COMPUTING RIEMANN–ROCH POLYNOMIALS AND CLASSIFYING HYPER-KÄHLER FOURFOLDS

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Abstract. We prove that a hyper-Kähler fourfold satisfying a mild topological assumption is of $\mathrm{K3}^{[2]}$ deformation type. This proves in particular a conjecture of O'Grady stating that hyper-Kähler fourfolds of $\mathrm{K3}^{[2]}$ numerical type are of $\mathrm{K3}^{[2]}$ deformation type. Our topological assumption concerns the existence of two integral degree-2 cohomology classes satisfying certain numerical intersection conditions.

There are two main ingredients in the proof. We first prove a topological version of the statement, by showing that our topological assumption forces the Betti numbers, the Fujiki constant, and the Huybrechts–Riemann–Roch polynomial of the hyper-Kähler fourfold to be the same as those of $\mathrm{K3}^{[2]}$ hyper-Kähler fourfolds. The key part of the article is then to prove the hyper-Kähler SYZ conjecture for hyper-Kähler fourfolds for divisor classes satisfying the numerical condition mentioned above.

Contents

1. Introduction

We work over the complex numbers. A hyper-Kähler manifold is a simply connected smooth compact Kähler manifold which carries a nowhere degenerate holomorphic 2-form

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(called a symplectic form) unique up to multiplication by a nonzero constant. This important class of manifolds is a generalization in all even dimensions of K3 surfaces.

There are two key conjectures about hyper-Kähler manifolds.

Conjecture 1.1. Any hyper-Kähler manifold can be deformed into a hyper-Kähler manifold with a Lagrangian fibration.

Let $2n$ be the dimension of X and assume that there exists a nonzero class $\mathsf{l} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^{2n} = 0$. One can deform X so that l becomes of type $(1,1)$, hence is the first Chern class of some nontrivial holomorphic line bundle L on X. For a very general deformation of this kind, X has Picard number one. In this case, its Kähler cone coincides with its positive cone ([H1], [H4, Proposition 3.2]) and, therefore, L or its dual is nef. Conjecture 1.1 is then implied by the following.

Conjecture 1.2 (hyper-Kähler SYZ conjecture). Let X be a hyper-Kähler manifold of dimension $2n$. Any nontrivial nef line bundle L on X such that $\int_X c_1(L)^{2n} = 0$ is semi-ample.

The existence of a class l as above is equivalent to the existence of a nonzero class in $H^2(X, \mathbf{Q})$ that is isotropic with respect to the Beauville–Bogomolov form q_X (see equation (3)). By Meyer's theorem, such isotropic classes exist when $b_2(X) \geq 5$.

A line bundle L as in Conjecture 1.2 would then define a fibration $f\colon X \to B$, with B normal. By results of Matsushita, B is projective, f is a Lagrangian hence equidimensional fibration, and any smooth fiber $X_b := f^{-1}(b)$ is an abelian variety, endowed with a canonical polarization which is the positive generator of the saturation of the image of the rank-1 restriction map $H^2(X, \mathbf{Z}) \to H^2(X_b, \mathbf{Z})$ (see Section 3).

It is conjectured that the base B of any Lagrangian fibration is smooth, and in fact $\mathbf{P}^n$; note that B is smooth if and only if f is flat. When X is projective and B is smooth, it was proved by Hwang in [Hw] that B is $\mathbf{P}^n$; this was extended to the non-projective case by Greb and Lehn in [GL]. When X is projective and $n = 2$, results of Ou in [Ou] and Huybrechts and Xu in [HX] imply that B is always $\mathbf{P}^2$.

Both conjectures are true for all known examples of hyper-Kähler manifolds: for Conjecture 1.1, see Section 3.3; the following stronger form of Conjecture 1.2 follows from work of Matsushita and others.

Theorem 1.3. *Let X be a hyper-Kähler manifold of dimension $2n$ of either* $\mathrm{K3}^{[n]}$*, generalized Kummer,* OG10*, or* OG6 *deformation type. Let L be a nef line bundle on X such that $\int_X c_1(L)^{2n} = 0$ and $c_1(L)$ is primitive in $H^2(X, \mathbf{Z})$. There exists a Lagrangian fibration $f\colon X \to \mathbf{P}^n$ such that $f^*\mathscr{O}_{\mathbf{P}^n}(1) \simeq L$.*

Indeed, a "rational version" of Conjecture 1.2, namely [Ma4, Conjecture 1.1], is known for a movable line bundle L with primitive isotropic first Chern class by [Ma4, Corollary 1.1] in the case of $\mathrm{K3}^{[n]}$ or generalized Kummer deformation type (based on [BM, M3, Y2]), by [MO, Theorem 2.2] in the case of OG10 deformation type, and by [MR, Theorem 7.2] in the case of OG6 deformation type. When L is nef, we can apply [Ma4, Claim 3.1 and Claim 3.2] to conclude.

One of our main results is that both conjectures are true in dimension 4 under an additional topological assumption that we now explain. We introduce another class $\mathsf{m} \in H^2(X, \mathbf{Z})$

with $q_X(\mathsf{l},\mathsf{m})>0$. The number

$$a := \frac{1}{n!}\int_X \mathsf{l}^n\mathsf{m}^n \tag{1}$$

is then a positive integer (see Lemma 2.2). For most of this article, *we make the assumption* $a=1$*, that is,* $\int_X \mathsf{l}^n\mathsf{m}^n = n!$ (the minimal possible value).

Assume m is the first Chern class of a line bundle M on X. When there is a Lagrangian fibration $f\colon X\to \mathbf{P}^n$ and $L=f^*\mathscr{O}_{\mathbf{P}^n}(1)$, the restriction of M to any smooth fiber is a polarization of "degree" a. So the condition $a=1$ means that these polarizations are principal. This set up was the starting point of this work. We prove in Section 3 that upon replacing m by $\mathsf{m}+r\mathsf{l}$, for a suitable integer r, we can assume $q_X(\mathsf{l},\mathsf{m})=1$ and $q_X(\mathsf{m})=0$. By mimicking the work [Ri] of Ríos Ortiz, we find that the Huybrechts–Riemann–Roch polynomial $P_{RR,X}$ (see Section 2.2 for the definition) then takes the very simple form

$$\forall k\in\mathbf{Z}\qquad P_{RR,X}(2k)=P_{RR,X}(q_X(k\mathsf{l}+\mathsf{m}))=\chi(X,L^k\otimes M)=\chi(\mathbf{P}^n,\mathscr{O}_{\mathbf{P}^n}(k+1)) \tag{2}$$

(see Theorem 3.1 for more properties of X).

We are naturally led to asking whether this conclusion still holds without assuming the existence of the Lagrangian fibration f.

Conjecture 1.4. Let X be a hyper-Kähler manifold of dimension $2n$ with classes $\mathsf{l},\mathsf{m}\in H^2(X,\mathbf{Z})$ such that

$$\int_X \mathsf{l}^{2n}=0\quad\text{and}\quad\int_X \mathsf{l}^n\mathsf{m}^n=n!.$$

Then the Huybrechts–Riemann–Roch polynomial of X satisfies

$$\forall k\in\mathbf{Z}\qquad P_{RR,X}(2k)=\chi(\mathbf{P}^n,\mathscr{O}_{\mathbf{P}^n}(k+1))=\binom{k+1+n}{n}.$$

Conjecture 1.4 is not strictly speaking implied by Theorem 3.1 and the SYZ conjecture. Indeed, the latter predicts that if L is a nef holomorphic line bundle, *some positive power* of L is generated by global sections and gives a Lagrangian fibration. The relation (2) proves Conjecture 1.4 when l is the first Chern class of a nef holomorphic line bundle L such that L itself is generated by global sections. The question of whether L is generated by global sections, if a power induces a Lagrangian fibration, was recently studied in [KV], where the authors give a sufficient condition for this to happen; unfortunately, their result does not apply in our situation.

Our other results exclusively deal with the case of dimension 4. In that dimension, there are two known deformation classes of hyper-Kähler manifolds, namely the deformations of the Hilbert square $S^{[2]}$ of a K3 surface S (called the K3$^{[2]}$ deformation type) and that of the generalized Kummer variety $\mathsf{K}_2(A)$ of an abelian surface A.

We say that a hyper-Kähler fourfold X is *of* K3$^{[2]}$ *numerical type* if, for some K3 surface S, there exists an isomorphism of abelian groups $\psi\colon H^2(X,\mathbf{Z})\xrightarrow{\sim} H^2(S^{[2]},\mathbf{Z})$ such that, for all $\alpha\in H^2(X,\mathbf{Z})$, we have $\int_X\alpha^4=\int_{S^{[2]}}\psi(\alpha)^4$ (this is equivalent to asking that the lattices $(H^2(X,\mathbf{Z}),q_X)$ and $(H^2(S^{[2]},\mathbf{Z}),q_{S^{[2]}})$ be isometric and the Fujiki constants be the same).

In [O], O'Grady conjectured that a hyper-Kähler fourfold of K3$^{[2]}$ numerical type is of K3$^{[2]}$ deformation type and provided strong evidence for this statement. One of our results implies O'Grady's conjecture under a much weaker topological hypothesis.

Theorem 1.5. *Let X be a hyper-Kähler fourfold. Assume there are classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^4 = 0$ and $\int_X \mathsf{l}^2\mathsf{m}^2 = 2$. Then X is of $\mathrm{K3}^{[2]}$ deformation type.*

Corollary 1.6 (O'Grady's conjecture)**.** *A hyper-Kähler fourfold of $\mathrm{K3}^{[2]}$ numerical type is of $\mathrm{K3}^{[2]}$ deformation type.*

Let us explain how we prove Theorem 1.5. The first step is to show Conjecture 1.4 in dimension 4 (see Theorem 4.3).

Theorem 1.7. *Let X be a hyper-Kähler fourfold with classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^4 = 0$ and $\int_X \mathsf{l}^2\mathsf{m}^2 = 2$. The Huybrechts–Riemann–Roch polynomial of X satisfies*

$$P_{RR,X}(2k) = \chi(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(k+1)) = \binom{k+3}{2}.$$

Furthermore, the Fujiki constant and the Hodge and Chern numbers of X are those of the Hilbert square of a K3 surface.

The conclusion of Theorem 1.7 is weaker than being of $\mathrm{K3}^{[2]}$ numerical type. Theorem 1.7 is the starting point for the second step of the proof, in which we establish Conjecture 1.2 in dimension 4 under the same assumptions on X made in Theorem 1.5. More precisely, our key result is the following.

Theorem 1.8. *Let X be a hyper-Kähler fourfold and let L be a nef line bundle on X. Set $\mathsf{l} := c_1(L) \in H^2(X, \mathbf{Z})$ and assume that there exists $\mathsf{m} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^4 = 0$ and $\int_X \mathsf{l}^2\mathsf{m}^2 = 2$. There exists a Lagrangian fibration $f\colon X \to \mathbf{P}^2$ with $f^*\mathscr{O}_{\mathbf{P}^2}(1) \simeq L$.*

As we mentioned earlier, hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$ deformation type satisfy the SYZ conjecture, so Theorem 1.8 is weaker than Theorem 1.5. In our approach, Theorem 1.8 (or rather some weaker versions of it) is a step towards proving Theorem 1.5, the precise logical relationships being as follows. We first consider the case of a very general hyper-Kähler fourfold X satisfying the assumptions of Theorem 1.8 and for which the class m is also of type $(1,1)$, that is, $\mathsf{m} = c_1(M)$ for some line bundle M on X. In such a case, we can assume that m also satisfies $\int_X \mathsf{m}^4 = 0$ and is in the boundary of the positive cone of X.

A study of the effective cone of X then shows that there are two cases.

- The first case, mostly studied in Section 6, is when L and M are both nef and $L \otimes M$ is ample. We show in Proposition 5.6 that in this case,
  - either, after possibly permuting L and M, Theorem 1.8 holds: the linear system $|L|$ induces a Lagrangian fibration to $\mathbf{P}^2$. But this contradicts what we had proved earlier in Section 3.2: that, if $|L|$ induces a Lagrangian fibration, L and M cannot be both nef;
  - or any divisor in the linear system $|L \otimes M|$ is irreducible and the image of the rational map $\varphi_{L\otimes M}\colon X \dashrightarrow \mathbf{P}^5$ is rationally connected. But this contradicts [V2, Theorem 4.2] showing that this situation cannot happen, at least when X is very general as above with Picard number 2.

  So this case in fact does not arise (Corollary 5.7).
- The second case, mostly studied in Section 7, is when X admits a divisorial contraction and M is not nef. In this case, we first prove Proposition 5.8 (a slightly weaker version of Theorem 1.8), namely the existence of a Lagrangian fibration $f\colon X \to \mathbf{P}^2$ with $f^*\mathscr{O}_{\mathbf{P}^2}(1) \simeq L^{k_L}$, for some positive integer k_L. We then use this weaker result to prove that X is of $\mathrm{K3}^{[2]}$ deformation type (see Section 8).

This completes the proof of Theorem 1.5: as explained at the beginning of Section 8, by deformation, one can always assume that the classes l and m are of type $(1,1)$ and the triple $(X, \mathsf{l}, \mathsf{m})$ satisfies the hypotheses made above. But, by Theorem 1.3, this also proves Theorem 1.8, once we know (by Theorem 1.5) that X is of $\mathrm{K3}^{[2]}$ deformation type. Note that our results rely, in the first case described above, on the classification work of [V2, Section 4], which extends to our context O'Grady's analysis in [O].

We end the article with Section 9, which includes boundedness results when we fix the dimension $2n$ and the integer a defined in (1), and an (incomplete) analysis of what happens when $n = 2$ and a is small.

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2. Review of hyper-Kähler manifolds

We recall in Section 2.1 the definitions of Beauville–Bogomolov–Fujiki forms and Fujiki constants for hyper-Kähler manifolds. We then define their Huybrechts–Riemann–Roch polynomials and review some of their elementary properties in Section 2.2.

2.1. The Beauville–Bogomolov–Fujiki form and the Fujiki constant.

Let X be a hyper-Kähler manifold of dimension $2n$. There exists a canonical integral nondivisible quadratic form q_X (the *Beauville–Bogomolov–Fujiki form*) on $H^2(X, \mathbf{Z})$ and a positive rational constant c_X (the *Fujiki constant*) such that

$$\forall \alpha \in H^2(X, \mathbf{R}) \qquad \int_X \alpha^{2n} = c_X\, q_X(\alpha)^n \tag{3}$$

(after extending q_X to a quadratic form on $H^2(X, \mathbf{R})$). Moreover, $q_X(h) > 0$ for all Kähler classes h (see [H2, Proposition 23.14]).

Assume $q_X(\alpha) = 0$. Then $\alpha^{n+1} = 0$ (see for example [H2, Proposition 24.1]) and, by comparing the coefficients of t^n in the relation

$$\int_X (t\alpha + \beta)^{2n} = c_X q_X(t\alpha + \beta)^n = c_X(2tq_X(\alpha, \beta) + q_X(\beta))^n,$$

we get

$$\forall \beta \in H^2(X, \mathbf{R}) \qquad \frac{1}{2^n}\binom{2n}{n} \int_X \alpha^n \beta^n = c_X q_X(\alpha, \beta)^n. \tag{4}$$

This is a particular case of the polarization of the Fujiki relation (3), which in dimension 4 takes the form

$$3\int_X \alpha_1\alpha_2\alpha_3\alpha_4 = c_X\big(q_X(\alpha_1, \alpha_2)q_X(\alpha_3, \alpha_4) + q_X(\alpha_1, \alpha_3)q_X(\alpha_2, \alpha_4) + q_X(\alpha_1, \alpha_4)q_X(\alpha_2, \alpha_3)\big) \tag{5}$$

for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^2(X, \mathbf{Z})$.

2.2. **The Huybrechts–Riemann–Roch polynomial.** By [H2, Corollary 23.18], there is a polynomial

$$P_{RR,X}(T) = \sum_{i=0}^{n} a_i T^i \in \mathbf{Q}[T]$$

of degree n such that, for each line bundle L on X, one has

$$\chi(X, L) = P_{RR,X}(q_X(c_1(L))). \tag{6}$$

This polynomial has the following properties:

(a) the constant term of $P_{RR,X}(T)$ is $\chi(X, \mathscr{O}_X) = n + 1$;
(b) the leading term of $P_{RR,X}(T)$ is $\frac{c_X}{(2n)!}T^n$;
(c) the coefficients of $P_{RR,X}(T)$ are all positive ([J, Theorem 1.1]).

The next two lemmas are elementary.

Lemma 2.1. *Let X be a hyper-Kähler manifold. For every class $\alpha \in H^2(X, \mathbf{Z})$, one has $P_{RR,X}(q_X(\alpha)) \in \mathbf{Z}$.*

Proof. Since the period map is surjective ([H2, Proposition 25.12]), the class α becomes the first Chern class of some holomorphic line bundle on a deformation X' of X, and (6) implies $P_{RR,X'}(q_{X'}(\alpha)) \in \mathbf{Z}$. Since the polynomial $P_{RR,X}$ and the form q_X are deformation invariant, the lemma follows. □

Lemma 2.2. *Let X be a hyper-Kähler manifold. Assume that there is a class $\mathsf{l} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^{2n} = 0$. For every $\mathsf{m} \in H^2(X, \mathbf{Z})$, the number*

$$a := \frac{1}{n!}\int_X \mathsf{l}^n \mathsf{m}^n \tag{7}$$

is an integer.

Proof. By (3), we have $q_X(\mathsf{l}) = 0$. Set

$$\forall k \in \mathbf{Z} \qquad P(k) := P_{RR,X}(q_X(k\mathsf{l} + \mathsf{m})) = P_{RR,X}(2kq_X(\mathsf{l}, \mathsf{m}) + q_X(\mathsf{m})). \tag{8}$$

Then P is a polynomial of degree n whose leading coefficient is (use (4) and property (b) above)

$$\frac{c_X}{(2n)!}(2q_X(\mathsf{l}, \mathsf{m}))^n = \frac{1}{(n!)^2}\int_X \mathsf{l}^n \mathsf{m}^n = \frac{a}{n!}. \tag{9}$$

By Lemma 2.1, the polynomial P takes integral values on integers, hence a is an integer. □

Remark 2.3. Under the hypotheses of Lemma 2.2, when a is divisible by no nontrivial nth powers, the sublattice $\mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$ of $H^2(X, \mathbf{Z})$ is saturated: this follows from the fact that $\frac{1}{n!}\int_X \mathsf{l}^n \alpha^n$ is an integer for all $\alpha \in H^2(X, \mathbf{Z})$.

3. Lagrangian fibrations

Let X be again a hyper-Kähler manifold of dimension $2n$. We assume in this section that there is a fibration $f\colon X \to \mathbf{P}^n$. Set $L := f^*\mathscr{O}_{\mathbf{P}^n}(1)$, with first Chern class $\mathsf{l} \in H^2(X, \mathbf{Z})$; it satisfies $q_X(\mathsf{l}) = 0$. Let $\mathsf{m} \in H^2(X, \mathbf{Z})$ be another class, not necessarily of type $(1, 1)$; we assume $q_X(\mathsf{l}, \mathsf{m}) > 0$.

Any smooth fiber $X_b := f^{-1}(b)$ is a Lagrangian complex torus of dimension n ([Ma1, Theorem 1], [AC, Theorem 1]). By [Ma3, Lemma 2.2], the hyperplane $\mathsf{l}^\perp$ is contained in the kernel of the restriction map $r_b\colon H^2(X,\mathbf{C})\to H^2(X_b,\mathbf{C})$. Since the restriction of a Kähler class on X is a Kähler class on X_b, the map r_b has rank exactly 1 and the rational class $r_b(\mathsf{m})$ is a positive multiple of a Kähler class, hence is an ample class on X_b. In particular, X_b is an abelian variety ([AC, Proposition 4]) and the "degree" of the polarization $r_b(\mathsf{m})=\mathsf{m}|_{X_b}$ is the positive integer

$$\frac{1}{n!}\int_{X_b}(\mathsf{m}|_{X_b})^n=\frac{1}{n!}\int_X \mathsf{l}^n\mathsf{m}^n=a$$

already considered in (7).

There are restrictions on the values that a can take (see Theorem 9.3 for restrictions on small values of a when $n=2$). For the moment, we prove that the existence of the Lagrangian fibration f imposes strong conditions on X when $a=1$, that is, when there is a class on X that induces a principal polarization on the smooth fibers of f.

3.1. **The Huybrechts–Riemann–Roch polynomial.** Keeping the notation and the hypotheses as above, we show that in the case $a=1$, the Huybrechts–Riemann–Roch polynomial of X is completely determined. The main idea of the proof is taken from [Ri]: the hypothesis $a=1$ is essential because it implies that a certain locally free sheaf of rank a on $\mathbf{P}^n$ can be written as $\mathscr{O}_{\mathbf{P}^n}(d)$ for some integer d.

Theorem 3.1. *Let X be a hyper-Kähler manifold of dimension $2n$ with a Lagrangian fibration $f\colon X\to\mathbf{P}^n$. Set $\mathsf{l}\coloneqq c_1(f^*\mathscr{O}_{\mathbf{P}^n}(1))\in H^2(X,\mathbf{Z})$ and assume that there exists $\mathsf{m}\in H^2(X,\mathbf{Z})$ such that $\int_X \mathsf{l}^n\mathsf{m}^n=n!$. Then, $q_X(\mathsf{l},\mathsf{m})=\pm1$, the quadratic form q_X is even, $c_X=(2n-1)!!$,*

$$P_{RR,X}(T)=\binom{\frac{T}{2}+n+1}{n},$$

and the sublattice $\mathbf{Z}\mathsf{l}\oplus\mathbf{Z}\mathsf{m}$ of $(H^2(X,\mathbf{Z}),q_X)$ is isomorphic to a hyperbolic plane.

For all known hyper-Kähler manifolds X, the lattice $(H^2(X,\mathbf{Z}),q_X)$ contains a hyperbolic plane.

Proof. Changing m into $-\mathsf{m}$ is necessary, we may assume $q_X(\mathsf{l},\mathsf{m})>0$. Consider the universal family $(\mathscr{X},\mathscr{L})\to\mathfrak{M}_{\mathsf{l}}$ over one component of the (non-Hausdorff) moduli space of marked hyper-Kähler manifolds with a fixed $(1,1)$-class l. The period map $\mathscr{P}\colon\mathfrak{M}_{\mathsf{l}}\to\mathbf{P}(\mathsf{l}^\perp\otimes\mathbf{C})$ is injective over very general points and the points in an arbitrary fiber correspond to the chambers of the decomposition of the positive cone. For a distinguished point $0\in\mathfrak{M}_{\mathsf{l}}$ we have $(\mathscr{X}_0,\mathscr{L}_0)\simeq(X,L)$, where $L\coloneqq f^*\mathscr{O}_{\mathbf{P}^n}(1)$. But there also exists a fiber $X'\coloneqq\mathscr{X}_{0'}$, with $0'\in\mathfrak{M}_{\mathsf{l}}$, whose rational Néron–Severi group $\mathrm{NS}(X')_{\mathbf{Q}}$ is generated by the classes l and m. In fact, since the lattice generated by l and m is saturated (Remark 2.3), the integral Néron–Severi group is generated by l and m. Furthermore, replacing $0'$ by another point in the same fiber over $\mathscr{P}(0')$, we may assume that $L'\coloneqq\mathscr{L}_{0'}$ is nef. Since $q_X(k\mathsf{l}+\mathsf{m})=2kq_X(\mathsf{l},\mathsf{m})+q_X(\mathsf{m})>0$ for $k\gg0$, the manifold X' is projective by [H1].

We apply [Ma4, Theorem 1.2, Claim 3.1 and Claim 3.2]: there exists a Lagrangian fibration $f'\colon X'\to\mathbf{P}^n$ such that $f'^*\mathscr{O}_{\mathbf{P}^n}(1)\simeq L'$. Upon replacing (X,L) by (X',L'), we may, since q_X, c_X, and $P_{RR,X}(T)$ are invariant by deformation, assume that X carries a line bundle M with first Chern class m.

Since L is nef, $q_X(\mathsf{l},\mathsf{m})>0$, and $\mathrm{NS}(X)$ is generated by l and m, we can replace M with $L^k\otimes M$, for $k\gg0$, and assume that M is ample on X. By [Ko, Theorem 10.32], R^if_*M

vanishes for $i > 0$. Because f is flat, the sheaf $\mathscr{M} := f_*M$ on $\mathbf{P}^n$ is locally free; since the restriction of M to a smooth generic fiber of f defines a principal polarization, the rank of $\mathscr{M}$ is $a = 1$. By the projection formula and (6), $\mathscr{M}$ satisfies

$$\forall k \in \mathbf{Z} \qquad \chi(\mathbf{P}^n, \mathscr{M}(k)) = \chi(X, L^k \otimes M) = P_{RR,X}(2kq_X(\mathsf{l}, \mathsf{m}) + q_X(\mathsf{m})). \tag{10}$$

Following [Ri, Section 3], we write $\mathscr{M} = \mathscr{O}_{\mathbf{P}^n}(d)$ for some integer d and, from (10), we deduce

$$\forall k \in \mathbf{Z} \qquad P_{RR,X}(2kq_X(\mathsf{l}, \mathsf{m}) + q_X(\mathsf{m})) = \chi(\mathbf{P}^n, \mathscr{O}_{\mathbf{P}^n}(d+k)) = \binom{d+k+n}{n},$$

so that

$$P_{RR,X}(T) = \binom{d + \frac{T - q_X(\mathsf{m})}{2q_X(\mathsf{l},\mathsf{m})} + n}{n}. \tag{11}$$

Set $\gamma := \frac{q_X(\mathsf{m})}{2q_X(\mathsf{l},\mathsf{m})}$. Since $P_{RR,X}(0) = n+1$, either $d - \gamma = 1$, or n is even and $d - \gamma = -n-2$.[1] Since the coefficients of $P_{RR,X}$ are positive and the coefficient of T^{n-1} in (11) is a positive multiple of $n + 2(d-\gamma) + 1$, the latter case is ruled out. So γ is an integer and $d = \gamma + 1$. Replacing M by $L^{-\gamma} \otimes M$, we may assume $\gamma = 0$ and $d = 1$, hence $q_X(\mathsf{m}) = 0$, so that (11) becomes

$$P_{RR,X}(T) = \binom{\frac{T}{2q_X(\mathsf{l},\mathsf{m})} + n + 1}{n}. \tag{12}$$

By Lemma 3.2 below (applied with $c = n!$, $c' = 1$, and $q = 2q_X(\mathsf{l}, \mathsf{m})$), we get $q_X(\mathsf{l}, \mathsf{m}) = 1$ and the quadratic form q_X is even. The value of c_X is then derived from (4) and the polynomial $P_{RR,X}(T)$ from (12). □

It remains to prove the arithmetical result used at the end of the proof above. We prove a bit more than what we actually used above but we will need this stronger statement for the proof of Theorem 9.3.

Lemma 3.2. *Let X be a hyper-Kähler manifold of dimension $2n$. Assume that there are positive integers c, c', and q such that c' is divisible by no nontrivial nth powers and the polynomial $P(T) := \frac{c}{c'} P_{RR,X}(qT)$ is monic with integral coefficients. Then, either $q = 1$, or $q = 2$ and the quadratic form q_X is even.*

Proof. Let $\alpha \in H^2(X, \mathbf{Z})$ and write $\frac{q_X(\alpha)}{q} =: \frac{r}{s}$, where r and s are relatively prime integers. One has

$$P_{RR,X}(q_X(\alpha)) = \frac{c'}{c} P\Big(\frac{q_X(\alpha)}{q}\Big) = \frac{c'}{c} P\Big(\frac{r}{s}\Big)$$

and this is an integer by Lemma 2.1. Since $P(T)$ is monic with integral coefficients, $s^n P(\frac{r}{s})$ is an integer congruent to r^n modulo s, hence prime to s^n. But $c's^nP(\frac{r}{s}) = cs^nP_{RR,X}(q_X(\alpha))$ is divisible by s^n, hence so is c'. Our hypothesis implies $s = \pm 1$, which proves that q divides all values that q_X takes on $H^2(X, \mathbf{Z})$. Since the integral bilinear form associated with q_X is not divisible, either $q = 1$, or $q = 2$ and the quadratic form q_X is even. □

[1] The equation $\binom{x+n}{n} = n+1$ is equivalent to the monic equation $\prod_{i=1}^n (x+i) = (n+1)!$, so any rational solution is in fact integral, and the only integral solutions are $x = 1$ and, when n is even, $x = -n-2$.

3.2. **The exceptional divisor.** We keep our hyper-Kähler manifold X of dimension $2n$ with a Lagrangian fibration $f\colon X \to \mathbf{P}^n$ and we set as above $L := f^*\mathscr{O}_{\mathbf{P}^n}(1)$, with first Chern class $\mathsf{l} \in H^2(X,\mathbf{Z})$. We assume further that there exists a line bundle M on X whose class m satisfies $\int_X \mathsf{l}^n\mathsf{m}^n = n!$ and $q_X(\mathsf{l},\mathsf{m}) > 0$.

As in the proof of Theorem 3.1, X is projective and we may assume $q_X(\mathsf{l},\mathsf{m}) = 1$ and $q_X(\mathsf{m}) = 0$. Then $f_*M = \mathscr{O}_{\mathbf{P}^n}(1)$ and $f_*(L^{-1}\otimes M) = \mathscr{O}_{\mathbf{P}^n}$, hence the linear system $|L^{-1}\otimes M|$ contains a single (effective) divisor E which still induces a principal polarization on the smooth fibers of f. It satisfies $q_X([E]) = -2$ and $q_X([E],\mathsf{m}) = -1$; in particular, M is not nef.

Lemma 3.3. *Assume that* $\mathrm{NS}(X) = \mathbf{Z}\mathsf{l}\oplus\mathbf{Z}\mathsf{m}$*. Then the divisor* $E \in |L^{-1}\otimes M|$ *is irreducible and reduced.*

Proof. For any integer $k \geq 2$, the linear system $|L^{-k}\otimes M|$ is empty, because any divisor in that linear system would have negative q_X-intersection with E, and would therefore contain E, but the linear system $|L^{-k+1}|$ is empty. It follows that E has no vertical components. The fact that E is irreducible and reduced then follows from the relation $q_X([E],\mathsf{l}) = 1$. □

Since $q_X([E]) < 0$, the prime divisor E is therefore exceptional in the sense of [B, déf. 3.10] and [M2, Definition 3.2] hence it spans an extremal ray in the effective cone $\mathrm{Eff}(X)$. Moreover, by [D, prop. 1.4 and rem. 4.3], there is a birational isomorphism $\varphi\colon X \overset{\sim}{\dashrightarrow} X'$ (with X' smooth hyper-Kähler) and a projective divisorial contraction $c\colon X' \to Y$ with exceptional divisor $\varphi(E)$; moreover, the general fibers of $c|_{\varphi(E)}$ are either smooth rational curves, or unions of two smooth rational curves meeting transversely at one point. The divisor E is uniruled, its class $-\mathsf{l}+\mathsf{m}$ is primitive (because $q_X(\mathsf{l},[E]) = 1$), the reflection

$$\alpha \longmapsto \alpha + q_X(\alpha, -\mathsf{l}+\mathsf{m})(-\mathsf{l}+\mathsf{m})$$

is integral and a monodromy operator that permutes l and m. Finally, the class in $H_2(X',\mathbf{Z}) \simeq H^2(X',\mathbf{Z})^\vee \simeq H^2(X,\mathbf{Z})^\vee$ of a general (curve) fiber of $\varphi(E) \to c(\varphi(E))$ is given by the linear form $q_X(-\mathsf{l}+\mathsf{m},\bullet)$; moreover, since $q_X(-\mathsf{l}+\mathsf{m},\mathsf{l}) = 1$, this general fiber cannot be the union of two homologous curves, hence it is a smooth rational curve ([M2, Corollary 3.6]).

3.3. **Examples.** The next two examples show that Theorem 3.1 applies to hyper-Kähler manifolds of $\mathrm{K3}^{[n]}$ deformation type (Example 3.4) or of OG10 deformation type (Example 3.5).

Example 3.4. Let S be a K3 surface with a primitive polarization h_S of degree $\mathsf{h}_S^2 = 2d$ and set $n = d+1$ (this is the genus of any curve in the linear system $|\mathsf{h}_S|$). Assume that the pair (S,h_S) is very general, so that $\mathrm{NS}(S) = \mathbf{Z}\mathsf{h}_S$. The smooth projective moduli space $\mathsf{M}_0(S) := M_S(0,\mathsf{h}_S,0)$ parametrizes pairs consisting of a curve $C \in |\mathsf{h}_S|$ and a torsion-free, rank-1 coherent sheaf on C of degree $n-1$, considered as a (torsion) sheaf on X. It is a hyper-Kähler variety of $\mathrm{K3}^{[n]}$ deformation type. There is a Lagrangian fibration $f\colon \mathsf{M}_0(S) \longrightarrow |\mathsf{h}_S| = \mathbf{P}^n$ that takes a sheaf to its support. The fiber of a smooth $C \in |\mathsf{h}_S|$ is the Jacobian $J^{n-1}(C)$.

The lattice $\mathrm{NS}(\mathsf{M}_0(S))$ is spanned by two isotropic vectors $\mathsf{l} := c_1(f^*\mathscr{O}_{\mathbf{P}^n}(1))$ and m which satisfy $q_X(\mathsf{l},\mathsf{m}) = 1$. Since the Fujiki constant is $(2n-1)!!$, formula (9) gives $a = q_X(\mathsf{l},\mathsf{m})^n = 1$. The fibers $J^{n-1}(C)$ have canonical theta divisors that fit together to define an effective divisor in $\mathsf{M}_0(S)$ which is the exceptional divisor E from Section 3.2.

Example 3.5 (Ríos Ortiz). According to [LSV], there is a hyper-Kähler manifold X of OG10 deformation type with a Lagrangian fibration $f\colon X \to \mathbf{P}^5$ and an f-ample effective divisor Θ on X that restricts to a principal polarization on the smooth fibers of f ([LSV, Proposition 5.3]).

Set $L := f^*\mathscr{O}_{\mathbf{P}^n}(1)$ and $M := \mathscr{O}_X(\Theta)$. Theorem 3.1 shows that the Fujiki constant c_X is equal to $9!! = 945$ (it was originally computed in [R]) and that $P_{RR,X}(T) = \binom{\frac{T}{2}+6}{5}$ ([Ri]).

The next two examples deal with the other known types of hyper-Kähler manifolds.

Example 3.6. Let A be an abelian surface with a polarization h_A of type $(1, d)$, with $d \geq 3$. Assume that the pair (A, h_A) is very general, so that $\mathrm{NS}(A) = \mathbf{Z}\mathsf{h}_A$. The smooth projective moduli space $\mathsf{M}_0(A) := M_A(0, \mathsf{h}_A, 0)$ parametrizes pairs consisting of a curve $C \subset A$ with class h_A and a torsion-free, rank-1 coherent sheaf on C of degree d. The fibers of its (surjective) Albanese map $\mathsf{M}_0(A) \to A \times \widehat{A}$ are all isomorphic to the same hyper-Kähler manifold $\mathsf{K}_0(A)$ of dimension $2n := 2d - 2$ which is of generalized Kummer deformation type ([Y1, Theorem 0.2(1)]). There is a Lagrangian fibration $f\colon \mathsf{K}_0(A) \to \mathbf{P}^n$ that takes a sheaf to its support. The fiber of a smooth $C \underset{\text{num}}{\equiv} \mathsf{h}_A$ is the kernel of the Abel–Jacobi map $J^d(C) \to A$.

By [Y1, Theorem 0.2(2)], the lattice $\mathrm{NS}(\mathsf{K}_0(A))$ is spanned by the two isotropic vectors $\mathsf{l} := c_1(f^*\mathscr{O}_{\mathbf{P}^n}(1))$ and $\mathsf{m} = c_1(M)$ which satisfy $q_X(\mathsf{l}, \mathsf{m}) = 1$. Since the Fujiki constant is $(n + 1)(2n - 1)!!$, formula (9) gives $a = n + 1$.

Example 3.7. Examples of hyper-Kähler manifolds of OG6 deformation type with a Lagrangian fibration are described in [R].

4. Conjecture 1.4 for hyper-Kähler fourfolds

The main result of this section is the proof of Conjecture 1.4 in dimension 4 (see Theorem 4.3). It is a simple consequence of the work [Gu] of Guan who gave a list of possible Betti numbers for hyper-Kähler fourfolds.

Let X be a hyper-Kähler fourfold. Following [J, Section 2.4], we set

$$A_X := \int_X \mathrm{td}^{1/2}(X) = \frac{1}{5760}(7c_2^2(X) - 4c_4(X)) = \frac{1}{8}\Big(7 - \frac{1}{432}c_4(X)\Big). \tag{13}$$

It is known that $c_4(X)$ is divisible by 12 (see, for example, [G, Proposition 2.4]), hence $288A_X$ is an integer.

Lemma 4.1. *Let X be a hyper-Kähler fourfold. Then,*

(a) *either $b_2(X) = 23$, $b_3(X) = 0$, and the Hodge numbers of X are those of the Hilbert square of a K3 surface, in which case $c_4(X) = 324$ and $A_X = \frac{25}{32}$,*
(b) *or $b_2(X) \leq 8$, in which case $c_4(X) \leq 144$ and $\frac{5}{6} \leq A_X \leq \frac{131}{144}$.*

In particular, if $t \in [0, \frac{1}{3})$, then $4A_X - t$ is an integer only when $t = \frac{1}{8}$ and $A_X = \frac{25}{32}$.

Proof. This follows from (13), the relation $c_4(X) = 3(4b_2(X) + 16 - b_3(X))$, and the list of possible Betti numbers given in [Gu, Main Theorem]. ☐

We now introduce classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that $q_X(\mathsf{l}) = 0$ and define $a = \frac{1}{2}\int_X \mathsf{l}^2\mathsf{m}^2$ as in (7). By Lemma 2.2 and (9), it is a nonnegative integer.

Lemma 4.2. *The number $\sqrt{2aA_X}$ is rational.*

Proof. By (9), we have $c_X q_X(\mathsf{l}, \mathsf{m})^2 = 3a$. Using Section 2.2 and [J, Lemma 5.7], we obtain

$$P_{RR,X}(T) = \frac{c_X}{24}T^2 + T\sqrt{\frac{2}{3}c_X A_X} + 3 = \frac{a}{8}\Big(\frac{T}{q_X(\mathsf{l}, \mathsf{m})}\Big)^2 + \sqrt{2aA_X}\Big(\frac{T}{q_X(\mathsf{l}, \mathsf{m})}\Big) + 3. \tag{14}$$

This implies that $\sqrt{2aA_X}$ is a rational number. $\square$

We are now ready to prove (a stronger form of) Conjecture 1.4 (which corresponds to the case $a = 1$) in dimension 4. Analogous, but weaker, results for low values a will be given in Theorem 9.3.

Theorem 4.3. *Let X be a hyper-Kähler fourfold with classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^4 = 0$ and $\int_X \mathsf{l}^2\mathsf{m}^2 = 2$. The Chern and Hodge numbers of X are those of the Hilbert square of a K3 surface and all the conclusions of Theorem 3.1 hold.*

Proof. Changing m into $\pm\mathsf{m} + r\mathsf{l}$, for some $r \in \mathbf{Z}$, if necessary, we may assume $-q_X(\mathsf{l}, \mathsf{m}) < q_X(\mathsf{m}) \le q_X(\mathsf{l}, \mathsf{m})$ or, equivalently, $\gamma \coloneqq \frac{q_X(\mathsf{m})}{q_X(\mathsf{l},\mathsf{m})} \in (-1, 1]$.

As in (8), we introduce the polynomial

$$P(k) \coloneqq P_{RR,X}(q_X(k\mathsf{l} + \mathsf{m})) = P_{RR,X}(2kq_X(\mathsf{l}, \mathsf{m}) + q_X(\mathsf{m})).$$

Using (14), we compute

$$\begin{aligned} P(k) &= \frac{1}{8}(2k + \gamma)^2 + \sqrt{2A_X}(2k + \gamma) + 3 \\ &= \frac{1}{2}k^2 + \Big(\frac{1}{2}\gamma + 2\sqrt{2A_X}\Big)k + \frac{1}{8}\gamma^2 + \gamma\sqrt{2A_X} + 3 \\ &=: \frac{1}{2}k^2 + bk + c. \end{aligned}$$

Since P takes integral values on integers, $\frac{1}{2} + b = P(1) - P(0)$ and $c = P(0)$ are integers; we write $b = \frac{1}{2} + b'$, with $b' \in \mathbf{Z}$.

We also note that

$$\begin{aligned} 4A_X - \frac{b^2}{2} &= 4A_X - \frac{1}{2}\Big(\frac{1}{2}\gamma + 2\sqrt{2A_X}\Big)^2 \\ &= 4A_X - \frac{1}{2}\Big(\frac{1}{4}\gamma^2 + 2\gamma\sqrt{2A_X} + 8A_X\Big) \\ &= 3 - c \end{aligned}$$

is an integer, hence so is $4A_X - \frac{1}{8}$. By Lemma 4.1, this is only possible when $A_X = \frac{25}{32}$ and the Chern and Hodge numbers of X are those of the Hilbert square of a K3 surface. We also have

$$\frac{1}{2} + b' = b = \frac{1}{2}\gamma + \frac{5}{2},$$

which implies that γ is an even integer. Since $\gamma \in (-1, 1]$, we obtain $\gamma = 0$, hence $q_X(\mathsf{m}) = 0$, $b = \frac{5}{2}$, and $c = 3$. By the very definition (8) of P, one has

$$P_{RR,X}(2kq_X(\mathsf{l}, \mathsf{m})) = P(k) = \frac{1}{2}(k^2 + 5k + 6) = \binom{k+3}{2}.$$

By Lemma 3.2 (applied with $c = 2$, $c' = 1$, and $q = 2q_X(\mathsf{l}, \mathsf{m})$), we get $q_X(\mathsf{l}, \mathsf{m}) = 1$, the quadratic form q_X is even, and all the conclusions of Theorem 3.1 hold. $\square$

5. On the SYZ conjecture in dimension 4

In this section we state our two main results, Proposition 5.6 and Proposition 5.8, on the SYZ conjecture for very general hyper-Kähler fourfolds when numerically we expect that there exists a principal polarization. We will prove them respectively in Section 6 and Section 7; we start this section by reviewing a few general results on semi-ample line bundles and Lagrangian fibrations in dimension 4.

5.1. Results of Kawamata, Fujino, Matsushita, Fukuda, Huybrechts and Xu. Let X be a hyper-Kähler manifold of dimension $2n$. As we noted in Section 2.1, given a nontrivial nef line bundle L with primitive class l such that $\int_X \mathsf{l}^{2n} = 0$, we have $\mathsf{l}^n \neq 0$ and $\mathsf{l}^{n+1} = 0$, hence the *numerical dimension* $\nu(X, L)$ is n.

The *Iitaka dimension* $\kappa(X, L)$, that is, the dimension of the image of the rational map $\varphi_{L^k} \colon X \dashrightarrow \mathbf{P}(H^0(X, L^k)^\vee)$ for k sufficiently large and divisible, satisfies $\kappa(X, L) \le \nu(X, L)$. If there is equality (that is, in our case, if $\kappa(X, L) = n$), the line bundle L is *good* in the sense of [Ka, Section 1] (the current terminology is *abundant*; see [Fu, Definition 2.2]). In that case, a theorem of Kawamata ([Ka, Theorem 6.1] in the algebraic case and [Fu, Theorem 4.8] for a simpler proof also valid in the analytic case) says that L is semi-ample: for k sufficiently large and divisible, the sections of L^k define a *morphism* $f \colon X \to B$ with connected fibers which, by [Ma1, Theorem 1], is a Lagrangian fibration.

When the dimension of X is 4, there are further results:

- assuming only $h^0(X, L^k) \ge 2$ for some $k > 0$ (that is, $\kappa(X, L) > 0$), the line bundle L is semi-ample ([Fuk, Theorem 1.5]);
- the base B of the Lagrangian fibration is isomorphic to $\mathbf{P}^2$ ([HX]).

Assume now that L is semi-ample. Since l is primitive, one can write $f^*\mathscr{O}_{\mathbf{P}^2}(1) = L^{k_L}$ for some positive integer k_L. By [Ma2, Theorem 1.3] and the projection formula, we have $R^q f_* L^{k_L} \simeq \Omega^q_{\mathbf{P}^2}(1)$ for all q. Since we know that $h^p(\mathbf{P}^2, \Omega^q_{\mathbf{P}^2}(1)) = 0$ except for $p = q = 0$, the Leray spectral sequence gives

$$h^0(X, L^{k_L}) = h^0(\mathbf{P}^2, \mathscr{O}_{\mathbf{P}^2}(1)) = 3 \quad \text{and} \quad h^i(X, L^{k_L}) = 0 \ \text{ for all } i > 0. \tag{15}$$

In particular, the map f is the map $\varphi_{L^{k_L}}$. We will need the following elementary observation.

Lemma 5.1. *Let L be a nef line bundle on a hyper-Kähler fourfold X with $\kappa(X, L) > 0$. We assume that its class $\mathsf{l} \in \mathrm{NS}(X)$ is primitive and satisfies $\int_X \mathsf{l}^4 = 0$. Then,*

(a) *either $h^0(X, L) \le 1$;*
(b) *or $k_L = 1$, $h^0(X, L) = 3$, and L is globally generated.*

Proof. Let us assume $k_L > 1$ and $h^0(X, L) \ge 2$. Let $\sigma, \tau \in H^0(X, L)$ be linearly independent sections. The $k_L + 1$ sections $\sigma^{k_L}, \sigma^{k_L - 1}\tau, \dots, \tau^{k_L}$ are then linearly independent in $H^0(X, L^{k_L})$. Since this space has dimension 3 (see (15)), we have $k_L = 2$ and the image of the map $f = \varphi_{L^{k_L}} \colon X \to \mathbf{P}^2$ is a conic. This contradicts the surjectivity of f. ☐

5.2. Cones of divisors. Let X be a hyper-Kähler manifold of dimension $2n$ with classes $\mathsf{l}, \mathsf{m} \in \mathrm{NS}(X)$ such that $q_X(\mathsf{l}) = q_X(\mathsf{m}) = 0$ and $q_X(\mathsf{l}, \mathsf{m}) > 0$. We assume moreover that $\mathrm{NS}(X) = \mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$.

The (closed) *positive cone* $\overline{\mathrm{Pos}}(X) \subset \mathrm{NS}(X) \otimes \mathbf{R}$ is defined as the closure of the set of classes of divisors with positive self-intersection and positive intersection with a Kähler class for the form q_X. Under our assumptions, after possibly changing signs, the positive cone is then

$$\overline{\mathrm{Pos}}(X) = \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}\mathsf{m}.$$

The (closed) *movable cone* $\overline{\mathrm{Mov}}(X) \subset \mathrm{NS}(X) \otimes \mathbf{R}$ is defined as the closure of the cone generated by classes of effective divisors whose base locus has codimension ≥ 2. We have an inclusion $\overline{\mathrm{Mov}}(X) \subset \overline{\mathrm{Pos}}(X)$. To determine the movable cone, we need to understand prime exceptional divisors, namely reduced and irreducible divisors with negative self-intersection for the quadratic form q_X (see [M1, Lemma 6.22]).

Lemma 5.2. *Let E be a prime exceptional divisor on X. We have $[E] = \pm(-\mathsf{l}+\mathsf{m})$ in $\mathrm{NS}(X)$.*

Proof. Let us write $[E] = t\mathsf{l} + u\mathsf{m}$, with $t, u \in \mathbf{Z}$. By [M2, Corollary 3.6], the class

$$-2\frac{q_X([E], \bullet)}{q_X([E])} = -\frac{q_X(t\mathsf{l} + u\mathsf{m}, \bullet)}{tuq_X(\mathsf{l}, \mathsf{m})}$$

is in $H_2(X, \mathbf{Z})$. By applying it to l and m, we get $|t| = |u| = 1$. Since $q_X([E]) < 0$, we have $t = -u$, as we wanted. □

By Lemma 5.2, after possibly permuting l and m, we can assume that the ray $\mathbf{R}_{\geq 0}\mathsf{l}$ is extremal for the movable cone $\overline{\mathrm{Mov}}(X)$. By [HT, Theorem 7], upon replacing X with a birational model if necessary, which by [H1, Theorem 4.6] does not change the deformation type, we can assume that this ray is also extremal for the nef cone $\mathrm{Nef}(X)$. Therefore, one can write

$$\mathrm{Nef}(X) = \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}(t_{\mathrm{nef}}\mathsf{l} + \mathsf{m}),$$
$$\overline{\mathrm{Mov}}(X) = \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}(t_{\mathrm{mov}}\mathsf{l} + \mathsf{m}),$$

where $t_{\mathrm{nef}} \geq t_{\mathrm{mov}} \geq 0$ are rational numbers. By [B, prop. 4.4], the movable and pseudo-effective cones are mutually q_X-dual, hence

$$\mathrm{Psef}(X) = \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}(-t_{\mathrm{mov}}\mathsf{l} + \mathsf{m}).$$

By Lemma 5.2, we deduce the following.

Lemma 5.3. *Either $t_{\mathrm{mov}} = 0$, or $t_{\mathrm{mov}} = 1$ and there is a prime effective divisor E in X with class $-\mathsf{l} + \mathsf{m}$.*

Note that, if $t_{\mathrm{mov}} = 1$, the nef cone coincides with the movable cone if and only if the class $\mathsf{l} + \mathsf{m}$ is nef; this class is then semi-ample by Kawamata's Base-Point-Free Theorem and some multiple of it defines a divisorial contraction $c\colon X \to Y$ with exceptional divisor E.

5.3. **The two key propositions.** In the rest of this article, we will be mostly concerned with triples $(X, \mathsf{l}, \mathsf{m})$ satisfying the following set of properties:

$$\begin{cases} X \text{ is a hyper-Kähler fourfold;} \\ \mathsf{l}, \mathsf{m} \in \mathrm{NS}(X); \\ \int_X \mathsf{l}^4 = 0 \text{ and } \int_X \mathsf{l}^2\mathsf{m}^2 = 2. \end{cases} \tag{16}$$

We will always denote by L and M the line bundles on X representing l and m.

By Theorem 4.3, properties (16) imply the following other set of properties (after possibly changing m into $-\mathsf{m}$ and adding to it a multiple of l):

$$(16') \quad \begin{cases} q_X(\mathsf{l}) = q_X(\mathsf{m}) = 0 \text{ and } q_X(\mathsf{l},\mathsf{m}) = 1; \\ \text{the Chern and Hodge numbers of } X \text{ are those of the Hilbert square of a K3 surface;} \\ \text{the canonical map } \mathrm{Sym}^2 H^2(X,\mathbf{Q}) \xrightarrow{\sim} H^4(X,\mathbf{Q}) \text{ is an isomorphism;} \\ P_{RR,X}(T) = \binom{\frac{T}{2}+3}{2}; \\ \text{the quadratic form } q_X \text{ is even;} \\ c_X = 3, \text{ so that } \int_X \alpha^4 = 3q_X(\alpha)^2 \text{ for all } \alpha \in H^2(X,\mathbf{Z}). \end{cases}$$

So when we assume (16), we will always assume that (16′) also holds.

Following [O, Section 3], we define the dual $q_X^\vee \in H^4(X,\mathbf{Q})$ of the quadratic form q_X. Since $b_2(X) = 23$, it satisfies ([O, Proposition 2.2])

$$(17) \qquad \int_X q_X^\vee \cdot q_X^\vee = 575 \quad , \qquad \forall \alpha, \beta \in H^2(X,\mathbf{Z}) \quad \int_X q_X^\vee \cdot \alpha \cdot \beta = 25 q_X(\alpha,\beta).$$

Finally, we have ([O, (3.0.45)])

$$(18) \qquad c_2(X) = \frac{6}{5} q_X^\vee.$$

For any cohomology class $\eta \in H^4(X,\mathbf{Q})$, we define a symmetric intersection matrix

$$(19) \qquad M_\eta := \begin{pmatrix} \int_X \eta \mathsf{l}^2 & \int_X \eta \mathsf{l}\mathsf{m} \\ \int_X \eta \mathsf{l}\mathsf{m} & \int_X \eta \mathsf{m}^2 \end{pmatrix}$$

with rational coefficients (which are integers if η is integral). We have for example

$$(20) \qquad M_{\mathsf{l}^2} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad M_{\mathsf{lm}} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad M_{\mathsf{m}^2} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{q_X^\vee} = \begin{pmatrix} 0 & 25 \\ 25 & 0 \end{pmatrix}.$$

The manifolds X satisfying (16) are all projective and, by the surjectivity of the period map, they form an irreducible 18-dimensional family. For very general members of this family, we show that the groups of Hodge classes are very simple.

Proposition 5.4. *A very general triple* $(X,\mathsf{l},\mathsf{m})$ *satisfying* (16) *has the following properties:*

(a) $\mathrm{NS}(X) = \mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$*;*

(b) *the group of degree-2 rational Hodge classes is*

$$\mathrm{Hdg}^2(X,\mathbf{Q}) = \mathrm{Sym}^2\mathrm{NS}(X)_{\mathbf{Q}} \oplus \mathbf{Q}q_X^\vee = \mathbf{Q}\mathsf{l}^2 \oplus \mathbf{Q}\mathsf{lm} \oplus \mathbf{Q}\mathsf{m}^2 \oplus \mathbf{Q}q_X^\vee.$$

Proof. Item (a) is classical. As for item (b), the isomorphism $\mathrm{Sym}^2 H^2(X,\mathbf{Q}) \simeq H^4(X,\mathbf{Q})$ from (16′) induces a decomposition

$$(21) \qquad H^4(X,\mathbf{Q}) = \mathrm{Sym}^2 H^2(X,\mathbf{Q})_{\mathrm{tr}} \oplus (H^2(X,\mathbf{Q})_{\mathrm{tr}} \otimes \mathrm{NS}(X)_{\mathbf{Q}}) \oplus \mathrm{Sym}^2\mathrm{NS}(X)_{\mathbf{Q}},$$

where $H^2(X,\mathbf{Q})_{\mathrm{tr}} := \mathrm{NS}(X)_{\mathbf{Q}}^\perp$. Since X is very general, the Mumford–Tate group of the Hodge structure on $H^2(X,\mathbf{Q})_{\mathrm{tr}}$ is the orthogonal group of the form q_X, so that the only Hodge classes in $\mathrm{Sym}^2 H^2(X,\mathbf{Q})_{\mathrm{tr}} \subset H^4(X,\mathbf{Q})$ are multiples of the class $q_X^\vee$. It follows from the decomposition (21) that the space of rational degree-2 Hodge classes on X is generated by $\mathrm{Sym}^2\mathrm{NS}(X)_{\mathbf{Q}}$ and $q_X^\vee$. □

We described in Section 5.2 the general structure of the cones of divisors. In our case, we use an idea similar to the one used in the proof of Proposition 5.4 to make this description more precise.

Proposition 5.5. *Let* $(X, \mathsf{l}, \mathsf{m})$ *be a triple satisfying* (16) *and such that* $\mathrm{NS}(X) = \mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$. *The nef and movable cones of* X *coincide. Equivalently,* X *has no nontrivial hyper-Kähler birational models.*

Proof. We follow the proof of [HT, Theorem 22], whose argument we briefly review. We assume for a contradiction that the nef and movable cones of X are different. By [WW, Theorem 1.1], there exists a Mukai flop on X, hence a Lagrangian plane $\mathbf{P}^2 \subset X$. Let $\ell \in H_2(X, \mathbf{Z})$ be the class of a line in $\mathbf{P}^2$.

Since the lattice $(\mathrm{NS}(X), q_X)$ is unimodular (by (16′), it is a hyperbolic plane), there exists $A \in \mathrm{NS}(X)$ such that

$$\forall B \in \mathrm{NS}(X) \qquad q_X(A, B) = B \cdot \ell. \tag{22}$$

Since the class ℓ is of type $(1, 1)$, it is orthogonal to $H^{2,0}(X)$, hence to the transcendental lattice as defined in [H5, Definition 3.2.5]. But the transcendental lattice is also the orthogonal complement of $\mathrm{NS}(X)$, hence (22) remains valid for all B in $\mathrm{NS}(X)^{\perp}_{\mathbf{Q}}$, hence for all B in $H^2(X, \mathbf{Q})$.

As explained in [HT, (8)] and [O, Section 3], upon replacing X with a very general deformation for which the class ℓ remains a Hodge class (hence for which the plane $\mathbf{P}^2$ deforms along; see [V1]), we can write

$$[\mathbf{P}^2] = tA^2 + uq_X^{\vee} \in H^4(X, \mathbf{Q}),$$

for some $t, u \in \mathbf{Q}$. We now compute the numbers

$$[\mathbf{P}^2] \cdot [\mathbf{P}^2], \qquad c_2(X) \cdot [\mathbf{P}^2], \qquad A^2 \cdot [\mathbf{P}^2]$$

in order to evaluate t, u, and $q_X(A)$.

Since $\mathbf{P}^2 \subset X$ is Lagrangian, we have $N_{\mathbf{P}^2/X} \simeq \Omega^1_{\mathbf{P}^2}$, and so (here we use $c_X = 3$ from (16′), and (17))

$$3 = c_2(N_{\mathbf{P}^2/X}) = [\mathbf{P}^2] \cdot [\mathbf{P}^2] = 3t^2 q_X(A)^2 + 50tuq_X(A) + 575u^2. \tag{23}$$

From the normal bundle exact sequence for $\mathbf{P}^2 \subset X$, we get (here we use (18) and (17))

$$-3 = c_2(X) \cdot [\mathbf{P}^2] = \frac{6}{5}\left(25tq_X(A) + 575u\right) = 30\left(tq_X(A) + 25u\right). \tag{24}$$

Finally, since $A|_{\mathbf{P}^2} = (A \cdot \ell)\ell$, we get from (22) the relations

$$q_X(A)^2 = (A \cdot \ell)^2 = (A|_{\mathbf{P}^2})^2 = A^2 \cdot [\mathbf{P}^2] = 3tq_X(A)^2 + 25uq_X(A). \tag{25}$$

Solving (23), (24), and (25) for t and u, we get the equation

$$92q_X(A)^2 + 20q_X(A) - 525 = 0,$$

which has no integral solutions. This a contradiction, which proves the proposition. ☐

Under the assumptions of Proposition 5.5, permuting l and m if necessary, we can assume that l is nef and write, as in Lemma 5.3 (by Proposition 5.5, we do not need to replace X by a birational model as in Section 5.2)

$$\begin{aligned} \overline{\mathrm{Pos}}(X) &= \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}\mathsf{m}, \\ \overline{\mathrm{Mov}}(X) = \mathrm{Nef}(X) &= \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}(t_0\mathsf{l} + \mathsf{m}), \\ \mathrm{Psef}(X) &= \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}(-t_0\mathsf{l} + \mathsf{m}), \end{aligned} \tag{26}$$

where $t_0 \in \{0,1\}$. In sum, there are two cases for a triple $(X, \mathsf{l}, \mathsf{m})$ satisfying (16) and such that $\mathrm{NS}(X) = \mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$:

(C1) either $t_0 = 0$, the class m is nef and all cones of divisors are equal;

(C2) or $t_0 = 1$ and Lemma 5.3 says that there exists a unique prime divisor E whose class is not in the positive cone. We have $E \in |L^{-1} \otimes M|$ and the sections of $(L \otimes M)^k$ define, for $k \gg 0$, the divisorial contraction $c\colon X \to Y$ of E. The discussion of Section 3.2 still applies: a general fiber of $E \to c(E)$ is a smooth rational curve and the reflection about the hyperplane $(-\mathsf{l} + \mathsf{m})^\perp$ is a monodromy operator that permutes l and m.

In short, the class m is nef in case (C1) and not nef in case (C2). In the former case, we have the following result.

Proposition 5.6. *Let $(X, \mathsf{l}, \mathsf{m})$ be a very general triple satisfying (16) and assume in addition that we are in case (C1). Then one of the following statements holds:*

(a) *$H^0(X, L) \neq 0$, $H^0(X, M) \neq 0$, and there is a Lagrangian fibration $f\colon X \to \mathbf{P}^2$ with $f^*\mathscr{O}_{\mathbf{P}^2}(1) \simeq L$ or M;*

(b) *at least one of $H^0(X, L)$ or $H^0(X, M)$ is trivial and the image of the rational map $\varphi_{L\otimes M}\colon X \dashrightarrow \mathbf{P}^5$ is rationally connected.*

We will prove Proposition 5.6 in Section 6.3. As a consequence of the results in Section 3.2 and [V2], we obtain that case (C1) actually does not occur.

Corollary 5.7. *Let $(X, \mathsf{l}, \mathsf{m})$ be a very general triple satisfying (16). Case (C1) does not occur.*

Proof. By [V2, Theorem 4.2], in case (C1), if $H^0(X, L)$ or $H^0(X, M)$ is 0, the image of the rational map $\varphi_{L\otimes M}\colon X \dashrightarrow \mathbf{P}^5$ cannot be rationally connected. Hence we are in case (a) of Proposition 5.6: there exists a Lagrangian fibration $f\colon X \to \mathbf{P}^2$ with $f^*\mathscr{O}_{\mathbf{P}^2}(1) \simeq L$ or M. The discussion of Section 3.2 then applies, showing that M and L cannot be both nef, contradicting (C1). □

In case (C2), we have the following result.

Proposition 5.8. *Let $(X, \mathsf{l}, \mathsf{m})$ be a very general triple satisfying (16) and assume in addition that we are in case (C2). There exist a positive integer k_L and a Lagrangian fibration $f\colon X \to \mathbf{P}^2$ such that $f^*\mathscr{O}_{\mathbf{P}^2}(1) \simeq L^{k_L}$.*

We will prove Proposition 5.8 in Section 7. We will then show in Section 8 that $k_L = 1$ and X is of $\mathrm{K3}^{[2]}$ deformation type, thus completing the proof of both Theorem 1.5 and Theorem 1.8.

We note that Corollary 5.7 and Proposition 5.8 do already imply a slightly weaker version of Theorem 1.8, where we have no control on k_L. We include the proof since it does not use the results of Section 8 and might apply to more general situations.

Corollary 5.9 (Hyper-Kähler SYZ conjecture)**.** *Let X be a hyper-Kähler fourfold and let L be a nef line bundle on X. Set $\mathsf{l} := c_1(L)$. Assume $\int_X \mathsf{l}^4 = 0$ and that there exists a class $\mathsf{m} \in H^2(X, \mathbf{Z})$ with $\int_X \mathsf{l}^2\mathsf{m}^2 = 2$. There exists a Lagrangian fibration $f\colon X \to \mathbf{P}^2$ with $f^*\mathscr{O}_{\mathbf{P}^2}(1) \simeq L^{k_L}$ for some positive integer k_L.*

Proof. We can argue as in the beginning of the proof of Theorem 3.1 and consider the moduli space $\mathfrak{M}_\mathsf{l}$ of marked hyper-Kähler manifolds with a fixed $(1,1)$-class l. Then there exist points $0, 0' \in \mathfrak{M}_\mathsf{l}$ such that $(\mathscr{X}_0, \mathscr{L}_0) = (X, L)$ and $(X', L') := (\mathscr{X}_{0'}, \mathscr{L}_{0'})$ has the property that its

Néron–Severi group is generated by the classes l and m, L' is nef, and X' is very general in the sense of Proposition 5.5.

By Proposition 5.8, some power L'^k defines a Lagrangian fibration $X' \to \mathbf{P}^2$. According to [Ma3, Lemma 2.4], this implies that the line bundle $\mathscr{L}_t$ is semi-ample for all fibers $\mathscr{X}_t$ of Picard rank one. In particular, for very general points $t \in \mathfrak{M}_{\mathsf{l}}$, one has $h^0(\mathscr{X}_t, \mathscr{L}_t^{k_t}) \geq 2$ for some positive integer k_t. Hence, the countable union of the closed sets $\{t \in \mathfrak{M}_{\mathsf{l}} \mid h^0(\mathscr{X}_t, \mathscr{L}_t^k) \geq 2\}$, for all $k \in \mathbf{Z}_{>0}$, contains all very general points. This is enough to conclude that there exists one k for which the corresponding set is all of $\mathfrak{M}_{\mathsf{l}}$ and, in particular $h^0(X, L^k) \geq 2$. Hence, L is semi-ample by Section 5.1. □

6. The case of two nef isotropic classes

Let $(X, \mathsf{l}, \mathsf{m})$ be a very general triple satisfying (16). By Proposition 5.4, we have $\mathrm{NS}(X) = \mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$. We assume in this section that we are in case (C1): the class m is isotropic and both l and m are nef. Our aim is to prove Proposition 5.6 (in Section 6.3).

As observed in Section 5.3, all cones of divisors are equal and the class $p\mathsf{l} + q\mathsf{m}$ is ample on X for all integers $p, q > 0$. By Kodaira vanishing, we get

$$h^0(X, L^p \otimes M^q) = \chi(X, L^p \otimes M^q) = P_{RR,X}(2pq) = \binom{pq+3}{2} \tag{27}$$

and in particular

$$h^0(X, L \otimes M) = 6, \quad h^0(X, L^2 \otimes M) = 10, \quad h^0(X, L^3 \otimes M^2) = 36.$$

The next two sections and the paper [V2] are devoted to the study of the induced rational map

$$\varphi_{L\otimes M} \colon X \dashrightarrow \mathbf{P}^5.$$

This map was studied by O'Grady in [O, Section 4] when X is of $\mathrm{K3}^{[2]}$ numerical type, and for a very general deformation of the pair $(X, L \otimes M)$.

6.1. Case where L and M both have nonzero sections.

The following lemma will allow us to apply Lemma 5.1. We keep the same hypotheses: $(X, \mathsf{l}, \mathsf{m})$ is a very general triple satisfying (16) and we are in case (C1).

Lemma 6.1. *Assume that $H^0(X, L) \neq 0$ and $H^0(X, M) \neq 0$. Then*

$$h^0(X, L) + h^0(X, M) \geq 3,$$

hence either $h^0(X, L) \geq 2$ or $h^0(X, M) \geq 2$.

Proof. Let σ be a nonzero section of L and let τ be a nonzero section of M. The divisors D_σ of σ and D_τ of τ have no common component, since the class of any effective divisor on X is an integral combination of l and m with nonnegative coefficients. It follows that

$$\Sigma_{\sigma\tau} \coloneqq D_\sigma \cap D_\tau$$

is a surface whose ideal sheaf $\mathscr{I}_{\Sigma_{\sigma\tau}}$ admits the Koszul resolution

$$0 \to (L \otimes M)^{-1} \to L^{-1} \oplus M^{-1} \to \mathscr{I}_{\Sigma_{\sigma\tau}} \to 0.$$

If we tensor it by $L \otimes M$, the associated exact sequence in cohomology gives

$$h^0(X, \mathscr{I}_{\Sigma_{\sigma\tau}}(L \otimes M)) = h^0(X, L) + h^0(X, M) - 1. \tag{28}$$

If we tensor the resolution by $L^2 \otimes M^2$, we get, using (27),

$$\begin{aligned} &h^0(X, \mathscr{I}_{\Sigma_{\sigma\tau}}(L^2 \otimes M^2)) = h^0(X, L^2 \otimes M) + h^0(X, L \otimes M^2) - h^0(X, L \otimes M) = 14, \\ &h^1(X, \mathscr{I}_{\Sigma_{\sigma\tau}}(L^2 \otimes M^2)) = 0. \end{aligned} \tag{29}$$

Using again (27), we get $h^0(X, L^2 \otimes M^2) = 21$, and, from (29), we deduce

$$h^0(\Sigma_{\sigma\tau}, (L^2 \otimes M^2)|_{\Sigma_{\sigma\tau}}) = 7. \tag{30}$$

Assume by contradiction $h^0(X, L) + h^0(X, M) = 2$. Then, by (28), we get

$$\mathrm{rk}\big(H^0(X, L \otimes M) \longrightarrow H^0(\Sigma_{\sigma\tau}, (L \otimes M)|_{\Sigma_{\sigma\tau}})\big) = 5. \tag{31}$$

In particular, the surface $\Sigma_{\sigma\tau}$ is not contained in the base locus of $|L \otimes M|$.

We prove now the following properties.

Claim 6.2. *In the notation used above,*

(a) *the surface $\Sigma_{\sigma\tau}$ is irreducible and reduced;*
(b) *the image of the rational map*

$$\varphi_{L\otimes M}|_{\Sigma_{\sigma\tau}} \colon \Sigma_{\sigma\tau} \dashrightarrow \mathbf{P}^5$$

is a surface.

Proof. To prove (a), we note that the surface $\Sigma_{\sigma\tau}$ has class lm and its associated matrix (as in (19)) is

$$M_{[\Sigma_{\sigma\tau}]} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

If $\Sigma_{\sigma\tau}$ is not irreducible or not reduced, there exist surfaces Σ_1, Σ_2 in X, such that

$$M_{[\Sigma_1]} = M_{[\Sigma_2]} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We show that no such decomposition exists under our assumptions.

Since $(X, \mathsf{l}, \mathsf{m})$ is very general, we may apply Proposition 5.4 and write

$$[\Sigma_i] = t_i\mathsf{l}^2 + u_i\mathsf{lm} + v_i\mathsf{m}^2 + w_i q_X^\vee$$

with $t_i, u_i, v_i \in \mathbf{Q}_{\geq 0}$ (because l and m are nef) and $w_i \in \mathbf{Q}$. Using (20), we find

$$t_i = v_i = 0, \quad 2u_i + 25w_i = 1,$$

so that

$$[\Sigma_i] = \frac{1}{2}\Big(1 - 25w_i\Big)\mathsf{lm} + w_i q_X^\vee.$$

We obtain $w_2 = -w_1$, so we can assume $w =: w_1 \geq 0$. We have

$$[\Sigma_i] = \frac{1}{2}\mathsf{lm} - (-1)^i w\Big(q_X^\vee - \frac{25}{2}\mathsf{lm}\Big), \quad \text{for } i \in \{1, 2\}.$$

Using (17), we get

$$\Sigma_1^2 = \Sigma_2^2 = \frac{1}{2} + 525w^2, \qquad \Sigma_1\Sigma_2 = \frac{1}{2} - 525w^2.$$

Since these numbers are integers, w is positive and $2 \cdot w^2 = 1 \cdot 3 \cdot 7 \cdot (5w)^2$ is an integer, which implies that the denominator of w is at most 5, so that $w \geq \frac{1}{5}$. Then,

$$wq_X^\vee = [\Sigma_1] + \frac{1}{2}(25w - 1)\mathsf{lm} \tag{32}$$

with $25w-1>0$. But this is impossible: by Proposition 5.5 and [H4, Proposition 3.2], the closure of the Kähler cone of X is the subset $\overline{\mathscr{K}}_X \subset H^{1,1}(X,\mathbf{R})$ consisting of those real $(1,1)$-classes ω satisfying $q_X(\omega)\geq 0$, $q_X(\omega,\omega_0)>0$ for a Kähler class ω_0, and $q_X(\omega,\mathsf{l})\geq 0$ and $q_X(\omega,\mathsf{m})\geq 0$. For all $\omega\in\overline{\mathscr{K}}_X$ and all effective classes $\eta\in \mathrm{Hdg}^4(X,\mathbf{Q})$, we have $\int_X \eta\omega^2\geq 0$. Furthermore, for all $\omega\in\overline{\mathscr{K}}_X$ such that $q_X(\omega)=0$, we have $\int_X q_X^\vee\omega^2=0$; so by (32) we have

$$0=\int_{\Sigma_1}\omega^2+\frac{1}{2}(25w-1)\int_X \mathsf{l}\mathsf{m}\omega^2,$$

where both terms on the right are nonnegative. Hence $\int_X \mathsf{l}\mathsf{m}\omega^2=0$; this implies that the class $\mathsf{l}\mathsf{m}$ is proportional to $q_X^\vee$, which is absurd.

Claim (b), namely the fact that the image $\varphi_{L\otimes M}(\Sigma_{\sigma\tau})$ is a surface, is proved as follows. If $\varphi_{L\otimes M}(\Sigma_{\sigma\tau})$ is a linearly nondegenerate curve in $\mathbf{P}^4$, it has degree $d\geq 4$. Let f be the class of a fiber of $\Sigma_{\sigma\tau} \dashrightarrow \varphi_{L\otimes M}(\Sigma_{\sigma\tau})$. We have

$$(\mathsf{l}+\mathsf{m})[\Sigma_{\sigma\tau}]=(\mathsf{l}+\mathsf{m})\mathsf{l}\mathsf{m}=\mathsf{l}^2\mathsf{m}+\mathsf{l}\mathsf{m}^2$$

and this class can be written as $d\mathsf{f}+\mathsf{e}$, where e is an effective curve class. This contradicts the fact that the class of any effective curve in X has the form $t\mathsf{l}^2\mathsf{m}+u\mathsf{l}\mathsf{m}^2$ with $t,u\in\frac{1}{2}\mathbf{Z}_{\geq 0}$. □

The claim gives us a contradiction, since we know by (31) that $\varphi_{L\otimes M}(\Sigma_{\sigma\tau})$ is a linearly nondegenerate surface in $\mathbf{P}^4$, while, by (30), it is contained in at least $\binom{4+2}{2}-7=8$ independent quadrics; this contradicts Castelnuovo's lemma, which says that it is contained in at most $\binom{2+1}{2}$ independent quadrics. □

6.2. **Case where either L or M has no nonzero sections.** We keep the same hypotheses: $(X,\mathsf{l},\mathsf{m})$ is a very general triple satisfying (16) and we are in case (C1). This section is devoted to the proof of the following result.

Proposition 6.3. *If either $H^0(X,L)=0$ or $H^0(X,M)=0$, the image of the rational map $\varphi_{L\otimes M}\colon X\dashrightarrow \mathbf{P}^5$ is rationally connected.*

Our aim is to prove that

$$Y:=\mathrm{Im}(\varphi_{L\otimes M})\subset \mathbf{P}^5$$

is rationally connected, that is, that a general pair of points of Y are joined by a rational curve. If $\dim(Y)<4$, this holds by [L, Theorem 1.4], so it suffices to prove Proposition 6.3 in the case $\dim(Y)=4$, that is, when $Y\subset\mathbf{P}^5$ is a hypersurface.

We will assume $H^0(X,L)=0$; the case $H^0(X,M)=0$ is of course analogous (just permute L and M).

Proposition 6.4. *If $H^0(X,L)=0$ and $\dim(Y)=4$, a plane section C_{0,W_3} of Y defined by a general vector subspace $W_3\subset H^0(X,L\otimes M)$ of dimension 3 is either of geometric genus 0 or a smooth cubic curve.*

Proposition 6.4 implies Proposition 6.3 as follows. Proposition 6.4 then shows that either Y is rationally connected since its general plane sections are rational curves, or it is a cubic hypersurface, hence it is uniruled. In the second case, we can apply [L, Theorem 1.4] again to the maximal rationally connected quotient of Y, which has dimension <4. It is thus rationally connected and is therefore a point by [GHS], proving again that Y is rationally connected.

We start the proof of Proposition 6.4 by introducing some notation. Set

$$\begin{aligned} W_6 &:= H^0(X, L \otimes M), \\ W_{10} &:= H^0(X, L^2 \otimes M), \\ W_{36} &:= H^0(X, L^3 \otimes M^2), \end{aligned}$$

with multiplication map

$$\mu\colon W_6 \otimes W_{10} \longrightarrow W_{36}.$$

Lemma 6.5. *If $H^0(X, L) = 0$, there are no rank-2 elements in* $\mathrm{Ker}(\mu)$.

Proof. A rank-2 element in $\mathrm{Ker}(\mu)$ is given by a nontrivial relation $\alpha\sigma = \beta\tau$ in $H^0(X, L^3 \otimes M^2)$ with $\alpha, \beta \in H^0(X, L \otimes M)$ linearly independent. Since $H^0(X, L) = 0$, any divisor in $|L \otimes M|$ is irreducible and reduced, hence the divisors of α and β have no common component. It follows that $\tau \in H^0(X, L^2 \otimes M)$ vanishes on the divisor of α hence can be written as the product of α by a section of L. This implies $\tau = 0$, which is absurd. □

Lemma 6.6. *Assume there are no rank-2 elements in* $\mathrm{Ker}(\mu)$. *For a general 3-dimensional vector subspace $W_3 \subset W_6$, the restriction $\mu|_{W_3}\colon W_3 \otimes W_{10} \to W_{36}$ of the map μ has rank ≥ 28.*

Proof. Let $\mathscr{S}_3 \to \mathrm{Gr}(3, W_6)$ be the rank-3 tautological subbundle. The natural sheaf inclusion $\mathscr{S}_3 \to W_6 \otimes \mathscr{O}_{\mathrm{Gr}(3,W_6)}$ induces a morphism $f\colon \mathbf{P}(\mathscr{S}_3 \otimes W_{10}) \to \mathbf{P}(W_6 \otimes W_{10})$ which makes $\mathbf{P}(\mathscr{S}_3 \otimes W_{10})$ a smooth birational model of the set of elements of rank at most 3 in $\mathbf{P}(W_6 \otimes W_{10})$. We set

$$R_3 := f^{-1}(\mathbf{P}(\mathrm{Ker}(\mu))).$$

Since, by assumption, there are no rank-2 elements in $\mathrm{Ker}(\mu)$, the scheme R_3 is isomorphic to the set of elements of rank ≤ 3 in $\mathbf{P}(\mathrm{Ker}(\mu))$.

The fiber of the natural morphism $\pi\colon R_3 \to \mathrm{Gr}(3, W_6)$ over a point $[W_3]$ is the space $\mathbf{P}(\mathrm{Ker}(\mu|_{W_3}))$. Arguing by contradiction, if the conclusion of the lemma does not hold, the fiber of π has dimension ≥ 2, hence $\dim(R_3) \geq 11$.

Let H be the restriction to R_3 of the line bundle $\mathscr{O}_{\mathbf{P}(\mathscr{S}_3 \otimes W_{10})}(1)$. We have a tautological inclusion $H^{-1} \to \pi^*\mathscr{S}_3 \otimes W_{10}$

$$W_{10}^\vee \otimes \mathscr{O}_{R_3} \longrightarrow H \otimes \pi^*\mathscr{S}_3. \tag{33}$$

Since there are no rank-2 elements in $\mathrm{Ker}(\mu)$, the morphism (33) is surjective, with kernel $\mathscr{K}$ a locally free sheaf of rank 7. Thus we have $c_i(\mathscr{K}) = 0$ for $i > 7$, that is, $s_i(H \otimes \pi^*\mathscr{S}_3) = 0$, for all $i > 7$, where the s_i denote the Segre classes.

We now deduce a contradiction. For any line bundle H on a variety and any vector bundle $\mathscr{E}$ of rank 3, we have the relation

$$s_i(\mathscr{E} \otimes H) = \sum_{j=0}^{i} (-1)^j \binom{i+2}{j} H^j s_{i-j}(\mathscr{E}). \tag{34}$$

In our case, one has $s_j(\mathscr{S}_3) = 0$ for $j \geq 4$, because of the exact sequence

$$0 \to \mathscr{S}_3 \to W_6 \otimes \mathscr{O}_{\mathrm{Gr}(3,W_6)} \to \mathscr{Q}_3 \to 0$$

on the Grassmannian, where $\mathscr{Q}_3$ is locally free of rank 3. The relations (34) thus give four linear relations involving $s_0(\pi^*\mathscr{S}_3), \ldots, s_3(\pi^*\mathscr{S}_3)$, namely

$$0 = s_8(\pi^*\mathscr{S}_3 \otimes H)$$

$$= \binom{10}{8} H^8 s_0(\pi^*\mathscr{S}_3) - \binom{10}{7} H^7 s_1(\pi^*\mathscr{S}_3) + \binom{10}{6} H^6 s_2(\pi^*\mathscr{S}_3) - \binom{10}{5} H^5 s_3(\pi^*\mathscr{S}_3),$$
$$0 = s_9(\pi^*\mathscr{S}_3 \otimes H)$$
$$= -\binom{11}{9} H^9 s_0(\pi^*\mathscr{S}_3) + \binom{11}{8} H^8 s_1(\pi^*\mathscr{S}_3) - \binom{11}{7} H^7 s_2(\pi^*\mathscr{S}_3) + \binom{11}{6} H^6 s_3(\pi^*\mathscr{S}_3),$$
$$0 = s_{10}(\pi^*\mathscr{S}_3 \otimes H)$$
$$= \binom{12}{10} H^{10} s_0(\pi^*\mathscr{S}_3) - \binom{12}{9} H^9 s_1(\pi^*\mathscr{S}_3) + \binom{12}{8} H^8 s_2(\pi^*\mathscr{S}_3) - \binom{12}{7} H^7 s_3(\pi^*\mathscr{S}_3),$$
$$0 = s_{11}(\pi^*\mathscr{S}_3 \otimes H)$$
$$= -\binom{13}{11} H^{11} s_0(\pi^*\mathscr{S}_3) + \binom{13}{10} H^{10} s_1(\pi^*\mathscr{S}_3) - \binom{13}{9} H^9 s_2(\pi^*\mathscr{S}_3) + \binom{13}{8} H^8 s_3(\pi^*\mathscr{S}_3).$$

We can assume $\dim(R_3) = 11$, replacing it by a proper algebraic subset if necessary. Multiplying these equations by adequate powers of H, we get the linear relations

$$0 = H^3 s_8(\pi^*\mathscr{S}_3 \otimes H), \qquad 0 = H^2 s_9(\pi^*\mathscr{S}_3 \otimes H),$$
$$0 = H s_{10}(\pi^*\mathscr{S}_3 \otimes H), \qquad 0 = s_{11}(\pi^*\mathscr{S}_3 \otimes H),$$

which, by expanding as above, give four linear relations between the four intersection numbers

$$H^8 s_3(\pi^*\mathscr{S}_3),\ H^9 s_2(\pi^*\mathscr{S}_3),\ H^{10} s_1(\pi^*\mathscr{S}_3),\ H^{11}$$

on R_3. Since these four linear relations are clearly independent, we conclude that these four intersections numbers vanish, which contradicts the fact that H is ample on R_3. □

Lemma 6.5 and Lemma 6.6 together imply the following result.

Corollary 6.7. *Assume $H^0(X, L) = 0$. For a general 3-dimensional vector subspace $W_3 \subset W_6$, the image $\mu(W_3 \otimes W_{10})$ has dimension ≥ 28.*

Proof of Proposition 6.4. We choose a general 3-dimensional vector subspace $W_3 \subset W_6$. The locus defined by the vanishing of the sections in W_3 has a mobile part $C_{W_3} \subset X$ which, by Bertini's theorem, is an irreducible curve which dominates the plane curve C_{0,W_3} via $\varphi_{L\otimes M}$. Consider the restriction maps

$$r^{p,q} \colon H^0(X, L^p \otimes M^q) \longrightarrow H^0(C_{W_3}, (L^p \otimes M^q)|_{C_{W_3}})$$

for $p, q > 0$. Let us set $W'_{p,q} := \operatorname{Im}(r^{p,q})$. We will estimate the dimension of $W'_{p,q}$, for $(p, q) \in \{(1, 1), (2, 1), (3, 2)\}$. First of all, we note that $W'_{1,1}$ has dimension 3.

We then claim that $W'_{3,2}$ has dimension ≤ 8. Indeed, we have the inclusion

$$\mu(W_3 \otimes W_{10}) \subset \operatorname{Ker}(r^{3,2})$$

and, by Corollary 6.7, the space on the left has dimension at least 28, while $h^0(X, L^3 \otimes M^2) = 36$.

Finally, we claim that

(a) either the space $W'_{2,1}$ has dimension ≥ 5,
(b) or it has dimension 4 and the image $\varphi_{L^2\otimes M}(C_{W_3})$ is a rational normal cubic curve in $\mathbf{P}^3$.

Indeed, since $\varphi_{L\otimes M}$ is a dominant rational map to a hypersurface $Y \subset \mathbf{P}^5$, the curve C_{W_3} can be chosen to pass through a general triple of points $x, y, z \in X$. Assume $\varphi_{L^2\otimes M}(C_{W_3})$ spans at most a $\mathbf{P}^3$ in $\mathbf{P}^9 = \mathbf{P}(W_{10}^\vee)$. This $\mathbf{P}^3$ then contains the projective plane spanned by the images of x, y, z, and thus this plane intersects the curve $\varphi_{L^2\otimes M}(C_{W_3})$ in at least a fourth point, unless $\varphi_{L^2\otimes M}(C_{W_3})$ is a rational normal curve of degree 3.

In the former case, we conclude that the variety $Y' := \varphi_{L^2\otimes M}(X)$, which spans $\mathbf{P}^9$, has the property that a general trisecant plane of Y' is 4-secant. This is absurd, since a general 1-dimensional linear section of Y' spans at least a $\mathbf{P}^6$.

In the latter case, the curve $\varphi_{L^2\otimes M}(C_{W_3})$ has degree ≤ 3. This curve cannot be a plane curve, since otherwise the projection of Y' through any of its points y would contain a line through any two points x, z, hence would be a projective space of dimension at most 4. Hence it must be a rational normal curve in $\mathbf{P}^3$, as claimed.

Let us now consider the multiplication map

$$W'_{1,1}\otimes W'_{2,1}\longrightarrow W'_{3,2}.$$

In case (a), the three spaces have respective dimensions 3, at least 5, and at most 8. Since the curve C_{W_3} is irreducible, we can apply Lemma 6.8 below: it says that this is possible only if the linear system $W'_{1,1}$ factors through an elliptic plane curve or a rational curve.

In case (b), we have $W'_{2,1} = \mathrm{Sym}^3(W''_2)$, for some 2-dimensional linear system W''_2 on C_{W_3}. We now consider the multiplication maps

$$\begin{aligned} W'_{1,1}\otimes W''_2 &\longrightarrow W''_{k_1},\\ W''_{k_1}\otimes W''_2 &\longrightarrow W''_{k_2},\\ W''_{k_2}\otimes W''_2 &\longrightarrow W'_{3,2},\end{aligned}$$

where W''_{k_1} and W''_{k_2} are spaces of sections of adequate line bundles on C_{W_3}, of respective dimensions k_1 and k_2. The Hopf lemma ([ACGH, p. 108]) gives

$$\begin{aligned} \dim(W''_{k_1}) &\geq \dim(W'_{1,1})+1,\\ \dim(W''_{k_2}) &\geq \dim(W''_{k_1})+1,\\ \dim(W'_{3,2}) &\geq \dim(W''_{k_2})+1.\end{aligned}$$

As $\dim(W'_{1,1}) = 3$ and $\dim(W'_{3,2}) \leq 8$, the three inequalities above cannot all be strict, hence one of them must be an equality. It is well known that this implies that the plane curve $C_{0,W_3} = \varphi_{L\otimes M}(C_{W_3})$ is rational, thus completing the proof of Proposition 6.4. ☐

We used above the following lemma, for which we could not find a reference.

Lemma 6.8. *Let C be a smooth connected projective curve and let H and H' be line bundles on C. Let $W_3 \subset H^0(C,H)$ and $W_k \subset H^0(C,H')$ be base-point-free linear systems on C, of respective linear dimensions 3 and k, with $k \geq 4$. Assume that the rank of the multiplication map*

$$\mu\colon W_3\otimes W_k \longrightarrow H^0(C, H\otimes H')$$

is at most $3+k$. Then, both linear systems factor through a morphism $C\to C_0$, where C_0 is either a rational curve or a degree-3 elliptic curve in $\mathbf{P}(W_3)$.

Proof. Denote by $\varphi_3\colon C\to\mathbf{P}^2$ the morphism induced by W_3 and by $\varphi_k\colon C\to \mathbf{P}^{k-1}$ the morphism induced by W_k. We first claim that the result is true if φ_k does not factor through φ_3. Indeed, assume there is a length-2 subscheme $z\subset C$ that imposes only one condition on W_3

and two conditions on W_k. Then z imposes two conditions on $\mathrm{Im}(\mu)$, hence the multiplication map

$$\mu_z\colon W_3(-z)\otimes W_k \longrightarrow H^0(C, H\otimes H')$$

has rank at most $k+1$, while $W_3(-z)$ has dimension 2. We are thus in the equality case of the Hopf lemma and we conclude that W_k is the pullback to C of $H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(k-1))$ via a morphism $\psi\colon C\to \mathbf{P}^1$. This case is easily concluded by studying the multiplication maps

$$W_3\otimes H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(k-2)) \longrightarrow H^0(C, H\otimes \psi^*\mathscr{O}_{\mathbf{P}^1}(k-2)),$$

with image W', and

$$W'\otimes H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(1)) \longrightarrow H^0(C, H\otimes \psi^*\mathscr{O}_{\mathbf{P}^1}(k-1)),$$

of rank $\leq 3+k$. By the Hopf lemma applied to both maps, $\dim(W')\in\{k+1, k+2\}$. Thus, one the two maps satisfies the equality in the Hopf lemma, hence φ_3 also factors through ψ. This proves the claim.

We now prove a similar claim with φ_3 and φ_k permuted. Assume there is a length-2 subscheme $z\subset C$ that imposes only one condition on W_k but two conditions on W_3. This produces a $(k-1)$-dimensional vector subspace $W_k(-z)\subset W_k$ of sections vanishing on z, with the property that the multiplication map

$$\mu_z\colon W_3\otimes W_k(-z) \longrightarrow H^0(X, H\otimes H'(-z))$$

has rank $\leq k+1$, which is the minimum allowed by the Hopf Lemma. We conclude that there is a morphism $\psi : C\to \mathbf{P}^1$ such that $W_3 = \psi^*H^0(\mathbf{P}^1, \mathscr{O}_{\mathbf{P}^1}(2))$. We now consider the multiplication maps

$$\mu_1'\colon W_2\otimes W_k \longrightarrow H^0(C, H'\otimes \psi^*\mathscr{O}_{\mathbf{P}^1}(1)),$$

with image W', and

$$\mu_2'\colon W_2\otimes W' \longrightarrow H^0(C, H'\otimes \psi^*\mathscr{O}_{\mathbf{P}^1}(2)) = H^0(C, H\otimes H').$$

We know that μ_2' has rank at most $3+k$, so either $\dim(W')=k+1$ and μ_1' satisfies the equality case of the Hopf lemma, or $\dim(W')=k+2$ and μ_2' satisfies the equality case of the Hopf lemma. In both cases, we conclude that both linear systems factor through ψ.

Using these two claims, we can now assume that both linear systems factor through the curve $C_0 \coloneqq \varphi_3(C)\subset \mathbf{P}(W_3)$, that C_0 is birationally isomorphic to its image in $\mathbf{P}(W_k)$, and that the normalization C_0' of C_0 is not rational, as otherwise the lemma is proved. We denote by H_0 and H_0' the line bundles on C_0' whose respective pullbacks to C are H and H'. For general points $x_1,\dots,x_{k-3}\in C_0'$, we have a 3-dimensional space $W_3'\subset W_k$ of sections vanishing at $x_1,\dots,x_{k-3}$, and the multiplication map

$$\mu'\colon W_3\otimes W_3' \longrightarrow H^0(C_0', H_0\otimes H_0')$$

has rank ≤ 6 since its image is contained in $\mathrm{Im}(\mu)$ and vanishes at $x_1,\dots,x_{k-3}$. We can thus assume that the kernel

$$K\coloneqq \mathrm{Ker}(\mu')\subset W_3\otimes W_3' \tag{35}$$

has dimension 3 (if it has dimension 4, we are in the equality case of the Hopf lemma, so we can ignore this case). The inclusion (35) induces a morphism

$$K\otimes \mathscr{O}_{\mathbf{P}(W_3)} \longrightarrow W_3'\otimes \mathscr{O}_{\mathbf{P}(W_3)}(1) \tag{36}$$

of rank-3 vector bundles on $\mathbf{P}(W_3)$. This morphism has rank at most 2 along C_0 and must have rank at least 2 generically on $\mathbf{P}(W_3)$ since otherwise, the image would be a rank-1 sheaf with at least three independent sections contained in $W_3'\otimes\mathscr{O}_{\mathbf{P}(W_3)}(1)$, hence a copy of $\mathscr{O}_{\mathbf{P}(W_3)}(1)$ and the elements of K would be rank-1 tensors in $W_3 \otimes W_3'$. Since we assumed that the curve C_0' is not rational, no quadratic equation vanishes on it, hence the morphism (36) also has generic rank 2 along the curve C_0.

If the curve C_0 has degree at most 3, the lemma is proved. Otherwise, the morphism (36) has rank at most 2 everywhere on $\mathbf{P}(W_3)$ and is generically of rank 2 along C_0, which means that the three polynomials of type $(1,1)$ on $\mathbf{P}(W_3) \times \mathbf{P}(W_3')$ given by K vanish on a surface Σ which contains the natural embedding of C_0' in $\mathbf{P}(W_3) \times \mathbf{P}(W_3')$ and is birationally isomorphic to $\mathbf{P}(W_3)$ by the first projection. The surface Σ is also birationally isomorphic to $\mathbf{P}(W_3')$ by the second projection since its image in $\mathbf{P}(W_3')$ contains the image of C_0' in $\mathbf{P}(W_3')$ which, being birationally isomorphic to C_0' for a generic choice of $x_1 \ldots, x_{k-3}$, is also not rational.

We claim that the surface $\Sigma \subset \mathbf{P}(W_3) \times \mathbf{P}(W_3')$ is the graph of an isomorphism $\mathbf{P}(W_3) \simeq \mathbf{P}(W_3')$. Indeed, as Σ is contained in three hypersurfaces of type $(1,1)$, it is an irreducible component of the complete intersection of two such hypersurfaces, but it is not the complete intersection of two such hypersurfaces (since there is a third equation of type $(1,1)$ vanishing on it). It follows that the class of Σ is of the form $(h_1 + h_2)^2 - e = h_1^2 + 2h_1h_2 + h_2^2 - e$, where $h_i := c_1(\mathrm{pr}_i^* \mathscr{O}_{\mathbf{P}^2}(1))$ and the class e is effective and nonzero on $\mathbf{P}(W_3) \times \mathbf{P}(W_3')$. Since the projections pr_1 and pr_2 are dominant and $(\Sigma \cdot h_1 \cdot h_2)^2 \geq (\Sigma \cdot h_1^2)(\Sigma \cdot h_2^2)$ (Hodge Index Theorem), the only possibility is $[\Sigma] = h_1^2 + h_1h_2 + h_2^2$, that is, $[\Sigma]$ is the class of the graph of an isomorphism $\mathbf{P}(W_3) \simeq \mathbf{P}(W_3')$. This implies that Σ itself is the graph of an isomorphism since we then have $\Sigma^* \mathscr{O}_{\mathbf{P}(W_3')}(1) = \mathscr{O}_{\mathbf{P}(W_3)}(1)$. This proves the claim. The claim implies that the line bundles H_0 and $H_0'(-x_1 - \cdots - x_{k-3})$ on C_0' coincide. As $x_1 \ldots, x_{k-3}$ are general points of C_0' and $k \geq 4$, this implies that C_0' is rational, which is a contradiction. □

6.3. **Proof of Proposition 5.6.** If $H^0(X,L)$ and $H^0(X,M)$ are both nonzero, by Lemma 6.1 and Lemma 5.1, after possibly permuting L and M, the line bundle L is globally generated and thus gives a Lagrangian fibration $f \colon X \to \mathbf{P}^2$ with $f^* \mathscr{O}_{\mathbf{P}^2}(1) \simeq L$. So we are in case (a) of Proposition 5.6.

If either $H^0(X,L)$ or $H^0(X,M)$ is zero, the image of the rational map $\varphi_{L \otimes M}$ is rationally connected by Proposition 6.3 and we are in case (b) of Proposition 5.6.

7. The divisorial contraction case

Let $(X, \mathsf{l}, \mathsf{m})$ be a very general triple satisfying (16). By Proposition 5.4, we have $\mathrm{NS}(X) = \mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$. We assume in this section that we are in case (C2): the class l is nef, while the class m is isotropic and not nef. Thus, we have $\overline{\mathrm{Mov}}(X) = \mathbf{R}_{\geq 0}\mathsf{l} + \mathbf{R}_{\geq 0}(\mathsf{l} + \mathsf{m})$ and there is a divisorial contraction $c \colon X \to Y$ defined by some positive power of the semi-ample line bundle $L \otimes M$, whose exceptional locus is the prime divisor E with class $-\mathsf{l} + \mathsf{m}$. Our aim is to prove Proposition 5.8: some tensor power of L defines a Lagrangian fibration on X.

The fourfold Y is Gorenstein with trivial canonical sheaf, rational singularities, and singular locus the surface $\Sigma := c(E)$. The class in $H_2(X, \mathbf{Z}) \simeq H^2(X, \mathbf{Z})^\vee$ of a general fiber of $E \to \Sigma$ is given by the linear form $q_X(-\mathsf{l} + \mathsf{m}, \bullet)$ which, since $q_X(-\mathsf{l} + \mathsf{m}, \mathsf{l}) = 1$, is nondivisible. The class of any curve contracted by c is a multiple of that class, hence all 1-dimensional fibers of c are irreducible, smooth rational curves.

Proposition 7.1. *The fibers of the contraction $c \colon X \to Y$ all have dimension at most 1, the varieties E and Σ are smooth, and the restriction $c_E := c|_E \colon E \to \Sigma$ is a $\mathbf{P}^1$-fibration.*

Proof. We argue by contradiction and assume that there is an integral surface $S \subset X$ such that $c(S)$ is a point in Y. Then $S \subset E$. We first claim that the class

$$[S'] \coloneqq 2(\mathsf{l}+\mathsf{m})(-\mathsf{l}+\mathsf{m}) - [S] \tag{37}$$

in $\mathrm{Hdg}^4(X, \mathbf{Z})$ is effective.

Indeed, by assumption, the line bundles $L^2 \otimes M^2$ and $L^3 \otimes M$ are big and nef on X. By the Kawamata–Viehweg vanishing theorem and (16′), we get

$$h^0(X, L^2 \otimes M^2) = 21 \quad \text{and} \quad h^0(X, L^3 \otimes M) = 15.$$

It follows that the restriction map

$$H^0(X, L^2 \otimes M^2) \to H^0(E, (L^2 \otimes M^2)|_E),$$

whose kernel equals $H^0(X, L^3 \otimes M)$, has rank at least 6. On the other hand, as $L \otimes M$ is numerically trivial on S, the restriction map

$$H^0(X, L^2 \otimes M^2) \to H^0(S, (L^2 \otimes M^2)|_S)$$

has rank at most 1. It follows that there exists a section of $L^2 \otimes M^2$ vanishing on S which does not vanish identically on E. Since the class of E is $-\mathsf{l}+\mathsf{m}$, the claim follows.

We now compute the intersection matrices $M_{[S]}$ and $M_{[S']}$ defined in (19). They are nonzero since there is an ample class in $\mathbf{Z}\mathsf{l} \oplus \mathbf{Z}\mathsf{m}$. Since the surface S is contracted by c, the line bundle $L \otimes M$ is numerically trivial on S and we get

$$M_{[S]} = \begin{pmatrix} t & -t \\ -t & t \end{pmatrix}$$

for some positive integer t. Hence, using (37), we get

$$M_{[S']} = \begin{pmatrix} 4-t & t \\ t & -4-t \end{pmatrix}.$$

Since $[S']$ is effective, we get $4 - t \geq 0$, hence $t \in \{1, 2, 3, 4\}$.

By Proposition 5.4, we can write

$$\begin{aligned} [S] &= \frac{t}{2}\left(\mathsf{l}^2 - \mathsf{l}\mathsf{m} + \mathsf{m}^2\right) + w\left(q_X^\vee - \frac{25}{2}\mathsf{l}\mathsf{m}\right), \\ [S'] &= -\left(2+\frac{t}{2}\right)\mathsf{l}^2 + \frac{t}{2}\mathsf{l}\mathsf{m} + \left(2-\frac{t}{2}\right)\mathsf{m}^2 - w\left(q_X^\vee - \frac{25}{2}\mathsf{l}\mathsf{m}\right), \end{aligned} \tag{38}$$

for some $w \in \mathbf{Q}$. Since $[S]$ is an integral class, we have $[S]^2 \in \mathbf{Z}$. Using (17), we get

$$2[S]^2 = t^2 + 525\, w^2 = t^2 + 3 \cdot 7\, (5w)^2,$$

so that $5w \in \mathbf{Z}$.

From (38), by the same reasoning used at the end of the proof of Lemma 6.1, for any class $\omega \in \overline{\mathscr{K}}_X \subset H^{1,1}(X, \mathbf{R})$ (in our situation this means in particular that $q_X(\omega, \mathsf{l}) \geq 0$ and $q_X(\omega, -\mathsf{l}+\mathsf{m}) \geq 0$) with $q_X(\omega) = 0$, we obtain

$$\begin{aligned} 0 &\leq \int_X [S]\omega^2 = \frac{t}{2}\int_X \mathsf{l}^2\omega^2 - \left(\frac{t+25w}{2}\right)\int_X \mathsf{l}\mathsf{m}\omega^2 + \frac{t}{2}\int_X \mathsf{m}^2\omega^2, \\ 0 &\leq \int_X [S']\omega^2 = -\left(2+\frac{t}{2}\right)\int_X \mathsf{l}^2\omega^2 + \left(\frac{t+25w}{2}\right)\int_X \mathsf{l}\mathsf{m}\omega^2 + \left(2-\frac{t}{2}\right)\int_X \mathsf{m}^2\omega^2. \end{aligned} \tag{39}$$

Moreover, since $q_X(\omega)=0$ and $c_X=3$, the Fujiki relation (5) implies, for all $\alpha,\beta\in H^2(X,\mathbf{Z})$,

$$\int_X \alpha\beta\omega^2 = 2q_X(\alpha,\omega)\,q_X(\beta,\omega).$$

Thus, from (39), we deduce

$$\frac{t+25w}{2}\le t \quad\text{and}\quad \frac{t+25w}{2}\ge t,$$

and thus

$$25w=t,\qquad \text{with } t\in\{1,2,3,4\} \text{ and } 5w\in\mathbf{Z},$$

which is impossible. This proves that c contracts no surfaces.

The other statements of the lemma then follow from [W, Theorem 1.3(ii)]. □

Since the line bundle L has intersection 1 with all fibers of c_E, we immediately deduce the following result.

Lemma 7.2. *Let $\mathscr{E}\coloneqq {c_E}_*L$. We have an isomorphism $E\simeq \mathbf{P}_\Sigma(\mathscr{E}^\vee)$ over Σ.*

By Lemma 7.2, the line bundle $(L\otimes M)|_E$ descends to a line bundle H_Σ on Σ, which is ample since some positive power of $L\otimes M$ is the pullback of an ample line bundle on Y. We will study in more detail in Section 8 the surface Σ, the polarization H_Σ, and the rank-2 vector bundle $\mathscr{E}$. For the moment, we just use their existence to finish the proof of Proposition 5.8. We need one last result before that.

Lemma 7.3. *One has $H^2(X,M^2)=H^4(X,M^2)=0$ and $h^0(X,M^2)\ge 3$.*

Proof. We first prove $H^2(X,M^{-2})=0$. Consider the exact sequence

$$0\to\mathscr{O}_X(-E)\to\mathscr{O}_X\to\mathscr{O}_E\to 0.$$

Tensoring by $(L\otimes M)^{-1}$, we obtain

$$0\to M^{-2}\to (L\otimes M)^{-1}\to (L\otimes M)^{-1}|_E\to 0.$$

Since $H^2(X,(L\otimes M)^{-1})=H^2(X,L\otimes M)=0$, it suffices to prove $H^1(E,(L\otimes M)^{-1}|_E)=0$. But, by Lemma 7.2, we have

$$H^1(E,(L\otimes M)^{-1}|_E)\simeq H^1(\Sigma,H_\Sigma^{-1})=0,$$

since H_Σ is ample on the smooth surface Σ, as we wanted.

By Serre duality, we get $H^2(X,M^2)=0$. We also have $H^4(X,M^2)=H^0(X,M^{-2})=0$.

Finally, the Riemann–Roch theorem takes the form

$$h^0(X,M^2)+h^2(X,M^2)+h^4(X,M^2)\ge\chi(X,M^2)=P_{RR,X}(0)=3, \tag{40}$$

completing the proof of the lemma. □

In fact, the divisor E is fixed in $|M^2|$, hence $h^0(X,M^2)=h^0(X,L\otimes M)=6$, but we will use again the Riemann–Roch argument (40) in the proof below.

Proof of Proposition 5.8. As observed in Section 5.1, we only have to show $h^0(X,L^2)\ge 2$. This inequality follows from Lemma 7.3 by deformation and specialization: as we explained in Section 3.2, the reflection that permutes l and m is a monodromy operator. This means that one can deform the pair (X,L) into the pair (X,M) through a family $(\mathscr{X},\mathscr{L})\to T$.

By semi-continuity, Lemma 7.3 implies $h^2(\mathscr{X}_t, \mathscr{L}_t^2) = h^4(\mathscr{X}_t, \mathscr{L}_t^2) = 0$ for $t \in T$ general. The Riemann–Roch argument of (40) then implies $h^0(\mathscr{X}_t, \mathscr{L}_t^2) \geq 3$ for $t \in T$ general. By upper semi-continuity again, we conclude $h^0(X, L^2) \geq 3$, as we wanted. □

8. Proofs of the main theorems

In this section, we prove Theorem 1.5 and Theorem 1.8. Let X be a hyper-Kähler fourfold. We assume that there are classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ with $\int_X \mathsf{l}^4 = 0$ and $\int_X \mathsf{l}^2\mathsf{m}^2 = 2$.

The sublattice spanned by l and m is indefinite, hence its q_X-orthogonal has signature $(2, b_2(X) - 4)$. By the surjectivity of the period map ([H1]), we can then deform X and assume that $(X, \mathsf{l}, \mathsf{m})$ is a very general triple satisfying (16) (hence also (16′)). After possibly permuting l and m, we can further assume that l is nef (see (26)). By Corollary 5.7, the class m cannot be nef, and we are in the situation of Proposition 5.8. Let us denote by $f\colon X \to \mathbf{P}^2$ the Lagrangian fibration such that $f^*\mathscr{O}_{\mathbf{P}^2}(1) \simeq L^{k_L}$, for some positive integer k_L. We start with the following vanishing result.

Lemma 8.1. *Under the above assumptions, let $p, q \in \mathbf{Z}$, with $q > 0$. For all $i \geq 3$, we have*

$$H^i(X, L^p \otimes M^q) = 0.$$

Proof. Choose $r \in \mathbf{Z}$ be such that $p + rk_L \geq q$. We can write

$$L^p \otimes M^q = L^{p+rk_L} \otimes L^{-rk_L} \otimes M^q = L^{p+rk_L} \otimes M^q \otimes f^*\mathscr{O}_{\mathbf{P}^2}(-r),$$

where $L^{p+rk_L} \otimes M^q$ is big and nef. By [Ko, Theorem 10.32] again, we obtain that $R^j f_*(L^p \otimes M^q)$ vanishes for all $j > 0$ and $f_*(L^p \otimes M^q)$ is a vector bundle on $\mathbf{P}^2$. In particular, this gives us what we need:

$$H^i(X, L^p \otimes M^q) = H^i(\mathbf{P}^2, f_*(L^p \otimes M^q)) = 0,$$

for all $i \geq 3$. □

We now study in more detail the surface Σ and the vector bundle $\mathscr{E}$. Recall that we have $(L \otimes M)|_E = c_E^* H_\Sigma$. Let us denote by $\mathsf{h} \in \mathrm{NS}(\Sigma)$ the class of H_Σ.

Lemma 8.2. *The pair (Σ, h) is a polarized K3 surface of degree 2 with $\mathrm{NS}(\Sigma) = \mathbf{Z}\mathsf{h}$.*

Proof. By [W, Theorem 1.4], we know that Σ is a symplectic surface. To show that it is a K3 surface, it is enough to compute $\chi(\Sigma, \mathscr{O}_\Sigma) = \chi(E, \mathscr{O}_E)$. By the Riemann–Roch Theorem, we have

$$\chi(E, \mathscr{O}_E) = \chi(X, \mathscr{O}_X) - \chi(X, \mathscr{O}_X(-E)) = 3 - 1 = 2,$$

as we wanted. Also,

$$\mathsf{h}^2 = \int_X (\mathsf{l} + \mathsf{m})^2(-\mathsf{l} + \mathsf{m})\mathsf{l} = 2.$$

Thus, we are left to show that the Néron–Severi group of Σ has rank one.

Let us consider the transcendental lattice $H^2(X, \mathbf{Z})_{\mathrm{tr}} \coloneqq \mathrm{NS}(X)^\perp \subset H^2(X, \mathbf{Z})$, which under our assumptions has rank 21. For all $\alpha \in H^2(X, \mathbf{Z})_{\mathrm{tr}}$, we have, by (5),

$$\int_X \alpha(-\mathsf{l} + \mathsf{m})(\mathsf{l} + \mathsf{m})^2 = 0.$$

Hence, we can write

$$\alpha|_E = c_E^* \mathsf{n}_\alpha$$

for a unique class $\mathsf{n}_\alpha \in \mathsf{h}^\perp \subset H^2(\Sigma, \mathbf{Z})$. Moreover, again by (5), for all $\alpha, \beta \in H^2(X, \mathbf{Z})_{\mathrm{tr}}$, we have

$$q_X(\alpha, \beta) = \int_X \alpha\beta\mathsf{l}(-\mathsf{l} + \mathsf{m}) = \int_\Sigma \mathsf{n}_\alpha \mathsf{n}_\beta.$$

Therefore, the morphism

$$\vartheta\colon H^2(X, \mathbf{Z})_{\mathrm{tr}} \longrightarrow \mathsf{h}^\perp \subset H^2(\Sigma, \mathbf{Z}), \qquad \alpha \longmapsto \mathsf{n}_\alpha$$

gives an isometry $\vartheta_{\mathbf{C}}\colon H^2(X, \mathbf{Z})_{\mathrm{tr}} \otimes \mathbf{C} \xrightarrow{\sim} \mathsf{h}^\perp \otimes \mathbf{C}$.

The morphism ϑ is a nonzero morphism of Hodge structures. Since the Hodge structure $H^2(X)_{\mathrm{tr}}$ is irreducible, ϑ is injective hence Σ has Picard number one. □

Lemma 8.3. *The vector bundle $\mathscr{E}$ is the unique spherical stable bundle on Σ with Mukai vector $(2, H_\Sigma, 1)$.*

Proof. By Lemma 8.2, the Picard group of Σ is generated by H_Σ. Hence, we can write the Mukai vector of $\mathscr{E}$ as $v(\mathscr{E}) = (2, sH_\Sigma, s')$, with $s, s' \in \mathbf{Z}$. By the Riemann–Roch Theorem, since $[E] = -\mathsf{l} + \mathsf{m}$, we have

$$\chi(X, L(-E)) = \chi(X, L^2 \otimes M^{-1}) = 0.$$

Hence, from the exact sequence

$$0 \to L(-E) \to L \to L|_E \to 0 \tag{41}$$

we get

$$s' + 2 = \chi(\Sigma, \mathscr{E}) = \chi(E, L|_E) = \chi(X, L) - \chi(X, L(-E)) = \chi(X, L) = 3,$$

and thus $s' = 1$. To compute s, we proceed similarly and use the exact sequence

$$0 \to M^{-1}(-E) \to M^{-1} \to M^{-1}|_E \to 0. \tag{42}$$

Again, since $\mathscr{O}_X(-E) = L \otimes M^{-1}$, we get

$$M^{-1}(-E) = L \otimes M^{-2}.$$

Hence

$$\begin{aligned} 5 - 2s = \chi(\Sigma, \mathscr{E} \otimes H_\Sigma^{-1}) &= \chi(E, M^{-1}|_E) \\ &= \chi(X, M^{-1}) - \chi(X, L \otimes M^{-2}) = \chi(X, M^{-1}) = 3, \end{aligned} \tag{43}$$

and thus $s = 1$.

To finish the proof, we only need to prove that $\mathscr{E}$ is stable; indeed, it is then automatically spherical, since its Mukai vector has square -2. Since $\mathscr{E}$ is a rank-2 vector bundle of slope $(H_\Sigma \cdot c_1(\mathscr{E}))/(H_\Sigma^2)\,\mathrm{rk}(\mathscr{E}) = 1/2$ and Σ has Picard number 1, it is enough to prove $H^0(\Sigma, \mathscr{E} \otimes H_\Sigma^{-1}) = 0$. This is a slight refinement of (43) above. Indeed,

$$H^0(\Sigma, \mathscr{E} \otimes H_\Sigma^{-1}) = H^0(E, (L \otimes (L \otimes M)^{-1})|_E) = H^0(E, M^{-1}|_E).$$

From (42), we obtain an exact sequence

$$H^0(X, M^{-1}) \to H^0(E, M^{-1}|_E) \to H^1(X, L \otimes M^{-2}).$$

Under our assumptions, m is contained in the closure of the positive cone and therefore $-\mathsf{m}$ cannot be effective; hence, $H^0(X, M^{-1}) = 0$. Therefore, we only have to show $H^1(X, L \otimes M^{-2}) = 0$ or, by Serre duality, $H^3(X, L^{-1} \otimes M^2) = 0$. This follows immediately from Lemma 8.1. □

Next, we use Lemma 8.3 to show $k_L = 1$ and study the induced map $f_E := f|_E\colon E \to \mathbf{P}^2$.

Lemma 8.4. *The restriction morphism*

$$H^0(X, L) \longrightarrow H^0(E, L|_E) \simeq \mathbf{C}^3$$

is an isomorphism. In particular, $k_L = 1$ and L is globally generated.

Proof. We consider again the exact sequence (41). By Lemma 8.1 and Serre duality, we get

$$h^i(X, L(-E)) = h^i(X, L^2 \otimes M^{-1}) = h^{4-i}(X, L^{-2} \otimes M) = 0,$$

for $i \in \{0, 1\}$, as we wanted. The equality $h^0(E, L|_E) = 3$ and the last statement follow since, by the Riemann–Roch Theorem, we have that $h^0(E, L|_E) \geq 3$, and then we apply Lemma 5.1. □

The moduli space $\mathsf{M}_0(\Sigma)$ was defined in Example 3.4. Let us denote by E_0 the exceptional divisor given by the universal family of curves in Σ over $\mathbf{P}^2 = |\mathsf{h}|$. Then E_0 is isomorphic to E over Σ since they are projective bundles associated with the same vector bundle $\mathscr{E}$.

To finish the proof of Theorem 1.5 and Theorem 1.8, we only need to show the following result, which can be seen as a version of [N, Theorem 1].

Proposition 8.5. *Let $(X, \mathsf{l}, \mathsf{m})$ be a very general triple satisfying (16) with m isotropic and l nef. Then X is isomorphic to $\mathsf{M}_0(\Sigma)$. In particular, X is of $\mathrm{K3}^{[2]}$ deformation type.*

Proof. By Lemma 8.4, the morphism f_E coincides with the composition

$$E \xrightarrow{\sim} E_0 \hookrightarrow \mathsf{M}_0(\Sigma) \xrightarrow{f_0} \mathbf{P}^2,$$

and thus E is isomorphic to the universal family of curves in Σ over $\mathbf{P}^2 = |\mathsf{h}|$. Let us consider the nonempty open subset $U \subset \mathbf{P}^2$ where all morphisms f, f_0 and their restrictions respectively to E and E_0 are smooth. Then the restriction $f_E|_{U_E}$ of f_E to $U_E \coloneqq f_E^{-1}(U)$ will factor via the relative Albanese variety $\mathrm{Alb}(U_E/U)$ (see [F, Proposition 1]): there exists a proper morphism

$$g\colon \mathrm{Alb}(U_E/U) \longrightarrow f^{-1}(U)$$

over U compatible with the inclusion of U_E.

For all points $u \in U$, the class of the curve $C_u \coloneqq f_E^{-1}(u)$ gives a principal polarization on the abelian surface $A_u \coloneqq f^{-1}(u)$. Since C_u lives in a very general K3 surface, A_u cannot be isomorphic to the product of two elliptic curves. Hence, A_u is isomorphic to the Jacobian of C_u and the principal polarization is the theta polarization. From this, we immediately deduce that g is an isomorphism. But the relative Albanese variety only depends on f_E and thus it is isomorphic to the relative Albanese variety of the universal family of curves in Σ over $\mathbf{P}^2 = |\mathsf{h}|$. This gives a birational morphism $X \overset{\sim}{\dashrightarrow} \mathsf{M}_0(\Sigma)$ which is then an isomorphism, since the nef and movable cones coincide. □

9. Further results

We keep the same setup: X is a hyper-Kähler manifold of dimension $2n$ with classes $\mathsf{l}, \mathsf{m} \in H^2(X, \mathbf{Z})$ such that $\int_X \mathsf{l}^{2n} = 0$ and

$$a \coloneqq \frac{1}{n!} \int_X \mathsf{l}^n \mathsf{m}^n$$

is a nonzero integer (Lemma 2.2); changing m into $-\mathsf{m}$ if necessary, we may assume that the integers $q_X(\mathsf{l}, \mathsf{m})$, hence also a, are positive. After dealing for most of this article with the case $a = 1$, we examine in this section the general case.

9.1. **Boundedness results.** We first prove a general boundedness result, assuming n and a are fixed (this is [K, Theorem 4.9]).

Proposition 9.1. *Let a be a fixed positive integer. The number of deformation types of hyper-Kähler manifolds X of fixed dimension $2n$ for which there are classes $\mathsf{l},\mathsf{m}\in H^2(X,\mathbf{Z})$ such that $\int_X \mathsf{l}^{2n}=0$ and $\int_X \mathsf{l}^n\mathsf{m}^n = an!$ is finite.*

Proof. As explained above, we may assume $q_X(\mathsf{l},\mathsf{m})>0$. Since we can always add to m a multiple of l (which changes $q_X(\mathsf{m})$ by adding multiples of $2q_X(\mathsf{l},\mathsf{m})$ but neither $q_X(\mathsf{l},\mathsf{m})$ nor a), we may further assume

$$-q_X(\mathsf{l},\mathsf{m}) < q_X(\mathsf{m}) \le q_X(\mathsf{l},\mathsf{m}). \tag{44}$$

We have, using (3), (44), and (9),

$$\int_X (\mathsf{l}+\mathsf{m})^{2n} = c_X q_X(\mathsf{l}+\mathsf{m})^n = c_X(2q_X(\mathsf{l},\mathsf{m})+q_X(\mathsf{m}))^n \le c_X 3^n q_X(\mathsf{l},\mathsf{m})^n = a3^n\frac{(2n)!}{2^n n!}.$$

By the surjectivity of the period map, there exists a deformation of X whose Néron–Severi group is generated by the class $\mathsf{l}+\mathsf{m}$, which is therefore ample (or antiample). One can then apply [H3, Corollary 1.2] to conclude. □

In dimension 4, we improve this result by allowing a to take infinitely many values.

Proposition 9.2. *The number of deformation types of hyper-Kähler fourfolds X for which there are classes $\mathsf{l},\mathsf{m}\in H^2(X,\mathbf{Z})$ such that $\int_X \mathsf{l}^4=0$ and the integer $a=\frac{1}{2}\int_X \mathsf{l}^2\mathsf{m}^2$ is positive and square-free, is finite.*

Proof. The proposition is a consequence of Lemma 4.2, whose notation we keep. Indeed, that lemma and (13) say that the integer $aA'_X := 12^2\cdot 2aA_X$ is a perfect (positive) square. Since a is square-free and $A'_X \le 262$ by Lemma 4.1, the integer a is a product of (distinct) prime numbers that are all < 262. It is therefore bounded (by the product of all these primes) and boundedness for X follows from Proposition 9.1. □

9.2. **Low values of a for hyper-Kähler fourfolds.** We obtain restrictions on the values that the integer a may take for hyper-Kähler fourfolds. The case $a=1$ was analyzed in Theorem 4.3 and is completely clarified by Theorem 1.5: it is only realized by hyper-Kähler manifolds of $\mathrm{K3}^{[2]}$ deformation type.

Theorem 9.3. *Let X be a hyper-Kähler fourfold with classes $\mathsf{l},\mathsf{m}\in H^2(X,\mathbf{Z})$ such that $\int_X \mathsf{l}^4=0$. Assume $a:=\frac{1}{2}\int_X \mathsf{l}^2\mathsf{m}^2\in\{2,\dots,8\}$. We are in one of the following cases:*

(a) *either $a=3$, $q_X(\mathsf{l},\mathsf{m})=1$, $c_X=9$, $P_{RR,X}(T)=3\binom{\frac{T}{2}+2}{2}$, and the quadratic form q_X is even; moreover,*
  - *either $(b_2(X),b_3(X),b_4(X))=(7,8,108)$;*
  - *or $(b_2(X),b_3(X),b_4(X))=(6,4,102)$;*
  - *or $(b_2(X),b_3(X),b_4(X))=(5,0,96)$.*

(b) *or $a=4$, the Chern and Hodge numbers of X are those of the Hilbert square of a K3 surface, and*
  - *either $q_X(\mathsf{l},\mathsf{m})=2$, $c_X=3$, the form q_X is even, and $P_{RR,X}(T)=\binom{\frac{T}{2}+3}{2}$;*
  - *or $q_X(\mathsf{l},\mathsf{m})=1$, $c_X=12$, and $P_{RR,X}(T)=\binom{T+3}{2}$.*

Proof. We follow the proof of Theorem 4.3, which dealt with the case $a = 1$: we may assume $\gamma := \frac{q_X(\mathsf{m})}{q_X(\mathsf{l},\mathsf{m})} \in (-1, 1]$, we introduce the polynomial

$$P(k) := P_{RR,X}(q_X(k\mathsf{l} + \mathsf{m})) = P_{RR,X}(2kq_X(\mathsf{l},\mathsf{m}) + q_X(\mathsf{m})),$$

and we compute

$$\begin{aligned} P(k) &= \frac{a}{2}k^2 + \Big(\frac{a}{2}\gamma + 2\sqrt{2aA_X}\Big)k + \frac{a}{8}\gamma^2 + \gamma\sqrt{2aA_X} + 3 \\ &=: \frac{a}{2}k^2 + bk + c. \end{aligned}$$

Since P takes integral values on integers, $\frac{a}{2} + b = P(1) - P(0)$ and $c = P(0)$ are integers; when a is odd, we write $b = \frac{1}{2} + b'$, with $b' \in \mathbf{Z}$. We also check

$$4A_X - \frac{b^2}{2a} = 3 - c \in \mathbf{Z}. \tag{45}$$

Finally, when a is not a perfect square, the fact that $\sqrt{2aA_X}$ is rational (Lemma 4.2), implies $A_X \neq \frac{25}{32}$ hence, by Lemma 4.1(b), $\frac{5}{6} \le A_X \le \frac{131}{144}$.

Case $a = 2$. By (45), we have $4A_X - \frac{b^2}{4} \in \mathbf{Z}$ and $b \in \mathbf{Z}$. By Lemma 4.1, this case does not happen.

Case $a = 3$. By (45), we have $4A_X - \frac{b^2}{6} = 4A_X - \frac{b'(b'+1)}{6} - \frac{1}{24} \in \mathbf{Z}$, hence $4A_X - \frac{1}{24} \in \mathbf{Z} + \{0, \frac{1}{3}\}$. Since a is not a perfect square, we have (as noted above) $\frac{5}{6} \le A_X \le \frac{131}{144}$, hence

$$\frac{10}{3} - \frac{1}{24} \le 4A_X - \frac{1}{24} \le \frac{131}{36} - \frac{1}{24}.$$

The only possibility is $4A_X - \frac{1}{24} = 3 + \frac{1}{3}$, that is, $A_X = \frac{27}{32}$. This implies

$$\frac{1}{2} + b' = b = \frac{3}{2}\gamma + \frac{9}{2},$$

so that γ is an even integer. As above, this implies $\gamma = q_X(\mathsf{m}) = 0$, $b = \frac{9}{2}$, and $c = 3$. Furthermore,

$$P_{RR,X}(2kq_X(\mathsf{l},\mathsf{m})) = \frac{3}{2}k^2 + \frac{9}{2}k + 3 = 3\binom{k+2}{2}.$$

By Lemma 3.2 (applied with $c = 2$, $c' = 3$, and $q = 2q_X(\mathsf{l},\mathsf{m})$), we get $q_X(\mathsf{l},\mathsf{m}) = 1$, the form q_X is even, $c_X = 9$, and $P_{RR,X}(T) = 3\binom{\frac{T}{2}+2}{2}$, as in the generalized Kummer case.

From (13), we obtain $c_4(X) = 108$ and $4b_2(X) - b_3(X) = 20$. The possible Betti numbers listed in the theorem then follow from [Gu].

Case $a = 4$. By (45), we have $4A_X - \frac{b^2}{8} \in \mathbf{Z}$ and $b \in \mathbf{Z}$. If $\frac{5}{6} \le A_X \le \frac{131}{144}$, we have $b \equiv 2 \pmod 4$ and $A_X = \frac{7}{8}$, which contradicts the fact that $\sqrt{2aA_X}$ is rational. Hence, by Lemma 4.1, we have $A_X = \frac{25}{32}$ and $b^2 \equiv 1 \pmod 8$, so that b is odd. Furthermore, $b = 2\gamma + 5$, hence γ is an integer, which can only be 0 or 1.

When $\gamma = q_X(\mathsf{m}) = 0$ and $b = 5$, we obtain $c = 3$ and

$$P_{RR,X}(2kq_X(\mathsf{l},\mathsf{m})) = P(k) = 2k^2 + 5k + 3 = \binom{2k+3}{2}.$$

When $\gamma = 1$ and $b = 7$, we get $q_X(\mathsf{l}, \mathsf{m}) = q_X(\mathsf{m})$ and $c = 6$, and

$$P_{RR,X}((2k+1)q_X(\mathsf{l}, \mathsf{m})) = P(k) = 2k^2 + 7k + 6 = \binom{(2k+1)+3}{2}.$$

In both cases, we have

$$P_{RR,X}(q_X(\mathsf{l}, \mathsf{m})T) = \binom{T+3}{2}.$$

By Lemma 3.2 (applied with $c = 2$, $c' = 1$, and $q = q_X(\mathsf{l}, \mathsf{m})$), we get $q_X(\mathsf{l}, \mathsf{m}) \in \{1, 2\}$.

Finally, since $A_X = \frac{25}{32}$, Lemma 4.1 implies that the Chern and Hodge numbers of X are those of the Hilbert square of a K3 surface.

Case $a = 5$**.** We obtain $4A_X - \frac{b^2}{10} \in \mathbf{Z}$ and $b \in \mathbf{Z} + \frac{1}{2}$ and, since a is not a perfect square, $\frac{5}{6} \leq A_X < 1$. We get $A_X = \frac{29}{32}$, which contradicts $\sqrt{2aA_X} \in \mathbf{Q}$.

Case $a = 6$**.** We obtain $4A_X - \frac{b^2}{12} \in \mathbf{Z}$ and $A_X \in \{\frac{5}{6}, \frac{15}{16}\}$, which contradicts $\sqrt{2aA_X} \in \mathbf{Q}$.

Case $a = 7$**.** We obtain $4A_X - \frac{b^2}{14} \in \mathbf{Z}$ and $A_X \in \{\frac{193}{224}, \frac{217}{224}\}$, which contradicts $\sqrt{2aA_X} \in \mathbf{Q}$.

Case $a = 8$**.** We obtain $4A_X - \frac{b^2}{16} \in \mathbf{Z}$ and $A_X = \frac{57}{64}$, which contradicts $\sqrt{2aA_X} \in \mathbf{Q}$. □

As we saw in Example 3.6, the first item of the case $a = 3$ in Theorem 9.3 is realized by hyper-Kähler fourfolds of generalized Kummer deformation type; we do not know whether the other two items occur. In the case $a = 4$, the first item is realized on hyper-Kähler fourfolds of $\mathrm{K3}^{[2]}$ deformation type with m divisible by 2.

Under a condition on some generalized Fujiki constants of X, Beckmann and Song prove in [BS] (see also [S]) that the only possible Betti numbers $b_2(X)$, $b_3(X)$, and $b_4(X)$ for X are as in case (a) of Theorem 9.3. Moreover, by [BS, Proposition 5.6], [BS, Conjecture 1.2] would imply that the Fujiki constant c_X is either 3 or 9. In particular, the second case in (b) should not occur.

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