THE FULLNESS CONJECTURES FOR PRODUCTS OF ELLIPTIC CURVES

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ABSTRACT. We prove all conjectures of Yves André's book [1, Ch. 7] in the case of products of elliptic curves, except for the Ogus conjecture where an assumption on the base field is necessary. The proofs given here are simpler and more uniform than the previous proofs in known cases.

Introduction. Let X be a product of elliptic curves over a field k. If $k = \mathbf{C}$, the Hodge conjecture is known for X: this is attributed to Tate (unpublished) by Grothendieck in [6, §3 c)]; a full proof was given by Imai in [7].

Let ℓ be a prime number invertible in k. The Tate conjecture for ℓ-adic cohomology is known for X in the following cases:

- k is a number field (Imai [7]);
- k is finite (Spieß [16]).

(In [7], Imai proves the Hodge conjecture and the Mumford-Tate conjecture for X, hence implicitly the Tate conjecture for X.)

In each case, the result is stronger: the Hodge, or Tate, classes, are generated by those of degree 2. The proofs, however, are different: for the Hodge conjecture and the Tate conjecture over a number field, Imai uses essentially a Tannakian argument plus results of Shimura-Taniyama while, over a finite field, Spieß' proof is obtained from an inequality on elliptic Weil numbers. On the other hand, the proof given by Imai involves a number of subcases and is especially delicate when dealing with elliptic curves with complex multiplication.

This prompted me to look for a simpler and unified proof, which would also cover two other similar conjectures mentioned in [1, Ch. 7]:

- The "de Rham-Betti" conjecture (aka the weak Grothendieck period conjecture);
- the Ogus conjecture.

For the readers' convenience, I recall these lesser-known conjectures in Section 1. The main result of this article is:

Theorem 1. *All the above-mentioned conjectures except perhaps the Ogus conjecture hold for X, in the strong sense that the cohomology classes coming from algebraic cycles are generated by those of degree 2. In particular, the Tate conjecture holds for X over any finitely generated field k.*

The Ogus conjecture holds (in the strong sense) over any number field k which is linearly disjoint from all real quadratic fields (e.g. $[k:\mathbf{Q}]$ odd).

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In the case of the de Rham-Betti realisation, this implies part of a conjecture of Rohrlich: if all elliptic curves involved in X have complex multiplication, then the multiplicative relations between its periods are generated by the "obvious" ones [1, 24.6.3.1].

Remark 1. Since isogenous abelian varieties have isomorphic motives (as follows from [5, Th. 3.1]), Theorem 1 immediately extends to abelian varieties isogenous to products of elliptic curves.

The formalism developed here gives this result more generally for any "enriched realisation" into a Tannakian category $\mathcal{A}$ verifying certain axioms: see Theorems 2 and 3. These axioms are a bit of a patchwork, but this is the best I could do.

To be able to use Tannakian arguments, one also needs the source category to be Tannakian. This is the case for Grothendieck motives (for a classical Weil cohomology) restricted to abelian varieties, in characteristic 0 thanks to work of Lieberman, but it is not known in characteristic > 0. However, Milne introduced in [13] a subcategory of "Lefschetz motives" (defined only for abelian varieties), which is semi-simple Tannakian in every characteristic and for any Weil cohomology: this is extremely convenient, and well-adapted to the present situation where the strong form of the conjectures is true.

The main condition on the enriched realisation is that the corresponding fullness conjecture (plus a semi-simplicity condition) should hold for products of 2 elliptic curves; this is known in all cases mentioned above [1, 7.1.7.5]. The other conditions are more technical, and are introduced to allow the classical method, outlined in [1, 7.6], to work smoothly. This method consists of a group-theoretic argument: it starts with the case of one elliptic curve (and its powers), and uses a principle due to Goursat, Kolchin and Ribet to pass from there to the general case.

The first case works well provided one knows that the (geometric) Tannakian group attached to any elliptic curve is *connected*. This is trivial for the Hodge conjecture and easy for the Tate conjecture; for the two other conjectures, I borrowed arguments from Yves André (see §§6.3 and 6.4).

The general case is where elliptic curves with complex multiplication, whose motivic Galois groups are abelian, have rendered the Goursat-Kolchin-Ribet principle delicate in previous works. However, nobody seems to have used the full strengh of Kolchin's version of this principle [11]: his theorem is powerful enough to create a streamlined proof when the coefficients of the Weil cohomology involved are $\mathbf{Q}$: see Theorem 2 for a weaker hypothesis. This is true, in particular, in the case of the Hodge conjecture, where this approach trivialises Imai's arguments. It is also true for the de Rham-Betti realisation. By wriggling around, one can also get the Tate conjecture over any finitely generated field and for any prime ℓ. For the Ogus conjecture, the condition given in Theorem 1 is a special case of that in Theorem 2.

One last, intriguing point: if one starts in this abstract setting with an enriched realisation functor into a Tannakian category $\mathcal{A}$, it is not clear how to extend this to base fields which are finite separable extensions l of k; but the method I use here is to reduce to the separable closure of k, so a definition of categories $\mathcal{A}(l)$

(and corresponding realisation functors) is necessary. It turns out that this can be done provided the Weil cohomology at hand is "of Galois type" (Definition 2): this applies to all the examples above, except for the Ogus conjecture where the Weil cohomology (de Rham cohomology) is rather of "differential type" (ibid.). Of course, in each of the above examples one can simply take for $\mathcal{A}(l)$ the "same" definition as for $\mathcal{A}(k)$, but I believe that the abstract construction given in Section 4 is interesting in itself. At least, it illustrates the fact that the slogan "all Weil cohomologies are equivalent" is overstated.

The present method remains unfortunately very special to products of elliptic curves and not prone to generalisation, as far as I have seen. It raises nevertheless interesting questions about the generality of the *result*. For example, let A be an abelian variety of a type for which one of the conjectures is known "in the strong sense" (for A and all its powers), e.g. one taken from the examples in [14, A.7] lifted to characteristic 0. Can one prove the same for the other conjectures?

This work was done in 2019, and was given a brief announcement in the algebraic geometry seminar of IMJ-PRG on June 20, 2019[1]. For some reason, I didn't release it until now. Since then, Mingmin Chen and Charles Vial have also proven the de Rham-Betti conjecture for products of elliptic curves in [2]. Their approach is quite different, and relies on Deligne's "Hodge = absolute Hodge" theorem for abelian varieties and on Wüstholz's analytic subgroup theorem (which is also implicitly used here when we apply [1, 7.5.3]). It can be hoped that each approach sheds light on the other.

1. **Review of the de Rham-Betti and the Ogus conjectures.** For both conjectures, k is a number field; let Y be a smooth projective k-variety. In the first case, we fix an embedding $k \hookrightarrow \mathbf{C}$ and define a *de Rham-Betti cycle of codimension* n as a pair $(\alpha, \beta) \in H^{2n}_{\mathrm{dR}}(Y/k) \times H^{2n}_B(Y, \mathbf{Q})$ such that $\alpha \otimes 1 \mapsto (2\pi i)^n \beta \otimes 1$ via the period isomorphism $H^{2n}_{\mathrm{dR}}(Y/k) \otimes_k \mathbf{C} \xrightarrow{\sim} H^{2n}_B(Y, \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$. The cycle classes of any algebraic cycle of codimension n yield a de Rham-Betti cycle and, conversely:

De Rham-Betti conjecture ([1, 7.5.1.1]). *Any de Rham-Betti cycle on Y is algebraic.*

In the second case, we consider de Rham cohomology of Y with extra structure: if v is a finite unramified place of k where Y has good reduction, we have the Berthelot isomorphism

$$H^{2n}_{\mathrm{dR}}(Y_v/k_v) \xrightarrow{\sim} H^{2n}_{\mathrm{cris}}(Y(v)) \otimes_{W(k(v))} k_v$$

where k_v (resp. $k(v)$) is the completion (resp. the residue field) of k at v, $Y_v = Y \otimes_k k_v$ and $Y(v)$ is the special fibre of a smooth model of Y at v. By transport of structure, the Frobenius automorphism of the right hand side provides the left hand side with an automorphism φ_v which is semi-linear with respect to the absolute Frobenius of k_v. An *Ogus cycle* is an element α of $H^{2n}_{\mathrm{dR}}(Y/k)$ such that, for almost all v, one has $\varphi_v(\alpha) = q_v^n \alpha$, where $q_v = |k(v)|$. The cycle class of any algebraic cycle of codimension n is an Ogus cycle and, conversely:

Ogus conjecture ([1, 7.4.1.2]). *Any Ogus cycle on Y is algebraic.*

[1]Except that the Ogus conjecture was optimistically announced to be proven for X over any number field, due to a incorrect weight argument.

2. **The set-up.** Let k be a field, and let $\mathcal{M}(k) = \mathcal{M}$ be the category of pure motives over k modulo algebraic equivalence, with coefficients in a field K of characteristic 0 (with the "contravariant" convention: the motive functor is contravariant). We write $\mathbb{L} \in \mathcal{M}$ for the Lefschetz motive. Let H be a Weil cohomology with coefficients in K. We have a $\otimes$-functor

$$H^* : \mathcal{M} \to \mathrm{Vec}_K^*$$

where Vec_K is the $\otimes$-category of finite-dimensional K-vector spaces and, for any additive $\otimes$-category $\mathcal{A}$, $\mathcal{A}^*$ is the $\otimes$-category of $\mathbf{Z}$-graded objects of $\mathcal{A}$, the commutativity constraint being given by the Koszul rule.

Definition 1 ([1, 7.1.1])**.** A (neutral) *enrichment* of H is a factorisation $H^* = \omega^* \circ \mathbf{H}^*$:

$$\mathcal{M} \xrightarrow{\mathbf{H}^*} \mathcal{A}^* \xrightarrow{\omega^*} \mathrm{Vec}_K^*$$

where $\mathcal{A}$ is a K-linear Tannakian category with fibre functor ω, and $\mathbf{H}^*$ is a $\otimes$-functor. We write $\mathbf{H} : \mathcal{M} \to \mathcal{A}$ for the underlying functor ($\mathbf{H}(M) = \bigoplus_{i\in\mathbf{Z}} \mathbf{H}^i(M)$). For $A \in \mathcal{A}$ and $p \in \mathbf{Z}$, we write $A(p) := A \otimes \mathbf{H}^2(\mathbb{L})^{\otimes -p}$.

Write $\mathbf{S}\mathrm{m}^{\mathrm{proj}}$ for the category of smooth projective k-varieties. For $X \in \mathbf{S}\mathrm{m}^{\mathrm{proj}}$ and $r \geq 0$, we are interested in the condition

(F)(X, r)**:** the map

$$A^r_{\mathrm{alg}}(X) \otimes K = \mathcal{M}(\mathbf{1}, h(X)(r)) \to \mathcal{A}(\mathbf{1}, \mathbf{H}^{2r}(X)(r))$$

is surjective.

The following is well-known (cf. [1, Ch. 7]):

Lemma 1. *Condition* **(F)**(X, r) *for all* $X \in \mathbf{S}\mathrm{m}^{\mathrm{proj}}$ *and all* $r \geq 0$ *is equivalent to the fullness of* $\mathbf{H}^*$. □

3. **The separably closed case.** For ease of exposition, we first consider the special case where k is separably closed.

Recall that Milne defined in [13] a rigid $\otimes$-category $\mathbf{LMot}$ of "Lefschetz motives"[2], modelled on abelian varieties. This category naturally embeds into $\mathcal{M}$ as a (nonfull) $\otimes$-subcategory, and numerical equivalence agrees with homological equivalence in $\mathbf{LMot}$ for any Weil cohomology H as above; in particular, $\mathbf{LMot}$ becomes Tannakian after changing the commutativity constraint as usual, and H induces a fibre functor H_L. See [10] for a definition of $\mathbf{LMot}$ modulo other adequate equivalence relations and over not necessarily algebraically closed fields.

Theorem 2. *Let* $X = \prod_{i\in I} E_i$ *be a product of elliptic curves over* k. *Assume that*

(A) *The Tannakian group associated to* $(\mathcal{A}(E_i), \omega)$ *is connected for any* E_i, *where* $\mathcal{A}(E_i)$ *is the full Tannakian subcategory of* $\mathcal{A}$ *generated by* $\mathbf{H}(E_i)$.
(B) $\mathbf{H}^1(E_i)$ *is semi-simple for all* i*;*
(C) **(F)**$(E_i \times E_j, 1)$ *holds for all* i, j*;*

[2] This terminology creates an ambiguity with the name "Lefschetz motive" given to $\mathbb{L}$; we hope this will not cause any confusion.

(D) *Let E_i, E_j be non-isogenous and have complex multiplication (be ordinary if* $\operatorname{char} k > 0$*). Write F_{ij} for the real quadratic subfield of* $\operatorname{End}(E_i) \otimes \operatorname{End}(E_j) \otimes \mathbf{Q}$. *Then $F_{ij} \otimes_{\mathbf{Q}} K$ is a field.*

Then

(i) $\mathbf{H}^*(X)$ *is semi-simple;*

(ii) *The composition* $\mathbf{LMot}_H(X) \hookrightarrow \mathcal{M} \xrightarrow{\mathbf{H}^*} \mathcal{A}^*$ *is fully faithful, where* $\mathbf{LMot}_H(X)$ *is the full Tannakian subcategory of* $\mathbf{LMot}_H$ *generated by the [Lefschetz] motive of X.*

In particular, $\mathbf{H}^*$ *is fully faithful on* $\mathcal{M}_H(X)$*, where* $\mathcal{M}_H$ *is the category of Grothendieck motives relative to H with coefficients in K and* $\mathcal{M}_H(X)$ *is its full pseudo-abelian ⊗-category generated by the motive of X.*

Proof. (i) follows from (B) and the fact that semi-simplicity is preserved under direct sum and tensor product. Let $i, j \in I$. By a standard rigidity argument, (C) yields an isomorphism

$$\mathbf{H}^* : \mathbf{LMot}_H(h^1(E_i), h^1(E_j)) \xrightarrow{\sim} \mathcal{A}(\mathbf{H}^1(E_i), \mathbf{H}^1(E_j)). \tag{3.1}$$

Write $G_{\mathbf{LMot}}(X)$ for the Tannakian group of $\mathbf{LMot}(X)$ with respect to H_L: it comes with a "Tate" character t_X and contains the Tannakian group $G_\omega(\mathbf{H}(X)) =: G_\omega(X)$, which is reductive by (i). We have to prove equality. By [12, Cor. 4.7], we have an isomorphism

$$G_{\mathbf{LMot}}(X) \xrightarrow{\sim} \prod_{\mathbb{G}_m} G_{\mathbf{LMot}}(X_\alpha), \tag{3.2}$$

where the X_α's are the isotypic components of X (up to isogeny) and $\prod_{\mathbb{G}_m}$ denotes the fibre product over $\mathbb{G}_m$ with respect to the t_{X_α}'s. Moreover, $G_{\mathbf{LMot}}(X_\alpha) = G_{\mathbf{LMot}}(E_i)$ for any E_i of type X_α. Therefore we are reduced to proving:

(a) $G_\omega(E_i) = G_{\mathbf{LMot}}(E_i)$ for any i;

(b) $G_\omega(X) \xrightarrow{\sim} \prod_{\mathbb{G}_m, j\in J} G_\omega(X_j)$, where $J \subseteq I$ is a subset of representatives of the isogeny classes of the E_i's.

By the double centraliser theorem, (3.1) implies that

$$KG_\omega(E_i \times E_j) = KG_{\mathbf{LMot}}(E_i \times E_j) \text{ in } \operatorname{End} H^1(E_i \times E_j) \quad \text{for all } i, j \in J. \tag{3.3}$$

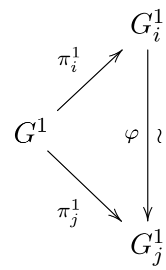
Note that t_{E_i} factors through $\det : \mathbf{GL}(H^1(E_i)) \to \mathbb{G}_m$ for all i. The reductive subgroups of $\mathbf{GL}_2$ which map surjectively to $\mathbb{G}_m$ by the determinant are $\mathbf{GL}_2$, $\mathbb{G}_m$ (its centre), maximal tori and the normalisers of maximal tori. These subgroups are distinguished by the K-subalgebra they generate in $\operatorname{End}_K H(E_i)$, except for the pair of $\mathbf{GL}_2$ and the normaliser of a maximal torus. Thus (3.3) proves (a) except when E_i has no complex multiplication, in which case we are saved by (A).

For (b), write for simplicity $G = G_\omega(X)$ and $G_j = G_\omega(E_j)$; write also M^1 for $\operatorname{Ker}(M \to \mathbb{G}_m)$, for all groups M appearing in the picture. Let $J' = \{j \in I \mid G_k^1 = \{1\}\}$: projecting onto $\prod_{i\in J-J'} G_i$, we may assume that $J' = \emptyset$.

It now suffices to prove that the inclusion $G^1 \subseteq \prod_i G_i^1$ is an equality. Assume the contrary and write $\pi_i^1 : G^1 \to G_i^1$ for the projection induced by π_i: since π_i

is faithfully flat, so is π_i^1. By Condition (A), all G_i's are connected, hence isomorphic to $\mathbf{GL}_2$ (no complex multiplication), $R_{\mathrm{End}(E_i)\otimes K/K}\mathbb{G}_m$ (complex multiplication/ordinary in positive characteristic) or $\mathbb{G}_m$ (supersingular). This ensures that all G_i^1's are (quasi-)simple, so we are in a position to apply Kolchin's result [11, Theorem]. By this theorem and the remarks following it, there are two possibilities:

(I) There exist two distinct indices $i \neq j$ such that G_i^1, G_j^1 are nonabelian (hence isomorphic to $\mathbf{SL}_2$), and an isomorphism $\varphi : G_i^1 \xrightarrow{\sim} G_j^1$ such that the diagram



commutes.

(II) There exist l distinct indices $j(1), \dots, j(l)$ with $l \geq 2$ and $G_{j(1)}^1, \dots, G_{j(l)}^1$, each commutative, and l faithfully flat K-homomorphisms $f_\lambda : G_{j(\lambda)}^1 \to G_{j(l)}^1$ $(1 \leq \lambda \leq l)$ such that

$$\prod_{1\leq\lambda\leq l} f_\lambda \circ \pi_{j(\lambda)}^1 = 1.$$

In Case (I), the image of G^1 in $G_i^1 \times G_j^1$ must be the diagonal, hence the image of G in $G_i \times_{\mathbb{G}_m} G_j$ must be the fibred diagonal, which contradicts (3.3). In Case (II), since $G_{j(\lambda)}^1$ is a 1-dimensional torus, the faithfully flat homomorphism f_λ is an isogeny for each λ. But Condition (D) implies that the K-tori $G_{j(\lambda)}^1$ and $G_{j(l)}^1$ are non-isomorphic, hence non-isogenous, a contradiction. □

4. **Weil cohomologies of Galois and differential type.** For the general case where k may not be separably closed, we need a special property of H. Let l/k be a finite separable extension, and write $H(l) = H(\operatorname{Spec} l)$: this is a commutative K-algebra of dimension $[l:k]$. If $\tilde{l}$ is a finite extension of l which is Galois over k, let $\Sigma = \mathrm{Hom}_k(l, \tilde{l})$; then the isomorphism $l \otimes_k \tilde{l} \xrightarrow{\sim} \prod_\Sigma \tilde{l}$ yields an isomorphism of K-algebras

$$H(l) \otimes_K H(\tilde{l}) \xrightarrow{\sim} \prod_\Sigma H(\tilde{l}), \tag{4.1}$$

If l is already Galois, then $Gal(l/k)$ acts on $H(l)$ and makes it a Galois K-algebra.

Lemma 2. *$H(l)$ is reduced (i.e. étale).*

Proof. Let R (resp. $\tilde{R}$) be the radical of $H(l)$ (resp. of $H(\tilde{l})$), $r = \dim R$, $\tilde{r} = \dim \tilde{R}$. The isomorphism (4.1) induces a surjection

$$H(l)/R \otimes_K H(\tilde{l})/\tilde{R} \longrightarrow\!\!\!\!\rightarrow \prod_{\Sigma} H(\tilde{l})/\tilde{R},$$

hence an inequality of dimensions

$$([l:k]-r)([\tilde{l}:k]-\tilde{r}) \geq |\Sigma|([\tilde{l}:k]-\tilde{r})$$

whence

$$[l:k]-r \geq |\Sigma|.$$

But $|\Sigma| = [l:k]$, hence $r = 0$. □

Definition 2. The Weil cohomology H is *of Galois type* (resp. *of differential type*) if the Galois algebra $H(l)$ is split (resp. is a field) for any finite Galois extension l/k.

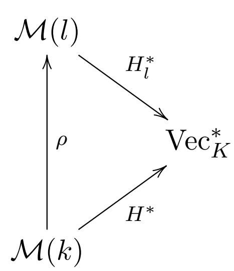
Example 1. If k or K is separably closed, any Weil cohomology with coefficients in K is of Galois type. Betti and l-adic cohomology are of Galois type, while de Rham and crystalline cohomology are of differential type. (As pointed out by Joseph Ayoub, tensoring de Rham cohomology with a nontrivial finite extension of k produces an example of a Weil cohomology which is neither of Galois nor of differential type.)

Let H be of Galois type. Then (4.1) shows that $H(l)$ is a split K-algebra for any finite separable extension l/k. More precisely, (4.1) descends to an isomorphism of K-algebras

$$H(l) \xrightarrow{\sim} \prod_{\Sigma} K. \tag{4.2}$$

Indeed, $G = Gal(\tilde{l}/k)$ permutes the idempotents of $H(\tilde{l})$, which implies that (4.1) is G-equivariant, and we take the G-invariants. Choosing a separable closure k_s of k, we may identify Σ with $\mathrm{Hom}_k(l, k_s)$; then (4.2) is natural in l.

Proposition 1. *Let $l \subseteq k_s$. Let $\rho : \mathcal{M}(k) \to \mathcal{M}(l)$ denote the extension of scalars functor.*
a) Suppose H of Galois type. Then there is a Weil cohomology H_l on $\mathbf{Sm}^{\mathrm{proj}}(l)$, with coefficients K, such that the diagram



is naturally $\otimes$-commutative.
b) Suppose H of differential type. Then there is a Weil cohomology H_l on $\mathbf{S}\mathrm{m}^{\mathrm{proj}}(l)$, with coefficients $H(l)$, such that the diagram

$$\begin{array}{ccc} \mathcal{M}(l) & \xrightarrow{H_l^*} & \mathrm{Vec}^*_{H(l)} \\ \uparrow \rho & & \uparrow \lambda \\ \mathcal{M}(k) & \xrightarrow[H^*]{} & \mathrm{Vec}^*_K \end{array}$$

is naturally $\otimes$-commutative, where λ is the functor "extension of scalars from K to $H(l)$".
In both cases, H_l is unique up to unique $\otimes$-isomorphism.

Proof. Let $X \in \mathbf{S}\mathrm{m}^{\mathrm{proj}}(l)$. Then $H^*(X)$ is an $H(l)$-algebra.
a) The inclusion of l in k_s defines a canonical element $\sigma \in \Sigma$. Define

$$H_l^*(X) = H^*(X) \otimes_{H(l)} K$$

where the homomorphism $H(l) \to K$ is induced by σ via (4.2).
b) We just provide $H_l^*(X)$ with its $H(l)$-algebra structure. □

Let $B \overset{f}{\leftarrow} A \overset{g}{\to} C$ be a 1-diagram in a 2-category $\mathcal{C}$. A 2-*push-out* of this diagram is a diagram $B \overset{g'}{\longrightarrow} D \overset{f'}{\longleftarrow} C$ and a 2-isomorphism $u : g'f \Rightarrow gf'$ which are universal for such data. There is another version where "2-isomorphism" is replaced by "2-morphism": we shall not need it here, but if $\mathcal{C} = \mathbf{Cat}$, it has the dual property to Mac Lane's comma construction. Then the 2-push-out exists, see here.

If $\mathcal{C}$ is the 2-category of abelian $\otimes$-categories and faithful, exact $\otimes$-functors, I guess this 2-push-out exists in general, but one ought to give it an explicit construction. The case I am interested in is

$$\begin{array}{ccc} \mathcal{M}_0(l) & & \\ \rho \uparrow & & \\ \mathcal{M}_0(k) & \xrightarrow{\mathbf{H}^*} & \mathcal{A}^*. \end{array}$$

where $\mathcal{M}_0(k) \subset \mathcal{M}(k)$ is the full subcategory of Artin motives. If $\mathcal{A}_l^*$ denotes this 2-push-out, it completes naturally the diagrams of Proposition 1 by its 2-universal property.

The category $\mathcal{A}_l^*$ is easy to describe in the Galois case: the functor $\mathcal{M}_0(k) \xrightarrow{\mathbf{H}^*} \mathcal{A}^*$ then induces a homomorphism

$$\pi : G_\omega \to G_k. \tag{4.3}$$

Let $G_\omega^l = \pi^{-1}(G_l)$; identifying $\mathcal{A}$ with $\mathrm{Rep}_K(G_\omega)$, we define $\mathcal{A}_l$ to be $\mathrm{Rep}_K(G_\omega^l)$. We write $\omega_l : \mathcal{A}_l \to \mathrm{Vec}_K$ for the corresponding fibre functor, so that $G_\omega^l = G_{\omega_l}$.

Lemma 3. *There is a unique factorisation of H_l^* through $\mathcal{A}_l^*$ (as a $\otimes$-functor); we denote it by $\mathbf{H}_l$.*

Proof. Let $X \in \mathbf{Sm}^{\rm proj}(l)$. The action of G_ω on $H(l)$ factors through (4.3), hence G_ω^l acts trivially on $H(l)$. Its action on $H^*(X)$ therefore induces an action on $H_l^*(X)$. □

Passing to the limit, we get a Weil cohomology H_s on $\mathbf{Sm}^{\rm proj}(k_s)$ and an enrichment $\mathbf{H}_s$ of H_s. We write $\mathcal{A}_s$ and ω_s for the corresponding Tannakian category and fibre functor.

Lemma 4. *Let $A \in \mathcal{A}$, and let A_s be its image in $\mathcal{A}_s$. Then A is semi-simple if and only if A_s is semi-simple. In particular, $\mathbf{H}(M)$ is semi-simple for any Artin motive M.*

Proof. Suppose A semi-simple. Let l/k be a finite Galois subextension of k_s/k. By Clifford's theorem [3], the restriction A_l of A to $\mathcal{A}_l$ is semi-simple. Since $\dim_K \omega(A) < \infty$, we can choose l large enough so that every simple summand of A_l remains simple in $\mathcal{A}_s$, which proves "only if". Conversely, suppose A_s semi-simple. Then A_l is semi-simple for l large enough, and the usual averaging argument [9, Lemma 3] then shows that A is semi-simple. The last statement follows, since $\mathbf{H}(M)_s$ is a trivial G_{ω_s}-module. □

5. **The general case.** We now relax the hypothesis that k is separably closed; we assume H to be of Galois type. We need:

Hypothesis 1. *For l/k as above, the canonical homomorphism (with $X = \operatorname{Spec} l$)*

$$K = A^0_{\rm alg}(X) \otimes K = \mathcal{M}(\mathbf{1}, h(X)) \to \mathcal{A}^*(\mathbf{1}, \mathbf{H}^*(X)) = \mathcal{A}(\mathbf{1}, \mathbf{H}^0(X))$$

is bijective.

Lemma 5. *Hypothesis 1 for $\mathbf{H}$ is equivalent to the surjectivity of* (4.3).

Proof. This follows from [4, Prop. 2.21 (a)] and Lemma 4. □

Theorem 3. *Let $\mathbf{H}$ verify Hypothesis 1. Let $X = \prod_i E_i$ be a product of elliptic curves over k. We keep the assumptions of Theorem 2 for $X_s := X \otimes_k k_s$ and $\mathbf{H}_s$. Then the conclusions of Theorem 2 hold for X and $\mathbf{H}$.*

Proof. The proof of (i) is the same as for Theorem 2, and its conclusion (ii) holds for X_s. Since taking invariants of continuous discrete actions of a profinite group is an exact functor on K-vector spaces, we get (ii) (over k) by taking Galois invariants, thanks to Hypothesis 1. □

Remark 2. The following condition

(C*) **(F)**$((E_i \times E_j)_l, 1)$ holds in $\mathcal{A}_l$ for all i, j and for any finite separable extension l/k.

implies Condition (C) for X_s in $\mathcal{A}_s$; it is actually equivalent to Condition (C), by taking Galois invariants. Similarly, Condition (B) for X and $\mathbf{H}$ is equivalent to the same for X_s and $\mathbf{H}_s$ by Lemma 4.

6. **Examples.** We take the examples of [1, Ch. 7]. In all cases, Hypothesis 1 is easily verified, and Conditions (B) of Theorem 2 and (C*) of Remark 2 hold by [1, th. 7.1.7.5]. Note that Condition (A) of Theorem 2 is implied by the stronger

Hypothesis 2. *The Tannakian group associated to* $(\mathcal{A}, \omega)$ *is connected.*

Hypothesis 2 for $\mathbf{H}_s$ is equivalent to the equality $G_\omega^0 = G_{\omega_s}$.

6.1. *The Hodge conjecture* [1, 7.2]. Here k is algebraically closed in $\mathbf{C}$, H is Betti cohomology, $\mathcal{A}$ is the category of polarisable pure $\mathbf{Q}$-Hodge structures and $K = \mathbf{Q}$. Condition (D) of Theorem 2 is automatic and Hypothesis 2 is classically verified (Mumford-Tate groups are connected). So Theorem 3 = Theorem 2 holds in this case.

6.2. *The Tate conjecture* [1, 7.3]. Here k is finitely generated over its prime field, H is ℓ-adic cohomology for some prime number ℓ invertible in k, $\mathcal{A}$ is the category of continuous representations of G_k on finite-dimensional $\mathbf{Q}_\ell$-vector spaces, and $K = \mathbf{Q}_\ell$. For $M \in \mathcal{M}$, $G_\omega(M)$ is the Zariski closure of the action of G_k on $H(M)$; it follows that the composition $G_k \to G_\omega(M)(k) \to \pi_0(G_\omega(M))$ is surjective, which implies Hypothesis 2.

Condition (D) of Theorem 2 does not always hold, but it does for a set of prime numbers in a suitable arithmetic progression depending on X, hence so does the (strong) ℓ-adic Tate conjecture for X. We are now going to prove it for *any* ℓ, in several steps.

6.2.1. k is finite. By the above, there exists at least one prime $\ell = \ell_0$ such that the ℓ_0-adic Tate conjecture holds for X, and $H^*_{\ell_0}(X)$ is semi-simple. By [18, Th. 2.9] (see Condition (c) of loc. cit.), the same then holds for any ℓ. This does not imply the strong form of the conjecture yet, so we refine this reasoning as follows. For $\ell = \ell_0$, algebraic cycles modulo ℓ_0-adic homological equivalence on X are generated by divisors. But by Condition (b) of loc. cit., they coincide with cycles modulo numerical equivalence, tensored with $\mathbf{Q}_{\ell_0}$. Hence cycles modulo numerical equivalence with $\mathbf{Q}$ coefficients are also generated by divisors, and this remains true a fortiori with $\mathbf{Q}_\ell$ coefficients for any ℓ. By the ℓ-adic Tate conjecture and still by loc. cit., they coincide with cycles modulo ℓ-adic homological equivalence, which shows that Tate cycles in $H^*_\ell(X)$ are generated by those of degree 2. We have recovered the results of [16] by a roundabout, but different method. (By [8], the ℓ_0-adic Tate conjecture actually implies that rational and numerical equivalences coincide on algebraic cycles on X with rational coefficients: this provides a shortcut to the above reasoning.)

6.2.2. k is a number field. Given a prime ℓ, write as above ω for the fibre functor corresponding to ℓ-adic cohomology. Write now $X = X_1 \times X_2$, where X_1 is the product of the CM factors of X and X_2 is the product of the other factors. As before we use an index s to indicate passage to the separable closure. By Theorem 2, we know that $G_{\omega_s}(X_{2,s}) \xrightarrow{\sim} G_{\mathbf{LMot}_s}(X_{2,s})$ and that $G_{\omega_s}(X_s) \xrightarrow{\sim} G_{\omega_s}(X_{1,s}) \times_{\mathbb{G}_m} G_{\omega_s}(X_{2,s})$; in view of (3.2), in order to conclude it suffices to prove that $G_{\omega_s}(X_{1,s}) \xrightarrow{\sim} G_{\mathbf{LMot}_s}(X_{1,s})$, and in view of 6.1 it suffices for this to know that $G_{\omega_s}(X_{1,s}) = G_H(X_{1,s})$, where G_H is the Tannakian group corresponding to the Hodge realisation (Mumford-Tate conjecture). This is proven in [7], and more generally for any CM abelian variety by Pohlmann [15] (see also Yu [19]).

6.2.3. The general case. Let k_0 be the field of constants of k. Write $X = X_1 \times X_2$ as above, where "CM" is meant to mean "ordinary" in positive charactristic. Similar to the previous reasoning, we may replace k by some finite separable extension, hence assume that X_1 is defined over k_0. Then the isomorphism

$$G_{\omega_s}(X_{1,s}) \xrightarrow{\sim} G_{\mathbf{LMot}_s}(X_{1,s})$$

over the separable closure of k_0 remains valid over the separable closure of k, because the left and the right hand sides do not change under such extension of scalars: this is clear for the right hand side, while for the left hand side it is true because $G_k \to G_{k_0}$ is surjective, hence G_k and G_{k_0} have same Zariski closure in $\mathbf{GL}(H^*_\ell(X_1))$.

6.3. *The weak Grothendieck period conjecture* [1, 7.5]. Here k is a number field embedded in $\mathbf{C}$, H is Betti cohomology, $\mathcal{A}$ is the category $\mathrm{Vec}_{k,\mathbf{Q}}$ of [1, 7.1.6] and $K = \mathbf{Q}$, so Condition (D) of Theorem 2 is automatic. To obtain Theorem 3, it suffices to prove Hypothesis 2, i.e. that any finite quotient G of the Tannakian group of $\mathrm{Vec}_{\bar{\mathbf{Q}},\mathbf{Q}}$ is trivial.

The following argument was kindly communicated by Y. André. Let (W, V, i) be the object of $\mathrm{Vec}_{\bar{\mathbf{Q}},\mathbf{Q}}$ corresponding to a representation of G. Then i is defined over $\bar{\mathbf{Q}}$ (indeed, W, V and their tensor constructions define a torsor under G, and i defines a complex point of this finite torsor). The choice of a basis of V then identifies (W, V, i) with a sum of copies of the unit object in $\mathrm{Vec}_{\bar{\mathbf{Q}},\mathbf{Q}}$.

Note that we get the statement of [2, Th. 1 (ii)]: there are no odd degree de Rham-Betti classes on X, since they are generated in degree 2.

Remark 3. It it tempting to try and reach the full Grothendieck period conjecture by the same technique, say for a product X of CM elliptic curves, reducing to the case of one such curve (Chudnovsky). However this is doomed to failure, unless one knows something on the closure Z of the canonical $\mathbf{C}$-point in the torsor $\mathfrak{P}$ of periods for X. Namely, a necessary condition is that Z is a sub-torsor of $\mathfrak{P}$, meaning that its stabiliser in $G_\omega(X)$ has same dimension as Z; conversely, this condition is inductively sufficient. Can one prove it?

6.4. *The Ogus conjecture* [1, 7.4]. Here k is a number field embedded in $\mathbf{C}$, $H = H_{\mathrm{dR}}$ is de Rham cohomology, $\mathcal{A}$ is the category $\mathbf{Og}(k)$ of [1, 7.1.5], $K = \mathbf{Q}$ [1, p. 72, footnote (4)] but $L = k$. Here (B) holds by [1, 7.4.1.4].

This case does not quite enter the above framework for two reasons:

- H is of differential type, not of Galois type;
- the Tannakian category $\mathcal{A}$ is not (a priori) neutral: more precisely, even if it turns out to be neutral, the fibre functor considered is not with coefficients in $K = \mathbf{Q}$ (unless $k = \mathbf{Q}$).

This first creates the issue of providing the categories $\mathcal{A}_l$ of Section 4; however, we may take $\mathcal{A}_l = \mathbf{Og}(l)$.

The second problem can be solved by extending scalars from $\mathbf{Q}$ to k, i.e. by replacing $\mathbf{Og}(k)$ by $\mathbf{Og}(k) \otimes_{\mathbf{Q}} k$; since $[k : \mathbf{Q}] < \infty$, this is still a Tannakian category and it is now neutralised by H^*_{dR}. Condition (A) is verified, thanks to the argument in [1, 7.4.3.1]. (More precisely, loc. cit. deals with a non CM elliptic curve, but the theorem of [17, A.2.4] covers the CM case similarly). Thus we get the Ogus conjecture provided Condition (D) holds with respect to $K = k$, in particular when k is as in Theorem 1.

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