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# A mathematical proof of a formula of Aspinwall and Morrison

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**Abstract.** We give a rigorous proof of Aspinwall–Morrison formula, which expresses the cubic derivatives of the Gromov–Witten as a series depending only on the number of rational curves in each homology class, for a Calabi–Yau threefold with only rigid immersed rational curves.

**Key words:** Calabi–Yau varieties, rational curves, Gromov–Witten potential

## 1. Introduction

Let  $X$  be a Calabi–Yau variety of dimension three, and let  $\phi: \mathbb{P}^1 \rightarrow X$  be a holomorphic immersion: the normal bundle  $N_\phi = \phi^*T_X/\phi_*T_{\mathbb{P}^1}$  splits into the direct sum of two line bundles,  $N_\phi = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ , and by the adjunction formula  $a + b = -2$ . We will assume that  $\phi(\mathbb{P}^1)$  is infinitesimally rigid, that is  $N_\phi$  has no holomorphic section, or equivalently  $a = b = -1$ . In this case, for any holomorphic map  $\psi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  of degree  $k$ , the deformations of the map  $\phi \circ \psi$  consist of maps  $\phi \circ \psi'$ , where  $\psi': \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a deformation of  $\psi$ . It follows by compactness of the Chow variety of curves in  $X$  of bounded degree, or by [4], that for any  $\alpha \in H_2(X, \mathbb{Z})$ , there is a neighbourhood  $V$  of  $\mathbb{P}^1$  in  $X$  such that the only rational curves of class  $\alpha$  such that  $d^0\alpha \leq kd^0A$  are supported on  $\phi(\mathbb{P}^1)$ , where the degree is counted with respect to any ample line bundle on  $X$ , and  $A = \phi_*([\mathbb{P}^1])$ .

Now consider a small general perturbation  $J_\epsilon$  of the pseudocomplex structure  $J$  of  $X$  and let  $\nu$  be small general  $C^\infty$  section of the bundle  $pr_1^*\Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^*(T_{X_\epsilon}^{1,0})$  on  $\mathbb{P}^1 \times X$ , where  $\Omega^{0,1}$  denotes complex  $(0, 1)$ -forms, and  $T_{X_\epsilon}^{1,0}$  denotes vector fields of type  $(1, 0)$  for the pseudocomplex structure  $J_\epsilon$ . Then it is known (cf. [4], [9], [12]) that the space  $W_{kA, J_\epsilon, \nu}$  of solutions to the equation

$$\bar{\partial}_\epsilon \psi = (\text{Id}, \psi)^* \nu \tag{1.1}$$

for  $\psi: \mathbb{P}^1 \rightarrow X$  such that  $\psi_*([\mathbb{P}^1]) = kA$ , is smooth, naturally oriented of dimension six and can be compactified with a boundary of dimension  $\leq 4$ . By compactness, for  $(J_\epsilon, \nu)$  close enough to  $(J, 0)$ , and for  $V$  as above the subspace  $W_{kA, J_\epsilon, \nu}^V$  of

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$W_{kA, J_\epsilon, \nu}$  consisting of maps  $\psi$  whose image is contained in  $V$  is a component (non necessarily connected) of  $W_{kA, J_\epsilon, \nu}$ .

Let  $x_1, x_2, x_3$  be three distinct points of  $\mathbb{P}^1$ , and consider the evaluation map

$$\begin{aligned}
 ev: W_{kA, J_\epsilon, \nu}^V &\rightarrow X^3 \\
 \psi &\mapsto (\psi(x_1), (\psi(x_2), (\psi(x_3))).
 \end{aligned}
 \tag{1.2}$$

Again the image of  $ev$  is six dimensional oriented, and can be compactified with a boundary of dimension  $\leq 4$ , so has a homology class in  $H_6(X^3)$  (which in fact is in the image of  $H_6(V^3) \rightarrow H_6(X^3)$ , which is generated by  $A \times A \times A$ ). This paper gives a proof of the following

**THEOREM 1.1** *This class is equal to  $A \times A \times A \in H_6(X^3)$ .*

In [10], Manin already proved this statement, admitting the possibility to apply Bott formula to stacks ( which may be only a formal point to verify) and using some ideas due to Kontsevich ([5]). It may be nevertheless interesting to have a proof close to Aspinwall and Morrison argument ([1]), and justifying a posteriori their computation.

This theorem is, as in the paper by Aspinwall and Morrison [1], a consequence of a more precise statement, namely that as a space of curves in  $\mathbb{P}^1 \times X$ , the component  $W_{kA, J_\epsilon, \nu}^V$  is homologous to any cycle in  $M_k := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, 1)))$ , Poincaré dual to the top Chern class of the bundle with fiber at  $\psi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , the space  $H^1((\phi \circ \psi)^*T_X)$ . Here we view  $M_k$  as a compactification of the space  $M_k^0$  parametrizing degree  $k$  coverings  $\psi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , and we identify it to a set of curves in  $\mathbb{P}^1 \times X$ , via  $\phi$ . This statement is quite natural, since this vector bundle, at least on  $M_k^0$ , is exactly the excess bundle for the too large family of holomorphic curves  $M_k$ . However, the proof shows that one has to be careful with the singular curves in  $\mathbb{P}^1 \times X$  parametrized by  $M_k - M_k^0$ , and especially with non reduced curves: for a special choice of  $\nu$  (and for  $J_\epsilon = J$ ) we will exhibit a section  $s$  of this bundle on  $M_k^0 \subset M_k$  such that  $W_{kA, J, \nu}^V$  identifies naturally to the zero set of  $s$ . However, this section is not even continuous at non reduced curves in  $M_k$ . The result is that, nevertheless, the closure of the zero locus of  $s$  in  $M_k$  has for homology class the Poincaré dual of the top Chern class of this bundle.

We mention at this point an essential difference between Manin’s computation [10] and ours: Manin works with the moduli space of stable maps to get a complicated, but more satisfactory from the point of view of moduli spaces, compactification of the space of smooth ramified covers of  $\mathbb{P}^1$ . As in [1], we work with the naive compactification  $M_k \cong \mathbb{P}^{2k+1}$ , on which the Chern classes computations are quite easy, but which is not a good moduli space at the boundary.

The Theorem 1.1 is one version of Aspinwall–Morrison formula [1], which we now explain: let  $\omega \in H^2(X, \mathbb{Z})$  such that  $Re\omega \cong \alpha$  is a sufficiently large kähler

class on  $X$ . The Gromov–Witten potential is the function on  $H^{\text{even}}(X)$  defined by the series (expected to be convergent for large  $\alpha$ )

$$\Psi_\omega(\eta) = \sum_{\substack{A \in H_2(X, \mathbb{Z}) \\ k \geq 3}} \frac{1}{k!} e^{-\int_A \omega} \Phi_A(\eta, \eta, \eta | \underbrace{\eta \dots \eta}_{k-3}) \tag{1.3}$$

([7], [13]) where the mixed Gromov–Witten invariants  $\Phi_A(\eta, \eta, \eta | \underbrace{\eta \dots \eta}_{k-3})$  ([12])

are defined as follows: for  $(J, \nu)$  generic,  $J$  a pseudocomplex structure,  $\nu$  a section of  $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^*(T_{X,J}^{1,0})$  on  $\mathbb{P}^1 \times X$  and  $A \in H_2(X, \mathbb{Z})$ , consider the evaluation map

$$ev_{k-3}: W_{A,J,\nu} \times \mathbb{P}^{1^{k-3}} \rightarrow X^k$$

$$(\psi, z_1, \dots, z_{k-3}) \mapsto (\psi(x_1), \psi(x_2), \psi(x_3), \psi(z_1), \dots, \psi(z_{k-3})), \tag{1.4}$$

the points  $x_i$  being fixed on  $\mathbb{P}^1$ . Then  $\text{Im } ev_{k-3}$  is as before oriented, smooth of real dimension  $6 + 2(k - 3)$ , and can be compactified with a boundary of codimension two, so defines a homology class in  $X^k$  on which one can integrate  $\eta^{\otimes k}$ , which gives the invariant. For  $A = 0, k > 3$ , one has  $\Phi_A(\eta, \eta, \eta | \underbrace{\eta \dots \eta}_{k-3}) = 0$ , essentially

because the map  $ev_{k-3}$  has positive dimensional fibers, at least when  $\nu = 0$ , and for  $A = 0, k = 3$  one has  $\Phi_A(\eta, \eta, \eta) = \int_X \eta^3$  because  $W_{A,J,0}$  identifies to the constant maps, and  $ev(W_{A,J,0})$  is then simply the diagonal in  $X^3$ .

Now assume that all generically immersed rational curves in  $X$  are immersed and infinitesimally rigid, and let  $n(A)$  be the number of immersed rational curves of class  $A \neq 0$ . Then all rational curves on  $X$  are multiple covers of immersed infinitesimally rigid curves, and we can apply the Theorem 1.1, which says that for  $l \geq 1, A \neq 0, W_{lA,J,\nu}$  is made of  $n(A)$  components whose contribution to  $\Phi_{lA}(\eta, \eta, \eta | \underbrace{\eta \dots \eta}_{k-3})$  is equal to

$$l^{k-3} \left( \int_A \eta \right)^{k-3} \int_{A \times A \times A} \eta^{\otimes 3} \tag{1.5}$$

It follows that

$$\begin{aligned} \Psi_\omega(\eta) &= \frac{1}{6} \int_X \eta^3 + \sum_{\substack{A \in H_2(X, \mathbb{Z}) - \{0\} \\ k \geq 3, l \geq 1}} n(A) \frac{1}{k!} e^{-\int_{lA} \omega} l^{k-3} \left( \int_A \eta \right)^k \\ &= \frac{1}{6} \int_X \eta^3 + \sum_{\substack{A \in H_2(X, \mathbb{Z}) - \{0\} \\ l \geq 1}} \frac{1}{l^3} n(A) e^{\int_{lA} -\omega + \eta} \end{aligned} \tag{1.6}$$

modulo a quadratic term in  $\eta$ . So if we consider the cubic derivatives  $\partial^3 \Psi_\omega / \partial t_i \partial t_j \partial t_k(\eta)$  w.r.t. linear coordinates on  $H^{\text{even}}(X)$  corresponding to a basis  $\eta_i$  of  $H^{\text{even}}(X)$ , we find

$$\begin{aligned} & \partial^3 \Psi_\omega / \partial t_i \partial t_j \partial t_k(\eta) \\ &= \int_X \eta_i \wedge \eta_j \wedge \eta_k + \sum_{\substack{A \in H_2(X, \mathbb{Z}) - \{0\} \\ l \geq 1}} \frac{n(A)}{l^3} e^{\int_{lA} -\omega + \eta} \int_{lA} \eta_i \int_{lA} \eta_j \int_{lA} \eta_k \\ &= \int_X \eta_i \wedge \eta_j \wedge \eta_k + \sum_{\substack{A \in H_2(X, \mathbb{Z}) - \{0\} \\ l \geq 1}} n(A) e^{\int_{lA} -\omega + \eta} \int_A \eta_i \int_A \eta_j \int_A \eta_k \\ &= \int_X \eta_i \wedge \eta_j \wedge \eta_k + \sum_{A \neq 0} n(A) e^{\int_A -\omega + \eta} / (1 - e^{\int_A -\omega + \eta}) \\ & \quad \int_A \eta_i \int_A \eta_j \int_A \eta_k \end{aligned} \tag{1.7}$$

which is Aspinwall–Morrison formula for the Yukawa couplings of the ‘A-model’ of  $X$ , at the point  $\omega - \eta$  (see [1], [15], [3],[12]).

**2. Choice of the Parameter  $\nu$**

We will assume in this section that  $\phi: \mathbb{P}^1 \rightarrow X$  is an embedding, and consider the general case in Section 4. We will use the following result ([8])

**THEOREM 2.1.** *Let  $\phi: \mathbb{P}^1 \hookrightarrow X$  such that  $N_\phi \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ ; then a neighbourhood  $V$  of  $\mathbb{P}^1$  in  $X$  is holomorphically isomorphic to a neighbourhood of the zero section of the total space  $W$  of the bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  on  $\mathbb{P}^1$ .*

Since the Theorem 1.1 is a local statement, we may assume from now on that  $X = W$ . Now let  $\pi: W \rightarrow \mathbb{P}^1$  be the natural projection, with fiber  $\pi^{-1}(x) = N_{\phi(x)}$ ; we get an inclusion

$$\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \subset T_W \tag{2.8}$$

as the vertical tangent space of  $\pi$  ( $T_W$  is the bundle of  $(1, 0)$ -vector fields on  $W$ ). We choose now two  $\mathcal{C}^\infty$  sections  $\sigma_1, \sigma_2$  of  $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ , and we define  $\nu = (\nu_1, \nu_2)$  where  $\nu_i = (\text{Id} \times \pi)^* \sigma_i$ .  $\nu$  is then a  $\mathcal{C}^\infty$  section of  $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^* T_W$  via the inclusion (2.8).

We study now the solutions to the equation

$$\bar{\partial} \psi = (\text{Id} \times \psi)^* \nu \tag{2.9}$$

for  $\psi: \mathbb{P}^1 \rightarrow W$  a  $\mathcal{C}^\infty$  map such that  $\psi_*([\mathbb{P}^1]) = kA$ , where  $A$  is the homology class of the zero section. Since by construction  $\pi_*(\nu)$  vanishes as a section of

$pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^* \circ \pi^* T_{\mathbb{P}^1}$  on  $\mathbb{P}^1 \times W$ , we get  $\bar{\partial}(\pi \circ \psi) = 0$ , so  $\pi \circ \psi$  is holomorphic, of degree  $k$ . Let  $\psi' = \pi \circ \psi$ ; then  $\psi$  is described by a couple  $(\psi_1, \psi_2)$ , where  $\psi_i$  are  $C^\infty$  sections of the bundle  $\psi'^* \mathcal{O}_{\mathbb{P}^1}(-1)$ . The equation (2.9) rewrites then simply as

$$\bar{\partial} \psi_i = (\text{Id} \times \psi')^* \sigma_i, \quad i = 1, 2 \tag{2.10}$$

Since  $H^0(\psi'^* \mathcal{O}_{\mathbb{P}^1}(-1)) = \{0\}$ ,  $\psi_i$  are determined by  $\psi'$  and exist if and only if  $(\text{Id} \times \psi')^* \sigma_i$ , which are  $(0, 1)$ -forms with values in  $\psi'^* \mathcal{O}_{\mathbb{P}^1}(-1)$ , vanish in  $H^1(\psi'^* \mathcal{O}_{\mathbb{P}^1}(-1))$ .

As in [1], let us introduce  $M_k = \mathbb{P}(H^0(\mathcal{O}_Q(k, 1)))$ , where  $Q = \mathbb{P}^1 \times \mathbb{P}^1$ ,  $\mathcal{O}_Q(k, 1) = pr_1^* \mathcal{O}_{\mathbb{P}^1}(k) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(1)$ .  $M_k$  is a compactification of the family of holomorphic maps of degree  $k$  from  $\mathbb{P}^1$  to  $\mathbb{P}^1$ : indeed the general member of  $M_k$  is a smooth curve in  $Q$ , isomorphic to  $\mathbb{P}^1$  by the first projection, and of degree  $k$  over  $\mathbb{P}^1$  by the second projection.

In  $M_k \times Q$  we consider as in [1] the universal divisor  $D$  defined as the zero set of the natural section of  $p_M^* \mathcal{O}_{M_k}(1) \otimes p_Q^* \mathcal{O}_Q(k, 1)$  corresponding to the identification  $H^0(\mathcal{O}_{M_k}(1))^* \cong H^0(\mathcal{O}_Q(k, 1))$ , where  $p_M$  and  $p_Q$  are the projections to  $M_k$  and  $Q$  respectively. Let  $pr_2 : Q \rightarrow \mathbb{P}^1$  be the second projection, and let  $E := R^1 p_{M*}(pr_2 \circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))|_D$ ; then since  $R^1 p_{M*}(pr_2 \circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \{0\}$  and  $R^2 p_{M*}(pr_2 \circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1)) = \{0\}$  we conclude by the long exact sequence associated to

$$\begin{aligned} 0 \rightarrow (pr_2 \circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{I}_D &\rightarrow (pr_2 \circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \\ &\rightarrow (pr_2 \circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1))|_{D \rightarrow 0} \end{aligned} \tag{2.11}$$

that  $E \cong R^2 p_{M*}((pr_2 \circ p_Q)^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \otimes \mathcal{I}_D)$ . Since  $\mathcal{I}_D \cong p_M^* \mathcal{O}_{M_k}(-1) \otimes p_Q^* \mathcal{O}_Q(-k, -1)$ , we get

**LEMMA 2.2.** ([1])  $E \cong \mathcal{O}_{M_k}(-1) \otimes H^2(Q, \mathcal{O}_Q(-k, -2))$ . In particular  $E$  is a vector bundle on  $M_k$  of rank  $k - 1$ .

Let  $M_k^0$  be the open set parametrizing smooth curves in  $Q$ , that is maps  $\psi' : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  of degree  $k$ ; in  $M_k^0$ , we have two sections of  $E$ , denoted by  $s_{\sigma_1}, s_{\sigma_2}$ , defined by  $s_{\sigma_i}(\psi') = \text{class of } (\text{Id} \times \psi')^*(\sigma_i) \text{ in } H^1(\psi'^*(\mathcal{O}_{\mathbb{P}^1}(-1))) \cong E_{\psi'}$ . We have shown that the solutions of (2.9) in  $M_k^0$  are in bijection with the zeroes of the section  $(s_{\sigma_1}, s_{\sigma_2})$  of  $E \times E$ ; since  $\dim_{\mathbb{C}} M_k = 2k + 1$ ,  $\text{rank}_{\mathbb{C}} E = k - 1$ , the zero set of  $(s_{\sigma_1}, s_{\sigma_2})$  is expected to be of real dimension 6, as we want.

### 3. Study of the Section $s_\sigma$

The behaviour of the section  $s_\sigma$  of  $E$  on  $M_k^0$ , for  $\sigma$  a  $C^\infty$  section of  $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$  on  $Q = \mathbb{P}^1 \times \mathbb{P}^1$  is easily described by the following

LEMMA 3.1.  $s_\sigma$  is of class  $\mathcal{C}^\infty$  on  $M_k^0$ .

*Proof.* By definition, for  $(C) \in M_k^0$ ,  $s_\sigma((C))$  is represented by a  $(0, 1)$ -form on  $C$ , which varies in a  $\mathcal{C}^\infty$  way with  $(C)$ . Now, we have the isomorphism  $E_{(C)} \cong H^1(C, pr_2^*(\mathcal{O}_{\mathbb{P}^1}(-1)|_C))$ , where  $C \subset Q$  corresponds to  $(C) \in M_k^0$ , and we have shown that the rank of this space is independent of  $(C)$ . This implies that  $s_\sigma$  is of class  $\mathcal{C}^\infty$ , because we have then the isomorphism  $E^* \cong R^0 p_{M*}(K_{D/M_k} \otimes (pr_2 \circ p_Q)^* \mathcal{O}_{\mathbb{P}^1}(-1)) \cong H^0(Q, \mathcal{O}_Q(k-2, 0)) \otimes \mathcal{O}_{M_k}(1)$ , and it is immediate to see that for a holomorphic section  $\eta$  of the right hand side, the function  $\langle s_\sigma, \eta \rangle$  is given by integrals over the curves  $C$  of forms varying in a  $\mathcal{C}^\infty$  way with  $(C)$ .

It is unfortunately not true that  $s_\sigma$  extends continuously over  $M_k$ . The rest of this section is devoted to the study of the singularities of  $s_\sigma$  and to the proof of the following

THEOREM 3.2. Let  $\sigma_1, \sigma_2$  be general  $\mathcal{C}^\infty$  section of  $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$  on  $Q = \mathbb{P}^1 \times \mathbb{P}^1$ ; let  $\bar{V}_{\sigma_1, \sigma_2}$  be the closure in  $M_k$  of the zero locus  $V(s_{\sigma_1}, s_{\sigma_2}) \subset M_k^0$  of the section  $(s_{\sigma_1}, s_{\sigma_2})$  of  $E \times E$  on  $M_k^0$ ; then  $V(s_{\sigma_1}, s_{\sigma_2})$  is smooth of dimension 6, and  $\bar{V}_{\sigma_1, \sigma_2} - V(s_{\sigma_1}, s_{\sigma_2})$  can be stratified by subsets contained in locally closed subvarieties of dimension  $\leq 4$  of  $M_k$ , so  $\bar{V}_{\sigma_1, \sigma_2}$  has a homology class in  $H_6(M_k)$ , which is Poincaré dual to the top Chern class of  $E \times E$ .

The proof of this theorem will be based on the following Proposition 3.3, for which we introduce a few notations: for any  $(C) \in M_k$ , one can write  $C = C' \cup V_C$ , where  $C' \subset Q$  is a smooth member of  $|\mathcal{O}_Q(l, 1)|$ ,  $l \leq k$  and the vertical part  $V_C = pr_1^{-1}(D_C)$  for some divisor  $D_C$  of degree  $k-l$  on  $\mathbb{P}^1$ . We will denote by  $D'_C$  the intersection  $C' \cap V_C$ , and by  $\psi_{C'} : C' \rightarrow \mathbb{P}^1$  the second projection, which is a morphism of degree  $l$ ; writing  $D'_C = \sum_i n_i p_i$  for distinct points  $p_i$  of  $C'$  we will denote by  $B_C$  the divisor  $\sum_i (n_i - 1) p_i$  that we will view as a divisor either on  $C'$  or on  $\mathbb{P}^1 \xrightarrow{pr_1^{-1}} C'$ . There is a natural structure of scheme on  $Z := \bigcup_{C \in M_k} D'_C \subset M_k \times \mathbb{P}^1$  defined as follows: choosing homogeneous coordinates  $Y_0, Y_1$  on  $\mathbb{P}^1$ , a section of  $\mathcal{O}_Q(k, 1)$  can be written as  $pr_1^* P_0 pr_2^* Y_0 + pr_1^* P_1 pr_2^* Y_1$ , where  $P_i \in H^0(\mathcal{O}_{\mathbb{P}^1}(k))$  depend algebraically on  $(C) \in M_k$ ;  $Z$  is then defined by  $P_0 = P_1 = 0$ . This induces immediately a scheme structure on  $B := \bigcup_{C \in M_k} B_C \subset M_k \times \mathbb{P}^1$ , the ideal being generated by the partial derivatives of  $P_0, P_1$  w. r. t. homogeneous coordinates on  $\mathbb{P}^1$ .

We have already used the isomorphism

$$\begin{aligned} E^* &\cong R^0 p_{M*}(K_{D/M_k} \otimes (pr_2 \circ p_Q)^* \mathcal{O}_{\mathbb{P}^1}(-1)) \\ &\cong H^0(Q, \mathcal{O}_Q(k-2, 0)) \otimes \mathcal{O}_{M_k}(1) \end{aligned} \quad (3.12)$$

which depends essentially on the choice of an isomorphism  $K_Q \cong \mathcal{O}_Q(-2, -2)$ . Since  $H^0(Q, \mathcal{O}_Q(k-2, 0)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-2))$  we get

$$E^* \cong R^0 \pi_{M*}(\pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(k-2)) \otimes \mathcal{O}_{M_k}(1), \quad (3.13)$$

where  $\pi_M, \pi_{\mathbb{P}^1}$  are the projections of  $M_k \times \mathbb{P}^1$  onto its factors. We have then the following

**PROPOSITION 3.3.** *Let  $\phi$  be a holomorphic section of  $R^0\pi_{M*}(\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1)$  on an open set  $U$  of  $M_k$ . Then the function  $\langle s_\sigma, \phi \rangle$  defined on  $U \cap M_k^0$  extends continuously on  $U$ .*

*Proof.* Let  $(C_0) \in U$  and  $F_{C_0}$  an equation for  $C_0 \subset Q$ . One can write  $F_{C_0} = pr_1^*P_{C_0} \cdot F'_{C_0}$  where  $P_{C_0}$  is an equation for  $D_{C_0}$  and  $F'_{C_0} \in |\mathcal{O}_Q(k-l_0, 1)|$  defines a smooth curve in  $Q$ ,  $l_0 = d^0D_{C_0}$ . Using a partition of unity on  $Q$ , one may assume that  $\sigma$  is compactly supported in a product of disks  $D_1 \times D_2$  with affine coordinates  $z_1, z_2$  such that  $D_i = \{z_i, |z_i| < 1\}$  and  $(0, 0) \in C_0 \cap D_1 \times D_2$ , and the inhomogeneous polynomials corresponding to  $P_{C_0}, F'_{C_0}$  satisfy

$$\begin{aligned} p_{C_0} &= z_1^l q_{C_0}(z_1), & q_{C_0}(z_1) &\neq 0 \text{ on } D_1 \\ f'_{C_0}(z_1) &= \tilde{h}_{C_0}(z_1) + z_2 \tilde{g}_{C_0}(z_1), \end{aligned} \tag{3.14}$$

where one of the polynomials  $\tilde{f}_{C_0}, \tilde{g}_{C_0}$  does not vanish on  $D_1$ , since  $f'_{C_0} = 0$  has no vertical component. We assume  $\tilde{g}_{C_0} \neq 0$  on  $D_1$ , the other case working similarly. By shrinking  $D_1$  we may even assume  $|q_{C_0}\tilde{g}_{C_0}| \geq c > 0$  on  $D_1$ . Let  $h_{C_0} = q_{C_0}\tilde{h}_{C_0}, g_{C_0} = q_{C_0}\tilde{g}_{C_0}$ ; a small generic deformation  $f_C$  of  $f_{C_0}$  can be written as

$$f_C = p_C(z_1)(h_C(z_1) + z_2 g_C(z_1)) + r_C(z_1), \tag{3.15}$$

where we can normalize  $f_C$  by imposing the condition  $g_C(0) = 1$ , and  $d^0p_C = l, p_C(z_1) = z_1^l + \sum_{i < l} \alpha_i z_1^i, d^0r_C \leq l-1, d^0h_C \leq k-l, d^0g_C \leq k-l$ ; the polynomials  $p_C, h_C, g_C, r_C$  vary holomorphically with  $(C)$  in a neighbourhood (that we still call  $U$ ) of  $(C_0)$ , and  $p_{C_0} = z_1^l, r_{C_0} = 0$ . The variety  $Z \cap U \times D_1$  is described by the equations  $p_C(z_1) = r_C(z_1)$  and the variety  $B \cap U \times D_1$  is described by the equations  $p_C(z_1) = r_C(z_1) = \partial p_C / \partial z_1(z_1) = \partial r_C / \partial z_1(z_1) = 0$ . The restriction to  $U \times D_1$  of a section  $\phi$  of  $\pi_{\mathbb{P}^1}^*\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B$  can then be written as

$$\begin{aligned} \phi(z_1, (C)) &= \phi_C^p(z_1)p_C + \psi_C^p(z_1)\frac{\partial p_C}{\partial z_1} + \phi_C^r(z_1)r_C \\ &+ \psi_C^r(z_1)\frac{\partial r_C}{\partial z_1}, \end{aligned} \tag{3.16}$$

where  $\phi_C^p, \psi_C^p, \phi_C^r, \psi_C^r$  are holomorphic functions of  $((C), z_1)$ . We can write  $\sigma = \psi(z_1, z_2)d\bar{z}_1$ , where  $\psi$  is a compactly supported function of class  $C^\infty$  in  $D_1 \times D_2$ . The couplings  $\gamma((C)) := \langle s_\sigma, \phi \rangle$  defined on  $U \cap M_k^0$  are obtained by taking the residue along  $C$  of the  $pr_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ -valued meromorphic form  $\phi \cdot \eta / f_C$ , and integrating over  $C$  the cup-product of this form with  $\sigma|_C$ ; hence  $\gamma((C))$  has



the following form

$$\begin{aligned}
 \gamma((C)) &= \gamma_C^p + \gamma_C^{p'} + \gamma_C^r + \gamma_C^{r'} \\
 &= \int_{D_1} \phi_C^p(z_1) \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \cdot \frac{1}{g_C(z_1)} \\
 &\quad + \psi_C^p(z_1) \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \frac{\partial_{z_1} p_C}{p_C g_C} \\
 &\quad + \phi_C^r(z_1) \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \frac{r_C}{p_C g_C} \\
 &\quad + \psi_C^r(z_1) \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \frac{\partial_{z_1} r_C}{p_C g_C} dz_1 \wedge d\bar{z}_1 \tag{3.17}
 \end{aligned}$$

and it suffices to show that each function  $\gamma_C^p, \gamma_C^{p'}, \gamma_C^r, \gamma_C^{r'}$  extends continuously at  $(C_0) \in U$ . This is in fact obvious for  $\gamma_C^p$  and  $\gamma_C^{r'}$  since the functions  $\phi_C^p(z_1) \psi(z_1, (-r_C - p_C h_C)/(p_C g_C))/g_C(z_1)$  and  $\phi_C^r(z_1) \psi(z_1, (-r_C - p_C h_C)/(p_C g_C))r_C/(p_C g_C)$  are bounded by a constant independant of  $(C)$  and are continuous along  $(C_0) \times D_1^*$ . To show that  $\gamma_C^{p'}$  extends continuously at  $(C_0)$ , consider the degree  $l$  covering  $\tilde{U} \xrightarrow{r} U, \tilde{U} \subset U \times D_1^l$  obtained by taking the roots of  $p_C$  (which are all in  $D_1$  for  $(C)$  close to  $(C_0)$ ), that is  $\tilde{U} = \{((C), \lambda_1, \dots, \lambda_l) | p_C = \prod_i (z_1 - \lambda_i)\}$ . It suffices to show that  $r^*(\gamma_C^{p'})$  extends continuously at  $((C_0), 0, \dots, 0) \in \tilde{U}$ ; but

$$\begin{aligned}
 r^*(\gamma_C^{p'}) &= \int_{D_1} \frac{\psi_C^p(z_1)}{g_C(z_1)} \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \\
 &\quad \times \left( \sum_{i=1}^{i=l} (1/z_1 - \lambda_i) \right) dz_1 \wedge d\bar{z}_1. \tag{3.18}
 \end{aligned}$$

For  $(C)$  close enough to  $(C_0)$ , the  $\lambda_i$ 's are close to zero, so we may assume that  $\psi(z_1, z_2) = 0$  outside  $|z_1 - \lambda_i| \leq 1$ . It follows that

$$\begin{aligned}
 r^*(\gamma_C^{p'}) &= \sum_i \int_{D_1} \frac{\psi_C^p(z_1 + \lambda_i)}{g_C(z_1 + \lambda_i)} \\
 &\quad \psi \left( z_1 + \lambda_i, \frac{-r_C - p_C h_C}{(p_C g_C)(z_1 + \lambda_i)} \right) \cdot \frac{1}{z_1} dz_1 \wedge d\bar{z}_1. \tag{3.19}
 \end{aligned}$$

But the function  $(\psi_C^p/g_C)(z_1 + \lambda_i) \psi(z_1 + \lambda_i, (-r_C - p_C h_C)/(p_C g_C)(z_1 + \lambda_i))$  is bounded by a constant on  $D_1$ , and the function  $1/z_1$  is  $L^1$  on  $D_1$ ; since for  $z_1 \neq 0$ , one has

$$\begin{aligned} & \lim_{\substack{(C) \rightarrow (C_0) \\ \lambda_i \rightarrow 0}} \frac{1}{z_1} \left( \frac{\psi_C^p}{g_C} \right) (z_1 + \lambda_i) \psi \left( z_1 + \lambda_i, \frac{-r_C - p_C h_C}{(p_C g_C)(z_1 + \lambda_i)} \right) \\ &= \frac{\psi_{C_0}^p}{g_{C_0}}(z_1) \psi \left( z_1, -\frac{h_{C_0}}{g_{C_0}}(z_1) \right) \cdot \frac{1}{z_1}, \end{aligned} \tag{3.20}$$

one may apply Lebesgue dominated convergence theorem in order to conclude that  $\lim_{(C) \rightarrow (C_0)} r^*(\gamma_C^{p'})$  exists and is equal to

$$l \int_{D_1} \frac{\psi_{C_0}^p}{g_{C_0}}(z_1) \psi \left( z_1, -\frac{h_{C_0}}{g_{C_0}}(z_1) \right) \cdot \frac{1}{z_1} dz_1 \wedge d\bar{z}_1. \tag{3.21}$$

The proof that  $\gamma_C^{r'}$  extends continuously at  $(C_0)$  works similarly: in fact, using the result for  $\gamma_C^{p'}$  it suffices to prove it for

$$\begin{aligned} \gamma_C^{r'} &= \int_{D_1} \psi_C^r(z_1) \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \partial_{z_1} \\ &\quad \times \frac{r_C + p_C}{p_C g_C} dz_1 \wedge d\bar{z}_1. \end{aligned} \tag{3.22}$$

Now we can write

$$\begin{aligned} \gamma_C^{r'} &= \int_{D_1} \psi_C^r(z_1) \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \\ &\quad \times \partial_{z_1} \frac{r_C + p_C}{r_C + p_C(g_C)} \times \frac{r_C + p_C}{p_C} dz_1 \wedge d\bar{z}_1, \end{aligned} \tag{3.23}$$

and because  $\psi$  is compactly supported in  $D_1 \times D_2$  the function

$$\psi_C^r(z_1) \psi \left( z_1, \frac{-r_C - p_C h_C}{p_C g_C} \right) \frac{r_C + p_C}{p_C g_C}$$

is bounded in  $D_1$ . But  $d^0 r_C \leq l - 1$  and  $\lim_{(C) \rightarrow (C_0)} r_C = 0$  so the polynomial  $p_C + r_C$  is normalized of degree  $l$  and has all its roots in  $D_1$  for  $(C)$  close to  $(C_0)$ ; as before we can introduce the cover  $\tilde{U} \xrightarrow{r} U$  parametrizing an ordering of the roots of  $r_C + p_C$ , so  $r^*(r_C + p_C) = \prod_{i=1}^l (z_1 - \lambda_i)$ , and we get

$$\begin{aligned} r^*(\gamma_C^{r'}) &= \sum_i \int_{D_1} \psi_C^r(z_1 + \lambda_i) \psi \left( z_1 + \lambda_i, \frac{-r_C - p_C h_C}{p_C g_C}(z_1 + \lambda_i) \right) \\ &\quad \times \frac{r_C + p_C}{p_C g_C}(z_1 + \lambda_i) \times \frac{1}{z_1} dz_1 \wedge d\bar{z}_1, \end{aligned} \tag{3.24}$$

and we can apply Lebesgue dominated convergence theorem since the integrand is bounded by  $M/|z_1|$  and converges weakly to the  $L^1$  function

$$\psi_{C_0}^r(z_1)\psi\left(z_1, \frac{-h_{C_0}}{g_{C_0}(z_1)}\right) \frac{1}{z_1 g_{C_0}(z_1)} \quad (3.25)$$

outside 0, when  $(C)$  tends to  $(C_0)$ . So the proposition is proved.

In fact, the proof of the proposition gives as well the interpretation of the limit of the functions  $\langle s_\sigma, \phi \rangle$ : we have the decomposition  $C_0 = C'_0 \cup pr_1^{-1}(D_{C_0})$ , with  $C'_0$  smooth and  $D'_{C_0} = \Sigma_i n_i p_i$ ,  $n_i \neq 0$ , where  $D'_{C_0}$  is the inverse image of  $D_{C_0}$  under the isomorphism  $pr_1: C'_0 \rightarrow \mathbb{P}^1$ . Let  $D''_{C_0} := \Sigma_i p_i$ ; denote by  $\mathcal{C}_{D'_{C_0}}^\infty(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1))$  the space of  $\mathcal{C}^\infty$  sections  $\tau$  of  $pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0}$  which satisfy the condition:  $\tau(p_i) = \tau(p_i) = \dots = (\partial_z)^{(n_i-1)}\tau(p_i) = 0$  for all  $p_i$  and for any coordinate  $z$  on  $C'_0$  at  $p_i$ ; similarly, let  $\mathcal{C}_{D''_{C_0}}^\infty(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1))$  the space of  $\mathcal{C}^\infty$  sections  $\tau$  of  $pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0}$  which satisfy the condition:  $\tau(p_i) = 0, \forall p_i$ . We have

LEMMA 3.4. *There are natural isomorphisms*

$$\begin{aligned} H^1(C_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C_0}) &\cong A_{C'_0}^{0,1}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0})/\bar{\partial}\mathcal{C}_{D'_{C_0}}^\infty(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)), \\ H^1(C'_0, pr_2^*\mathcal{O}(-1)|_{C'_0} \otimes \mathcal{I}_{D''_{C_0}}) \\ &\cong A_{C'_0}^{0,1}(pr_2^*\mathcal{O}(-1)|_{C'_0})/\bar{\partial}\mathcal{C}_{D''_{C_0}}^\infty(pr_2^*\mathcal{O}(-1)). \end{aligned} \quad (3.26)$$

*Proof.* Consider the exact sequence of coherent sheaves on  $C_0$

$$\begin{aligned} 0 \rightarrow pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0} \otimes \mathcal{I}_{D'_{C_0}} &\rightarrow pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C_0} \\ &\rightarrow pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{V_{C_0}} \rightarrow 0. \end{aligned} \quad (3.27)$$

It is easy to see that the last sheaf has trivial cohomology, and it follows that

$$H^1(C_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C_0}) \cong H^1(C'_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0} \otimes \mathcal{I}_{D'_{C_0}}) \quad (3.28)$$

so we are reduced to prove the existence of natural isomorphisms

$$\begin{aligned} H^1(C'_0, pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0} \otimes \mathcal{I}_{D'_{C_0}}) \\ &\cong A_{C'_0}^{0,1}(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0})/\bar{\partial}\mathcal{C}_{D'_{C_0}}^\infty(pr_2^*\mathcal{O}_{\mathbb{P}^1}(-1)), \\ H^1(C'_0, pr_2^*\mathcal{O}(-1)|_{C'_0} \otimes \mathcal{I}_{D''_{C_0}}) \\ &\cong A_{C'_0}^{0,1}(pr_2^*\mathcal{O}(-1)|_{C'_0})/\bar{\partial}\mathcal{C}_{D''_{C_0}}^\infty(pr_2^*\mathcal{O}(-1)) \end{aligned} \quad (3.29)$$

which is immediate because we have the fine resolution

$$\begin{aligned}
 0 \rightarrow pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0} \otimes \mathcal{I}_{D'_0} &\rightarrow \mathcal{A}_{D'_0}^0 (pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)) \\
 &\xrightarrow{\bar{\partial}} \mathcal{A}_{C'_0}^{0,1}(pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0}) \rightarrow 0,
 \end{aligned}
 \tag{3.30}$$

where  $\mathcal{A}_{D'_0}^0, \mathcal{A}^{0,1}$  are now the sheaves of  $\mathcal{C}_{D'_0}^\infty$  sections and of  $(0, 1)$ -forms respectively. One gets similarly the second isomorphism.

Now, by the Lemma 3.4,  $\sigma|_{C'_0}$  gives a class  $s_\sigma(C_0) \in H^1(C'_0, pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0} \otimes \mathcal{I}_{D'_0})$ , and this group is naturally a quotient of  $E_{(C_0)} = H^1(C_0, pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{C_0})$ . It is immediate to verify that  $H^1(C'_0, pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)|_{C'_0} \otimes \mathcal{I}_{D'_0})$  identifies to the dual of  $H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_{B_{C_0}}) \subset H^0(\mathcal{O}_{\mathbb{P}^1}(k-2))$  (modulo the choice of an isomorphism  $K_Q \cong \mathcal{O}_Q(-2, -2)$  and of an equation for  $C_0$ ) and the computation of the limits in the proof of the Proposition 3.3 shows

LEMMA 3.5. *Let  $\phi$  be a local holomorphic section of  $R^0 \pi_{M*}(\pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1)$  near  $(C_0)$ ; then*

$$\lim_{(C) \rightarrow (C_0)} \langle s_\sigma, \phi \rangle = \langle s_\sigma((C_0)), \phi((C_0)) \rangle.
 \tag{3.31}$$

Now we can show the following Proposition 3.6, which shows the first part of the Theorem 3.2; for each sequence  $d. = (d_1, \dots, d_k)$  of integers, with  $\sum_i id_i \leq k$ , we denote by  $M_k^d.$  the smooth locally closed subvariety of  $M_k$  consisting of curves  $C = C' \cup V_C$ , such that  $C'$  is a smooth member of  $|\mathcal{O}_Q(k - \sum_i id_i, 1)$  and  $V_C = pr_1^{-1}(D_C)$  where  $D_C$  has  $d_i$  points of multiplicity  $i$  for each  $i$ . The  $M_k^d.$ 's form a stratification of  $M_k$  and  $M_k^0 = M_k^{(0, \dots, 0)}$ . On each  $M_k^d.$ ,  $\sigma$  gives a section of the bundle  $E^d.$  with fiber at  $C$  the space  $H^1(C', pr_2^*(\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{I}_{D'_C}))$ , that we will denote by  $s_\sigma^d.$  As in Lemma 3.1, it is immediate to prove that  $s_\sigma^d.$  is of class  $C^\infty$  on  $M_k^d.$ . We have

PROPOSITION 3.6. *Let  $\sigma_1, \sigma_2$  be two  $C^\infty$  sections of  $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$  on  $Q$ . Then  $\bar{V}_{\sigma_1, \sigma_2}$  is contained in  $\sqcup_d V(s_{\sigma_1}^d., s_{\sigma_2}^d.)$ ; if  $\sigma_i$  are general, for each  $d.$ ,  $V(s_{\sigma_1}^d., s_{\sigma_2}^d.)$  is smooth of real dimension  $6 - 2\sum_i d_i$ .*

*Proof.* Let  $(C) \in M_k^d.$ , and let  $D_C = \sum_i n_i p_i, B_C = \sum_i (n_i - 1)p_i$ . Consider  $H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_{B_C}) \otimes \mathcal{O}_{M_k}(1)_{(C)} \subset E_{(C)}^*$ . In a neighbourhood  $U$  of  $(C)$ , we can find a holomorphic subbundle  $F$  of  $E^*$  whose sheaf of sections is contained in  $R^0 \pi_{M*}(\pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1)$  and such that  $F_{(C)} = H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_{B_C}) \otimes \mathcal{O}_{M_k}(1)_{(C)}$ . Let  $E/(F^\perp) \cong F^*$  be the corresponding quotient; the Proposition 3.3 shows that the projection  $p_F(s_\sigma)$  of  $s_\sigma$  in  $F^*$  extends continuously. Furthermore, by definition of  $F$  and by the Lemma 3.5, we have  $F^*|_{M_k^d. \cap U} = E_k^d.$

and we have the equality in  $U \cap M_k^d$

$$p_F(s_\sigma)|_{M_k^d} = s_\sigma^d \tag{3.32}$$

Now we have on  $U \cap M_k^0$ ,  $V(s_{\sigma_1}, s_{\sigma_2}) \subset V(p_F(s_{\sigma_1}), p_F(s_{\sigma_2}))$  for  $\sigma_1, \sigma_2$  as above and by continuity of  $p_F(s_{\sigma_i})$ , we get

$$\overline{V}_{\sigma_1, \sigma_2} \cap U \subset V(p_F(s_{\sigma_1}), p_F(s_{\sigma_2})) \tag{3.33}$$

Finally, the equality (3.32) gives

$$\overline{V}_{\sigma_1, \sigma_2} \cap U \cap M_k^d \subset V(s_{\sigma_1}^d, s_{\sigma_2}^d) \cap U \tag{3.34}$$

which shows the first part of the proposition.

Now note that the real dimension of  $M_k^d$  is equal to  $2(2(k - \sum_i id_i + 1) - 1 + \sum_i d_i)$ , and the rank over  $R$  of  $E^d \times E^d$  is equal to  $4(k - 1 - \sum_i (i - 1)d_i)$ . Since  $s_{\sigma_i}^d$  are of class  $C^\infty$  over  $M_k^d$ , the fact that  $V(s_{\sigma_1}^d, s_{\sigma_2}^d)$  is smooth of real dimension  $6 - 2(\sum_i d_i)$  for general  $\sigma_1, \sigma_2$  follows from the following

LEMMA 3.7. *There exists a finite number of  $C^\infty$  sections  $\sigma_i$  of  $pr_1^* \Omega^{0,1}(\mathbb{P}^1) \otimes pr_2^* \mathcal{O}_{\mathbb{P}^1}(-1)$  on  $Q$  such that the corresponding sections  $s_{\sigma_i}^d$  generate  $E^d$  on  $M_k^d$  for any sequence  $d$ .*

*Proof.* Since  $M_k$  is compact, it suffices to check it locally on  $M_k$ . Now let  $(C) \in M_k$ ; for  $\sigma$  supported away from  $\text{Sing } C$ , one shows exactly as in 3.1 that  $s_\sigma$  extends as  $C^\infty$  section of  $E$  at  $(C)$ . Next, using Lemma 3.4, one checks easily that the values at  $(C)$  of such sections  $s_\sigma$  generate the fiber  $E_{(C)}$ . So they generate  $E$  in a neighbourhood  $U$  of  $(C)$  and its quotients  $E^d$  in  $U \cap M_k^d$ .

It follows from this proposition that for general  $(\sigma_1, \sigma_2)$ ,  $\overline{V}_{\sigma_1, \sigma_2}$  has a homology class  $[\overline{V}_{\sigma_1, \sigma_2}] \in H_6(M_k, \mathbb{Z})$ , which is defined using the natural orientation of  $V_{\sigma_1, \sigma_2}$  coming from the complex structure on  $M_k$  and  $E \times E$ . Now we have

PROPOSITION 3.8.  $[\overline{V}_{\sigma_1, \sigma_2}]$  is Poincarè dual to the top Chern class of  $E \times E$ .

*Proof.* We show first the existence of a continuous section  $(s'_1, s'_2)$  of  $E \times E$  with zero locus equal to  $\sqcup_d V((s_{\sigma_1}^d, s_{\sigma_2}^d))$ : consider the coherent subsheaf  $(E^*)' = R^0 \pi_{M^*}(\pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_B) \otimes \mathcal{O}_{M_k}(1) \subset E^*$ ; let  $F$  be a holomorphic vector bundle on  $M_k$  such that there exists a surjective morphism  $\phi': F \rightarrow (E^*)'$ . We denote by  $\phi$  the composition of  $\phi'$  with the inclusion  $(E^*)' \subset E^*$ . Putting hermitian metrics on  $F$  and  $E^*$ , we construct a  $C^\infty$  complex linear endomorphism  $\Phi = \phi \circ \phi^t: E^* \rightarrow E^*$ , which has the property:  $\forall (C) \in M_k, \text{Im } \Phi_{(C)} = \text{Im } \phi_{(C)} = H^0(\mathcal{O}_{\mathbb{P}^1}(k-2) \otimes \mathcal{I}_{B_C}) \otimes \mathcal{O}_{M_k}(1)_{(C)}$ . Also, by construction, for any  $C^\infty$  section  $\tau$  of  $E^*$ ,  $\Phi(P)$  can be written locally as  $\sum_j f_j \tau_j$  where  $f_j$  are  $C^\infty$  complex functions and  $\tau_j$  are sections of  $(E^*)'$ . It follows from the Proposition 3.3 that for any such  $\tau$ , the function  $\langle s_\sigma, \tau \rangle$  is continuous on  $M_k$ , which means that  $s' = \Phi^*(s_\sigma)$  is a continuous section of  $E$ . Furthermore, for  $(C) \in M_k^d$ ,  $s'$  vanishes at  $(C)$  if and only if  $s_\sigma^d$  vanishes at

(C), by Lemma 3.5. Applying this construction to the couple  $(\sigma_1, \sigma_2)$  we get a continuous section  $(s'_1, s'_2)$  of  $E \times E$  which vanishes exactly on  $\sqcup_d V((s^d_{\sigma_1}, s^d_{\sigma_2}))$ .

Notice that  $(s'_1, s'_2)$  is smooth when  $(s_{\sigma_1}, s_{\sigma_2})$  is, so  $(s'_1, s'_2)$  is smooth on  $M_k^0$ ; furthermore, since the map  $\Phi^*$  is  $\mathbb{C}$ -linear the orientation of  $V(s_{\sigma_1}, s_{\sigma_2})$  corresponding to the section  $(s'_1, s'_2)$  coincides with the one given by the section  $(s_{\sigma_1}, s_{\sigma_2})$ .

Now, using approximation by smooth sections, we can construct a  $C^\infty$  section  $(s''_1, s''_2)$  of  $E \times E$ , which is equal to  $(s'_1, s'_2)$  outside an arbitrarily small neighbourhood of  $M_k - M_k^0$ , and such that the zero locus  $V(s''_1, s''_2)$  is contained in the union of  $V(s_{\sigma_1}, s_{\sigma_2})$  and of an arbitrarily small neighbourhood of  $\sqcup_{d \neq (0, \dots, 0)} V((s^d_{\sigma_1}, s^d_{\sigma_2}))$ . Using the fact that  $\dim V((s^d_{\sigma_1}, s^d_{\sigma_2})) \leq 4$  for  $d \neq (0, \dots, 0)$ , by Proposition 3.6, any homology class of dimension  $2 \dim M_k - 6$  can be represented by a subvariety  $W$  of  $M_k$  which does not meet a small neighbourhood of  $\sqcup_{d \neq (0, \dots, 0)} V((s^d_{\sigma_1}, s^d_{\sigma_2}))$ . So  $W$  may be chosen to meet  $V(s_{\sigma_1}, s_{\sigma_2})$  transversally and only in the open set where  $(s_{\sigma_1}, s_{\sigma_2})$  and  $(s''_1, s''_2)$  coincide, and then the intersection number  $W \cdot \bar{V}_{\sigma_1, \sigma_2} = W \cdot V(s''_1, s''_2)$  is simply the top Chern class of  $E \times E$  evaluated on  $W$ , which proves the Proposition 3.8, hence also the Theorem 3.2.

#### 4. Proof of the Theorem 1.1

The homology class that we want to compute is defined as follows: let  $(J_\epsilon, \nu)$  be a small general deformation of  $(J, 0)$ , where  $J$  is the original complex structure; there is a component  $W_{kA, J_\epsilon, \nu}^V$  of  $\bar{W}_{kA, J_\epsilon, \nu}$  made of curves contained in a given small neighbourhood  $V$  of  $\mathbb{P}^1 \subset X$  (cf. Introduction); one can construct a compactification  $\bar{W}_{kA, J_\epsilon, \nu}^V$  of  $W_{kA, J_\epsilon, \nu}^V$ , such that the points of the boundary parametrize curves in  $\mathbb{P}^1 \times X$ , which are limits of graphs of functions  $\psi \in W_{kA, J_\epsilon, \nu}^V$ . One has then a family of curves

$$\begin{array}{ccc}
 D & \xrightarrow{(p_2, p_3)} & \mathbb{P}^1 \times V \subset \mathbb{P}^1 \times X \\
 p_1 \downarrow & & \\
 \bar{W}_{kA, J_\epsilon, \nu}^V & & 
 \end{array} \tag{4.35}$$

which induces the family of threefolds

$$\begin{array}{ccc}
 D \times_{p_1} D \times_{p_1} D & \xrightarrow{(p_2^3, p_3^3)} & \mathbb{P}^1 \times V^3 \subset \mathbb{P}^1 \times X^3 \\
 p_1^3 \downarrow & & \\
 \bar{W}_{kA, J_\epsilon, \nu}^V & & 
 \end{array} \tag{4.36}$$

The class that we want to compute is the class of  $p_3^3((p_2^3)^{-1}((x_1, x_2, x_3)))$ , for  $x_1, x_2, x_3$  three distinct generic points of  $\mathbb{P}^1$ . Now we do this computation with  $W_{kA, J_\epsilon, \nu}^V$  replaced by  $V(s_{\sigma_1}, s_{\sigma_2})$ , that we have identified set theoretically to a component of  $W_{kA, J, \nu}^V$  for special  $\nu$  in Section 2; as before we identify  $V$  to a neighbourhood of the zero section of the bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ , and call  $\pi: V \rightarrow \mathbb{P}^1$  the projection; we may assume that  $\pi$  induces an isomorphism  $\pi_*: H_*(V) \cong H_*(\mathbb{P}^1)$  hence an isomorphism  $\pi_*^3: H_*(V^3) \rightarrow H_*(\mathbb{P}^1^3)$ . Now, by construction, for  $(C) \in V(s_{\sigma_1}, s_{\sigma_2})$ , the associated map  $\psi: \mathbb{P}^1 \rightarrow V$  solution of the equation (2.9), satisfies  $\pi \circ \psi = \psi_{(C)}$ , where  $\psi_{(C)}: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is the map determined by  $C \subset Q = \mathbb{P}^1 \times \mathbb{P}^1$ . It follows that the image under  $(\text{Id}, \pi)$  of the family (4.35) is simply the restriction to  $V(s_{\sigma_1}, s_{\sigma_2}) \subset M_k^0$  of the divisor  $D$  of Section 2.

$$\begin{array}{ccc} D|_{V(s_{\sigma_1}, s_{\sigma_2})} & \xrightarrow{(p_2, \pi \circ p_3)} & \mathbb{P}^1 \times \mathbb{P}^1 \\ p_1 \downarrow & & \\ V(s_{\sigma_1}, s_{\sigma_2}). & & \end{array} \tag{4.37}$$

Since we know that  $\overline{V}_{\sigma_1, \sigma_2} \subset M_k$  has for homology class the Poincaré dual of the top Chern class of  $E \times E$ , with  $E \cong \mathcal{O}_{M_k}^{k-1} \otimes \mathcal{O}_{M_k}(1)$ , we find as in [1] that  $[\overline{V}_{\sigma_1, \sigma_2}]$  is the homology class of a  $\mathbb{P}^3 \subset M_k \cong \mathbb{P}^{2k+1}$ . It is then immediate to conclude that  $(\pi \circ p_3)_*^3([p_2^3)^{-1}((x_1, x_2, x_3))])$  is equal to the fundamental homology class of  $\mathbb{P}^1^3$ .

In order to complete the proof of the Theorem 1.1, it remains to verify that the computation of the class of  $p_3^3((p_2^3)^{-1}((x_1, x_2, x_3)))$  (for generic  $J_\epsilon, \nu$ ) can be done using  $V(s_{\sigma_1}, s_{\sigma_2})$ , that is we have to verify the following points

LEMMA 4.1.  $W_{kA, J, \nu}^0$  is smooth along  $V(s_{\sigma_1}, s_{\sigma_2})$ , for  $\nu$  as in Section 2 and generic  $\sigma_i$ .

In other words we have to identify ‘schematically’  $W_{kA, J, \nu}^V$  and  $V(s_{\sigma_1}, s_{\sigma_2})$ .

LEMMA 4.2. The orientation of  $V(s_{\sigma_1}, s_{\sigma_2})$  as the zero set of a section of a complex vector bundle on  $M_k$  coincide with the natural orientation of  $W_{kA, J, \nu}^V$  (defined in [9], Chapter 3).

LEMMA 4.3. For  $(J_n, \nu_n)$  a sequence of generic deformations of  $(J, 0)$  converging to  $(J, \nu)$ ,  $\overline{W}_{kA, J_n, \nu_n}^V$  converges to  $\overline{V}_{\sigma_1, \sigma_2}$ .

(That is we have to exclude the existence of a limit component which would be made of curves in  $\mathbb{P}^1 \times X$  with a vertical component).

*Proof of Lemma 4.1.* We want to show that for  $(C) \in V(s_{\sigma_1}, s_{\sigma_2})$  defining  $\psi_{(C)}: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  such that  $(\text{Id} \times \psi_{(C)})^*((\sigma_1, \sigma_2) = (\overline{\partial}\psi_1, \overline{\partial}\psi_2)$ ,  $\psi_i \in \mathcal{C}^\infty(\psi_{(C)}^*(\mathcal{O}_{\mathbb{P}^1}(-1)))$ , and  $\psi: \mathbb{P}^1 \rightarrow V$ ,  $\psi = (\psi_{(C)}, \psi_1, \psi_2)$ , where  $V$  is identified to an open set of  $N_\phi$  as in Section 2, the tangent space at  $(C)$  of  $V(s_{\sigma_1}, s_{\sigma_2})$  and at  $\psi$  of  $W_{kA, J, \nu}^V$  coincide. But the last space is the kernel of the linearized equation

$$D_\psi := D(\bar{\partial} - (\text{Id}, \psi)^* \nu) : \mathcal{C}^\infty(\psi^* T_X) \rightarrow A_{\mathbb{P}^1}^{0,1}(\psi^* T_X). \tag{4.38}$$

The bundle  $T_{X|V}$  fits into the exact sequence

$$0 \rightarrow \pi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \rightarrow T_{X|V} \rightarrow \pi^*(T_{\mathbb{P}^1}) \rightarrow 0 \tag{4.39}$$

and  $\nu = ((\text{Id} \times \pi)^* \sigma_1, (\text{Id} \times \pi)^* \sigma_2)$ . Since  $\pi \circ \psi = \psi_C$  is holomorphic, it is immediate to verify that  $D_{\psi|_{\mathcal{C}^\infty(\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))}}$  is simply the  $\bar{\partial}$  operator, and that the induced quotient map  $\bar{D}_\psi : \mathcal{C}^\infty(\psi^*(T_{\mathbb{P}^1})) \rightarrow A_{\mathbb{P}^1}^{0,1}(\psi^*(T_{\mathbb{P}^1}))$  is also the  $\bar{\partial}$ -operator. Since  $\bar{\partial} : \mathcal{C}^\infty(\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) \rightarrow A_{\mathbb{P}^1}^{0,1}(\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)))$  is injective, and  $\bar{\partial} : \mathcal{C}^\infty(\psi^*(T_{\mathbb{P}^1})) \rightarrow A_{\mathbb{P}^1}^{0,1}(\psi^*(T_{\mathbb{P}^1}))$  is surjective, we get an exact sequence

$$0 \rightarrow \text{Ker } D_\psi \rightarrow \text{Ker } \bar{\partial}_{\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))} \xrightarrow{\beta} \text{Coker } \bar{\partial}_{\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))} \rightarrow 0 \tag{4.40}$$

and identifying the second term to  $T_{M_k(C)}$  and the last term to  $H^1(\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) = (E \times E)_{(C)}$ , it is immediate to verify that  $\beta$  is equal to the linearization of  $(s_{\sigma_1}, s_{\sigma_2})$  at  $(C)$ , which proves Lemma 4.1.

*Proof of Lemma 4.2.* The orientation of the variety  $W_{kA,J,\nu}^V$  at the point  $\psi$  corresponding to  $(C)$  is described as follows (cf. [9]): Replacing  $\mathcal{C}^\infty$  sections of the bundles  $\psi^* T_X, \Omega^{0,1}(\psi^* T_X)$  by sections with  $L^1$  derivatives up to order  $k$ , the operator  $D_\psi$  gives a Fredholm operator (surjective at a smooth point)

$$D_\psi : W^{k,1}(\psi^* T_X) \rightarrow W^{k-1,1}(\Omega^{0,1}(\psi^* T_X)). \tag{4.41}$$

The observation is that both spaces have natural (continuous) complex structures and that the  $\mathbb{C}$ -antilinear part of  $D_\psi$  is of order 0, hence is compact. So there is a natural (linear) homotopy from  $D_\psi$  to its  $\mathbb{C}$ -linear part  $D_\psi^L$  in the space of Fredholm operators from  $W^{k,1}(\psi^* T_X)$  to  $W^{k-1,1}(\Omega^{0,1}(\psi^* T_X))$ . The orientation on  $T_{W_{kA,J,\nu}^V}$  at the point  $\psi$  is obtained by using the real line bundle  $\text{Det}_t := \bigwedge_{\mathbb{R}}^{\max} \text{Ker } D_t \otimes (\bigwedge_{\mathbb{R}}^{\max} \text{Coker } D_t)^*$  on  $[0, 1]$ , where  $D_t = (1 - t)D_\psi + tD_\psi^L$ . Since for  $t = 1$ ,  $D_1 = D_\psi^L$  is complex linear  $\text{Det}_1$  is naturally oriented, hence  $\text{Det}_0 = \bigwedge_{\mathbb{R}}^{\max} T_{W_{kA,J,\nu}^V}$  is also naturally oriented.

Now as mentioned above, the operator  $D_\psi$  induces the complex linear operators

$$\bar{\partial} : W^{k,1}(\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1)^2)) \rightarrow W^{k-1,1}(\Omega^{0,1}(\psi^*(\mathcal{O}_{\mathbb{P}^1}(-1)^2))) \tag{4.42}$$

and

$$\bar{\partial} : W^{k,1}(\psi^*(T_{\mathbb{P}^1})) \rightarrow W^{k-1,1}(\Omega^{0,1}(\psi^*(T_{\mathbb{P}^1}))). \tag{4.43}$$



So its complex linear part satisfies the same property, as do all the operators  $D_t$ . It follows that for each  $t$  we have an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ker } D_t \rightarrow \text{Ker } \bar{\partial}_{\psi_C^* T_{\mathbb{P}^1}} \xrightarrow{\beta_t} \text{Coker } \bar{\partial}_{\psi_C^* (\mathcal{O}_{\mathbb{P}^1}(-1)^2)} \\ \rightarrow \text{Coker } D_t \rightarrow 0, \end{aligned} \tag{4.44}$$

hence a canonical isomorphism

$$\text{Det}_t \cong \bigwedge_{\mathbb{R}}^{\max} \text{Ker } \bar{\partial}_{\psi_C^* T_{\mathbb{P}^1}} \otimes \left( \bigwedge_{\mathbb{R}}^{\max} \text{Coker } \bar{\partial}_{\psi_C^* (\mathcal{O}_{\mathbb{P}^1}(-1)^2)} \right)^*, \tag{4.45}$$

which is easily seen to be continuous. The right hand side has a natural orientation coming from the complex structure on  $\text{Ker } \bar{\partial}$  and  $\text{Coker } \bar{\partial}$ . But for  $t = 1$ , the exact sequence (4.44) is an exact sequence of complex vector spaces and complex linear maps, so the isomorphism (4.45) for  $t = 1$  is compatible with the complex orientation. On the other hand, for  $t = 0$ , the isomorphism (4.45) induces on the left hand side (which is equal to  $\bigwedge_{\mathbb{R}}^{\max} T_{W_{kA,J,\nu}}^V$  at  $\psi$ ) the orientation of  $V(s_{\sigma_1}, s_{\sigma_1})$ , given by the complex structure on  $M_k$  and the complex structure on  $E \times E$ . So Lemma 4.2 is proved.

*Proof of Lemma 4.3.* We use the following version of the compacity theorem (cf. [4], [12])

**THEOREM 4.4.** *Assume  $(J_n, \nu_n)$  converges to  $(J, \nu)$  and let  $\psi_n \in W_{kA, J_n, \nu_n}^V$ ; then one can extract a subsequence  $\psi_{n_k}$  such that the graph of  $\psi_{n_k}$  in  $\mathbb{P}^1 \times X$  converges to the connected union of the graph of  $\psi_0 \in W_{\eta, J, \nu}^V$ , and of a vertical components  $t_i \times C_i$ , where  $t_i \in \mathbb{P}^1$  and  $C_i \subset U$  is holomorphic.*

Necessarily  $C_i$  must be equal to  $\mathbb{P}^1 \subset X$  since its class may take only finitely values, and we may assume that there is no rational curve in  $V$  having one of these classes, excepted for  $\mathbb{P}^1$ . So we must have  $\eta = lA$ ,  $l \leq k$  and the “limit”  $\psi_0$  corresponds to  $(C_0) \in V_l(s_{\sigma_1}, s_{\sigma_2}) \subset M_l^0$ . Now assume that there is a six dimensional family of limit graphs consisting of reducible curves; this would imply that for some  $l < k$ , there is an open set  $K$  of  $V_l(s_{\sigma_1}, s_{\sigma_2})$  such that for  $(C) \in K$ , the corresponding map  $\psi: \mathbb{P}^1 \rightarrow V$  meets  $\mathbb{P}^1$ ; writing  $\psi = (\psi_C, \psi_1, \psi_2)$  as above, this means that  $(\psi_1, \psi_2)$  vanishes at some point  $t \in C$ . But then, since by definition  $\bar{\partial}\psi_i = (\text{Id} \times \psi_C)^* \sigma_i$  we would have  $(\text{Id} \times \psi_C)^*(\sigma_1, \sigma_2) = 0$  in  $H^1(C, \psi_C^*(\mathcal{O}_{\mathbb{P}^1}(-1)) \oplus \psi_C^*(\mathcal{O}_{\mathbb{P}^1}(-1))(-t))$ , and by Lemma 3.4 the curve  $C \cup t \times \mathbb{P}^1$  would be in the zero set of the section  $(s_{\sigma_1}, s_{\sigma_2})$  on  $M_{l+1}$ . (Notice that by the Proposition 3.3,  $(s_{\sigma_1}, s_{\sigma_2})$  is continuous at reduced curves of  $M_{l+1}$ ). On the other hand,  $C \cup t \times \mathbb{P}^1$  belongs to the stratum  $M_{l+1}^{(1,0,\dots,0)}$  of  $M_{l+1}$ , and we have proved that for general  $(\sigma_1, \sigma_2)$  the intersection  $\bar{V}_{\sigma_1, \sigma_2} \cap M_{l+1}^{(1,0,\dots,0)}$  is at most four dimensional, which contradicts the fact that it would contain a 6 dimensional subvariety of  $M_{l+1}$ .

So we have proved the Theorem 1.1 for embedded rigid  $\mathbb{P}^1 \subset X$ . It remains to see what happens if  $\mathbb{P}^1 \xrightarrow{j} X$  is only an immersion: but we can replace  $X$  by a neighbourhood  $V$  of  $\mathbb{P}^1$  in its normal bundle, with the complex structure induced by an exponential map  $V \rightarrow X$ , which is a local diffeomorphism. The only thing that we have to verify is that we can choose the parameter  $\nu$  on  $\mathbb{P}^1 \times V$ , of the form  $((\text{Id} \times \pi)^*(\sigma_1), (\text{Id} \times \pi)^*(\sigma_2))$ , as in section 2, satisfying the transversality conclusion of the Proposition 3.6, and coming from  $\mathbb{P}^1 \times X$ : but it suffices to choose  $\sigma_i$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  vanishing over  $pr_2^{-1}(U_p)$  for an adequate (small) neighbourhood  $U_p$  in  $\mathbb{P}^1$  of any  $p \in \mathbb{P}^1$  such that  $j^{-1}(j(p)) \neq \{p\}$ . It is not difficult to show that the conclusion of the Proposition 3.6 still holds for a general couple  $(\sigma_1, \sigma_2)$  satisfying such a vanishing assumption.

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