Annali di Matematica (2004)

Digital Object Identifier (DOI) 10.1007/s10231-003-0092-2

Claire Voisin

Remarks on filtrations on Chow groups and the Bloch conjecture

À la mémoire de Fabio Bardelli

Received: June 10, 2002; in final form: November 6, 2002
Published online: February 11, 2004 – © Springer-Verlag 2004

0. Introduction

It has been conjectured by Bloch and Beilinson that there exists a decreasing filtration $F^i_{BB}CH^k(X)_{\mathbb{Q}}$ on the Chow groups with rational coefficients of any smooth complex projective variety X, satisfying the following properties:

1. (functoriality) $F^{\cdot}_{BB}$ is stable under correspondences, i.e. if $\Gamma \in CH^k(X \times Y)$, the induced morphism
$$\Gamma_* : CH^l(X)_{\mathbb{Q}} \to CH^{k+l-dim\,X}(Y)_{\mathbb{Q}}$$
satisfies: $\Gamma_* F^i_{BB}CH^l(X)_{\mathbb{Q}} \subset F^i_{BB}CH^{k+l-dim\,X}(Y)_{\mathbb{Q}}$.
2. (graded) If $\Gamma \in CH^k(X \times Y)_{hom}$ is a correspondence homologous to 0, the associated morphism Γ_* satisfies
$$Gr^i_{F_{BB}}\Gamma_* = 0 : Gr^i_{F_{BB}}CH^l(X)_{\mathbb{Q}} \to Gr^i_{F_{BB}}CH^{k+l-dim\,X}(Y)_{\mathbb{Q}}. \qquad (0.1)$$
Another version which we shall adopt says that the vanishing (0.1) holds once the Künneth component of $[\Gamma]$, which lies in
$$Hom\,(H^{2l-i}(X,\mathbb{Q}), H^{2k+2l-2dim\,X-i}(Y,\mathbb{Q})),$$
vanishes. In other words, the graded pieces $Gr^i_{F_{BB}}CH^l(X)_{\mathbb{Q}}$ are governed by the cohomology groups $H^{2l-i}(X,\mathbb{Q})$.
3. (finiteness) The filtration ends-up: $F^{k+1}_{BB}CH^k(X)_{\mathbb{Q}} = 0$, $\forall k$.

Several filtrations have been constructed, which satisfy some of the properties stated above. Hiroshi Saito [11] proposes considering the filtration $F^i_H CH^k(X)_{\mathbb{Q}}$

C. Voisin: Institut de mathématiques de Jussieu, CNRS, UMR 7586,
e-mail: Claire.Voisin@math.jussieu.fr

constructed inductively, starting with $F_H^1 CH^k(X)_{\mathbb{Q}} = CH^k(X)_{hom,\mathbb{Q}}$ and defining

$$F_H^{i+1} CH^k(X) =< \Gamma_* F_H^i CH^l(Y),\ [\Gamma] = 0 \,\mathrm{in}\, H^{2\gamma}(Y \times X, \mathbb{Z}) >_{\mathbb{Q}},$$

where $k = \gamma + l - dim\, Y,\ \gamma = codim\, \Gamma$. A more direct way to define it is to put

$$F_H^i CH^k(X)_{\mathbb{Q}} =< \Gamma_*(z_1 \cdot \ldots \cdot z_i),\ \Gamma \in CH(Y \times X),\ z_j \in CH(Y)_{hom,\mathbb{Q}}) >_{\mathbb{Q}} .$$

This filtration must clearly be contained in any Bloch–Beilinson filtration, if there is one, because of property 2. Also, it satisfies property 1 and the first version of property 2 but nothing is known about 3.

Shuji Saito [10] constructs a filtration $F_S^i CH^k(X)$ similar in spirit but closer to satisfying 3. Roughly speaking, it is defined as follows: one starts with $F_S^1 CH^k(X)_{\mathbb{Q}} = CH^k(X)_{hom,\mathbb{Q}}$ and one defines

$$F_S^{i+1} CH^k(X) =< \Gamma_* F_S^i CH^l(Y),\ [\Gamma]_{2\gamma-2k+i,2k-i} = 0 >_{\mathbb{Q}},$$

where $k = \gamma + l - dim\, Y,\ \gamma = codim\, \Gamma$ and

$$[\Gamma]_{2\gamma-2k+i,2k-i} \in Hom\, (H^{2l-i}(Y, \mathbb{Q}), H^{2k-i}(X, \mathbb{Q}))$$

denotes the Künneth component of type $(2\gamma - 2k + i, 2k - i)$ of the cohomology class of Γ. This filtration obviously contains the previous one, and again, it must clearly be contained in any Bloch–Beilinson filtration, if there is one, because of (the strengthened version) of property 2. It also satisfies properties 1 and 2, and, assuming the Lefschetz standard conjecture, the following version of 3:

$$F_S^{k+1} CH^k(X)_{\mathbb{Q}} = F_S^{k+2} CH^k(X)_{\mathbb{Q}} = F_S^l CH^k(X)_{\mathbb{Q}},\ l \geq k + 1.$$

(Notice that in the precise version of [10], one does not need the Lefschetz conjecture to get the stationarity above.)

Our purpose in this paper is of several kinds. First of all we introduce a filtration $F^i_{naïve}$ on the group $CH_0(X)_{\mathbb{Q}}$ of 0-cycles with rational coefficients of any smooth projective variety X. The group $F^i_{naïve} CH_0(X)$ is simply defined as the subgroup of $CH_0(X)$ consisting of elements annihilated by all 0-correspondences from X to a variety of dimension $< i$. This filtration satisfies properties 1 (functoriality) and 3 (finiteness) but we cannot say much about 2. We shall come back to this in the last section. In the first section we show the following:

Proposition 1. *Assume there exists a Bloch–Beilinson filtration. Assume furthermore the Lefschetz standard conjecture is true. Then we must have*

$$F^i_{naïve} = F^i_{BB}$$

on the groups $CH_0(X)_{\mathbb{Q}} = CH^n(X)_{\mathbb{Q}},\ n = dim\, X.$

We turn next to a filtration introduced by Nori [8]. This is an increasing filtration that we shall denote by $\mathcal{N}_r CH^k(X)_{\mathbb{Q}}$. It satisfies the properties that

$$\mathcal{N}_0 CH^k(X)_{\mathbb{Q}} = CH^k(X)_{alg,\mathbb{Q}},$$

where the right-hand side is the subgroup of cycles algebraically equivalent to 0, and that for $r \geq n-k$, $n = dim\ X$ one has

$$\mathcal{N}_r CH^k(X)_{\mathbb{Q}} = CH^k(X)_{hom,\mathbb{Q}}.$$

We show the following:

Proposition 2. *Assume a Bloch–Beilinson filtration exists. Assume furthermore the Lefschetz standard conjecture holds. Then one has, for $r > 0$,*

$$F^r_{BB} CH^k(X)_{\mathbb{Q}} \subset \mathcal{N}_{k-r} CH^k(X)_{\mathbb{Q}}.$$

In particular, for $k = r$ we recover a fact observed by Jannsen [4], namely that under the same assumptions, we have

$$F^k_{BB} CH^k(X)_{\mathbb{Q}} \subset CH^k(X)_{alg,\mathbb{Q}}. \tag{0.2}$$

In the second and third sections of this paper, we formulate two conjectures, which together will appear to imply the Bloch conjecture [1] for surfaces. One is a weak form of the inclusion 0.2:

Conjecture 1. For some $N > 0$ we have

$$F^N_H CH^k(X)_{\mathbb{Q}} \subset CH^k(X)_{alg,\mathbb{Q}},$$

where $F^N_H CH$ is the Hiroshi Saito filtration introduced above.

We shall show the following:

Theorem 1. *Assume the generalized Hodge conjecture holds. Assume furthermore that Conjecture 4 is satisfied. Then if X is a smooth projective variety satisfying the condition that $H^{i,0}(X) = 0,\ i > 0$, we have*

$$CH_0(X)_{hom,\mathbb{Q}} = 0.$$

This is a converse to the higher-dimensional version of Mumford's theorem [6].

The second conjecture is related to the fact that the Hodge structure on the transcendental part of the cohomology group $H^n(X,\mathbb{Q})$, $n = dim\ X$ is naturally a polarized Hodge structure. It follows from this that if

$$\Gamma \in CH^n(X \times Y),\ n = dim\ X = dim\ Y,$$

the morphism of Hodge structures

$$[\Gamma]_* : H^n(X,\mathbb{Q})_{tr} \to H^n(Y,\mathbb{Q})_{tr}$$

satisfies

$$Im\,[\Gamma]_* = Im\,[\Gamma \circ {}^t\Gamma]_* \subset H^n(Y,\mathbb{Q})_{tr}. \tag{0.3}$$

Here ${}^t\Gamma \in CH^n(Y \times X)$ is the transposed cycle and $[{}^t\Gamma]_* : H^n(Y,\mathbb{Q})_{tr} \to H^n(X,\mathbb{Q})_{tr}$ is the transpose of $[\Gamma]_* : H^n(X,\mathbb{Q})_{tr} \to H^n(Y,\mathbb{Q})_{tr}$ with respect to the intersection pairing.

The equality (0.3) together with the Bloch–Beilinson conjecture then lead to the following conjecture:

Conjecture 2. Let $\Gamma \in CH^n(X \times Y)$, $n = dim\, X = dim\, Y$ be a correspondence. Then

$$Im\, F^n\Gamma_* = Im\, F^n(\Gamma \circ {}^t\Gamma)_* \subset F^nCH^n(Y)_{\mathbb{Q}}.$$

Here $F^n\Gamma_*$ and $F^n(\Gamma \circ {}^t\Gamma)_*$ denote, respectively, the restrictions of Γ_* and $\Gamma \circ {}^t\Gamma_*$ to $F^nCH^n(X)_{\mathbb{Q}}$ and $F^nCH^n(Y)_{\mathbb{Q}}$, the filtration being the naïve filtration.

We shall explain several consequences of this conjecture in the case where $n = dim\, 2$ (the conjecture is well known for curves). Recalling that $F^3_{naïve}CH_0(X)$ is made of cycles annihilated by all 0-correspondences from X to a surface, first of all it allows us to study the naïve filtration for 0-cycles on a threefold by the following results:

Proposition 3. *Assume Conjecture 2 holds. Then if $\Gamma \in CH^2(X \times S)$ is a correspondence between a smooth complex projective threefold and a smooth complex projective surface, and if $Y \stackrel{j}{\hookrightarrow} X$ is the inclusion of any smooth ample surface, we have*

$$CH_0(X) = j_*CH_0(Y) + Ker\, \Gamma_*.$$

Proposition 4. *Under the same assumption, let $\Gamma \in CH^2(X \times \mathcal{S})$, where $\pi : \mathcal{S} \to B$ is a family of smooth surfaces parametrized by a smooth connected quasiprojective basis B. Then the kernel $Ker\, (F^2\Gamma_{b*} : F^2CH_0(X) \to F^2CH_0(S_b))$ is constant, where $F^2CH_0 = Ker\, alb$ and $S_b = \pi^{-1}(b),\ \Gamma_b = \Gamma_{|X \times S_b}$.*

Another consequence of Conjecture 2 is the following rigidity statement:

Proposition 5. *Assume Conjecture 2. Then if B is a smooth connected quasiprojective variety, $\mathcal{S} \to B$ is a smooth family of surfaces, Σ is a surface, and*

$$\Gamma \in CH^2(\mathcal{S} \times \Sigma)$$

is a codimension 2 cycle, the subgroup

$$Im\, F^2\Gamma_{b*} \subset F^2CH^2(\Sigma)_{\mathbb{Q}}$$

is a constant subgroup, i.e. is independent of $b \in B$.

Finally we show that Conjectures 4 and 2 together imply Bloch's conjecture for correspondences between surfaces

Theorem 2. *Assume Conjectures 4 and 2. Then if $\Gamma \in CH^2(S \times T)$ is a correspondence between smooth surfaces, such that $[\Gamma]_* : H^{2,0}(S) \to H^{2,0}(T)$ is equal to 0, the morphism*

$$F^2\Gamma_* : F^2CH_0(S) \to F^2CH_0(T)$$

is equal to 0.

1. On some filtrations on Chow groups

1.1. A naïve filtration on $CH_0(X)$

We define a decreasing filtration $F^i_{naïve}$ on $CH_0(X)$, where X is smooth and projective, by putting

$$F^i_{naïve}CH_0(X) = \cap_\Gamma Ker\,\Gamma_*,$$

where the intersection is taken over all correspondences $\Gamma \in CH^l(X \times Y)$ with

$$l = dim\,Y < i.$$

The obvious facts concerning this filtration are the following:

Lemma 1. *$F^i_{naïve}$ is stable under 0-correspondences, i.e. correspondences*

$$\Gamma \in CH^l(X \times Y),\ \ l = dim\,Y.$$

Proof. Indeed, if Γ is such a correspondence and $\Gamma' \in CH^k(Y \times Z)$ with $k = dim\,Z < i$, the composed correspondence $\Gamma' \circ \Gamma$ belongs to $CH^k(X \times Z)$ and hence annihilates $F^i_{naïve}CH_0(X)$. This proves that

$$\Gamma_*\big(F^i_{naïve}CH_0(X)\big) \subset F^i_{naïve}CH_0(Y).$$

□

Lemma 2. *We have the equalities*

$$F^1_{naïve}CH_0(X) = CH_0(X)_{hom}$$
$$F^2_{naïve}CH_0(X)_{\mathbb{Q}} = Ker\,alb_X \subset CH_0(X)_{hom,\mathbb{Q}},$$

where $alb_X : CH_0(X)_{hom} \to Alb(X)$ is the Albanese morphism.

Proof. The first equality is obvious: it suffices to consider the correspondences from each component of X to a point.

The second fact is more difficult, but classically known (cf. [7]). We recall the argument here: first of all the inclusion

$$Ker\,alb_X \subset F^2_{naïve}CH_0(X)$$

follows from the fact that for any correspondence,

$$\Gamma \in CH^1(X \times C),$$

where C is a smooth curve, the morphism

$$\Gamma_* : CH_0(X)_{hom} \to CH_0(C)_{hom}$$

factors through the Albanese morphism.

The reverse inclusion is proven as follows: let $C \stackrel{j}{\hookrightarrow} X$ be the inclusion of a smooth curve which is the complete intersection of ample hypersurfaces. By the Lefschetz theorem, the restriction provides an inclusion

$$j^* : H^1(X, \mathbb{Z}) \to H^1(C, \mathbb{Z}).$$

Tensorized by $\mathbb{Q}$, this morphism is the inclusion of a sub-Hodge structure and using the polarization on the right-hand side, we get a splitting

$$H^1(C, \mathbb{Q}) = H^1(X, \mathbb{Q}) \oplus L \tag{1.4}$$

into a direct sum of rational sub-Hodge structures. Now let γ be the degree 2 rational Hodge class on $C \times X$ whose associated morphisms (or Künneth components)

$$\gamma_* : H^l(C, \mathbb{Q}) \to H^l(X, \mathbb{Q})$$

vanish for $l \neq 1$ and are equal to the projector on the first factor in (1.4) for $l = 1$. There exists a divisor $D \in CH^1(C \times X)_{\mathbb{Q}}$ such that $[D] = \gamma$. Now we claim that $Ker\ D^* : CH_0(X)_{\mathbb{Q}} \to CH_0(C)_{\mathbb{Q}}$ is contained in $Ker\ alb_X \otimes \mathbb{Q}$. This is because, by assumption, $j_* \circ [D]^*$ acts as the identity on $Alb(X) \otimes \mathbb{Q}$.

This fact implies that

$$F^2_{naïve} CH_0(X)_{\mathbb{Q}} \subset Ker\ D^* \subset Ker\ alb_X,$$

as claimed. □

Lemma 3. *We have* $F^{n+1}_{naïve} CH_0(X)_{\mathbb{Q}} = 0$*, where* $n = dim\ X$.

Proof. Indeed, it suffices to consider the cycle $\Gamma \in CH^n(X \times X)$ which is given by the diagonal. Clearly $Ker\ \Gamma_* = 0$. On the other hand, we have, by definition,

$$F^{n+1}_{naïve} CH_0(X) \subset Ker\ \Gamma_*.$$

□

In conclusion this filtration satisfies the properties 1 and 3 of the Bloch–Beilinson filtration, but we cannot say anything about property 2. The simplest question to ask would be an analogue of Mumford's theorem for this filtration, namely:

Conjecture 3. Let X be a n-dimensional smooth complex projective variety such that $F^n_{naïve} CH_0(X) = 0$. Then we have $H^{n,0}(X) = 0$.

In the case $n = 2$, the answer is affirmative since we know that $F^2_{naïve} = Ker\ alb$, so that the conjecture above is Mumford's theorem in this case. The first interesting case is $F^3_{naïve} CH_0(X)$ for X a threefold. We shall come back to this in Section 3.

We now show:

Proposition 6. *Assume the Lefschetz standard conjecture* [5] *hold. Assume a Bloch–Beilinson filtration* F_{BB} *exists. Then we have*

$$F^i_{naïve} CH_0(X)_{\mathbb{Q}} = F^i_{BB} CH_0(X)_{\mathbb{Q}}.$$

Proof. The inclusion,

$$F^i_{BB}CH_0(X)_{\mathbb{Q}} \subset F^i_{naïve}CH_0(X)_{\mathbb{Q}},$$

follows from the finiteness property 3 of the Bloch–Beilinson filtration. Indeed, if $z \in F^i_{BB}CH_0(X)_{\mathbb{Q}}$ and $\Gamma \in CH^l(X \times Y)$ with $l = dim\, Y < i$ we have

$$\Gamma_* z \in F^i_{BB}CH_0(Y)_{\mathbb{Q}}$$

and this group is 0 because $dim\, Y < i$.

The reverse inclusion is proved by induction on $n = dim\, X$. We choose an ample smooth hypersurface $Y \overset{j}{\hookrightarrow} X$. Since we assume the Lefschetz standard conjecture, there exists a cycle $\Gamma \in CH^{n-1}(Y \times X)_{\mathbb{Q}}$ such that

$$[\Gamma]_* \circ j^* : H^l(X, \mathbb{Q}) \to H^l(X, \mathbb{Q})$$

is the identity for $l < n$. Dualizing, we get that

$$j_* \circ [{}^t\Gamma]_* : H^l(X, \mathbb{Q}) \to H^l(X, \mathbb{Q})$$

is the identity for $l > n$. If $\Gamma_j \subset Y \times X$ is the graph of j, the cycle $\Gamma_j \circ {}^t\Gamma \in CH^n(X \times X)$ has its cohomology class γ acting as $j_* \circ [{}^t\Gamma]_*$ on the cohomology groups of X. So it acts as the identity on $H^l(X, \mathbb{Q})$ for $l > n$. Now if a Bloch–Beilinson filtration exists, property 2 says that the cycle $\Gamma_j \circ {}^t\Gamma$ must act on the graded pieces

$$Gr^i_{F_{BB}}CH_0(X)_{\mathbb{Q}},$$

as the identity for $i < n$.

It remains to see that this implies that $F^i_{naïve}CH_0(X)_{\mathbb{Q}} \subset F^i_{BB}CH_0(X)_{\mathbb{Q}}$ for all i. We do it now by induction on i. Assume this is proved for $i - 1$. Let $z \in F^i_{naïve}CH_0(X)_{\mathbb{Q}}$. So by induction on i, $z \in F^{i-1}_{BB}CH_0(X)_{\mathbb{Q}}$. Because we know that $j_* \circ {}^t\Gamma_*$ acts as the identity on $Gr^{i-1}_{F_{BB}}CH_0(X)_{\mathbb{Q}}$, we have that

$$z = j_* \circ {}^t\Gamma_* z \text{ modulo } F^i_{BB}CH_0(X)_{\mathbb{Q}}.$$

Now, since ${}^t\Gamma_* z \in F^i_{naïve}CH_0(Y)_{\mathbb{Q}}$, induction on $dim\, X$ shows that

$${}^t\Gamma_* z \in F^i_{BB}CH_0(Y)_{\mathbb{Q}}.$$

Hence we get that $j_* \circ {}^t\Gamma_* z \in F^i_{BB}CH_0(X)_{\mathbb{Q}}$, and also $z \in F^i_{BB}CH_0(X)_{\mathbb{Q}}$. □

1.2. Nori's and Bloch–Beilinson's filtrations

Recall that the (increasing) Nori filtration $\mathcal{N}_r CH^k(X)$ is defined by

$$\mathcal{N}_r CH^k(X) = < \Gamma_* z, \Gamma \in CH^{k+r}(Y \times X),\ z \in CH_r(Y)_{hom} > .$$

Here we consider all possible smooth projective Y and all possible correspondences Γ and look at the group generated by the cycles above. We clearly have

$$\mathcal{N}_0 CH^k(X) = CH^k(X)_{alg},$$

the subgroup of cycles algebraically equivalent to 0. We also have

$$\mathcal{N}_r CH^k(X) = CH^k(X)_{hom},$$

for $r \geq n-k$, since for $r = n-k$, one can take for Γ the diagonal cycle in $X \times X$. Furthermore the Nori filtration is obviously stable under correspondences. Nori [8] proves that the graded pieces of his filtration are generally non-trivial modulo torsion, thus refining Griffiths' result [3] on the non-triviality of the Griffiths group $CH^k(X)_{hom}/CH^k(X)_{alg}$. We prove the following relation between Nori's and Bloch–Beilinson's filtrations:

Proposition 7. *Assume the Lefschetz standard conjecture is true and that a Bloch–Beilinson filtration exists. Then we have, for any $r > 0$,*

$$F^r_{BB} CH^k(X)_{\mathbb{Q}} \subset \mathcal{N}_{k-r} CH^k(X)_{\mathbb{Q}}.$$

For $r = k$ we then get the following, which was proved by Jannsen [4]:

Corollary 1. *In particular we have the inclusion*

$$F^k_{BB} CH^k(X)_{\mathbb{Q}} \subset CH^k(X)_{alg,\mathbb{Q}}.$$

Proof of Proposition 7. The proof is again by induction on *dim* X. There are two possibilities:

a) $k-r \geq n-k$. In this case we have $\mathcal{N}_{k-r} CH^k(X) = CH^k(X)_{hom}$ and $F^r CH^k(X) \subset CH^k(X)_{hom}$ because $r > 0$. Hence there is nothing to prove in this case.

b) $k-r < n-k$. In this case, we have, as well,

$$2k - r < n.$$

We choose an ample smooth hypersurface $Y \stackrel{j}{\hookrightarrow} X$. Since we assume the Lefschetz standard conjecture, there exists a cycle $\Gamma \in CH^{n-1}(Y \times X)_{\mathbb{Q}}$ such that

$$[\Gamma]_* \circ j^* : H^l(X, \mathbb{Q}) \to H^l(X, \mathbb{Q})$$

is the identity for $l < n$. In particular $[\Gamma]_* \circ j^*$ acts as the identity on $H^l(X, \mathbb{Q})$ for $l \leq 2k - r$. Now if a Bloch–Beilinson filtration exists, the correspondence $\Gamma \circ {}^t\Gamma_j$ must act as the identity on the graded pieces

$$Gr^l_{F_{BB}} CH^k(X)_{\mathbb{Q}}$$

for $2k - l \leq 2k - r$, hence $\Gamma \circ {}^t\Gamma_j$ must act bijectively on the group $F^r_{BB} CH^k(X)_{\mathbb{Q}}$. It follows that any $z \in F^r_{BB} CH^k(X)_{\mathbb{Q}}$ can be written as

$$z = \Gamma_* \circ j^* z',$$

for some $z' \in F^r_{BB}CH^k(X)_{\mathbb{Q}}$. Now, by induction on *dim X*, we may assume that $j^*z' \in \mathcal{N}_{k-r}CH^k(Y)_{\mathbb{Q}}$, hence it follows that

$$z = \Gamma_* \circ j^*z' \in \mathcal{N}_{k-r}CH^k(X)_{\mathbb{Q}}. \qquad \square$$

2. A generalized Nori conjecture and its consequences

We have seen in Proposition 7 that if a Bloch–Beilinson filtration exists, and the Lefschetz conjecture holds, we must have the inclusion

$$F^k_{BB}CH^k(X)_{\mathbb{Q}} \subset CH^k(X)_{alg,\mathbb{Q}}$$

for $r > 0$. We do not know if a Bloch–Beilinson filtration exists, but we have the Hiroshi Saito filtration described in the introduction, which has to be contained in any Bloch–Beilinson filtration. So we can make the following conjecture, which is a weak converse to Proposition 7:

Conjecture 4. For some $N > 0$ (depending on X), we have

$$F^N_H CH^k(X)_{\mathbb{Q}} \subset CH^k(X)_{alg,\mathbb{Q}}.$$

We call this a weak generalized Nori conjecture because it extends to any codimension and weakens the following conjecture by Nori [8]:

Conjecture 5. The kernel of the Abel–Jacobi map for codimension 2 cycles is contained modulo torsion in the group of cycles algebraically equivalent to 0.

Indeed, it is a standard fact that $F^N_H CH^k(X)$ is contained in the kernel of the Abel–Jacobi map for $N \geq 2$, (the product of two cycles homologous to 0 is Abel–Jacobi equivalent to 0), so that Conjecture 4 is a weakening of Nori's Conjecture 5 in the case of codimension 2 cycles.

In this section we prove the following statement:

Theorem 3. *Assume that the generalized Hodge conjecture is true. Assume that Conjecture* 4 *holds. Then, if X is a smooth projective complex variety satisfying the condition*

$$H^{i,0}(X) = 0,\ i > 0,$$

we have

$$CH_0(X)_{hom} = 0.$$

This "theorem" says that the converse to the generalized Mumford theorem [6] is implied by Conjecture 4. Indeed, the generalized Mumford theorem says, conversely, that the vanishing

$$CH_0(X)_{hom} = 0$$

implies the vanishing

$$H^{i,0}(X) = 0,\ i > 0.$$

The proof of the theorem starts with the following, presumably standard, lemma:

Lemma 4. *Assume the generalized Hodge conjecture holds. Then if X is a smooth projective complex variety satisfying the condition*

$$H^{i,0}(X) = 0,\ i > 0,$$

there exists a cycle

$$\Gamma \in CH^n(X \times X)_{\mathbb{Q}},\ n = dim\ X$$

such that

$$[\Gamma] = 0 \text{ in } H^{2n}(X \times X, \mathbb{Q}),$$
$$\Gamma_* = Id \text{ on } CH_0(X)_{hom,\mathbb{Q}}.$$

Proof. It suffices to consider the case where X is connected. Because $H^{i,0}(X) = 0$, $i > 0$, the generalized Hodge–Grothendieck conjecture implies that there exists a variety Y of dimension $n - 1$, and a morphism

$$j : Y \to X$$

such that

$$j_* : H^{l-2}(Y, Q) \to H^l(X, \mathbb{Q})$$

is surjective for $l > 0$. Consider the morphism

$$(j, j) : Y \times Y \to X \times X.$$

It provides a surjective morphism of Hodge structures

$$(j, j)_* : H^{l-2}(Y, \mathbb{Q}) \otimes H^{2n-l-2}(Y, \mathbb{Q}) \to H^l(X, \mathbb{Q}) \otimes H^{2n-l}(X, \mathbb{Q}),$$

for $l > 0$ and $2n - l > 0$.

Let $x \in X,\ \Delta_X \subset X \times X$ be the diagonal and

$$\Delta'_X = \Delta_X - x \times X - X \times x \in CH^n(X \times X)_{\mathbb{Q}}.$$

Clearly Δ'_X acts as the identity on $CH_0(X)_{hom,\mathbb{Q}}$. On the other hand, the class $[\Delta'_X]$ is the sum of the Künneth components of $[\Delta_X]$ whose type is different from $(0, 2n)$ or $(2n, 0)$. It follows then from the above, and from the fact that the Hodge structures we consider are polarized, that there is a rational Hodge class

$$\gamma' \in H^{2n-4}(Y \times Y, \mathbb{Q}) \cap H^{n-2,n-2}(Y \times Y)$$

such that

$$(j, j)_*\gamma' = [\Delta'_X].$$

Because the Hodge conjecture is supposed to be true, there is a cycle

$$\Gamma' \in CH^{n-2}(Y \times Y, \mathbb{Q})$$

such that $[\Gamma'] = \gamma'$. The cycle

$$\Gamma = \Delta'_X - (j, j)_*\Gamma' \in CH^n(X \times X)_{\mathbb{Q}}$$

then satisfies our conclusion, since $(j, j)_*\Gamma'$ acts trivially on $CH_0(X)_{\mathbb{Q}}$ because it is supported on $j(Y) \times j(Y)$. □

Proof of Theorem 3. Let X be as in the theorem. Using the above lemma, we get a cycle $\Gamma \in CH^n(X \times X)$ such that $[\Gamma] = 0$ and $\Gamma_* = Id$ on $CH_0(X)_{hom,\mathbb{Q}}$. By the definition of the Hiroshi Saito filtration, $[\Gamma] = 0$ implies that

$$\Gamma^{\circ N} \in F_H^N CH^n(X \times X)_{\mathbb{Q}},$$

for any $N > 0$. If Conjecture 4 is true, then we have

$$\Gamma^{\circ N} \in CH^n(X \times X)_{alg,\mathbb{Q}},$$

for some N.

Now we recall the following result, due independently to Voevodsky [12] and the author [13]:

Theorem 4. *Let $z \in CH(X \times X)_{alg,\mathbb{Q}}$ be a cycle algebraically equivalent to 0. Then for some $M > 0$, we have*

$$z^{\circ M} = 0 \text{ in } CH(X \times X)_{\mathbb{Q}}.$$

Applying this theorem to $\Gamma^{\circ N}$ we conclude that some power

$$\Gamma^{\circ MN}$$

vanishes in $CH^n(X \times X)_{\mathbb{Q}}$. In particular,

$$\Gamma_*^{\circ MN} = 0 \text{ in } End\,(CH_0(X)_{\mathbb{Q}}).$$

Now because $\Gamma_* = Id$ on $CH_0(X)_{hom,\mathbb{Q}}$, it follows that

$$CH_0(X)_{hom,\mathbb{Q}} = 0.$$

Finally Roitman's theorem [9] implies that $CH_0(X)_{hom} = 0$. □

3. Polarizations

In this section, we would like to point out a problem which is the missing ingredient to extend Theorem 3 to the case of 0-correspondences between varieties inducing the 0-maps between spaces of holomorphic forms. The starting point is the observation that if a Bloch–Beilinson filtration exists for a correspondence

$$\Gamma \in CH^n(X \times Y),\ n = dim\, X = dim\, Y,$$

the morphism

$$F^n\Gamma_* : F_{BB}^n CH_0(X)_{\mathbb{Q}} \to F_{BB}^n CH_0(Y)_{\mathbb{Q}}$$

is determined by the morphism of Hodge structures

$$[\Gamma]_* : H^n(X,\mathbb{Q}) \to H^n(Y,\mathbb{Q}).$$

Furthermore, if the generalized Hodge conjecture is true, it is even governed by the morphism of Hodge structures

$$[\Gamma]_* : H^n(X,\mathbb{Q})_{tr} \to H^n(Y,\mathbb{Q})_{tr}.$$

(Here the subscript *tr* stands for the transcendental part and means the orthogonal with respect to the intersection pairing of the maximal sub-Hodge structure which has no $(n,0)$-part. Equivalently, this is the smallest sub-Hodge structure which is rational and contains over $\mathbb{C}$ the space $H^{n,0}$.) This fact is proved in the same way as Lemma 4.

Next we recall that the intersection pairing on $H^n(X,\mathbb{Q})$ induces a polarized Hodge structure on $H^n(X,\mathbb{Q})_{prim}$, where the primitive cohomology group $H^n(X,\mathbb{Q})_{prim}$ is defined as the kernel of the morphism

$$L = c_1(H)\cup : H^n(X,\mathbb{Q}) \to H^{n+2}(X,\mathbb{Q}),$$

for a given ample line bundle H on X. The fact that the Hodge structure is polarized means that if h is the hermitian form deduced from $<,>$ by the rule

$$h(\alpha,\beta) = (-1)^n < \alpha, \overline{\beta} >,$$

first of all the Hodge decomposition is orthogonal with respect to h and secondly, the restriction of h to each $H^{p,q}$ is of a definite sign.

Note now that we have the inclusion

$$H^n(X,\mathbb{Q})_{tr} \subset H^n(X,\mathbb{Q})_{prim},$$

for any choice of polarization H, because $H^n(X,\mathbb{Q})_{prim}$ is also the orthogonal of the sub-Hodge structure

$$Im \cup c_1(H) : H^{n-2}(X,\mathbb{Q}) \to H^n(X,\mathbb{Q}),$$

which has no $(n,0)$-part. It follows that $<,>$ induces a polarized Hodge structure on $H^n(X,\mathbb{Q})_{tr}$ as well. Now we have the following lemma:

Lemma 5. *Let $\phi : H \to H'$ be a morphism of Hodge structures which are polarized. Then*

$$Im\,\phi = Im\,\phi \circ {}^t\phi,$$

where ${}^t\phi : H' \to H$ is the transpose of ϕ with respect to the polarizations. More precisely, there exists a morphism $\psi : H \to H'$ of Hodge structures such that

$$\phi = \phi \circ {}^t\phi \circ \psi.$$

Proof. Let $L = Ker\,\phi$. This is a sub-Hodge structure of H. Because H is polarized, the pairing on H remains non degenerate after restriction to L, because the associated hermitian form remains non degenerate after restriction to each $L^{p,q}$, and it follows that we have a decomposition

$$H = L \oplus L^{\perp}.$$

Furthermore $L^{\perp}$ is also a sub-Hodge structure of H.

Similarly we have an orthogonal decomposition

$$H' = L' \oplus L'^{\perp},$$

where $L' := Im\, \phi$. The morphism ϕ now induces an isomorphism of Hodge structures $\phi_0 : L^{\perp} \cong L'$ and we can write

$$\phi = j_{L'} \circ \phi_0 \circ \pi_{L^{\perp}}, \tag{3.5}$$

where $j_{L'}$ is the inclusion of L' into H', and $\pi_{L^{\perp}}$ is the orthogonal projection onto $L^{\perp}$. The three morphisms here are morphisms of Hodge structures. It follows from (3.5) that

$$\phi \circ {}^t\phi = j_{L'} \circ \phi_0 \circ {}^t\phi_0 \circ \pi_{L'}, \tag{3.6}$$

where ${}^t\phi_0 : L' \to L^{\perp}$ is the transpose of ϕ_0 with respect to the restricted pairings. Now there obviously exists a morphism of Hodge structures

$$\psi_0 : L^{\perp} \to L',$$

such that

$$\phi_0 = \phi_0 \circ {}^t\phi_0 \circ \psi_0, \tag{3.7}$$

namely it suffices to take $\psi_0 = {}^t\phi_0^{-1}$. Then equations (3.7) and (3.5) give

$$\phi = j_{L'} \circ \phi_0 \circ {}^t\phi_0 \circ \psi_0 \circ \pi_{L^{\perp}},$$

which is obviously equal to

$$j_{L'} \circ \phi_0 \circ {}^t\phi_0 \circ \pi_{L'} \circ \psi,$$

where $\psi = j_{L'} \circ \psi_0 \circ \pi_{L^{\perp}} : H \to H'$. Hence applying (3.6) again we have written

$$\phi = \phi \circ {}^t\phi \circ \psi.$$

□

Now let $\Gamma \in CH^n(X \times Y)_{\mathbb{Q}}$, $n = dim\, X = dim\, Y$, and let us apply the above to

$$[\Gamma]_* : H^n(X, \mathbb{Q})_{tr} \to H^n(Y, \mathbb{Q})_{tr}.$$

The lemma above gives a morphism of Hodge structures

$$\psi : H^n(X, \mathbb{Q})_{tr} \to H^n(Y, \mathbb{Q})_{tr}$$

such that

$$[\Gamma]_* = [\Gamma]_* \circ {}^t[\Gamma]_* \circ \psi.$$

If the Hodge conjecture is true, this ψ is equal to $[\Psi]_*$ for some cycle

$$\Psi \in CH^n(X \times Y)_{\mathbb{Q}}.$$

Furthermore we know that

$$^t[\Gamma]_* = [^t\Gamma]_*,$$

where $^t\Gamma \in CH^n(Y \times X)_{\mathbb{Q}}$ is the transpose of Γ. In conclusion we have the equality

$$[\Gamma]_* = [\Gamma \circ {}^t\Gamma \circ \Psi]_* : H^n(X, \mathbb{Q})_{tr} \to H^n(Y, \mathbb{Q})_{tr}.$$

According to Bloch and Beilinson, we should have the corresponding equality

$$F^n\Gamma_* = F^n(\Gamma \circ {}^t\Gamma \circ \Psi)_* : F^n_{BB}CH^n(X)_{\mathbb{Q}} \to F^n_{BB}CH^n(Y)_{\mathbb{Q}}$$

and, in particular,

$$Im\ F^n\Gamma_* = Im\ F^n(\Gamma \circ {}^t\Gamma)_* \subset F^n_{BB}CH^n(Y)_{\mathbb{Q}}.$$

Recall that here $F^n_{BB}CH^n(Y)_{\mathbb{Q}}$ should be equal to $F^n_{naïve}CH^n(Y)_{\mathbb{Q}}$, which is well defined. We thus are led to the following conjecture:

Conjecture 6. Let $\Gamma \in CH^n(X \times Y)_{\mathbb{Q}}$, $n = dim\ X = dim\ Y$. Then

$$Im\ F^n\Gamma_* = Im\ F^n(\Gamma \circ {}^t\Gamma)_* \subset F^n_{naïve}CH^n(Y)_{\mathbb{Q}},$$

where $F^n\Gamma_*$ denotes the restriction of Γ_* to $F^n_{naïve}CH_0$.

Remark 1. A stronger form of the conjecture would ask the existence of a cycle $\Psi \in CH^n(X \times Y)_{\mathbb{Q}}$ such that

$$F^n\Gamma_* = F^n(\Gamma \circ {}^t\Gamma \circ \Psi)_* : F^n_{naïve}CH^n(X)_{\mathbb{Q}} \to F^n_{naïve}CH^n(Y)_{\mathbb{Q}}.$$

It turns out that, in the case $n = 2$ that we will be considering below, these two versions are equivalent. Note also that when $n = 1$, the conjecture is true.

Remark 2. For $n = 2$, the conjecture also has the following consequence: for any correspondence $\Gamma \in CH^2(X \times Y)_{\mathbb{Q}}$, where X and Y are smooth complex projective surfaces, we have the equality $Ker\ F^2\Gamma_* = Ker\ F^2({}^t\Gamma \circ \Gamma)_*$. Indeed, as noticed above, the conjecture for $n = 2$ implies the existence of a $\Psi \in CH^2(X \times Y)_{\mathbb{Q}}$ such that

$$F^2\Gamma_* = F^2(\Gamma \circ {}^t\Gamma \circ \Psi)_*.$$

This implies the equality of the actions of the transposed correspondences on the F^2 level, namely

$$F^2\, {}^t\Gamma_* = F^2({}^t\Psi \circ \Gamma \circ {}^t\Gamma)_* : F^2CH_0(Y)_{\mathbb{Q}} \to F^2CH_0(X)_{\mathbb{Q}},$$

as follows from the Bloch–Srinivas decomposition (cf [2]). This last equality obviously implies that $Ker\ F^2\, {}^t\Gamma_* = Ker\ F^2(\Gamma \circ {}^t\Gamma)_*$.

Example 1. Let X, Y be two smooth projective surfaces which are fibered over a 1-dimensional smooth basis B. Let $\Gamma \subset X \times_B Y$ be a 2-cycle. So $\Gamma \subset X \times Y$ is the union over $b \in B$ of the one-dimensional correspondences Γ_b, and of finitely many cycles supported on products $X_b \times Y_b$. Such a correspondence Γ satisfies Conjecture 6 because each Γ_b does.

The first consequence of Conjecture 6 for surfaces is the following:

Proposition 8. *Assume Conjecture* 6 *holds for* $n = 2$. *Let* $\Gamma \in CH^2(X \times S)$ *be a* 0*-correspondence, where* X *is a smooth projective threefold and* S *is a smooth projective surface. Then for any very ample smooth surface*

$$Y \overset{j}{\hookrightarrow} X,$$

we have $CH_0(X) = j_*CH_0(Y) + Ker\,\Gamma_*$, *or equivalently*

$$Im\,\Gamma_* = Im\,(\Gamma_* \circ j_*).$$

Proof. Let $(Y_t)_{t\in\mathbb{P}^1}$ be a Lefschetz pencil of surfaces, with $Y_0 = Y$. Let $\Gamma_t \in CH^2(Y_t \times S)$ be the restriction of Γ. Then the cycle $\Gamma_t \circ {}^t\Gamma_t \in CH^2(S \times S)$ is constant because t varies in $\mathbb{P}^1$. It follows that $Im\,F^2(\Gamma_t \circ {}^t\Gamma_t)_*$ is constant. If Conjecture 6 is true, this is also equal to $Im\,F^2\Gamma_{t*}$, hence $Im\,F^2\Gamma_{t*}$ is independent of t for smooth Y_t and this easily implies the same for any t. On the other hand, let $C = Y_0 \cap Y_\infty$ be the base curve of the pencil. C is ample in each Y_t, hence by the Lefschetz theorem, we have

$$CH_0(Y_t) = F^2CH_0(Y_t) + k_{t*}CH_0(C),$$

where k_t is the inclusion of C in Y_t. It follows that

$$\Gamma_{t*}(CH_0(Y_t)) = Im\,F^2\Gamma_{t*} + \Gamma_*CH_0(C),$$

which is independent of t. Hence $Im\,\Gamma_*$, which is generated by the $Im\,\Gamma_{t*}$ must be equal to $Im\,\Gamma_{t*}$ for any t. □

Next we have the following two rigidity statements:

Proposition 9. *Assume Conjecture* 6 *holds for surfaces. Let* $\mathcal{S} \overset{\pi}{\to} B$ *be a family of smooth complex projective surfaces parametrized by a smooth connected quasiprojective basis* B. *Let* $\Gamma \in CH^2(\mathcal{S} \times \Sigma)$, *where* Σ *is a smooth complex projective surface. Then*

$$Im\,F^2\Gamma_{b*} \subset F^2CH_0(\Sigma)_{\mathbb{Q}}$$

is a constant subgroup, independent of $b \in B$.

Here of course $\Gamma_b \in CH^2(S_b \times \Sigma)$ is the restriction of Γ to $S_b \times \Sigma$, $S_b = \pi^{-1}(b)$.

Proposition 10. *Assume Conjecture* 6 *holds for surfaces. Let* $\mathcal{S} \overset{\pi}{\to} B$ *be as above, and let* $\Gamma \in CH^2(X \times \mathcal{S})$, *where* X *is any smooth projective complex variety. Then*

$$Ker\,F^2\Gamma_{b*} \subset F^2CH_0(X)_{\mathbb{Q}} = Ker\,alb_X \otimes \mathbb{Q}$$

is a constant subgroup of $F^2CH_0(X)_{\mathbb{Q}}$, *independent of* $b \in B$.

Here $\Gamma_b \in CH^2(X \times S_b)$ is the restriction of Γ to $X \times S_b$, $S_b = \pi^{-1}(b)$.

Proof of Proposition 9. If Conjecture 6 holds, we have

$$Im\ F^2\Gamma_{b*} = Im\ F^2(\Gamma_b \circ {}^t\Gamma_b)_*.$$

It follows that it suffices to prove the result when $\mathcal{S} = \Sigma \times B$ and each cycle $\Gamma_b \in CH^2(\Sigma \times \Sigma)$ is self-adjoint, i.e. satisfies ${}^t\Gamma_b = \Gamma_b$.

Fix now $0 \in B$. Since B is irreducible, each cycle $\Gamma_b - \Gamma_0$ is algebraically equivalent to 0 in $\Sigma \times \Sigma$. So by Theorem 4, some power

$$(\Gamma_b - \Gamma_0)^{\circ N}$$

vanishes in $CH^2(\Sigma \times \Sigma)_{\mathbb{Q}}$, which implies that

$$F^2(\Gamma_b - \Gamma_0)_*^{\circ N} : F^2CH_0(\Sigma)_{\mathbb{Q}} \to F^2CH_0(\Sigma)_{\mathbb{Q}}$$

vanishes. Now since $\Gamma_b - \Gamma_0$ is self-adjoint, Conjecture 6 implies that

$$Im\ F^2(\Gamma_b - \Gamma_0)_* = Im\ F^2(\Gamma_b - \Gamma_0)_*^{\circ 2} = \ldots = Im\ F^2(\Gamma_b - \Gamma_0)_*^{\circ N}.$$

Hence $F^2(\Gamma_b - \Gamma_0)_*$ already vanishes, and a fortiori

$$Im\ F^2\Gamma_{b*} = Im\ F^2\Gamma_{0*}.$$

□

Proof of Proposition 10. We note to begin with that we can assume that X is a surface because, by the Lefschetz theorem, the group

$$Ker\ \left(F^2\Gamma_{b*} : F^2CH_0(X)_{\mathbb{Q}} \to F^2CH_0(S_b)_{\mathbb{Q}}\right)$$

is generated by the groups $Ker\ (F^2\Gamma^Y_{b*} : F^2CH_0(Y)_{\mathbb{Q}} \to F^2CH_0(S_b)_{\mathbb{Q}})$, for all ample surfaces $Y \subset X$, where Γ^Y_b denotes the restriction of Γ_b to $Y \times S_b$. Next by Remark 2, Conjecture 6 for surfaces implies that

$$Ker\ F^2({}^t\Psi \circ \Psi)_* = Ker\ F^2\Psi_*$$

for any correspondence Ψ between surfaces. Hence, if Conjecture 6 is true for surfaces, it suffices to prove the statement in the case where X is a surface, $\mathcal{S} = X \times B$, and $\Gamma_b \in CH^2(X \times X)_{\mathbb{Q}}$ is self-adjoint. (We replace for this the cycle Γ_b by ${}^t\Gamma_b \circ \Gamma_b$.) Now we conclude as before: the cycle $\Gamma_b - \Gamma_0$ is algebraically equivalent to 0 for any $b \in B$, so, by Theorem 4, some power of it is rationally equivalent to 0, hence we have

$$F^2(\Gamma_b - \Gamma_0)_*^{\circ N} = 0 : F^2CH_0(X)_{\mathbb{Q}} \to F^2CH_0(X)_{\mathbb{Q}}.$$

On the other hand, Conjecture 6 implies that $Im\ F^2(\Gamma_b - \Gamma_0)_*^{\circ N} = Im\ F^2(\Gamma_b - \Gamma_0)_*$, so that $F^2(\Gamma_b - \Gamma_0)_*^{\circ N} = 0$ implies $F^2\Gamma_{b*} = F^2\Gamma_{0*}$ and a fortiori

$$Ker\ F^2\Gamma_{b*} = Ker\ F^2\Gamma_{0*}.$$

□

Propositions 8 and 10 say the following about the naïve filtration on $CH_0(X)$, where X is a threefold: recall that

$$F^3_{naïve}CH_0(X) = \cap_\Gamma Ker\,\Gamma_*,$$

where Γ runs over all 0-correspondences between X and a smooth surface. Proposition 8 says that, assuming Conjecture 6, for a given correspondence $\Gamma \in CH^2(X \times S)$, where S is a surface, we have

$$CH_0(X) = j_*CH_0(Y) + Ker\,\Gamma_*,$$

where $Y \stackrel{j}{\to} X$ is a smooth hyperplane section of X. Next, Proposition 10 says that, assuming Conjecture 6, if we have a family of correspondences Γ_b between X and a varying surface S_b parametrized by an irreducible quasiprojective basis B, the kernel of $F^2\Gamma_{b*}$ is constant. But there are countably many such families of correspondences $(\Gamma_{n,b})_{b\in B_n}$ such that any correspondence between X and a surface is one fiber $\Gamma_{n,b}$. Hence it follows from the above that, assuming Conjecture 6, we have

$$F^3_{naïve}CH_0(X) = \cap_n Ker\,\Gamma_{n*},$$

and that for each n, we have

$$CH_0(X) = j_*CH_0(Y) + Ker\,\Gamma_{n*}.$$

Unfortunately we cannot, however, conclude from this that

$$CH_0(X) = j_*CH_0(Y) + \cap_n Ker\,\Gamma_n* = j_*CH_0(Y) + F^3_{naïve}CH_0(X). \quad (3.8)$$

Indeed, $Im\ j_*$ could be a complement of each $Ker\,\Gamma_{n*}$, which can even be assumed to be decreasing, without being a complement of their intersection.

Remark 3. Notice that by Mumford's theorem [6], the equality (3.8) would imply Conjecture 3 for threefolds. Indeed, if $F^3_{naïve}CH_0(X) = 0$, this equality would say that $CH_0(X)$ is supported on a surface, and the generalized Mumford theorem then allows us to conclude that $H^{3,0}(X) = 0$.

We conclude this section by proving that Conjectures 4 and 6 together imply Bloch's conjecture for surfaces.

Theorem 5. *Assume Conjecture* 4 *for codimension* 2 *cycles and Conjecture* 6 *for surfaces hold. Then if* $\Gamma \in CH^2(S \times T)$ *is a correspondence between surfaces such that*

$$[\Gamma]^* : H^{2,0}(T) \to H^{2,0}(S)$$

vanishes, the morphism

$$F^2\Gamma_* : F^2CH_0(S) \to F^2CH_0(T)$$

is equal to 0.

Proof. Using the Chow–Künneth decomposition for surfaces [7] and the Lefschetz theorem on (1, 1)-classes, the assumption on Γ implies that there exists a cycle $\Gamma' \in CH^2(S \times T)$ such that:

1. $[\Gamma'] = 0$ in $H^4(S \times T, \mathbb{Q})$.
2. $F^2\Gamma'_* = F^2\Gamma_* : F^2CH_0(S) \to F^2CH_0(T)$.

So it suffices to show that $F^2\Gamma'_* = 0$. Next, using Conjecture 6, we may replace Γ' by ${}^t\Gamma' \circ \Gamma'$, and hence assume that Γ' is self-adjoint in $CH^2(S \times S)$ and homologous to 0. But if Conjecture 4 holds, $\Gamma'^{\circ N} = 0$ is algebraically equivalent to 0 up to torsion for sufficiently large N. Then by Theorem 4, some multiple of Γ' is equal to 0 in $CH^2(S \times S)_{\mathbb{Q}}$. In particular,

$$(F^2\Gamma'_*)^{\circ N'} = 0$$

for some integer N'. But as already used, if Conjecture 6 is true, the self-adjointness of Γ' now implies that $F^2\Gamma'_* = 0$. □

In conclusion, we just established relations between several conjectures, which does not actually constitute a theorem. However we have split Bloch's conjecture into two parts: Conjecture 4 which might be easier than Bloch's conjecture itself since it concerns a filtration on a countable group, and Conjecture 6 which seems to be a very interesting geometric problem.

References

1. Bloch, S.: Lectures on algebraic cycles. Duke Univ. Math. Ser. IV (1980)
2. Bloch, S., Srinivas, V.: Remarks on correspondences and algebraic cycles. Am. J. Math. **105**, 1235–1253 (1983)
3. Griffiths, Ph.: On the periods of certain rational integrals I, II. Ann. Math. **90**, 460–541 (1969)
4. Jannsen, U.: Equivalence relations on algebraic cycles. In: The arithmetic and geometry of algebraic cycles, pp. 225–260 (Banff, AB, 1998) NATO Sci. Ser. C, Math. Phys. Sci. 548. Dordrecht: Kluwer Acad. Publ. 2000
5. Kleiman, S.: The standard conjectures. In: Motives, Proceedings of Symposia in Pure Mathematics, Volume 55 (1994), Part I, 3–20
6. Mumford, D.: Rational equivalence of zero-cycles on surfaces. J. Math. Kyoto Univ. **9**, 195–204 (1968)
7. Murre, J.: On the motive of an algebraic surface. J. Reine Angew. Math. **409**, 190–204 (1990)
8. Nori, M.: Algebraic cycles and Hodge theoretic connectivity. Invent. Math. **111**, 349–373 (1993)
9. Roitman, A.: The torsion of the group of zero-cycles modulo rational equivalence. Ann. Math. **111**, 553–569 (1980)
10. Saito, S.: Motives and Filtrations on Chow groups. Invent. Math. **125**, 149–196 (1996)
11. Saito, H.: Generalization of Abel's theorem and some finiteness properties of 0-cycles on surfaces. Compos. Math. **84**, 289–332 (1992)
12. Voevodsky, V.: Nilpotence theorem for cycles algebraically equivalent to 0. Int. Math. Res. Not. (1995), 187–190
13. Voisin, C.: Remarks on zero-cycles of self-products of varieties. In: Moduli of vector bundles, Proc. of the Taniguchi symposium on vector bundles, December 1994, ed. by M. Maruyama, pp. 265–285. Decker 1996