On the generalised Hodge and Tate conjectures for products of elliptic curves

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Cohomological realizations of motives

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1. Review of the generalised conjectures

1.1. The Hodge and Tate conjectures. k field, X/k smooth projective variety, H^* Weil cohomology theory with coefficients in F: cycle class map

$$CH^n(X) \otimes F \xrightarrow{\mathrm{cl}^n} H^{2n}(X)(n)(n \geq 0).$$

Hodge conjecture (HC): $k = \mathbf{C}$, $H = H_B$ ($F = \mathbf{Q}$):

$$\operatorname{Im} \mathrm{cl}^n = \{(n, n)\text{-classes}\}.$$

Tate conjecture (TC): $k = \mathbf{F}_q$, $H = l$-adic cohomology ($F = \mathbf{Q}_l$):

$$\operatorname{Im} \mathrm{cl}^n = \{\text{Galois invariant classes}\}.$$

(We write $H_l^n(X) := H_{\text{ét}}^n(X, \mathbf{Q}_l)$ for l-adic cohomology, $l \nmid q$.)

For the generalised conjectures, need *coniveau filtration*:

$$\begin{aligned} N^r H^n(X) &= \bigcup_{\mathrm{codim}_X(Z)\geq r} \mathrm{Im}\left(H^n_Z(X) \to H^n(X)\right) \\ &= \bigcup_{\mathrm{codim}_X(Z)\geq r} \mathrm{Ker}\left(H^n(X) \to H^n(X - Z)\right). \end{aligned}$$

Remark 1. $n = r/2$: $N^r H^{2r}(X) = \mathrm{Im}\,\mathrm{cl}^r$ by semi-purity.

Theorem 2 (Deligne)**.**

$$N^r H_B^i(X) = \bigcup_{f:Y\to X} \mathrm{Im}\left(H_B^{i-2r}(Y)(-r) \xrightarrow{f_*} H_B^i(X) \right)$$

$$N^r H_l^i(X) = \bigcup_{f:Y\to X} \mathrm{Im}\left(H_l^{i-2r}(Y)(-r) \xrightarrow{f_*} H_l^i(X) \right)$$

where $f : Y \to X$ *runs through morphisms of smooth projective varieties such that* $\dim X - \dim Y = r$*.*

Non-trivial theorem! Uses mixed Hodge theory over $\mathbf{C}$, and Weil II over $\mathbf{F}_q$ (plus de Jong).

Variant with correspondences:

$$N^r H^i(X) = \bigcup_{\gamma \in \mathrm{Corr}^r(X,Y)} \mathrm{Im}\left(H^{i-2r}(Y)(-r) \xrightarrow{\gamma^*} H^i(X) \right)$$

$$\mathrm{Corr}^r(X, Y) = CH^{\dim Y - r}(X \times Y) \otimes F.$$

1.2. The generalised conjectures of Grothendieck.

Generalised Hodge conjecture (GHC): If $k = \mathbf{C}$, $N^r H^i_B(X)$ is the largest Hodge substructure of $H^i_B(X)$ which is effective of coniveau $\geq r$.

Generalised Tate conjecture (GTC): If $k = \mathbf{F}_q$, $N^r_{\bar{\mathbf{F}}_q} H^i_l(X)$ is the largest Galois submodule of $H^i_l(X)$ in which all eigenvalues of [the geometric] Frobenius are algebraic integers divisible by q^r.

(Need to take coniveau filtration over $\bar{\mathbf{F}}_q$, not over $\mathbf{F}_q$!)

In GHC, a pure Hodge structure V is *effective of coniveau* $\geq r$ if all its Hodge numbers (p, q) verify $p \geq r, q \geq r$. This is Grothendieck's corrected form of Hodge's general conjecture. The generalised Tate conjecture appears in [Brauer III, 10.3].

By Remark 1, GHC $\Rightarrow$ HC and GTC $\Rightarrow$ TC. "Essential surjectivity results" imply converse implications:

- Over $\mathbf{F}_q$, Honda's theorem implies (TC $\Rightarrow$ GTC).
- Over $\mathbf{C}$, theorem of Hazama-Abdulali implies (HC $\Rightarrow$ GHC) for X's such that $H_B^*(X)$ is purely of CM type.

Honda's theorem: : for any Weil number α, there exists an abelian $\mathbf{F}_q$-variety A such that α is an eigenvalue of Frobenius acting on $H^1_l(A)$.

Hazama-Abdulali theorem: for any effective (polarisable) Hodge structure H of weight n of CM type, there exists an abelian variety A of CM type such that H is a direct summand of $H^n_B(A)$.

(Serre proved this previously, but only up to a twist.)

Precise statements:

Theorem 3. $k = \mathbf{F}_q$: *assume Frobenius action on* $H_l^n(X)$ *is semi-simple (e.g.,* $X =$ *abelian variety). If TC holds in codimension* $n-r$ *for all products* $A \times X$*,* A *abelian variety, then GTC holds for* $N^r H_l^n(X)$*.*

Theorem 4. $k = \mathbf{C}$: *assume that the Hodge structure* $H_B^n(X)$ *is of CM type (e.g.,* $X =$ *CM abelian variety of CM type). If HC holds in codimension* $n-r$ *for all products* $A \times X$*,* A *abelian variety of CM type, then GHC holds for* $N^r H_B^n(X)$*.*

Theme of the talk: can we make Theorems 3 and 4 effective?

Idea: test on products of elliptic curves $X = \prod E_i$ because

- $k = \mathbf{C}$: HC is true for X by Tate-Imai-K. Murty.
- $k = \mathbf{F}_q$: TC is true for X by Spieß.

Principle: given $X = \prod E_i$, find out exactly what abelian varieties A show up in Theorems 3 and 4. If we get only products of elliptic curves, we win. If not, get new (and interesting) problem.

The point: this is very computable!

2. Elliptic curves in general position

Definition 5. $S = (E_1, \ldots, E_m)$ family of elliptic curves over a field F; $\bar{S}$ set of isogeny classes of S, and $\bar{S}_0 \subseteq \bar{S}$ subset consisting of

- CM isogeny classes if char $F = 0$;
- ordinary isogeny classes if char $F > 0$.

$K_1, \ldots, K_n$ the endomorphism fields of elements of $\bar{S}_0$ (quadratic imaginary). We say that S is *in general position* if the K_i are linearly disjoint over $\mathbf{Q}$.

Lemma 6. *If* $n \leq 3$ *in Definition 5, then* S *is in general position.*

Proof. Clear for $n \leq 2$. For $n = 3$, K_3 cannot lie in the biquadratic extension $L = K_1 K_2$, as the third quadratic subfield of L is real. $\square$

Theorem 7. *a) $E_1, \ldots, E_m$ elliptic curves in general position over $\mathbf{F}_q$. Then GTC holds for $X = \prod E_i$.*
b) $E_1, \ldots, E_m$ elliptic curves in general position over $\mathbf{C}$. Then GHC holds for $X = \prod E_i$.

b) proven by Abdulali in case all E_i are CM.

Corollary 8. *GTC (resp. GHC) holds for $N^1H^3(X)$ for any product X of elliptic curves.* $\square$

3. Four elliptic curves in special position

K_1, K_2, K_3 distinct imaginary quadratic fields, $K = K_1K_2K_3$: $[K : \mathbf{Q}] = 8$ and K contains exactly one other imaginary quadratic field K_0.

$K_i \leftrightarrow$ unique isogeny class E_i of CM elliptic curves. Reducing mod p yields 4 isogeny classes of ordinary elliptic curves over $\bar{\mathbf{F}}_p$ for any prime p. We say that $(E_0, \ldots, E_3)$ are *in special position*, with associated CM field K.

Let $B = \prod_{i=0}^{3} E_i$. By Corollary 8 and TC (*resp.* HC) for B, first open case of GTC or GHC is for $N^1H^4(B)$.

Theorem 9. *In the above situation, there exists an absolutely simple 4-dimensional abelian variety A, with complex multiplication by K, and a free $K \otimes F$-module $H \subset H^6(A^2 \times B)$ of rank 1 consisting of Hodge (resp. Tate) cycles, such that GHC (resp. GTC) holds for $N^1H^4(B)$ if and only if H consists of algebraic cycles.*

Moreover, HC (resp. TC) holds for A and all its powers.

4. Tannakian review

4.1. The Hodge realisation. $k = \mathbf{C}$: homological equivalence = numerical equivalence for abelian varieties (Lieberman). Hence thick subcategory $\mathcal{M}_{\mathrm{num}}^{\mathrm{ab}}$ of pure numerical motives $\mathcal{M}_{\mathrm{num}}$ generated by motives of abelian varieties is Tannakian.
String of Tannakian categories and $\otimes$-functors:

$$\langle \mathbb{L} \rangle \subset \mathbf{Lef} \subset \mathcal{M}_{\mathrm{num}}^{\mathrm{ab}} \to \mathbf{PHS}^* \to \mathbf{Vec}_{\mathbf{Q}}^*$$

$\mathbb{L}$ = Lefschetz motive, $\mathbf{Lef}$ = subcategory of correspondences defined by intersection products of divisor classes (Milne), $\mathbf{PHS}^*$ = graded polarisable Hodge structures, $\mathbf{Vec}_{\mathbf{Q}}^*$ = graded $\mathbf{Q}$-vector spaces.

Dually, string of Tannakian groups over $\mathbf{Q}$:

$$\mathbb{G}_m \xrightarrow{w} \mathrm{MT} \to G_{\mathrm{Mot}} \to L \xrightarrow{t} \mathbb{G}_m$$

w weight cocharacter, MT = Mumford-Tate group, G_{Mot} = motivic Galois group, L = Lefschetz group, t Tate character (composition $= -2$).

A abelian variety:

$$\begin{array}{ccccccc} \mathbb{G}_m \xrightarrow{w} \mathrm{MT}(A) \hookrightarrow G_{\mathrm{Mot}}(A) \hookrightarrow & L(A) & \hookrightarrow & GL(H^1_B(A)) \times \mathbb{G}_m \\ & & \overset{t}{\searrow} & \downarrow p_2 \\ & & & \mathbb{G}_m \end{array}$$

So: $(\mathrm{MT}(A) = L(A)) \iff \bigoplus_{n\geq 0} H_B^{2n}(A^i)^{(n,n)}$ generated in degree 1 for any $i > 0 \Rightarrow$ HC for all powers of A.

Milne: $A \simeq_{\mathbf{Q}} \prod A_i^{n_i}$ semi-simple decomposition of $A \Rightarrow (L(A), t) \simeq \prod_i (L(A_i), t)$ (fibre product over characters t).

Case of products of elliptic curves:

(1) E elliptic curve: $\mathrm{MT}(E) = L(E)$ (direct computation).
(2) $E_1, \ldots, E_n$ non-isogenous elliptic curves: $(\mathrm{MT}(\prod E_i), t) = \prod_i (\mathrm{MT}(E_i), t)$.
(3) X product of elliptic curves: $\mathrm{MT}(X) = L(X)$.

4.2. The Tate realisation. $k = \mathbf{F}_q$, $H = H_l$ $(l \nmid q)$. Here, "homological equivalence = numerical equivalence" is open for abelian varieties (except Clozel's theorem for certain l's). So, Milne replaces the "motivic Galois group" by an *ad hoc* defined group:

$$\begin{array}{ccccccccc} \mathbb{G}_m & \xrightarrow{w} & P(A) & \hookrightarrow & M(A) & \hookrightarrow & L(A) & \hookrightarrow & \mathrm{End}^0(A)^* \times \mathbb{G}_m \\ & & & & & & & \searrow^{t} & \downarrow p_2 \\ & & & & & & & & \mathbb{G}_m \end{array}$$

- $L(A)(\mathbf{Q}) = \{\alpha \in C(A)^* \mid \alpha\alpha^\dagger \in \mathbf{Q}^*\}$: $C(A)$ centre of $\mathrm{End}^0(A)$, $\dagger$ restriction of any Rosati involution to $C(A)$;
- $M(A) = \{\alpha \in L(A) \mid \alpha$ acts trivially on cycles modulo numerical equivalence$\}$.
- $P(A) =$ Zariski closure of π_A, the Frobenius endomorphism of A.

Case of a product X of elliptic curves: same as above ($P(X) = L(X)$) thanks to Spieß's theorem:

Theorem 10. *Let $n \geq 1$ and $\beta_1, \dots, \beta_{2n}$ Weil numbers of X such that $\beta_1 \dots \beta_{2n} = q^n$. Then, up to a permutation of $\{1, \dots, 2n\}$, we have $\beta_{2i-1}\beta_{2i} = \zeta_i q$ for $i = 1, \dots, n$, ζ_i roots of unity.*

(Exercise: prove this along the same lines as over $\mathbf{C}$.)

5. Proof of Theorem 7 (sketch)

$K_1, \dots, K_n$ the imaginary quadratic fields corresponding to the ordinary/CM isogeny classes of the E_i; $K = K_1 \dots K_n$ compositum of the K_i: $G = Gal(K/\mathbf{Q}) \simeq (\mathbf{Z}/2)^n$, with basis of characters $(\chi_1, \dots, \chi_n)$ $(\chi_i \leftrightarrow K_i)$. $(\sigma_1, \dots, \sigma_n)$ dual basis of G; $c = \sigma_1 \dots \sigma_n$ (complex conjugation). Set $H_i = \operatorname{Ker} \chi_i$.

The main lemma:

Lemma 11. *For any* $(\varepsilon_1, \dots, \varepsilon_n) \in (\mathbf{Z}/2)^n$,

$$\bigcup_{i=1}^{n} c^{\varepsilon_i} H_i \neq G.$$

Proof. Clear if all ε_i are 0 as c does not belong to the LHS. General case: up to permutation, may assume $\varepsilon_1 = \cdots = \varepsilon_r = 0$ and $\varepsilon_{r+1} = \cdots = \varepsilon_n = 1$. As we just saw, $g = \sigma_1 \dots \sigma_r \notin H_1 \cup \cdots \cup H_r$. But $g \notin cH_{r+1} \cup \cdots \cup cH_n$, since $gc^{-1} = \sigma_{r+1} \dots \sigma_n$. □

M simple direct summand of $H^*(X)$ ($H = H_l$ or H_B). May view M as a simple representation of $P(X)$ or $\mathrm{MT}(X)$.

Over $\mathbf{F}_q$:

Lemma 12. *$\beta_1, \ldots, \beta_m$ (some) Weil numbers attached to X.*
a) If all β_i are ordinary and no two of them are conjugate up to a root of unity, then the ideal $(\beta_1 \ldots \beta_m) \subset O_K$ is not divisible by (p).
b) In general, suppose

$$(\beta_1 \ldots \beta_m) = (q\beta)$$

β some algebraic integer. Then $\exists\, i \neq j$ such that

$$(\beta_i \beta_j) = (q).$$

Sketch of proof:
$p \mid q$ is totally decomposed in K. Pick a prime divisor $\mathfrak{p}$ of p in O_K, and let $\mathfrak{p}_i = \mathfrak{p} \cap K_i$. $\forall i\, \exists!\, \alpha_i \in K_i$ (Weil number) such that $\alpha_i O_{K_i} = \mathfrak{p}_i^r$; then

$$(\alpha_i) := \alpha_i O_K = \mathfrak{p}^{rN_i}$$

with

$$N_i = \sum_{g \in H_i} g \in \mathbf{Z}[G].$$

For a), by assumption, may write

$$\beta_1 \dots \beta_m = \alpha_1^{m_1 c^{\varepsilon_1}} \dots \alpha_n^{m_n c^{\varepsilon_n}}$$

for some $\varepsilon_i \in \mathbf{Z}/2$ and some integers $m_i \geq 0$. Thus

$$(\beta_1 \dots \beta_m) = \mathfrak{p}^{r(m_1 c^{\varepsilon_1} N_1 + \dots + m_n c^{\varepsilon_n} N_n)}.$$

By Lemma 11, the inequality

$$N \leq r(m_1 c^{\varepsilon_1} N_1 + \dots + m_n c^{\varepsilon_n} N_n), \quad N := \sum_{g \in G} g$$

is false in $\mathbf{N}[G]$ (for the partial ordering given componentwise). Since $(p) = \mathfrak{p}^N$, this concludes.
For b), need to handle supersingular Weil numbers, which is not hard.

Lemma 12 implies: M is (up to a twist) a direct summand of $H_l^*(Y)$ with $Y = \prod_{i \in J} E_i$, $J \subseteq \{1, \ldots, m\}$, hence can apply TC to $Y \times X$. This proves Theorem 7.

Over C: $\mathrm{MT}(X) = \prod_{s\in\bar{S}}(\mathrm{MT}(E_s), t) = \prod_{s\in\bar{S}_0}(\mathrm{MT}(E_s), t) \times_{\mathbb{G}_m} \prod_{s\in\bar{S}-\bar{S}_0}(\mathrm{MT}(E_s), t)$.

- $s \in \bar{S}_0 \leftrightarrow K_i$: $\mathrm{MT}(E_s) = R_{K_i/\mathbf{Q}}\mathbb{G}_m$.
- $s \notin \bar{S}_0$: $\mathrm{MT}(E_s) \simeq GL_2$.

Have $M \otimes \bar{\mathbf{Q}} = \bigoplus_\alpha W^\alpha$, W^α absolutely simple, permuted by $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ and

$$W^\alpha = \bigotimes_{s \in \bar{S}} W_s^\alpha$$

W_s^α absolutely simple representation of $\mathrm{MT}(E_s)$,

If $s \notin \bar{S}_0$: W_s^α of the form $\mathrm{Sym}^a(M_s) \otimes \det(M_s)^b$, $M_s = H^1(E_s)$: defined over $\mathbf{Q}$. Therefore $W_s^\alpha = W_s$ independent of α and

$$M = M_1 \otimes M_2$$

with

- M_1 simple representation of $\prod_{s \in \bar{S}_0} (\mathrm{MT}(E_s), t)$,
- $M_2 = \otimes_{s \notin \bar{S}_0} W_s$.

Moreover $\det(M_s) = \mathbf{Q}(-1)$ and coniveau of $W_s = b$ (because $(0, a)$ is a Hodge number of $\mathrm{Sym}^a(M_s)$). And $\mathrm{Sym}^a(M_s)$ direct summand of $H^a_B(E^a_s)$.

Hence reduced to handle M_1:

- $E = \mathrm{End}(M_1) =$ CM subfield of K and $\dim_E M_1 = 1$;
- $M_1 \leftrightarrow \varphi : \Sigma_E \to \mathbf{Z}$ such that $\varphi(x) + \varphi(cx) = n$ ($\Sigma_E = \mathrm{Hom}_{\mathbf{Q}}(E, \bar{\mathbf{Q}})$, $n =$ weight of M_1).

Lift φ to Σ_K and conclude by similar use of Lemma 11 as over $\mathbf{F}_q$.

(*Question*: give uniform group-theoretic proof over $\mathbf{C}$ and $\mathbf{F}_q$.)

6. Proof of Theorem 9 (sketch)

Will only describe algebraic situation in $G = Gal(K/\mathbf{Q})$. Same notation as before:

- (χ_1, χ_2, χ_3) basis of $X(G)$ corresponding to K_1, K_2, K_3.
- $(\sigma_1, \sigma_2, \sigma_3)$ dual basis of G; $c = \sigma_1\sigma_2\sigma_3$ (complex conjugation).
- K_0 4-th quadratic imaginary field $\leftrightarrow \chi_0 = \chi_1\chi_2\chi_3$.
- $H_i = \operatorname{Ker} \chi_i$.
- $N_i = \sum_{g \in H_i} g$, $N = \sum_{g \in G} g$.

Definition 13. A *CM type* of (G, c) is a section of the projection $G \to G/\langle c \rangle$.

CM-types $\leftrightarrow$ elements $x \in \mathbf{N}[G]$ such that $(1 + c)x = N$.

Lemma 14. *Up to multiplication by an element of G, the distinct CM types of (G, c) are given by N_i ($i = 0, \ldots, 3$) and $\rho = 1 + \sigma_1 + \sigma_2 + \sigma_3$.*

(Proof: combinatorial computations.)

Then ρ defines the abelian variety A of Theorem 9.

Relation

$$N_1 + N_2 + N_3 + cN_0 = 2\rho + N$$

$\Rightarrow$ relation between Weil numbers:

(1) $$\alpha_1\alpha_2\alpha_3\alpha_0^c = \zeta q \beta^2$$

$\alpha_i \leftrightarrow E_i$, $\beta \leftrightarrow A$, ζ root of unity.

(So Lemma 11 is false in this case!) But also:

$$c\rho + \sigma_1\rho + \sigma_2\rho + \sigma_3\rho = 2N_0 + N$$

hence

(2) $$\beta^{c+\sigma_1+\sigma_2+\sigma_3} = \zeta' q \alpha_0^2.$$

Similar "mirror relations" found by Mestre for two 4-dimensional simple abelian varieties over $\mathbf{F}_2$ (with different and non-Galois fields of endomorphisms!)

To get a "new" Hodge (or Tate) class: (1) and Hodge analogue shows that $\Psi^2 H^1(A) \subset H^2(A^2)$ is direct summand of $H^4(B)(1)$ (recall: $B = \prod_{i=0}^{3} E_i$).

(Since A is CM/ordinary, think of $H^1(A)$ as representation of a torus: then $\Psi^2 H^1(A)$ makes sense as a representation. Over $\mathbf{C}$, even makes sense as numerical motive!)

Symmetrically, $\Psi^2 H^1(B)$ direct summand of $H^4(A)(1)$ by (2) (and Hodge analogue).

Mumford-Tate groups:

$$\begin{array}{ccc} \mathrm{MT}(A \times B) & \xrightarrow{p_2} & \mathrm{MT}(B) \\ {\scriptstyle p_1}\big\downarrow & & \\ \mathrm{MT}(A) & & \end{array}$$

p_1, p_2 isogenies of degree 2! (computation within $\mathbf{Z}[G]$).

$\Rightarrow \operatorname{rk} \mathrm{MT}(A) = \operatorname{rk} \mathrm{MT}(B) = 5$. Easy: $\operatorname{rk} L(A) = 5$; hence $\mathrm{MT}(A) = L(A)$ and HC holds for A and its powers. Similarly, $P(A) = L(A)$ over $\mathbf{F}_q$.

On the other hand, $L(A \times B) = L(A) \times_{\mathbb{G}_m} L(B)$, much bigger than $\mathrm{MT}(A \times B)$...