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A CORRECTION ON "A CONJECTURE OF CLEMENS ON RATIONAL CURVES ON HYPERSURFACES"

CLAIRE VOISIN

1.

The purpose of this note is to correct a mistake in the proof of the main theorem of [3]:

Theorem 1. *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d. Let $k \leq n-3$; then the following hold:*

i) If $d \geq 2n-1-k$, any k-dimensional subvariety Y of X has a desingularization $\tilde{Y}$ with an effective canonical bundle.

ii) If $d > 2n-1-k$, and Y is as above, the canonical map of $\tilde{Y}$ is generically one to one on its image.

Recall that Ein [1] proved the following:

Theorem 2. *Let $X \subset \mathbb{P}^n$ be a general hypersurface of degree d and $k \leq n-1$. Then the following hold:*

i) If $d \geq 2n-k$, any k-dimensional subvariety Y of X has a desingularization $\tilde{Y}$ with an effective canonical bundle.

ii) If $d > 2n-k$, and Y is as above, the canonical map of $\tilde{Y}$ is generically one to one on its image.

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Ein's theorem follows from the fact that if $\mathcal{X} \subset \mathbb{P}^n \times S^d$ is the universal hypersurface, $S^d = H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, with special smooth fiber X_F, $F \in S^d$, then the bundle $T_{\mathcal{X}}(1)_{|X_F}$ is generated by global sections. Then $\bigwedge^{n-1-k} T_{\mathcal{X}}(n-1-k)_{|X_F}$ is also generated by global sections. On the other hand we have

$$\bigwedge^{n-1-k} T_{\mathcal{X}}(n-1-k)_{|X_F} \cong \Omega_{\mathcal{X}}^{N+k}(n-1-k-d+n+1)_{|X_F},$$

with $N = dim\, S^d$. Hence if $n-1-k-d+n+1 \leq 0$, the bundle $\Omega_{\mathcal{X}}^{N+k}{}_{|X_F}$ is generated by global sections. If we have an étale map $U \to S^d$ and a universal (reduced, irreducible) subscheme $\mathcal{Y} \subset \mathcal{X}_U$ of relative dimension k, with desingularization $\tilde{\mathcal{Y}}$, then we will get by restriction non-zero sections of

$$\Omega_{\tilde{\mathcal{Y}}}^{N+k}{}_{|\tilde{Y}_t} \cong K_{\tilde{Y}_t}.$$

The case of strict inequality follows in the same way.

What we proposed to do in [3] for improving these inequalities was to study sections of the bundle $\bigwedge^2 T_{\mathcal{X}}(1)_{|X_F}$. When $n-1-k \geq 2$, they will provide, by wedge-product with sections of $T_{\mathcal{X}}(1)_{|X_F}$, sections of

$$\bigwedge^{n-1-k} T_{\mathcal{X}}(n-2-k)_{|X_F} \cong \Omega_{\mathcal{X}}^{N+k}(n-2-k-d+n+1)_{|X_F}.$$

So if now $2n-1-k-d \leq 0$, and $\mathcal{Y} \subset \mathcal{X}_U$ is as above, by restriction one can hope to get non-zero sections of

$$\Omega_{\tilde{\mathcal{Y}}}^{N+k}{}_{|\tilde{Y}_t} \cong K_{\tilde{Y}_t},$$

(respectively of $K_{\tilde{Y}_t}(-1)$ if the inequality is strict). We claimed in [3] that for generic F, the space $H^0(\bigwedge^2 T_{\mathcal{X}}(1)_{|X_F})$, viewed as a space of sections of a line bundle on the Grassmannian of codimension two subspaces of $T_{\mathcal{X}|X_F}$ has no base points on the set of $Gl(n+1)$ invariant codimension two subspaces of $T_{\mathcal{X}|X_F}$, i.e., subspaces $V \subset T_{\mathcal{X},(x,F)}$ containing the tangent space to the $Gl(n+1)$-orbit of (x, F), where $Gl(n+1)$ acts in the natural way on $\mathcal{X} \subset \mathbb{P}^n \times S^d$.

However this statement is false, as was pointed out to me by K. Amerik, whom I thank very much for her observation. Her counterexample is the following : assume that $n+1 \leq d \leq 2n-3$, so that the variety of lines in generic X_F is non-empty of dimension $2n-3-d$, and the subvariety $P_{X_F} \subset X_F$ covered by the lines is of dimension

$k = 2n - 2 - d \leq n - 3$. We have a corresponding universal subvariety $\mathcal{P} \subset \mathcal{X}$ of relative dimension k, which is obviously $Gl(n+1)$-invariant. If the statement were true, since $T_{\mathcal{X}}(1)_{|X_F}$ is globally generated, there would be sections of

$$\bigwedge^{n-1-k} T_{\mathcal{X}}(n-2-k)_{|X_F} \cong \Omega_{\mathcal{X}}^{N+k}(1)_{|X_F},$$

which do not vanish by restriction in

$$H^0(\Omega_{\tilde{\mathcal{P}}}^{N+k}(1)_{|\tilde{P}_F}) \cong H^0(K_{\tilde{P}_F}(1)),$$

and this is absurd since $\tilde{P}_F$ is covered by lines.

In fact, there are other counterexamples, in any degree $d \geq n+2$, showing that the base locus of $H^0(\bigwedge^2 T_{\mathcal{X}}(1)_{|X_F})$ is somewhat large : choose an integer r such that $1 \leq 2n-2-(d-r) \leq n-3$, and positive integers $l_1, \ldots, l_r$ such that $\sum_i l_i = d$. For generic X, the subvariety $P_{l_1,\ldots,l_r,X}$ of X made of points x such that there exists a line $\Delta \subset \mathbb{P}^n$, with $\Delta \cap X = l_1 x + l_2 x_2 + \ldots + l_r x_r$, $x_2, \ldots, x_r \in X$, is of dimension $k = 2n-2-(d-r)$. Let $\mathcal{P}_{l_1,\ldots,l_r} \stackrel{j}{\hookrightarrow} \mathcal{X}$ be the corresponding universal subvariety, and

$$\tilde{\mathcal{P}}_{l_1,\ldots,l_r} \stackrel{\tau}{\to} \mathcal{P}_{l_1,\ldots,l_r}$$

be a desingularization. If the statement were true, there would be for generic F a section σ of

$$\bigwedge^{n-1-k} T_{\mathcal{X}}(n-2-k)_{|X_F} \cong \Omega_{\mathcal{X}}^{N+k}(-r+1)_{|X_F},$$

which does not vanish by restriction in

$$H^0(\Omega_{\tilde{\mathcal{P}}}^{N+k}(-r+1)_{|\tilde{P}_F}) \cong H^0(K_{\tilde{P}_F}(-r+1)).$$

This is absurd for the following reason: the points $x_2, \ldots, x_r$ give a correspondence from $P_{l_1,\ldots,l_r,X}$ to X; that is a generically finite smooth cover

$$P'_{l_1,\ldots,l_r,X} \stackrel{r}{\to} \tilde{P}_{l_1,\ldots,l_r,X}$$

parametrizing the r-uples $(x_1, \ldots, x_r)$ such that

$$\Delta \cap X = l_1 x + l_2 x_2 + \ldots + l_r x_r.$$

Let

$$j_i : P'_{l_1,\ldots,l_r,X} \to X,\ (x_1, \ldots, x_r) \mapsto x_i,$$

so that $j_1 = j \circ \tau \circ r$. Now for any point of $P'_{l_1,\dots,l_r,X}$ the corresponding points x_i of X satisfy the condition $\sum_i l_i x_i \equiv H^{n-1}.X$, where $H = c_1(\mathcal{O}_X(1))$, and $\equiv$ is rational equivalence. Adapting the arguments of [4] to this (higher dimensional) situation, we conclude the following:

Lemma 1. *For any $s \in H^0(\Omega^{N+k}_{\mathcal{X}}{}_{|X_F})$ with $k > 0$, we have*

$$\sum_i l_i j_i^* s = 0, \text{ in } H^0(\Omega^{N+k}_{\mathcal{P}'_{l_1,\dots,l_r}|P'_{l_1,\dots,l_r,X_F}}) \cong H^0(K_{P'_{l_1,\dots,l_r,X_F}}).$$

Applying this to $s = f.\sigma$, where $f \in H^0(\mathcal{O}_X(r-1))$ vanishes at $x_2,\dots,x_r$ but not at x_1, and $j_1^*\sigma$ does not vanish at a point of $P'_{l_1,\dots,l_r,X_F}$ parametrizing $(x_1,\dots,x_r)$, we get a contradiction.

2.

We will correct the proof of Theorem 1 as follows: first of all by Yheorem 2, we have only to study the case $d = 2n-k-1$, $k \le n-3$ in i) and $d = 2n-k$, $k \le n-3$ in ii). What remains true is the following: Assume we have a universal subscheme

$$\mathcal{Y} \subset \mathcal{X}_U$$

of relative dimension k, with desingularization $\tilde{\mathcal{Y}}$, which we may assume to be $Gl(n+1)$-invariant for some lift of the $Gl(n+1)$-action to $\mathcal{X}_U$. Assume in case i) that the restriction map

$$\begin{aligned} H^0(\bigwedge^{n-1-k} T_{\mathcal{X}}(n-2-k)_{|X_F}) \cong & H^0(\Omega^{N+k}_{\mathcal{X}}{}_{|X_F}) \\ & \to H^0(\Omega^{N+k}_{\tilde{\mathcal{Y}}}{}_{|\tilde{Y}_F}) \cong H^0(K_{\tilde{Y}_F}) \end{aligned}$$

vanishes (otherwise $K_{\tilde{Y}_F}$ is effective and we are done). Then for a smooth point (y,F) of $\mathcal{Y}$ the tangent space

$$T_{\mathcal{Y},(y,F)} \subset T_{\mathcal{X}_U,(y,F)}$$

is in the base-locus of $H^0(\bigwedge^{n-1-k} T_{\mathcal{X}}(n-2-k)_{|X_F})$, and since $T_{\mathcal{X}}(1)_{|X_F}$ is globally generated it follows that any codimension-two subspace $V \subset T_{\mathcal{X}_U,(y,F)}$ containing $T_{\mathcal{Y},(y,F)}$ is in the base-locus of $H^0(\bigwedge^2 T_{\mathcal{X}}(1)_{|X_F})$. Similarly, in case ii) assume that the restriction map

$$\begin{aligned} H^0(\bigwedge^{n-1-k} T_{\mathcal{X}}(n-2-k)_{|X_F}) \cong & H^0(\Omega^{N+k}_{\mathcal{X}}(-1)_{|X_F}) \\ & \to H^0(\Omega^{N+k}_{\tilde{\mathcal{Y}}}(-1)_{|\tilde{Y}_F}) \cong H^0(K_{\tilde{Y}_F}(-1)) \end{aligned}$$

vanishes (otherwise $K_{\tilde{Y}_F(-1)}$ is effective and we are done). Then for a smooth point (y, F) of $\mathcal{Y}$, any codimension two subspace $V \subset T_{\mathcal{X}_U,(y,F)}$ containing $T_{\mathcal{Y},(y,F)}$ is in the base-locus of $H^0(\bigwedge^2 T_{\mathcal{X}}(1)_{|X_F})$. Now recall from [3] the following lemma.

Lemma 2. *Let $(x, F) \in \mathcal{X}$, and $V \subset T_{\mathcal{X},(x,F)}$ be a codimension-two subspace which is in the base-locus of $H^0(\bigwedge^2 T_{\mathcal{X}}(1)_{|X_F})$. Then $V \cap S^d_x$ contains the ideal $I_\Delta(d)$ of a line Δ containing x.*

Here $S^d_x = H^0(\mathcal{I}_x(d)) \subset S^d$ is naturally contained in $T_{\mathcal{X},(x,F)}$ as the vertical tangent space of the first projection $pr_1 : \mathcal{X} \to \mathbb{P}^n$. It follows easily from this lemma that under our assumptions, in case i) or ii), the tangent space $T_{\mathcal{Y},(y,F)}$ at a smooth point of $\mathcal{Y}$ has to contain $I_\Delta(d)$ for a line Δ containing x. Clearly Δ is unique, since otherwise $T_{\mathcal{Y},(y,F)}$ would contain S^d_x, and since $pr_{1*} : T_{\mathcal{Y},(y,F)} \to T_{\mathbb{P}^n,y}$ is surjective by $Gl(n+1)$-equivariance, $T_{\mathcal{Y},(y,F)}$ would be equal to $T_{\mathcal{X}_U,(y,F)}$.

Hence under our assumptions, there is a morphism $\phi : \mathcal{Y} \to Grass(1, n)$, such that:

- The line $\Delta_{y,F} = \phi((y, F))$ passes through y.
- The ideal $I_{\Delta_{y,F}}$ is contained in $T_{\mathcal{Y},(y,F)}$ (and more precisely in the vertical tangent space $T^{vert}_{\mathcal{Y},(y,F)}$ with respect to pr_1).

Now we prove

Lemma 3. *The differential ϕ_* of ϕ at (y, F) vanishes on $I_{\Delta_{y,F}} \subset T_{\mathcal{Y},(y,F)}$.*

Proof. The inclusion $I_{\Delta_{y,F}} \subset T_{\mathcal{Y},(y,F)}$ defines a distribution $\mathcal{I} \subset T_{\mathcal{Y}}$, which is in fact contained in the integrable distribution $T^{vert}_{\mathcal{Y}} = Ker\, pr_{1*}$. The bracket induces then a $\mathcal{O}$-linear map

$$\Psi : \bigwedge^2 \mathcal{I} \to T^{vert}_{\mathcal{Y}}/\mathcal{I} \subset T^{vert}_{\mathcal{X}}{}_{|\mathcal{Y}}/\mathcal{I},$$

with fiber at (y, F)

$$\psi : \bigwedge^2 I_{\Delta_{y,F}} \to H^0(\mathcal{O}_{\Delta_{y,F}}(d)(-y)),$$

such that $Im\psi \subset T^{vert}_{\mathcal{Y},(y,F)}\, mod.\, I_{\Delta_{y,F}}$.

Now note that since $y \in \Delta_{(y,F)}$, $\phi_*(T^{vert}_{\mathcal{Y},(y,F)})$ is contained in $H^0(N_{\Delta_{(y,F)}/\mathbb{P}^n}(-y))$. In the sequel we will denote $\Delta_{y,F}$ by Δ. Recall that there is a natural bilinear map that we will denote by $(a, b) \mapsto a \cdot b$:

$$I_\Delta \otimes H^0(N_{\Delta/\mathbb{P}^n}(-y)) \to H^0(\mathcal{O}_\Delta(d)(-y)).$$

It is easy to see that ψ is described by

$$\psi(A\wedge B)=A\cdot\phi_*(B)-B\cdot\phi_*(A),\ A,\,B\in I_{\Delta_{y,F}}.$$

In particular, assume that $A\in I^2_\Delta$ satisfies $\phi_*(A)\neq 0$; then $T^{vert}_{\mathcal{Y},(y,F)}\,mod.\,I_\Delta$ would contain the elements $B\cdot\phi_*(A)$ for any $B\in I_\Delta$, and would be equal to $H^0(\mathcal{O}_\Delta(d)(-y))$, which is absurd because this would imply that $T^{vert}_{\mathcal{Y},(y,F)}=T^{vert}_{\mathcal{X},(y,F)}$. Hence ϕ_* vanishes on I^2_Δ and gives a map

$$\phi: I_\Delta/I^2_\Delta\to H^0(N_{\Delta/\mathbb{P}^n}(-y)).$$

Denoting by K the $(n-1)$-dimensional vector space $H^0(N_{\Delta/\mathbb{P}^n}(-y))$, we have a natural isomorphism

$$I_\Delta/I^2_{\Delta_{y,F}}\cong H^0(\mathcal{O}_\Delta(d-1))\otimes K^*,$$

such that the bilinear map, used above and factorized by I^2_Δ, is the contraction map

$$H^0(\mathcal{O}_\Delta(d-1))\otimes K^*\otimes K\to H^0(\mathcal{O}_\Delta(d-1)),$$

taken into account the isomorphism

$$H^0(\mathcal{O}_\Delta(d)(-y))\cong H^0(\mathcal{O}_\Delta(d-1)).$$

Hence the resulting map

$$\overline{\psi}:\bigwedge^2(I_\Delta/I^2_\Delta)\to H^0(\mathcal{O}_\Delta(d)(-y))$$

identifies with

$$\bigwedge^2(H^0(\mathcal{O}_\Delta(d-1))\otimes K^*)\to H^0(\mathcal{O}_\Delta(d-1)),$$

$$A\wedge B\mapsto\, < A,\phi(B)> - < B,\phi(A)> .$$

Finally we use

Lemma 4. *Let $\phi: W\otimes K^*\to K$ be a linear map. If $\phi\neq 0$, then the map*

$$\overline{\psi}:\bigwedge^2(W\otimes K^*)\to W,$$

$$A\wedge B\mapsto\, < A,\phi(B)> - < B,\phi(A)>$$

has at least a hyperplane of W for image .

Proof. Let $L = Ker\,\phi$, $I = Im\,\phi$ and $G = Im\,\overline{\psi}$; then G contains the elements $< A, B >$ for $A \in L$, $B \in I$. It follows that L is contained in $G \otimes K^* + W \otimes I^{\perp}$, so that we have

$$rk\,\phi \geq dim\,(W/G) \otimes (K^*/I^{\perp}) = (dim\,W/G) rk\,\phi.$$

Hence if $rk\,\phi > 0$, then $dim\,W/G \leq 1$. q.e.d.

Applying this to $W = H^0(\mathcal{O}_{\Delta}(d-1))$, we conclude that if $\phi_* \neq 0$, the image of ψ contains at least a hyperplane in $H^0(\mathcal{O}_{\Delta}(d)(-y))$, so that $T^{vert}_{\mathcal{Y},(y,F)} \subset T^{vert}_{\mathcal{X},(y,F)}$ is at least a hyperplane, which contradicts the fact that the codimension of $\mathcal{Y}$ in $\mathcal{X}$ is at least 2. Hence Lemma 3 is proved.

q.e.d.

From Lemma 3 we conclude that under our assumptions the following hold: for $(y, F) \in \mathcal{Y}$, we have $y \times F + I_{\Delta_{y,F}} \subset \mathcal{Y}$ and $\Delta_{y,G}$ is independent of $G \in F + I_{\Delta_{y,F}}$. Indeed, from the fact that ϕ_* vanishes on $I_{\Delta_{y,F}}$, one concludes that the distribution $\mathcal{I}$ is integrable, and since ϕ is constant along the leaves of the corresponding foliation, the leaves must be the affine spaces $y \times F + I_{\Delta_{y,F}}$.

Now the codimension of $T^{vert}_{\mathcal{Y},y}$ in $S^d_y = T^{vert}_{\mathcal{X},y}$ is equal to the codimension of $\mathcal{Y}$ in $\mathcal{X}$, that is $n-k-1$. Thus the image of the restriction map

$$T^{vert}_{\mathcal{Y},(y,F)} \rightarrow H^0(\mathcal{O}_{\Delta}(d)(-y))$$

has also codimension $n-k-1$, and therefore has dimension $d-n+k+1$ which is equal to $n \leq d-2$ in case i) and to $n+1 \leq d-2$ in case ii). But recall that $\mathcal{Y}$ is invariant under $Gl(n+1)$ so that $T^{vert}_{\mathcal{Y},(y,F)}$ contains the elements of $T_{S^d} \oplus T_{\mathbb{P}^n,y}$ tangent to the orbit of (y, F) and projecting to 0 in $T_{\mathbb{P}^n,y}$, that is the element $F \in S^d_y$ and $I_y J^{d-1}_F$. Finally we may assume that F is generic in the affine space $F + I_{\Delta_{y,F}}$ so that if $X_0, \ldots, X_n$ are the coordinates in $\mathbb{P}^n$ with $X_i(y) = 0$, $i \geq 1$ and $X_{i|\Delta_{y,F}} = 0$, $i \geq 2$, then the elements $X_1 \frac{\partial F}{\partial X_i}_{|\Delta_{y,F}}$, $i \geq 2$, are generic and in particular independent modulo the space generated by $F_{|\Delta_{y,F}}$, $X_1 \frac{\partial F}{\partial X_0}_{|\Delta_{y,F}}$, $X_1 \frac{\partial F}{\partial X_1}_{|\Delta_{y,F}}$, which depends only on $F_{|\Delta_{y,F}}$.

The conditions $dim\,< F, I_y J^{d-1}_F >_{|\Delta_{y,F}} \leq n$ in case i), and $dim\,< F, I_y J^{d-1}_F >_{|\Delta_{y,F}} \leq n+1$ in case ii) imply now that

$$dim\,< F_{|\Delta_{y,F}}, X_1 \frac{\partial F}{\partial X_0}_{|\Delta_{y,F}}, X_1 \frac{\partial F}{\partial X_1}_{|\Delta_{y,F}} > \leq 1 \textit{ in case i)},$$

$$dim < F_{|\Delta_{y,F}}, X_1\frac{\partial F}{\partial X_0}_{|\Delta_{y,F}}, X_1\frac{\partial F}{\partial X_1}_{|\Delta_{y,F}} > \leq 2 \ in\ case\ ii).$$

Thus $F_{|\Delta_{y,F}} = \alpha X_1^d$ in case i), and $F_{|\Delta_{y,F}} = X_1^l Z^{d-l}$ in case ii), for some linear form Z on $\Delta_{y,F}$ and some $l \geq 1$ which obviously will be independent of $(y,F) \in \mathcal{Y}$. Comparing dimensions we see that in case i),Y_F has to be a component of the variety $P_{d,F} \subset X_F$ made of points through which passes a line osculating X_F to order d, while in case ii) Y_F has to be a component of the variety $P_{l,d-l,F} \subset X_F$ made of points x through which passes a line Δ with $\Delta \cap X_F = lx + (d-l)x'$. Note that by the arguments explained in Section 1 the corresponding varieties $\mathcal{P}_d$,(resp. $\mathcal{P}_{l,d-l}$) of $\mathcal{X}$ actually satisfy the condition that the restriction map

$$H^0(\Omega^{N+k}_{\mathcal{X}}{}_{|X_F}) \to H^0(\Omega^{N+k}_{\tilde{\mathcal{P}}_d}{}_{|\tilde{P}_{d,F}})$$

vanishes, (resp. the restriction map

$$H^0(\Omega^{N+k}_{\mathcal{X}}(-1)_{|X_F}) \to H^0(\Omega^{N+k}_{\tilde{\mathcal{P}}_{l,d-l}}(-1)_{|\tilde{P}_{l,d-l,F}})$$

vanishes).

So to finish the proof of Theorem 1, it suffices to show

Proposition 1. *Assume $n-3 \geq k_d = 2n-1-d \geq 0$ (for case i) or $n-3 \geq k_{l,d-l} = 2n-d \geq 0$ (for case ii); then for generic F, the k_d-dimensional variety $P_{d,F}$ admits a desingularization $\tilde{P}_{d,F}$, the canonical map of which is generically one to one on its image. Similarly the $k_{l,d-l}$-dimensional variety $P_{l,d-l,F}$ admits a desingularization $\tilde{P}_{l,d-l,F}$, the canonical map of which is generically one to one on its image.*

Let $G \subset \mathbb{P}^n \times Grass(1,n)$ be the set $\{(x,\Delta)/x \in \Delta\}$, and let $\mathbb{P} \stackrel{\pi}{\to} G$ be the pull-back of the universal $\mathbb{P}^1$ bundle on $Grass(1,n)$. Then there is a natural section τ of π given by $\tau(x,\Delta) = x \in \Delta$, and a corresponding line subbundle $\mathcal{L}$ of the bundle $\mathcal{E}_d = \pi_*\mathcal{O}(d)$, with fiber at (x,Δ) the set of polynomials of degree d on Δ vanishing to order d at x. Let $\mathcal{F}_d = \mathcal{E}_d/\mathcal{L}$. Now let F be a section of $\mathcal{O}_{\mathbb{P}^n}(d)$; then there is an induced section σ_F of $\mathcal{F}_d$, and by definition $P_{d,F}$ is the image by the first projection of $V(\sigma_F)$. Since $\mathcal{F}_d$ is generated by the sections σ_F, $V(\sigma_F)$ is smooth of the right dimension for generic F, and one verifies that $pr_1 : V(\sigma_F) \to P_{d,F}$ is a desingularization (one uses here the inequality $n-3 \geq k_d = 2n-1-d \geq 0$).

Similarly, to desingularize $P_{l,d-l,F}$, let Y be the blow-up of $\mathbb{P}^n \times \mathbb{P}^n$ along the diagonal. There is a natural map

$$f : Y \to Grass(1,n),\ (x,y) \mapsto < x,y >,$$

and there are two sections

$$\tau_1,\ \tau_2,\ \tau_1((x,y)) = x \in < x,y >,\ \tau_2((x,y)) = y \in < x,y >$$

of the induced $\mathbb{P}^1$ bundle $\mathbb{P} \xrightarrow{\pi} Y$ on Y. There is then a line subbundle $\mathcal{L}$ of the bundle $\mathcal{E}_d = \pi_*\mathcal{O}_{\mathbb{P}}(d)$, with fiber at (x,y) the set of polynomials f of degree d on Δ vanishing to order l at x and to order $d-l$ at y (when $x = y$, f should vanish to order d at x). Let $\mathcal{F}_d = \mathcal{E}_d/\mathcal{L}$. Now let F be a section of $\mathcal{O}_{\mathbb{P}^n}(d)$; there is an induced section σ_F of $\mathcal{F}_d$, and by definition $P_{l,d-l,F}$ is the image by the first projection of $V(\sigma_F)$. Since $\mathcal{F}_d$ is generated by the sections σ_F, $V(\sigma_F)$ is smooth of the right dimension for generic F and one verifies that $pr_1 : V(\sigma_F) \to P_{l,d-l,F}$ is a desingularization (one uses here the inequality $n-3 \geq k_{l,d-l} = 2n-d \geq 0$).

In both cases it suffices to show that the canonical map of $V(\sigma_F)$ is of degree one on its image.

In the case of $P_{d,F}$ the canonical bundle of $V(\sigma_F)$ is equal to $K_G + c_1(\mathcal{F}_d)$. Now note that G is the universal $\mathbb{P}^1$-bundle on $Grass(1,n)$, via pr_2 so that $Pic\,G$ is generated by $H = pr_1^*(\mathcal{O}_{\mathbb{P}^n}(1))$ and $L = pr_2^*(\mathcal{O}_{Grass}(1))$. It is easy to show that $K_G = -2H - nL$.

Next $\mathcal{E}_d$ is the pull-back via pr_2 of the corresponding bundle over $Grass(1,n)$, hence has determinant equal to $\frac{d(d+1)}{2}L$. Finally the natural section of $\mathbb{P} \xrightarrow{\pi} G$ is simply given by the evaluation map $\pi_*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{E}_1 \to \tau^*(\mathcal{O}_{\mathbb{P}}(1))$, and since $\tau^*(\mathcal{O}_{\mathbb{P}}(1)) = H$, its kernel $\mathcal{L}_1$ is of class $L - H$. Clearly $\mathcal{L} \cong \mathcal{L}_1^{\otimes d}$, hence $\mathcal{L}$ is of class $d(L-H)$. So we have

$$\begin{aligned} K_{V(\sigma_F)} &= K_G + c_1(\mathcal{F}_d) \\ &= -2H - nL + \frac{d(d+1)}{2}L - d(L-H) \\ &= (d-2)H + (\frac{d(d-1)}{2} - n)L. \end{aligned}$$

Since $n - 3 \geq 2n - 1 - d \geq 0$, we have $n \geq 3$, $d \geq n+2 \geq 5$, hence $d - 2 > 0$, $\frac{1}{2}d(d-1) - n > 0$, which implies that $K_{V(\sigma_F)}$ is very ample.

In the case of $P_{l,d-l}$, $f : Y \to Grass(1,n)$ identifies Y with the self-product of the tautological $\mathbb{P}^1$-bundle on $Grass(1,n)$, hence its Picard group is generated by $H_1 = pr_1^*(\mathcal{O}_{\mathbb{P}^n}(1))$, $H_2 = pr_2^*(\mathcal{O}_{\mathbb{P}^n}(1))$, and $L = f^*(\mathcal{O}_{Grass}(1))$. One computes easily that $K_Y = -2H_1 - 2H_2 + (-n+1)L$.

Next the two sections τ_1, τ_2 correspond to the evaluation maps

$$\mathcal{E}_1 \to \tau_1^*(\mathcal{O}_{\mathbb{P}}(1)),\ \mathcal{E}_1 \to \tau_2^*(\mathcal{O}_{\mathbb{P}}(1)),$$

with $\tau_1^*(\mathcal{O}_{\mathbb{P}}(1)) = H_1$, and $\tau_2^*(\mathcal{O}_{\mathbb{P}}(1)) = H_2$, so their kernels $\mathcal{L}_1$, $\mathcal{L}_2$ have for class $L - H_1$ and $L - H_2$ respectively. Clearly $\mathcal{L} \cong \mathcal{L}_1^{\otimes l} \otimes \mathcal{L}_1^{\otimes d-l}$, and hence is of class $l(L - H_1) + (d - l)(L - H_2$. Thus

$$\begin{aligned} K_{V(\sigma_F)} =& K_Y + c_1(\mathcal{F}_d) \\ =& -2H_1 - 2H_2 + (-n+1)L \\ &+ \frac{d(d+1)}{2}L - dL + lH_1 + (d-l)H_2. \end{aligned}$$

So if $l \geq 2$, and $d - l \geq 2$, we conclude easily that the canonical map of $V(\sigma_F)$ is of degree one on its image.

If $l = 1$ or $d - l = 1$, say $d - l = 1$ for example, we construct another desingularization of $P_{l,d-l}$ as follows: Let as above $G \subset \mathbb{P}^n \times Grass(1, n)$ be the set $\{(x, \Delta)/x \in \Delta\}$. Let $\mathbb{P} \xrightarrow{\pi} G$ be the pull-back of the universal $\mathbb{P}^1$ bundle on $Grass(1, n)$, and τ be the natural section of π. There is a natural rank-two subbundle $\mathcal{K}$ of $\mathcal{E}_d$, whose fiber at (x, Δ) is the set of polynomials of degree d on Δ vanishing to order $d - 1$ at x. In fact, if $\mathcal{L}_1$ is as above the kernel of the evaluation map

$$\mathcal{E}_1 \to \tau^*\mathcal{O}_{\mathbb{P}}(1) = H,$$

$\mathcal{K}$ is isomorphic to $\mathcal{L}_1^{\otimes d-1} \otimes \mathcal{E}_1$.

Now if F is a section of $\mathcal{O}_{\mathbb{P}^n}(d)$, there is an induced section σ_F of $\mathcal{F} = \mathcal{E}_d/\mathcal{K}$, and by definition $P_{d-1,1,F}$ is the image by the first projection of $V(\sigma_F)$. Since $\mathcal{F}$ is generated by the sections σ_F, $V(\sigma_F)$ is smooth of the right dimension for generic F, and one verifies that $pr_1 : V(\sigma_F) \to P_{d-1,1,F}$ is a desingularization. We have then

$$\begin{aligned} K_{V(\sigma_F)} =& K_G + c_1(\mathcal{F}) \\ =& -2H - nL + \frac{d(d+1)}{2}L - 2(d-1)c_1(\mathcal{L}_1) - c_1(\mathcal{E}_1) \\ =& (2d-4)H + (\frac{d(d+1)}{2} - n - 1 - 2(d-1))L. \end{aligned}$$

Using the inequalities $d \geq n + 3 \geq 6$, we immediately see that $K_{V(\sigma_F)}$ is very ample. So Proposition 1 is proved. q.e.d.

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Université Pierre et Marie Curie
4, Place Jussien, Paris, France