Zeta and L-functions of triangulated motives

Bruno Kahn

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Aim of talk: associate to a Voevodsky motive M over a global field K a Dirichlet series

$$L^{\text{near}}(M,s) = \prod_{\mathfrak{p}\text{ finite}} L^{\text{near}}_{\mathfrak{p}}(M,s)$$

with

- $L^{\text{near}}(M,s)$ is absolutely convergent for $\Re(s) \gg 0$ (explicit).
- $L^{\text{near}}_{\mathfrak{p}}(M,s) \in \mathbf{Q}(N(\mathfrak{p})^{-s})$.
- If $M' \to M \to M'' \to M[1]$ exact triangle in $DM_{\text{gm}}(K,\mathbf{Q})$,

$$L^{\text{near}}_{\mathfrak{p}}(M,s) = L^{\text{near}}_{\mathfrak{p}}(M',s)L^{\text{near}}_{\mathfrak{p}}(M'',s).$$

- If $M = M(X)^*$, X smooth projective,

$$L^{\text{near}}_{\mathfrak{p}}(M,s) = \zeta(\mathcal{X}_{\mathfrak{p}},s) \text{ if } X \text{ has good reduction at } \mathfrak{p}$$

  ($\mathcal{X}_{\mathfrak{p}}$ special fibre of a good model of X at $\mathfrak{p}$).
- If $K = \mathbf{F}_q(C)$: $L(M,s) \in \mathbf{Q}(q^{-s})$; functional equation.

Remark 0.1. X/K smooth projective, $i \geq 0$: Serre's L-function

$$L^{\mathrm{Serre}}(H^i(X), s)) = \prod_{\mathfrak{p}} L^{\mathrm{Serre}}_{\mathfrak{p}}(H^i(X), s),$$

$$L^{\mathrm{Serre}}_{\mathfrak{p}}(H^i(X), s) = \det(1 - \varphi_{\mathfrak{p}} N(\mathfrak{p})^{-s} \mid H^i(\bar{X}, \mathbf{Q}_l)^{I_{\mathfrak{p}}})^{-1}$$

$I_{\mathfrak{p}}$ inertia at $\mathfrak{p}$ (well-defined modulo weight-monodromy conjecture). So

$$L^{\mathrm{near}}_{\mathfrak{p}}(M(X)^*, s) = \prod_{i=0}^{2d} L^{\mathrm{Serre}}_{\mathfrak{p}}(H^i(X), s)^{(-1)^i}$$

if X has good reduction at $\mathfrak{p}$.
But cannot expect Serre's L-function extends to Euler-Poincaré characteristic on $DM_{\mathrm{gm}}(K)$, because of the invariants under inertia. So, $L^{\mathrm{near}} =$ "best triangulated approximation" of Serre's L-function.

Since L^{near} differs from L^{Serre} only at finitely many Euler factors, maybe one can use it to study the Beilinson conjectures.

1. Crash-review of Voevodsky's motives

1.1. Grothendieck's pure motives.

$\sim$ adequate equivalence relation on algebraic cycles

$$\mathrm{Sm}^{\mathrm{proj}}(K) \to \mathrm{Cor}_{\sim}(K, \mathbf{Q}) \xrightarrow{\natural} \mathcal{M}_{\sim}^{\mathrm{eff}}(K, \mathbf{Q}) \xrightarrow{\mathbb{L}^{-1}} \mathcal{M}_{\sim}(K, \mathbf{Q})$$

($\natural$ pseudo-abelian completion, $\mathbb{L}^{-1}$: inverting the Lefschetz motive).
$\mathcal{M}_{\sim}(K, \mathbf{Q})$ rigid $\mathbf{Q}$-linear $\otimes$-category.

1.2. Voevodsky's triangulated motives over a field.

$$\begin{aligned} &\mathrm{Sm}(K) \to \mathrm{SmCor}(K) \to K^b(\mathrm{SmCor}(K), \mathbf{Q}) \\ &\quad \to K^b(\mathrm{SmCor}(K), \mathbf{Q})/\langle HI + MV \rangle \\ &\quad \xrightarrow{\natural} DM_{\mathrm{gm}}^{\mathrm{eff}}(K, \mathbf{Q}) \xrightarrow{\mathbf{Z}(-1)} DM_{\mathrm{gm}}(K, \mathbf{Q}). \end{aligned}$$

$DM_{\mathrm{gm}}(K, \mathbf{Q})$ rigid $\mathbf{Q}$-linear $\otimes$-triangulated category.

1.3. Relationship.

$$\mathcal{M}_{\text{rat}}(K, \mathbf{Q}) \to DM_{\text{gm}}(K, \mathbf{Q})$$

fully faithful $\otimes$-functor.
Voevodsky's construction extends over a base.

1.4. 6 operations.
Need a 2-functor

$$\mathbb{D} : \{\mathbf{Z} - \text{schemes ess. of finite type}\}^{op} \to \{\text{triangulated categories}\}$$

with

(i) S regular: $\mathbb{D}(S) = DM_{\text{gm}}(S, \mathbf{Q})$.
(ii) A theory of six operations $f^*, f_*, f^!, f_!, \otimes, \underline{\text{Hom}}$.
(iii) l-adic realisations: l prime invertible on S, $D^b_c(S, \mathbf{Q}_l)$ Ekedahl's triangulated category of l-adic sheaves: covariant functor

$$R^l : \mathbb{D}(S) \to D^b_c(S, \mathbf{Q}_l)$$

commuting with the 6 operations.

By Voevodsky and Ayoub, given a 2-functor $\mathbb{D}$, to have 6 operations one only needs a few of them plus certain axioms, esp. glueing complementary closed/open subsets.

Ayoub proves that $\mathbb{D}(S) = DA_c^{\text{ét}}(S, \mathbf{Q})$ (defined using étale sheaves without transfers) verifies (ii), resp. (iii). Same as Cisinski-Déglise's Beilinson motives.

For $S \mapsto DM_{\text{gm}}(S, \mathbf{Q})$, axioms are not easy. Done by Cisinski-Déglise for S normal, but not in general. They also prove that $\mathbb{D} = DA_c^{\text{ét}}$ verifies (i), even for normal schemes. (More direct variant of this proof by Ayoub.)

In sequel I take $\mathbb{D}(S) = DA_c^{\text{ét}}(S, \mathbf{Q})$, so have (i), (ii) and (iii).

2. Zeta functions

2.1. Traces in rigid categories. $\mathcal{M}$ rigid F-linear $\otimes$-category (char $F = 0$). We assume $\operatorname{End}_{\mathcal{M}}(\mathbf{1}) = F$.

$M \in \mathcal{M}$, $f \in \operatorname{End}(M)$: recall the *trace of* f:

$$\mathbf{1} \xrightarrow{\eta} M^* \otimes M \xrightarrow{1 \otimes f} M^* \otimes M \xrightarrow{\sigma} M \otimes M^* \xrightarrow{\epsilon} \mathbf{1}$$

$\operatorname{tr}(f) \in \operatorname{End}_{\mathcal{M}}(\mathbf{1}) = F$.

Lemma 2.1 (The trace formula)**.** *$\mathcal{N}$ rigid E-linear $\otimes$-category ($E \supseteq F$), $R : \mathcal{M} \to \mathcal{N}$ K-linear $\otimes$-functor. Then $\forall M \in \mathcal{M}$, $\forall f \in \operatorname{End}(M)$:*

$$\operatorname{tr}(R(f)) = R(\operatorname{tr}(f)) \quad (= \operatorname{tr}(f), \text{ computed in } E).$$

Proof. Trivial. □

2.2. The zeta function. $M \in \mathcal{M}$, $f \in \mathrm{End}(M)$.

Definition 2.2.

$$Z(M,f,t) = \exp(\sum_{n\geq 1} \mathrm{tr}(f^n)\frac{t^n}{n}) \in F[[t]].$$

Theorem 2.3 (K.). *$\mathcal{M}$ abelian semi-simple "of homological origin":*

(i) $Z(M,f,t) \in F(t)$*;* $\deg(Z(M,f,t) = \chi(M) := \mathrm{tr}(1_M)$.

(ii) *If f invertible,* functional equation

$$Z(M^*, {}^t f^{-1}, t^{-1}) = (-t)^{\chi(M)} \det(f) Z(M,f,t)$$

where $\det(f) =$ *value at $t = \infty$ of* $(-t)^{\chi(M)} Z(M,f,t)^{-1}$.

(Other formula for det: $\det(1 - ft) = Z(M,f,t)^{-1} \in F(t)$.)

2.3. Example: numerical motives. Here $F = \mathbf{Q}$, $\mathcal{M} = \mathcal{M}_{\mathrm{num}}(k, \mathbf{Q})$, k a field.

Theorem 2.4 (Jannsen)**.** *$\mathcal{M}$ is abelian semi-simple.*

Moreover $\mathcal{M}$ is of "homological origin" thanks to homological equivalence, so Theorem 2.3 applies.

Example 2.5. $k = \mathbf{F}_q$: every $M \in \mathcal{M}$ has its *Frobenius endomorphism* F_M and

$$Z(h(X), F_{h(X)}, t) = Z(X, t)$$

if X smooth projective.

2.4. Voevodsky's motives over a finite field. k field: the triangulated $\otimes$-category $\mathbb{D}(k) = DM_{\mathrm{gm}}(k, \mathbf{Q})$ is rigid by de Jong's theorem ($\Leftarrow$ it is generated by the $M(X)$, X smooth projective). So $Z(M, f, t)$ makes sense here.

If $k = \mathbf{F}_q$, every $M \in \mathbb{D}(k)$ has its Frobenius endomorphism F_M.

Lemma 2.6. *E/k finite extension of degree n, $f : \operatorname{Spec} E \to \operatorname{Spec} k$.*
a) For $M \in \mathbb{D}(k)$, $F_{f^*M} = f^* F_M^n$.
b) For $M \in \mathbb{D}(E)$, $F_{f_*M} = f_* F_M$.
c) For $M \in \mathbb{D}(E)$:

$$\operatorname{tr}(f_* F_M^m) = \begin{cases} 0 & \text{if } n \nmid m \\ \operatorname{tr}(F_M^{m/n}) & \text{if } n \mid m. \end{cases}$$

For c), idea to avoid l-adic realisation (Ayoub). Do as for induced representations: $\operatorname{tr}(f_* F_M^m) = \operatorname{tr}(f^* f_* F_M^m)$ because f^* monoidal. Write $f^* f_* M = \bigoplus_{\sigma \in G} \sigma^* M$, $G = Gal(E/k)$. Then $f^* f_* F_M$ permutes the $\sigma^* M$ in the obvious way. □

Awkward problem: would like to define

$$Z(M,t) = Z(M, F_M, t)$$

but this causes compatibility problems with l-adic realisation (philosophy: $S \mapsto DM_{\mathrm{gm}}(S, \mathbf{Q})$ is a "homology theory" but to compute L-functions you use cohomology with compact supports).
Solution: slightly artificial definition of zeta function.

Definition 2.7. For $M \in DM_{\mathrm{gm}}(\mathbf{F}_q, \mathbf{Q})$:

$$\begin{aligned} Z(M,t) &= Z(M^*, F_{M^*}, t) = Z(M, F_M^{-1}, t) \\ \zeta(M,s) &= Z(M, q^{-s}). \end{aligned}$$

Theorem 2.8. *a)* $M' \to M \to M'' \to M'[1]$ *exact triangle:*

$$\zeta(M,s) = \zeta(M',s)\zeta(M'',s).$$

b) $\zeta(M,s) \in \mathbf{Q}(q^{-s})$, *degree* $\chi(M)$.
c) Functional equation

$$\zeta(M^*,-s) = (-q^{-s})^{\chi(M)} \det(F_M)^{-1}\zeta(M,s).$$

d) Identities

$$\zeta(M[1],s) = \zeta(M,s)^{-1}, \qquad \zeta(M(1),s) = \zeta(M,s-1).$$

e) $f : X \to \mathbf{F}_q$ *scheme of finite type:*

$$\zeta(f_!\mathbf{Z},s) = \zeta(X,s).$$

(In e), $f_! : \mathbb{D}(X) \to \mathbb{D}(\mathbf{F}_q) = DM_{\mathrm{gm}}(\mathbf{F}_q,\mathbf{Q})$. It is for this formula that I take the weird definition of $\zeta(M,s)$.)

Sketch of proofs. a) uses theorem of J. Peter May on additivity of traces: $\mathcal{T}$ rigid $\otimes$-triangulated category [coming from a model structure], $M' \to M \to M'' \xrightarrow{+1}$ exact triangle in $\mathcal{T}$. Any commutative diagram

$$\begin{array}{ccccccc} M' & \to & M & \to & M'' & \to & M'[1] \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & & & \downarrow{\scriptstyle f'[1]} \\ M' & \to & M & \to & M'' & \to & M'[1] \end{array}$$

may be completed into

$$\begin{array}{ccccccc} M' & \to & M & \to & M'' & \to & M'[1] \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'[1]} \\ M' & \to & M & \to & M'' & \to & M'[1] \end{array}$$

so that

$$\operatorname{tr}(f) = \operatorname{tr}(f') + \operatorname{tr}(f'').$$

Want to apply this with $\mathcal{T} = DM_{\mathrm{gm}}(\mathbf{F}_q, \mathbf{Q})$, $f' = F_{M'}^{-1}$, $f = F_M^{-1}$. Would like $f'' = F_{M''}^{-1}$. Given May's f'',

$$(f'' - F_{M''}^{-1})^2 = 0.$$

Is the trace of nilpotent endomorphisms 0? Yes, thanks to the l-adic realisation.

b) (rationality) and c) (functional equation): commutative diagram

$$\begin{array}{ccc} K_0(\mathcal{M}_{\mathrm{rat}}(\mathbf{F}_q, \mathbf{Q})) & \xrightarrow{\Phi} & K_0(DM_{\mathrm{gm}}(\mathbf{F}_q, \mathbf{Q})) \\ \downarrow & \searrow & \downarrow \\ K_0(\mathcal{M}_{\mathrm{num}}(\mathbf{F}_q, \mathbf{Q})) & \longrightarrow & \{\text{zeta functions}\} \end{array}$$

Φ bijective by Bondarko (relying on de Jong), so reduce to pure numerical motives.

d) (shift and twist): trivial.

e) (classical zeta function): $f : X \to \operatorname{Spec} \mathbf{F}_q$, $g : Z \to \operatorname{Spec} \mathbf{F}_q$ closed subscheme, $h : U \to \operatorname{Spec} \mathbf{F}_q$ open complement; exact triangle

$$h_! \mathbf{Z} \to f_! \mathbf{Z} \to g_! \mathbf{Z} \xrightarrow{+1}$$

If we had resolution of singularities, we could reduce to X smooth projective and then use $\mathcal{M}_{\mathrm{rat}}(\mathbf{F}_q, \mathbf{Q})$. (This works if $\dim X \leq 2$). de Jong's theorem not quite sufficient (see next slide). So, need to use the l-adic realisation and the Grothendieck-Verdier trace formula.

To avoid l-adic realisation, can almost use twisting lemma (learned from A. Pacheco): if $\pi : U \to V$ Galois étale covering of degree m, $G = Gal(\pi)$

$$\frac{1}{m}\sum_{\sigma\in G}|U^{(\sigma)}(\mathbf{F}_q)| = |V(\mathbf{F}_q)|$$

$$\frac{1}{m}\sum_{\sigma\in G}\sharp(M^c(U^{(\sigma)})) = \sharp(M^c(V))$$

$U^{(\sigma)}$ twist of U (viewed as G-torsor over U by 1-cocycle of $Z^1(\mathbf{F}_q, G)$ sending φ to σ).

de Jong's equivariant alteration theorem not quite sufficient to conclude.

2.5. Zeta functions of motives over a base. S = $\mathbf{Z}$-scheme of finite type.

Definition 2.9. $M \in \mathbb{D}(S)$:

$$\zeta(M, s) = \prod_{x \in S_{(0)}} \zeta(i_x^* M, s)$$

$S_{(0)}$ = set of closed points of S.

Theorem 2.10. *a) This defines a Dirichlet series, absolutely convergent for $\Re(s) \gg 0$.*
b) If $f : S \to T$ is a morphism,

$$\zeta(M, s) = \zeta(f_! M, s).$$

c) If $T = \operatorname{Spec} \mathbf{F}_q$ in b), $\zeta(M, s) \in \mathbf{Q}(q^{-s})$.
d) If S smooth projective of dimension d in c), functional equation

$$\zeta(M^*, d - s) = (-q^{-s})^{\chi(f_! M)} \det(F_{f_! M})^{-1} \zeta(M, s)$$

with $M^* := \underline{\operatorname{Hom}}(M, \mathbf{Z})$.

Sketch of proof. 2 steps:

1) Prove b) via the l-adic realisation (but almost have a proof purely using $\mathbb{D}$). c) and d) follow from Theorem 2.8 c) and the 6 functors formalism.

2) If $S \to \operatorname{Spec} \mathbf{Z}$ is not dominant, done. If dominant, 1) reduces us to $S = \operatorname{Spec} \mathbf{Z}$, crucial case.
$f : X \to \operatorname{Spec} \mathbf{Z}$ smooth scheme of finite type: $\zeta(f_! \mathbf{Z}, s) = \zeta(X, s)$ and Serre proved (elementarily) absolute convergence for $\Re(s) > \dim X$. Since the $f_! \mathbf{Z}$ "generate" $\mathbb{D}(\mathbf{Z})$, should suffice. But they generate only up to idempotents (the devil is in the idempotents).
Thus need a more sophisticated and expensive argument: uses l-adic realisation, Bondarko's isomorphism, Weil conjecture (Riemann hypothesis) + Deligne's generic constructibility theorem (SGA 4 1/2, th. finitude). □

2.6. A theorem of Serre. (Lectures on $N_X(p)$).
K number field: for $M \in \mathbb{D}(O_K)$ and $\mathfrak{p} \subset O_K$, define

$$N_M(\mathfrak{p}) = \mathrm{tr}(F_{M^*_\mathfrak{p}})$$

the number of points of M modulo $\mathfrak{p}$.

Theorem 2.11. *Let $M \in \mathbb{D}(O_K)$. Suppose that $\zeta(M, s)$ is not a finite product of Euler factors. Then the set*

$$\{\mathfrak{p} \mid N_M(\mathfrak{p}) = 0\}$$

has a density ϵ, with

$$\epsilon \leq 1 - \frac{1}{b_\infty(M)^2}$$

where $b_\infty(M) = \sum_i \dim H^i_l(M_K)$.

Proof Same as Serre's. $H_l(M) \in D_c^b(O_K[1/l], \mathbf{Q}_l)$ l-adic realisation of M. By Deligne's generic base change theorem, $\exists$ open subset $U \subseteq \operatorname{Spec} O_K[1/l]$ such that $H_l(M)_{|U}$ commutes with any base change. In particular, may compute

$$\operatorname{tr}(F_{M_{\mathfrak{p}}^*} \mid H_l^*(M_{\mathfrak{p}})), \mathfrak{p} \in U$$

as traces of [conjugacy class of] arithmetic Frobenius $\varphi_{\mathfrak{p}} \in Gal(\bar{K}/K)$ acting on $H_l^*(M_K)$. Statement then reduces to

Theorem 2.12 (Serre)**.** *G compact group, K locally compact field of characteristic 0, $\rho : G \to GL_n(K)$, $\rho' : G \to GL_{n'}(K)$ two continous K-linear representations of G. Then*

(i) *either* $\mathrm{tr}_\rho = \mathrm{tr}_{\rho'}$*;*

(ii) *or the set* $\{g \in G \mid \mathrm{tr}_\rho(g) \neq \mathrm{tr}_{\rho'}(g)\}$ *has a Haar density* $\geq \frac{1}{(n+n')^2}$.

3. L-functions over global fields

3.1. Motives with good reduction.

Definition 3.1. $S/\mathbf{Z}$ essentially of finite type:

$$\mathbb{D}^{\mathrm{proj}}(S) = \langle f_! \mathbf{Z} \mid f : X \to S \text{ smooth projective} \rangle.$$

Example 3.2. $S = \operatorname{Spec} k$: $\mathbb{D}^{\mathrm{proj}}(k) = \mathbb{D}(k)$ (by de Jong).

Definition 3.3. S a trait (spectrum of a dvr), $j : \eta \hookrightarrow S$ generic point: $M \in \mathbb{D}(\eta)$ *has good reduction* if $M \in \text{ess-im}(\mathbb{D}^{\mathrm{proj}}(S) \xrightarrow{j^*} \mathbb{D}(\eta))$.

Lemma 3.4. *$i : x \to S$ immersion of the closed point, $M \in \mathbb{D}(S)$.*
a) $\exists$ natural transformation

$$u_M : i^* M(-1)[-2] \to i^! M.$$

b) If $M \in \mathbb{D}^{\mathrm{proj}}(S)$, u_M isomorphism.

(Proof of a) uses 6 operations. Proof of b) uses "absolute purity" theorem of Cisinski-Déglise, relying on Quillen's localisation theorem for algebraic K-theory.)

3.2. The total L-function. K global field, $C_K = \operatorname{Spec} O_K$, O_K ring of integers (in char. 0), or smooth projective model (in char. p), $j : \operatorname{Spec} K \to C_K$ inclusion of the generic point.
$M \in \mathbb{D}(K)$: would like to define an L-function of M as the zeta function of $j_* M$.
This object exists but in the "large" category $DA^{\text{ét}}(C_K, \mathbf{Q})$ (it is not constructible). However,

$$2 - \varinjlim_{U \subseteq C_K} \mathbb{D}(U) \xrightarrow{\sim} \mathbb{D}(K)$$

which leads to:

Definition 3.5. x closed point of C_K, $S_x = \operatorname{Spec} \mathcal{O}_{C_K,x}$, $i_x : x \to S_x$, $j_x : \operatorname{Spec} K \to S_x$.
For $M \in \mathbb{D}(K)$,

$$L_x^{\mathrm{tot}}(M,s) = \zeta(i_x^*(j_x)_*M, s)$$
$$L^{\mathrm{tot}}(M,s) = \prod_{x \in C_K} L_x^{\mathrm{tot}}(M,s).$$

Theorem 3.6. *$L^{\mathrm{tot}}(M,s)$ is an absolutely convergent Dirichlet series, for $\Re(s) \gg 0$.*

Proof: M has good reduction at x for almost all $x \in C_K$. More precisely, $\exists U \subseteq C_K$ and $\mathcal{M} \in \mathbb{D}^{\mathrm{proj}}(U)$ such that $j_U^* \mathcal{M} = M$ for $j_U : \operatorname{Spec} K \to U$. For $x \in U$, let $j_{U,x} : S_x \to U$ and $\mathcal{M}_x = j_{U,x}^* \mathcal{M}$. Localisation exact triangle

$$(i_x)_* i_x^! \mathcal{M}_x \to \mathcal{M}_x \to (j_x)_* j_x^* \mathcal{M}_x \xrightarrow{+1}$$

apply i_x^*:

$$i_x^! \mathcal{M}_x \to i_x^* \mathcal{M}_x \to i_x^* (j_x)_* M \xrightarrow{+1}$$

Thus

$$L_x^{\mathrm{tot}}(M, s) = \frac{\zeta(i_x^* \mathcal{M}_x, s)}{\zeta(i_x^! \mathcal{M}_x, s)} = \frac{\zeta(i_x^* \mathcal{M}_x, s)}{\zeta(i_x^* \mathcal{M}_x, s+1)}$$

by Lemma 3.4.

But $\prod_{x \in U} \zeta(i_x^* \mathcal{M}_x, s) = \zeta(\mathcal{M}, s)$ convergent by Theorem 2.10, so we win.

3.3. The nearby L-function.

Lemma 3.7. *$f = \sum_{n=1}^{\infty} a_n n^{-s}$ convergent Dirichlet series with complex coefficients, with $a_1 = 1$. Then the equation*

$$f(s) = g(s)/g(s+1)$$

has a unique solution as a convergent Dirichlet series (with first coefficient 1), namely

$$g(s) = \prod_{m=0}^{\infty} f(s+m).$$

Moreover, g has the same absolute convergence abscissa as f.
If the coefficients of f belong to $F \subseteq \mathbf{C}$, so do those of g.

Definition 3.8. $M \in \mathbb{D}(K)$:

$$L^{\text{near}}(M, s) = \prod_{x \in C_K} L_x^{\text{near}}(M, s)$$

given by the rule

$$L_x^{\text{tot}}(M, s) = \frac{L_x^{\text{near}}(M, s)}{L_x^{\text{near}}(M, s+1)}$$

cf. Lemma 3.7.

Theorem 3.9. *a)* $\forall x \in C_K,\ L_x^{\text{near}}(M,s) \in \mathbf{Q}(N(x)^{-s})$.
b) $L^{\text{near}}(M,s)$ *convergent Dirichlet series.*
c) If M *has good reduction at* x *and* $\mathcal{M}_x$ *is a good model at* x*, then*

$$L_x^{\text{near}}(M,s) = \zeta(i_x^* \mathcal{M}_x, s).$$

d) If K *function field over* $\mathbf{F}_q$*,* $L^{\text{near}}(M,s) \in \mathbf{Q}(q^{-s})$*, and functional equation between* $L^{\text{near}}(M,s)$ *and* $L^{\text{near}}(M^*, 1-s)$.

For a), two proofs:
(i) Pass to l-adic realisation:

$$L_x^{\text{tot}}(M, s) = L(i_x^* R(j_x)_* R^l(M), s).$$

If V l-adic representation of G_K, need to show that

$$L(i_x^* R(j_x)_* V, s) = f(N(x)^{-s})/f(N(x)^{-s-1})$$

for some $f \in \mathbf{Q}(t)$.
We have

$$L(i_x^* R(j_x)_* V, s) = \frac{\det(1 - \varphi_x N(x)^{-s} \mid H^1(I_x, V))}{\det(1 - \varphi_x N(x)^{-s} \mid H^0(I_x, V))}.$$

This is an Euler-Poincaré characteristic, so may assume V *semi-simple*. Then I_x acts by a finite quotient by the l-adic monodromy theorem, thus

$$H^1(I_x, V) = V_{I_x}(-1) \simeq V^{I_x}(-1)$$

and

$$L(i_x^* R(j_x)_* V, s) = \frac{L^{\text{Serre}}(V, s)}{L^{\text{Serre}}(V, s+1)} \text{ (in the semi-simple case).}$$

(ii) $\Upsilon_x : \mathbb{D}(K) \to \mathbb{D}(\kappa(x))$ Ayoub's "unipotent" specialisation system: exact triangle

$$i_x^*(j_x)_*M \to \Upsilon_x(M) \to \Upsilon_x(M)(-1) \xrightarrow{+1}$$

hence $L_x^{\text{near}}(M, s) = \zeta(\Upsilon_x(M), s)$.

Remark 3.10. First proof gives other explicit formula for $L_x^{\text{near}}(M, s)$:

$$L_x^{\text{near}}(M, s) = L_x^{\text{Serre}}(R^l(M)^{ss}, s)$$

$R^l(M)^{ss}$ semi-simplification of $R^l(M)$.
(Since action of inertia factors through finite quotient, "Serre L-function" could be replaced by "Artin L-function".)

Example. E elliptic curve $/K$ with multiplicative reduction at x, $V = H^1(\bar{E}, \mathbf{Q}_l)$:

$$L_x^{\mathrm{Serre}}(H^1(E), s) = \det(1 - N(x)^{-s}\varphi_x \mid V^{I_x})^{-1}$$
$$L_x^{\mathrm{near}}(H^1(E), s) = L_x^{\mathrm{Serre}}(H^1(E), s) \times \det(1 - N(x)^{-s}\varphi_x \mid V/V^{I_x})^{-1}.$$

Extra poles are explicitly computable...

Discussions with J. Ayoub: $K = \mathbf{F}_q(C)$, explicit functional equation for $L^{\mathrm{near}}(M,s)$ with exponential term of the form $A(M)^s$,

$$\begin{aligned} A(M) &= q^{\chi(M)(2g-2)+\deg \mathfrak{f}(M)} \\ g &= \text{ genus of } C \\ \chi(M) &= \operatorname{tr}(1_M) \\ \mathfrak{f}(M) &= \sum_{x\in C} a_x(M) x \\ a_x(M) &= \text{ Artin conductor of } M \text{ at } x \end{aligned}$$

To define $a_x(M)$ without l-adic realisation, use Ayoub's "full" specialisation system $\Psi_x : \mathbb{D}(K) \to \mathbb{D}(\kappa(x))$: $\Psi_x(M)$ carries action of wild inertia P_x, define $sw_x(M)$ from character

$$P_x \ni g \mapsto \operatorname{tr}(g \mid \Psi_x(M)).$$

(To be continued...)