COMPARISON OF SOME FIELD INVARIANTS

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In all this paper, p is a prime number and F is a field of characteristic $\neq p$. The invariants we are interested in are:

- The p-cohomological dimension $\mathrm{cd}_p(F)$ [16].
- The diophantine dimension $\mathrm{dd}(F) := \inf\{i \mid F \text{ is } C_i\}$ [5].
- (For $p = 2$) The quadratic diophantine dimension $\mathrm{dd}_q(F) := \inf\{i \mid F \text{ is } C_i^q\}$, where C_i^q is the condition introduced by Pfister [13].
- (For $p = 2$) The u-invariant $u(F)$ [8, ch. 11].
- The λ_p-invariant [7]: for an element $c \in {}_pBr(F) = H^2(F, \mu_p)$, $\lambda_p(c) = \inf\{n \mid c \text{ is a sum of } n \text{ classes of algebras of degree } p\}$; $\lambda_p(F) = \sup\{\lambda_p(c) \mid c \in {}_pBr(F)\}$.
- The λ'_p-invariant [7]: for c as above, $\lambda'_p(c) = \log_p \operatorname{ind} c$, where $\operatorname{ind} c$ is the Schur index of any central simple algebra representing c; $\lambda'_p(F) = \sup\{\lambda'_p(c) \mid c \in {}_pBr(F)\}$.

To muddy water a little more, we shall also consider the stable λ_p and λ'_p-invariants

$$\tilde{\lambda}_p(F) = \sup\{\lambda_p(E) \mid E/F \text{ finite separable, } ([E : F], p) = 1\}$$

$$\tilde{\lambda}'_p(F) = \sup\{\lambda'_p(E) \mid E/F \text{ finite separable, } ([E : F], p) = 1\}$$

and (for $p = 2$) the stable u-invariant:

$$\tilde{u}(F) = \sup\{u(E) \mid E/F \text{ finite separable, } [E : F] \text{ odd}\}.$$

Evidently, $\lambda_p(F) \leq \tilde{\lambda}_p(F)$, $\lambda'_p(F) \leq \tilde{\lambda}'_p(F)$ and $u(F) \leq \tilde{u}(F)$.

In proposition 1 below, we get some relationships between these invariants. This is applied in theorem 1 to get a universal bound for the length of the decomposition of a central simple algebra A of exponent 2 as a sum of symbols in the Brauer group of certain fields, purely in terms of the index of A. (Such an explicit bound is not known for algebras of odd prime exponent.) This boomerangs to provide a converse to a bound in proposition 1 (corollary 1). We then give some conjectures on what the sharp bound should be (conjecture 2) and on

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another relationship between the invariants of proposition 1 (conjecture 1). In the appendix, we give a construction of divided powers in certain quotients of Milnor K-theory, as these divided powers are used in the course of the proof of proposition 1. This has been known for a long time but has not appeared in print, to the best of our knowledge.

These results were found several years ago. At the time, the Milnor conjecture was not yet proven [20], so some were conditional to it. For the skeptical reader's convenience, we stress the results which depend on this conjecture with an asterisque (*).

1. The case $p = 2$

Proposition 1. *We have*

(1) $\mathrm{dd}_q(F) \leq \mathrm{dd}(F)$.
(2) $\tilde{u}(F) \leq 2^{\mathrm{dd}_q(F)}$.
(3) $2^{\mathrm{cd}_2(F)} \leq \tilde{u}(F)$. *(*)*
(4) $\lambda_2'(F) \leq \lambda_2(F)$, *with equality if* $\mathrm{cd}_2(F) = 2$.
(5) $\tilde{\lambda}_2'(F) \leq \tilde{\lambda}_2(F)$, *with equality if* $\mathrm{cd}_2(F) = 2$.
(6) $2\lambda_2(F) + 2 \leq u(F)$, *with equality if* $\mathrm{cd}_2(F) = 2$.
(7) $2\tilde{\lambda}_2(F) + 2 \leq \tilde{u}(F)$, *with equality if* $\mathrm{cd}_2(F) = 2$.
(8) $\mathrm{cd}_2(F) \leq 2\tilde{\lambda}_2(F) + 2$ *if F is not formally real* ($\leq 2\tilde{\lambda}_2(F) + 1$ *if* $-1 \in F^{*2}$). *(*)*

In summary:

$$\mathrm{cd}_2(F) \leq \log_2 \tilde{u}(F) \leq \mathrm{dd}_q(F) \leq \mathrm{dd}(F)(^*)$$

$$\mathrm{cd}_2(F) \leq 2\tilde{\lambda}_2(F) + 2 \leq \tilde{u}(F) \textit{ if } F \textit{ is not formally real.}(^*)$$

Proof. 1) $C_i \Rightarrow C_i^q$. 2) $u(F) \leq 2^{\mathrm{dd}_q(F)}$ is obvious; by [13], $\mathrm{dd}_q(E) \leq \mathrm{dd}_q(F)$ for any E algebraic over F. 3) If $2^i > u(F)$, then $I^i F = 0$, hence $H^i(F, \mathbf{Z}/2) = 0$ by the Milnor conjecture. If this is true for any finite extension of F of odd degree, then $\mathrm{cd}_2(F) < i$ [16]. 4) The inequality is obvious, and the equality was proven in [7] (Merkurjev's theorem). 5) Follows from 4). 6) This was proven in [7]. 7) Follows from 6).

For 8), we first assume $\sqrt{-1} \in F$. Then reduced power operations $x \mapsto x^{[i]}$ exist in $K^M(F)/2 = H^*(F, \mathbf{Z}/2)$ (see appendix). It suffices to show that $x = (a_1, b_1, \dots, a_i, b_i) = 0$ in $H^{2i}(E, \mathbf{Z}/2)$ for $i = 2\lambda_2(E) + 1$ and $[E : F]$ odd. Consider $c = (a_1, b_1) + \dots + (a_i, b_i) \in H^2(F, \mathbf{Z}/2)$. Then $x = c^{[i]}$. On the other hand, c is a sum of $i - 1$ symbols, hence $c^{[i]} = 0$.

If F is not formally real, then its absolute Galois group is torsion-free, hence $\mathrm{cd}_2(F(\sqrt{-1})) = \mathrm{cd}_2(F)$ [17]. On the other hand,

$H^q(F(\sqrt{-1}),\mathbf{Z}/2)$ is generated for any q by symbols of the form $(a_1,\dots,a_{q-1},b)$ where all a_i are in F^*. So $H^q(F(\sqrt{-1}),\mathbf{Z}/2)=0$ for $q>2\lambda_2(F)+2$. Repeating this argument for all separable odd degree extensions of F, we get what we want. □

Proposition 2. *The bounds in proposition 1 are optimal, except perhaps 8) when $\sqrt{-1}\notin F$.*

Proof. 1), 2) and 3): on $F=\mathbf{C}((t_1))\dots((t_n))$, the Pfister form $\ll t_1,\dots,t_n\gg$ is anisotropic; $\mathrm{cd}_2(F)=n$. 4), 5), 6), 7): take $\mathrm{cd}_2(F)=2$. 8) For $\sqrt{-1}\in F$, take $F=\mathbf{C}((t_1))\dots((t_{2n+1}))$: then $\mathrm{cd}_2(F)=2n+1$. On the other hand, $H^2(F,\mathbf{Z}/2)=\Lambda^2(\langle t_1,\dots,t_{2n+1}\rangle)$. The theory of alternating forms shows that any element in this alternating square is a sum of $[\frac{2n+1}{2}]=n$ decomposable tensors. □

In the same vein:

Proposition 3. *If F is a function field in n variables over an algebraically closed field k, then $\lambda_2(F)\geq[\frac{n}{2}]$.*

Proof. Complete F at a closed point x of a smooth model over k. Then $F_x\simeq k((t_1)\dots((t_n))$ and $H^2(F,\mathbf{Z}/2)\to H^2(F_x,\mathbf{Z}/2)$ is surjective. On the other hand, the argument above shows that $\lambda_2(F_v)=[\frac{n}{2}]$. □

This bound is not optimal (there are division algebras of exponent 2 and index 4 over $\mathbf{C}(t_1,t_2,t_3)$).

2. An application

Fix a ground field k. For any prime power $d\geq 1$, let

$$\lambda_p(d)=\sup\{\lambda_p(A)\mid A \text{ is a simple algebra of degree } d \text{ and exponent } p \text{ containing } k\}.$$

Theorem 1. *a) If k is algebraically closed, then $\lambda_2(d)\leq 2^{d^2+3d/2-2}-1$. b) If k is finite, then $\lambda_2(d)\leq 2^{d^2+3d/2-1}-1$.*

Proof. Let A be a generic central simple algebra of exponent 2 and degree d. Then $\lambda_2(d)=\lambda_2(A)$ [19]. If F is the centre of A, then $\lambda_2(A)\leq\frac{\tilde u(F)-2}{2}$ and $\tilde u(F)\leq 2^{\mathrm{dd}(F)}$ by proposition 1. It remains to find an explicit bound for $\mathrm{dd}(F)$.

We may construct A and F as follows. First take the division ring of left fractions A_0 of $k\{M,D\}$, where M is a generic square matrix of rank d and D is a generic diagonal matrix of the same rank. Then A_0 is a division algebra of degree d over its centre F_0, and $\mathrm{trdeg}(F_0/k)=d^2+d$.

Next, observe that ind $A^{\otimes 2} \leq d/2$ [1] (in fact there is equality here); let D be the associated division algebra, X its Severi-Brauer variety and $F = F_0(X)$. Then $A = A_0 \otimes_{F_0} F$ is generic of degree d and exponent 2 [3]. Now $\operatorname{trdeg}(F/F_0) = \dim X = d/2 - 1$. This gives

$$\operatorname{trdeg}(F/k) = d^2 + 3d/2 - 1.$$

In case a), we get $\operatorname{dd}(F) \leq d^2 + 3d/2 - 1$; in case b) we get $\operatorname{dd}(F) \leq d^2 + 3d/2$. This gives what we claimed. □

As a corollary, we obtain a converse to the bound of proposition 1 4):

Corollary 1. *For k as in theorem 1 and F containing k,*

$$\lambda_2(F) \leq 2^{2^{2\lambda'_2(F)} + 3\cdot 2^{\lambda'_2(F)-1} - 2} - 1 \text{ in case a)}$$

$$\lambda_2(F) \leq 2^{2^{2\lambda'_2(F)} + 3\cdot 2^{\lambda'_2(F)-1} - 1} - 1 \text{ in case b).}$$

□

These bounds look horrendous and are probably way too large, see conjecture 2 below. It is also annoying not to have any explicit bound for $\lambda_2(d)$ over $\mathbf{Q}$ or $\mathbf{Q}(i)$.

3. The case $p > 2$

In this case, much less is known. For example, even assuming the Kato conjecture, I don't know of an argument showing that $\operatorname{cd}_p(F) \leq \operatorname{dd}(F)$. (For $\operatorname{dd}(F) \leq 2$, this is true thanks to the reduced norm of central simple algebras.) It is also unknown whether $\lambda'_p(F) = \lambda_p(F)$ when $\operatorname{cd}_p(F) = 2$. Finally, while the generic argument does give that $\lambda_p(d)$ is finite for any d, I don't know of any explicit bound for it. The inequality $\operatorname{cd}_p(F) \leq 2\lambda_p(F) + 1$ is true, however, assuming the Kato conjecture (same proof).

Using index reduction methods, one can probably produce fields of cohomological dimension 2 with prescribed λ_p-invariant, including $\lambda_p = \infty$. For $p = 2$, this follows from Merkurjev's construction of fields with given even u-invariant [10] and proposition 1 6).

4. Some conjectures for $p = 2$

Conjecture 1. *If F is not formally real, $\tilde{u}(F) \leq 2^{2\lambda_2(F)+2}$.*

The evidence for this conjecture is meagre: it is true for $\lambda_2(F) = 1$ by Elman-Lam [4]. The first test would be to understand the case $\lambda_2(F) = 2$. In this respect, Saltman has proven that $\lambda_2(F) = 2$ for F a function field in one variable over $\mathbf{Q}_p$ (p odd) [15]; Hoffmann-van

Geel have used this result to prove that $\tilde{u}(F) \leq 22$ [6] and Parimala-Suresh have refined this bound to $\tilde{u}(F) \leq 10$ [12]. These results at least do not contradict the statement of conjecture 1. Maybe the bound is not correct; in any case I conjecture that $\tilde{u}(F)$ is bounded in terms of $\lambda_2(F)$.

Conjecture 2. $\lambda_2(d) \leq d/2$.

This conjecture is at least true for $d = 1, 2, 4, 8$ by results of Wedderburn, Albert [1] and Tignol [18]. Here is a related conjecture:

Conjecture 3. *Let A be a central simple algebra of exponent 2 over F, and let E/F be a finite extension.*
a) If $[E : F]$ is odd, then $\lambda_2(A_E) = \lambda_2(A)$.
b) If $[E : F] = 2$, then $\lambda_2(A_E) \geq \lambda_2(A)/2$.

One can easily check that this conjecture is true for $\mathrm{ind}(A) \leq 8$ (by using the same results as above).

Proposition 4. *Conjecture 3 implies conjecture 2.*

Proof. Let $\mathrm{ind}(A) = d$. We may suppose that A is division. By a), we may further assume that A contains a maximal subfield E which is filtered by quadratic extensions. Let K/F be a quadratic subfield of E/F: then $\mathrm{ind}(A_K) = d/2$ and the result follows by induction on d. □

Remark 1. It might even be true that $\lambda_2(d) = d/2$. In [19, cor. 2.10], Tignol proves that $\lambda_2(d) \geq \log_2 d + 1$ for $d \geq 8$.

Appendix A. Divided powers in Milnor K-theory

Theorem 2 (Papy [11], Revoy [14]). *Let M be a module over a commutative ring R. Then there exists a unique collection of maps (divided power operations)*

$$\Lambda^{2p}(M) \to \Lambda^{2ip}(M)$$
$$x \mapsto x^{[i]}$$

with the following properties:

(1) $s^{[0]} = 1$, $s^{[1]} = s$.
(2) $(st)^{[n]} = s^n t^{[n]}$.
(3) $s^{[m]} t^{[n]} = \binom{m+n}{n} (st)^{[m+n]}$.
(4) $(s+t)^{[n]} = \sum_{p+q=n} s^{[p]} t^{[q]}$.
(5) $(s^{[p]})^{[q]} = \frac{(pq)!}{p! q!^p} s^{[pq]}$.
(6) $s^{[p]} = 0$ *if s is a symbol (decomposable tensor) and $p \geq 2$.*

If $2M=0$, then the divided power operations are defined on the whole of $\Lambda^*(M)$, with the same properties. □

Remark 2. The statement when $2M=0$ does not appear in [14], but the proof is similar and actually simpler.

The following proposition was observed by Serre and Rost in the early nineties.

Proposition 5. *With notation as in theorem 2, let I be a graded ideal of $\Lambda(M)$ generated by symbols. Then the divided power operations of theorem 2 induce operations on $\Lambda(M)/I$.*

Proof. Let $x\in\Lambda^i(M)$ (i even if $2M\neq 0$), $y\in I$ be a symbol of degree i and $n\geq 2$. By Theorem 2 (1), (4) and (6), we have

$$(x+y)^{[n]}=x^{[n]}+yx^{[n-1]}\equiv x^{[n]}\pmod I$$

hence the result since I is generated by symbols. □

If F is a field and p is an odd prime, proposition 5 applies to $K_*^M(F)/p$, since then it is the quotient of the exterior algebra on F^*/F^{*p} by an ideal generated by symbols. This remains true for $K_*^M(F)$ and $K_*^M(F)/2$ when F has characteristic 2, but not in general as we have the identity $\{x,x\}=\{x,-1\}$ in $K_2(F)$. For $K_*^M(F)/2$, this will be true as soon as $-1\in F^{*2}$.

We get divided powers on the quotient of $K_*^M(F)$ by the ideal generated by $\{-1\}$ in all cases. This provides a hilarious proof of part of a theorem of Bass-Tate on the Milnor K-theory of a global field F [2]: by a theorem of Lenstra [9], $K_2(F)$ consists of symbols. Hence, by the same argument as in the proof of proposition 1 8), $K_4^M(F)$ is generated by $\{-1\}$ and $K_i^M(F)$ is of exponent 2 for all $i\geq 4$. (Bass and Tate's result is much sharper: they prove that $K_i^M(F)\simeq(\mathbf{Z}/2)^{r_1}$ for $i\geq 3$, where r_1 is the number of real embeddings of F.)

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