

## A FEW LOCALISATION THEOREMS

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### Abstract

Given a functor  $T: \mathcal{C} \rightarrow \mathcal{D}$  carrying a class of morphisms  $S \subset \mathcal{C}$  into a class  $S' \subset \mathcal{D}$ , we give sufficient conditions for  $T$  to induce an equivalence on the localised categories. These conditions are in the spirit of Quillen's Theorem A. We give some applications in algebraic and birational geometry.

### Introduction

Let  $T: \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $S \subset \mathcal{C}$ ,  $S' \subset \mathcal{D}$  two classes of morphisms containing identities and stable under composition, such that  $T(S) \subseteq S'$ . This induces the situation

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{D} \\ P \downarrow & & Q \downarrow \\ S^{-1}\mathcal{C} & \xrightarrow{\bar{T}} & S'^{-1}\mathcal{D} \end{array} \quad (0.1)$$

where  $P$  and  $Q$  are localisation functors. In this note, we offer an answer to the following question:

*Question 0.1.* Give sufficient conditions for  $\bar{T}$  to be an equivalence of categories.

The answer, Theorem 2.1, is in the spirit of Quillen's Theorem A [15, Th. A] which we recall here for motivation: in the above situation, forgetting  $S$  and  $S'$ , if for all  $d \in \mathcal{D}$  the category  $d \setminus T$  (see § 1.1) is  $\infty$ -connected, then  $T$  is a weak equivalence.

### Background

In [9, Th. 3.8], we proved that  $\bar{T}$  is an equivalence of categories when  $\mathcal{D}$  is the category of smooth varieties over a field of characteristic 0,  $\mathcal{C}$  is its full subcategory consisting of smooth projective varieties, and we take for  $S$  and  $S'$  either birational morphisms or “stable birational morphisms” (i.e. dominant morphisms such that the corresponding function field extension is purely transcendental). When we started revising [9], it turned out that we needed similar localisation theorems in other situations. At this stage it was becoming desirable to understand these localisation

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theorems more abstractly, and indeed we got two non-overlapping, technical (and very ugly) statements.

The first author then discussed these results with Georges Maltsiniotis, and they arrived at Corollary 4.4 below. Using Proposition 5.10 below, one can easily see that the hypotheses of Corollary 4.4 are verified in the case of Theorem 3.8 of [9]. However, they are not verified in some of the other geometric situations mentioned above.

“Catching” the latter situations led to Theorem 2.1. Thus we had two sets of abstract hypotheses implying that  $\bar{T}$  is an equivalence of categories:

- hypotheses (0), (1) and (2) of Theorem 2.1;
- hypotheses (0) and (1') of Corollary 4.4.

To crown all, Maltsiniotis gave us an argument showing that  $(0) + (1') \Rightarrow (0) + (1) + (2)$ : this is the content of Theorem 4.3 a) and the proof we give is essentially his.

In the same period, Joël Riou proved a localisation theorem of a similar nature (Theorem 5.2). It turns out that Hypotheses (0), (1) and (2) are implied by Riou's hypotheses (and actually by less); see Theorem 5.3.

After stating and proving the main theorem, Theorem 2.1, we prove a “relativisation” theorem, Theorem 4.3, which leads to Corollary 4.4 mentioned above. We then give a number of conditions which imply the hypotheses of Theorem 2.1 in § 5. In § 6 we show that the fact that  $\bar{T}$  is an equivalence of categories in (0.1) is stable under adjoining products and coproducts. We then give some algebro-geometric applications (hyperenvelopes, cubical hyperresolutions, etc.) in § 7, and finally, in § 8, the birational applications we alluded to; these will be used to simplify the exposition of the revision [10] of [9].

Even though Maltsiniotis did not wish to appear as a coauthor of this note, we want to stress his essential contributions in bringing the results here to their present form. Let us also mention that Hypotheses (0), (1) and (2) imply much more than Theorem 2.1; they actually yield the existence of an “absolute” derived functor (in the sense of Quillen [14, § 4.1, Def. 1]) associated to any functor  $F: \mathcal{D} \rightarrow \mathcal{E}$  such that  $FT(S)$  is invertible. This will be developed in a forthcoming work of Maltsiniotis and the first author, where a different proof of Theorem 2.1 will be given [8]; see § 3 where a weaker result is already given. In [8], we also hope to lift Theorem 2.1 to the “Dwyer-Kan localisation” [3] by suitably reinforcing its hypotheses. Finally, we wish to thank the referee for a very helpful comment regarding Theorem 5.3 (see Lemma 5.7).

## 1. Notation

### 1.1. Comma categories

Recall [12, Ch. II, §6] that to a diagram of categories and functors

$$\begin{array}{ccc} & & \mathcal{B} \\ & & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array}$$

one associates a category  $F \downarrow G$ , the (ordered) “2-fibred product” of  $F$  and  $G$ :

$$\begin{array}{ccc} F \downarrow G & \xrightarrow{F'} & \mathcal{B} \\ \downarrow G' & & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C}. \end{array}$$

An object of  $F \downarrow G$  is a triple  $(a, b, f)$  where  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$  and  $f$  is a morphism from  $F(a)$  to  $G(b)$ . A morphism from  $(a, b, f)$  to  $(a', b', f')$  is a pair of morphisms  $\varphi: a \rightarrow a'$ ,  $\psi: b \rightarrow b'$  such that the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{f} & G(b) \\ \downarrow F(\varphi) & & \downarrow G(\psi) \\ F(a') & \xrightarrow{f'} & G(b') \end{array}$$

commutes. Composition of morphisms is defined in the obvious way.

This notation is subject to the following abbreviations:

- $G = Id_{\mathcal{B}}: F \downarrow G = F \downarrow \mathcal{B}$ .
- Dually,  $F = Id_{\mathcal{A}}: F \downarrow G = \mathcal{A} \downarrow G$ .
- If  $\mathcal{B}$  is the point category and  $G$  has image  $c: F \downarrow G = F \downarrow c = F/c = \mathcal{A}/c$  (the latter notation being used only when there is no possible ambiguity).
- Dually, if  $\mathcal{A}$  is the point category and  $F$  has image  $c: F \downarrow G = c \downarrow G = c \setminus G = c \setminus \mathcal{B}$ .

The category  $F \downarrow G$  should not be confused with its full subcategory  $F \times_c G$  or  $\mathcal{A} \times_c \mathcal{B}$  (1-fibred product), consisting of those triples  $(a, b, f)$  such that  $f$  is an identity.

### 1.2. Path groupoid

For any category  $\mathcal{E}$ , the category obtained by inverting all arrows of  $\mathcal{E}$  is denoted by  $\Pi_1(\mathcal{E})$ ; this is the *path groupoid* of  $\mathcal{E}$ .

### 1.3. Connectedness

A category  $\mathcal{E}$  is *n-connected* if (the geometric realisation of) its nerve is *n-connected*;  $-1$ -connected is synonymous to “non-empty”. For  $n \leq 1$ ,  $\mathcal{E}$  is *n-connected* if and only if  $\Pi_1(\mathcal{E})$  is *n-connected*. Thus  $0$ -connected means that  $\mathcal{E}$  is nonempty and any two of its objects may be connected by a zig-zag of arrows (possibly not all pointing in the same direction) and  $1$ -connected means that  $\Pi_1(\mathcal{E})$  is equivalent to the point (category with one object and one morphism).

If  $\mathcal{E}$  is *n-connected* for any  $n$ , we say that it is  *$\infty$ -connected* (this notion is apparently weaker than “contractible”).

### 1.4. Cofinal functors

According to [12, Ch. IX, § 3, p. 217], a functor  $L: \mathcal{J}' \rightarrow \mathcal{J}$  is called *cofinal* if, for all  $j \in \mathcal{J}$ , the category  $L \downarrow j = L/j$  is  $0$ -connected.

*Comment 1.1.* The following theorem is dual to [12, loc. cit., Th. 1, p. 217 and Ex. 5, p. 218]:

**Theorem 1.2.** *Let  $L: \mathcal{J}' \rightarrow \mathcal{J}$  be a functor between small categories. The following conditions are equivalent:*

- (i)  *$L$  is cofinal.*
- (ii) *For any functor  $F: \mathcal{J} \rightarrow X$  such that  $\varprojlim_{\mathcal{J}'} FL$  exists, the limit  $\varprojlim_{\mathcal{J}} F$  exists and the canonical morphism*

$$\varprojlim_{\mathcal{J}'} FL \rightarrow \varprojlim_{\mathcal{J}} F$$

*is an isomorphism.*

- (iii) *Same statement by limiting to  $X = \text{Set}$  and  $F$  co-representable.*

We shall use this theorem in the proof of Theorem 4.3.

## 2. The main localisation theorem

### 2.1. The categories $I_d$

With the notation of the introduction, consider  $S$  and  $S'$  as subcategories of  $\mathcal{C}$  and  $\mathcal{D}$  with the same objects, and let  $T_S: S \rightarrow S'$  be the functor induced by  $T$ . Set, for all  $d \in \mathcal{D}$ ,

$$I_d = d \downarrow T_S = d \setminus S;$$

see 1.1. Thus:

- An object of  $I_d$  is a pair  $(c, s)$  where  $c \in \mathcal{C}$  and  $s: d \rightarrow T(c)$  belongs to  $S'$ . We summarise this with the notation  $d \xrightarrow{s} T(c)$ , or sometimes  $s$ , or even  $c$  if this does not cause any confusion.
- If  $d \xrightarrow{s} T(c)$ ,  $d \xrightarrow{s'} T(c')$  are two objects of  $I_d$ , a morphism from the first to the second is a morphism  $\sigma: c \rightarrow c'$  belonging to  $S$  and such that the diagram

$$\begin{array}{ccc} & T(c) & \\ & \uparrow s & \downarrow T(\sigma) \\ d & & \\ & \downarrow s' & \\ & T(c') & \end{array}$$

commutes, composition of morphisms being the obvious one.

### 2.2. Categories of diagrams

Let  $E$  be a small category. In the category  $\mathcal{C}^E = \mathbf{Hom}(E, \mathcal{C})$ , one may consider the following class of morphisms  $S(E)$ : if  $c, c' \in \mathcal{C}^E$ , a morphism  $s: c \rightarrow c'$  belongs to  $S(E)$  if and only if for all  $e \in E$ ,  $s(e): c(e) \rightarrow c'(e)$  belongs to  $S$ .  $S'(E)$ , a class

of morphisms in  $\mathcal{D}^E$ , is defined similarly. This gives a meaning to the notation  $I_d$  for  $d \in \mathcal{D}^E$ .

We shall be interested in the case where  $E = \Delta^n$ , corresponding to the totally ordered set  $\{0, \dots, n\}$ , thus  $\mathcal{C}^{\Delta^n}$  can be identified with the category of sequences of  $n$  composable arrows  $(f_n, \dots, f_1)$  of  $\mathcal{C}$ . For  $n = 0$ , this is just the category  $\mathcal{C}$ .

**2.3. Statement of the theorem**

With notation as in §§ 2.1 and 2.2, the theorem is as follows:

**Theorem 2.1** (Simplicial theorem). *Suppose verified the following assumptions:*

- (0) *For all  $d \in \mathcal{D}$ ,  $I_d$  is 1-connected.*
- (1) *For all  $f \in \mathcal{D}^{\Delta^1}$ ,  $I_f$  is 0-connected.*
- (2) *For all  $(f_2, f_1) \in \mathcal{D}^{\Delta^2}$ ,  $I_{(f_2, f_1)}$  is -1-connected.*

*Then  $\bar{T}$  is an equivalence of categories.*

**2.4. Preparatory lemmas**

Before proving Theorem 2.1, we shall establish a few lemmas. The first is trivial:

**Lemma 2.2.** *For all  $d \in \mathcal{D}$ , the composite functor*

$$I_d \rightarrow \mathcal{C} \xrightarrow{P} S^{-1}\mathcal{C},$$

*where the first functor sends  $d \xrightarrow{s} T(c)$  to  $c$ , inverts all arrows of  $I_d$ , hence induces a functor*

$$\mathbf{F}(d): \Pi_1(I_d) \rightarrow S^{-1}\mathcal{C}.$$

For  $d \in \mathcal{D}$  and for  $c, c' \in \Pi_1(I_d)$ , denote by  $\gamma_{c,c'}$  the unique morphism from  $c$  to  $c'$ , as well as its image in  $Ar(S^{-1}\mathcal{C})$  by the functor  $\mathbf{F}(d)$ . Let  $f: d_0 \rightarrow d_1$  be a morphism of  $\mathcal{D}$ . For  $(c_1, c_0, g) \in Ob(I_{d_1}) \times Ob(I_{d_0}) \times Ob(I_f)$ , set

$$\varphi_f(c_1, c_0, g) = \gamma_{c_1, rg}^{-1} \circ g \circ \gamma_{c_0, dg} \in S^{-1}\mathcal{C}(c_0, c_1),$$

where  $dg, rg$  denote respectively the domain and the range of  $g$ . If  $g, g' \in I_f$ , a morphism  $g \rightarrow g'$  yields a commutative diagram

$$\begin{array}{ccc} dg & \xrightarrow{g} & rg \\ \sigma \downarrow & & \tau \downarrow \\ dg' & \xrightarrow{g'} & rg' \end{array}$$

with  $\sigma \in Ar(I_{d_0}), \tau \in Ar(I_{d_1})$ . We then have

$$\begin{aligned} \varphi_f(c_1, c_0, g') &= \gamma_{c_1, rg'}^{-1} \circ g' \circ \gamma_{c_0, dg'} = \gamma_{c_1, rg'}^{-1} \circ \tau \circ g \circ \sigma^{-1} \circ \gamma_{c_0, dg'} \\ &= \gamma_{c_1, rg}^{-1} \circ g \circ \gamma_{c_0, dg} = \varphi_f(c_1, c_0, g) \end{aligned}$$

in view of the fact that  $\sigma = \gamma_{dg, dg'}$  and  $\tau = \gamma_{rg, rg'}$  in  $S^{-1}\mathcal{C}$ .

Since  $I_f$  is 0-connected, a canonical map

$$\varphi_f: Ob(I_{d_1}) \times Ob(I_{d_0}) \rightarrow Ar(S^{-1}\mathcal{C})$$

can be deduced such that  $d\varphi_f(c_1, c_0) = c_0$  and  $r\varphi_f(c_1, c_0) = c_1$ . Observe the formula

$$\varphi_f(c'_1, c'_0) = \gamma_{c'_1, c_1}^{-1} \varphi_f(c_1, c_0) \gamma_{c'_0, c_0}. \tag{2.1}$$

In other words,  $\varphi_f$  defines a functor  $\Pi_1(I_{d_0}) \times \Pi_1(I_{d_1}) \rightarrow (S^{-1}\mathcal{C})^{\Delta^1}$  lifting the functors  $\mathbf{F}(d_0)$  and  $\mathbf{F}(d_1)$  via the commutative diagram

$$\begin{array}{ccc} \Pi_1(I_{d_0}) \times \Pi_1(I_{d_1}) & \xrightarrow{\varphi_f} & (S^{-1}\mathcal{C})^{\Delta^1} \\ & \searrow \mathbf{F}(d_0) \times \mathbf{F}(d_1) & \downarrow (d,r) \\ & & S^{-1}\mathcal{C} \times S^{-1}\mathcal{C}. \end{array}$$

**Lemma 2.3.**

a) If  $f = 1_d$  for some  $d \in \mathcal{D}$ , then  $\varphi_f(c, c) = 1_c$  for all  $c \in Ob(I_d)$ .

b) If  $f_1: d_0 \rightarrow d_1$  and  $f_2: d_1 \rightarrow d_2$ , then

$$\varphi_{f_2 f_1}(c_2, c_0) = \varphi_{f_2}(c_2, c_1) \varphi_{f_1}(c_1, c_0)$$

for all  $(c_0, c_1, c_2) \in Ob(I_{d_0}) \times Ob(I_{d_1}) \times Ob(I_{d_2})$ .

c) If  $f \in S'$ ,  $\varphi_f(c_1, c_0)$  is invertible in  $S^{-1}\mathcal{C}$  for all  $(c_0, c_1) \in Ob(I_{d_0}) \times Ob(I_{d_1})$ .

*Proof.* a) is obvious. To prove b), let us use hypothesis (2) to find  $g_1: c_0 \rightarrow c_1$  and  $g_2: c_1 \rightarrow c_2$  respectively in  $I_{f_1}$  and  $I_{f_2}$ . Then  $\varphi_{f_1}(c_1, c_0) = g_1$ ,  $\varphi_{f_2}(c_2, c_1) = g_2$  and  $\varphi_{f_2 f_1}(c_2, c_0) = g_2 g_1$ . Hence b) is true for this particular choice of  $(c_0, c_1, c_2)$ , and it follows from (2.1) that it remains true for all other choices.

Let us prove c). Choose a commutative diagram ( $-1$ -connectedness of  $I_f$ )

$$\begin{array}{ccc} d_0 & \xrightarrow{s_0} & T(c'_0) \\ f \downarrow & & T(g) \downarrow \\ d_1 & \xrightarrow{s_1} & T(c'_1) \end{array}$$

where  $s_0, s_1 \in S'$ . Since  $S'$  is stable under composition, this diagram shows (using  $s_1 f$ ) that  $g$  defines an object of  $I_{1_{d_0}}$ ; moreover, obviously  $\varphi_{1_{d_0}}(c'_1, c'_0) = g$ . From a) and (2.1) (applied with  $c_0 = c_1$ ), it follows that  $g$  is invertible. On the other hand,  $g = \varphi_f(c'_1, c'_0)$  also; reapplying (2.1), we get the desired conclusion.  $\square$

**2.5. Proof of Theorem 2.1**

We start by defining a functor

$$F: \mathcal{D} \rightarrow S^{-1}\mathcal{C}$$

as follows: for all  $d \in Ob(\mathcal{D})$ , choose an object  $d \xrightarrow{s_d} T(c_d)$  of  $I_d$ . Set

$$\begin{aligned} F(d) &= c_d \\ F(f) &= \varphi_f(c_{d_1}, c_{d_0}) \text{ for } f: d_0 \rightarrow d_1. \end{aligned}$$

Lemma 2.3 shows that  $F$  is indeed a functor, and that it inverts the arrows of  $S'$ ; hence it induces a functor

$$\bar{F}: S'^{-1}\mathcal{D} \rightarrow S^{-1}\mathcal{C}.$$

For  $c \in Ob(S^{-1}\mathcal{C})$ , there is an isomorphism

$$\gamma_{c, c_{T(c)}}: \bar{F}\bar{T}(c) \xrightarrow{\sim} c.$$

Formula (2.1) shows that it is natural in  $c$ : it is first checked for the morphisms of  $\mathcal{C}$ , then naturality passes automatically to  $S^{-1}\mathcal{C}$ . On the other hand, for  $d \in Ob(S'^{-1}\mathcal{D})$ , there is an isomorphism

$$s_d: d \xrightarrow{\sim} \bar{T}\bar{F}(d).$$

The definitions of  $\varphi_f$  and (2.1) show again that this isomorphism is natural in  $d$  (by the same method).

It follows that  $\bar{F}$  is quasi-inverse to  $\bar{T}$ . □

### 3. Towards Kan extensions

Keep the setting of (0.1) and hypotheses of Theorem 2.1, and let  $F: \mathcal{D} \rightarrow \mathcal{E}$  be another functor. We assume:

**Hypothesis 3.1.** *The functor  $FT$  inverts  $S$ , i.e., there exists a functor  $G: S^{-1}\mathcal{C} \rightarrow \mathcal{E}$  and a natural isomorphism*

$$FT \simeq GP.$$

Under this hypothesis, let us define

$$RF := G\bar{T}^{-1}: S'^{-1}\mathcal{D} \rightarrow \mathcal{E}$$

where  $\bar{T}^{-1}$  is a chosen quasi-inverse to  $\bar{T}$ .

We construct a natural transformation  $\eta: F \Rightarrow RF \circ Q$  as follows:

Let  $d \in \mathcal{D}$  and  $d \xrightarrow{s} T(c) \in I_d$ . Then  $s$  defines

$$F(d) \xrightarrow{F(s)} FT(c) \simeq GP(c) \xrightarrow{G\bar{T}^{-1}Q(s)^{-1}} G\bar{T}^{-1}Q(d) = RF \circ Q(d).$$

Since  $I_d$  is 0-connected, this morphism  $\eta_d$  does not depend on the choice of  $s$ . Then the  $-1$ -connectedness of the categories  $I_f$  shows that  $\eta$  is indeed a natural transformation.

It will be shown in [8] that  $(RF, \eta)$  is in fact a left Kan extension [12, Ch. X, § 3] (= right total derived functor à la Quillen [14, § 4.1, Def. 1]) of  $F$  along  $Q$ , but this requires the full force of the hypotheses of Theorem 2.1.

#### 4. A relativisation theorem

##### 4.1. Two lemmas on comma categories

**Lemma 4.1** (“Theorem A”). *Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a final functor (§ 1.4). Then  $F$  induces a bijection on the sets of connected components. In particular,  $\mathcal{A}$  is 0-connected if and only if  $\mathcal{B}$  is 0-connected.*

*Proof.* (See also [13, Ex. 1.1.32].) Surjectivity is obvious. For injectivity, let  $a_0, a_1 \in \mathcal{A}$  be such that  $F(a_0)$  and  $F(a_1)$  are connected. By the surjectivity of  $F$ , any vertex of a chain linking them is of the form  $F(a)$ . To prove that  $a_0$  and  $a_1$  are connected, it can therefore be reduced to the case where  $F(a_0)$  and  $F(a_1)$  are directly connected, say by a morphism  $f: F(a_0) \rightarrow F(a_1)$ . But the two objects

$$F(a_0) \xrightarrow{f} F(a_1), \quad F(a_1) \xrightarrow{=} F(a_1)$$

of  $F/F(a_1)$  are connected by assumption, which implies that  $a_0$  and  $a_1$  are connected in  $\mathcal{A}$ .  $\square$

**Lemma 4.2.** *Let*

$$\begin{array}{ccc} F \downarrow G & \xrightarrow{F'} & \mathcal{B} \\ \downarrow G' & & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array}$$

be a “2-cartesian square” of categories.

a) *For all  $b \in \mathcal{B}$ , the functor*

$$G_*: F'/b \rightarrow F/G(b)$$

$$\left[ \begin{array}{ccc} F(a) & \xrightarrow{f} & G(b') \\ & G(\varphi) \downarrow & \\ & & G(b) \end{array} \right] \mapsto [F(a) \xrightarrow{G(\varphi)f} G(b)]$$

has a right adjoint/right inverse  $G^!$  given by

$$G^!([F(a) \xrightarrow{f} G(b)]) = \left[ \begin{array}{ccc} F(a) & \xrightarrow{f} & G(b) \\ & G(1_b) \downarrow & \\ & & G(b) \end{array} \right].$$

In particular,  $G_*$  is a weak equivalence.

b) *Suppose that  $F/c$  is nonempty for all  $c \in \mathcal{C}$ . Then  $F'$  is surjective.*

c) *Suppose moreover that  $F$  is cofinal. Then  $F'$  induces a bijection on connected components.*



*Proof.* a) is confirmed immediately; the fact that  $G_*$  is a weak equivalence then follows from [15, p. 92, Cor. 1]. b) is obvious. It remains to prove c): by a), the categories  $F'/b$  are 0-connected. The conclusion then follows from Lemma 4.1 applied to  $F'$ .  $\square$

**4.2. The theorem**

For all  $d \in \mathcal{D}$ , let now  $J_d := d \setminus \mathcal{D}$ .

**Theorem 4.3.**

a) Suppose the following conditions hold for all  $d \in \text{Ob}(\mathcal{D})$ :

(0)  $I_d$  is 1-connected.

(1') The obvious functor  $\Phi_d: I_d \rightarrow J_d$  is cofinal (§ 1.4).

Then, for all  $n \geq 0$  and all  $(d_0 \rightarrow \dots \rightarrow d_n) \in \mathcal{D}^{\Delta^n}$ , the category  $I_{(d_0 \rightarrow \dots \rightarrow d_n)}$  is 0-connected.

b) Suppose that, for all  $d \in \mathcal{D}$  all  $j \in J_d$ ,  $I_d$  and  $I_d/j$  are  $\infty$ -connected. Then, for all  $n \geq 0$  and all  $(d_0 \rightarrow \dots \rightarrow d_n) \in \mathcal{D}^{\Delta^n}$ , the category  $I_{(d_0 \rightarrow \dots \rightarrow d_n)}$  is  $\infty$ -connected.

*Proof.*

a) The proof proceeds by induction on  $n$ , the case  $n = 0$  following from Hypothesis (0). Consider the obvious forgetful functors

$$I_{(d_0 \rightarrow \dots \rightarrow d_n)} \xrightarrow{u} I_{(d_1 \rightarrow \dots \rightarrow d_n)} \xrightarrow{v} I_{d_1}, \quad I_{(d_0 \rightarrow \dots \rightarrow d_n)} \xrightarrow{w} I_{d_0}.$$

It can be immediately verified that the diagram

$$\begin{array}{ccc} I_{(d_0 \rightarrow \dots \rightarrow d_n)} & \xrightarrow{u} & I_{(d_1 \rightarrow \dots \rightarrow d_n)} \\ w \downarrow & & \downarrow f_1^* \circ \Phi_{d_1} \circ v \\ I_{d_0} & \xrightarrow{\Phi_{d_0}} & J_{d_0}. \end{array} \tag{4.1}$$

induces an isomorphism of categories

$$I_{(d_0 \rightarrow \dots \rightarrow d_n)} = \Phi_{d_0} \downarrow (f_1^* \circ \Phi_{d_1} \circ v),$$

i.e. is 2-cartesian. Here,  $f_1: d_0 \rightarrow d_1$ . Hypothesis (1') then implies that Lemma 4.2 c) can be applied with  $F = \Phi_{d_0}$ . Therefore  $u$  induces a bijection on connected components, hence the conclusion.

b) Let us use Diagram (4.1) again. It follows from Lemma 4.2 a) that  $u/x$  is  $\infty$ -connected for all  $x \in I_{(d_1 \rightarrow \dots \rightarrow d_n)}$ . By Quillen's Theorem A [15, Th. A],  $u$  is a weak equivalence; by induction on  $n$ ,  $I_{(d_1 \rightarrow \dots \rightarrow d_n)}$  is  $\infty$ -connected, hence so is  $I_{(d_0 \rightarrow \dots \rightarrow d_n)}$ .  $\square$

**Corollary 4.4** (Normand theorem). *Suppose the following conditions verified for all  $d \in \text{Ob}(\mathcal{D})$ :*

(0)  $I_d$  is 1-connected.

(1') The obvious functor  $\Phi_d: I_d \rightarrow J_d$  is cofinal (§ 1.4).

Then  $\bar{T}$  is an equivalence of categories.

*Proof.* This follows from Theorem 4.3 a) and Theorem 2.1.  $\square$

*Remark 4.5.* There is an  $n$ -connected version of Quillen's Theorem A for any  $n$  (cf. Maltiniotis [13, 1.1.34], Cisinski [1]). Using it, one may replace  $\infty$ -connected by  $n$ -connected in the hypothesis and conclusion of Theorem 4.3 b) (same proof).

## 5. Complements

### 5.1. A relative version

**Corollary 5.1.** *Suppose that  $T$  is fully faithful.*

- a) *If Conditions (0), (1), (2) of Theorem 2.1 are satisfied, they are also satisfied for all  $c \in \mathcal{C}$  for the functor  $c \setminus \mathcal{C} \rightarrow T(c) \setminus \mathcal{D}$  induced by  $T$ .*
- b) *Same result with Conditions (0), (1') of Theorem 4.3 a).*

*In particular, in case a) or b), the functor*

$$S^{-1}(c \setminus \mathcal{C}) \rightarrow S'^{-1}(T(c) \setminus \mathcal{D})$$

*induced by  $T$  is an equivalence of categories.*

*Proof.* For  $\delta = [T(c) \rightarrow d] \in T(c) \setminus \mathcal{D}$ , the full faithfulness of  $T$  implies that the forgetful functors

$$\delta \setminus (c \setminus \mathcal{C}) \rightarrow d \setminus \mathcal{C}, \quad \delta \setminus (c \setminus S) \rightarrow d \setminus S$$

are isomorphisms of categories. The case is similar when dealing with categories of type  $I_f$  and  $I_{(f_2, f_1)}$ .  $\square$

### 5.2. Riou's theorem

A statement similar to Corollary 4.4 was obtained independently by Joël Riou:

**Theorem 5.2** (Riou [16, II.2.2]). *Suppose that*

- (i)  *$T$  is fully faithful;  $S = S' \cap \mathcal{C}$ .*
- (ii) *In  $\mathcal{D}$ , push-outs of arrows of  $S'$  exist and belong to  $S'$ .*
- (iii) *If  $s \in S'$  and the domain of  $s$  is in  $T(\mathcal{C})$ , so is its range.*
- (iv) *For any  $d \in \mathcal{D}$ ,  $I_d \neq \emptyset$ .*

*Then  $\bar{T}$  is an equivalence of categories.*

(Riou's hypotheses are actually dual to these; we write them as above for easy comparison with the previous results. Also, Riou does not assume that  $S'$  is stable under composition.)

Riou's proof is in the style of the proof of Theorem 2.1, but more direct because push-outs immediately provide a functor. Actually, as we realised when reading Gillet–Soulé [4], his hypothesis (iii) is not necessary, as is shown by the following.

**Theorem 5.3.** *Assume that the hypotheses (i), (ii), (iv) of Theorem 5.2 are verified. Then:*

- a)  $T^E: \mathcal{C}^E \rightarrow \mathcal{D}^E$  and  $S(E), S'(E)$  (cf. § 2.2).
- b) For any  $d \in \mathcal{D}$ ,  $I_d$  is 1-connected (and even  $\infty$ -connected; see Lemma 5.7).
- c) In the situation of a), the hypotheses of Theorem 2.1 are verified; in particular,  $\bar{T}^E$  is an equivalence of categories.

*Proof.* For a), it suffices to prove (iv); for this, we argue by induction on  $|E|$ , the case  $E = \{0\}$  being Hypothesis (iv).

Suppose that  $|E| > 0$ , and let  $d_\bullet \in \mathcal{D}^E$ . Let  $e \in E$  be a maximal element,  $E' = E - \{e\}$  and  $d'_\bullet$  the restriction of  $d_\bullet$  to  $\mathcal{D}^{E'}$ . By induction, pick an object  $d'_\bullet \xrightarrow{s'_\bullet} T^{E'}(c'_\bullet)$  in  $I_{d'_\bullet}$ .

Let  $F$  be the set of those maximal elements of  $E'$  which are  $< e$ . If  $F = \emptyset$ , we just pick  $d_e \xrightarrow{s_e} T(c_e)$  in  $I_{d_e}$  (by (iv)) and adjoin it to the previous object. If  $F$  is not empty, let  $d'$  be the push-out of the maps  $d_f \xrightarrow{s_f} T(c_f)$  (for  $f \in F$ ) along the maps  $d_f \rightarrow d_e$ . By (ii), the map  $d_e \rightarrow d'$  is in  $S'$ . Pick  $d' \rightarrow T(c_e)$  in  $I_{d'}$  by (iv), and define  $s_e$  as the composition  $d_e \rightarrow d' \rightarrow T(c_e)$ . By (i), the compositions

$$T(c_f) \rightarrow d' \rightarrow T(c_e)$$

define morphisms  $\sigma_{f,e}: c_f \rightarrow c_e$  in  $S$ , and we are done. In diagram form:

$$\begin{array}{ccccc}
 d_f & \xrightarrow{s_f} & T(c_f) & & \\
 f_{f,e} \downarrow & & \downarrow & \searrow T(\sigma_{f,e}) & \\
 d_e & \xrightarrow{\quad} & d' & \xrightarrow{\quad} & T(c_e). \\
 & \searrow s_e & & & 
 \end{array}$$

For b), let  $s, s' \in I_d$ . Taking their push-out, we get a new object  $d' \in \mathcal{D}$ ; applying (iv) to  $I_{d'}$ , we then get a new object  $s'' \in I_d$  and maps  $s \rightarrow s'', s' \rightarrow s''$ . In particular,  $I_d$  is 0-connected.

A similar argument shows that the first axiom of calculus of fractions holds in  $I_d$  (for the collection of all morphisms of  $I_d$ ). Therefore, in  $\Pi_1(I_d)$ , any morphism may be written as  $u_2^{-1}u_1$  for  $u_1, u_2$  morphisms of  $I_d$ . To prove that  $I_d$  is 1-connected, it therefore suffices to show that, given two morphisms  $u_1, u_2 \in I_d$  with the same domain and range,  $u_1$  and  $u_2$  become equal in  $\Pi_1(I_d)$ .

The following proof is inspired from reading [4, pp. 139–140]. Let  $s: d \rightarrow T(c)$  and  $s': d \rightarrow T(c')$  be the domain and range of  $u_1$  and  $u_2$ . Consider the push-out diagrams

$$\begin{array}{ccc}
 d & \xrightarrow{s} & T(c) \\
 s \downarrow & & a \downarrow \\
 T(c) & \xrightarrow{a} & d' \\
 T(u_1) \downarrow & & g_1 \downarrow \\
 T(c') & \xrightarrow{f} & d''
 \end{array}
 \qquad
 \begin{array}{ccc}
 d & \xrightarrow{s} & T(c) \\
 s \downarrow & & a \downarrow \\
 T(c) & \xrightarrow{a} & d' \\
 T(u_2) \downarrow & & g_2 \downarrow \\
 T(c') & \xrightarrow{f} & d''
 \end{array}$$

Here  $d''$  and  $f$  are common to the two diagrams because  $T(u_1)s = T(u_2)s$ . For

the same reason, we have  $g_1a = g_2a$  (vertically), hence (in the lower squares)

$$fT(u_1) = g_1a = g_2a = fT(u_2).$$

Choose  $d'' \xrightarrow{s''} T(c'')$  in  $I_{d''}$  by 1). Then  $s''f = T(\sigma)$  for some  $\sigma \in S$ . Hence

$$\sigma u_1 = \sigma u_2$$

and  $u_1 = u_2$  in  $\Pi_1(I_d)$ , as required.

To show c), for any  $n \geq 0$ , consider the ordered set  $E \times \Delta^n$ . Then a) and b) show that, for any  $d_\bullet \in \mathcal{D}^{E \times \Delta^n} = (\mathcal{D}^E)^{\Delta^n}$ ,  $I_{d_\bullet}$  is 1-connected. In particular, the hypotheses of Theorem 2.1 hold.  $\square$

*Remark 5.4.* It is not clear whether the conditions of Theorem 5.2 imply Condition (1') of Theorem 4.3 a).

*Remark 5.5.* We shall use Theorem 5.3 in the geometric applications.

*Remark 5.6.* Even though the categories  $I_d$  are 1-connected under the conditions of Theorem 5.2, they are not filtering in general (for example, they are not filtering in the geometric case considered by Riou). A natural question is whether they are  $\infty$ -connected. We would like to thank the referee for providing an affirmative answer and sketching an argument, which we reproduce in the lemma below.

**Lemma 5.7** (Referee's lemma). *Let  $(\mathcal{C}, \mathcal{D}, T, S, S')$  be as in the introduction. Suppose that for any finite partially ordered set (poset)  $E$  and for any  $d_\bullet \in \mathcal{D}^E$ , the category  $I_{d_\bullet}$  is 0-connected. Then for any finite poset  $E$  and any  $d_\bullet \in \mathcal{D}^E$ ,  $I_{d_\bullet}$  is  $\infty$ -connected.*

*Proof.* We will use the following sufficient condition for a simplicial set to be  $\infty$ -connected: if  $X$  is a simplicial set such that any map from the nerve of a finite partially ordered set (poset) is simplicially homotopic to a constant map, then  $X$  is  $\infty$ -connected. (This can be proven, for example, using the fact that for any integer  $k \geq 1$ , the iterated subdivision  $Sd^k(\partial\Delta^n)$  is the nerve of a finite poset and from the fact that Kan's  $Ex^\infty(X)$  is a fibrant replacement of  $X$ .)

From this, it follows that if  $\mathcal{C}$  is a small category such that for any finite poset  $E$ , the category of functors  $\mathcal{C}^E$  is 0-connected, then  $\mathcal{C}$  is  $\infty$ -connected (using the fact that the functor "nerve" is fully faithful and commutes with finite products). Now let  $d \in \mathcal{D}$ . We note that for a finite poset  $E$ , a functor  $u$  from  $E$  to  $I_d$  is the same as a functor  $v$  from  $E$  to  $S$  with a morphism of functors from  $d_E$  to  $T_S^E(v)$ , where  $d_E$  denotes the constant functor from  $E$  to  $S'$ , with value  $d$ . In other words, we have an equivalence of categories  $I_d^E \simeq I_{d_E}$ . Hence by a) of Theorem 5.3, we can apply b) to  $T^E$  and conclude that  $I_d^E$  is 0-connected, which proves that  $I_d$  is  $\infty$ -connected.

We may then apply this conclusion to the collection

$$(\mathcal{C}^E, \mathcal{D}^E, T^E, S(E), S'(E))$$

for any finite poset  $E$ , since for another finite poset  $F$  we have  $(\mathcal{C}^E)^F \simeq \mathcal{C}^{E \times F}$ , etc.  $\square$

**5.3. Weakening the hypotheses**

This subsection has grown out of exchanges with Maltiniotis.

For  $d \in \mathcal{D}$ , let us write  $J_d = d \backslash T$  as in Theorem 4.3. The projections  $J_d \rightarrow \{d\} \subset \mathcal{D}$  define a fibred category  $J$  over  $\mathcal{D}$ . Similarly, the  $I_d$  define a fibred category  $I$  over  $S'$  (viewed as before as a category).

Now replace  $S$  and  $S'$  by their strong saturations  $\langle S \rangle$  and  $\langle S' \rangle$ . (Recall that the strong saturation  $\langle S \rangle$  of  $S$  is the collection, containing  $S$ , of morphisms  $u \in \mathcal{C}$  such that  $u$  becomes invertible in  $S^{-1}\mathcal{C}$ .) We have similarly a fibred category  $\langle I \rangle$  over  $\langle S' \rangle$ . For any  $d$ , we have obvious inclusions

$$I_d \subseteq \langle I \rangle_d \subseteq J_d.$$

We are interested in a collection of subcategories  $I'_d$  of  $\langle I \rangle_d$  which form a fibred category over  $S'$ . Concretely, this means that for any  $s: d \rightarrow d'$  in  $S'$ , the pull-back functor

$$s^*: \langle I \rangle_{d'} \rightarrow \langle I \rangle_d$$

sends  $I'_{d'}$  into  $I'_d$ .

**Definition 5.8.** A fibred category  $I' \rightarrow S'$  as above is called a *weak replacement* of  $I$ .

If  $E$  is a small category, we have the fibred category  $I(E)$  over  $S'(E)$  and we define a weak replacement of  $I(E)$  similarly; namely, a collection of subcategories  $I'(E)_d$  of  $\langle I(E) \rangle_d$  respected by pull-backs under morphisms of  $S(E)$ .

**Theorem 5.9** (Variant of Theorem 2.1). *Suppose given, for  $n = 0, 1, 2$ , a weak replacement  $I'(\Delta^n)$  of  $I(\Delta^n)$ . Suppose moreover that:*

- For any  $f: d_0 \rightarrow d_1$ , the face functors  $J_f \rightarrow J_{d_0}$  and  $J_f \rightarrow J_{d_1}$  send  $I'_f$  to  $I'_{d_0}$  and  $I'_{d_1}$ .
- For any  $(f_2, f_1)$ , the face functors  $J_{f_2, f_1} \rightarrow J_{f_2}$ ,  $J_{f_2, f_1} \rightarrow J_{f_1}$  and  $J_{f_2, f_1} \rightarrow J_{f_2 f_1}$  send  $I'_{(f_2, f_1)}$  respectively to  $I'_{f_2}$ ,  $I'_{f_1}$  and  $I'_{f_2 f_1}$ .
- For any  $d \in \mathcal{D}$ ,  $I'_d$  contains at least one object of the form  $[1_d \rightarrow T(1_c)]$ .

(The last condition is verified, for example, if the degeneracy functor  $J_d \rightarrow J_{1_d}$  sends  $I'_d$  to  $I'_{1_d}$ .)

Finally, suppose that the  $I'_d$  have the same connectivity properties as in Theorem 2.1. Then  $\bar{T}$  is an equivalence of categories.

*Proof.* It can be verified by inspection that the proof of Theorem 2.1 holds with these data. □

It was Maltiniotis' remark that Corollary 4.4 still holds with a weak replacement of  $I$  rather than  $I$ . Presumably it can be verified that Theorem 4.3 still holds with weak replacements of the  $I(\Delta^n)$ , provided they satisfy simplicial compatibilities similar to those of Theorem 5.9.

**5.4. Sufficient conditions for (0), (1), (1') and (2)**

**Proposition 5.10.**

- a) The following conditions imply the conditions of Theorem 4.3 b) (hence, a fortiori, conditions (0) and (1') of Theorem 4.3 a)): for any  $d \in \mathcal{D}$  and  $j \in J_d$ 
  - (a1)  $I_d$  is cofiltering;
  - (a2)  $I_d/j$  is (nonempty and) cofiltering.
- b) The following conditions imply (a1) and (a2):
  - (b1) given  $s \in S'$ ,  $T(f)s = T(g)s \Rightarrow f = g$  ( $f, g \in \text{Ar}(\mathcal{C})$ );
  - (b2)  $I_d$  is nonempty;
  - (b3) for any  $(i, j) \in I_d \times J_d$ , the 1-fibred product  $I_d/i \times_{I_d} I_d/j$  is nonempty.
- c) In b), conditions (b2) and (b3) are consequences of the following: finite products exist in  $\mathcal{C}$ ,  $T$  commutes with them and, for any  $d \in \mathcal{D}$ , there is a family of objects  $K_d \subset J_d$  such that:
  - (c1)  $K_d \neq \emptyset$ ;  $I_d \subseteq K_d$ ; for any  $k \in K_d$ ,  $I_d/k \neq \emptyset$ .
  - (c2) If  $k \in K_d$  and  $j \in J_d$  then  $j \times k \in K_d$ . (Note that the assumption on finite products implies that they exist in  $J_d$  for any  $d \in \mathcal{D}$ .)

*Proof.* a) is “well-known”; see [13, Prop. 2.4.9].

- b) (b1) implies that  $I_d$ , hence also  $I_d/j$ , are ordered; (b2) and (b3) (the latter applied with  $j \in I_d$ ) then imply that  $I_d$  is cofiltering and (b3) implies a fortiori that  $I_d/j$  is nonempty for any  $j \in J_d$ ; since  $I_d$  is cofiltering,  $I_d/j$  is automatically cofiltering.
- c) Clearly (c1)  $\Rightarrow$  (b2). For (b3), let  $(i, j) \in I_d \times J_d$ . By hypothesis,  $i \times j \in K_d$ , hence  $I_d/i \times j \neq \emptyset$  and there is an  $i'$  such that  $i'$  maps to  $i \times j$ , which exactly means that  $i' \in I_d/i \times_{I_d} I_d/j$ .  $\square$

For the next proposition, we need to introduce a definition relative to the pair  $(\mathcal{D}, S')$ :

**Definition 5.11.** Given a diagram

$$\begin{array}{ccc} d & \xrightarrow{s} & d' \\ f \downarrow & & \\ d_1 & & \end{array}$$

with  $s \in S'$ , we say that  $s$  is *in good position with respect to  $f$*  if the push-out

$$\begin{array}{ccc} d & \xrightarrow{s} & d' \\ f \downarrow & & f' \downarrow \\ d_1 & \xrightarrow{s_1} & d'_1 \end{array}$$

exists and  $s_1 \in S'$ .

**Proposition 5.12.** *Suppose that the following conditions are verified:*

- (d1) Morphisms of  $S'$  are epimorphisms within  $S'$ .

(d2) If  $f \in S'$  in Definition 5.11, then any  $s \in S'$  is in good position with respect to  $f$ .

(d3) If  $s \in S'$  is in good position with respect to  $gf$ , then it is in good position with respect to  $f$ .

(d4)  $T$  is fully faithful and  $S = S' \cap \mathcal{C}$ .

(d5) For any  $f: d \rightarrow d_1$  in  $\mathcal{D}$ , there exists  $s \in I_d$  in good position with respect to  $f$ . Then for all  $m \geq 0$  and all  $d_\bullet \in \mathcal{D}^{\Delta^m}$ ,  $I_{d_\bullet}$  is ordered and filtering, hence  $\infty$ -connected. In particular, the hypotheses of Theorem 2.1 are verified.

*Proof.* We first show that  $I_{d_\bullet}$  is nonempty. For  $m = 0$ , this is (d5) applied to  $f = 1_{d_0}$ . Suppose  $m > 0$ ; we argue by induction on  $m$ . Applying (d5) and (d3) to  $f_m \circ \dots \circ f_1$ , we find  $s_0 \in I_{d_0}$  and a commutative (pushout) diagram

$$\begin{array}{ccccccc} d_0 & \xrightarrow{f_1} & d_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_m} & d_m \\ s_0 \downarrow & & s'_1 \downarrow & & & & s'_m \downarrow \\ T(c_0) & \xrightarrow{f'_1} & d'_1 & \xrightarrow{f'_2} & \dots & \xrightarrow{f'_m} & d'_m \end{array}$$

with  $s'_1, \dots, s'_m \in S'$ . By induction,  $I_{(d'_1 \rightarrow \dots \rightarrow d'_m)}$  is nonempty, which shows, using (d4), that  $I_{d_\bullet}$  is nonempty. (d1) then implies that it is ordered.

Let us prove that they are filtering. Using (d2), we see that the push-out  $d_\bullet \rightarrow d'_\bullet$  of two objects  $d_\bullet \xrightarrow{s_\bullet} T(c_\bullet)$ ,  $d_\bullet \xrightarrow{s'_\bullet} T(c'_\bullet)$  of  $I_{(f_m, \dots, f_1)}$  exists as a diagram in  $\mathcal{D}$ ; using the non-emptiness of  $I_{d'_\bullet}$ , we conclude.  $\square$

*Remark 5.13.* This proposition (with its proof) may be seen as an easier variant of Theorem 5.3.

### 5.5. Another variant of Theorem 2.1

Keep notation as in Theorem 4.3. As in § 2.1, let  $S$  (resp.  $S'$ ) denote the subcategory of  $\mathcal{C}$  (resp. of  $\mathcal{D}$ ) with the same objects but with only arrows in  $S$  (resp.  $S'$ ). Consider the category

$$Id_S \downarrow T = \{(d, c, s) \mid d \in S', c \in S, s: d \rightarrow T(c)\}.$$

We have a projection functor

$$\begin{aligned} p_1: Id_S \downarrow T &\rightarrow S' \\ (d, c, s) &\mapsto d. \end{aligned}$$

For  $d \in S'$  we then define

$$\underline{I}_d = p_1 \downarrow d$$

so that objects of  $\underline{I}_d$  are diagrams

$$\begin{array}{ccc} u & \xrightarrow{s} & T(c) \\ j \downarrow & & \\ d & & \end{array} \tag{5.1}$$

with  $s, j \in S'$ , and morphisms are the obvious ones (in  $S$ ).

We have the same definition for categories of diagrams. Then:

**Theorem 5.14.** *Suppose the following assumptions verified:*

- (0) For all  $d \in \mathcal{D}$ ,  $\underline{I}_d$  is 1-connected.
- (1) For all  $f \in \mathcal{D}^{\Delta^1}$ ,  $\underline{I}_f$  is 0-connected.
- (2) For all  $(f_2, f_1) \in \mathcal{D}^{\Delta^2}$ ,  $\underline{I}_{(f_2, f_1)}$  is  $-1$ -connected.

Suppose moreover that the following 2/3 property holds:

- (\*) If  $s \in S'$  and  $st \in S'$ , then  $t \in S'$ .

Then  $\bar{T}$  is an equivalence of categories.

*Proof.* First, the arguments of § 2.4 are mimicked line by line. The only place where the added datum  $j$  creates a difficulty is in the analogue of Lemma 2.3 c). We then argue as follows: let  $f: d_0 \rightarrow d_1 \in S'$ . By the  $-1$ -connectedness of  $\underline{I}_f$ , we have a commutative diagram

$$\begin{array}{ccccc} d_0 & \xleftarrow{j_0} & u_0 & \xrightarrow{s_0} & T(c'_0) \\ f \downarrow & & t \downarrow & & T(g') \downarrow \\ d_1 & \xleftarrow{j_1} & u_1 & \xrightarrow{s_1} & T(c'_1). \end{array}$$

Note that  $j_1 t = f j_0 \in S'$ , thus  $t \in S'$  by (\*), and therefore  $s_1 t \in S'$ . So we have another commutative diagram

$$\begin{array}{ccccc} d_0 & \xleftarrow{j_0} & u_0 & \xrightarrow{s_0} & T(c'_0) \\ 1_{d_0} \downarrow & & 1_{u_0} \downarrow & & T(g') \downarrow \\ d_0 & \xleftarrow{j_0} & u_0 & \xrightarrow{s_1 t} & T(c'_1) \end{array}$$

describing an object of  $\underline{I}_{1_{d_0}}$ . From this point, the proof proceeds as in the proof of Lemma 2.3 c).

The analogue of § 2.5 is now as follows: for each  $d \in \mathcal{D}$  choose an object  $(u_d, c_d, j_d, s_d) \in \underline{I}_d$  and define a functor  $F: \mathcal{D} \rightarrow S^{-1}\mathcal{C}$  by  $F(d) = c_d$ ,  $F(f) = \varphi_f(c_{d_1}, c_{d_0})$  as in § 2.5. The natural isomorphism  $\bar{F}\bar{T} \Rightarrow Id_{S^{-1}\mathcal{C}}$  is defined as in § 2.5; on the other hand, the isomorphism  $Id_{S'^{-1}\mathcal{D}} \Rightarrow \bar{T}\bar{F}$  is defined on an object  $d \in S'^{-1}\mathcal{D}$  by  $s_d j_d^{-1}$ ; it is easy to verify that it is natural.  $\square$

## 6. Adding finite products or coproducts

In this section, we show that the property for  $\bar{T}$  to be an equivalence of categories in (0.1) is preserved by adjoining finite products or coproducts. We shall only treat the case of coproducts, since that of products is dual.

We shall say that a category  $\mathcal{C}$  has finite coproducts (or that  $\mathcal{C}$  is a category with finite coproducts) if all finite coproducts are representable in  $\mathcal{C}$ . This is the case if and only if  $\mathcal{C}$  has an initial object (empty coproduct) and the coproduct of any two objects exists in  $\mathcal{C}$ .



**Proposition 6.1.** *Let  $\mathcal{C}$  be a category. There exists a category  $\mathcal{C}^{\amalg}$  with finite coproducts and a functor  $I: \mathcal{C} \rightarrow \mathcal{C}^{\amalg}$  with the following 2-universal property: any functor  $F: \mathcal{C} \rightarrow \mathcal{E}$  where  $\mathcal{E}$  has finite coproducts extends through  $I$ , uniquely up to natural isomorphism, to a functor  $F^{\amalg}: \mathcal{C}^{\amalg} \rightarrow \mathcal{E}$  which commutes with finite coproducts. We call  $\mathcal{C}^{\amalg}$  the finite coproduct envelope of  $\mathcal{C}$ .*

*Proof.* We shall only give a construction of  $\mathcal{C}^{\amalg}$ : objects are families  $(C_i)_{i \in I}$  where  $I$  is a finite set and  $C_i \in \mathcal{C}$  for all  $i \in I$ . A morphism  $\varphi: (C_i)_{i \in I} \rightarrow (D_j)_{j \in J}$  is given by a map  $f: I \rightarrow J$  and, for all  $i \in I$ , a morphism  $\varphi_i: C_i \rightarrow D_{f(i)}$ . Composition is defined in the obvious way.  $\square$

**Proposition 6.2** ([13, 1.3.6 and 2.1.8]). *Let  $\mathcal{C}$  be a category with finite coproducts and  $S$  a family of morphisms of  $\mathcal{C}$  stable under coproducts. Then  $S^{-1}\mathcal{C}$  has finite coproducts and the localisation functor  $\mathcal{C} \rightarrow S^{-1}\mathcal{C}$  commutes with them.*  $\square$

**Corollary 6.3.** *Let  $\mathcal{C}$  be a category and  $S$  a family of morphisms of  $\mathcal{C}$ . In  $\mathcal{C}^{\amalg}$ , consider the following family  $S^{\amalg}$  (see proof of Proposition 6.1):  $s: (C_i)_{i \in I} \rightarrow (D_j)_{j \in J}$  is in  $S^{\amalg}$  if and only if the underlying map  $f: I \rightarrow J$  is bijective and  $s_i: C_i \rightarrow D_{f(i)}$  belongs to  $S$  for all  $i \in I$ . Then we have an equivalence of categories*

$$(S^{-1}\mathcal{C})^{\amalg} \simeq (S^{\amalg})^{-1}\mathcal{C}^{\amalg}.$$

*Proof.* By Proposition 6.2,  $(S^{\amalg})^{-1}\mathcal{C}^{\amalg}$  has finite coproducts, hence it is enough to show that any functor  $F: S^{-1}\mathcal{C} \rightarrow \mathcal{E}$ , where  $\mathcal{E}$  has finite coproducts, factors canonically through a functor from  $(S^{\amalg})^{-1}\mathcal{C}^{\amalg}$  which commutes with finite coproducts. Let  $P: \mathcal{C} \rightarrow S^{-1}\mathcal{C}$  be the localisation functor; then  $F \circ P$  factors through  $\mathcal{C}^{\amalg}$ . The resulting functor inverts morphisms of  $S$  and commutes with coproducts, hence also inverts morphisms of  $S^{\amalg}$ . Thus we get a functor  $(S^{\amalg})^{-1}\mathcal{C}^{\amalg} \rightarrow \mathcal{E}$ , which obviously commutes with finite coproducts.  $\square$

**Theorem 6.4.** *In the situation of (0.1), if  $\bar{T}$  is an equivalence of categories, then so is  $\overline{T^{\amalg}}$ , where  $T^{\amalg}: \mathcal{C}^{\amalg} \rightarrow \mathcal{D}^{\amalg}$  is the functor induced by  $T$ . Moreover,  $\overline{T^{\amalg}} = (\bar{T})^{\amalg}$ .*  $\square$

## 7. Applications in algebraic geometry

Let  $k$  be a field. We denote by  $\text{Sch}(k)$  the category of reduced separated  $k$ -schemes of finite type.

### 7.1. Hyperenvelopes (Gillet–Soulé [4])

In this example,  $k$  is of characteristic 0. We take for  $\mathcal{D}^{\text{op}}$  the category of simplicial reduced  $k$ -schemes of finite type, and for  $\mathcal{C}^{\text{op}}$  the full subcategory consisting of smooth simplicial  $k$ -schemes.

For  $S$  and  $S'$  we take *hyperenvelopes* as considered by Gillet and Soulé in [4, 1.4.1]: recall that a map  $f: X_{\bullet} \rightarrow Y_{\bullet}$  in  $\mathcal{D}$  is a hyperenvelope if and only if, for any extension  $F/k$ , the induced map of simplicial sets  $X_{\bullet}(F) \rightarrow Y_{\bullet}(F)$  is a trivial Kan fibration (see loc. cit. for another equivalent condition).

**Theorem 7.1.** *In the above situation, the conditions of Theorem 5.3 are satisfied. In particular,  $\bar{T}^E$  is an equivalence of categories for any finite ordered set  $E$ .*

*Proof.* (i) is true by definition; (ii) is proved (or remarked) in [4, p. 136] and (iv) is proved in [4, Lem. 2, p. 135] (which, of course, uses Hironaka's resolution of singularities). The last assertion follows from Theorem 5.3.  $\square$

### 7.2. Proper hypercovers (Deligne–Saint Donat [SGA4.II])

Here  $k$  is any field. We take the same  $\mathcal{C}$  and  $\mathcal{D}$  as in the previous example, but we let  $S'$  be the collection of proper hypercovers (defined from proper surjective morphisms as in [SGA4.II, Exp. Vbis, (4.3)]).

**Theorem 7.2.** *In the above situation, the conditions of Theorem 5.3 are satisfied. In particular,  $\bar{T}^E$  is an equivalence of categories for any finite ordered set  $E$ .*

The proof is exactly the same as for Theorem 7.1, replacing the use of Hironaka's theorem in the proof of (iv) by that of de Jong's alteration theorem [7].

### 7.3. Cubical hyperresolutions (Guillén–Navarro Aznar [5])

In this example,  $k$  is again of characteristic 0. We are not going to give a new proof of the main theorem of [5, Th. 3.8], but merely remark that its proof in loc. cit. can be viewed as proving a special case of Theorem 2.1. Namely, in this situation,  $\mathcal{D}$  is a category of diagrams of schemes of a certain type,  $\mathcal{C}$  is the category of cubical hyperresolutions of objects of  $\mathcal{D}$ ,  $T$  maps a hyperresolution to the diagram it resolves,  $S'$  consists of identities and  $S$  consists of arrows mapping to identities; the categories  $I_d, I_f$  then reduce to the fibre categories of  $T$ . Guillén and Navarro Aznar prove that, on the level of  $S^{-1}\mathcal{C}$ ,  $I_d$  is 1-connected for any  $d \in \mathcal{D}$  and that  $I_f$  is 0-connected for any  $f \in \mathcal{D}^{\Delta^1}$ . The  $-1$ -connectedness of the  $I_{f_2, f_1}$  is then automatic in this special case, because Lemma 3.8.6 of [5] shows that the first two conditions already imply that  $\bar{T}$  is faithful.

### 7.4. Jouanolou's device (Riou [16, Prop. II.16])

Here  $\mathcal{C}$  is the category of smooth affine schemes over some regular scheme  $R$ ,  $\mathcal{D}$  is the category of smooth  $R$ -schemes,  $S'$  consists of morphisms of the form  $Y \rightarrow X$  where  $Y$  is a torsor under a vector bundle on  $X$  and  $S = S' \cap \mathcal{C}$ . Riou checks that the hypotheses of Theorem 5.2 are verified by taking the opposite categories, hence that the inclusion functor  $T: \mathcal{C} \rightarrow \mathcal{D}$  induces an equivalence on localised categories.

### 7.5. Closed pairs

Here we take for  $\mathcal{C}$  the category whose objects are closed embeddings  $i: Z \rightarrow X$  of proper  $k$ -schemes such that  $X - Z$  is dense in  $X$ , and a morphism from  $(X, Z)$  to  $(X', Z')$  is a morphism  $f: X \rightarrow X'$  such that  $f(X - Z) \subseteq X' - Z'$ . We take  $\mathcal{D} = \text{Sch}(k)$ , and for  $T$  the functor  $T(X, Z) = X - Z$ . Finally, we take for  $S'$  the isomorphisms of  $\mathcal{D}$  and  $S := T^{-1}(S')$ .

**Theorem 7.3** (cf. [6, Lem. 2.3.4]). *In the above situation, the conditions of Proposition 5.10 b) are satisfied. In particular,  $\bar{T}$  is an equivalence of categories.*

*Proof.* It is sufficient to prove (b1) and the conditions of Proposition 5.10 c). In (b1),  $T(f)s = T(g)s \Rightarrow T(f) = T(g)$  is trivial since  $s$  is by definition an isomorphism. On the other hand,  $T$  is faithful by a classical diagonal argument, since all schemes are separated.

In Proposition 5.10 c), the assertion on finite products is clear (note that  $(X_1, Z_1) \times (X_2, Z_2) = (X_1 \times X_2, Z_1 \times X_2 \cup X_1 \times Z_2)$ ). For  $U \in \text{Sch}(k) = \mathcal{D}$ , we define  $K_U$  as the full subcategory of  $J_U$  consisting of immersions  $U \hookrightarrow X - Z$ .

Nagata's theorem implies that  $I_U$  is nonempty; in particular,  $K_U$  is nonempty. Let  $\kappa = (U \hookrightarrow X - Z)$  be an object of  $K_U$ , and let  $\bar{U}$  be the closure of  $U$  in  $X$ . Then  $(\bar{U}, \bar{U} - U)$  defines an object of  $I_U/\kappa$ , and (c1) is verified. As for (c2), it is trivial since the product of an immersion with any morphism remains an immersion.  $\square$

### 7.6. Another kind of closed pairs

Here we assume that  $\text{char } k = 0$ . For  $n \geq 0$ , we define  $\mathcal{D}_n^{\text{op}}$  to be the category whose objects are closed embeddings  $i: Z \rightarrow X$  with  $X$  an (irreducible) variety of dimension  $n$ ,  $X - Z$  dense and *smooth*; a morphism from  $(X, Z)$  to  $(X', Z')$  is a map  $f: X \rightarrow X'$  such that  $f^{-1}(Z') = Z$ . We define  $\mathcal{C}_n^{\text{op}}$  as the full subcategory of  $\mathcal{D}_n^{\text{op}}$  consisting of pairs  $(X, Z)$  such that  $X$  is smooth.

We take for  $S'$  the set of morphisms  $s: (X, Z) \rightarrow (X', Z')$  such that  $s|_{X-Z}$  is an isomorphism onto  $X' - Z'$ , and  $S = S' \cap \mathcal{C}_n$ .

**Lemma 7.4.** *If  $f$  and  $s$  have the same domain in  $\mathcal{D}_n$ , with  $s \in S'$ , then  $s$  is always in good position with respect to  $f$ .*

*Proof.* Translating in the opposite category, we have to see that if  $f: (X_1, Z_1) \rightarrow (X, Z)$  and  $s: (\tilde{X}, \tilde{Z}) \rightarrow (X, Z)$  are maps in  $\mathcal{D}^{\text{op}}$  with  $s \in S'$ , then the fibre product  $(\tilde{X}_1, \tilde{Z}_1)$  of  $f$  and  $s$  exists and the pull-back map  $s': (\tilde{X}_1, \tilde{Z}_1) \rightarrow (X_1, Z_1)$  is in  $S'$ . Indeed, note that  $(\tilde{X}_1, \tilde{Z}_1)$  is given by the same formula as in the proof of Theorem 7.3, provided it exists, namely  $\tilde{X}_1 = X_1 \times_X \tilde{X}$  and  $\tilde{Z}_1 = X_1 \times_X \tilde{Z} \cup Z_1 \times_X \tilde{X}$ . The thing to prove is that  $\tilde{X}_1 - \tilde{Z}_1$  is still dense in  $\tilde{X}_1$ , which will imply in particular that  $\tilde{X}_1$  is a variety. It is sufficient to prove separately that  $\tilde{Z} \times_X X_1$  and  $\tilde{X} \times_X Z_1$  are nowhere dense in  $\tilde{X}_1$ , which we leave to the reader.  $\square$

**Theorem 7.5.** *In the above situation, the conditions of Proposition 5.12 are satisfied. In particular,  $\bar{T}$  is an equivalence of categories. Moreover,  $S^{-1}\mathcal{C}$  is not changed if we replace  $S$  by the subset of  $S' \cap \mathcal{C}_n$  generated by blow-ups with smooth centres.*

*Proof.* (d1) is true because two morphisms from the same source which coincide on a dense open subset are equal. (d2) and (d3) are immediately verified thanks to Lemma 7.4. (d4) is clear and (d5) follows from Hironaka's resolution theorem. The last statement of Theorem 7.5 also follows from Hironaka's theorem that any resolution of singularities may be dominated by a composition of blow-ups with smooth centres.  $\square$

## 8. Applications in birational geometry

We shall reserve the word "variety" to mean an integral scheme in  $\text{Sch}(k)$ , and denote their full subcategory by  $\text{Var}(k)$ ; we usually abbreviate with  $\text{Sch}$  and  $\text{Var}$ .

Recall [EGA, (2.3.4)] that a *birational morphism*  $s: X \rightarrow Y$  in Sch is a morphism such that every irreducible component  $Z'$  of  $Y$  is dominated by a unique irreducible component  $Z$  of  $X$  and the induced map  $s|_Z: Z \rightarrow Z'$  is a birational map of varieties.

**Definition 8.1.** We denote by  $S_b$  the multiplicative system of birational morphisms in Sch, by  $S_o$  the subsystem consisting of open immersions and by  $S_b^p$  the subsystem consisting of proper birational morphisms.

We shall also say that a morphism  $f: X \rightarrow Y$  in Sch is *dominant* if its image is dense in  $Y$ , or equivalently if every irreducible component of  $Y$  is dominated by some irreducible component of  $X$ .

**Lemma 8.2.** a) Let

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{\sigma_1} \\ \xrightarrow{\sigma_2} \end{array} & Y \\ & s \downarrow & \\ & & Z \end{array}$$

be a diagram in Sch, with  $X$  reduced,  $Y$  separated and  $s, \sigma_1, \sigma_2 \in S_b$ . Suppose that  $s\sigma_1 = s\sigma_2$ . Then  $\sigma_1 = \sigma_2$ .

b) Let  $f, g: Y \rightarrow Z, h: X \rightarrow Y \in \text{Ar}(\text{Sch})$  be such that  $fh = gh$ . Suppose that  $h$  is dominant. Then  $f = g$ .

*Proof.* a) Recall from [EGA, (5.1.5)] the kernel scheme  $\text{Ker}(\sigma_1, \sigma_2) \subset X$ : it is the inverse image scheme of the diagonal  $\Delta_Y(Y) \subset Y \times_k Y$  via the morphism  $(\sigma_1, \sigma_2)$ . Since  $Y$  is separated,  $\text{Ker}(\sigma_1, \sigma_2)$  is a closed subscheme of  $X$  and, by definition of birational morphisms, it contains all the generic points of  $X$ . Hence  $\text{Ker}(\sigma_1, \sigma_2) = X$  since  $X$  is reduced, and  $\sigma_1 = \sigma_2$ .

b) is obvious, since by assumption  $h^{-1}(\text{Ker}(f, g)) = X$ . □

**Definition 8.3.** Let  $\mathcal{C}$  be a subcategory of Sch. In particular, we write Var for the full subcategory of varieties and Sm for the full subcategory of smooth varieties.

a) We denote by  $\mathcal{C}^{\text{qp}}$  (resp.  $\mathcal{C}^{\text{prop}}, \mathcal{C}^{\text{proj}}$ ) the full subcategory of  $\mathcal{C}$  consisting of quasiprojective (resp. proper, projective) objects.

b) We denote by  $\mathcal{C}_{\text{sm}}$  the non-full subcategory of  $\mathcal{C}$  with the same objects, but where a morphism  $f: X \rightarrow Y$  is in  $\mathcal{C}_{\text{sm}}$  if and only if  $f$  maps the smooth locus of  $X$  into the smooth locus of  $Y$ .

The following proposition is the prototype of our birational results.

**Proposition 8.4.** In the commutative diagram

$$\begin{array}{ccc} S_b^{-1} \text{Var}^{\text{prop}} & \xrightarrow{A} & S_b^{-1} \text{Var} \\ \uparrow C & & \uparrow D \\ S_b^{-1} \text{Var}^{\text{proj}} & \xrightarrow{B} & S_b^{-1} \text{Var}^{\text{qp}} \end{array}$$

all functors are equivalences of categories. The same holds if the subscript sm is added everywhere.

*Proof.* We first prove that  $A$  and  $B$  are equivalences of categories. For this, we apply Proposition 5.10 b) with  $\mathcal{C} = \text{Var}^{\text{prop}}$  (resp.  $\text{Var}^{\text{proj}}$ ),  $\mathcal{D} = \text{Var}$  (resp.  $\text{Var}^{\text{qp}}$ ),  $T$  the obvious inclusion,  $S = S_b$  and  $S' = S_b$ :

- Condition (b1) holds because  $T$  is fully faithful and birational morphisms are dominant (see Lemma 8.2 b)).
- (b2) is true by Nagata’s theorem in the proper case and tautologically in the projective case.
- For (b3) we use the “graph trick”: we are given  $i: X \rightarrow \bar{X}$  and  $j: X \rightarrow Y$  where  $\bar{X}$  and  $Y$  are proper (resp. projective) and  $i$  is birational. Let  $\bar{X}'$  be the closure of the diagonal image of  $X$  in  $\bar{X} \times Y$ . Then  $X \rightarrow \bar{X}'$  is still birational,  $\bar{X}'$  is proper (resp. projective) and the projections  $\bar{X}' \rightarrow \bar{X}$ ,  $\bar{X}' \rightarrow Y$  give the desired object of  $I_X/i \times_{I_X} I_X/j$ .

We now prove that  $D$  is an equivalence of categories, which will also imply that  $C$  is an equivalence of categories. This time we apply Proposition 5.12 with  $\mathcal{C} = (\text{Var}^{\text{qp}})^{\text{op}}$ ,  $\mathcal{D} = \text{Var}^{\text{op}}$ ,  $T$  the obvious inclusion and  $S = S_o$ ,  $S' = S_o$ :

- Condition (d1) is clear (open immersions are monomorphisms even in  $\text{Var}$ ).
- (d2) means that the intersection of two dense open subsets in a variety is dense, which is true.
- (d3) means that if  $(gf)^{-1}(U) \neq \emptyset$ , then  $g^{-1}(U) \neq \emptyset$ , which is true.
- (d4) is clear.
- In (d5), we have a morphism  $f: X_1 \rightarrow X$  of varieties and want to find a quasi-projective dense open subset  $U \subseteq X$  such that  $f^{-1}(U) \neq \emptyset$ : take  $U$  containing  $f(\eta_{X_1})$  (any point has an affine neighbourhood).

The proofs with indices  $\text{sm}$  are the same. □

In what follows  $\text{Sm} \subset \text{Var}$  is the subcategory consisting of smooth varieties.

**Proposition 8.5.** *In the commutative diagram*

$$\begin{array}{ccc}
 S_b^{-1} \text{Sm}^{\text{prop}} & \xrightarrow{A'} & S_b^{-1} \text{Sm} \\
 \uparrow C' & & \uparrow D' \\
 S_b^{-1} \text{Sm}^{\text{proj}} & \xrightarrow{B'} & S_b^{-1} \text{Sm}^{\text{qp}}
 \end{array}$$

$D'$  is an equivalence of categories. Under resolution of singularities, this is true of the three other functors.

*Proof.* The same as that of Proposition 8.4, except that for  $A'$  and  $B'$ , we need to desingularise a compactification of a smooth variety using Hironaka’s theorem. □

**Proposition 8.6.** *If  $k$  is perfect, in the commutative diagram*

$$\begin{array}{ccc} S_b^{-1} \text{Sm} & \xrightarrow{E} & S_b^{-1} \text{Var}_{\text{sm}} \\ \uparrow G & & \uparrow H \\ S_b^{-1} \text{Sm}^{\text{qp}} & \xrightarrow{F} & S_b^{-1} \text{Var}_{\text{sm}}^{\text{qp}} \end{array}$$

*all functors are equivalences of categories.*

*Proof.* The case of  $H$  has been seen in Proposition 8.4, and the case of  $G = D'$  has been seen in Proposition 8.5. We now prove that  $E$  and  $F$  are equivalences of categories. Here we apply Proposition 5.12 with  $\mathcal{C} = \text{Sm}^{\text{op}}$  (resp.  $(\text{Sm}^{\text{qp}})^{\text{op}}$ ),  $\mathcal{D} = \text{Var}_{\text{sm}}^{\text{op}}$  (resp.  $(\text{Var}_{\text{sm}}^{\text{qp}})^{\text{op}}$ ),  $T$  the obvious inclusion and  $S = S' = S_o$ . Note that open immersions automatically respect smooth loci. Let us check the conditions:

- (d1), (d2) and (d3) and (d4) are clear (see proof of Proposition 8.4).
- It remains to prove (d5): if  $f: X_1 \rightarrow X$  is a morphism in  $\text{Var}_{\text{sm}}$ , then  $f(\eta_{X_1})$  is contained in the smooth locus  $U$  of  $X$ , hence  $U \rightarrow X$  is in good position with respect to  $f$ . □

**Proposition 8.7.** *Under resolution of singularities, all functors in the commutative diagram*

$$\begin{array}{ccc} S_b^{-1} \text{Sm}^{\text{prop}} & \xrightarrow{I} & S_b^{-1} \text{Var}_{\text{sm}}^{\text{prop}} \\ \uparrow K & & \uparrow L \\ S_b^{-1} \text{Sm}^{\text{proj}} & \xrightarrow{J} & S_b^{-1} \text{Var}_{\text{sm}}^{\text{proj}} \end{array}$$

*are equivalences of categories.*

*Proof.* The case of  $K = C'$  has been seen in Proposition 8.5 and the case of  $L$  in Proposition 8.4. The case of the other functors is then implied by the previous propositions (the reader should draw a commutative cube of categories in order to confirm that enough equivalences of categories have been proven). □

**Proposition 8.8.** *Propositions 8.4 to 8.7 remain true if we replace all categories in sight by their finite coproduct envelopes (see Proposition 6.1) and  $S_b$  by  $S_b^{\text{II}}$  (ibid.).*

*Proof.* This follows from Theorem 6.4. □

*Remarks 8.9.* a) Note that even though Proposition 8.8 says that  $D^{\text{II}}$  induces an equivalence of categories on localisations, where  $D$  is the functor of Proposition 8.4,  $(D^{\text{II}}, S_o^{\text{II}})$  does not satisfy the (dual) simplicial hypotheses of Theorem 2.1. Indeed, let  $X$  be a non-quasiprojective variety over  $k$  that we assume algebraically closed for simplicity. By Kleiman's theorem [11], there exists a finite set  $\{x_1, \dots, x_n\}$  of closed points of  $X$  which is contained in no affine open subset, hence also in no quasi-projective open subset. Thus, if  $Y = \coprod_n \text{Spec } k$  and  $f: Y \rightarrow X$  is the map defined by the  $x_i$ , then  $I_f$  is empty. This shows that the simplicial hypotheses are not preserved by finite product envelope.

- b) Also, while  $(D, S_o)$  satisfies the dual simplicial hypotheses, it does not satisfy the dual of hypothesis (1') of Corollary 4.4; this is obvious from Chow's lemma. This shows that the hypotheses of Corollary 4.4 are strictly stronger than those of Theorem 2.1.

*Remark 8.10.* To summarise Propositions 8.5, 8.6 and 8.7 under resolution of singularities, we have the following equivalences of categories:

$$S_b^{-1} \text{Sm}^{\text{proj}} \simeq S_b^{-1} \text{Sm}^{\text{prop}} \simeq S_b^{-1} \text{Sm}^{\text{qp}} \simeq S_b^{-1} \text{Sm} \\ \simeq S_b^{-1} \text{Var}_{\text{sm}}^{\text{proj}} \simeq S_b^{-1} \text{Var}_{\text{sm}}^{\text{prop}} \simeq S_b^{-1} \text{Var}_{\text{sm}}^{\text{qp}} \simeq S_b^{-1} \text{Var}_{\text{sm}}.$$

(The superscript qp could also be replaced by "affine", as the proofs show.)

We shall show in [10] that

$$S_b^{-1} \text{Sm}^{\text{proj}}(X, Y) = Y(k(X))/R$$

for any two smooth projective varieties  $X, Y$ , where  $R$  is Manin's  $R$  equivalence.

*Remark 8.11.* On the other hand, the functor  $S_b^{-1} \text{Sm} \rightarrow S_b^{-1} \text{Var}$  is neither full nor faithful, even under resolution of singularities. Indeed, take  $k$  of characteristic 0. Let  $X$  be a proper irreducible curve of geometric genus  $> 0$  with one nodal singular point  $p$ . Let  $\tilde{\pi}: \tilde{X} \rightarrow X$  be its normalisation,  $U = X - \{p\}$ ,  $\tilde{U} = \tilde{\pi}^{-1}(U)$ ,  $\pi = \tilde{\pi}|_{\tilde{U}}$  and  $j: U \rightarrow X$ ,  $\tilde{j}: \tilde{U} \rightarrow \tilde{X}$  the two inclusions. We assume that  $\tilde{\pi}^{-1}(p)$  consists of two rational points  $p_1, p_2$ . Finally, let  $f_i: \text{Spec } k \rightarrow \tilde{X}$  be the map given by  $p_i$ .

$$\begin{array}{ccc} \text{Spec } F & \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} & \tilde{X} \xrightarrow{\tilde{\pi}} X \\ & & \uparrow \tilde{j} \quad \uparrow j \\ & & \tilde{U} \xrightarrow{\pi} U \end{array}$$

In  $S_b^{-1} \text{Var}$ ,  $\tilde{\pi}$  is an isomorphism so that  $f_1 = f_2$ . We claim that  $f_1 \neq f_2$  in  $S_b^{-1} \text{Sm}^{\text{prop}} \xrightarrow{\sim} S_b^{-1} \text{Sm}$ . Otherwise, since  $R$ -equivalence is a birational invariant of smooth proper varieties [2, Prop. 10], we would have  $p_1 = p_2 \in \tilde{X}(k)/R$ . But this is false since  $\tilde{X}$  has nonzero genus. We thank A. Chambert-Loir for his help in finding this example.

More generally, it is well-known that for any integral curve  $C$  and any two closed points  $x, y \in C$ , there exists a proper birational morphism  $s: C \rightarrow C'$  such that  $s(x) = s(y)$  (cf. [17, Ch. IV, § 1, no. 3] when  $F$  is algebraically closed). This shows that any two morphisms  $f, g: X \rightrightarrows C$  such that  $f(\eta_X)$  and  $g(\eta_X)$  are closed points become equal in  $S_b^{-1} \text{Var}$ . This can be used to show that the functor  $S_b^{-1} \text{Sm} \rightarrow S_b^{-1} \text{Var}$  does not have a right adjoint.

Non-fullness holds even if we restrict to normal varieties. Indeed, let us take  $k = \mathbf{R}$  and let  $X$  be the affine cone with equation  $x_1^2 + \dots + x_n^2 = 0$  (for  $n \geq 3$  this is a normal variety). Let  $\tilde{X}$  be a desingularisation of  $X$  (for example obtained by blowing up the singular point) and  $\bar{X}$  a smooth compactification of  $\tilde{X}$ . Then  $\bar{X}(\mathbf{R}) = \emptyset$  by a valuation argument, hence  $S_b^{-1} \text{Sm}^{\text{proj}}(\mathbf{R})(\text{Spec } \mathbf{R}, \bar{X}) = \emptyset$  by Remark 8.10. On the

other hand,  $X(\mathbf{R}) \neq \emptyset$ , hence

$$S_b^{-1} \text{Var}^{\text{proj}}(\mathbf{R})(\text{Spec } \mathbf{R}, \bar{X}) = S_b^{-1} \text{Var}^{\text{proj}}(\mathbf{R})(\text{Spec } \mathbf{R}, X) \neq \emptyset.$$

We are indebted to Mahé for pointing out this example. For  $n \geq 4$ , this singularity is even terminal in the sense of Mori's minimal model programme, as Beauville pointed out (which seems to mean unfortunately that we cannot insert this programme in our framework).

*Remark 8.12.* Let  $n \geq 0$ . Replacing all the subcategories  $\mathcal{C}$  of  $\text{Sch}$  used above by their full subcategories  $\mathcal{C}_n$  consisting of schemes of dimension  $\leq n$ , it can readily be shown that all corresponding equivalences of categories remain valid, with the same proofs. This raises the question whether the induced functor  $S_b^{-1}\mathcal{C}_n \rightarrow S_b^{-1}\mathcal{C}_{n+1}$  is fully faithful for some (or all) choices of  $\mathcal{C}$ . It can be proven [10] that this is true at least for  $\mathcal{C} = \text{Sm}^{\text{proj}}$  in characteristic zero, hence for the other  $\mathcal{C}$ s which become equivalent to it after inverting birational morphisms as in Remark 8.10. However the proof is indirect and consists in observing that the morphisms are still given by the formula of Remark 8.10. It is an interesting question whether such a result can be proven by methods in the spirit of the present paper.

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