4]. It would be wrong for ordinary K-theory in general. Axiom **COH5** follows from [51, Th. 7.3]. Just as in example 7.3 (6), we get algebraic K^B-theory with finite coefficients by smashing with a Moore spectrum.

7.5. **More on hypercohomology and excision.**

This subsection can be considered as a sequel to subsection 7.1.

7.5.1. Let ν be a Grothendieck topology on $\mathcal{S}_k$. Suppose that we now start with a substratum of complexes C. Sheafifying C for the ν-topology, we get a complex of ν-sheaves $\mathcal{C}$. Choosing an injective right Cartan-Eilenberg resolution $\mathcal{I}$ of $\mathcal{C}$, the augmentation $\mathcal{C} \to Tot(\mathcal{I})$ yields an augmentation

$$C(X) \to \Gamma(X, \mathcal{C}) \to \Gamma(X, Tot(\mathcal{I})).$$

Similarly, if ν has enough points, the Godement resolution construction of 7.1.3 gives a natural transformation

$$C(X) \to \mathbb{H}^{\cdot}(X_\nu, \mathcal{C}). \tag{7.2}$$

By analogy with 7.1.4, we may ask the question:

Question. When is (7.2) a quasi-isomorphism?

This problem has no simple solution in general; however we shall explain that it has one when ν is either the Zariski or the Nisnevich topology.

In both cases, an obvious necessary condition is that the cohomology theory h^* associated to C satisfies Zariski (resp. étale) excision, since $\mathbb{H}^{\cdot}(X_\nu, \mathcal{C})$ does by 7.1.7. The remarkable fact is that this condition is sufficient:

Theorem 7.5.1. *Suppose that $\nu = \mathrm{Zar}$ (resp. $\nu = \mathrm{Nis}$). Then* (7.2) *is a quasi-isomorphism if and only if the cohomology theory h^* associated to C satisfies Zariski (resp. étale) excision. In other terms, a cohomology theory with supports which admits a substratum made of complexes satisfies Zariski (resp. étale) excision if and only if it can be defined by Zariski (resp. Nisnevich) hypercohomology of a complex of sheaves.*

This theorem is either a consequence or an easy analogue of

Theorem 7.5.2 (Brown-Gersten–Thomason–Nisnevich)**.** *Suppose that $\nu = \mathrm{Zar}$ (resp. $\nu = \mathrm{Nis}$) and let C be a substratum of spectra over $\mathcal{S}_k$. Then* (7.2) *is a quasi-isomorphism if and only if the cohomology theory h^* associated to C satisfies Zariski (resp. étale) excision. In other terms, a cohomology theory with supports which admits a substratum made of spectra satisfies Zariski (resp. étale) excision if and only if it can be defined by Zariski (resp. Nisnevich) hypercohomology of a sheaf of spectra.*

Proof. See [5] and [50, 2.5] for the Zariski case, [38] for the Nisnevich case. □

By 7.1.4, a complex of sheaves $\mathcal{C}$ defining h^* can be chosen as the sheaf associated to the presheaf $U \mapsto C(U)$, where C is a substratum defining h^*.

Corollary 7.5.3. *Under the conditions of Theorem 7.5.2, there is for any $X \in \mathcal{S}_k$ a spectral sequence*

$$E_2^{p,q} = H^p(X_\nu, \mathcal{H}^q) \Rightarrow h^{p+q}(X).$$

where $\nu = \mathrm{Zar}$ or Nis and $\mathcal{H}^q$ is the ν sheaf associated to the presheaf $U \mapsto h^q(U)$.

Proof. This is just the hypercohomology spectral sequence for the cohomology of $\mathbb{H}^{\cdot}(X_\nu, \mathcal{C})$. □

Example 7.5.4. Let μ be a Grothendieck topology on $\mathcal{S}_k$ which is finer than ν, and let $\alpha : \mu \to \nu$ be the corresponding morphism of sites. Let $\mathcal{D}$ be a complex of sheaves (or sheaf of spectra) for the μ-topology, and take $\mathcal{C} = R\alpha_*\mathcal{D}$ (or $\mathbb{R}^{\cdot}\alpha\mathcal{D}$). There is a canonical quasi-isomorphism (or weak equivalence)

$$\mathbb{H}^{\cdot}(X_\nu, \mathcal{C}) \simeq \mathbb{H}^{\cdot}(X_\mu, \mathcal{D})$$

and we recover the Leray spectral sequence for the morphism α.

Note that, just as the spectral sequence of example 5.1.3 (3), this spectral sequence is defined for arbitrary, not necessarily smooth, $X \in \mathcal{S}_k$. The two spectral sequences have a priori nothing to do with each other. In other words, the comment in [50, last § of p. 467] misses the point. The Bloch-Ogus–Gabber theorem implies that, when X is smooth, they have isomorphic E_2-terms. Moreover, they actually coincide in this case for many theories (Deligne, unpublished, cf [2, footnote p. 195], Gillet-Soulé, [17]). See also Paranjape [41].

7.5.2. Let h^* be the cohomology theory with supports associated to some complex of Nisnevich sheaves $\mathcal{C}$. As seen above, h^* satisfies axiom **COH1**. By 7.1.2, h^* only depends, up to isomorphism, on the class of $\mathcal{C}$ in the *derived category* $\mathcal{D}(\mathrm{Nis})$ of the category of Nisnevich sheaves. Indeed, the Nisnevich cohomological dimension of a scheme of finite Krull dimension is finite [38]. For the convenience of the reader, we reformulate axioms **COH3** and **COH5** purely in terms of $\mathcal{C}$ (viewed in $\mathcal{D}(\mathrm{Nis})$):

D3 Let $\pi : \mathbf{A}^1_k \to \operatorname{Spec} k$ be the structural map. Then $\mathcal{C} \xrightarrow{\sim} R\pi_*(\mathcal{C}_{|\mathbf{A}^1})$.

To formulate axiom **D5**, note that, if h^* satisfies étale excision and axiom **COH5**, the associated cohomology theory e^* satisfies étale excision as well by remark 5.4.1 (2). If h^* is given by a substratum C, then e^* is given by the substratum $D(X) = \operatorname{Ker}(C(\mathbf{P}^1_X) \xrightarrow{s_\infty^*} C(X))$. By Theorem 7.5.2, both h^* and e^* are given by Nisnevich hypercohomology of complexes of sheaves $\mathcal{C}$ and $\mathcal{D}$.

D5 a) There exists an object $\mathcal{D} \in \mathcal{D}(\mathrm{Nis})$ and, for all $X \in \mathcal{S}_k$, a map

$$\operatorname{Pic} X \to \operatorname{Hom}_{\mathcal{D}(\mathrm{Nis}_{|X})}(\mathcal{D}_{|X}, \mathcal{C}_{|X})$$

natural in X.
b) For $X = \mathbf{P}^1_k$, the map of a) induces a morphism

$$\mathcal{D}_{|\mathbf{P}^1} \xrightarrow{[\mathcal{O}(1)]-[\mathcal{O}]} \mathcal{C}_{|\mathbf{P}^1}.$$

Consider the adjoint map

$$\mathcal{D} \xrightarrow{\alpha} R\tilde{\pi}_*\mathcal{C}_{|\mathbf{P}^1}.$$

Then the map

$$\mathcal{C} \oplus \mathcal{D} \xrightarrow{(\varepsilon,\alpha)} R\tilde{\pi}_*\mathcal{C}_{|\mathbf{P}^1}$$

is a (quasi-)isomorphism, where ε is the unit (adjunction) map.

If $\mathcal{C}$ is a sheaf of spectra, one should replace $R\pi_*$ and $R\tilde{\pi}_*$ by $\mathbb{R}^{\cdot}\pi$ and $\mathbb{R}^{\cdot}\tilde{\pi}$ in axioms **D3** and **D5**.

8. A SELECTION OF COROLLARIES

8.1. **Multiplying by a fixed variety.**

Let T be a (not necessarily smooth) k-variety. The following theorem gives concrete illustrations of item (1) in subsection 5.5.

Theorem 8.1.1. *Let Y be the spectrum of a semi-local ring of a smooth, connected k-variety, as in Proposition 2.1.2. Let n be prime to $\operatorname{char} k$ and $i \in \mathbf{Z}$. Then, with notation* *as in Proposition 2.1.2, there are universally exact sequences:*

$$0 \to H^q(Y \times_k T, \mu_n^{\otimes i}) \xrightarrow{e} H^q_{\eta \times_k T}(Y \times_k T, \mu_n^{\otimes i}) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} H^{q+1}_{x \times_k T}(Y \times_k T, \mu_n^{\otimes i}) \xrightarrow{d_1^{1,q}} \dots \tag{8.1}$$

$$0 \to G_q(Y \times_k T) \xrightarrow{e} G_q(k(Y) \otimes_k T) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} G_{q-1}(k(x) \otimes_k T) \xrightarrow{d_1^{1,q}} \dots \tag{8.2}$$

$$0 \to K_q^B(Y \times_k T) \xrightarrow{e} K_q^{B,\eta \times_k T}(Y \times_k T) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} K_{q-1}^{B,x \times_k T}(Y \times_k T) \xrightarrow{d_1^{1,q}} \dots \tag{8.3}$$

where (8.1) *is étale cohomology. If, moreover, T is smooth, there are universally exact sequences:*

$$0 \to H^q(Y \times_k T, \mu_n^{\otimes i}) \xrightarrow{e} H^q(k(Y) \otimes_k T, \mu_n^{\otimes i}) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} H^{q-1}(k(x) \otimes_k T, \mu_n^{\otimes (i-1)}) \xrightarrow{d_1^{1,q}} \dots \tag{8.4}$$

$$0 \to K_q(Y \times_k T) \xrightarrow{e} K_q(k(Y) \otimes_k T) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} K_{q-1}(k(x) \otimes_k T) \xrightarrow{d_1^{1,q}} \dots \tag{8.5}$$

Proof. After item (1) of subsection 5.5, the exactness of (8.1) and (8.3), as well as the same sequence as (8.3) with G instead of K, follows from examples 7.3 (1), 7.3 (6) and 7.4 (6). Universal exactness follows from section 6. We have (8.2) by purity of G-theory [43, prop. 7.3.2]. When T is smooth we have purity for étale cohomology, hence (8.4), and the K-groups with support identify with G-groups with support, hence (8.3) yields (8.5). □

Remarks 8.1.2.

(1) We could of course state (8.3) and (8.5) for K-theory with finite coefficients.

(2) The reader is invited to apply this principle to other examples (e.g. $\mathcal{K}$-cohomology, compare [7, th. 5.2.5]).

(3) In the étale case, the use of item (1) of subsection 5.5 can be replaced by the isomorphisms
$$\mathbb{H}^*_{ét}(Y \times_k T, \mu_n^{\otimes i}) \simeq \mathbb{H}^*_{ét}(Y, Rf_*((\mu_n^{\otimes i})_{|T}))$$
where $f : T \to \operatorname{Spec} k$ is the structural map, noting that the complex of sheaves $Rf_*((\mu_n^{\otimes i})_{|T})$ is defined over k. In Appendix B, we shall prove an analogue of Theorem 8.1.1, replacing the projection $Y \times_k T \to Y$ by a not necessarily constant map $X \xrightarrow{\pi} Y$, provided
- π is proper and smooth;
- $\dim Y = 1$.

One may ask whether the condition $\dim Y = 1$ is necessary. This issue is being investigated by Panin.

8.2. **Galois action.**

Proposition 8.2.1. *Let $\mathcal{S}_k$ satisfy assumption 5.1.6. Let R be a ring and h^* a cohomology theory with supports on $\mathcal{S}_k$ with values in the category of R-modules. Let $X, t_1, \dots, t_r, Y$*

be as in Theorem 6.2.1. Suppose h^* is given by a substratum C which is strictly effaceable at $t_1, \dots, t_r$. Finally, let M be a left R-module. Then, for and for any q, s, the complex

$$0 \to Tor_s^R(M, h^q(Y)) \xrightarrow{e} \coprod_{x \in Y^{(0)}} Tor_s^R(M, h_x^q(Y)) \xrightarrow{d_1^{0,q}} \coprod_{x \in Y^{(1)}} Tor_s^R(M, h_x^{q+1}(Y)) \xrightarrow{d_1^{1,q}} \dots$$

is exact.

Proof. This is an immediate consequence of Theorem 6.2.1. □

Theorem 8.2.2. *Let ℓ/k be a finite Galois extension and $G = Gal(\ell/k)$. Assume $\mathcal{S}_k$ satisfies assumption 5.1.6. Let h^* be a cohomology theory with supports satisfying axioms COH1 and either COH3 or COH5, plus COH6 if k is finite. Suppose h^* is given by a substratum C. Let X, $t_1, \dots, t_r$, Y be as in Theorem 6.2.1, with X smooth; denote by Y_ℓ the pull-back of Y over ℓ. Then, at least if C is given by complexes of abelian groups, the complex*

$$0 \to H_n(G, h^q(Y_\ell)) \to \coprod_{x \in Y_\ell^{(0)}} H_n(G, h_x^q(Y_\ell)) \to \coprod_{x \in Y_\ell^{(1)}} H_n(G, h_x^{q+1}(Y_\ell)) \to \dots$$

is exact for all $q, n \geq 0$.

Proof. Consider the new cohomology theory with support and substratum h_ℓ^*, C_ℓ given by

$$h_Z^q(X)_\ell = h_{Z_\ell}^q(X_\ell)$$
$$C(X)_\ell = C(X_\ell).$$

Then h_ℓ^* naturally takes its values in the category of $\mathbf{Z}[G]$-modules, and clearly satisfies the same set of axioms as h^*. If C is given by complexes of abelian groups, then C_ℓ takes its values in the category of complexes of $\mathbf{Z}[G]$-modules. The claim then follows from Corollary 6.2.4 and Theorem 6.2.5, applying the functor $H_n(G, ?)$ to a universally exact sequence just as for Proposition 8.2.1. □

If C is given by spectra, then C_ℓ takes its values in the category $\mathcal{E}^G$ of G-spectra. We can then get away similarly if functorial factorizations similar to those in $\mathcal{E}$ (see subesction 5.2) are available in $\mathcal{E}^G$, provided with a suitable closed model category structure. This is closely related to Thomason's unfinished approach to model structures on functor categories, as outlined in Weibel [54].

Theorem 8.2.2 applies in particular to étale cohomology. It also applies to algebraic K-theory provided one fixes the remark of the last paragraph. In the former case, specializing to coefficients twisted roots of unity and using purity, we get the following, which was needed in [30] (precisely for a finite base field!):

Corollary 8.2.3. *Let $X, t_1, \dots, t_r, Y$ be as in Theorem 8.2.2, with X irreducible; denote by Y_ℓ the pull-back of Y over ℓ. Then, for all $q, n \geq 0$, the complex*

$$0 \to H_n(G, H^q(Y_\ell, \mu_m^{\otimes i})) \to H_n(G, H^q(\ell(Y), \mu_m^{\otimes i})) \to \coprod_{x \in Y_\ell^{(1)}} H_n(G, H^{q-1}(\ell(x), \mu_m^{\otimes (i-1)})) \to \dots$$

is exact. □

Remark 8.2.4. One might be tempted to extend this result to the case of any Galois étale covering (not only those coming from the base field) by using Proposition 2.2.4, but this fails. The point is that Proposition 2.2.4 will give homotopies, but not necessarily G-equivariant homotopies.

8.3. **Zariski cohomology and Nisnevich cohomology.**

Theorem 8.3.1. (Nisnevich [39, th. 0.12]) *a) Let h^* be a cohomology theory with supports satisfying axioms COH1, COH2 and also axiom COH6 if the base field k is finite. For $i \in \mathbf{Z}$, let $\mathcal{H}^i_{\mathrm{Zar}}$ (resp. $\mathcal{H}^i_{\mathrm{Nis}}$) be the sheaf associated to the presheaf $U \mapsto h^i(U)$ on the big Zariski (resp. Nisnevich) site of $\operatorname{Spec} k$. Then, for all smooth $X \in \mathcal{S}_k$ and $n \geq 0$, the natural map*

$$H^n_{\mathrm{Zar}}(X, \mathcal{H}^i_{\mathrm{Zar}}) \to H^n_{\mathrm{Nis}}(X, \mathcal{H}^i_{\mathrm{Nis}})$$

is bijective.

This theorem applies notably to algebraic K-theory, étale cohomology and all examples listed in subsections 7.3 and 7.4.

Proof. We need a lemma:

Lemma 8.3.2. *Let $X \in \mathcal{S}_k$ and $\mathcal{M}$ be the category of étale morphisms $U \xrightarrow{f} X$. For $x \in X$ and $i \in \mathbf{Z}$, let h^i_x denote the presheaf on $\mathcal{M}$*

$$f \mapsto \coprod_{y \in f^{-1}(x)} h^i_y(U).$$

Then h^i_x is a sheaf for the Nisnevich topology on $\mathcal{M}$.

Proof. For simplicity, let us write $h^i_x(U)$ instead of $h^i_x(f)$. It is enough to show that, if f covers X at x, i.e. if there exists $x' \in f^{-1}(x)$ such that $k(x) \xrightarrow{\sim} k(x')$, then the sequence

$$0 \to h^i_x(X) \xrightarrow{\varphi} \coprod_{y \in f^{-1}(x)} h^i_y(U) \xrightarrow{\psi} \coprod_{z \in (f \times_X f)^{-1}(x)} h^i_z(U \times_X U)$$

is exact. Write $f^{-1}(x) = \{x'\} \cup T$. From étale excision it is easy to deduce that the map

$$h^i_x(X) \to h^i_{x'}(U)$$

is bijective. This shows that φ is split injective. It is now enough to show that the quotient complex

$$0 \to 0 \to \coprod_{y \in T} h^i_y(U) \xrightarrow{\psi'} \coprod_{z \in (f \times_X f)^{-1}(x)} h^i_z(U \times_X U)$$

is exact, i.e. that ψ' is injective. But we can decompose the set $(f \times_X f)^{-1}(x)$ into

$$\{(x', x')\} \cup \{x'\} \times T \cup T \times \{x'\} \cup T'.$$

Here we note that, for any $y \in f^{-1}(x)$, the schemes $x' \times_X y$ and $y \times_X x'$ are spectra of fields, because $k(x') = k(x)$; we abbreviate these schemes by (x', y) and (y, x'). By étale excision again, the maps

$$h^i_y(U) \to h^i_{(x',y)}(U \times_X U)$$
$$h^i_y(U) \to h^i_{(y,x')}(U \times_X U)$$

given respectively by the first and the second projection are bijective. The injectivity of ψ' follows. □

Proof of theorem 8.3.1. a) Let $\tilde{X}_{\text{Zar}}$ be the restriction of the big Zariski site of X to the category of schemes étale over X and $\alpha : X_{\text{Nis}} \to \tilde{X}_{\text{Zar}}$ be the natural projection. It is obvious that $\alpha^*\mathcal{H}^i_{\text{Zar}} = \mathcal{H}^i_{\text{Nis}}$. Therefore, applying α^* to the resolution (analogous to) (2.1) of $\mathcal{H}^i_{\text{Zar}}$ yields a resolution of $\mathcal{H}^i_{\text{Nis}}$. For $x \in X$, one has clearly an isomorphism of functors

$$\alpha^* i^{\text{Zar}}_{x*} \simeq i^{\text{Nis}}_{x*}\alpha^*$$

hence α^*(2.1) can be identified to the complex of Nisnevich sheaves

$$0 \to \coprod_{x \in X^{(0)}} i^{\text{Nis}}_{x*} h^q_x(X) \to \coprod_{x \in X^{(1)}} i^{\text{Nis}}_{x*} h^{1+q}_x(X) \to \cdots \to \coprod_{x \in X^{(p)}} i^{\text{Nis}}_{x*} h^{p+q}_x(X) \to \dots \tag{8.6}$$

It is clear that i^{Nis}_{x*} is an exact functor for all $x \in X$. Therefore, the n-th Nisnevich cohomology of the p-th term of this complex is

$$\coprod_{x \in X^{(p)}} H^n_{\text{Nis}}(x, h^{p+q}_x(X)).$$

But the Nisnevich cohomological dimension of a field is 0, hence this group is 0 for $n > 0$. It follows that the terms of (8.6) are acyclic. Finally, Lemma 8.3.2 shows that the global sections of (8.6) are (the analogue of) (2.1). □

Corollary 8.3.3. *Under the assumptions of Theorem 8.3.1, the Zariski and Nisnevich Brown-Gersten spectral sequences of Corollary 7.5.3 coincide.*

Indeed, they are compatible and their E_2-terms coincide. □

8.4. **Shapiro's lemma.**

Theorem 8.4.1. *Let h^* be a strictly effaceable cohomology theory with supports and let $i \in \mathbf{Z}$. let $\mathcal{H}^i$ denote the Zariski sheaf associated to the presheaf h^i. Let $f : Y \to X$ be a finite morphism, with Y smooth. Then $R^q f_* \mathcal{H}^i = 0$ for $q > 0$.*

Proof. We can compute $R^q f_* \mathcal{H}^i$ by using the (flasque) Cousin resolution $Cous$ of $\mathcal{H}^i$ over Y. But the stalk of $f_* Cous$ at a point $x \in X$ is none other than the "stalk" of $Cous$ at $f^{-1}(x)$, i.e. $\Gamma(\mathcal{O}_{Y, f^{-1}(x)}, Cous)$, which is exact. □

8.5. **Birational invariance.**

Theorem 8.5.1. *Let h^* be a cohomology theory with supports satisfying axioms COH1, COH2 and also axiom COH6 if the base field k is finite. Let $X \in \mathcal{S}_k$ be smooth and let $H^*(X, \mathcal{H}^i)$ denote either of the groups $H^*_{\mathrm{Zar}}(X, \mathcal{H}^i_{\mathrm{Zar}}), H^*_{\mathrm{Nis}}(X, \mathcal{H}^i_{\mathrm{Nis}})$ of Theorem 8.3.1 (they coincide by this theorem). Then, for all $i \in \mathbf{Z}$, $H^0(X, \mathcal{H}^i)$ is a birational invariant of smooth proper varieties $X \in \mathcal{S}_k$.*

Proof. a) By Corollary 5.1.11, the functor $X \mapsto h^i(X)$ satisfies “codimension 1 purity” for regular local rings of a smooth variety in the sense of [7, def. 2.1.4 (b)] (a cohomology class which is unramified at points of codimension 1 is unramified everywhere locally). The claim now follows from [7, prop. 2.1.8]. □

8.6. **Rational invariance.**

Let $\mathcal{S}_k = Sm/k$, and let h^* be a cohomology theory with supports on $\mathcal{P}_k$. Assume h^* satisfies axioms **COH1** (étale excision) and **COH2** (key lemma), the latter for all $V \in Sm/k$. If k is finite, assume h^* also satisfies axiom **COH6**. We then have the following theorem:

Theorem 8.6.1. *Let X, Y be two smooth integral k-varieties, with respective function fields $k(X), k(Y)$, and let $p : X \to Y$ be a proper morphism. Assume that the generic fibre X_η of p is $k(Y)$-birational to d-dimensional projective space $\mathbf{P}^d_{k(Y)}$. Then, for any $i \in \mathbf{Z}$, the map*

$$H^0(Y, \mathcal{H}^i) \xrightarrow{p^*} H^0(X, \mathcal{H}^i)$$

is an isomorphism.

Proof. For any smooth integral k-variety Z, with generic point η, we have by definition

$$h^i_\eta(Z) = \varinjlim_{U \subseteq Z} h^i(U)$$

where U runs through the nonempty open subsets of Z. We denote this group by $h^i(k(Z))$. Corollary 5.1.11 yields an exact sequence

$$0 \to H^0(Z, \mathcal{H}^i) \to h^i(k(Z)) \to \coprod_{x \in Z^{(1)}} h^{i+1}_x(Z). \tag{8.7}$$

We may replace $\mathbf{P}^d_{k(Y)}$ by the d-fold self-product $(\mathbf{P}^1_{k(Y)})^d$ in the assumption of Theorem 8.6.1. By hypothesis, there exists a birational map

$$(\mathbf{P}^1_Y)^d \dashrightarrow X$$

over Y. Since $(\mathbf{P}^1_Y)^d$ is regular and p is proper, this rational map extends to a Y-morphism

$$U \xrightarrow{f} X$$

where U is an open subset of $(\mathbf{P}^1_Y)^d$ containing all points of codimension 1 (valuative criterion of properness, cf [26, th. II. 4.7]). The exact sequence (8.7) then shows that the restriction map

$$H^0((\mathbf{P}^1_Y)^d, \mathcal{H}^i) \to H^0(U, \mathcal{H}^i)$$

is an isomorphism.

Let ζ (resp. ξ) be the generic point of X (resp. $(\mathbf{P}^1_Y)^d$ and U). We have a commutative diagram

$$\begin{array}{ccccc}
h^i(k(X)) & \xrightarrow{f^*} & h^i(k(\mathbf{P}^1_Y)^d) & = & h^i(k(\mathbf{P}^1_Y)^d) \\
\| & & \| & & \| \\
h^i_\zeta(X) & \xrightarrow{f^*} & h^i_\xi(U) & = & h^i_\xi((\mathbf{P}^1_Y)^d) \\
\bigcup & & \bigcup & & \bigcup \\
H^0(X,\mathcal{H}^i) & \longrightarrow & H^0(U,\mathcal{H}^i) & = & H^0((\mathbf{P}^1_Y)^d,\mathcal{H}^i) \\
{\scriptstyle p^*}\nwarrow & & \uparrow & & \nearrow \\
& & H^0(Y,\mathcal{H}^i) & &
\end{array}$$

in which the vertical inclusions follow from (8.7). Since f is birational, f^* is an isomorphism. It is thus enough to prove Theorem 8.6.1 in the case $X = (\mathbf{P}^1_Y)^d$. By induction on d, we may assume $d = 1$.

We first deal with the special case $Y = \operatorname{Spec} k$. To begin with, the natural map

$$h^i(k) \to h^i(k(\mathbf{P}^1_k))$$

is injective. If k is infinite, this follows from the classical section argument, since any open subset of $\mathbf{P}^1_k$ contains a rational point. If k is finite, axiom **COH6** provides a variant of this argument, since any open subset of $\mathbf{P}^1_k$ contains two closed points of coprime degrees.

On the other hand, since $\mathbf{P}^1_k$ is of dimension one, we have an exact sequence

$$h^i(\mathbf{P}^1_k) \to h^i(k(\mathbf{P}^1_k)) \to \coprod_{x \in (\mathbf{P}^1_k)^{(1)}} h^{i+1}_x(\mathbf{P}^1_k)$$

hence, from (8.7):

$$H^0(\mathbf{P}^1_k, \mathcal{H}^i) = \operatorname{Im}(h^i(\mathbf{P}^1_k) \to h^i(k(\mathbf{P}^1_k))).$$

The map $h^i(\mathbf{P}^1_k) \to h^i(k(\mathbf{P}^1_k))$ obviously factors through $h^i(\mathbf{A}^1_k)$. By axiom **COH2**, it even factors through $h^i(k)$, hence Theorem 8.6.1 in this case.

In the general case, let $\eta = \operatorname{Spec} k(Y)$ denote the generic point of Y. Note that any smooth $k(Y)$-variety is a filtering inverse limit of smooth k-varieties, with affine transition morphisms: we may therefore extend h^* to $\mathcal{P}_{k(Y)}$ (corresponding to $\mathcal{S}_{k(Y)} := Sm/k(Y)$) by direct limits. This cohomology theory with supports obviously satisfies axiom **COH1**; it also satisfies **COH2** because we assumed the original h^* verified it for all smooth varieties.

We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathbf{P}^1_Y, \mathcal{H}^i) & \longrightarrow & H^0(\mathbf{P}^1_{k(Y)}, \mathcal{H}^i) & \longrightarrow & \coprod_{y\in(\mathbf{P}^1_Y)^{(1)}-(\mathbf{P}^1_{k(Y)})^{(1)}} h^{i+1}_y(\mathbf{P}^1_Y) \\ & & \uparrow & & \wr\uparrow & & \uparrow \\ 0 & \longrightarrow & H^0(Y, \mathcal{H}^i) & \longrightarrow & h^i(k(Y)) & \longrightarrow & \coprod_{x\in Y^{(1)}} h^{i+1}_x(Y). \end{array}$$

To conclude the proof, it is sufficient to show that the right vertical map in this diagram is injective. But this map factors through

$$\coprod_{x\in Y^{(1)}} h^{i+1}_x(Y) \to \coprod_{x\in Y^{(1)}} h^{i+1}_{\eta_x}(\mathbf{P}^1_Y)$$

where $\eta_x = \operatorname{Spec} k(\mathbf{P}^1_x)$.

Let $x \in Y^{(1)}$: we have to see that the map $h^{i+1}_x(Y) \to h^{i+1}_{\eta_x}(\mathbf{P}^1_Y)$ is injective. We may replace Y by $Y' = \operatorname{Spec} \mathcal{O}_{Y,x}$. By definition, we have

$$h^{i+1}_{\eta_x}(\mathbf{P}^1_{Y'}) = \varinjlim_Z h^{i+1}_{\mathbf{P}^1_x - Z}(\mathbf{P}^1_{Y'} - Z)$$

where Z runs through proper closed subsets of $\mathbf{P}^1_x$.

Suppose first that $k(x)$ is infinite. Then $\mathbf{P}^1_x - Z$ contains a $k(x)$-rational point; since $\mathbf{P}^1(Y') \to \mathbf{P}^1(k(x))$ is surjective, we may lift it to a section s of $\mathbf{P}^1_{Y'} \to Y'$, which does not meet Z. Then the composite

$$h^{i+1}_x(Y) \to h^{i+1}_{\mathbf{P}^1_x - Z}(\mathbf{P}^1_{Y'} - Z) \xrightarrow{s^*} h^{i+1}_x(Y)$$

is the identity and $h^{i+1}_x(Y) \to h^{i+1}_{\mathbf{P}^1_x - Z}(\mathbf{P}^1_{Y'} - Z)$ is split injective.

Suppose now that $k(x)$ is finite, hence k is finite. Then $\mathbf{P}^1_x - Z$ contains in any case two closed points x_1, x_2 of coprime degrees d_1, d_2. Extending scalars to the residue fields k_1, k_2 and using axiom **COH6**, we get as above compositions

$$\begin{aligned} h^{i+1}_x(Y) \to h^{i+1}_{\mathbf{P}^1_x - Z}(\mathbf{P}^1_{Y'} - Z) &\to h^{i+1}_{(\mathbf{P}^1_x - Z)_{k_i}}((\mathbf{P}^1_{Y'} - Z)_{k_i}) \\ &\xrightarrow{s_i^*} h^{i+1}_{x_i}(Y_{x_i}) \xrightarrow{\operatorname{Cor}_{k_i/k}} h^{i+1}_x(Y) \qquad (i = 1, 2) \end{aligned}$$

with values d_1, d_2. Since these integers are coprime, we get that $h^{i+1}_x(Y) \to h^{i+1}_{\mathbf{P}^1_x - Z}(\mathbf{P}^1_{Y'} - Z)$ is split injective once again. □

Appendix A. Etale cohomology: the non-torsion case

The aim of this appendix is to extend Theorem 2.2.7 to all complexes of sheaves coming from the small étale site of k.

A.1. **Proper base change.**

The following is a version of the proper base change theorem involving non-torsion coefficients:

Proposition A.1.1. *Let V be an excellent Noetherian scheme of finite Krull dimension and $\tilde{\pi} : X \to V$ be a proper normal morphism* [EGA4, (6.8.1)]*. Then, for any complex of étale sheaves $\mathcal{C}^{\cdot}$ over V and any geometric point $\overline{x}$ of V, there is a quasi-isomorphism*

$$(R\tilde{\pi}_*\tilde{\pi}^*\mathcal{C}^{\cdot})_{\overline{x}} \simeq R\Gamma(X_{\overline{x}}, \tilde{\pi}^*\mathcal{C}^{\cdot}).$$

Proof. Since $\tilde{\pi}$ is proper and V is Noetherian, $R\tilde{\pi}_*$ has finite l-cohomological dimension for any prime l, and the argument of [SGA4-III, exposé X, proof of th. 4.1] shows that its rational cohomological dimension is bounded by $\dim V$. Hence, by comparison of hypercohomology spectral sequences, it is enough to prove this when $\mathcal{C}^{\cdot}$ consists of one sheaf C placed in degree 0. We proceed as Deninger in [10], first reducing to the case where C is **Z**-constructible. Using the excellence of V, we can further reduce as in *loc. cit.* to the case where $C = \tau_* F$, where $\tau : V' \to V$ is finite, V' is normal and F is a constant sheaf given by a finitely generated **Z**-module. Passing to the strict henselization A of V at a geometric point $\overline{x}$, we get a commutative diagram of cartesian squares:

$$\begin{array}{ccccc}
\operatorname{Spec}\kappa' & \xrightarrow{\iota'} & \operatorname{Spec}A' & \longrightarrow & V' \\
{\scriptstyle\tau_\kappa}\downarrow & & {\scriptstyle\tau_A}\downarrow & & {\scriptstyle\tau}\downarrow \\
\operatorname{Spec}\kappa & \xrightarrow{\iota} & \operatorname{Spec}A & \longrightarrow & V
\end{array}$$

Here $\kappa = \kappa(\overline{x})$, $\operatorname{Spec}A' = \operatorname{Spec}A \times_V V'$ and $\operatorname{Spec}\kappa' = \operatorname{Spec}\kappa \times_V V'$, so that $\kappa' = \kappa \otimes_A A'$. Since A' is finite over A, it is a product of strictly henselian local rings and κ' is an Artin local ring with the same residue fields as A'. Since V' is normal, so are A' and any of its connected components. Let B be such a component. Since $\tilde{\pi}$ is proper, it is of finite type, hence $\tilde{\pi}_B$ and $\tilde{\pi}_{\kappa(\overline{y})}$ are normal by [EGA4, (6.8.3)], where for any ring R and morphism $\operatorname{Spec}R \to V$, we denote by $\tilde{\pi}_R : X_R \to \operatorname{Spec}R$ the pull-back of $\tilde{\pi}$. In particular, X_B and $X_{\kappa(\overline{y})}$ are normal, where $\overline{y}$ is the closed point of $\operatorname{Spec}B$. Hence, by [10, (2.3)], we have:

$$(R^q(\tilde{\pi}_B)_*F)_{\overline{y}} \simeq H^q(X_{\kappa(\overline{y})}, F).$$

On the other hand, letting κ'' be the product of the residue fields of A', the morphism $X_{\kappa''} \to X_{\kappa'}$ is radicial hence induces an isomorphism of étale cohomology [36, ch. II, remark 3.17]. It follows that

$$\iota'^*(R^q(\tilde{\pi}_{A'})_*F) \simeq R^q(\tau_\kappa)_*F.$$

Since τ is finite, $R^q(\tau_A)_* = R^q(\tau_\kappa)_* = 0$ for all $q > 0$ [36, ch. II, cor. 3.6] and the latter isomorphism implies

$$\iota^* R^q(\tau_A \circ \tilde{\pi}_{A'})_*F \simeq R^q(\tau_\kappa \circ \tilde{\pi}_{\kappa'})_*(\iota'')^*F.$$

$$\begin{array}{ccccccc}
X_{\kappa''} & \longrightarrow & X_{\kappa'} & \xrightarrow{\iota''} & X_{A'} & \longrightarrow & X_{V'} \\
{\scriptstyle \tilde\pi_{\kappa''}}\downarrow & & {\scriptstyle \tilde\pi_{\kappa'}}\downarrow & & {\scriptstyle \tilde\pi_{A'}}\downarrow & & \downarrow \\
\operatorname{Spec}\kappa'' & \longrightarrow & \operatorname{Spec}\kappa' & \xrightarrow{\iota'} & \operatorname{Spec}A' & \longrightarrow & V' \\
 & & {\scriptstyle \tau_\kappa}\downarrow & & {\scriptstyle \tau_A}\downarrow & & {\scriptstyle \tau}\downarrow \\
 & & \operatorname{Spec}\kappa & \xrightarrow{\iota} & \operatorname{Spec}A & \longrightarrow & V
\end{array}$$

But using now the finiteness of $X_{V'} \to X_V$ and the induced maps and arguing as in [10, proof of (2.4)], we get an isomorphism

$$\iota^* R^q(\tilde\pi_A)_*(\tau_* F) \simeq R^q(\tilde\pi_\kappa)_*(\iota')^*(\tau_* F),$$

as desired. (See diagram below.)

$$\begin{array}{ccc}
X_{\kappa'} & \xrightarrow{\iota''} & X_{A'} \\
{\scriptstyle \text{finite}}\downarrow & & {\scriptstyle \text{finite}}\downarrow \\
X_\kappa & \xrightarrow{\iota'} & X_A \\
{\scriptstyle \tilde\pi_\kappa}\downarrow & & {\scriptstyle \tilde\pi_A}\downarrow \\
\operatorname{Spec}\kappa & \xrightarrow{\iota} & \operatorname{Spec}A
\end{array}$$

Remark A.1.2. We can use Artin's example in [SGA4-III, exposé XII, $\S$ 2] to show that, in general, one cannot extend Proposition A.1.1 to more general complexes of sheaves over X than those of the form $D^\cdot = \tilde\pi^* \mathcal{C}^\cdot$. To be specific, take $V = \operatorname{Spec} R$ where R is an complete discrete valuation ring and $X = \mathbf{P}^1_R$. Let Y be the projective curve over R with equation $zy^2 = x(x-z)(x-\pi z)$, where π is a uniformizing parameter of R, that we view as a two-fold covering $Y \xrightarrow{\tau} X$ via the function x/z. Let $D^\cdot = \tau_* \mathbf{Z}[0]$. Then $H^1(X, D^\cdot) = H^1(Y, \mathbf{Z}) = 0$ while $H^1(X_0, D^\cdot) = H^1(Y_0, \mathbf{Z}) \simeq \mathbf{Z}$ where X_0, Y_0 are the special fibres of X and Y.

A.2. **An integral Chern class.**

Definition A.2.1. (compare [13, proof of Lemma 2]) Let $i \in \mathbf{Z}$.
a) For any prime number p, we denote by $\mathbf{Q}_p/\mathbf{Z}_p(i)$ the extension by 0 of the étale sheaf $\mathbf{Q}_p/\mathbf{Z}_p(i) = \varinjlim \mu_{p^n}^{\otimes i}$ from $\operatorname{Spec} \mathbf{Z}[1/p]$ to $\operatorname{Spec} \mathbf{Z}$. This defines a sheaf over the big étale site of $\operatorname{Spec} \mathbf{Z}$.
b) We define $\mathbf{Q}/\mathbf{Z}(i) = \bigoplus \mathbf{Q}_p/\mathbf{Z}_p(i)$.

Remark. Note that with this definition, $\mathbf{Q}/\mathbf{Z}(0)$ does not in general coincide with $\mathbf{Q}/\mathbf{Z}$!

Let

$$\mathbf{Q}/\mathbf{Z}(0)[-1] \to \mathbf{Z} \tag{A.1}$$

be the morphism in the derived category of big étale sheaves over $\operatorname{Spec}\mathbf{Z}$ defined as follows: for a prime p, we have an exact sequence of sheaves $0 \to \mathbf{Z} \to \mathbf{Z}[1/p] \to \mathbf{Q}_p/\mathbf{Z}_p \to 0$ hence a morphism $\mathbf{Q}_p/\mathbf{Z}_p[-1] \to \mathbf{Z}$ in the derived category. Letting j_p be the open immersion $\operatorname{Spec}\mathbf{Z}[1/p] \to \operatorname{Spec}\mathbf{Z}$, we get a corresponding morphism

$$\mathbf{Q}_p/\mathbf{Z}_p(0)[-1] := (j_p)_!\mathbf{Q}_p/\mathbf{Z}_p[-1] \to (j_p)_!\mathbf{Z} = (j_p)_!(j_p)^*\mathbf{Z}.$$

Composing with the adjunction map $(j_p)_!(j_p)^*\mathbf{Z} \to \mathbf{Z}$, we get a morphism $\mathbf{Q}_p/\mathbf{Z}_p(0)[-1] \to \mathbf{Z}$ over $\operatorname{Spec}\mathbf{Z}$. The desired morphism is the sum of these morphisms for all primes p.

Definition A.2.2. (compare [13, Appendix B]) Let X be a scheme and L a line bundle on X. To L we associate a morphism in $D(X_{ét})$

$$\mathbf{Q}/\mathbf{Z}(-1)[-3] \xrightarrow{C_1(L)} \mathbf{Z}$$

as follows. The class of L in $H^1(X_{ét}, \mathbb{G}_{>})$ corresponds to a morphism in $D(X_{ét})$

$$\mathbf{Z} \xrightarrow{[L]} \mathbb{G}_{>}[\nVdash].$$

On the other hand, for all primes p, the sheaf $\mathbb{G}_{>}$ is p-divisible away from the locus where p is not invertible. This yields for all $n \geq 1$ an isomorphism:

$$\mathbb{G}_{>} \overset{\mathbb{L}}{\otimes} \mathbf{Z}/\ltimes(\nvdash) \xrightarrow{\sim} \mathbf{Z}/\ltimes(\nVdash)[\nVdash].$$

These fit together to give a “Kummer” isomorphism

$$\mathbb{G}_{>} \overset{\mathbb{L}}{\otimes} \mathbf{Q}/\mathbf{Z}(\nvdash) \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}(\nVdash)[\nVdash].$$

$C_1(L)$ is then the composition of the morphisms in the sequence

$$\mathbf{Q}/\mathbf{Z}(-1)[-3] \to \mathbb{G}_{>} \overset{\mathbb{L}}{\otimes} \mathbf{Q}/\mathbf{Z}(-\nVdash)[-\nvDash] \xrightarrow{\sim} \mathbf{Q}/\mathbf{Z}(\nvdash)[-\nVdash] \to \mathbf{Z}$$

where the first morphism is $[L]$ tensored by $\mathbf{Q}/\mathbf{Z}(-1)[-3]$, the second one is the Kummer isomorphism twisted and shifted and the last morphism is (A.1). For two line bundles L and L' on X, we have $C_1(L \otimes L') = C_1(L) + C_1(L')$; in particular, if L is trivial then $C_1(L) = 0$.

The last claim of the definition is obvious from the construction of C_1 and the additivity of line bundle classes in $H^1(X_{ét}, \mathbb{G}_{>})$.

A.3. **Cohomology of $\mathbf{P}^1$.**

Theorem A.3.1. (compare [SGA5, exposé VII, th. 2.2.1] and [13, Lemma 3]) *Let V be as in Proposition A.1.1 and let $\tilde{\pi} : \mathbf{P}^1_V \to V$ be the natural projection. Then, for any complex of sheaves $\mathcal{C}^{\cdot}$ over the small étale site of V, there is a natural isomorphism in $D(V_{ét})$:*

$$\mathcal{C}^{\cdot} \oplus \mathcal{C}^{\cdot} \overset{L}{\otimes} \mathbf{Q}/\mathbf{Z}(-1)[-3] \xrightarrow{\sim} R\tilde{\pi}_*\tilde{\pi}^*\mathcal{C}^{\cdot}.$$

This isomorphism is the adjunction of a morphism

$$\tilde{\pi}^*\mathcal{C}^{\cdot} \oplus \tilde{\pi}^*\mathcal{C}^{\cdot} \overset{L}{\otimes} \tilde{\pi}^*\mathbf{Q}/\mathbf{Z}(-1)[-3] \to \tilde{\pi}^*\mathcal{C}^{\cdot}$$

in which the first component is the identity and the second one is given by tensoring (in the derived sense) the Chern class map $\tilde{\pi}^*\mathbf{Q}/\mathbf{Z}(-1)[-3] \xrightarrow{C_1(\mathcal{O}(1))} \tilde{\pi}^*\mathbf{Z}$ *of definition A.2.2 by* $\tilde{\pi}^*\mathcal{C}^{\cdot}$.

Proof. By Proposition A.1.1, we have an isomorphism

$$(R\tilde{\pi}_*\tilde{\pi}^*\mathcal{C}^{\cdot})_{\overline{x}} \simeq R\Gamma(\mathbf{P}^1_{\kappa(\overline{x})}, \tilde{\pi}^*_{\overline{x}}\mathcal{C}^{\cdot})$$

for any geometric point $\overline{x}$ of V. To prove Theorem A.3.1, we may therefore assume that $V = \operatorname{Spec} \kappa$, where κ is a separably closed field. We first remark:

Lemma A.3.2. *Let* $\mathcal{C}^{\cdot}$ *be a complex of abelian groups and* $D^{\cdot}$ *be a bounded above complex of étale sheaves over* $\mathbf{P}^1_\kappa$. *Then the natural morphism in the derived category of abelian groups*

$$\mathcal{C}^{\cdot} \overset{L}{\otimes} R\Gamma(\mathbf{P}^1_\kappa, D^{\cdot}) \longrightarrow R\Gamma(\mathbf{P}^1_\kappa, \mathcal{C}^{\cdot} \overset{L}{\otimes} D^{\cdot})$$

is an isomorphism.

Proof. The argument of [36, ch. VI, Lemma 8.7], which consists of reducing to the case where $\mathcal{C}^{\cdot}$ is a single finitely generated free $\mathbf{Z}$-module placed in degree 0, applies (compare *loc. cit.*, remark 8.14). Note that it is not necessary to assume that $\mathcal{C}^{\cdot}$ is bounded above, since the Tor-dimension of $\mathbf{Z}$ is finite.

Applying Lemma A.3.2 to $D^{\cdot} = \mathbf{Z}[0]$ ($\mathbf{Z}$ placed in degree 0), it now suffices to prove Theorem A.3.1 in the case $\mathcal{C}^{\cdot} = \mathbf{Z}[0]$. In this case, it follows from

Lemma A.3.3. *a) We have* $H^1(\mathbf{P}^1_\kappa, \mathbf{Z}) = 0$ *and* $H^q(\mathbf{P}^1_\kappa, \mathbf{Q}_p/\mathbf{Z}_p) = 0$ *for* $q > 0$ *if char*$(\kappa) = p > 0$.
b) There is an isomorphism in $D(Ab)$:

$$\kappa^* \oplus \mathbf{Z}[-1] \xrightarrow{\sim} R\Gamma(\mathbf{P}^1_\kappa, \mathbb{G}_m)$$

whose first component is the adjunction of the map $\tilde{\pi}^*\kappa^* \to \mathbb{G}_m$ *in* $D((\mathbf{P}^1_\kappa)_{ét})$ *and the second one is the adjunction of* $\tilde{\pi}^*\mathbf{Z} \xrightarrow{[\mathcal{O}(1)]} \mathbb{G}_m$ *(compare definition A.2.2).*

Proof. The vanishing of $H^1(\mathbf{P}^1_\kappa, \mathbf{Z})$ follows from the normality of $\mathbf{P}^1_\kappa$. Suppose char$(\kappa) = p > 0$. We have

$$H^q(\mathbf{P}^1_\kappa, \mathbb{G}_a) = H^q_{Zar}(\mathbf{P}^1_\kappa, \mathcal{O}_{\mathbf{P}^1_\kappa}) = \begin{cases} \kappa & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

[36, ch. II, prop. 3.7 and remark 3.8] and [26, ch III, th. 5.1]. Using the Artin-Schreier exact sequence $0 \to \mathbf{Z}/p \to \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \to 0$ [36, ch. II, example 2.18 (c)], this implies that

$$H^q(\mathbf{P}^1_\kappa, \mathbf{Z}/p) = 0 \text{ if } q > 0$$

for p equal to the characteristic of κ. Using the exact sequences

$$0 \to \mathbf{Z}/p^n \to \mathbf{Z}/p^{n+1} \to \mathbf{Z}/p \to 0$$

it follows that $H^q(\mathbf{P}^1_\kappa, \mathbf{Z}/p^n) = 0$ for all n, hence $H^q(\mathbf{P}^1_\kappa, \mathbf{Q}_p/\mathbf{Z}_p) = 0$ for $q > 0$, as claimed. Finally, we have

$$H^q(\mathbf{P}^1_\kappa, \mathbb{G}_m) = \begin{cases} \kappa^* & \text{if } q = 0 \\ \operatorname{Pic} \mathbf{P}^1_\kappa = \mathbf{Z} & \text{if } q = 1 \\ 0 & \text{if } q > 1 \end{cases}$$

[36, ch. III, example 2.23 (b)], where the second isomorphism is induced by the degree map. Since $\mathcal{O}(1)$ generates $\operatorname{Pic} \mathbf{P}^1_\kappa$, the map $\mathbf{Z} \to \mathbb{G}_m[1]$ defined by its class induces an isomorphism

$$\mathbf{Z} = H^0(\mathbf{P}^1_\kappa, \mathbf{Z}) \longrightarrow H^1(\mathbf{P}^1_\kappa, \mathbb{G}_m).$$

inverse to the former. The last claim of Lemma A.3.3 follows.

We now finish to prove Theorem A.3.1 for $\mathcal{C}^\cdot = \mathbf{Z}[0]$ in the case $V = \operatorname{Spec} \kappa$. By *e.g.* [10, (2.1)], $H^q(\mathbf{P}^1_\kappa, \mathbf{Q}) = 0$ for $q > 0$. From the exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$ we derive

$$H^{q-1}(\mathbf{P}^1_\kappa, \mathbf{Q}/\mathbf{Z}) \xrightarrow{\sim} H^q(\mathbf{P}^1_\kappa, \mathbf{Z}) \quad \text{for } q > 1.$$

In view of Lemma A.3.3 a), this implies that (A.1) induces isomorphisms

$$H^{q-1}(\mathbf{P}^1_\kappa, \mathbf{Q}/\mathbf{Z}(0)) \xrightarrow{\sim} H^q(\mathbf{P}^1_\kappa, \mathbf{Z}) \quad \text{for } q > 1.$$

Using the Kummer exact sequence $1 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 1$ [36, ch. II, example 2.18 (b)], Lemma A.3.3 b) implies that $H^q(\mathbf{P}^1_\kappa, \mu_n) = 0$ for $q \neq 0, 2$ and n prime to the characteristic of κ; in particular, $H^q(\mathbf{P}^1_\kappa, \mathbf{Q}/\mathbf{Z}(0)) = 0$ for $q \neq 0, 2$. We therefore have $H^q(\mathbf{P}^1_\kappa, \mathbf{Z}) = 0$ for $q \neq 0, 3$.

Clearly, $\mathbf{Z} \to H^0(\mathbf{P}^1_\kappa, \mathbf{Z})$ is an isomorphism; it remains to see that the adjoint to $C_1(\mathcal{O}(1))$ induces an isomorphism $\mathbf{Q}/\mathbf{Z}(-1) \xrightarrow{\sim} H^3(\mathbf{P}^1_\kappa, \mathbf{Z})$. To do this, we follow the definition of C_1. According to definition A.2.2 and Lemma A.3.2, the map

$$\mathbf{Q}/\mathbf{Z}(-1) \longrightarrow H^3(\mathbf{P}^1_\kappa, \mathbf{Z})$$

can be decomposed as follows:

$$\begin{aligned} \mathbf{Q}/\mathbf{Z}(-1) \to H^1(R\Gamma(\mathbf{P}^1_\kappa, \mathbb{G}_m) \overset{\mathbb{L}}{\otimes} \mathbf{Q}/\mathbf{Z}(-1)) \xrightarrow{\sim} \mathbb{H}^1(\mathbf{P}^1_\kappa, \mathbb{G}_m \overset{\mathbb{L}}{\otimes} \mathbf{Q}/\mathbf{Z}(-1)) \\ \xrightarrow{\sim} H^2(\mathbf{P}^1_\kappa, \mathbf{Q}/\mathbf{Z}(0)) \xrightarrow{\sim} H^3(\mathbf{P}^1_\kappa, \mathbf{Z}). \end{aligned}$$

By Lemma A.3.2, the second map is an isomorphism. According to definition A.2.2, the third map is an isomorphism and as seen above the fourth one is an isomorphism too. It remains to see that the first map is an isomorphism. But it is obtained by tensoring the isomorphism of Lemma A.3.3 b) by $\mathbf{Q}/\mathbf{Z}(-1)$. □

Appendix B. The one-dimensional case

In this section, we prove a version of Gersten's conjecture for regular one-dimensional schemes, not necessarily in the presence of a base field. The proof mimics Gillet's in [16]. As in section 5, we shall axiomatize the situation. The axioms necessary to make the proof work turn out to be much more costly than those in section 5.

B.1. **Some axioms.**

Let A be a semi-local principal domain, $S = \operatorname{Spec} A$ and $\mathcal{S}_S$ the category of regular schemes separated and *quasifinite* over S. By Zariski's main Theorem, a connected object $X \to S$ of $\mathcal{S}_S$ is of the form $\operatorname{Spec} B$, where B is either a finite extension of one residue field of S or a localization of a finite extension of A at some maximal ideals. We give ourselves a "cohomology theory"

$$h^i : \mathcal{S}_S \to \mathcal{A}\ (i \in \mathbf{Z}),$$

a collection of contravariant functors to some abelian category $\mathcal{A}$, satisfying the following axioms:

(i) **Additivity.** h^* is additive.

(ii) **Transfers.** For a finite morphism $f : Y \to X$ in $\mathcal{S}_S$, there is given a map $f_* : h^{*-2c}(Y) \to h^*(X)$, where $c = \operatorname{codim}_X Y$; this collection of maps makes h^* a covariant functor.

(iii) **Purity.** For $X \in \mathcal{S}_S$ of dimension 1, $Z \xrightarrow{i} X$ a (reduced) closed subset of dimension 0 and $U \xrightarrow{j} X$ the complementary open subset, there is an exact sequence

$$\dots \to h^{i-2}(Z) \xrightarrow{i_*} h^i(X) \xrightarrow{j^*} h^i(U) \xrightarrow{\partial} h^{i-1}(Z) \to \dots$$

Moreover, if $f : X' \to X$ is a finite and flat map, the square

$$\begin{array}{ccc} h^i(U') & \xrightarrow{\partial'} & h^{i-1}(Z') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f'_*} \\ h^i(U) & \xrightarrow{\partial} & h^{i-1}(Z) \end{array}$$

commutes, where $Z' = f^{-1}(Z)_{\text{red}}$, $U' = X' - Z'$ and $f' : Z' \to Z$ is the map induced by f (no multiplicities!)

(iv) **Action of units.** For any $X \in \mathcal{S}_S$ there is a pairing

$$\Gamma(X, \mathcal{O}_X^*) \times h^*(X) \to h^{*+1}(X)$$

which is contravariant in X and satisfies the projection formula for finite flat maps. Moreover,

- for X, Z, U, ∂ as in (iii), we have, for $(f, \alpha) \in \Gamma(U, \mathcal{O}_X^*) \times h^*(X)$:

$$\partial(f \cdot j^*\alpha) = \sum_{z \in Z} v_z(f) i_z^*(\alpha)$$

where i_z is the inclusion $z \hookrightarrow X$ (here we used the additivity of h^*).

- In the situation of (iii), given $f \in \Gamma(X, \mathcal{O}_X^*)$, the following diagram anticommutes:

$$\begin{array}{ccc} h^i(U) & \xrightarrow{\partial} & h^{i-1}(Z) \\ {\scriptstyle \cdot f}\downarrow & & \downarrow{\scriptstyle \cdot f} \\ h^{i+1}(U) & \xrightarrow{\partial} & h^i(Z) \end{array}$$

(v) **Rigidity.** Let $X \in \mathcal{S}_S$ of dimension 1, $x \in X$ a closed point and X_x^h the henselization of X at x. Let $h^*(X_x^h) := \varinjlim h^*(U)$, where U runs through all étale neighbourhoods of x. Then $h^*(X_x^h) \to h^*(x)$ is an isomorphism.

Examples B.1.1.

(1) Algebraic K-theory verifies all axioms except (v); algebraic K-theory with coefficients $\mathbf{Z}/n$, where n is invertible in A, satisfies all axioms including (v) [47].

(2) Let $\mathcal{C}$ be a complex of sheaves over the small étale site of S; assume that its cohomology sheaves are all locally constant constructible, torsion invertible in A. Then $h^*(X) = \coprod_{n\in\mathbf{Z}} H^*_{\acute{e}t}(X, \mathcal{C}(\backslash))$ satisfies all the axioms. Axiom (i) is a general property of étale cohomology. Axioms (ii) and (iii) follow from cohomological purity in dimension 1 [SGA5, exposé I, th. 5.1] and the existence of trace maps [SGA4-III, exposé XVIII]. Axiom (iv) is folklore: see [42, Lemma 3] for a detailed proof. Axiom (v) can be deduced from proper base change as in [SGA4-III, exposé XII, cor. 5.5]. Note that cohomological purity and proper base change for complexes of sheaves follow from the same for sheaves plus comparison of hypercohomology spectral sequences.

B.2. **The result.**

Theorem B.2.1. *Let R be a ring and h^* a cohomology theory with values in R-modules, satisfying axioms (i)–(v). Let Z be the set of closed points of S and η its generic point. Then, for all $i \in \mathbf{Z}$, the sequence*

$$0 \to h^i(S) \to h^i(\eta) \xrightarrow{\partial} h^{i-1}(Z) \to 0$$

is universally exact.

Proof. For convenience we use ring-theoretic notation. Let $\mathcal{R}$ be the radical of A and F its field of fractions, so that $Z = V(\mathcal{R})$ and $\eta = \operatorname{Spec} F$ and the sequence of Theorem B.2.1 can be rewritten

$$0 \to h^i(A) \to h^i(F) \xrightarrow{\partial} h^{i-1}(A/\mathcal{R}) \to 0. \tag{B.1}$$

Write $h^{i-1}(A/\mathcal{R})$ as a direct limit of finitely presented R-modules. Let M be such a module. We construct an R-linear map $f_M : M \to h^i(F)$ such that $\partial \circ f_M = \iota$, where ι is the map $M \to h^{i-1}(A/\mathcal{R})$. In view of the long cohomology exact sequence of which (B.1) is part (axiom (iii)), this will show that the restriction of (B.1) to M is a split exact sequence of R-modules.

Let A^h be the henselization of A along $\mathcal{R}$. Then A^h splits as a finite product of henselian discrete valuation rings. By axioms (i) and (v), the natural map

$$h^{i-1}(A^h) \to h^{i-1}(A^h/\mathcal{R}A^h)$$

is an isomorphism.

Consider the commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}_R(M,h^{i-1}(A^h)) & \xrightarrow{\sim} & \mathrm{Hom}_R(M,h^{i-1}(A^h/\mathcal{R}A^h))\\ \uparrow & & \uparrow\\ \varinjlim \mathrm{Hom}_R(M,h^{i-1}(A')) & \longrightarrow & \varinjlim \mathrm{Hom}_R(M,h^{i-1}(A'/\mathcal{R}A'))\end{array}$$

where A' runs through the quasi-finite A-subalgebras of A^h. Since M is finitely presented, the two vertical maps are isomorphisms, hence so is also the bottom horizontal one. Therefore there exists an A' and an R-linear map

$$\sigma : M \to h^{i-1}(A')$$

such that the diagram

$$\begin{array}{ccc}h^{i-1}(A') & \longrightarrow & h^{i-1}(A'/\mathcal{R}A')\\ {\scriptstyle\sigma}\uparrow & & \uparrow\\ M & \xrightarrow{\iota} & h^{i-1}(A/\mathcal{R})\end{array}$$

commutes.

Let A_1 be the integral closure of A in the total ring of fractions F' of A'. Then $A_1 \subseteq A'$ and A' is a semi-localization of A_1 at some of its maximal ideals. Since A' is étale over A, F'/F is separable and A_1 is finite over A [46,]. Then the diagram

$$\begin{array}{ccc}h^i(F') & \xrightarrow{\partial} & h^{i-1}(A_1/\mathcal{R}A_1)\\ {\scriptstyle\mathrm{Cor}}\downarrow & & {\scriptstyle\mathrm{Cor}}\downarrow\\ h^i(F) & \xrightarrow{\partial} & h^{i-1}(A/\mathcal{R})\end{array} \tag{B.2}$$

commutes (axiom (iii)). Let $\mathcal{R}_1 = \mathcal{R}A_1$. Write $\mathcal{R}_1 = \mathcal{R}'\mathcal{R}''$, where $\mathcal{R}' + \mathcal{R}'' = A_1$, $\mathcal{R}'A' = \mathcal{R}_1A'$ and $\mathcal{R}''A' = A'$. Then $A_1/\mathcal{R}_1 \cong A_1/\mathcal{R}' \times A_1/\mathcal{R}'' \cong A'/\mathcal{R}A' \times A_1/\mathcal{R}'' \cong A/\mathcal{R} \times A_1/\mathcal{R}''$ and the composite $A/\mathcal{R} \to A_1/\mathcal{R}_1 \cong A/\mathcal{R} \times A_1/\mathcal{R}'' \xrightarrow{pr_1} A/\mathcal{R}$ is the identity. Accordingly, diagram (B.2) becomes:

$$\begin{array}{ccc}h^i(F') & \xrightarrow{(\partial',\partial'')} & h^{i-1}(A/\mathcal{R}) \oplus h^{i-1}(A_1/\mathcal{R}'')\\ {\scriptstyle\mathrm{Cor}}\downarrow & & \left(\begin{smallmatrix} Id \\ ? \end{smallmatrix}\right)\downarrow\\ h^i(F) & \xrightarrow{\partial} & h^{i-1}(A/\mathcal{R})\end{array} \tag{B.3}$$

By the Chinese remainder theorem, choose $f \in E'^*$ such that:

- $f \equiv 1 \pmod{\mathcal{R}''}$;
- f generates $\mathcal{R}'$.

For $\beta \in h^{i-1}(A')$, write $\overline{\beta}$ for its image in $h^{i-1}(A'/\mathcal{R}A') = h^{i-1}(A/\mathcal{R})$. Applying axiom (iv), we get:

Lemma B.2.2. *For all $\beta \in h^{i-1}(A')$, one has*

$$\partial'(\beta \cdot (f)) = \overline{\beta};$$
$$\partial''(\beta \cdot (f)) = 0.\square$$

Corollary B.2.3. *The map $f_M : M \to h^i(F)$ given by $f_M(\alpha) = \operatorname{Cor}_{F'/F}(\sigma(\alpha) \cdot (f))$ has the required properties.*

Proof. This follows from diagram (B.3) and Lemma B.2.2. □

B.3. **Corollaries.**

Notation B.3.1. For any scheme X, we denote by $\mathcal{LC}^+(X)$ the category of bounded below complexes of sheaves over the small étale site of X, whose cohomology sheaves are locally constant constructible, torsion prime to the residue characteristics of X.

Corollary B.3.2. *With notation as in Theorem B.2.1, let B be a finite, étale, Galois A-algebra, with Galois group G. Let $C^\cdot \in \mathcal{LC}^+(B)$. Let E be the total ring of fractions of B. Then the complex of $\mathbf{Z}[G]$-modules*

$$0 \to h^i(B) \to h^i(E) \xrightarrow{\partial} h^{i-1}(B/\mathcal{R}B) \to 0$$

is universally exact for all $i \in \mathbf{Z}$.

Proof. Apply th. B.2.1 to $R\pi_* C^\cdot$ and $R = \mathbf{Z}[G]$, where $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ is the natural map. □

Corollary B.3.3. *With notation as in Theorem B.2.1, let X be a proper and smooth A-scheme, X_F its generic fibre and X_0 its closed fibre. Let $C^\cdot \in \mathcal{LC}^+(X)$. Then the complex*

$$0 \to H^i(X_{ét}, C^\cdot) \to H^i(X_{F,ét}, C^\cdot) \xrightarrow{\partial} H^{i-1}(X_{0,ét}, C^\cdot(-1)) \to 0$$

is universally exact for all $i \in \mathbf{Z}$.

Proof. Let $\pi : X \to \operatorname{Spec} A$ be the structure map. By [36, cor. VI.4.2], $R\pi_* C^\cdot$ is in $\mathcal{LC}^+(A)$. It is clear that $H^0(F, R\pi_* C^\cdot) \cong H^0(X_F, C^\cdot)$, since $\operatorname{Spec} F \to \operatorname{Spec} A$ is an open immersion. Moreover, the proper base change theorem [36, cor. VI.2.3] shows that $H^0(A/\mathcal{R}, R\pi_* C^\cdot(-1)) \cong H^0(X_0, C^\cdot(-1))$. Therefore the complex of Corollary B.3.3 can be rewritten

$$0 \mapsto H^i_{ét}(A, R\pi_* C^\cdot) \to H^i_{ét}(F, R\pi_* C^\cdot) \xrightarrow{\partial} H^{i-1}(A/\mathcal{R}, R\pi_* C^\cdot(-1)) \to 0$$

and we can apply th. B.2.1. □

Remark B.3.4. We can combine corollaries B.3.2 and B.3.3.

Appendix C. Unbounded complexes

In this appendix, we extend the notion of injective resolution from bounded below to unbounded complexes of objects of a suitable abelian category. These results are similar to those of Spaltenstein [45] (see also [3]).

C.1. **Fibrant complexes.**

Let $\mathcal{A}$ be an abelian category and let $C(\mathcal{A})$ be the category of complexes of objects of $\mathcal{A}$. We set up the following definition:

Definition C.1.1. a) A morphism

$$C^{\cdot} \to D^{\cdot}$$

in $C(\mathcal{A})$ is a *trivial cofibration* if it is both a monomorphism and a quasi-isomorphism (i.e. it induces an isomorphism on cohomology).
b) An object $F^{\cdot} \in C(\mathcal{A})$ is *fibrant* if it has the following property: given a trivial cofibration $C^{\cdot} \to D^{\cdot}$, any morphism from $C^{\cdot}$ to $F^{\cdot}$ extends to a morphism from $D^{\cdot}$ to $F^{\cdot}$.

Fibrant complexes are closely related with K-injective complexes in the sense of J. Bernstein. It can be shown that the latter are those complexes which are homotopy equivalent to the former (compare [45, prop. 1.5]).

Proposition C.1.2. *a) If $F^{\cdot}$ is fibrant, then F^n is injective for any $n \in \mathbf{Z}$.*
b) If $F^{\cdot}$ is bounded below, the converse is true.

Proof. a) Let $A \overset{\varphi}{\hookrightarrow} B$ be a monomorphism in $\mathcal{A}$, and let $f : A \to F^n$ be a homomorphism. Let $C^{\cdot}, D^{\cdot}$ be the complexes such that $C^i = D^i = 0$ for $i \neq n, n+1$, $C^n = C^{n+1} = A$, $D^n = D^{n+1} = B$ and the differentials $C^n \to C^{n+1}$ and $D^n \to D^{n+1}$ are given by the identity. The monomorphism φ induces an obvious monomorphism of acyclic complexes $C^{\cdot} \hookrightarrow D^{\cdot}$, and f induces a morphism of complexes $f' : C^{\cdot} \to F^{\cdot}$. Applying the defining property, f' extends to a morphism $\tilde{f}' : D^{\cdot} \to F^{\cdot}$, whose restriction to $D^n = B$ defines an extension of f to B.

b) It is convenient to give a lemma:

Lemma C.1.3. *Let $F^{\cdot} \in C(\mathcal{A})$ be such that F^n is injective for some $n \in \mathbf{Z}$. Let $C^{\cdot} \hookrightarrow D^{\cdot}$ be a trivial cofibration, and let $f : C^{\cdot} \to F^{\cdot}$ be a homomorphism. Assume that $f^{n-1} : C^{n-1} \to F^{n-1}$ extends to $\tilde{f}^{n-1} : D^{n-1} \to F^{n-1}$ such that:*
(i) $\tilde{f}^{n-1}(B^{n-1}(D^{\cdot})) \subseteq B^{n-1}(F^{\cdot})$;
(ii) $\tilde{f}^{n-1}(Z^{n-1}(D^{\cdot})) \subseteq Z^{n-1}(F^{\cdot})$.
Then there exists $\tilde{f}^n : D^n \to F^n$ extending f^n such that
a) $\tilde{f}^n d = d\tilde{f}^{n-1}$;
b) $\tilde{f}^n(Z^n(D^{\cdot})) \subseteq Z^n(F^{\cdot})$.

Proof. We define $\tilde{f}^n$ first on $B^n(D^{\cdot})$, then on $Z^n(D^{\cdot})$ and finally on all of D^n.

- On $B^n(D^{\cdot})$, define $\tilde{f}^n$ by $\tilde{f}^n(dy) = d\tilde{f}^{n-1}(y)$ for $y \in D^{n-1}$. By assumption (ii), this does not depend on the choice of y.
- On $Z^n(D^{\cdot})$, define $\tilde{f}^n$ as the unique map whose restriction to $B^n(D^{\cdot})$ is as above and whose restriction to $Z^n(C^{\cdot})$ is f^n. This is well-defined by the quasi-isomorphism assumption.
- On D^n, choose for $\tilde{f}^n$ any extension of the above, applying the injectiveness of F^n.

One checks readily that $\tilde{f}^n$ indeed verifies conditions a) and b). □

Proof of Proposition C.1.2 b). Lemma C.1.3 implies that if its conditions a) and b) are satisfied for some $n \in \mathbf{Z}$ then they are satisfied for $n+1$. In case $F^{\cdot}$ is bounded below, conditions a) and b) are trivially satisfied for $n << 0$. $\square$

Lemma C.1.4. *a) Let $C^{\cdot} \in C(\mathcal{A})$ be acyclic and $F^{\cdot} \in C(\mathcal{A})$ be fibrant. Then any morphism $C^{\cdot} \xrightarrow{f} F^{\cdot}$ is homotopic to 0.*
b) Let $F^{\cdot}$ be fibrant and $C^{\cdot}$ be arbitrary. Then, any quasi-isomorphism $F^{\cdot} \xrightarrow{f} C^{\cdot}$ has a homotopy left inverse. If $C^{\cdot}$ is itself fibrant, f is a homotopy equivalence.

Proof. a) Applying the defining property of "fibrant" to the monomorphism $F^{\cdot} \hookrightarrow C(f)$, where $C(f)$ is the mapping cone of f, we get that the identity $F^{\cdot} \to F^{\cdot}$ extends to a morphism $C(f) \to F^{\cdot}$ (note that, since $C^{\cdot}$ is acyclic, $F^{\cdot} \hookrightarrow C(f)$ is a quasi-isomorphism). Since the composite $C^{\cdot} \xrightarrow{f} F^{\cdot} \to C(f)$ is homotopic to 0, we get that f itself is homotopic to 0.

b) (cf [24, proof of Lemma 4.5]) Note that the mapping cone $C(f)$ is acyclic. Applying a), we see that the morphism $C(f) \to F^{\cdot}[1]$ is homotopic to 0, and the conclusions easily follow. $\square$

Lemma C.1.5. *Consider a commutative square of objects of $C(\mathcal{A})$*

$$\begin{array}{ccc} C^{\cdot} & \xrightarrow{\alpha'} & F'^{\cdot} \\ \psi\downarrow & & \varphi\downarrow \\ D^{\cdot} & \xrightarrow{\alpha} & F^{\cdot} \end{array}$$

in which ψ is a trivial cofibration and $F^{\cdot}$ is fibrant. Let $\beta : D^{\cdot} \to F'^{\cdot}$ be a morphism extending α'. Then the two morphisms

$$\alpha, \varphi \circ \beta : D^{\cdot} \to F$$

are homotopic.

Proof. By assumption, $\alpha - \varphi \circ \beta$ factors through the acyclic complex $D^{\cdot}/C^{\cdot}$. The conclusion now follows from Lemma C.1.4 a). $\square$

C.2. **Homotopy limits.**

Definition C.2.1. Let $(F_n^{\cdot}, \varphi_{n+1,n})_{n\geq 0}$ be a projective system of objects of $C(\mathcal{A})$. Its *homotopy limit* $\operatorname{holim} F_n^{\cdot}$ is the total complex associated to the double complex (with vertical length 1)

$$\prod F_n^{\cdot} \xrightarrow{D} \prod F_n^{\cdot}$$

where $(\prod F_n^{\cdot})^q := \prod F_n^q$, the differential being defined componentwise, and where $D_n : \prod F_n^q \to \prod F_n^q$ is defined as $(-1)^q(Id - \varphi_{n+1,n}^q)$.

One readily checks that D anticommutes with the differentials of $\prod F_n$, so that the construction indeed defines a complex. It is clear than holim is a functor. By definition and the two spectral sequences associated to the double complex, we have:

Proposition C.2.2. *a) There is a short exact sequence of complexes*

$$0 \to \varprojlim F_n^{\cdot} \to \operatorname{holim} F_n^{\cdot} \to \varprojlim{}^1 F_n^{\cdot}[1] \to 0$$

hence a long exact sequence of cohomology groups

$$\cdots \to H^q(\varprojlim F_n^{\cdot}) \to H^q(\operatorname{holim} F_n^{\cdot}) \to H^{q-1}(\varprojlim{}^1 F_n^{\cdot}) \to H^{q+1}(\varprojlim F_n^{\cdot}) \to \dots$$

b) Assume that countable products are exact in $\mathcal{A}$. Then there are short exact sequences

$$0 \to \varprojlim{}^1 H^{q-1}(F_n^{\cdot}) \to H^q(\operatorname{holim} F_n^{\cdot}) \to \varprojlim H^q(F_n^{\cdot}) \to 0. \qquad \square$$

Corollary C.2.3. *Assume that countable products are exact in $\mathcal{A}$. Let $(C_n^{\cdot}), (F_n^{\cdot})$ be two projective systems in $C(\mathcal{A})$, and let (f_n) be a morphism from $(C_n^{\cdot})$ to $(F_n^{\cdot})$. Suppose that each f_n is a quasi-isomorphism (resp. a trivial cofibration). Then $\operatorname{holim} f_n$ is a quasi-isomorphism (resp. a trivial cofibration).* □

Lemma C.2.4. *For all $n \geq 0$, let $\varphi_n : \operatorname{holim} F_n^{\cdot} \to F_n^{\cdot}$ be given on $(\operatorname{holim} F_n^{\cdot})^q = \prod F_n^q \oplus \prod F_n^{q-1}$ by $(p_n, 0)$, where p_n is the n-th projection. Then:*
a) The φ_n are morphisms of complexes.
b) For all $n \geq 0$, φ_n and $\varphi_{n+1,n} \circ \varphi_{n+1}$ are homotopic.

Proof. a) is trivial. To see b), check that the map of degree -1

$$S_n : \operatorname{holim} F_n \longrightarrow F_n$$

given by $S_n^q = (0, (-1)^q p_n^{q-1})$ is a homotopy between φ_n and $\varphi_{n+1,n} \circ \varphi_{n+1}$. □

Lemma C.2.5. *Let $(F_n^{\cdot}, \varphi_{n+1,n})_{n\geq 0}$ be as in definition C.2.1, and let $C^{\cdot} \in C(\mathcal{A})$.*
a) Let $f : C^{\cdot} \to \operatorname{holim} F_n$ be a morphism. Then $\varphi_n \circ f = 0$ for all n if and only if $f = (0, (g_n))$, where, for all n, g_n is a morphism from $C^{\cdot}$ to $F_n^{\cdot}[1]$.
b) Let $f_n : C^{\cdot} \to F_n^{\cdot}$ be a family of morphisms such that, for all $n \geq 0$, f_n and $\varphi_{n+1,n} \circ f_{n+1}$ are homotopic. Then there exists a morphism $f : C^{\cdot} \to \operatorname{holim} F_n^{\cdot}$ such that, for any $n \geq 0$, $\varphi_n \circ f = f_n$.

Proof. a) is a simple computation. b) Choose for all n a homotopy s_n between f_n and $\varphi_{n+1,n} \circ f_{n+1}$. Define f by

$$f^q = ((f_n^q), ((-1)^q s_n^q))).$$

One checks easily that f is a morphism of complexes, and that the identity $\varphi_n \circ f = f_n$ holds. □

Proposition C.2.6. *Let $(F_n^{\cdot}, \varphi_{n+1,n})$ be an inverse system in $C(\mathcal{A})$. If each $F_n^{\cdot}$ is fibrant, then $\operatorname{holim} F_n^{\cdot}$ is fibrant.*

Proof. Let $C^{\cdot} \xrightarrow{\alpha} D^{\cdot}$ be a trivial cofibration, and let $f : C^{\cdot} \to \operatorname{holim} F_n$ be a morphism. By the fibrancy of F_n, we may extend $\varphi_n \circ f$ to a morphism $\tilde{f}_n$ for each n. Lemma C.1.5 shows that $\tilde{f}_n$ and $\varphi_{n+1,n} \circ \tilde{f}_{n+1}$ are homotopic for any n; therefore, applying Lemma C.2.5 b), we can find a $\tilde{f} : D^{\cdot} \to \operatorname{holim} F_n^{\cdot}$ such that $\varphi_n \circ \tilde{f} = f_n$ for all n. We have

$$\varphi_n \circ (f - \tilde{f} \circ \alpha) = 0 \qquad \text{for all } n.$$

By Lemma C.2.5 a), we can write $f - \tilde{f} \circ \alpha = (0, (g_n))$, where each g_n is a morphism from $C^{\cdot}$ to $F_n^{\cdot}[1]$. Applying the fibrancy of F_n again, we can extend each g_n to a $\tilde{g}_n : D^{\cdot} \to F_n^{\cdot}[1]$. Then $\tilde{f} + (0, (\tilde{g}_n)) : D^{\cdot} \to \operatorname{holim} F_n$ extends f. □

Corollary C.2.7. *If $f : C^{\cdot} \to D^{\cdot}$ is a morphism between fibrant complexes, then the mapping cone of f is fibrant.* □

C.3. **Resolutions.**

Theorem C.3.1. *Suppose that $\mathcal{A}$ verifies axiom AB5 and has a generator in the sense of* [23, 1.5 and 1.6], *and that countable products are exact in $\mathcal{A}$. Then there exists a functor $F : C(\mathcal{A}) \to C(\mathcal{A})$ and a natural transformation $\varepsilon : Id \to F$ such that, for any $C^{\cdot} \in C(\mathcal{A})$,*

(i) *$F(C^{\cdot})$ is fibrant;*
(ii) *$\varepsilon_{C^{\cdot}}$ is a trivial cofibration.*

Proof. By [23, th. 1.10.1], the assumptions imply the existence of a functor $I : \mathcal{A} \to \mathcal{A}$ and a natural transformation $\eta : Id \to I$ such that, for all $A \in \mathcal{A}$

(i) $I(A)$ is injective
(ii) η_A is a monomorphism.

We first construct F on bounded below complexes. If $C^{\cdot}$ is such a complex, the construction of [24, proof of Lemma 4.6 1)] embeds $C^{\cdot}$ into a bounded below complex of injectives by a trivial cofibration; by Proposition C.1.2 b), the latter complex is fibrant. We note that we can make this construction functorial by using the functor I above.

Suppose now $C^{\cdot}$ arbitrary. For any $n \in \mathbf{Z}$, let $C^{\cdot} \to C^{\cdot}_{\geq n}$ be the canonical truncation of $C^{\cdot}$ at level n. Recall that $C^{\cdot}_{\geq n}$ is defined by

$$(C^{\cdot}_{\geq n})^q = \begin{cases} C^q & \text{if } q \geq n \\ B^q(C^{\cdot}) & \text{if } q = n-1 \\ 0 & \text{if } q < n-1 \end{cases}$$

with the same differentials as $C^{\cdot}$, except that $B^n(C^{\cdot}) \to C^n$ is the canonical injection. We have $H^q(C^{\cdot}_{\geq n}) = \begin{cases} H^q(C^{\cdot}) & \text{if } q \geq n \\ 0 & \text{if } q < n. \end{cases}$ Note also that, for each $q \in \mathbf{Z}$, the inverse system $(C^{\cdot}_{\geq n})^q$ is stationary, hence $\varprojlim{}^1 C_{\geq n} = 0$ and

$$C^{\cdot} \xrightarrow{\sim} \varprojlim C^{\cdot}_{\geq n} \xrightarrow{\sim} \operatorname{holim} C^{\cdot}_{\geq n}$$

(isomorphisms) by Proposition C.2.2 a).

By the first part of the proof, we have a functorial trivial cofibration $C^{\cdot}_{\geq n} \to F(C^{\cdot}_{\geq n})$ for all n. This gives rise to a chain of morphisms

$$C^{\cdot} \xrightarrow{\sim} \varprojlim C^{\cdot}_{\geq n} \xrightarrow{\sim} \operatorname{holim} C^{\cdot}_{\geq n} \to \operatorname{holim} F(C^{\cdot}_{\geq n}). \tag{C.1}$$

We define $F(C^{\cdot})$ as $\operatorname{holim} F(C^{\cdot}_{\geq n})$ and $\varepsilon_{C^{\cdot}}$ as the composition of this chain. By Proposition C.2.6, $F(C^{\cdot})$ is fibrant and, by Corollary C.2.3, $\varepsilon_{C^{\cdot}}$ is a trivial cofibration, as desired. □

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