UNRAMIFIED COHOMOLOGY OF QUADRICS, II

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Abstract

We compute the unramified cohomology of quadrics of dimension 4 *in degree* 4 *over an arbitrary field of characteristic different from* 2. *We find that it is related to classical invariants of a more elementary nature, such as the group of spinor norms and the projective special orthogonal group modulo Manin's* R*-equivalence.*

Contents

1. Introduction

In this paper, we continue our investigation of the unramified cohomology of quadrics,

DUKE MATHEMATICAL JOURNAL
Vol. 106, No. 3, © 2001
Received 8 April 1999. Revision received 28 March 2000.
2000 *Mathematics Subject Classification*. Primary 11E04, 12G05; Secondary 11E88, 14F42.

initiated in [11]. Let F be a field of characteristic not equal to 2. For any F-scheme X, let $H^i X$ denote the étale cohomology group $H^i_{\text{ét}}(X, \mathbf{Z}/2)$, and let $H^i(X, i-1)$ denote the group $H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(i-1))$, where $\mathbf{Q}/\mathbf{Z}(i-1) = \varinjlim_{(n,\text{char } F)=1} \mu_n^{\otimes i-1}$. We recall from [11] the *unramified cohomology groups*

$$H^i_{\text{nr}}\big(F(X)/F\big) = \operatorname{Ker}\left(H^i F(X) \longrightarrow \coprod_{x \in X^{(1)}} H^{i-1} F(x)\right),$$

$$H^i_{\text{nr}}\big(F(X)/F, i-1\big) = \operatorname{Ker}\left(H^i F(X, i-1) \longrightarrow \coprod_{x \in X^{(1)}} H^{i-1}\big(F(x), i-2\big)\right)$$

for a smooth, proper, geometrically integral variety X over F. Extension of scalars from F to $F(X)$, the function field of X, determines homomorphisms

$$\eta_2^i : H^i F \longrightarrow H^i_{\text{nr}}\big(F(X)/F\big),$$

$$\eta^i : H^i(F, i-1) \longrightarrow H^i_{\text{nr}}\big(F(X)/F, i-1\big).$$

We are interested in the kernel and cokernel of η_2^i and η^i when X is a *quadric*. If X is the quadric defined by a quadratic form q, we sometimes write $\eta^i_{2,q}$ for precision. We then have $\dim X = \dim q - 2$. For the reader's convenience, let us briefly recall the results of [11].

(a) For any quadric X and any i, $\operatorname{Ker} \eta_2^i \xrightarrow{\sim} \operatorname{Ker} \eta^i$. For $i \le 4$, there is an exact sequence

$$0 \longrightarrow \big(\operatorname{Ker} \eta_2^i\big)_0 \longrightarrow \operatorname{Coker} \eta_2^i \longrightarrow \operatorname{Coker} \eta^i, \tag{1}$$

where

$$\big(\operatorname{Ker} \eta_2^i\big)_0 = \big\{\alpha \in \operatorname{Ker} \eta_2^i \mid \alpha \cdot (-1) = 0\big\}.$$

Let us mention here that the construction of this exact sequence relies on the Milnor conjecture. From the proof of this conjecture by Voevodsky, it follows that (1) is in fact valid for any i (cf. [11, proof of Prop. 7.4]).

(b) For $i \le 4$ and $\dim X > 2^{i-2} - 2$, $\operatorname{Ker} \eta_2^i$ consists of symbols, that is, of elements of the form $(a_1, \dots, a_i)$ for $a_1, \dots, a_i \in F^*$. A symbol $(a_1, \dots, a_i)$ lies in $\operatorname{Ker} \eta^i_{2,q}$ if and only if q is similar to a subform of the Pfister form $\langle\langle a_1, \dots, a_i \rangle\rangle$. In particular, it is

- trivial for $\dim X > 2^i - 2$,
- at most $\mathbf{Z}/2$ for $\dim X > 2^{i-1} - 2$.

Although we feel that the above must remain true for any i, it is not a straightforward consequence of the Milnor conjecture and we do not know a proof of it, say, for $i = 5$.

(c) For $i \leq 2$, $\operatorname{Coker} \eta^i = 0$ for all quadrics. The group $\operatorname{Coker} \eta^3$ is zero, except when X is an anisotropic Albert quadric (see Section 1.3 for a definition), in which case this group is isomorphic to $\mathbf{Z}/2$. Then the map $\operatorname{Coker} \eta_2^3 \to \operatorname{Coker} \eta^3$ is bijective. As for $\operatorname{Coker} \eta^4$, our main results are as follows. This group is zero for $\dim X = 1$ or $\dim X > 10$. For $\dim X > 4$, it embeds canonically into the 2-torsion of $\mathrm{CH}^3(X)$, which is itself at most $\mathbf{Z}/2$ (Karpenko).

Here we deal mostly with the cases $i = 4$, $\dim X = 2, 3, 4$. The cases where $i = 4$, $5 \leq \dim X \leq 10$ are dealt with in another paper (see [12]). This paper also contains the results on real quadrics announced in [11].

Let us now describe the results of the current paper.

1.1. The sequence (1)

A vexing issue is whether the map $\operatorname{Coker} \eta_2^i \to \operatorname{Coker} \eta^i$ is always surjective. We can prove the following theorem.

THEOREM 1

Suppose that F contains all 2-primary roots of unity. Then, for any quadric and any $i \geq 0$, the map $\operatorname{Coker} \eta_2^i \to \operatorname{Coker} \eta^i$ is surjective.

It seems quite difficult to make a descent from Theorem 1, even when F contains a fourth root of unity. The minimal possible counterexample to surjectivity is when $i = 4$ and X is a virtual Albert quadric (see Section 1.3 for a definition). To the best of our efforts, we have not been able to decide what happens in this case. At least we are able to prove surjectivity if we assume in addition that the 2-cohomological dimension of F is less than or equal to four (see Theorem 5).

1.2. $\operatorname{Ker} \eta_2^4$

Let us say that a subgroup A of $H^i F$ is *generated by its symbols* if every element of A is a sum of symbols, each of which belongs to A.

Obviously, A is generated by its symbols if it consists of symbols. The results recalled above then imply that $\operatorname{Ker} \eta_2^i$ is generated by its symbols for $i \leq 4$ and $\dim X > 2^{i-2} - 2$. For $i \leq 3$, this covers all quadrics, and for $i = 4$, it covers all quadrics of dimension greater than two. The following theorem deals with the remaining cases for $i = 4$.

THEOREM 2

For any quadric X of dimension less than or equal to two over F,

- $\operatorname{Ker} \eta_2^4 = H^1 F \cdot \operatorname{Ker} \eta_2^3$,
- $\operatorname{Ker} \eta_2^4$ *is generated by its symbols.*

To prove Theorem 2, we use the results of [17]. We note that, amusingly, the computation of $\operatorname{Ker}\eta_2^4$ for a 3-dimensional Pfister neighbour, which is the main step in [6] and [23], can be easily deduced from [17, (9.2)] and Theorem 2.1. We also provide an alternate proof of Theorem 2, in characteristic zero, by using the results of [10] (see Remark 2.7).

Note the following corollary to Theorem 2 and the results recalled before it (cf. [5, Prop. 3.2]).

COROLLARY 1
Let $i \leq 4$, $n \geq 0$ and q , ϕ be, respectively, a quadratic form and an n-fold Pfister form over F. Then

$$e^n(\phi)\operatorname{Ker}\eta^i_{2,q} \subseteq \operatorname{Ker}\eta^{n+i}_{2,\phi\otimes q}.$$

Proof
We argue as in [5]. We have to prove that, for $x \in \operatorname{Ker}\eta^i_{2,q}$, $e^n(\phi)\cdot x \in \operatorname{Ker}\eta^{n+i}_{2,\phi\otimes q}$. By Theorem 2 we may assume that x is a symbol, say, $x = e^i(\tau)$ for an i-fold Pfister form τ. Since $x_{F(q)} = 0$, we have $\tau_{F(q)} \sim 0$ (see [21]); hence q is similar to a subform of τ (see [1, Satz 1.3]). Then $\phi \otimes q$ is similar to a subform of the $(n+i)$-fold Pfister form $\phi\otimes\tau$. Therefore $(\phi\otimes\tau)_{F(\phi\otimes q)} \sim 0$ and $e^{n+i}(\phi\otimes\tau)_{F(\phi\otimes q)} = (e^n(\phi)\cdot x)_{F(\phi\otimes q)} = 0$. □

1.3. Coker η_2^4 *and* Coker η^4
Here we need to assume that char $F = 0$. We have the following results.

THEOREM 3
Let X be a quadric of dimension 2 *or* 3. *Then* Coker $\eta^4 = 0$.

By [11, Prop. A.1 and A.2], the same result holds for $\dim X = 1$ or 0 (the latter with a suitable definition of Coker η^4).

Assume now that $\dim X = 4$. For the reader's convenience, we recall the classification of (anisotropic) 4-dimensional quadrics used in [11]. Let X be such a quadric, let q be a quadratic form defining it, and let $d = d_\pm q \in F^*/F^{*2}$.

- *Neighbours*: $d \neq 1$, $q_{F(\sqrt{d})}$ is hyperbolic. (These are the 6-dimensional qs that are Pfister neighbours.)
- *Intermediate quadrics*: $d \neq 1$, $q_{F(\sqrt{d})}$ is isotropic without being hyperbolic. (These forms did not receive a name in [11].)
- *(Anisotropic) Albert quadrics*: $d = 1$.
- *Virtual Albert quadrics*: $d \neq 1$, $q_{F(\sqrt{d})}$ is anisotropic.

THEOREM 4

Let X be a 4*-dimensional quadric.*

(a) *If X is neighbour or intermediate, then* $\operatorname{Coker}\eta^4 = 0$.

(b) *If X is an Albert quadric, then cup product by the generator e of* $\operatorname{Coker}\eta^3$ *gives an isomorphism*

$$F^*/\operatorname{Sn}(X) \xrightarrow{\sim} \operatorname{Coker}\eta^4.$$

Here $\operatorname{Sn}(X)$ *is the group of spinor norms of X, that is, the subgroup of* F^* *generated by products of two nonzero values of q, where q is any quadratic form defining X.*

(c) *If X is a virtual Albert quadric of discriminant d, let* $E = F(\sqrt{d})$. *Then there is an exact sequence*

$$\operatorname{Coker}\eta_E^4 \xrightarrow{\operatorname{Cor}_{E/F}} \operatorname{Coker}\eta^4 \longrightarrow \operatorname{PSO}(q, F)/R \longrightarrow 0,$$

where $\operatorname{Coker}\eta_E^4 = \operatorname{Coker}\eta_{X_E}^4$ *and* $\operatorname{PSO}(q, F)/R$ *is the group of rational points of the projective special orthogonal group of q, modulo R-equivalence. Here q is any quadratic form defining X.*

In [20], Merkurjev shows that if q is a quadratic form of dimension less than or equal to six, the group $\operatorname{PSO}(q)$ is not R-trivial if and only if q is a virtual Albert form. It is therefore striking to see the group $\operatorname{PSO}(q, F)/R$ appear as a quotient of $\operatorname{Coker}\eta^4$ in the latter case. This is the first known cohomological description of this group.

COROLLARY 2

In Theorem 4 the map $\operatorname{Coker}\eta_2^4 \to \operatorname{Coker}\eta^4$ *is surjective, except perhaps in the case of a virtual Albert quadric.*

Proof

Indeed, the only issue is that of an Albert quadric. By [11], $\operatorname{Coker}\eta_2^3 \to \operatorname{Coker}\eta^3$ is then bijective, so cup product by the element e of (b) factors through $\operatorname{Coker}\eta_2^3$. □

We have not computed the kernel of the map $F^* \to \operatorname{Coker}\eta_2^4$ in Theorem 4(b); this would be an interesting exercise.

THEOREM 5

Let X be a virtual Albert quadric. Then the cokernel of the map

$$\operatorname{Coker}\eta_2^4 \longrightarrow \operatorname{Coker}\eta^4$$

is a subgroup of $\operatorname{Ker}\eta_2^5/(-1)\cdot\operatorname{Ker}\eta_2^4$. *In particular, it is zero if* $cd_2(F) < 5$.

For the proof of Theorems 3, 4, and 5, we use the results of [10]. This is the only reason why we need to assume that char $F = 0$, as the spectral sequences of [10] rely on results of Voevodsky which require resolution of singularities. Although he declined to be a coauthor of this paper, it was Markus Rost who first suggested that Theorems 3 and 4(b) should hold. He also explained the proof of Theorem 3 that we give here, assuming the existence of spectral sequences as in [10].

The structure of this paper is as follows. Theorem 2 is proven in Section 2. Theorem 3 is proven in Section 3, which also contains material for the next section. Theorem 4 is proven in Section 4; the most daunting case is that of a virtual Albert quadric; Theorems 1 and 5 are proven in Section 5. Finally, we construct in the appendix a spectral sequence analogous to those of [10] for an *affine* quadric; this spectral sequence is used in Section 4 for the virtual Albert case.

2. Proof of Theorem 2

Let X be a smooth, projective quadric of dimension greater than or equal to one. Choose a conic C traced over X, and let D be the quaternion algebra associated with C. If C is defined by a 3-dimensional form q, then $D = C_0(q)$. Recall from [17] the algebraic K-theory $K_*(X, D)$ of X with operators D (denoted there by $K_*^D(X)$); this is the K-theory of the exact category of locally free $\mathscr{O}_X$-modules provided with a left action of D. We have a Brown-Gersten-Quillen spectral sequence

$$E_2^{p,q} = H^p\big(X, \mathscr{K}_{-q}^D\big) \Longrightarrow K_{-p-q}(X, D), \tag{2}$$

where $H^p(X, \mathscr{K}_{-q}^D)$ denotes the pth cohomology group of an appropriate Gersten complex. We also have a Swan-style isomorphism

$$K_*(D)^n \oplus K_*\big(C_0 \otimes_F D\big) \xrightarrow{\sim} K_*(X, D) \tag{3}$$

described in [17, (4.1)]. (Here $n = \dim X$ and $C_0 = C_0(q)$, where q is any quadratic form defining X.)

THEOREM 2.1

There is a natural isomorphism

$$\operatorname{Ker} \eta_2^4 \xrightarrow{\sim} \frac{H^0\big(X, \mathscr{K}_2^D\big)}{K_2 D}.$$

Proof

Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & H^4F(C) & \hookrightarrow & H^4F(C \times X) & & \\
 & & \uparrow & & \uparrow & & \\
 & & H^4F & \xrightarrow{\eta_2^4} & H^4F(X) & & \\
 & & \uparrow{\scriptstyle \cdot[D]} & & \uparrow{\scriptstyle \cdot[D]} & & \\
0 & \longrightarrow & K_2F & \longrightarrow & K_2F(X) & \xrightarrow{d} & \coprod_{x \in X^{(1)}} K_1F(x) \\
 & & \uparrow{\scriptstyle \mathrm{Nrd}} & & \uparrow{\scriptstyle \mathrm{Nrd}} & & \uparrow{\scriptstyle \mathrm{Nrd}} \\
 & & K_2D & \longrightarrow & K_2D(X) & \xrightarrow{d'} & \coprod_{x \in X^{(1)}} K_1D(x)
\end{array}
$$

In this diagram both rows are complexes, and the maps labeled d and d' have (by definition) respective kernels $H^0(X, \mathscr{K}_2)$ and $H^0(X, \mathscr{K}_2^D)$. The top horizontal map is injective because X is isotropic over $F(C)$. The columns are exact at H^4F and $H^4F(X)$ by the 1-dimensional case, and at K_2F and $K_2F(X)$ by [17, Th. 2]. The bottom right square commutes by [17, (3.14)]. Finally, the reduced norm on K_1 of quaternion algebras is injective by Wang's theorem, so that the bottom right vertical arrow is injective. These remarks and an easy diagram chase give the map we want.

The third row of the above diagram is exact. At K_2F it follows from [24, Th. 3.6] and at $K_2F(X)$ it follows from [24, Cor. 5.6], since X is a complete rational variety. This implies that our map is an isomorphism. □

PROPOSITION 2.2 (see [17, §11])
Let $n = \dim X$. *Then*
(a) *the norm homomorphism* $H^n(X, \mathscr{K}_{n+1}^D) \xrightarrow{N} K_1D$ *is an isomorphism;*
(b) *the natural map* $H^n(X, \mathscr{K}_{n+1}^D) \to K_1(X, D)^{(n)}$ *is an isomorphism.*
In particular, all differentials arriving at $H^n(X, \mathscr{K}_{n+1}^D)$ *in the spectral sequence (2) are zero.*

(The results of [17, §11] in fact concern the case $n = 3$, but the proofs are easily seen to carry over for any n.)

COROLLARY 2.3
If $n = 2$, *the map* $K_2(X, D) \to H^0(X, \mathscr{K}_2^D)$ *is surjective. Therefore there is a*

canonical isomorphism

$$\frac{K_2(X,D)^{(0/1)}}{K_2D} \xrightarrow{\sim} \frac{H^0\big(X,\mathscr{K}_2^{\,D}\big)}{K_2D}.$$

Suppose $\dim X = 2$; let $d = d_\pm X$, and let $E = F(\sqrt{d})$. Then $C_0 \simeq D \otimes_F E$; hence $C_0 \otimes_F D$ is a simple algebra similar to E. On the other hand, X is hyperbolic on $E(X)$; hence $E(X)$ splits $D(X) := D \otimes_F F(X)$ and so is contained in $D(X)$. This allows us to define a map $\nu : K_*E \to K_*D(X)$ as the composite

$$K_*E \longrightarrow K_*E(X) \longrightarrow K_*D(X).$$

PROPOSITION 2.4
The composite

$$K_*D \oplus K_*D \oplus K_*E \xrightarrow{\sim} K_*(X,D) \longrightarrow K_*D(X)$$

maps a triple (f,g,v) to $f_{D(X)} + g_{D(X)} + \nu(v)$. Here the first map is Swan's isomorphism (3).

Proof
This follows from an elementary computation in the spirit of [17, (4.3), (4.11)]. □

COROLLARY 2.5
The composition

$$K_2E \hookrightarrow K_2(X,D) \twoheadrightarrow K_2(X,D)^{(0/1)} \twoheadrightarrow \frac{K_2(X,D)^{(0/1)}}{K_2D}$$

is surjective, where the first map is given by Swan's isomorphism.

PROPOSITION 2.6
Let X be a 2-dimensional quadric defined by a quadratic form of discriminant $d \neq 1$ and $E = F(\sqrt{d})$. Then the corestriction map

$$\mathrm{Cor}_{E/F} : \mathrm{Ker}\,\big(H^4E \longrightarrow H^4E(X)\big) \longrightarrow \mathrm{Ker}\,\big(H^4F \longrightarrow H^4F(X)\big)$$

is surjective.

Proof
By Theorem 2.1 and Corollaries 2.3 and 2.5, it suffices to prove that the map

$$K_2\big(E \otimes_F E\big) \xrightarrow{1\otimes N_{E/F}} K_2(E)$$

is surjective, which is obvious. □

Proof of Theorem 2
The statement is true for quadrics of dimension zero by the long exact sequence for a quadratic extension (see [1, Satz 4.5], [21]). If $\dim X = 1$, then by [19, Prop. 3.15] and the Milnor conjecture in degrees 2 and 3 (see [16], [21]), one has $\operatorname{Ker}\eta^4 = c(q)\cdot H^2F$, where q is a quadratic form defining X and $c(q)$ is its Clifford invariant. Theorem 2 follows from this by reapplying [16]. If $\dim X = 2$ and X is defined by a 2-fold Pfister form, then $\operatorname{Ker}\eta^4_X = \operatorname{Ker}\eta^4_C$, where C is a conic traced on X [11, Prop. 2.5 c)], so Theorem 2 follows from the 1-dimensional case.

Finally, let X be defined by $q = \langle d, -a, -b, ab\rangle$, with $d \notin F^{*2}$, and let $E = F(\sqrt{d})$. Note that q_E is the 2-fold Pfister form $\langle\langle a, b\rangle\rangle$, so $\operatorname{Ker}(H^4E \to H^4E(X)) = (a, b)\cdot H^2E$ by the above. Let $(a, b)\cdot x \in \operatorname{Ker}(H^4E \to H^4E(X))$, with $x \in H^2E$. By [16], x is a sum of symbols and, by [2, Cor. 5.3], it is even a sum of symbols of the form (y, e), with $y \in E^*$ and $e \in F^*$. Now

$$\operatorname{Cor}_{E/F}(a, b, y, e) = \big(a, b, N_{E/F}y, e\big).$$

This proves both (a) and (b), since

$$\big(a, b, N_{E/F}y\big) = \operatorname{Cor}_{E/F}(a, b, y) \in \operatorname{Ker}\big(H^3F \longrightarrow H^3F(X)\big). \qquad \square$$

Remark 2.7
In characteristic zero, one can give an alternative proof of Theorem 2 by using the results of [10]. By [10, Cor. 5.5], there is (for any quadric X) an exact sequence after localisation at two:

$$0 \longrightarrow H^1\big(X, \mathcal{K}_3\big) \longrightarrow K_2\big(E_1\big) \xrightarrow{\delta} \operatorname{Ker}\eta^4 \longrightarrow H^2\big(X, \mathcal{K}_3\big) \longrightarrow H^2\big(\overline{X}, \mathcal{K}_3\big),$$

where E_1 is a certain étale extension of F associated to X. If $\dim X = 2$, we have by [10, Prop. 6.2, Lemma 8.2, and Cor. 8.6] (see also [10, Cor. 6.3]),

$$E_1 = E = F\big(\sqrt{d}\big),$$
$$\delta(\{a, b\}) = \operatorname{Cor}_{E/F}\big((a, b)\cdot c(q)\big).$$

On the other hand, the map $H^2(X, \mathcal{K}_3) \to H^2(\overline{X}, \mathcal{K}_3)$ is injective (cf. [21, Lemma 2.6]).

3. Coker η^4: Some preparations. Proof of Theorem 3

Let X be a quadric over F. Recall that, by [10, Th. 4.4], there is a spectral sequence for any $n \geq 0$,

$$E_2^{p,q}(X, n) = H^{p-q}_{\text{ét}}\big(F, \operatorname{CH}^q(X_s)\otimes \mathbf{Z}(n-q)\big) \Longrightarrow H^{p+q} \qquad (4)$$

with maps $H^{p+q} \to H^{p+q}_{\text{ét}}(X, \mathbf{Z}(n))$ which are bijective for $p+q \leq 2n$ and injective for $p+q = 2n+1$. Here $H^{*}_{\text{ét}}(X, \mathbf{Z}(n))$ is étale motivic cohomology and $X_s = F_s \otimes_F X$, where F_s is a separable closure of F. These spectral sequences are compatible with products and transfer.

Suppose $n \leq 3$. Then one has $H^{n+1}_{\text{ét}}(F, \mathbf{Z}(n)) \otimes \mathbf{Z}_{(2)} = 0$ (Hilbert's theorem 90); from this one deduces that

$$E_2^{n+q+1,q}(X, n+q) \otimes \mathbf{Z}_{(2)} = 0 \tag{5}$$

for any $q \geq 0$ (see [10]). (In fact the restriction on n is not necessary in view of Voevodsky's proof of the Milnor conjecture (see [26]), but we do not need this here.)

Specialising (see [10, Cor. 5.5]) to quadrics, dimension by dimension, and taking [10, Prop. 6.2, Lemma 8.2, and Cor. 8.6] into account, we obtain the complexes below after localisation at two.

3.1. The cases $\dim X = 2, 3$*; proof of Theorem 3*

If $\dim X = 2$, Coker η^4 is a subgroup of the homology of

$$H^2\big(X, \mathscr{K}_3\big) \xrightarrow{\xi^4} F^* \xrightarrow{\delta} H^3 E. \tag{6}$$

Here $E = F(\sqrt{d})$, ξ^4 is the map $H^2(X, \mathscr{K}_3) \to H^2(\overline{X}, \mathscr{K}_3)^{G_F} \simeq F^*$, and $\delta(x) = (x)_E \cdot c(q_E)$, where q is a quadratic form defining X; δ is the differential $d_2^{3,2}(X, 3)$ of the above spectral sequence. (Note that $\mathrm{CH}^3(X) = 0$.) By [4, Prop. 2.3], (6) is exact.

If $\dim X = 3$, Coker η^4 is a subgroup of the homology of

$$H^2\big(X, \mathscr{K}_3\big) \xrightarrow{\xi^4} F^* \xrightarrow{\delta} H^3 F. \tag{7}$$

Notation is as above, except that $\delta(x) = (x) \cdot c(q)$. (Note that $\mathrm{CH}^1(X) \to \mathrm{CH}^1(\overline{X})$ is surjective and $\mathrm{CH}^3(X)_{\text{tors}} = 0$ (see [13]).) By [11, computation before Cor. 5.3] and [18, Cor. to Th. 7], (7) is exact.

We have proven Theorem 3. □

3.2. The case $\dim X = 4$

If $\dim X = 4$, there is a complex

$$0 \longrightarrow \operatorname{Coker} \eta^4 \longrightarrow \mathrm{CH}^3(X)_{\text{tors}} \longrightarrow H^6\big(X, \mathbf{Z}(3)\big), \tag{8}$$

and the kernel of the first map is a subgroup of the homology of

$$H^2\big(X, \mathscr{K}_3\big) \xrightarrow{\xi^4} E^* \xrightarrow{\delta} H^3 F. \tag{9}$$

Notation is as above, except that $\delta(x) = \mathrm{Cor}_{E/F}((x)\cdot c(q_E))$. (Note that $\mathrm{CH}^1(X) \to \mathrm{CH}^1(\overline{X})$ is surjective.) By [13, Cor. 4.5, Th. 7.3, and Remark 7.2], $\mathrm{CH}^3(X)_{\mathrm{tors}} = \mathbf{Z}/2$ if X is intermediate or a neighbour, and zero otherwise.

In the Albert and virtual Albert cases, we need the following more precise fact (see [13]). The differential $d_3^{3,2}(X,3)$ of the spectral sequence (4) acts as

$$\mathrm{Ker}\,\delta \xrightarrow{d_3^{3,2}(X,3)} H^5(F,3); \tag{10}$$

one has $\mathrm{Im}\,\xi^4 \subseteq \mathrm{Ker}\,d_3^{3,2}(X,3)$ and the kernel of the first map in (8) is isomorphic to $\mathrm{Ker}\,d_3^{3,2}(X,3)/\,\mathrm{Im}\,\xi^4$. We see that, in fact, $d_3^{3,2}(X,3) = 0$ for any 4-dimensional quadric X.

We need the following lemma here only for $n = 1, 2$ ($n = 2$ for the proof of Proposition 4.5, $n = 1$ for the proof of Lemma 4.17).

LEMMA 3.1

Let X be a 4-dimensional quadric, let h be a hyperplane section of X, and let $\mathrm{cl}^1(h)$ be its class in $H^2(X,\mathbf{Z}(1))$. Let $\pi : X \to \mathrm{Spec}\,F$ be the structural map, and let $\beta : \mathbf{Z}/2_{\text{ét}} \simeq \mathbf{Z}/2(n)_{\text{ét}} \to \mathbf{Z}(n)[1]_{\text{ét}}$ be the integral Bockstein. Then the sequence

$$H^{n-1}(E,\mathbf{Z}/2) \xrightarrow{A} H^{n+2}\big(F,\mathbf{Z}(n)\big) \xrightarrow{B} H^{n+4}\big(X,\mathbf{Z}(n+1)\big)$$

is exact after localisation at two for $0 \le n \le 3$, where $A(x) = \beta\,\mathrm{Cor}_{E/F}(x\cdot c(q_E))$ and $B(y) = (\pi^* y)\cdot \mathrm{cl}^1(h)$.

Proof

Let $F^pH^m(X,\mathbf{Z}(n))$ be the (decreasing) filtration induced on $H^m(X,\mathbf{Z}(n))$ by the spectral sequence (4). By multiplicativity, the image of B is contained in $F^{n+2}H^{n+4}(X,\mathbf{Z}(n+1))$. The factor group

$$\frac{F^{n+2}H^{n+4}\big(X,\mathbf{Z}(n+1)\big)}{F^{n+3}H^{n+4}\big(X,\mathbf{Z}(n+1)\big)} = E_\infty^{n+2,2}(X,n+1)$$

is a subquotient of $E_2^{n+2,2}(X,n+1)$. For $n \le 4$, the latter is zero by Hilbert's theorem 90 (see (5); remember that everything is localised at 2); hence $F^{n+2}H^{n+4}(X,\mathbf{Z}(n+1)) = F^{n+3}H^{n+4}(X,\mathbf{Z}(n+1))$. Dividing by F^{n+4}, we therefore get an induced map

$$H^{n+2}\big(F,\mathbf{Z}(n)\big) \xrightarrow{\bar{B}} E_\infty^{n+3,1}(X,n+1).$$

For $n \le 3$, the differential hitting the corresponding E_r-term is zero for $r > 2$, by dimension counting. So, in this range, there is a commutative diagram

$$\begin{array}{ccc} & & \dfrac{E_2^{n+3,1}\big(X, n+1\big)}{\operatorname{Im} d_2^{n+1,2}} \\ & \nearrow & \uparrow \\ H^{n+2}\big(F, \mathbf{Z}(n)\big) & \xrightarrow{\bar{B}} & E_\infty^{n+3,1}\big(X, n+1\big) \end{array} \tag{11}$$

where the vertical arrow is injective. We have an isomorphism

$$E_2^{n+3,1}(X, n+1) \simeq H^{n+2}\big(F, \mathrm{CH}^1\left(X_s\right) \otimes \mathbf{Z}(n)\big) \simeq H^{n+2}\big(F, \mathbf{Z}(n)\big).$$

This isomorphism identifies the oblique map in (11) with the natural projection. Indeed, by multiplicativity, it suffices to check this for $n = 0$. In that case we have $d_2^{n+1,2} = 0$, and we are left to prove that the composition

$$H^2(F, \mathbf{Z}) \xrightarrow{B} H^4\big(X, \mathbf{Z}(1)\big) \xrightarrow{e} H^2\big(F, \mathrm{CH}^1(X_s)\big)$$

is the identity (identifying $\mathrm{CH}^1(X_s)$ with $\mathbf{Z}$ via its generator h), where e is the edge homomorphism of the weight 1 spectral sequence. This is clear, again by multiplicativity. Lemma 3.1 follows, using [10, Cor. 8.6 and 6.3] for the values of the d_2 differentials. □

4. Proof of Theorem 4

4.1. Generalities

Let X be a 4-dimensional quadric defined by the 6-dimensional quadratic form q. We constantly use the Clifford algebra $C(q)$ and even the Clifford algebra $C_0(q)$ of q, for which we refer to [15, Chap. V] or [14]. Let $d = d_{\pm}q$ and $E = F[t]/(t^2 - d)$, so that

$$E = \begin{cases} F\big(\sqrt{d}\big) & \text{if } d \notin F^{*2}, \\ F \times F & \text{if } d \in F^{*2}. \end{cases}$$

Recall that $C(q)$ is a central simple algebra of degree 8 over F and that $C_0(q)$ is an Azumaya algebra of degree 4 over E; if $d = 1$, then $C(q) \simeq M_2(A)$ and $C_0(q) \simeq A \times A$, where A is central simple of degree 4 over F. In general, by scaling q (e.g., ensuring $1 \in D(q)$), we may and do assume that $\operatorname{ind} C(q) \leq 4$.

PROPOSITION 4.1

In (9), we have

(a) $\operatorname{Ker}\delta = \{a \in E^* \mid N_{E/F}(a) \in F^{*2} \operatorname{Nrd} C(q)^*\}$;

(b) $\operatorname{Im}\xi^4 = F^* \operatorname{Nrd}_{C_0(q)/E} C_0(q)^*$.

Proof

(a) For $a \in E^*$, we have

$$\delta(a) = \operatorname{Cor}_{E/F}\left((a) \cdot c(q_E)\right) = \left(N_{E/F}(a)\right) \cdot c(q).$$

Since $\operatorname{ind} C(q) \leq 4$, the sequence

$$F^{*2} \operatorname{Nrd} C(q)^* \longrightarrow F^* \xrightarrow{\cdot c(q)} H^3 F$$

is exact by [18, Cor. to Th. 7], hence the claim.

(b) By the same arguments as in [18, pp. 74–75], we have

$$K_1(X)^{(2)} = F^*h^2 \oplus F^*h^3 \oplus K_1\big(C_0(q)\big),$$

where h is the class of a hyperplane section. Over the separable closure F_s, we get

$$K_1\big(X_s\big)^{(2)} = F_s^*h^2 \oplus F_s^*h^3 \oplus F_s^* P_1 \oplus F_s^* P_2,$$

where P_1, P_2 are the classes of two conjugate plane sections. Therefore

$$\big(K_1\big(X_s\big)^{(2)}\big)^{G_F} \simeq F^*h^2 \oplus F^*h^3 \oplus E^*,$$

and for this decomposition, the map $K_1(X)^{(2)} \to (K_1(X_s)^{(2)})^{G_F}$ induced by extension of scalars has matrix $\left(\begin{smallmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Nrd} \end{smallmatrix}\right)$. (Remember that $C_0(q)$ is an Azumaya algebra over E.) On the other hand, $H^2(X_s, \mathcal{K}_3) = F_s^* \otimes \mathrm{CH}^2(X_s) = F_s^* P_1 \oplus F_s^* P_2$, and with this decomposition, the projection $K_1(X_s)^{(2)} \to H^2(X_s, \mathcal{K}_3)$ has matrix $\left(\begin{smallmatrix} \mathrm{Id} & 0 & \mathrm{Id} & 0 \\ \mathrm{Id} & 0 & 0 & \mathrm{Id} \end{smallmatrix}\right)$. Hence the map $F^*h^2 \oplus F^*h^3 \oplus E^* \simeq (K_1(X_s)^{(2)})^{G_F} \to H^2(X_s, \mathcal{K}_3)^{G_F} \simeq E^*$ has matrix $(\,\Delta\ 0\ \mathrm{Id}\,)$, where Δ is the embedding $F^* \to E^*$. Finally, the composite map Φ has matrix

$$\begin{pmatrix} \Delta & 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Nrd} \end{pmatrix} = \begin{pmatrix} \Delta & 0 & \mathrm{Nrd} \end{pmatrix}.$$

By [18, Prop. 2], $K_1(X)^{(2)} \to H^2(X, \mathcal{K}_3)$ is surjective; hence Φ has the same image as ξ^4. The conclusion follows. □

4.2. The neighbour case

It was proved in [11, Th. 6(3)].

4.3. The intermediate case

The following lemma lifts a result of Szyjewski [23, Prop. 3.3.6(b) and 5.4.6] from mod 2 cohomology to motivic cohomology. For any smooth variety X and $n \geq 0$, let

$$\mathrm{cl}^n : \mathrm{CH}^n(X) \longrightarrow H^{2n}_{\text{ét}}\big(X, \mathbf{Z}(n)\big)$$

denote the étale motivic cycle map.

LEMMA 4.2

Let Y be the 3-dimensional quadric defined by any neighbour of the 3-fold Pfister form $\langle\langle a, b, c\rangle\rangle$. Let $\pi : Y \to \operatorname{Spec} F$ be the structure map, and let e be the nonzero torsion element of $\mathrm{CH}^2(Y)$ (see [23, Prop. 3.3.6(b)], [13, Th. 6.1]). Then

$$\mathrm{cl}^2(e) = \pi^*\beta(a, b, c).$$

Here $(a, b, c) \in H^3(F, \mathbf{Z}/2) = H^3(F, \mathbf{Z}/2(2))$ is the cup product of the Kummer classes of a, b, c, and $\beta : H^3(F, \mathbf{Z}/2(2)) \to H^4(F, \mathbf{Z}(2))$ is the integral Bockstein.

Proof

The spectral sequence (4), together with [10, Cor. 8.6], yields another exact sequence

$$0 \longrightarrow H^4\big(F, \mathbf{Z}(2)\big) \xrightarrow{\pi^*} H^4\big(Y, \mathbf{Z}(2)\big) \longrightarrow \mathrm{CH}^2\left(Y_{F_s}\right)^{G_F}.$$

Let $K = F(Y)$. We get a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & H^4\big(K, \mathbf{Z}(2)\big) & \xrightarrow{\pi^*} & H^4\big(Y_K, \mathbf{Z}(2)\big) & \longrightarrow & \mathrm{CH}^2\left(Y_{K_s}\right)^{G_K} \\
 & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & H^4\big(F, \mathbf{Z}(2)\big) & \xrightarrow{\pi^*} & H^4\big(Y, \mathbf{Z}(2)\big) & \longrightarrow & \mathrm{CH}^2\left(Y_{F_s}\right)^{G_F}
\end{array}$$

The right vertical arrow is an isomorphism. The left one coincides via the Bockstein map with

$$H^3\big(F, \mathbf{Q}/\mathbf{Z}(2)\big) \longrightarrow H^3\big(K, \mathbf{Q}/\mathbf{Z}(2)\big).$$

The kernel of this map is 2-torsion by the usual transfer argument. Since $H^3(F, \mathbf{Z}/2)$ injects into $H^3(F, \mathbf{Q}/\mathbf{Z}(2))$ (see [16]), it coincides with the kernel of

$$H^3(F, \mathbf{Z}/2) \longrightarrow H^3(K, \mathbf{Z}/2)$$

which is generated by (a, b, c) (see Arason [1, Satz 5.6]). All this implies that

$$\operatorname{Ker}\left(H^4\big(Y, \mathbf{Z}(2)\big) \longrightarrow H^4\big(Y_K, \mathbf{Z}(2)\big)\right) = \big\langle \pi^*\beta(a, b, c)\big\rangle.$$

Recall from [9] or [10] the short exact sequence

$$0 \longrightarrow \mathrm{CH}^2(Y) \xrightarrow{\mathrm{cl}^2} H^4\big(Y, \mathbf{Z}(2)\big) \longrightarrow H^0\big(Y, \mathscr{H}^3\big(\mathbf{Q}/\mathbf{Z}(2)\big)\big) \longrightarrow 0.$$

Since Y_K is isotropic, $e_K = 0$; hence $\mathrm{cl}^2(e) = \pi^*\beta(a, b, c)$ since cl^2 is injective. □

Remark 4.3
Lemma 4.2 shows incidentally that, if ψ is any neighbour of a 3-fold Pfister form φ, the map $\mathrm{CH}^2(X_\varphi)_{\mathrm{tors}} \to \mathrm{CH}^2(X_\psi)_{\mathrm{tors}}$ is an isomorphism.

LEMMA 4.4
Let $q = \langle\langle a, b\rangle\rangle \perp \langle -c, cd\rangle$ be an anisotropic 6-dimensional quadratic form, and let $E = F(\sqrt{d})$. Then the equation

$$(a, b, c) = \mathrm{Cor}_{E/F}\left(x \cdot (a, b)_E\right) \in H^3(F, \mathbf{Z}/2)$$

has no solution in $x \in H^1(E, \mathbf{Z}/2)$.

Proof
Let x be a solution; then

$$\big(a, b, cN(x)\big) = 0.$$

By [21], this implies that the 3-fold Pfister form $\langle\langle a, b, cN(x)\rangle\rangle$ is hyperbolic or that

$$\langle\langle a, b\rangle\rangle \cong cN(x)\langle\langle a, b\rangle\rangle.$$

Therefore

$$\begin{aligned} q &\cong cN(x)\langle\langle a, b\rangle\rangle \perp \langle -c, cd\rangle \\ &\cong cN(x)\big(\langle\langle a, b\rangle\rangle \perp N(x)\langle -1, d\rangle\big) \\ &\cong cN(x)\big(\langle\langle a, b\rangle\rangle \perp \langle -1, d\rangle\big), \end{aligned}$$

and the last form is isotropic, thereby contradicting the hypothesis. □

PROPOSITION 4.5
Let X be intermediate. Then
(a) *the map* $\mathrm{CH}^3(X)_{\mathrm{tors}} \to H^6(X, \mathbf{Z}(3))$ *of* (8) *is injective;*
(b) *in (9),* $\mathrm{Ker}\,\delta / \mathrm{Im}\,\xi^4 = 0$.

Proof
(a) Let q' be a 5-dimensional subform of a form q defining X which is a Pfister

neighbour, and let Y be the corresponding hyperplane section of X. Let i be the corresponding closed immersion. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^2(Y) & \xrightarrow{\mathrm{cl}^2} & H^4\big(Y, \mathbf{Z}(2)\big) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ \mathrm{CH}^3(X) & \xrightarrow{\mathrm{cl}^3} & H^6\big(X, \mathbf{Z}(3)\big) \end{array}$$

where both vertical maps are Gysin maps. Recall that $\mathrm{CH}^2(Y)_{\mathrm{tors}}$ and $\mathrm{CH}^3(X)_{\mathrm{tors}}$ are both isomorphic to $\mathbf{Z}/2$ [13]. Let e be the nonzero torsion element of $\mathrm{CH}^2(Y)$. We show that $i_* \,\mathrm{cl}^2(e) \neq 0$. This shows both that $i_*(e)$ is the generator e' of $\mathrm{CH}^3(X)_{\mathrm{tors}}$ and that $\mathrm{cl}^3(e') \neq 0$.

By Lemma 4.2, $\mathrm{cl}^2(e) = \pi_Y^*\beta(a,b,c) = i^*\pi_X^*\beta(a,b,c)$, where π_Y (resp., π_X) is the structural map for Y (resp., X). Then, by the projection formula,

$$i_* \,\mathrm{cl}^2(e) = \pi_X^*\beta(a,b,c) i_* i^*(1) = \pi_X^*\beta(a,b,c)\,\mathrm{cl}^1(h).$$

By Lemma 3.1 (applied for $n = 2$), this element is zero if and only if $\beta(a,b,c) = \beta\,\mathrm{Cor}_{E/F}(x \cdot c(q_E))$, with $x \in H^1(E, \mathbf{Z}/2)$, that is, if and only if $(a,b,c) = \mathrm{Cor}_{E/F}(x \cdot (a,b))$. Since X is anisotropic, this equation has no solution by Lemma 4.4.

(b) By scaling q, we may assume that $\operatorname{ind} C(q) = 2$. Let A be a quaternion algebra similar to $C(q)$. Let $a \in E^*$ be such that $N_{E/F}(a) = \mathrm{Nrd}(b)$, with $b \in A^*$. Choose a maximal subfield $L \subset A$ containing b, so that $\mathrm{Nrd}(b) = N_{L/F}(b)$. Let $K = EL$. By an easy consequence of [25, Cor. 2.10], there is a pair $(\lambda, u) \in F^* \times K^*$ such that $a = \lambda N_{K/E}(u)$; hence $a = \lambda\,\mathrm{Nrd}(u)$, where u is viewed as an element of $E \otimes_F A \sim C_0(q)$. We are therefore through by Proposition 4.1(b). □

Theorem 4 in the intermediate case follows from (8), (9), and Proposition 4.5.

4.4. *The Albert case*

LEMMA 4.6 (Rost)
Let q be an Albert form. Then

$$\mathrm{Sn}(q) = \big\{ f \in F^* \mid f^2 \in \mathrm{Nrd}\, C(q)^* \big\}.$$

This lemma can be found in [14, Prop. 16.6]; for the reader's convenience we give a self-contained proof. Let

$$S\Gamma(q) = \big\{ x \in C_0(q)^* \mid xVx^{-1} = V \big\}$$

be the special spinor group of q, where $V \subset C_1(q)$ is the underlying vector space of q (see [11, §4]). Let $x \mapsto x^t$ be the unique involution of $C(q)$ whose restriction to V is the identity, and let $\mathrm{sn}(x) = xx^t$. As is well known, $\mathrm{sn}(x) \in F^*$ for $x \in S\Gamma(q)$ and

$$\mathrm{Sn}(q) = \mathrm{sn}\big(S\Gamma(q)\big).$$

We have

$$C_0(q) \simeq A \times A, \quad C(q) \simeq M_2(A)$$

for a biquaternion algebra A; in particular, $\mathrm{Nrd}\, C(q)^* = \mathrm{Nrd}\, A^*$. By [11, Cor. 4.3], there is an exact sequence

$$1 \longrightarrow S\Gamma(q) \xrightarrow{(p_1,\mathrm{sn})} A^* \times F^* \xrightarrow{\omega} F^*,$$

where p_1 is given by the first projection $C_0(q) \to A$ and $\omega(a, f) = \mathrm{Nrd}(a)/f^2$. Lemma 4.6 follows tautologically from this exact sequence.

LEMMA 4.7
Let X be a nonhyperbolic Albert quadric. Then $d_3^{2,2}(X, 2) = 0$ in the motivic spectral sequence (4).

Of course, the lemma also holds if X is hyperbolic, but we do not have to use this fact.

Proof
Recall the complex

$$\mathrm{CH}^2(X) \xrightarrow{\xi^3} \mathrm{CH}^2\big(X_s\big)^{G_F} \xrightarrow{d_2^{2,2}(X,2)} \mathrm{Br}(F)$$

from [10, 5.3]. By [10, 5.3] there is an exact sequence

$$0 \longrightarrow \mathrm{Coker}\,\eta^3 \longrightarrow H \xrightarrow{d_3^{2,2}} H^4(F, 2), \tag{12}$$

with $H := \mathrm{Ker}\, d_2^{2,2}/\,\mathrm{Im}\,\xi^3$.

By a computation similar to that of Proposition 4.1, we have

$$\mathrm{Ker}\, d_2^{2,2}(X, 2) = \big\{aP_1 + bP_2 \mid a \equiv b \pmod 2\big\},$$
$$\mathrm{Im}\,\xi^3 = \big\{aP_1 + bP_2 \mid a \equiv b \pmod m\big\}$$

as subgroups of $\mathrm{CH}^2(X_s)^{G_F} = \mathbf{Z}P_1 \oplus \mathbf{Z}P_2$. Here

$$m = \begin{cases} 4 & \text{if } X \text{ is anisotropic,} \\ 2 & \text{if } X \text{ is isotropic.} \end{cases}$$

Indeed, the first equality follows trivially from $d_2^{2,2}(P_1) = d_2^{2,2}(P_2) = c(X) \neq 0$ (see [10, Th. 8.3]); the other one follows from $\mathrm{CH}^2(X) = \langle h^2, mP_1\rangle$ (see [13]). (Recall that $h^2 = P_1 + P_2$.) This shows that

$$H \simeq \begin{cases} \mathbf{Z}/2 & \text{if } X \text{ is anisotropic,} \\ 0 & \text{if } X \text{ is isotropic.} \end{cases}$$

On the other hand, if X is anisotropic, then $\operatorname{Coker} \eta^3 \simeq \mathbf{Z}/2$ by [11, Th. 5]; and if X is isotropic, then $\operatorname{Coker} \eta^3 = 0$. This shows that the first map in (12) is bijective, hence the claim. □

PROPOSITION 4.8

Let X be an anisotropic Albert quadric. Then

(a) $d_3^{3,2}(X, 3) = 0$ *in (10);*

(b) $\mathrm{CH}^3(X)_{\mathrm{tors}} = 0$.

Proof

(a) We have a commutative diagram of exact sequences (defining H')

$$\begin{array}{ccccccccc} & & \operatorname{Im} \xi^3 \otimes F^* & \longrightarrow & \operatorname{Ker} d_2^{2,2}(X, 2) \otimes F^* & \longrightarrow & H \otimes F^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & \operatorname{Im} \xi^4 & \longrightarrow & \operatorname{Ker} d_2^{3,2}(X, 3) & \longrightarrow & H' & \longrightarrow & 0 \end{array} \tag{13}$$

where the vertical arrows are given by cup product. This first yields an exact sequence

$$\operatorname{Coker} \lambda \longrightarrow \operatorname{Coker} \mu \longrightarrow \operatorname{Coker} \nu \longrightarrow 0. \tag{14}$$

Proposition 4.1 and the computation in the proof of Lemma 4.7 identify the left square of (13) with

$$\begin{array}{ccc} \{(x, y) \in F^* \times F^* \mid xy^{-1} \in F^{*4}\} & \longrightarrow & \{(x, y) \in F^* \times F^* \mid xy^{-1} \in F^{*2}\} \\ \downarrow & & \downarrow \\ \{(x, y) \in F^* \times F^* \mid xy^{-1} \in N\} & \longrightarrow & \{(x, y) \in F^* \times F^* \mid xy^{-1} \in F^{*2}N\} \end{array}$$

where $N := \operatorname{Nrd} C(q)^*$, and hence the first map of (14) with

$$N/F^{*4} \longrightarrow F^{*2}N/F^{*2}.$$

This shows that ν is surjective. By Lemma 4.7 and the multiplicativity of the spectral sequences, this proves (a).

(b) This follows from Karpenko [13]. □

Remark 4.9

For future reference, we give a precise description of the map ν in diagram (13). By the computation in the proof of Lemma 4.7, the generator e of H is represented by $2P_2 \in \operatorname{Ker} d_2^{2,2}(X,2)$. Therefore, for $f \in F^*$, $\nu(e \otimes f)$ is represented by $(1, f^2)$ in $\operatorname{Ker} d_2^{3,2}(X,3) \subseteq F^* \times F^*$ (cf. (9)).

Proof of Theorem 4 in the case of an Albert quadric

In view of Proposition 4.8, (8), (9), and (10), the group H' in (13) can be identified with Coker η^4. In the course of the proof of Proposition 4.8(a), we have seen that the map ν in (13) is surjective; hence we are left to prove that $\operatorname{Ker} \nu = \operatorname{Sn}(q)$. This is a consequence of the following diagram chase. We have $H \otimes F^* \xleftarrow{\sim} F^*/F^{*2}$, induced by cup product with the generator e of $H \xleftarrow{\sim} \operatorname{Coker} \eta^3 (\simeq \mathbf{Z}/2)$. Let $e \otimes f \in \operatorname{Ker} \nu$. Lift f to $(2P_2) \otimes f \in \operatorname{Ker} d_2^{2,2}(X,2) \otimes F^*$ (see Remark 4.9). Then $\mu((2P_2) \otimes f) = (1, f^2) \in \operatorname{Ker} d_2^{3,2}(X,3)$, viewed as a subgroup of $F^* \times F^*$ (cf. Proposition 4.1(a)). By assumption, $(1, f^2) \in \operatorname{Im} \xi^4$, which means that $f^2 \in N$. But this condition is equivalent to $f \in \operatorname{Sn}(q)$ by Lemma 4.6. □

4.5. The virtual Albert case

In this subsection we let

$$H' := \frac{\operatorname{Ker} d_2^{3,2}(X,3)}{\operatorname{Im} \xi^4}$$

denote the homology of the exact sequence (9). Recall, for a quadratic form q, the group of similarities of q

$$G(q) = \{a \in F^* \mid aq \simeq q\}.$$

LEMMA 4.10

Let q be an even-dimensional form of discriminant $d \neq 1$, and let $E = F(\sqrt{d})$. Then $G(q) \subseteq N_{E/F}(E^*)$.

Proof

Let $a \in G(q)$. Then $\langle 1, -a\rangle \otimes q \sim 0$. The lemma follows from taking the Clifford invariant of both sides. (The left one is (a, d).) □

LEMMA 4.11

Let X be virtual Albert, let d be its discriminant, and let $E = F(\sqrt{d})$. Then, in (9),

$$\operatorname{Ker} \delta = N_{E/F}^{-1}\big(G(q)\big).$$

Proof
By the description of $\delta = d_2^{3,2}(X, 3)$ (see Section 3),

$$\operatorname{Ker}\delta = \big\{a \in E^* \mid N_{E/F}(a)\cdot c(q) = 0\big\}.$$

Let $a \in \operatorname{Ker}\delta$, and let

$$\varphi = \big\langle 1, -N(a)\big\rangle \otimes q.$$

Then φ is a 12-dimensional form in I^2F. We have

$$e^2(\varphi) = e^1\big(\big\langle 1, -N(a)\big\rangle\big)\cdot e^1(q) = \big(N(a)\big)\cdot(d) = 0,$$

so $\varphi \in I^3F$. We also have $\langle\langle N(a), d\rangle\rangle \sim 0$; hence $\varphi \sim \langle 1, -N(a)\rangle \otimes (q \perp \langle 1, -d\rangle)$. Now

$$\begin{aligned} e^3(\varphi) &= e^3\big(\big\langle 1, -N(a)\big\rangle \otimes \big(q \perp \langle 1, -d\rangle\big)\big)\\ &= e^1\big(\big\langle 1, -N(a)\big\rangle\big)\cdot e^2\big(q \perp \langle 1, -d\rangle\big)\\ &= \big(N(a)\big)\cdot c(q)\\ &= 0. \end{aligned}$$

Hence $\varphi \in I^4F$ and, by the Arason-Pfister Hauptsatz, $\varphi \sim 0$. Therefore

$$q \cong N(a)q,$$

which shows that $\operatorname{Ker}\delta \subseteq N^{-1}(G(q))$. The opposite inclusion is proved by reversing this argument. □

LEMMA 4.12
Let $E^1 = \{x \in E^* \mid N_{E/F}(x) = 1\}$. Then there is an exact sequence

$$1 \longrightarrow E^1 \cap \big(F^* \operatorname{Nrd}_E C_0(q)^*\big) \longrightarrow E^1 \xrightarrow{\theta} H' \longrightarrow \operatorname{PSO}(q, F)/R \longrightarrow 0,$$

where R denotes R-equivalence.

Proof
By [20, p. 203, Cor. to Lemma 5 and bottom of same page], we have

$$\frac{\operatorname{PSO}(q, F)}{R} \simeq \frac{P\operatorname{Sim}_+(q)(F)}{R} \simeq \frac{G(q)}{F^{*2}N_{E/F}\big(\operatorname{Nrd} C_0(q)^*\big)}.$$

By Lemma 4.11 and Proposition 4.1, $H' \simeq N^{-1}(G(q))/F^*\operatorname{Nrd}_E C_0(q)$. Therefore the norm induces a map from H' to $P\operatorname{Sim}_+(q)(F)/R$. (This is the map at the right of the exact sequence.) This map is surjective by Lemma 4.10. The exactness at the other terms is obvious. □

We now want to prove that $d_3^{3,2}(X,3) = 0$ for a virtual Albert quadric X. We deduce this from the vanishing of $d_3^{3,2}(U,2)$ for the affine quadric

$$U = X \setminus Z,$$

where Z is a nonsingular hyperplane section of X and $d_3^{3,2}(U,2)$ is a differential in an analogue of (4) for U. The use of U was suggested by Rost. Although U is not a geometrically cellular variety in the sense of [10], we see that the analogues of the spectral sequences of [10] for U are particularly simple.

We start by considering a 4-dimensional quadric X that is either Albert or virtual Albert, and we choose an affine subquadric U as above. Let

$$\tilde{\mathbf{Z}} = f_* \mathbf{Z}/\mathbf{Z},$$

where f is the projection $\operatorname{Spec} E \to \operatorname{Spec} F$. As a G_F-module, $\tilde{\mathbf{Z}}$ is a free abelian group of rank 1; if $d \neq 1$, the Galois action is given by $gx = \varepsilon(g)x$, where $\varepsilon : G_F \to \mathbf{Z}/2$ is the character corresponding to the quadratic extension E/F. Otherwise, the Galois action is trivial. We have the following lemma.

LEMMA 4.13
For all $p \geq 0$, the map

$$\mathrm{CH}^{p-1}(Z_s) \xrightarrow{i_*} \mathrm{CH}^p(X_s)$$

is injective. Its cokernel $\mathrm{CH}^p(U_s)$ is
- $\mathbf{Z}$ *for $p = 0$,*
- $\tilde{\mathbf{Z}}$ *for $p = 2$,*
- *zero otherwise.*

Proof
The claim is obvious for $p = 0$. Suppose $p > 0$. Let h be the class of a hyperplane section of X in $\mathrm{CH}^1(X_s)$. By the projection formula, the map i_* sends h^i to h^{i+1} for all $i \geq 0$. We now go case by case.
- $p = 1$. The claim is clear, since $\mathrm{CH}^1(X_s)$ has basis h.
- $p = 2$. The group $\mathrm{CH}^2(X_s)$ has basis (P_1, P_2) with $P_1 + P_2 = h^2$. The claim follows.
- $p = 3, 4$. The claim is clear, since $\mathrm{CH}^{p-1}(Z_s)$ (resp., $\mathrm{CH}^p(X_s)$) is generated by $(1/2)h^{p-1}$ (resp., $(1/2)h^p$). □

PROPOSITION 4.14

For all $n \geq 0$, there is a spectral sequence

$$E_2^{p,q}(U,n) \Longrightarrow H^{p+q}$$

with maps $H^{p+q} \to H^{p+q}_{\text{ét}}(U, \mathbf{Z}(n))$ which are bijective for $p+q \leq 2n$. The E_2-terms are as follows:

$$E_2^{p,q}(U,n) = \begin{cases} H^p_{\text{ét}}\big(F, \mathbf{Z}(n)\big) & \text{for } q = 0, \\ H^{p-2}_{\text{ét}}\big(F, \tilde{\mathbf{Z}}(n-2)\big) & \text{for } q = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For the proof, see the appendix.

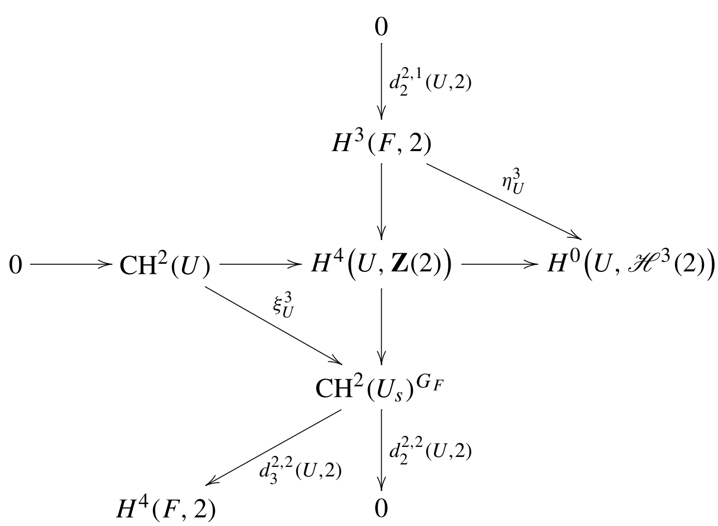
Remark 4.15

After the fact, we see that the E_2-terms can also be written as

$$E_2^{p,q}(U,n) = H^{p-q}_{\text{ét}}\big(F, \mathrm{CH}^q(U_s) \otimes \mathbf{Z}(n-q)\big)$$

just as in [10] for a geometrically cellular variety. This formula is perhaps more mnemotechnical; however, the reader should be aware that it is a sort of "miracle." (It would definitely not hold if the Chow groups of U_s had torsion.)

Take $n = 2, 3$. Let us reproduce for U part of the diagrams corresponding to those in [10, 5.3 and 5.4]. (Here, all groups are localised at 2.)



(15)

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \big\downarrow d_2^{3,1}(U,3) & & & \\
 & & & H^4(F,3) & \overset{\eta_U^4}{\searrow} & & \\
 & & & \big\downarrow & & & \\
0 \longrightarrow & H^2\big(U, \mathscr{K}_3\big) & \longrightarrow & H^5\big(U, \mathbf{Z}(3)\big) & \longrightarrow & H^0\big(U, \mathscr{H}^4(3)\big) & \\
 & & \underset{\xi_U^4}{\searrow} & \big\downarrow & & & \\
 & & & E^*/F^* & & & \\
 & & \overset{d_3^{3,2}(U,3)}{\swarrow} & \big\downarrow d_2^{3,2}(U,3) & & & \\
 & H^5(F,3) & & 0 & & &
\end{array}
\tag{16}
$$

If $d = 1$, the action of G_F on $\mathrm{CH}^2(X_s)$, $\mathrm{CH}^2(U_s)$, and $\mathrm{CH}^1(Z_s)$ is trivial.

LEMMA 4.16
Suppose X is a nonhyperbolic Albert quadric. Then
(a) *the composition*

$$\operatorname{Ker} d_2^{2,2}(X,2) \subset \mathrm{CH}^2\left(X_s\right) \longrightarrow \mathrm{CH}^2\left(U_s\right)$$

has cokernel $\mathbf{Z}/2$;
(b) $\operatorname{Coker} \xi_U^3 \neq 0$.

Proof
(a) By Lemma 4.13, the map $\mathrm{CH}^2(X_s) \to \mathrm{CH}^2(U_s)$ sends both basis elements P_1, P_2 to the same generator of $\mathrm{CH}^2(U_s)$. The claim then follows from the computation of $\operatorname{Ker} d_2^{2,2}(X,2)$ in the proof of Lemma 4.7. (Note that this computation uses only the fact that $c(q) \neq 0$ for a quadratic form q defining X.)

(b) Consider the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
\mathrm{CH}^1(Z) & \longrightarrow & \mathrm{CH}^2(X) & \longrightarrow & \mathrm{CH}^2(U) & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow \xi_X^3 & & \big\downarrow \xi_U^3 & & \\
\mathrm{CH}^1(Z_s) & \longrightarrow & \mathrm{CH}^2(X_s) & \longrightarrow & \mathrm{CH}^2(U_s) & \longrightarrow & 0
\end{array}
$$

By [13] (cf. [11, Prop. 1.1(b)]), the left vertical map is surjective; hence $\operatorname{Coker} \xi_X^3 \xrightarrow{\sim} \operatorname{Coker} \xi_U^3$. The former is isomorphic to $\mathbf{Z}/4$ or $\mathbf{Z}/2$ according to whether X is anisotropic or isotropic (see [13]). □

LEMMA 4.17

Suppose X is a nonhyperbolic Albert quadric. Then the cokernel of $\operatorname{Coker}\eta_X^3 \to \operatorname{Coker}\eta_U^3$ *contains a nonzero element of order* 2.

Proof

By the "purity" exact sequence in $\mathscr{H}$-cohomology, we have a diagram

$$\begin{array}{ccccccccc}
 & H^4(F,\mathbf{Z}(2)) & = & H^4(F,\mathbf{Z}(2)) & & H^3(F,\mathbf{Z}(1)) & \xrightarrow{\alpha} & H^3(F,\mathbf{Z}(1))\otimes \mathrm{CH}^1(X) \\
 & \downarrow{\scriptstyle \eta_X^3} & & \downarrow{\scriptstyle \eta_U^3} & & \downarrow{\scriptstyle \eta_Z^2}\wr & & \downarrow{\scriptstyle \beta} \\
0 \to & H^0(X,\mathscr{H}^4(\mathbf{Z}(2))) & \to & H^0(U,\mathscr{H}^4(\mathbf{Z}(2))) & \to & H^0(Z,\mathscr{H}^3(\mathbf{Z}(1))) & \longrightarrow & H^1(X,\mathscr{H}^4(\mathbf{Z}(2))) \\
 & \downarrow & & \downarrow & & & & \\
 & \operatorname{Coker}\eta_X^3 & \longrightarrow & \operatorname{Coker}\eta_U^3 & & & & \\
 & \downarrow & & \downarrow & & & & \\
 & 0 & & 0 & & & &
\end{array}$$

In this diagram, the middle row is exact; the map η_Z^2 is bijective by [1, comments after Satz 4.1] and [11, Th. 4]. The map α (resp., β) is defined by tensoring with h (resp., by $\beta(x\otimes y) = \pi^* x \cdot y$, where π is the structural map of X). One checks easily that the square involving η_Z^2, α, and β commutes.

The diagram then yields an exact sequence

$$0 \longrightarrow \operatorname{Coker}\eta_X^3 \longrightarrow \operatorname{Coker}\eta_U^3 \longrightarrow \operatorname{Ker}(\beta\alpha) \longrightarrow 0.$$

Consider the map $B : H^3(F,\mathbf{Z}(1)) \to H^5(X,\mathbf{Z}(2))$ of Lemma 3.1. Letting $G^p H^5(X,\mathbf{Z}(2))$ denote the (decreasing) filtration defined on $H^5(X,\mathbf{Z}(2))$ by the coniveau spectral sequence, we have $\operatorname{Im} B \subseteq G^1 H^5(X,\mathbf{Z}(2))$. By dimension counting, $E_\infty^{1,4} \hookrightarrow E_2^{1,4} \simeq H^1(X,\mathscr{H}^4(\mathbf{Z}(2)))$ and, again by multiplicativity, the diagram

$$\begin{array}{ccc}
H^3(F,\mathbf{Z}(1)) & \xrightarrow{B} & G^1H^5(X,\mathbf{Z}(2)) \\
\downarrow{\scriptstyle \beta\alpha} & & \downarrow \\
H^1(X,\mathscr{H}^4(\mathbf{Z}(2))) & \longleftarrow & E_\infty^{1,4}
\end{array}$$

commutes. In particular, $\operatorname{Ker} B \subseteq \operatorname{Ker}\beta\alpha$. By Lemma 3.1, $\operatorname{Ker} B = \{0, c(q)\}$. Lemma 4.17 follows. □

LEMMA 4.18
Suppose X is a nonhyperbolic Albert quadric. Then $d_3^{2,2}(U,2)=0$.

Proof
Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Coker}\eta_X^3 & \longrightarrow & H(X) & \longrightarrow & \operatorname{Im} d_3^{2,2}(X,2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Coker}\eta_U^3 & \longrightarrow & \operatorname{Coker}\xi_U^3 & \longrightarrow & \operatorname{Im} d_3^{2,2}(U,2) & \longrightarrow & 0. \end{array}$$

Here $H(X)=\operatorname{Ker} d_2^{2,2}(X,2)/\operatorname{Im}(\xi_X^3)$; we have $d_2^{2,2}(U,2)=0$ because its target group is zero. By Lemma 4.7 we have $\operatorname{Im} d_3^{2,2}(X,2)=0$. Therefore the snake lemma yields an exact sequence

$$\operatorname{Coker}\eta_X^3 \longrightarrow \operatorname{Coker}\eta_U^3 \longrightarrow \operatorname{Coker}\gamma \longrightarrow \operatorname{Im} d_3^{2,2}(U,2) \longrightarrow 0.$$

But Lemma 4.16 shows that $\operatorname{Coker}\gamma$ is isomorphic to $\mathbf{Z}/2$. Lemma 4.17 now shows that $\operatorname{Im} d_3^{2,2}(U,2)=0$, as desired. (This proof also shows that the cokernel of $\operatorname{Coker}\eta_X^3 \to \operatorname{Coker}\eta_U^3$ is exactly of order 2.) □

PROPOSITION 4.19
Let X be virtual Albert. Then
(a) $\mathrm{CH}^3(X)_{\mathrm{tors}}=0$;
(b) $d_3^{3,2}(X,3)=0$.

Proof
(a) This follows from Karpenko [13].

(b) Diagram (16), the analogous diagram for X (see [10, (5.4)]), and functoriality give a commutative diagram

$$\begin{array}{ccc} \operatorname{Ker} d_2^{3,2}(X,3) & \longrightarrow & E^*/F^* \\ \downarrow{\scriptstyle d_3^{3,2}(X,3)} & & \downarrow{\scriptstyle d_3^{3,2}(U,3)} \\ H^5(F,3) & = & H^5(F,3) \end{array}$$

which shows that it is sufficient to prove the vanishing of $d_3^{3,2}(U,3)$, where U is the complement of a hyperplane section as above. By a multiplicativity and transfer argument just as in [10, Lemma 6.1], to do this it is enough to show that

$$d_3^{2,2}\big(U_K,2\big) : \mathrm{CH}^2\left(U_s\right)^{G_K} \longrightarrow H^4(K,2)$$

is zero for any finite extension K/F. If $E \not\subseteq K$, then $\mathrm{CH}^2(U_s)^{G_K} = 0$ and this is trivial. If X_K is hyperbolic, then the computation of Lemma 4.13 shows that $\mathrm{CH}^2(U_K) \to \mathrm{CH}^2(U_s)^{G_K}$ is bijective, hence the result again (cf. (15)). Finally, the remaining case is covered by Lemma 4.18. □

COROLLARY 4.20
The map $H^i(F, \mathbf{Z}(3)) \to H^i(X, \mathbf{Z}(3))$ *is injective for* $i = 5, 6$.

Proof
For $i = 5$, this is not a corollary of Proposition 4.19; it follows from [10, Cor. 8.6], which shows that the differential $d_2^{3,1}(X, 3)$ is zero. For $i = 6$, it follows from Proposition 4.19. □

For the next propositions, we have to describe precisely the restriction and corestriction

$$\mathrm{Res}_{E/F} : H^1\big(F, \mathrm{CH}^2\left(X_s\right) \otimes \mathbf{Z}(1)\big) \longrightarrow H^1\big(E, \mathrm{CH}^2\left(X_s\right) \otimes \mathbf{Z}(1)\big),$$
$$\mathrm{Cor}_{E/F} : H^1\big(E, \mathrm{CH}^2\left(X_s\right) \otimes \mathbf{Z}(1)\big) \longrightarrow H^1\big(F, \mathrm{CH}^2\left(X_s\right) \otimes \mathbf{Z}(1)\big)$$

under the identification of these groups used in Proposition 4.1; this is the most confusing part of this paper. We recall these identifications from [10]. We have

$$H^1\big(F, \mathrm{CH}^2\left(X_s\right) \otimes \mathbf{Z}(1)\big) = H^0\big(F, \mathrm{CH}^2\left(X_s\right) \otimes F_s^*\big).$$

Taking the basis (P_1, P_2) of $\mathrm{CH}^2(X_s)$ consisting of the classes of two plane sections conjugate under G_F, the coefficient group gets identified with $(\mathbf{Z}P_1 \oplus \mathbf{Z}P_2) \otimes F_s^*$. Since G_F permutes P_1 and P_2, Galois invariants get identified with E^* via the map

$$x \longmapsto P_1 \otimes \{x\} + P_2 \otimes \{\bar{x}\}.$$

(We use the K-theoretic notation {} in order to avoid conflict between additive and multiplicative notation.)

On the other hand, the identification of $H^1(E, \mathrm{CH}^2(X_s) \otimes \mathbf{Z}(1)) = H^0(E, \mathrm{CH}^2(X_s) \otimes F_s^*)$ with $E^* \times E^*$ used in Section 4.4 is given by the map

$$(x, y) \longmapsto P_1 \otimes \{x\} + P_2 \otimes \{y\}.$$

The following lemma is now clear.

LEMMA 4.21
Under the identifications above,

(a) *the action of* $\mathrm{Gal}(E/F)$ *on* $E^* \times E^*$ *is given by* $(x, y) \mapsto (\bar{y}, \bar{x})$*;*

(b) *the restriction is given by the map*

$$\begin{aligned} r : E^* &\longrightarrow E^* \times E^*, \\ x &\longmapsto (x, \bar{x}); \end{aligned}$$

(c) *the corestriction is given by the map*

$$\begin{aligned} c : E^* \times E^* &\longrightarrow E^*, \\ (x, y) &\longmapsto x\bar{y}. \end{aligned}$$

PROPOSITION 4.22

The diagram

$$\begin{array}{ccccc} E^* & \xrightarrow{\varphi} & H_E \otimes E^* & \xrightarrow{\nu} & H'_E \\ {\scriptstyle\rho}\downarrow & & & & \downarrow{\scriptstyle c} \\ E^1 & & \xrightarrow{\quad\theta\quad} & & H' \end{array} \tag{17}$$

commutes, where $E^1 = \mathrm{Ker}\, N_{E/F}$ *as before,* $H' = \mathrm{Ker}\,\delta / \mathrm{Im}\,\xi^4$*,* H'_E *is the corresponding group over* E*,* $H_E = \mathrm{Ker}\, d_2^{2,2}(X_E, 2)/\,\mathrm{Im}\,\xi^3_{X_E}$*,* θ *is the map of Lemma 4.12,* $\varphi(x) = e \otimes x$ *(*e *is the generator of* H_E*),* ν *is the map of diagram (13),* ρ *is the map* $x \mapsto \bar{x}/x$ *(*$\bar{x}$ *is the conjugate of* x *under* $\mathrm{Gal}(E/F)$*), and* c *is induced by the map in Lemma 4.21(c).*

Proof

By Remark 4.9, we have $\nu\varphi(x) = (1, x^2)$ for $x \in E^*$; hence

$$c\big(\nu\varphi(x)\big) \equiv \bar{x}^2 \quad \big(\mathrm{mod}\, \mathrm{Im}\,\xi^4\big).$$

On the other hand, $\theta\rho(x)$ is represented by $\bar{x}/x$ in $\mathrm{Ker}\, d_2^{3,2}(X, 2)$. But

$$\frac{\bar{x}}{x} \equiv \bar{x}^2 \quad \big(\mathrm{mod}\, \mathrm{Im}\,\xi^4\big)$$

since

$$x\bar{x} \in F^* \subseteq \mathrm{Im}\,\xi^4$$

by Proposition 4.1. □

PROPOSITION 4.23

Let $x \in \mathrm{Ker}\,\delta$*. Then, with notation as in Lemma 4.21(b) and Proposition 4.22,*

$$r(x) \equiv \nu\phi(xy) \quad \big(\mathrm{mod}\, \mathrm{Im}\,\xi^4_E\big),$$

where $y \in F^*$ *is such that* $N_{E/F}(x)y^2 = N_{E/F}(xy) \in \mathrm{Nrd}\, C(q)^*$ *(cf. Proposition 4.1).*

Proof
We have

$$r(x) = (x, \bar{x}) \qquad \text{and} \qquad \nu\phi(xy) = \big(1, x^2y^2\big).$$

We claim that $(x, \bar{x}) \equiv (1, x^2y^2) \pmod{\operatorname{Im}\xi_E^4}$, or $(x, \bar{x}/x^2y^2) \in \operatorname{Im}\xi_E^4$. By Proposition 4.1, this is equivalent to

$$\frac{x}{\bar{x}/x^2y^2} = \frac{x^3y^2}{\bar{x}} = \frac{(xy)^4}{N_{E/F}(x)y^2} \in \operatorname{Nrd} C(q_E)^*.$$

Clearly, $(xy)^4 \in \operatorname{Nrd} C(q_E)^*$, and $N_{E/F}(x)y^2 \in \operatorname{Nrd} C(q)^* \subseteq \operatorname{Nrd} C(q_E)^*$. □

Proof of Theorem 4 in the case of a virtual Albert quadric
By Proposition 4.19, (8), (9), and (10), $\operatorname{Coker}\eta^4$ is identified with $H' = \operatorname{Ker}\delta/\operatorname{Im}\xi^4$ in (9), just as $\operatorname{Coker}\eta^4_{X_E}$ is identified with H'_E by the Albert quadric case. Putting Lemma 4.12 and Proposition 4.19 together, we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
E^* & \xrightarrow{\nu\phi} & \operatorname{Coker}\eta_E^4 & & & & \\
\downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\operatorname{Cor}_{E/F}} & & & & \\
E^1 & \xrightarrow{\theta} & \operatorname{Coker}\eta^4 & \longrightarrow & \dfrac{\operatorname{PSO}(q, F)}{R} & \longrightarrow & 0
\end{array} \tag{18}$$

The exactness of the sequence of Theorem 4 follows from (18) and Hilbert's theorem 90 (classical style), which asserts that ρ is surjective. □

5. Proofs of Theorems 1 and 5

5.1. Proof of Theorem 1

This should have been already noticed in [11]. Just observe that, in the diagram of [11, top of p. 867], the zero can be removed from $H^i(F, \mu_m^{\otimes(i-1)})_0$ because, under the assumption, the Bockstein that follows is zero (cf. [11, proof of Prop. 7.3]). We get the claim by passing to the limit over m in this diagram.

5.2. Proof of Theorem 5

LEMMA 5.1
Let X be a smooth variety such that $\operatorname{CH}^1 X \to \operatorname{CH}^1 X_s$ *and* $\operatorname{CH}^2 X/2 \to (\operatorname{CH}^2 X_s/2)^{G_F}$

are surjective. Then there is an exact sequence

$$H^4F \xrightarrow{\eta_2^4} H^0\big(X, \mathscr{H}^4\big) \xrightarrow{d_2^{2,3}} H^2\big(X, \mathscr{H}^3\big) \xrightarrow{\varphi} H^5X,$$

where $d_2^{2,3}$ *is a differential in the coniveau spectral sequence for* $\mathbf{Z}/2$ *coefficients and* φ *is an edge homomorphism in the same spectral sequence. Therefore,*

$$\operatorname{Coker} \eta_2^4 \xrightarrow{\sim} \operatorname{Ker} \varphi.$$

Proof
The sequence is obviously exact if we replace H^4F by H^4X, by dimension counting. The assumption on X implies that the map

$$H^4F \oplus \mathrm{CH}^1 X \otimes H^2F \oplus \mathrm{CH}^2 X \xrightarrow{\left(f^*, \mathrm{cl}^1 \cdot f^*, \mathrm{cl}^2\right)} H^4X$$

is surjective, as one sees from the Hochschild-Serre spectral sequence (cf. [8, Lemma 3]). It is then clear that $\mathrm{Im}(H^4X \to H^0(X, \mathscr{H}^4)) = \mathrm{Im}(H^4F \to H^0(X, \mathscr{H}^4))$, since the two other summands map to zero in $H^0(X, \mathscr{H}^4)$. □

The hypothesis of Lemma 5.1 is satisfied for quadrics of dimensions greater than four and for a virtual Albert quadric (but not for any other anisotropic 4-dimensional quadric).

The Hochschild-Serre spectral sequence defines a 3-step filtration on H^5X,

$$H^5X = F^0H^5 \supseteq F^1H^5 \supseteq F^2H^5 \supseteq 0,$$

where $F^{0/1}H^5$ is a subgroup of $H^1(F, \mathrm{CH}^2 X_s/2)$, $F^{1/2}H^5$ is isomorphic to $H^3F \otimes \mathrm{CH}^1 X$, and F^2H^5X is isomorphic to H^5F. To compute $\operatorname{Coker} \eta_2^4$, we filter $\operatorname{Ker} \varphi$ accordingly. Let us call

- $\bar{\varphi}_1$ the composite

$$H^2\big(X, \mathscr{H}^3\big) \xrightarrow{\varphi} H^5X \longrightarrow F^{0/2}H^5 = H^5X/H^5F,$$

- $\bar{\varphi}$ the composite

$$H^2\big(X, \mathscr{H}^3\big) \xrightarrow{\bar{\varphi}_1} F^{0/2}H^5 \longrightarrow F^{0/1}H^5 \subseteq H^1\big(F, \mathrm{CH}^2 X_s/2\big).$$

LEMMA 5.2
Let X *be virtual Albert. Then there is a commutative diagram with exact row and column (defining* $(\operatorname{Ker} \bar{\varphi})_1$*):*

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & \operatorname{Ker}\eta^4 & & & & \\
 & & \downarrow & & & & \\
0 & \longrightarrow & (\operatorname{Ker}\bar{\varphi})_1 & \longrightarrow & \operatorname{Ker}\bar{\varphi} & \longrightarrow & H^3F \\
 & & \downarrow & & & & \\
 & & \operatorname{Coker}\eta^4 & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

Proof
Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & \{\pm 1\} & & H^3F \\
 & & & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{Ker}\eta^4 & \longrightarrow & H^2(X,\mathscr{K}_3) & \xrightarrow{\xi^4} & E^* & \xrightarrow{d_2^{3,2}} & H^4\big(F,\mathbf{Z}(2)\big) \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\
0 & \longrightarrow & \operatorname{Ker}\eta^4 & \longrightarrow & H^2(X,\mathscr{K}_3) & \xrightarrow{\xi^4} & E^* & \xrightarrow{d_2^{3,2}} & H^4\big(F,\mathbf{Z}(2)\big) \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}\bar{\varphi} & \longrightarrow & H^2(X,\mathscr{H}^3) & \xrightarrow{\bar{\varphi}} & \dfrac{E^*}{E^{*2}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In this diagram, all columns are exact; the left one is by the exact sequence

$$0 \longrightarrow H^2\big(X,\mathscr{K}_3\big)/2 \longrightarrow H^2\big(X,\mathscr{H}^3\big) \longrightarrow {}_2\operatorname{CH}^3(X) \longrightarrow 0$$

(see [3, (3.2)]) since here ${}_2\operatorname{CH}^3(X) = 0$ (Karpenko), and the right one is by Hilbert's theorem 90 in weight two. The rows are exact, except the first two at E^*, where their homology is $\operatorname{Coker}\eta^4$ by the previous sections. A diagram chase yields a map from

$\operatorname{Ker}\bar{\varphi}$ to H^3F; it is well defined because $-1 \in \operatorname{Im}\xi^4$ by Proposition 4.1. For the same reason, the map $(\operatorname{Ker}\varphi)_1 \to \operatorname{Coker}\eta^4$ induced by another diagram chase is well defined; it is surjective because $\operatorname{Coker}\eta^4$ has exponent 2. A last diagram chase gives an exact sequence

$$\{\pm 1\} \longrightarrow \operatorname{Ker}\eta^4 \longrightarrow \big(\operatorname{Ker}\bar{\varphi}\big)_1.$$

But the left map is zero; indeed, there is an element of order 2 in $H^2(X, \mathscr{K}_3)$ mapping to -1 via ξ^4. To see this, go back to the proof of Proposition 4.1 and observe that $\Phi(-1, 1, 1) = -1$. This diagram chase actually shows that the column in the diagram of Lemma 5.2 is exact at $\operatorname{Ker}(\bar{\phi})_1$. □

LEMMA 5.3
The sequence

$$H^5\big(X, \mathbf{Z}(3)\big) \oplus H^5\big(F, \mathbf{Z}(3)\big) \xrightarrow{\binom{2}{\pi^*}} H^5\big(X, \mathbf{Z}(3)\big) \longrightarrow H^5X/H^5F$$

is exact.

Proof
We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^5\big(F, \mathbf{Z}(3)\big) & \xrightarrow{2} & H^5\big(F, \mathbf{Z}(3)\big) & \longrightarrow & H^5(F) & \longrightarrow & {}_2H^6\big(F, \mathbf{Z}(3)\big) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^5\big(X, \mathbf{Z}(3)\big) & \xrightarrow{2} & H^5\big(X, \mathbf{Z}(3)\big) & \longrightarrow & H^5(X) & \longrightarrow & {}_2H^6\big(X, \mathbf{Z}(3)\big) \\
 & & & & \downarrow & & \\
 & & & & H^5X/H^5F & &
\end{array}$$

The right vertical arrow is injective by Corollary 4.20. A diagram chase now completes the proof of the lemma. □

LEMMA 5.4
With assumptions and notation as in Lemma 5.2, we have $(\operatorname{Ker}\bar{\varphi})_1 = \operatorname{Ker}\bar{\varphi}_1$.

Proof
The Hochschild-Serre spectral sequence gives an exact sequence

$$0 \longrightarrow H^3\big(F, \mathrm{CH}^1 X_s/2\big) \longrightarrow H^5X/H^5F \longrightarrow H^1\big(F, \mathrm{CH}^2 X_s/2\big).$$

As $\bar{\varphi}$ factors as $H^2(X, \mathscr{H}^3) \to H^5X/H^5F \to H^1(F, \mathrm{CH}^2\, X_s/2)$, the above exact sequence gives another map

$$\operatorname{Ker} \bar{\varphi} \longrightarrow H^3\big(F, \mathrm{CH}^1\, X_s/2\big) \simeq H^3(F)$$

whose kernel is $\operatorname{Ker} \bar{\varphi}_1$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
H^2\big(X, \mathscr{K}_3\big) & \xrightarrow{e} & \dfrac{H^5\big(X, \mathbf{Z}(3)\big)}{H^5\big(F, \mathbf{Z}(3)\big)} & \xrightarrow{e'} & E^* & \xrightarrow{d_2^{3,2}} & H^4\big(F, \mathbf{Z}(2)\big) \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} \\
H^2\big(X, \mathscr{K}_3\big) & \xrightarrow{e} & \dfrac{H^5\big(X, \mathbf{Z}(3)\big)}{H^5\big(F, \mathbf{Z}(3)\big)} & \xrightarrow{e'} & E^* & \xrightarrow{d_2^{3,2}} & H^4\big(F, \mathbf{Z}(2)\big) \\
\downarrow & & \downarrow & & \downarrow & & \\
H^2\big(X, \mathscr{H}^3\big) & \xrightarrow{\bar{e}} & \dfrac{H^5X}{H^5F} & \xrightarrow{\bar{e}'} & \dfrac{E^*}{E^{*2}} & & \\
\downarrow & & & & \downarrow & & \\
0 & & & & 0 & &
\end{array}$$

To understand the proof, the reader should compare this diagram with the one used in the proof of Lemma 5.2. In the present diagram, e is induced by an edge map from the coniveau spectral sequence and e' is an edge map from the motivic spectral sequence, so that $\xi^4 = e' \circ e$ (see [10, 5.4] and the diagram in the proof of Lemma 5.2). Similarly, we have $\bar{e}' \circ \bar{e} = \bar{\varphi}$. The top two rows are exact at E^*, and the third column from the left is exact. Moreover, the second column from the left is also exact by Lemma 5.3. Finally, e' is injective by [10, 5.4]. We need to show that, if $x \in H^2(X, \mathscr{H}^3)$ is such that $\bar{\varphi}(x) = 0$ and furthermore maps to zero in H^3F by the diagram chase of Lemma 5.2, then $\bar{e}(x) = 0$ and conversely. This follows from a straightforward diagram chase. □

The group $\operatorname{Ker} \bar{\varphi}_1$ maps to H^5F, and we have the following lemma.

LEMMA 5.5
The composition

$$\operatorname{Ker} \eta^4 \longrightarrow \operatorname{Ker} \bar{\varphi}_1 \longrightarrow H^5F$$

is given by cup product by (-1).

Proof
Looking at the definition of the map $\operatorname{Ker} \eta^4 \to H^2(X, \mathscr{K}_3)$ stemming from the diagram

in [10, 5.4], we see that the above composition coincides with the composition of the middle row in the following commutative diagram:

$$\begin{CD}
H^4 F @>{\bar\beta}>> H^5\big(F, \mu_4^{\otimes 3}\big) @>>> H^5 F \\
@A{\|}AA @AAA @A{\|}AA \\
H^4 F @>{\beta}>> H^5\big(F, \mathbf{Z}(3)\big) @>>> H^5 F \\
@. @VVV @VVV \\
@. H^5\big(X, \mathbf{Z}(3)\big) @>>> H^5 X
\end{CD}$$

The top composition is cup product by (-1) by the Milnor conjecture and [7, Lemma 1]. □

Proof of Theorem 5
From the coniveau spectral sequence, the composition

$$H^2\big(X, \mathscr{H}^3\big) \longrightarrow H^5 X \longrightarrow H^0\big(X, \mathscr{H}^5\big)$$

is evidently zero. Therefore the image of the composition $\operatorname{Ker} \bar\varphi_1 \to H^5 F$ is contained in $\operatorname{Ker} \eta^5$. Moreover, the spectral sequence now gives an exact sequence

$$\operatorname{Ker} \bar\varphi_1 \longrightarrow \operatorname{Ker} \eta^5 \xrightarrow{\varepsilon} H^1\big(X, \mathscr{H}^4\big).$$

Taking the previous lemmas into account, we therefore get a cross of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \operatorname{Coker} \eta_2^4 & & & & \\
 & & & & \downarrow & \overset{\kappa}{\searrow} & & & \\
0 & \longrightarrow & \operatorname{Ker} \eta^4 & \longrightarrow & \operatorname{Ker} \bar\varphi_1 & \longrightarrow & \operatorname{Coker} \eta^4 & \longrightarrow & 0 \\
 & & & \underset{\cdot(-1)}{\searrow} & \downarrow & & & & \\
 & & & & \operatorname{Ker} \eta^5 & & & & \\
 & & & & \downarrow{\scriptstyle \varepsilon} & & & & \\
 & & & & H^1\big(X, \mathscr{H}^4\big) & & & &
\end{array}$$

This diagram and the lemma of the 700th (see [22]) give back the exact sequence (1) plus an isomorphism

$$\operatorname{Coker}\kappa \simeq \operatorname{Ker}\varepsilon/(-1)\cdot\operatorname{Ker}\eta^4. \tag{19}$$

□

Appendix. A spectral sequence for the étale motivic cohomology of an affine quadric

In this appendix, we prove Proposition 4.14. Recall that

$$\tilde{\mathbf{Z}} := f_*\mathbf{Z}/\mathbf{Z},$$

where f is the projection $\operatorname{Spec} E \to \operatorname{Spec} F$. Arguing as in [10], it suffices to show the following.

PROPOSITION A.1
Let $M(U)$ be the motive of U, viewed in the category $DM^{\text{eff}}_{-,\text{ét}}(F)$. Then

$$M(U) \simeq \mathbf{Z} \oplus \tilde{\mathbf{Z}}(2)[4],$$

where $\tilde{\mathbf{Z}}(i) = \tilde{\mathbf{Z}} \otimes \mathbf{Z}(i)$.

Proof
There is an exact triangle

$$M(U) \longrightarrow M(X) \longrightarrow M(Z)(1)[2] \longrightarrow M(U)[1].$$

By [10, Cor. 3.6], we have

$$M(X) \simeq \coprod_{p\geq 0} \operatorname{CH}^p\left(X_s\right)^* \otimes \mathbf{Z}(p)[2p],$$
$$M(Z) \simeq \coprod_{p\geq 0} \operatorname{CH}^p\left(Z_s\right)^* \otimes \mathbf{Z}(p)[2p],$$

where * denotes $\mathbf{Z}$-dual. Consequently, the above exact triangle decomposes as a direct sum of exact triangles (for $p \geq 0$):

$$M^{(p)}(U) \longrightarrow \operatorname{CH}^p\left(X_s\right)^* \otimes \mathbf{Z}(p)[2p] \longrightarrow \operatorname{CH}^{p-1}\left(Z_s\right)^* \otimes \mathbf{Z}(p)[2p] \longrightarrow M^{(p)}(U)[1].$$

By Lemma 4.13, these triangles identify $M^{(p)}(U)$ with

- $\mathbf{Z}$ for $p = 0$,
- $\tilde{\mathbf{Z}}^* \otimes \mathbf{Z}(2)[4] \simeq \tilde{\mathbf{Z}}(2)[4]$ for $p = 2$,
- zero otherwise.

Proposition A.1 follows. □

Acknowledgments. This work was performed during several visits by the first author to the Tata Institute for Fundamental Research, Mumbai; he wishes to express his thanks to this institute for its excellent working conditions and stimulating environment. The second author wishes to thank the Alexander von Humboldt Stiftung for support and the Universität Regensburg for its hospitality when the work was in progress.

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