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Abstract

We show that the torsion in the group of indecomposable $(2,1)$-cycles on a smooth projective variety over an algebraically closed field is isomorphic to a twist of its Brauer group, away from the characteristic. In particular, this group is infinite as soon as $b_2-\rho>0$. We derive a new insight into Roĭtman's theorem on torsion 0-cycles over a surface.

Introduction

Let X be a smooth projective variety over an algebraically closed field k. The group

$$C(X)=H^1(X,\mathcal{K}_2)\simeq CH^2(X,1)\simeq H^3(X,\mathbf{Z}(2))$$

has been widely studied. Its most interesting part is the *indecomposable quotient*

$$H^1_{\mathrm{ind}}(X,\mathcal{K}_2)\simeq CH^2_{\mathrm{ind}}(X,1)\simeq H^3_{\mathrm{ind}}(X,\mathbf{Z}(2)),$$

defined as the cokernel of the natural homomorphism

$$\operatorname{Pic}(X)\otimes k^*\xrightarrow{\theta}C(X). \tag{1}$$

It vanishes for $\dim X\leqslant 1$.

Let $\operatorname{Br}(X)=H^2_{\text{ét}}(X,\mathbb{G}_m)$ be the Brauer group of X: it sits in an exact sequence

$$0\to \mathrm{NS}(X)\otimes\mathbf{Q}/\mathbf{Z}\to H^2_{\text{ét}}(X,\mathbf{Q}/\mathbf{Z}(1))\to \operatorname{Br}(X)\to 0. \tag{2}$$

Here we write $A(n)$ for $\varinjlim_{(m,p)=1} {}_mA\otimes\mu_m^{\otimes n}$ for a prime-to-p torsion abelian group A, and we set for $n\geqslant 0$, $i\in\mathbf{Z}$,

$$H^i(X,\mathbf{Q}_p/\mathbf{Z}_p(n))=\varinjlim_s H^{i-n}_{\text{ét}}(X,\nu_s(n))$$

where p is the exponential characteristic of k and, if $p>1$, $\nu_s(n)$ is the sth sheaf of logarithmic Hodge–Witt differentials of weight n [Ill79, Mil88, GS88]. (See [Ill79, p. 629, (5.8.4)] for the p-primary part in characteristic p in (2).)

Theorem 1. *There are natural isomorphisms*

$$\beta':\operatorname{Br}(X)\{p'\}(1)\xrightarrow{\sim}H^3_{\mathrm{ind}}(X,\mathbf{Z}(2))\{p'\},$$
$$\beta_p:H^2(X,\mathbf{Q}_p/\mathbf{Z}_p(2))\xrightarrow{\sim}H^3_{\mathrm{ind}}(X,\mathbf{Z}(2))\{p\}$$

where $\{p\}$ (respectively, $\{p'\}$) denotes p-primary torsion (respectively, prime-to-p torsion.)

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Theorem 1 gives an interpretation of the Brauer group (away from p)[1] in terms of algebraic cycles. In view of (2), it also implies the following corollary.

Corollary 1. *If $b_2 - \rho > 0$, $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))$ is infinite. In characteristic zero, if $p_g > 0$ then $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))$ is infinite.*

To my knowledge, this is the first general result on indecomposable $(2,1)$-cycles. It relates to the following open question.

Question 1 (See also Remark 1). Is there a surface X such that $b_2 - \rho > 0$ but $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2)) \otimes \mathbf{Q} = 0$?

Many examples of complex surfaces X for which $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))$ is not torsion have been given; see, for example, [CDKL14] and the references therein. In most of them, one shows that a version of the Beilinson regulator with values in a quotient of Deligne cohomology takes non-torsion values on this group. On the other hand, there are examples of complex surfaces X with $p_g > 0$ for which the regulator vanishes rationally [Voi94, Theorem 1.6], but there seems to be no such X for which one can decide whether $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2)) \otimes \mathbf{Q} = 0$.

Question 1 evokes Mumford's non-representability theorem for the Albanese kernel $T(X)$ in the Chow group $CH_0(X)$ under the given hypothesis. It is of course much harder, but not unrelated. The link comes through the *transcendental part of the Chow motive of X*, introduced and studied in [KMP07]. If we denote this motive by $t_2(X)$ as in [KMP07], we have

$$T(X)_{\mathbf{Q}} = \mathrm{Hom}_{\mathbf{Q}}(t_2(X), \mathbb{L}^2) = H^4(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}}$$

[KMP07, Proposition 7.2.3]. Here, all groups are taken in the category $\mathbf{Ab} \otimes \mathbf{Q}$ of abelian groups modulo groups of finite exponent and $\mathrm{Hom}_{\mathbf{Q}}$ denotes the refined Hom group on the category $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(k, \mathbf{Q})$ of effective Chow motives with $\mathbf{Q}$ coefficients (see § 2 for all this), while $\mathbb{L}$ is the Lefschetz motive; to justify the last term, note that Chow correspondences act on motivic cohomology, so that motivic cohomology of a Chow motive makes sense. We show the following result.

Theorem 2 (See Proposition 3). *If X is a surface, we have an isomorphism in $\mathbf{Ab} \otimes \mathbf{Q}$:*

$$H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))_{\mathbf{Q}} \simeq H^3(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}}.$$

Corollary 2 [CR85, Proposition 2.15]. *In Theorem 2, assume that k has infinite transcendence degree over its prime subfield. If $T(X) = 0$, then $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))$ is finite.*

Proof. Under the hypothesis on k, $T(X) = 0 \iff t_2(X) = 0$ [KMP07, Corollary 7.4.9b]. Thus, $T(X) = 0 \Rightarrow H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))_{\mathbf{Q}} = 0$ by Theorem 2. This means that $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))$ has finite exponent, hence is finite by Theorem 1 and the known structure of $\mathrm{Br}(X)$. □

[1] The group $H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2))$ is very different from $\mathrm{Br}(X)\{p\}$. Suppose that k is the algebraic closure of a finite field $\mathbf{F}_q$ over which X is defined. In [Mil88, Remark 5.6], Milne proves

$$\det(1 - \gamma t \mid H^i(X, \mathbf{Q}_p(n))) = \prod_{v(a_{ij}) = v(q^n)} (1 - (q^n/a_{ij})t)$$

where γ is the 'arithmetic' Frobenius of X over $\mathbf{F}_q$ and the a_{ij} are the eigenvalues of the 'geometric' Frobenius acting on the crystalline cohomology $H^i(X/W) \otimes \mathbf{Q}_p$ (or, equivalently, on l-adic cohomology for $l \neq p$ by Katz and Messing). We get $V_p(\mathrm{Br}(X)\{p\})$ for $i = 2, n = 1$ and $V_p(H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)))$ for $i = 2, n = 2$.

Remark 1. (1) For $l \neq p$, $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))\{l\}$ finite $\iff b_2 - \rho = 0$ by Theorem 1. Under Bloch's conjecture, this implies that $t_2(X) = 0$ [KMP07, Corollary 7.6.11], hence $T(X) = 0$ and (by Theorem 2) $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))$ is finite. This provides conjectural converses to Corollaries 1 (for a surface) and 2.

(2) The quotient of $H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))_{\mathrm{tors}}$ by its maximal divisible subgroup is dual to $\mathrm{NS}(X)_{\mathrm{tors}}$, at least away from p: we leave this to the interested reader.

In § 4 we apply Theorem 2 to give a proof of Roĭtman's theorem that $T(X)$ is uniquely divisible, up to a group of finite exponent. This proof is related to Bloch's [Blo79], but avoids Lefschetz pencils; we feel that $t_2(X)$ gives a new understanding of the situation.

1. Proof of Theorem 1

This proof is an elaboration of the arguments of Colliot-Thélène and Raskind in [CR85], completed by Gros and Suwa [GS88, ch. IV] for $l = \operatorname{char} k$. We use motivic cohomology as it smooths the exposition and is more inspirational, but stress that these ideas go back to [Blo79, Pan82, CR85, GS88]. We refer to [Kah12, § 2] for an exposition of ordinary and étale motivic cohomology and the facts used below, especially to [Kah12, Theorem 2.6] for the comparison with étale cohomology of twisted roots of unity and logarithmic Hodge–Witt sheaves.

Multiplication by l^s on étale motivic cohomology yields 'Bockstein' exact sequences

$$0 \to H^i_{\text{ét}}(X, \mathbf{Z}(n))/l^s \to H^i_{\text{ét}}(X, \mathbf{Z}/l^s(n)) \to {}_{l^s}H^{i+1}_{\text{ét}}(X, \mathbf{Z}(n)) \to 0$$

for any prime l, $s \geqslant 1$, $n \geqslant 0$ and $i \in \mathbf{Z}$. Since $\varprojlim^1 H^i_{\text{ét}}(X, \mathbf{Z}(n))/l^s = 0$, one gets in the limit exact sequences:

$$0 \to H^i_{\text{ét}}(X, \mathbf{Z}(n))^{\widehat{}} \xrightarrow{a} H^i_{\text{ét}}(X, \hat{\mathbf{Z}}(n)) \xrightarrow{b} \hat{T}(H^{i+1}_{\text{ét}}(X, \mathbf{Z}(n))) \to 0 \tag{3}$$

where $\hat{T}(-) = \mathrm{Hom}(\mathbf{Q}/\mathbf{Z}, -)$ denotes the total Tate module. This first yields the following result.

Proposition 1. *For $i \neq 2n$, $\mathrm{Im}\, a \otimes \mathbf{Z}[1/p]$ is finite in (3) $\otimes \mathbf{Z}[1/p]$ and $H^i_{\text{ét}}(X, \mathbf{Z}(n)) \otimes \mathbf{Z}[1/p]$ is an extension of a finite group by a divisible group. If $p > 1$, $H^i_{\text{ét}}(X, \mathbf{Z}(n)) \otimes \mathbf{Z}_{(p)}$ is an extension of a group of finite exponent by a divisible group, and is divisible if $i = n$. In particular, $H^n_{\text{ét}}(X, \mathbf{Z}(n))$ is an extension of a finite group of order prime to p by a divisible group.*

Proof. This is the argument of [CR85, 1.8 and 2.2]. Let us summarise it: $H^i_{\text{ét}}(X, \mathbf{Z}(n))$ is 'of weight 0' and $H^i_{\text{ét}}(X, \hat{\mathbf{Z}}(n))$ is 'of weight $i - 2n$' by Deligne's proof of the Weil conjectures. It follows that $a \otimes \mathbf{Z}[1/p]$ has finite image in every l-component, hence has finite image by Gabber's theorem [Gab83]. One derives the structure of $H^i_{\text{ét}}(X, \mathbf{Z}(n)) \otimes \mathbf{Z}[1/p]$ from this.

At the referee's request, we give more details. Since X is defined over a finitely generated field, motivic cohomology commutes with filtering inverse limits of smooth schemes (with affine transition morphisms) and l-adic cohomology is invariant under algebraically closed extensions, to show that a has finite image we may assume that k is the algebraic closure of a finitely generated field k_0 over which X is defined. If $i \neq 2n$ and $l \neq p$, then $H^i_{\text{ét}}(X, \mathbf{Z}_l(n))^U$ is finite for any open subgroup U of $\mathrm{Gal}(k/k_0)$ [CR85, 1.5], while $H^i_{\text{ét}}(X, \mathbf{Z}(n)) = \bigcup_U H^i_{\text{ét}}(X, \mathbf{Z}(n))^U$. Thus the image $I(l)$ of the composition $H^i_{\text{ét}}(X, \mathbf{Z}(n)) \to H^i_{\text{ét}}(X, \mathbf{Z}(n))^{\widehat{}}_l \xrightarrow{a_l} H^i_{\text{ét}}(X, \mathbf{Z}_l(n))$ is contained in the (finite) torsion subgroup of $H^i_{\text{ét}}(X, \mathbf{Z}_l(n))$, hence this composition factors through $H^i_{\text{ét}}(X, \mathbf{Z}(n))/l^s$ for $s \gg 0$, implying that $\mathrm{Im}\, a_l = I(l)$ is finite, and 0 for almost all l by [Gab83]. The conclusion now follows by Lemma 1 below.

If $l = p$, the group $H^i_{\text{ét}}(X, \mathbf{Q}_p(n))^U$ is still 0 for $i \neq 2n$ by [GS88, II.2.3]. The group $H^i_{\text{ét}}(X, \mathbf{Z}_p(n))$ has the structure of an extension of a finitely generated pro-étale group by a unipotent quasi-algebraic group by [IR83, ch. IV, Theorem 3.3(b)], hence its torsion has finite exponent independent of k. Therefore $H^i_{\text{ét}}(X, \mathbf{Z}_p(n))^U$ has bounded exponent when U varies, hence (as above) $\operatorname{Im} a_p$ has finite exponent, and the first claim. For the second one, $H^n_{\text{ét}}(X, \mathbf{Z}_p(n))$ is always torsion-free by [Ill79, ch. II, Corollary 2.17]. □

Lemma 1. *Let A be an abelian group such that $\hat{A} = \varprojlim A/m$ has finite exponent. Then A is an extension of $\hat{A}$ by a divisible group.*

Proof. This is the argument of [CR85, Theorem 1.8], that we reproduce here. First, $\hat{A} \xrightarrow{\sim} A/m_0$ for some $m_0 \geqslant 1$, hence $A \to \hat{A}$ is surjective. Now $A/m \xrightarrow{\sim} A/m_0$ for any multiple m of m_0, hence $\operatorname{Ker}(A \to \hat{A}) = mA$ for any such m; thus this kernel is divisible as claimed. □

Remark 2. In characteristic p, the torsion subgroup of $H^i_{\text{ét}}(X, \mathbf{Z}_p(n))$ may well be infinite for $i > n$ (compare [Ill79, ch. II, § 7]), and then so is the quotient of $H^i_{\text{ét}}(X, \mathbf{Z}(n)) \otimes \mathbf{Z}_{(p)}$ by its maximal divisible subgroup.

Consider now the case $n = 2$. Recall that $H^i(X, \mathbf{Z}(2)) \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathbf{Z}(2))$ for $i \leqslant 3$ from the Merkurjev–Suslin theorem (cf. [Kah12, (2–6)]).

For $l \neq p$, let

$$\begin{aligned} H^2_{\text{ind}}(X, \mu_{l^n}^{\otimes 2}) &= \operatorname{Coker}(\operatorname{Pic}(X) \otimes \mu_{l^n} \to H^2_{\text{ét}}(X, \mu_{l^n}^{\otimes 2})), \\ H^2_{\text{ind}}(X, \mathbf{Z}_l(2)) &= \operatorname{Coker}(\operatorname{Pic}(X) \otimes \mathbf{Z}_l(1) \to H^2_{\text{ét}}(X, \mathbf{Z}_l(2))). \end{aligned}$$

Lemma 2. *For $l \neq p$, there is a canonical isomorphism $H^2_{\text{ind}}(X, \mathbf{Z}_l(2)) \simeq T_l(\operatorname{Br}(X))(1)$. In particular, this group is torsion-free.*

Proof. Straightforward from the Kummer exact sequence. □

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \operatorname{Pic}(X) \otimes \mu_{l^s} & \longrightarrow & H^2_{\text{ét}}(X, \mu_{l^s}^{\otimes 2}) & \longrightarrow & H^2_{\text{ind}}(X, \mu_{l^s}^{\otimes 2}) & \longrightarrow 0 \\ & {\scriptstyle\text{surjective}}\downarrow & & {\scriptstyle\alpha_s}\downarrow & & & \\ 0 \longrightarrow & {}_{l^s}(\operatorname{Pic}(X) \otimes k^*) & \longrightarrow & {}_{l^s}H^3(X, \mathbf{Z}(2)) & \longrightarrow & {}_{l^s}H^3_{\text{ind}}(X, \mathbf{Z}(2)) & \longrightarrow 0 \end{array} \tag{4}$$

where the upper row is exact and the lower row is a complex. This diagram is equivalent to the one in [CR85, 2.8], but the proof of its commutativity is easier, as a consequence of the compatibility of Bockstein boundaries with cup-product in hypercohomology. This yields maps

$$H^2_{\text{ind}}(X, \mu_{l^s}^{\otimes 2}) \xrightarrow{\beta_s} {}_{l^s}H^3_{\text{ind}}(X, \mathbf{Z}(2)), \tag{5}$$

an inverse limit commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \operatorname{NS}(X) \otimes \mathbf{Z}_l(1) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{Z}_l(2)) & \xrightarrow{\pi} & H^2_{\text{ind}}(X, \mathbf{Z}_l(2)) & \longrightarrow 0 \\ & {\scriptstyle\text{surjective}}\downarrow & & {\scriptstyle\hat{\alpha}}\downarrow & & {\scriptstyle\hat{\beta}}\downarrow & \\ 0 \longrightarrow & T_l(\operatorname{Pic}(X) \otimes k^*) & \longrightarrow & T_l(H^3(X, \mathbf{Z}(2)) & \longrightarrow & T_l(H^3_{\text{ind}}(X, \mathbf{Z}(2)) & \longrightarrow 0 \end{array} \tag{6}$$

(note that $\mathrm{Pic}(X)\otimes\mu_{l^s} \xrightarrow{\sim} \mathrm{NS}(X)\otimes\mu_{l^s}$) and a direct limit commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Pic}(X)\otimes\mu_{l^\infty} & \longrightarrow & H^2(X,\mathbf{Q}_l/\mathbf{Z}_l(2)) & \longrightarrow & \mathrm{Br}(X)\{l\}(1) & \longrightarrow 0 \\ & \wr\downarrow & & \alpha_l\downarrow & & \beta_l\downarrow & \\ 0 \longrightarrow & (\mathrm{Pic}(X)\otimes k^*)\{l\} & \longrightarrow & H^3(X,\mathbf{Z}(2))\{l\} & \longrightarrow & H^3_{\mathrm{ind}}(X,\mathbf{Z}(2))\{l\} & \longrightarrow 0 \end{array} \tag{7}$$

where β_l defines the map β' in Theorem 1. Note that the left vertical map in (7) is injective because $\mathrm{Tor}(\mathrm{Pic}(X), k^*\otimes\mathbf{Z}[1/l])\{l\} = 0$.

Lemma 3. *If X is defined over a subfield k_0 with algebraic closure k, the map π of (6) has a G-equivariant section after $\otimes\mathbf{Q}$, where $G = \mathrm{Gal}(k/k_0)$. In particular, if k_0 is finitely generated, then $H^2_{\mathrm{ind}}(X,\mathbf{Q}_l(2))^U = 0$ for any open subgroup U of G.*

Proof. Let $d = \dim X$; we may assume $d > 1$. If $d = 2$, the perfect Poincaré pairing $H^2_{\text{ét}}(X, \mathbf{Q}_l(1))\times H^2_{\text{ét}}(X,\mathbf{Q}_l(1)) \to \mathbf{Q}_l$ restricts to the perfect intersection pairing $\mathrm{NS}(X)\otimes\mathbf{Q}_l\otimes\mathrm{NS}(X)\otimes\mathbf{Q}_l \to \mathbf{Q}_l$; the promised section is then given by the orthogonal complement of $\mathrm{NS}(X)\otimes\mathbf{Q}_l(1)$ in $H^2_{\text{ét}}(X,\mathbf{Q}_l(2))$. If $d > 2$, let $L \in H^2(X,\mathbf{Q}_l)$ be the class of a smooth hyperplane section defined over k_0. The hard Lefschetz theorem and Poincaré duality provide a perfect pairing on $H^2_{\text{ét}}(X,\mathbf{Q}_l(1))$:

$$(x,y) \mapsto x\cdot L^{d-2}\cdot y$$

which restricts to a similar pairing on $\mathrm{NS}(X)\otimes\mathbf{Q}_l$. The Hodge index theorem for divisors [SGA6, Proposition 7.4, p. 665] implies that the latter pairing is also non-degenerate, so we get the desired section in the same way. The last claim now follows from the vanishing of $H^2(X,\mathbf{Q}_l(2))^U$; see the proof of Proposition 1. □

We shall use the following fact, which is proved in [CR85, 2.7] (and could be re-proved here with motivic cohomology in the same fashion).

Lemma 4. *In (1), $N := \mathrm{Ker}\,\theta$ has no l-torsion.*

Proposition 2 (Cf. [CR85, Remark 2.13]). *β_s is surjective in (5) and $\hat\beta$ is bijective in (6); N is uniquely divisible; the lower row of (7) is exact and β_l is bijective.*

Proof. Since $\mathrm{Pic}(X)\otimes k^*$ is l-divisible, Lemma 4 yields exact sequences

$$0 \to {}_{l^s}(\mathrm{Pic}(X)\otimes k^*) \to {}_{l^s}A \to N/l^s \to 0, \tag{8}$$

$$0 \to {}_{l^s}A \to {}_{l^s}H^3(X,\mathbf{Z}(2)) \to {}_{l^s}H^3_{\mathrm{ind}}(X,\mathbf{Z}(2)) \to 0, \tag{9}$$

where $A = \mathrm{Im}\,\theta$, and (9) implies the surjectivity of β_s, hence of $\hat\beta$ since the groups $H^2_{\mathrm{ind}}(X,\mu_{l^s}^{\otimes 2})$ are finite. Since α_s is surjective in (4), we also get that all groups in (8) and (9) are finite. Now the upper row of (6) is exact; in its lower row, the homology at $T_l(H^3(X,\mathbf{Z}(2))$ is isomorphic to $N\hat{_l}$ by taking the inverse limit of (8) and (9). A snake chase then yields an exact sequence

$$H^2(X,\mathbf{Z}(2))\hat{_l} \simeq \mathrm{Ker}\,\hat\alpha \to \mathrm{Ker}\,\hat\beta \to N\hat{_l} \to 0$$

where $\mathrm{Ker}\,\hat\alpha$ is finite by Proposition 1.

If, as in the proof of Proposition 1, k is the algebraic closure of a finitely generated field k_0 over which X is defined and U is an open subgroup of $\mathrm{Gal}(k/k_0)$, we have an isomorphism

$$(\mathrm{Ker}\,\hat{\beta})^U \otimes \mathbf{Q} \xrightarrow{\sim} (N_l^{\wedge})^U \otimes \mathbf{Q}.$$

On the one hand, $(\mathrm{Ker}\,\hat{\beta})^U \otimes \mathbf{Q} = 0$ by Lemma 3 because $\mathrm{Ker}\,\hat{\beta}$ is a subgroup of $H^2_{\mathrm{ind}}(X, \mathbf{Z}_l(2))$; on the other hand, since N/l is finite,

$$N_l^{\wedge} = \bigcup_U (N_l^{\wedge})^U.$$

Indeed, a finite set of generators $\{n_i\}$ of N modulo lN also generates N modulo l^sN for all $s \geqslant 1$, and an open subgroup U of G fixing all the n_i also fixes $N_l^{\wedge}$ (so the union is in fact stationary).

This gives $N_l^{\wedge} \otimes \mathbf{Q} = 0$, hence $N_l^{\wedge} = 0$ by Lemma 4; thus $\mathrm{Ker}\,\hat{\beta}$ is finite, hence 0 by Lemma 2. This also shows the l-divisibility of N, which thanks to (8) and (9) implies the exactness of the lower row of (4), hence of (7). Now α_l is surjective, and also injective since $\mathrm{Ker}\,\alpha_l \simeq H^2(X, \mathbf{Z}(2)) \otimes \mathbf{Q}_l/\mathbf{Z}_l$ is 0 by Proposition 1. Hence β_l is bijective. □

The case of p-torsion is similar and easier: by Proposition 1, we have an isomorphism

$$H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)) \xrightarrow{\sim} H^3(X, \mathbf{Z}(2))\{p\}$$

and $H^3(X, \mathbf{Z}(2))\{p\} \xrightarrow{\sim} H^3_{\mathrm{ind}}(X, \mathbf{Z}(2))\{p\}$ since k^* is uniquely p-divisible, hence also $\mathrm{Pic}(X) \otimes k^*$. This concludes the proof of Theorem 1.

2. Refined Hom groups

Let $\mathcal{A}$ be an additive category; write $\mathcal{A} \otimes \mathbf{Q}$ for the category with the same objects as $\mathcal{A}$ and Hom groups tensored with $\mathbf{Q}$, and $\mathcal{A} \boxtimes \mathbf{Q}$ for the pseudo-abelian envelope of $\mathcal{A} \otimes \mathbf{Q}$. If $\mathcal{A}$ is abelian, then $\mathcal{A} \otimes \mathbf{Q} = \mathcal{A} \boxtimes \mathbf{Q}$ is still abelian and is the localisation of $\mathcal{A}$ by the Serre subcategory $\mathcal{A}_{\mathrm{tors}}$ of objects A such that $n1_A = 0$ for some integer $n > 0$ (e.g. [BK, Proposition B.3.1]).

For $\mathcal{A} = \mathbf{Ab}$, the category of abelian groups, one has a chain of natural functors

$$\mathbf{Ab} \xrightarrow{a} \mathbf{Ab} \otimes \mathbf{Q} \xrightarrow{b} \mathbf{Vec}_{\mathbf{Q}}$$

where $\mathbf{Vec}_{\mathbf{Q}}$ is the category of $\mathbf{Q}$-vector spaces and the second functor is induced by 'tensoring objects with $\mathbf{Q}$'. The functor b is fully faithful when restricted to the full subcategory of $\mathbf{Ab} \otimes \mathbf{Q}$ given by finitely generated abelian groups, but it is not faithful in general; for example, $a(\mathbf{Q}/\mathbf{Z}) \neq 0$ while $ba(\mathbf{Q}/\mathbf{Z}) = 0$. Thus a retains torsion information that is lost when composing it with b. For simplicity, we shall write

$$a(A) = A_{\mathbf{Q}}, \quad ba(A) = A \otimes \mathbf{Q} \tag{10}$$

for the image of an abelian group $A \in \mathbf{Ab}$ respectively in $\mathbf{Ab} \otimes \mathbf{Q}$ and $\mathbf{Vec}_{\mathbf{Q}}$.

Let F be an additive functor (covariant or contravariant) from $\mathcal{A}$ to $\mathbf{Ab}$, the category of abelian groups. It then induces a functor

$$F_{\mathbf{Q}} : \mathcal{A} \boxtimes \mathbf{Q} \to \mathbf{Ab} \otimes \mathbf{Q}.$$

In particular, we get a bifunctor

$$\mathrm{Hom}_{\mathbf{Q}} : (\mathcal{A} \boxtimes \mathbf{Q})^{\mathrm{op}} \times \mathcal{A} \boxtimes \mathbf{Q} \to \mathbf{Ab} \otimes \mathbf{Q}$$

which refines the bifunctor Hom of $\mathcal{A} \boxtimes \mathbf{Q}$.

We shall apply this to $\mathcal{A} = \mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(k)$, the category of effective Chow motives with integral coefficients: the category $\mathcal{A} \boxtimes \mathbf{Q}$ is then equivalent to the category $\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(k, \mathbf{Q})$ of Chow motives with rational coefficients.

3. Chow–Künneth decomposition of $\mathcal{K}_2$-cohomology

In this section, X is a connected surface. Its Chow motive $h(X) \in \mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(k, \mathbf{Q})$ then enjoys a refined Chow–Künneth decomposition

$$h(X) = h_0(X) \oplus h_1(X) \oplus h_2^{\mathrm{alg}}(X) \oplus t_2(X) \oplus h_3(X) \oplus h_4(X) \tag{11}$$

[KMP07, Propositions 7.2.1 and 7.2.3]. The projectors defining this decomposition act on the groups $H^i(X, \mathbf{Z}(2))_{\mathbf{Q}}$; we propose to compute the corresponding direct summands $H^i(M, \mathbf{Z}(2))_{\mathbf{Q}}$. To be more concrete, we shall express this in terms of the $\mathcal{K}_2$-cohomology of X.

We keep the notation

$$H^1_{\mathrm{ind}}(X, \mathcal{K}_2) = \mathrm{Coker}(\mathrm{Pic}(X) \otimes k^* \to H^1(X, \mathcal{K}_2))$$

to which we adjoin

$$H^0_{\mathrm{ind}}(X, \mathcal{K}_2) = \mathrm{Coker}(K_2(k) \to H^0(X, \mathcal{K}_2)).$$

To relate to the notation in § 1, recall that $H^2(k, \mathbf{Z}(2)) = K_2(k)$ and $H^2(X, \mathbf{Z}(2)) = H^0(X, \mathcal{K}_2)$.

We shall also need a smooth connected hyperplane section C of X, appearing in the construction of (11) [Mur90, Sch94], and its own Chow–Künneth decomposition attached to the choice of a rational point:

$$h(C) = h_0(C) \oplus h_1(C) \oplus h_2(C). \tag{12}$$

The projectors defining (12) have integral coefficients, while those defining (11) only have rational coefficients in general.

The following proposition extends the computations of [KMP07, 7.2.1 and 7.2.3] to weight-2 motivic cohomology.

Proposition 3. (a) *We have the following table for $H^i(M, \mathbf{Z}(2))$:*

$M =$	$h_0(C)$	$h_1(C)$	$h_2(C)$
$i = 2$	$K_2(k)$	$H^0_{\mathrm{ind}}(C, \mathcal{K}_2)$	0
$i = 3$	0	$V(C)$	k^*
$i > 3$	0	0	0

where $V(C) = \mathrm{Ker}(H^1(C, \mathcal{K}_2) \xrightarrow{N} k^*)$ is Bloch's group.

(b) *We have the following table for $H^i(M, \mathbf{Z}(2))$, where all groups are taken in $\mathbf{Ab} \otimes \mathbf{Q}$ (see § 2):*

$M =$	$h_0(X)$	$h_1(X)$	$h_2^{\mathrm{alg}}(X)$	$t_2(X)$	$h_3(X)$	$h_4(X)$
$i = 2$	$K_2(k)$	A	0	B	0	0
$i = 3$	0	$\mathrm{Pic}^0(X) k^*$	$\mathrm{NS}(X) \otimes k^*$	$H^1_{\mathrm{ind}}(X, \mathcal{K}_2)$	0	0
$i = 4$	0	0	0	$T(X)$	$\mathrm{Alb}(X)$	$\mathbf{Z}$
$i > 4$	0	0	0	0	0	0

Here

$$\begin{aligned}\mathrm{Pic}^0(X)k^* &= \mathrm{Im}(\mathrm{Pic}^0(X)\otimes k^* \to H^1(X,\mathcal{K}_2)),\\ A &= \mathrm{Im}(H^0_{\mathrm{ind}}(X,\mathcal{K}_2)\to H^0_{\mathrm{ind}}(C,\mathcal{K}_2)),\\ B &= \mathrm{Ker}(H^0_{\mathrm{ind}}(X,\mathcal{K}_2)\to H^0_{\mathrm{ind}}(C,\mathcal{K}_2)).\end{aligned}$$

Proof. We proceed by exclusion as in the proof of [KMP07, Theorem 7.8.4]. Let us start with (a). We use the notation (10) of § 2.

– For $i>3$, $H^i(M,\mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^i(C,\mathbf{Z}(2))_{\mathbf{Q}}=0$.
– One has $h_2(C)=\mathbb{L}$, hence

$$H^i(h_2(C),\mathbf{Z}(2))_{\mathbf{Q}}=H^{i-2}(k,\mathbf{Z}(1))_{\mathbf{Q}}=\begin{cases}k^*_{\mathbf{Q}} & \text{if } i=3,\\ 0 & \text{otherwise.}\end{cases}$$

– One has

$$H^i(h_0(C),\mathbf{Z}(2))_{\mathbf{Q}}=H^i(k,\mathbf{Z}(2))_{\mathbf{Q}}=\begin{cases}K_2(k)_{\mathbf{Q}} & \text{if } i=2,\\ 0 & \text{if } i>2.\end{cases}$$

– The case of $M=h_1(C)$ follows from the two previous ones by exclusion.

Let us turn to (b).

– For $i>4$, $H^i(M,\mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^i(X,\mathbf{Z}(2))_{\mathbf{Q}}=0$.
– One has $h_4(X)=\mathbb{L}^2$, hence

$$H^i(h_4(X),\mathbf{Z}(2))_{\mathbf{Q}}=H^{i-4}(k,\mathbf{Z})_{\mathbf{Q}}=\begin{cases}\mathbf{Z}_{\mathbf{Q}} & \text{if } i=4,\\ 0 & \text{otherwise.}\end{cases}$$

– One has $h_3(X)=h_1(X)(1)$, hence

$$H^i(h_3(X),\mathbf{Z}(2))_{\mathbf{Q}}=H^{i-2}(h_1(X),\mathbf{Z}(1))_{\mathbf{Q}}.$$

As $h_1(X)$ is a direct summand of $h_1(C)$, $H^{i-2}(h_1(X),\mathbf{Z}(1))_{\mathbf{Q}}$ is a direct summand of $H^{i-2}(C,\mathbf{Z}(1))_{\mathbf{Q}}$. This group is 0 for $i\neq 3,4$. For $i=3$, one has $H^1(C,\mathbf{Z}(1))_{\mathbf{Q}}=H^1(h_0(C),\mathbf{Z}(1))_{\mathbf{Q}}$, hence

$$H^1(h_1(C),\mathbf{Z}(1))_{\mathbf{Q}}=H^1(h_1(X),\mathbf{Z}(1))_{\mathbf{Q}}=0.$$

For $i=4$, $H^2(h_1(X),\mathbf{Z}(1))_{\mathbf{Q}}=\mathrm{Alb}(X)_{\mathbf{Q}}$ (cf. [Mur90]).
– One has $h_2^{\mathrm{alg}}(X)=\mathrm{NS}(X)(1)$, hence

$$\begin{aligned}H^i(h_2^{\mathrm{alg}}(X),\mathbf{Z}(2))_{\mathbf{Q}} &= (H^{i-2}(k,\mathbf{Z}(1))\otimes \mathrm{NS}(X))_{\mathbf{Q}}\\ &= \begin{cases}(\mathrm{NS}(X)\otimes k^*)_{\mathbf{Q}} & \text{if } i=3,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

– One has

$$H^i(h_0(X),\mathbf{Z}(2))_{\mathbf{Q}}=H^i(k,\mathbf{Z}(2))_{\mathbf{Q}}=\begin{cases}K_2(k)_{\mathbf{Q}} & \text{if } i=2,\\ 0 & \text{if } i>2.\end{cases}$$

– As $h^1(X)$ is a direct summand of $h^1(C)$, $H^i(h^1(X),\mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^i(C,\mathbf{Z}(2))_{\mathbf{Q}}$; this group is therefore 0 for $i>3$. This completes row $i=4$ by exclusion.

– The action of refined Chow–Künneth projectors respects the homomorphism $(\mathrm{Pic}(X)\otimes k^*)_{\mathbf{Q}} \to H^3(X,\mathbf{Z}(2))_{\mathbf{Q}}$. As the action of π_2^{tr} (defining $t_2(X)$) is 0 on $\mathrm{Pic}(X)_{\mathbf{Q}}$, we get $H^3(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \simeq H^1_{\mathrm{ind}}(X,\mathcal{K}_2)_{\mathbf{Q}}$, which completes row $i=3$ by exclusion.
– The construction of π_2^{tr} [KMP07, proof of 2.3] shows that the composition

$$h(C) \xrightarrow{i_*} h(X) \to t_2(X)$$

is 0. Hence the composition

$$H^i(t_2(X),\mathbf{Z}(2))_{\mathbf{Q}} \to H^i(X,\mathbf{Z}(2))_{\mathbf{Q}} \xrightarrow{i^*} H^i(C,\mathbf{Z}(2))_{\mathbf{Q}}$$

is 0 for all i. Applying this for $i=2$, we see that $H^2(t_2(X),\mathbf{Z}(2))_{\mathbf{Q}} \subseteq B_{\mathbf{Q}}$. On the other hand, $H^2(h_1(X),\mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^2(h_1(C),\mathbf{Z}(2))_{\mathbf{Q}}$, hence injects in $A_{\mathbf{Q}}$. By exclusion, we have $H^2(t_2(X),\mathbf{Z}(2))_{\mathbf{Q}} \oplus H^2(h_1(X),\mathbf{Z}(2))_{\mathbf{Q}} \simeq H^0_{\mathrm{ind}}(X,\mathbf{Z}(2))_{\mathbf{Q}}$, hence row $i=2$. □

Remark 3. Let us clarify the 'reasoning by exclusion' that has been used repeatedly in this proof. Let F be a functor from smooth projective varieties to $\mathbf{Ab}\otimes\mathbf{Q}$, provided with an action of Chow correspondences. Then F automatically extends to $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(k,\mathbf{Q})$, and we wish to compute the effect of a Chow–Künneth decomposition of $h(X)$ on $F(X)$. The reasoning above is as follows in its simplest form.

Suppose that we have a motivic decomposition $h(X) = M \oplus M'$, hence a decomposition $F(X) = F(M) \oplus F(M')$. Suppose that we know an exact sequence

$$0 \to A \to F(X) \to B \to 0$$

and an isomorphism $F(M) \simeq A$. Then $F(M') \simeq B$.

Of course this reasoning is incorrect as it stands; to justify it, one should check that if π is the projector with image M yielding the decomposition of $h(X)$, then $F(\pi)$ does have image A. This can be checked in all cases of the above proof, but such a verification would be tedious, double the length of the proof and probably make it unreadable. I hope the reader will not disagree with this expository choice.

4. Generalisation

In this section, we take the gist of the previous arguments. For convenience we pass from effective Chow motives $\mathcal{M}^{\mathrm{eff}}_{\mathrm{rat}}(k,\mathbf{Q})$ to all Chow motives $\mathcal{M}_{\mathrm{rat}}(k,\mathbf{Q})$. Since étale motivic cohomology has an action of Chow correspondences and verifies the projective bundle formula, it yields well-defined contravariant functors

$$H^i_{\text{ét}} : \mathcal{M}_{\mathrm{rat}}(k,\mathbf{Q}) \to \mathbf{Ab}\otimes\mathbf{Q}$$

such that $H^i_{\text{ét}}(X,\mathbf{Z}(n))_{\mathbf{Q}} = H^{i-2n}_{\text{ét}}(h(X)(-n))$ for any smooth projective k-variety X and $i,n\in\mathbf{Z}$. We also have (contravariant) realisation functors

$$H^i_l : \mathcal{M}_{\mathrm{rat}}(k,\mathbf{Q}) \to \mathcal{C}_l\otimes\mathbf{Q}$$

extending l-adic cohomology for $l \neq \operatorname{char} k$, where $\mathcal{C}_l$ denotes the category of $l\mathbf{Z}$-adic inverse systems of abelian groups [SGA5, V.3.1.1]. For $l = \operatorname{char} k$ we use logarithmic Hodge–Witt cohomology as in Theorem 1 [Mil88, § 2], [GS88].

Definition 1. Let $M \in \mathcal{M}_{\mathrm{rat}}(k, \mathbf{Q})$. If $i \in \mathbf{Z}$, we say that M *is pure of weight* i if $H^j_l(M) = 0$ for all $j \neq i$ and all primes l.

For example, if $h(X) = \bigoplus_{i=0}^{2d} h_i(X)$ is a Chow–Künneth decomposition of the motive $h(X)$ of a d-dimensional smooth projective variety X, then $h_i(X)$ is pure of weight i. If $d = 2$, the motive $t_2(X)(-2)$ is pure of weight -2 as a direct summand of $h_2(X)(-2)$.

Theorem 3. *Let M be pure of weight i. Then $H^j_{\text{ét}}(M)$ is uniquely divisible for $j \neq i, i+1$. If, moreover, $i \neq 0$, then $H^i_{\text{ét}}(M)$ is uniquely divisible and $H^{i+1}_{\text{ét}}(M)\{l\} \simeq H^i_l(M) \otimes \mathbf{Q}/\mathbf{Z}$.*

(An object $A \in \mathbf{Ab} \otimes \mathbf{Q}$ is *uniquely divisible* if multiplication by n is an automorphism of A for any integer $n \neq 0$.)

Proof. As in § 1, we have Bockstein exact sequences in $\mathcal{C}_l \otimes \mathbf{Q}$,

$$0 \to H^j_{\text{ét}}(M)/l^* \xrightarrow{a} H^j_l(M) \to {}_{l^*}H^{j+1}_{\text{ét}}(M) \to 0,$$

which yields the first statement. For the second one, the weight argument of [CR85] (developed in the proof of Proposition 1 above) yields $\operatorname{Im} a = 0$. □

Let X be a surface. Applying Theorem 3 to $M = t_2(X)(-2)$ as above, we get that $H^i_{\text{ét}}(t_2(X), \mathbf{Z}(2))$ is uniquely divisible for $i \neq 3$ and

$$H^3_{\text{ét}}(t_2(X), \mathbf{Z}(2))\{l\} \simeq H^3_{\mathrm{tr}}(X, \mathbf{Z}_l(2)) \otimes \mathbf{Q}/\mathbf{Z} \simeq \mathrm{Br}(X)\{l\}$$

in $\mathbf{Ab} \otimes \mathbf{Q}$, recovering a slightly weaker version of Theorem 1 in view of Proposition 3. For $i = 4$, the exact sequence [Kah12, (2–7)]

$$0 \to CH^2(X) \to H^4_{\text{ét}}(X, \mathbf{Z}(2)) \to H^0(X, \mathcal{H}^3_{\text{ét}}(\mathbf{Q}/\mathbf{Z}(2))) \to 0$$

shows that $CH^2(X) \xrightarrow{\sim} H^4_{\text{ét}}(X, \mathbf{Z}(2))$ since $\dim X = 2$, whence

$$T(X) = H^4(t_2(X), \mathbf{Z}(2)) \xrightarrow{\sim} H^4_{\text{ét}}(t_2(X), \mathbf{Z}(2)),$$

yielding a proof of Roĭtman's theorem up to small torsion.

Remark 4. This argument is not integral because the projector π_2^{tr} defining $t_2(X)$ is not an integral correspondence. It is, however, l-integral for any l prime to a denominator D of π_2^{tr}. This D is essentially controlled by the degree of the Weil isogeny

$$\mathrm{Pic}^0_{X/k} \to \mathrm{Pic}^0_{C/k} = \mathrm{Alb}(C) \to \mathrm{Alb}(X)$$

where C is the ample curve involved in the construction of π_2^{tr}. If one could show that various Cs can be chosen so that the corresponding degrees have gcd equal to 1, one would deduce a full proof of Roĭtman's theorem from the above.

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