J. K-Theory **7** (2011), 409–427
doi:10.1017/is011003002jkt147

Relatively unramified elements in cycle modules

by

BRUNO KAHN

Abstract

In a recent paper, Merkurjev showed that for a smooth proper variety X over a field k, the functor $M_* \mapsto A^0(X, M_0)$ from cycle modules to abelian groups is corepresented by a cycle module constructed on the Chow group of 0-cycles of X. We show that if “proper" is relaxed, the result still holds by replacing the Chow group of 0-cycles by the 0-th Suslin homology group of X.

Key Words: Cycle modules, unramified cohomology, motivic homology
Mathematics Subject Classification 2010: 14C15, 14F42, 19E15

1. Introduction

Let X be a smooth proper variety over a field k, and let M_* be a cycle module over k in the sense of Rost [12]. The group $A^0(X, M_0)$ is an important birational invariant of X. In particular, if X is rational, this group is reduced to $M_0(k)$ for any cycle module M. In [10, Th. 2.10], Merkurjev proves:

1.1 Theorem *There is an isomorphism, natural in M*

$$A^0(X, M_0) \simeq \mathrm{Hom}_{\mathbf{CM}}(K^X, M)$$

where **CM** *is the category of cycle modules over k and* K^X *is the cycle module given by the formula*

$$K_n^X(F) = A_0(X_F, K_n^M)$$

for any function field F/k.

In this note we extend this theorem, when k is perfect, by relaxing the properness hypothesis on X. The replacement of K^X involves Suslin homology, or more accurately motivic homology in the sense of Voevodsky [15]. That Suslin homology should arise is clear via Déglise’s correspondence between cycle modules and homotopy invariant Nisnevich sheaves with transfers [7]: if $\mathcal{F}$ is such a sheaf, then we have

$$H^0(X, \mathcal{F}) = \mathrm{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(M(X), \mathcal{F}[0]) = \mathrm{Hom}_{\mathbf{HI}}(h_0^{\mathrm{Nis}}(X), \mathcal{F})$$

where $\mathbf{DM}^{\mathrm{eff}}_{-}$ is Voevodsky's triangulated category of motivic complexes and **HI** is the category of homotopy invariant Nisnevich sheaves with transfers.

It is of interest to formulate the generalisation of Merkurjev's theorem without reference to Déglise's theory, and this is what we do now. We first introduce the corepresenting object:

1.2 Proposition *Suppose k perfect. The assignment*

$$F \mapsto (H_{-n}(X_F, \mathbf{Z}(-n)))_{n\in\mathbf{Z}}$$

defines a cycle module denoted by H^X. If X is projective, we have $H^X = K^X$.

For $n = 0$ and $F = k$, we get Suslin homology $H_0^S(X) = H_0(X, \mathbf{Z}(0))$.

1.3 Theorem *There is an isomorphism, natural in M*

$$A^0(X, M_0) \simeq \mathrm{Hom}_{\mathbf{CM}}(H^X, M).$$

Here is a complement to Theorem 1.3:

1.4 Theorem *The isomorphism of Theorem 1.3 comes with another isomorphism*

$$\mathrm{Hom}_{\mathbf{CM}}(H^X, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{PST}}(h_0^{\mathrm{Nis}}(X), \mathcal{M}_0)$$

where $\mathcal{M}_0(U) = A^0(U, M_0)$ is viewed as a presheaf with transfers.

Theorem 1.4 refines [10, Th. 2.11], see Corollary 4.7.

Finally, in [10, 2.3], Merkurjev associates to a cycle module M and a k-scheme of finite type Y a new cycle module M^Y defined by

$$M_*^Y(F) = A_0(Y_F, M_*).$$

(In case $M = K$, we get the cycle module K^Y as above.) We elucidate the structure of M^Y in terms of Déglise's theory:

1.5 Theorem *Let Y be a scheme of finite type over a not necessarily perfect field k. Then*

(i) *Suppose Y quasi-projective. Then, for any function field F/k, $K_n^Y(F) = H^c_{-n}(Y_F, \mathbf{Z}(-n))$ where the right hand side is defined in terms of Bloch's higher Chow groups.*

(ii) *Assume Hironaka resolution of singularities, but relax "quasi-projective". Then we have a morphism $H^Y \to K^Y$ which is the isomorphism of Proposition 1.2 if Y is smooth and projective.*

(iii) *Assume Hironaka resolution of singularities again, and suppose Y smoothable. For any cycle module M, let $\mathcal{D}(M)$ be the associated homotopic module in the sense of [7, Def. 1.15] (see [7, Th. 3.4]). Then*

$$\mathcal{D}(M^Y) = \mathcal{D}(K^Y) \otimes \mathcal{D}(M)$$

where the tensor product is taken in the category of homotopic modules [7, 1.14]. In particular, if $K^Y \simeq K$, then $M^Y \simeq M$ for any M.

If we transport the tensor product of homotopic modules to **CM** via Déglise's equivalence of categories, Theorem 1.5 (iii) says that $M^Y = K^Y \otimes M$. For Y smooth projective, we thus get that the stable birational invariance of M^Y is in some sense dual to that of $A^0(Y,M)$.

While the formulations of Proposition 1.2 and Theorem 1.3 do not involve Déglise's theory, their proofs definitely do, as do those of Theorems 1.4 and 1.5.[1]

Note that from Theorem 1.3 we get canonical maps

$$\mathrm{Ext}^i_{\mathbf{CM}}(H^X, M) \to A^i(X, M_0) \quad (i > 0)$$

(Yoneda Ext), but these are far from being isomorphisms in general, e.g. for $X = \mathbf{P}^1$. One might hope to describe the higher A^i by replacing H^X by a complex of cycle modules, but Voevodsky's theory dictates that the situation is more complicated: using Gersten's conjecture [12, Th. (6.1)] and a translation via Déglise's theory, this would more or less correspond to the hope that the natural functor $D^-(\mathbf{HI}) \to \mathbf{DM}^{\mathrm{eff}}_-$ is an equivalence of categories, which is false.

Proposition 1.2 is proven in §2, Theorem 1.3 is proven in §3, Theorem 1.4 is proven in §4 and Theorem 1.5 is proven in §5. To prove Theorem 1.3, the method is similar to Merkurjev's: we construct two mutually inverse natural transformations. The main differences are that the relations defining $H_0(X,\mathbf{Z})$ are different from those defining $CH_0(X)$, and that the definition of the cycle module H^X is not as straighforward as that of K^X. Also, when X is not proper, H^X is usually not (-1)-connected: for example, $H_d(\mathbb{G}_m^d, \mathbf{Z}(d)) = \mathbf{Z}$ for any d. This makes the construction of one natural transformation a little more subtle, and the proofs slightly different.

In §6, we complement this work by giving a vanishing range for motivic homology and computing $H_d(X,\mathbf{Z}(d))$ for $d = \dim X$ when X has a smooth compactification with complement a divisor with normal crossings (Proposition 6.8).

Acknowledgements

I wish to thank Frédéric Déglise for helpful comments.

[1] Déglise's work is also used in [10, proof of Lemma 2.2].

Convention

All morphisms of cycle modules are of degree 0: one recovers morphisms of arbitrary degree by using the shift functor.

2. Proof of Proposition 1.2

2.A. Cycle modules and motivic complexes

The usual way to produce cycle modules is from a cohomology theory of Bloch-Ogus type. Namely, if $(Y,Z) \mapsto H^i_Z(Y,n)$ is a twisted cohomology theory with supports in the sense of Bloch and Ogus [2, (1.1)], defined on the category of pairs (Y,Z) with Y smooth and satisfying suitable axioms, then, for any $r \in \mathbf{Z}$, the assignment

$$F \mapsto (H^n(F,n+r))_{n\in\mathbf{Z}} \tag{2.1}$$

will verifiy the axioms of a cycle module, where $H^i(F,n) := \varinjlim_U H^i(U,n)$ with U running through the smooth varieties with function field F.

We shall only give an example. Given an object $C \in \mathbf{DM}_-^{\mathrm{eff}}$, define

$$H^i_Z(Y,n) = \begin{cases} H^i_Z(Y,C(n)) & \text{if } n \geq 0 \\ H^{2n+i-1}_{Z\times(\mathbf{A}^{-n}-\{0\})}((Y\times(\mathbf{A}^{-n}-\{0\}),Y\times\{1\};C) & \text{if } n \leq 0. \end{cases} \tag{2.2}$$

Here the cohomology is Nisnevich cohomology. If we take $C = \underline{C}_*(X)$ in (2.2), the formula becomes

$$H^i_Z(Y,n) = \operatorname{Hom}_{\mathbf{DM}_{\mathrm{gm}}}(M^Z(Y),M(X)(n))$$

for any $n \in \mathbf{Z}$ (cf. [13, §9]). The same formula is true for any C by computing in $\mathbf{DM}$, the homotopy category of $\mathbf{Z}(1)$-spectra, rather than in $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ [11, p. 247].[2]

By [6, 6.2.1], (2.1) gives a cycle module. More concretely, we get the data and rules of cycle premodules [12, §1] as follows. Data D1 and D2 (covariance and contravariance for finite extensions) are respectively given by contravariance of cohomology and the transfers. Datum D3 (K^M_*-module structure) is best obtained in $\mathbf{DM}$ from the cup-products

$$H^n(U,\mathbf{Z}(n)) \otimes H^i(U,m) \to H^{n+i}(U,n+m)$$

stemming from the isomorphisms

$$\mathbf{Z}(n) \otimes C(m) \xrightarrow{\sim} C(n+m)$$

[2] The category $\mathbf{DM}$ is constructed in [4, Ex. 6.25].

together with the Suslin-Voevodsky isomorphisms [14, Th. 3.4]

$$H^n(F,\mathbf{Z}(n)) \simeq K_n^M(F). \tag{2.3}$$

Datum D4 (residues) is obtained from the Gysin exact sequences

$$H^{i-2}(Z,n-1) \to H^i(U,n) \to H^i(U-Z,n) \xrightarrow{\partial} H^{i-1}(Z,n-1))$$

for Z a smooth divisor in U. This uses the cancellation theorem [17], [1, prop. in 6.1] (recall that k is perfect).

Rules R1a, R1b and R1c (functoriality and exchange for D1 and D2) come from the functoriality of cohomology and finite correspondences. Similarly for rules R2 (behaviour with respect to product). Finally, rules R3 (behaviour of D4 with respect to the other data) come from the functoriality of the Gysin maps [6, Lemmas 5.4.7 and 5.4.8].

By [12, Th. (2.3)], to get a cycle module we are left to check the conditions (FDL) (finite support) and (WL) (weak reciprocity) of loc. cit. The first one is obvious from the definition of residues, and the second one follows from the homotopy invariance of motivic cohomology.

2.1 Remark All cycle modules arise in this way: if we start from a cycle module M, we may recover it by taking $C = \mathcal{M}_0[0]$ in (2.2) (see Theorem 1.4 for $\mathcal{M}_0$).

2.B. Proof of Proposition 1.2

We get the cycle module of Proposition 1.2 by taking $C = \underline{C}_*(X)$ in (2.2) and $r = 0$ in (2.1).

If X is (smooth) projective of dimension d, Poincaré duality yields (for $F = k$):

$$H_n(X,\mathbf{Z}(n)) \simeq H^{2d-n}(X,\mathbf{Z}(d-n)).$$

If $n > 0$, this group is 0 since $2d - n > 2(d-n)$. If $n \leq 0$, in the coniveau spectral sequence

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} H^{q-p}(k(x),\mathbf{Z}(d-n-p)) \Rightarrow H^{p+q}(X,\mathbf{Z}(d-n))$$

the only nonzero E_1-term with $p+q = 2d-n$ is for $(p,q) = (d,d-n)$ (because $E_1^{p,q} = 0$ for $p > d$ or $q-p > d-n-p$). Hence $H_n(X,\mathbf{Z}(n))$ is the 0-th homology group of the complex

$$\cdots \to \bigoplus_{x\in X_{(1)}} H^{-n+1}(k(x),\mathbf{Z}(-n+1)) \to \bigoplus_{x\in X_{(0)}} H^{-n}(k(x),\mathbf{Z}(-n)) \to 0$$

that is, $A_0(X, K^M_{-n})$.

Thus $H_n^X = 0$ for $n < 0$ if X is projective, but not necessarily in general. Under resolution of singularities, one can show that $H_n^X = 0$ for $n < -d$ in general (see Proposition 6.7).

In the sequel we shall use the following trivial lemma:

2.2 Lemma *For any smooth X, any cycle module M and any $n \in \mathbf{Z}$, we have*

$$H_n(X \times \mathbb{G}_m, \mathbf{Z}(n)) \simeq H_n(X, \mathbf{Z}(n)) \oplus H_{n-1}(X, \mathbf{Z}(n-1))$$
$$A^0(X \times \mathbb{G}_m, M_n) \simeq A^0(X, M_n) \oplus A^0(X, M_{n-1}).$$

3. Two natural transformations

3.A. In one direction

The map

$$\mathrm{Hom}_{\mathbf{DM}}(M(X), M(X)) = H^0_{\mathrm{Nis}}(X, \underline{C}_*(X)) \\ \to H^0_{\mathrm{Nis}}(k(X), \underline{C}_*(X)) = H_0^X(k(X))$$

sends the identity map of $M(X)$ to a "generic element" $\eta_X \in H_0^X(k(X))$. If $f: H^X \to M$ is a morphism of cycle modules, $f(\eta_X) \in M_0(k(X))$ defines an element.

3.1 Lemma $f(\eta_X) \in A^0(X, M_0)$.

Proof: It suffices to do it in the universal case $M = H^X$, $f = Id$. For this, it is enough to see that the composition

$$H^0(X, \underline{C}_*(X)) \to H^0(k(X), \underline{C}_*(X)) \to \bigoplus_{x \in X^{(1)}} H^{-1}(M(k(x))(1), \underline{C}_*(X))$$

is 0. But the last map is obtained as a limit from the Gysin homomorphisms

$$\mathrm{Hom}(M(U), M(X)) \to \mathrm{Hom}(M(Z)(1)[1], M(X))$$

where U runs through the open subsets of X and Z runs through the smooth divisors in U. □

Lemma 3.1 yields a natural transformation

$$\psi_{X,M} : \mathrm{Hom}_{\mathbf{CM}}(H^X, M) \to A^0(X, M_0). \tag{3.1}$$

3.B. In the opposite direction

Let M be a cycle module, U,V two smooth varieties and $\Gamma \in c(V,U)$ a finite correspondence in the sense of [15, §2.1]. Assume Γ irreducible: thus, Γ is a closed subvariety of $V \times U$ such that the projection $p : \Gamma \to V$ is finite and surjective on a connected component of V. Letting $q : \Gamma \to U$ denote the other projection, covariant and contravariant functoriality of cycle cohomology [12, §§5 and 12] yield a composition

$$\Gamma^* : A^0(U,M_0) \xrightarrow{q^*} A^0(\Gamma,M_0) \xrightarrow{p_*} A^0(V,M_0)$$

which extends by linearity to a pairing

$$A^0(U,M_0) \times c(V,U) \to A^0(V,M_0).$$

Taking $U = X$, $V = \operatorname{Spec} k$, this defines a pairing

$$A^0(X,M_0) \times Z_0(X) \to M_0(k)$$

where $Z_0(X)$ is the group of 0-cycles of X. In particular, if $\alpha \in A^0(X,M_0)$, we get a map

$$\tilde{f}^\alpha : Z_0(X) \to M_0(k).$$

3.2 Lemma *The composition*

$$c(\mathbf{A}^1,X) \xrightarrow{s_0^*-s_1^*} Z_0(X) \xrightarrow{\tilde{f}^\alpha} M_0(k)$$

is 0*, where* $s_0,s_1 : \operatorname{Spec} k \to \mathbf{A}^1$ *are the inclusions of* 0 *and* 1.

Proof: Let $\Gamma \in c(\mathbf{A}^1,X)$ be a finite correspondence. For $i = 0,1$, the diagram

$$\begin{array}{ccc} A^0(X,M_0) & \xrightarrow{\Gamma^*} & A^0(\mathbf{A}^1,M_0) \\ {\scriptstyle (s_i^*\Gamma)^*}\downarrow & & \downarrow{\scriptstyle s_i^*} \\ M_0(k) & = & M_0(k) \end{array}$$

commutes (because $(s_i^*\Gamma)^* = (\Gamma \circ s_i)^* = s_i^* \circ \Gamma^*$, [5, Prop. 6.5]); the conclusion follows from the homotopy invariance of cycle cohomology [12, Prop. (8.6)]. □

Lemma 3.2 yields an induced map

$$f_0^\alpha : H_0^S(X_F) := \operatorname{Coker}\left(c(\mathbf{A}^1_F,X_F) \xrightarrow{s_0^*-s_1^*} Z_0(X_F) \right) \to M_0(F)$$

for any function field F/k. We extend this morphism to

$$f_n^\alpha : H_{-n}(X_F, \mathbf{Z}(-n)) \to M_n(F)$$

for any $n \in \mathbf{Z}$ as follows. If $n \geq 0$, the element

$$\alpha^n \in A^0(X \times \mathbb{G}_m^n, M_n) \tag{3.2}$$

corresponding to α under the decomposition

$$A^0(X \times \mathbb{G}_m^n, M_n) = \bigoplus_{i=0}^{n} A^0(X, M_i)^{\binom{n}{i}}$$

(see Lemma 2.2) yields a map

$$f_0^{\alpha^n} : H_0^S((X \times \mathbb{G}_m^n)_F) \to M_n(F).$$

We define f_n^α as the component of $f_0^{\alpha^n}$ on $H_{-n}(X_F, \mathbf{Z}(-n))$ (see Lemma 2.2 again).

If $n < 0$, we proceed à la Bass (cf. negative K-groups). For any $n \in \mathbf{Z}$, suppose f_{n+1}^α is defined. We then get a map

$$H_{n+1}^X(F(t)) \xrightarrow{(f_{n+1}^\alpha)_*} M_{n+1}(F(t)).$$

Using the isomorphism $N_{n+1}(F(t)) \simeq N_{n+1}(F) \oplus \bigoplus_{x \in \mathbf{A}_F^1} N_n(F(x))$ valid for any cycle module N (and natural in N), we get a map $\bar{f}_n^\alpha(F) : H_n^X(F) \to M_n(F)$, defined as the component of $f_{n+1}^\alpha(F(t))$ at $x = 0$. If $n < 0$, this defines $f_n^\alpha(F)$; if $n \geq 0$, we have $\bar{f}_n^\alpha(F) = f_n^\alpha(F)$ from the above definition of f_n^α.

3.3 Proposition *$(f_n^\alpha)_{n \in \mathbf{Z}}$ is a morphism of cycle modules.*

Proof: We have to show that f_*^α commutes with the data D1, D2, D3, D4 of [12, p. 329]. For D1 (contravariance for field extensions) and D2 (covariance for finite field extensions), this is already true on the level of $\tilde{f}^\alpha$ (see before Lemma 3.2). For D3 (cup-product with units) and D4 (residues), it follows from the definition of f_n^α from f_0^α, for $n \geq 0$ and $n < 0$, and the compatibility of these two definitions. □

Proposition 3.3 yields a homomorphism

$$\varphi_{X,M} : A^0(X, M_0) \to \mathrm{Hom}_{\mathbf{CM}}(H^X, M). \tag{3.3}$$

3.C. Compositions

3.4 Lemma *The composition $\psi_{X,M} \circ \varphi_{X,M}$ ((3.1) and (3.3)) is the identity.*

Proof: For $\alpha \in A^0(X, M_0)$, we have to prove that $f_0^\alpha(k(X))(\eta_X) = \alpha$. It suffices to show that $\tilde{f}_\alpha(k(X)) : Z_0(X_{k(X)}) \to M_0(k(X))$ sends the generic point η_X to α. But, by construction, it sends η_X to the specialisation at η_X of the pull-back of $\alpha \in A^0(X, M_0)$ in $A^0(X_{k(X)}, M_0)$, which is obviously equal to α. □

3.5 Lemma *The composition $\varphi_{X,M} \circ \psi_{X,M}$ is the identity.*

Proof: Let $f : H^X \to M$. We have to prove that $f^{f(\eta_X)} = f$. We first prove that $f_0^{f(\eta_X)} = f_0$. This amounts to showing, for any closed point $x \in X_{(0)}$, the identity

$$\tilde{f}^{f(\eta_x)}(x) = f(\eta_X)_x.$$

But the left hand side equals the right hand side by definition.

We now prove that $f_n^{f(\eta_X)} = f_n$ for all $n \in \mathbf{Z}$. For $n \geq 0$, we observe that, with the notation of (3.2), $f(\eta_X)^n$ is the image of $f(\eta_{X \times \mathbb{G}_m^n})$: this is checked inductively on n from the formula

$$\eta_{X \times \mathbb{G}_m} = \eta_X \cdot \eta_{\mathbb{G}_m}$$

stemming from the product

$$H_0^X(k(X)) \otimes H_0^{\mathbb{G}_m}(k(\mathbb{G}_m)) \to H_0^{X \times \mathbb{G}_m}(k(X \times \mathbb{G}_m))$$

and the formula

$$\partial_0 f(\eta_X \cdot \eta_{\mathbb{G}_m}) = f(\partial_0(\eta_X \cdot \eta_{\mathbb{G}_m})) = f(\eta_X)$$

coming from the identity

$$\partial_0(\eta_{\mathbb{G}_m}) = 1$$

where ∂_0 is the “residue" map $A^0(X \times \mathbb{G}_m, N_1) \to A^0(X, N_0)$ for any cycle module N.

For $n < 0$, we proceed by Bass induction from the case $n = 0$. □

Theorem 1.3 is proven.

4. Proof of Theorem 1.4

To prove Theorem 1.4, we proceed in two steps. The first one gives a weaker result: it is not logically necessary but is enlightening in its own way. In the second step, we obtain Theorem 1.4 as a straightforward application of Déglise’s results.

4.A. Generation by units

In [10], Merkurjev proceeds differently from here: instead of proving that $\varphi_{X,M} \circ \psi_{X,M}$ is the identity, he shows that $\psi_{X,M}$ is injective. This is based on the fact that K^X is "generated" by the generic element via units. We shall show here that this is still the case when X is not proper, which yields half of Theorem 1.4.

4.1 Definition Let N be a cycle module and $n \in \mathbf{Z}$. We say that N_{n+1} is *generated by* N_n *via units* if, for any F/k, the map

$$\bigoplus_{[E:F]<\infty} N_n(E) \otimes E^* \xrightarrow{f} N_{n+1}(F)$$

is surjective, where the component of f at E is $n \otimes u \mapsto N_{E/F}(n \cdot u)$.

4.2 Lemma *Let $\varphi : N \to M$ be a morphism of cycle modules, and let $n \in \mathbf{Z}$.*
a) If $\varphi_{n+1} = 0$ (resp. φ_{n+1} is a mono, an epi, an iso), the same is true of φ_n.
b) Suppose that N_{n+1} is generated by N_n via units. If $\varphi_n = 0$ (resp. is epi), the same is true of φ_{n+1}.

Proof: a) We argue as for the construction of f^α in §3.B: let F/k be a function field. Then $\varphi_n(F)$ is a direct summand of $\varphi_{n+1}(F(t))$ (say, via the closed point 0 of $\mathbf{A}^1_F$).

b) This is obvious. □

4.3 Proposition *Let F/k be a function field extension. For all $n \geq 0$, the map*

$$\bigoplus_{x \in X_{(0)}} K_n^M(F(x)) \to H_{-n}(X_F, \mathbf{Z}(-n))$$

given on the component x by $u \mapsto N_{F(x)/F}(x \cdot u)$ (where x is viewed as an element of $H_0(X_F)$) is surjective.

Proof: We may assume $F = k$. Recall (cf. §3.B) that $H_{-n}(X, \mathbf{Z}(-n))$ is functorially a direct summand of $H_0^S(X \times \mathbb{G}_m^n)$. I claim that there is a commutative diagram

$$\begin{tikzcd}
\bigoplus_{\xi \in (X \times \mathbb{G}_m^n)_{(0)}} \mathbf{Z} \arrow[r] \arrow[d] & H_0^S(X \times \mathbb{G}_m^n) \arrow[d] \\
\bigoplus_{x \in X_{(0)}} K_n^M(k(x)) \arrow[r] & H_{-n}(X, \mathbf{Z}(-n))
\end{tikzcd}$$

in which the right vertical map is the projection recalled above, the top horizontal map is the natural map from 0-cycles and the bottom horizontal map is the one of

Proposition 4.3. To define the left vertical map, map $\xi \in (X \times \mathbb{G}_m^n)_{(0)}$ to its image $x \in X_{(0)}$; the component of ξ on $\mathbb{G}_m^n$ then defines a sequence $(\lambda_1, \ldots, \lambda_n) \in k(x)^{*n}$, and we take as the image of ξ the symbol $\{\lambda_1, \ldots, \lambda_n\} \in K_n^M(k(x))$. □

4.4 Corollary *For any cycle module M, the map*

$$A^0(X,M) \simeq \mathrm{Hom}_{\mathbf{CM}}(H^X, M) \to \mathrm{Hom}_{\mathit{fields}/k}(H_0^X, M_0)$$

is injective, where fields$/k$ denotes the category of function fields over k. □

4.B. A refinement

For any cycle module M, $X \mapsto A^0(X, M_0)$ defines a homotopy invariant Nisnevich sheaf with transfers $\mathcal{M}_0$: the structure of presheaf with transfers is described in the beginning of §3.B ([7, Prop. 6.5] gives the functoriality), homotopy invariance is proven in [12, Prop. (8.6)] and the sheaf property is proven in [6, proof of 6.10]. We note:

4.5 Proposition *The functor*

$$\begin{aligned} \mathbf{CM} &\to \mathbf{HI} \\ M &\mapsto \mathcal{M}_0 \end{aligned}$$

has a left adjoint Φ.

Proof: Let $\mathcal{F} \in \mathbf{HI}$. For $n \in \mathbf{Z}$, define (cf. [7, 1.16])

$$\mathcal{F}_n = \begin{cases} \mathbb{G}_m^{\otimes n} \otimes \mathcal{F} & \text{if } n \geq 0 \\ \underline{\mathrm{Hom}}(\mathbb{G}_m^{\otimes n}, \mathcal{F}) & \text{if } n \leq 0 \end{cases}$$

where tensor product and internal Hom are computed in **HI**. By [6, 6.2.1], $\Phi(\mathcal{F})_n(F) = \mathcal{F}_n(\operatorname{Spec} F)$ defines a cycle module, which is easily seen to be the desired left adjoint computed at $\mathcal{F}$. □

4.6 Proposition *Let X be a smooth variety. For $\mathcal{F} = h_0^{\mathrm{Nis}}(X)$, we have $\Phi(\mathcal{F}) = H^X$.*

Proof: This is the content of [7, 1.3]. □

Propositions 4.5 and 4.6 together prove Theorem 1.4, since the inclusion functor $\mathbf{HI} \to \mathbf{PST}$ is fully faithful. From this, we get the following generalisation of [10, Th. 2.11]:

4.7 Corollary *For a smooth variety X, the following conditions are equivalent:*

(i) *For every cycle module M, the map $M_0(k) \to A^0(X, M_0)$ is an isomorphism.*

(ii) *The map $H^X \to K$ induced by the projection $X \to \operatorname{Spec} k$ is an isomorphism, where K is Milnor K-theory.*

(iii) *The degree map $H_0(X_F, \mathbf{Z}) \to \mathbf{Z}$ is an isomorphism for any function field F.*

(iv) *The element η_X from §3.A is defined over k.*

These conditions are verified if X is $\mathbf{A}^1$-trivial, i.e. if for any function field F, all points of $X(F)$ are $\mathbf{A}^1$-equivalent where $\mathbf{A}^1$-equivalence is the equivalence relation generated by $x_0 \sim x_1$ if there is a morphism $f : \mathbf{A}^1 \to X$ with $f(0) = x_0$ and $f(1) = x_1$.

Proof: (i) $\Rightarrow$ (iv): apply (i) to $M = H^X$.

(iv) $\Rightarrow$ (ii): via the isomorphism (3.3), η_X corresponds to the identity map of H^X. If this map factors through K, $H^X \to K$ must be an isomorphism.

(ii) $\iff$ (iii): this follows from Theorem 1.4.

(ii) $\iff$ (i): this follows from Theorem 1.3.

It remains to prove the last assertion. But the hypothesis implies (iii), by definition of Suslin homology. □

5. Proof of Theorem 1.5

5.A. Two Borel-Moore motivic homology theories

They are defined as follows:

a) In terms of Bloch's higher Chow groups:

$$H_i^c(Y, \mathbf{Z}(n)) := CH_n(Y, 2i - n).$$

b) In terms of quasi-finite correspondences:

$$H_i^c(Y, \mathbf{Z}(n)) := \begin{cases} \operatorname{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}(n)[i], \underline{C}_*^c(Y)) & \text{if } n \geq 0 \\ \operatorname{Hom}_{\mathbf{DM}_-^{\mathrm{eff}}}(\mathbf{Z}[i], \underline{C}_*^c(Y)(-n)) & \text{if } n \leq 0 \end{cases}$$

cf. [15, prop. 4.2.9].

Here are the features of these theories:

1. Over any k, the theory in a) verifies the localisation theorem if Y is quasi-projective [3].

2. Over any k, there is a canonical map for the theory in b):

$$H_i(Y, \mathbf{Z}(n)) \to H_i^c(Y, \mathbf{Z}(n))$$

induced by the map of complexes $\underline{C}_*(Y) \to \underline{C}_*^c(Y)$ (see [15, 4.1]).

3. If k admits resolution of singularities (in the sense of Hironaka), then the theories in a) and b) coincide for any quasi-projective Y [15, prop. 4.2.9]. Moreover, the theory of b) verifies localisation for any Y (not necessarily quasi-projective).

Moreover, motivic cohomology of smooth schemes coincides with higher Chow groups over any k by [16], but we will not use this.

5.B. *Proof of (i)*

Here we use Definition a) of §5.A and assume Y quasi-projective. Let $\vec{Z} = (\emptyset = Z_0 \subset \cdots \subset Z_d = Y)$ be an increasing chain of closed subsets, with $\dim Z_p \leq p$. The localisation theorem then yields a spectral sequence

$$E^1_{p,q}(\vec{Z}) = H^c_{p+q}(Z_p - Z_{p-1}, \mathbf{Z}(n)) \Rightarrow H^c_{p+q}(Y, \mathbf{Z}(n)).$$

Passing to the the colimit, we get a niveau spectral sequence

$$E^1_{p,q} = \bigoplus_{y \in Y_{(p)}} \varinjlim_{U_y} H^c_{p+q}(U_y, \mathbf{Z}(n)) \Rightarrow H^c_{p+q}(Y, \mathbf{Z}(n))$$

where the $\varinjlim_{U_y}$ are taken over the open neighbourhoods of y in $\overline{\{y\}}$, for $y \in Y$. Passing from homological to cohomological higher Chow groups, we may rewrite these colimits as:

$$\varinjlim_{U_y} H^c_{p+q}(U_y, \mathbf{Z}(n)) = \varinjlim_{U_y} H^{p-q}(U, \mathbf{Z}(p-n)) = H^{p-q}(k(y), \mathbf{Z}(p-n))$$

which gives the final form of the niveau spectral sequence.

In particular, $E^1_{p,q} = 0$ for $q < n$ and

$$E^1_{p,n} = \bigoplus_{y \in Y_{(p)}} K^M_{p-n}(k(y))$$

(see (2.3)). This shows that $H^c_n(Y, \mathbf{Z}(n))$ is canonically isomorphic to $A_0(Y, K^M_{-n})$.

5.C. *Proof of (ii)*

Here we assume that k admits resolution of singularities, but relax the quasi-projectiveness assumption. Then all the above may be performed using Definition b) of §5.A. Moreover, we get a map from motivic to Borel-Moore motivic homology, which yields the promised map of cycle modules $H^Y \to K^Y$. The fact that it gives back the isomorphism of Proposition 1.2 when Y is smooth projective follows from the proof of the latter.

5.D. *Proof of (iii)*

This follows from (i) and the special case $i = 0$ of [7, Cor. 9.1].

6. Complement: vanishing range of motivic homology

6.A. *A vanishing in any characteristic*

6.1 Proposition *Assume k perfect. For any smooth scheme X and any $n \in \mathbf{Z}$, one has*

$$H_i(X, \mathbf{Z}(n)) = 0 \text{ for } i < n.$$

If X is projective (and $n \geq 0$), $i < 2n$ is sufficient.
If U is an open subset of X, the map

$$H_n(U, \mathbf{Z}(n)) \to H_n(X, \mathbf{Z}(n))$$

is surjective for all $n \in \mathbf{Z}$. If the closed complement Z is of codimension ≥ 2, it is bijective. If Z is smooth and purely of codimension 1, we have exact sequences

$$H_{n-1}(Z, \mathbf{Z}(n-1)) \to H_n(U, \mathbf{Z}(n)) \to H_n(X, \mathbf{Z}(n)) \to 0.$$

Proof: We distinguish two cases: $n \geq 0$ and $n \leq 0$. If $n \geq 0$, we have in $\mathbf{DM}_{\mathrm{gm}}$ and $\mathbf{DM}_-^{\mathrm{eff}}$

$$\begin{aligned} H_i(X, \mathbf{Z}(n)) &= \mathrm{Hom}(\mathbf{Z}(n)[i], M(X)) \\ &= \mathrm{Hom}(\mathbf{Z}(n)[2n], M(X)[2n-i]) \subset H^{2n-i}_{\mathrm{Nis}}(\mathbf{P}^n, \underline{C}_*(X)) \end{aligned}$$

since $\mathbf{Z}(n)[2n]$ is a direct summand of $M(\mathbf{P}^n)$. Since the complex $\underline{C}_*(X)$ is concentrated in nonpositive degrees, $H^j_{\mathrm{Nis}}(\mathbf{P}^n, \underline{C}_*(X)) = 0$ for $j > \dim \mathbf{P}^n = n$ (by the known Nisnevich cohomological dimension), which yields the desired bound.

If $n \leq 0$, we have

$$\begin{aligned} H_i(X, \mathbf{Z}(n)) &= \mathrm{Hom}(\mathbf{Z}[i], M(X)(-n)) \\ &= \mathrm{Hom}(\mathbf{Z}[i-n], M(X)(-n)[-n]) \subset H^{n-i}(k, C_*(X \times \mathbb{G}_m^{-n})) \end{aligned}$$

since $\mathbf{Z}(-n)[-n]$ is a direct summand of $M(\mathbb{G}_m^{-n})$. Since $C_*(X \times \mathbb{G}_m^{-n})$ is concentrated in nonpositive degrees, this group vanishes for $n - i > 0$, as desired.

The surjectivity and exactness statements in the end of Proposition 6.1 are proven similarly, using Gysin exact sequences.

If X is projective of dimension d, we have by Poincaré duality

$$H_i(X, \mathbf{Z}(n)) \simeq H^{2d-i}(X, \mathbf{Z}(d-n)).$$

This group is known to vanish as soon as $2d - i > 2(d-n)$ (by the coniveau spectral sequence), i.e. for $i < 2n$. □

6.B. Application to birational invariance

6.2 Definition A rational map $f : X \dashrightarrow Y$ between two varieties is *coproper* if the projection $\bar{\Gamma}_f \to X$ is proper, where $\bar{\Gamma}_f \subset X \times Y$ is the graph of f. (If $U \subseteq X$ is a defining open subset for f, $\bar{\Gamma}_f$ is the closure of the graph of $f : U \to Y$ in $X \times Y$.)

6.3 Example Any morphism is coproper. If Y is proper, f is coproper. If $f : Y \to X$ is a birational morphism, the rational map $f^{-1} : X \dashrightarrow Y$ is coproper if and only if f is proper.

6.4 Lemma *Let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two composable rational maps. If f and g are coproper, so is $g \circ f$.*

Proof: Let $U_f \subseteq X$ be the domain of f and $U_g \subseteq Y$ be the domain of g: then g and f are composable if and only if $f(U_f) \cap U_g \neq \emptyset$. Then the domain of $g \circ f$ contains $U_{g \circ f} := f^{-1}(U_g)$.

Let $\Gamma_f \subset U_f \times Y$, $\Gamma_g \subset U_g \times Z$ and $\Gamma_{g \circ f} \subset U_{g \circ f} \times Z$ be the graphs of the respective morphisms, and let $\bar{\Gamma}_f \subset X \times Y$, $\bar{\Gamma}_g \subset Y \times Z$, $\bar{\Gamma}_{g \circ f} \subset X \times Z$ be their respective closures. We know that $\bar{\Gamma}_f \to X$ and $\bar{\Gamma}_g \to Y$ are proper, and we must deduce that $\bar{\Gamma}_{g \circ f} \to X$ is proper. The trick is to introduce the “double graph"

$$\Gamma_{g,f} = \{(x, f(x), g \circ f(x))\} \subset U_{g \circ f} \times U_g \times Z$$

and to compare its closure $\bar{\Gamma}_{g,f} \subset X \times Y \times Z$ with $\bar{\Gamma}_f$ and $\bar{\Gamma}_{g \circ f}$.

Since $\Gamma_{g,f} \subset U_{g \circ f} \times \Gamma_g$, we also have $\bar{\Gamma}_{g,f} \subset X \times \bar{\Gamma}_g$; moreover, since $p_{XY}(\Gamma_{f,g}) \subseteq \Gamma_f$, we have $p_{XY}(\bar{\Gamma}_{g,f}) \subset \bar{\Gamma}_f$. In the commutative diagram

$$\begin{tikzcd}
\Gamma_{g,f} \arrow[r, "j"] \arrow[d, "j'"'] & \bar{\Gamma}_{g,f} \arrow[r, "i"] \arrow[d, "p"'] & X \times \bar{\Gamma}_g \arrow[r] \arrow[d, "q"'] & X \times Y \times Z \arrow[d, "p_{XY}"'] \\
\Gamma_f \arrow[r, "j''"] & \bar{\Gamma}_f \arrow[r] \arrow[d, "\pi"'] & X \times Y \arrow[r, equal] & X \times Y \\
 & X & &
\end{tikzcd}$$

i is a closed immersion and q is (by hypothesis) proper, hence p is also proper. Moreover, j, j', j'' are open immersions, hence p is birational. Since (by hypothesis) π is also proper birational, we find that $\pi \circ p$ is proper birational. On the other hand, we have a similar commutative diagram:

$$\begin{tikzcd}
\Gamma_{g,f} \arrow[r, "j_1"] \arrow[d, "j_1'"'] & \bar{\Gamma}_{g,f} \arrow[r] \arrow[d, "p_1"'] & X \times Y \times Z \arrow[d, "p_{XZ}"'] \\
\Gamma_{g\circ f} \arrow[r, "j_1''"] & \bar{\Gamma}_{g\circ f} \arrow[r] \arrow[d, "\pi_1"'] & X \times Z. \\
& X &
\end{tikzcd}$$

Since $\pi_1 \circ p_1 = \pi \circ p$ is proper, so is p_1; but p_1 is birational, hence surjective, and therefore π_1 is also proper, as requested. □

6.5 Corollary *a) Let $f : X \dashrightarrow Y$ be a coproper rational map between varieties, with X smooth. Then f defines a map $f_* : H_n(X,\mathbf{Z}(n)) \to H_n(Y,\mathbf{Z}(n))$ for any $n \in \mathbf{Z}$.*
b) If Y is also smooth and $g : Y \dashrightarrow Z$ is another coproper rational map composable with f, then $(g \circ f)_* = g_* \circ f_*$.
c) If $f : X \to Y$ is a birational map with X,Y smooth and if f and f^{-1} are coproper, f_* is an isomorphism.
d) If $f : X \to Y$ is a proper birational morphism between two smooth varieties, then f_* is an isomorphism.
e) If $f : X \dashrightarrow Y$ is a birational map between two smooth proper varieties, then f_* is an isomorphism.

Proof: This is standard, cf. [9, Ch. II, Th. 8.19]:

a) By the valuative criterion of properness, f is defined on an open subset $U \subseteq X$ such that $X - U$ has codimension ≥ 2. The claim then follows from Proposition 6.1. b) is because in the proof of Lemma 6.4, $X - U_{g\circ f}$ is of codimension ≥ 2 in X. c) follows from b). Via Example 6.3, d) and e) are special cases of c). □

6.6 Remark Of course, the argument of Corollary 6.5 applies to any functor F on the category of smooth varieties such that $F(U) \xrightarrow{\sim} F(X)$ if X is smooth and $U \subseteq X$ is a dense open subset such that $X - U$ has codimension ≥ 2 in X.

6.C. A vanishing under resolution of singularities

6.7 Proposition (cf. [7, Prop. 8.2]) *Let X be smooth of dimension d. If* $\operatorname{car} k = 0$, *$H_i(X,\mathbf{Z}(n)) = 0$ for $n > d$ and is finitely generated for $n = d$. If* $\operatorname{car} k = p > 0$, *the same is true up to p-torsion.*

Proof: If X is projective, the vanishing is obvious from Poincaré duality (see end of proof of Proposition 6.1). We get from this case to the general case in a well-known manner by means of the Gysin exact sequences, from Hironaka's resolution of singularities in characteristic 0 and from Gabber's refinement of de Jong's theorem in characteristic p. □

(Is there a proof which avoids resolution?)

6.8 Proposition *Let X be a smooth scheme of dimension d. Assume that X is an open subscheme of a smooth projective $\bar{X}$, with complement D a normal crossing divisor. Let $D_1,\dots,D_r$ be the irreducible components of D; for $J\subseteq\{1,\dots,r\}$, write $D_J := \bigcap_{i\in J} D_i$ ($D_\emptyset := \bar{X}$). Then*

$$H_d(X,\mathbf{Z}(d)) \simeq \operatorname{Coker}\left(\bigoplus_{|J|=d-1} CH^0(D_J) \xrightarrow{\delta} \bigoplus_{|J|=d} CH^0(D_J)\right)$$

where δ is induced by the obvious restriction maps.

Proof: We shall do a slightly more general computation. Let $n\geq 0$. (Co)analogously to [8, 3.3], we have a spectral sequence of cohomological type:

$$E_1^{p,q} = \bigoplus_{|J|=p} H_q^{D_J}(\bar{X},\mathbf{Z}(n)) \Rightarrow H_{q-p}(X,\mathbf{Z}(n))$$

where J runs through the subsets of $\{1,\dots,r\}$, and $H_q^{D_J}(\bar{X},\mathbf{Z}(n))$ denotes motivic homology of $\bar{X}$ with supports in D_J. By purity, we may rewrite the E_1-terms as

$$E_1^{p,q} = \bigoplus_{|J|=p} H_{q-2p}(D_J,\mathbf{Z}(n-p)).$$

From Proposition 6.1, we find

$$E_1^{p,q} = 0 \text{ for } q<2n \text{ or } q-p<n.$$

Moreover, for $i>0$,

$$E_1^{n+i,2n+i} = \bigoplus_{|J|=n+i} H_{-i}(D_J,\mathbf{Z}(-i)) = \bigoplus_{|J|=n+i} CH_{-i}(D_J) = 0$$

so the only E_1-term contributing to $H_n(X,\mathbf{Z}(n))$ is

$$E_1^{n,2n} = \bigoplus_{|J|=n} CH_0(D_J) = \bigoplus_{|J|=n} CH^{d-n}(D_J)$$

and $H_n(X,\mathbf{Z}(n))$ is a quotient of this direct sum. The previous E_1-term is

$$E_1^{n-1,2n} = \bigoplus_{|J|=n-1} CH_1(D_J) = \bigoplus_{|J|=n-1} CH^{d-n}(D_J)$$

so $E_2^{n,2n}$ is the cokernel of the obvious restriction map

$$\bigoplus_{|J|=n-1} CH^{d-n}(D_J) \to \bigoplus_{|J|=n} CH^{d-n}(D_J).$$

For $n = d$, there is nothing above the row $q = 2d$ and $E_2^{d,2d} = E_\infty^{d,2d}$. For $n < d$, the term $E_2^{n-2,2n+1}$ is a subquotient of

$$E_1^{n-2,2n+1} = \bigoplus_{|J|=n-2} H_5(D_J,\mathbf{Z}(2)) = \bigoplus_{|J|=n-2} A^{d-n-1}(D_J, K_{d-n}^M)$$

corresponding to 3-cycles with coefficients in units. We stop this analysis here and encourage the interested reader to pursue it. □

6.9 Remark With the notation of Proposition 6.8, we also have exact sequences

$$\bigoplus_{i=1}^r H_{n-1}(D_i^o,\mathbf{Z}(n-1)) \xrightarrow{\delta} H_n(X,\mathbf{Z}(n)) \to H_n(\bar{X},\mathbf{Z}(n)) \to 0$$

for all $n \in \mathbf{Z}$, where $D_i^o = D_i \setminus \bigcup_{j\neq i} D_j$ and δ is induced by the Gysin maps. Note that the right hand side is 0 for $n < 0$. This may be obtained from the Gysin exact sequence relative to the open immersion $X \subset \bar{X} - \bigcup_{i\neq j} D_i \cap D_j$, with complement $\coprod_i D_i^o$ (by Proposition 6.1, $H_n(\bar{X} - \bigcup_{i\neq j} D_i \cap D_j,\mathbf{Z}(n)) \xrightarrow{\sim} H_n(\bar{X},\mathbf{Z}(n))$), or as the dual via Theorem 1.3 of the exact sequence

$$0 \to A^0(\bar{X}, M_n) \to A^0(X, M_n) \to \bigoplus_{i=1}^r A^0(D_i^o, M_{n-1})$$

valid for any cycle module M.

References

1. A. Beilinson, V. Vologodsky *A DG guide to Voevodsky's motives*, Geom. funct. anal. **17** (2007), 1709–1787.

2. S. Bloch, A. Ogus *Gerten's conjecture and the homology of schemes*, Ann. Sci. ÉNS **7** (1974), 181–201 (1975).

3. S. Bloch *The moving lemma for higher Chow groups*, J. Algebraic Geom. **3** (1994), 537–568.

4. D.-C. Cisinski, F. Déglise *Local and stable homological algebra in Grothendieck abelian categories*, Homology, Homotopy Appl. **11** (2009), 219–260.

5. F. Déglise *Transferts sur les groupes de Chow à coefficients*, Math. Zeit. **252** (2006), 315–343.

6. F. Déglise *Motifs génériques*, Rend. Mat. Sem. Univ. Padova **119** (2008), 173–244.

7. F. Déglise *Modules homotopiques*, preprint, 2009.

8. H. Esnault, B. Kahn, M. Levine, E. Viehweg *The Arason invariant and mod* 2 *algebraic cycles*, J. Amer. Math. Soc. **11** (1988), 73–118.

9. R. Hartshorne Algebraic geometry, Springer, 1977.

10. A. S. Merkurjev *Unramified elements in cycle modules*, J. London Math. Soc. **78** (2008), 51–64.

11. F. Morel *On the motivic π_0 of the sphere spectrum*, *in* Axiomatic, enriched and motivic homotopy theory (J.P.C Greenlees, ed.), NATO Sci. Series II. (Math. Phys. Chem.) **131**, Kluwer, 2004, 219–260.

12. M. Rost *Chow groups with coefficients*, Doc. Math. **1** (1996), 319–393.

13. E. Friedlander, V. Voevodsky *Bivariant cycle cohomology*, *in* Cycles, transfers and motivic cohomology theories, Ann. of Math. Studies **143**, Princeton, 2000, 138–187.

14. A. Suslin, V. Voevodsky *Bloch-Kato conjecture and motivic cohomology with finite coefficients*, *in* The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci. **548**, Kluwer, 2000.

15. V. Voevodsky *Triangulated categories of motives over a field*, *in* Cycles, transfers and motivic cohomology theories, Ann. of Math. Studies **143**, Princeton, 2000, 188–238.

16. V. Voevodsky *Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic*, Int. Math. Res. Not. **2002**, 351–355.

17. V. Voevodsky *Cancellation theorem*, Doc. Math. Extra Volume: Andrei A. Suslin's Sixtieth Birthday (2010) 671–685.

BRUNO KAHN `kahn@math.jussieu.fr`

Institut de Mathématiques de Jussieu
UMR 7586
Case 247
4 place Jussieu
75252 Paris Cedex 05
France

Received: March 29, 2010